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THE PRINCIPLE OF LIMITING AMPLITUDE

By

Kôji KUBOTA and Taira SHIROTA

1. Introduction.

We shall be concerned with the initial value problem

\[
\left[ \frac{\partial^2}{\partial t^2} - \Delta + q(x) \right] u(x, t) = f_2(x)e^{-i\omega t} \quad (t>0),
\]

\[
u(x, 0) = f_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = f_1(x),
\]

where \( x \) is a point of 3-dimensional Euclidean space \( R^3 = E \), \( \Delta \) denotes the Laplace operator in \( E \), \( \sigma \) is a positive number, and \( q(x) \) is a real-valued function defined on \( E \). The principle of limiting amplitude for the problem (1.1) consists in the following: The solution \( u(x, t) \) of the problem (1.1) is such that at any point \( x \in E \) we have

\[
\lim_{t \to \infty} u(x, t)e^{i\omega t} = u_+(x, \sigma) \quad (\sigma>0),
\]

where \( u_+(x, \sigma) \) is the solution of the equation \( (-\Delta + q)u = \sigma u + f_2 \) with the Sommerfeld radiation condition:

\[
u_+(x, \sigma) = 0(|x|^{-1}), \quad \frac{\partial u_+}{\partial |x|}(x, \sigma) - i\sqrt{\sigma} u_+(x, \sigma) = 0(|x|^{-1})
\]
at infinity. In [1] O. A. Ladyzhenskaya gave this proof for the case that \( q \) and \( f_2 \) have compact supports, \( f_0 = f_1 = 0 \), and the operator \( A \) has no eigenvalue (For the definition see §2). But in this case it should be assumed, in addition, that there exists no solution \( \omega \in L^2(E) \) of the equation \( (-\Delta + q)\omega = 0 \) satisfying the conditions \( \omega = 0(|x|^{-1}) \), \( \frac{\partial \omega}{\partial x_j} = 0(|x|^{-2}) \) at infinity, although it is not assumed there. In [2] D. M. Ŕidus gave this proof for the case that \( q \in C^1(E) \), \( q = 0(|x|^{-2-\alpha}) (\alpha > \frac{1}{6}) \) at infinity, and \( f_0 \in \mathfrak{D}(A) \), \( f_1 \in \mathfrak{D}(A^1) \), \( f_2 \in L^2(E) \), \( f_j = 0 \) \((|x|^{-3-r}) (r > 0) (j = 0, 1, 2) \) at infinity, assuming that \( q(x) \geq 0 \) in \( E \), which guarantee the absence of not only eigenvalues of the operator \( A \), but also the above-mentioned functions \( \omega \). In [3] K. Asano and T. Shirota constructed such a function \( \omega \) for the case that \( q \in C_0^\infty(E) \) and the inequality
does not hold in $E$. In [5], by using such a function $\omega$ the present authors constructed a solution $u(x, t)$ of (1.1) with $q \in C_0^2(E)$, $f_0 \in C_0^2(E)$, $f_1 \in C_0^1(E)$ and $f_2 = 0$ such that

$$u(x, t) = \omega(x) \quad \text{for} \quad |x| \leq t.$$  

The so constructed solution $u(x, t)$ of (1.1) is special but interesting to us, because that it give a relation of the principle of limiting amplitude and characteristics. They also remarked in [5] that, in the case that $q$ and $f_j$ ($j = 0, 1, 2$) have compact supports and the operator $A$ has no eigenvalue, the principle of limiting amplitude for (1.1) is valid if and only if the before-mentioned functions $\omega$ do not exist. Therefore it seems natural for us to conjecture the following: The principle of limiting amplitude for (1.1) with the condition $q = 0(|x|^{-2-\alpha})$ ($\alpha > 0$) ($|x| \to \infty$) is valid if and only if the operator $A$ has no eigenvalue and the above-mentioned functions $\omega$ do not exist.

In the present paper we establish that the above conjecture is true for $q = 0(|x|^{-3-\alpha})$ ($|x| \to \infty$) with some differentiability conditions. In particular the condition $q = 0(|x|^{-3-\alpha})$ ($\alpha > 0$) ($|x| \to \infty$) is used in proving the equality (5.5), Lemma 8, Lemma 10, and the inequality (5.20). The main results of this paper were presented in our note [5].

### 2. Results.

For simplicity and clearness we restrict ourselves in what follows to the problem

$$(2.1) \quad \left[ \frac{\partial^2}{\partial t^2} - \Delta + q(x) \right] u(x, t) = 0 \quad (t > 0),$$

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = f(x).$$

Throughout the present paper $q(x)$ and $f(x)$ are assumed to satisfy the following conditions $(C_1)$, $(C_2)$ and $(C_3)$:

$(C_1)$ $q(x)$ is a locally H"older continuous real-valued function defined on $E$ and behaves like $0(|x|^{-2-\alpha})$ ($\alpha > 0$) at infinity, i.e. there exist positive numbers $\alpha, C_0, R_0$ such that

$$|q(x)| \leq C_0 |x|^{-2-\alpha} \quad \text{for} \quad |x| \geq R_0.$$  

Under the condition $(C_1)$ the operator $-\Delta + q$ defined on $C_0^\infty(E)$ has the unique self-adjoint extension in $L^2(E)$, which we denote by $A$. It is known that the domain $\mathcal{D}(A)$ of $A$ is the Sobolev space $W_2^2(E)$. 

(C$_2$) $A$ has no eigenvalue. 
Under the condition (C$_2$) $A$ is positive definite. By $\mathfrak{D}(A^{\frac{1}{2}})$ we denote the domain of the self-adjoint operator $A^{\frac{1}{2}}$.

(C$_3$) $f$ belongs to $\mathfrak{D}(A^{\frac{1}{2}})$ and behaves like $0(|x|^{-3-\gamma})$ ($\gamma > 0$) at infinity.
In what follows we denote $\int_E \varphi(x)\psi(x)dx$ by $\langle \varphi, \psi \rangle$.
We have the following.

**Theorem 1.** Assume that $\langle f, \omega \rangle = 0$ for all solutions $\omega \in L^2(E)$ of the equation $(-\Delta + q)\omega = 0$ such that $\omega = 0 (|x|^{-1})$, $\frac{\partial \omega}{\partial x_j} = 0 (|x|^{-2})$ (|x|→∞). Then the solution $u(t) = u(x, t)$ of the problem (2.1) is such that we have

\[ (2.2) \quad \lim_{t \to \infty} \langle u(t), \varphi \rangle_{L^2(E)} = 0 \quad \text{for all } \varphi \in L^2(E), \]

and

\[ (2.3) \quad \lim_{t \to \infty} \|u(t)\|_{L^2(K)} = 0 \quad \text{for all compact } K \subset E. \]

Furthermore, suppose that $f \in \mathfrak{D}(A)$ and $\sup_{x \in K} |Vq(x)| < \infty$, where $Vq(x)$ denotes grad $q(x)$. Then we have

\[ (2.4) \quad \lim_{t \to \infty} \sup_{x \in K} |u(x, t)| = 0 \quad \text{for all compact } K \subset E. \]

The main result is the following.

**Theorem 2.** Assume that $q \in C^2(E)$ and $q = 0 (|x|^{-3-\gamma})$, $D^\beta q = 0 (|x|^{-2-\delta})$ (|x|→∞) ($|\beta| = 1, 2$). Then the solution of (2.1) is such that for any $\varphi \in L^2(E)$ satisfying $\varphi = 0 (|x|^{-1-\delta})$ ($\delta > \frac{1}{2}$) (|x|→∞) we have

\[ (2.5) \quad \lim_{t \to \infty} \langle u(t), \varphi \rangle = 4\pi \langle \varphi, \omega \rangle \cdot \langle f, \omega \rangle \cdot \langle q, \omega \rangle^{-2}, \]

where $\omega$ is the one mentioned in Theorem 1.

By virtue of Theorem 2 and theorem 6 in [2] we have

**Corollary.** Let $q(x)$ be as in Theorem 2. Then the solution $u(x, t)$ of (2.1) is such that we have

\[ \lim_{t \to \infty} \sup_{x \in K} |u(x, t)| = 0 \quad \text{for all compact } K \subset E \]

if and only if such functions $\omega$ as in Theorem 1 do not exist.

3. **Proof of Theorem 1.**

Since we can’t find the Theorem in the literatures, for the sake of the
completeness of description we shall give the proof.

Let us define an operator $T$ for functions in $L^6(E)$ by

$$T\varphi(x) = -\frac{1}{4\pi} \int_E \frac{q(y)\varphi(y)}{|x-y|} dy$$

($\varphi \in L^6(E)$).

**Lemma 1.** 1) $T$ is a compact operator on $L^6(E)$ to $L^6(E)$ and the adjoint operator $T^*$ of $T$ with respect to the bilinear form $\langle , \rangle$ is a compact operator on $L^\frac{6}{\alpha}(E)$ to $L^\frac{6}{\alpha}(E)$ given as follows:

$$T^*\phi(x) = -\frac{1}{4\pi} q(x) \int_E \frac{\phi(y)}{|x-y|} dy$$

($\phi \in L^\frac{6}{\alpha}(E)$).

2) By $M, M'$ we denote the subspaces $\{\omega \in L^6(E); (I-T)\omega = 0\}$, $\{\omega' \in L^\frac{6}{\alpha}(E); (I-T^*)\omega' = 0\}$ of $L^6(E)$, $L^\frac{6}{\alpha}(E)$ respectively, where $I$ denotes the identity operator. Then we have

$$\omega = 0 (|x|^{-1}), \quad \frac{\partial \omega}{\partial x_j} = 0 (|x|^{-2}) \ (|x| \to \infty) \quad \text{for } \omega \in M,$$

and

$$\omega' = 0 (|x|^{-3-\alpha}) \ (|x| \to \infty) \quad \text{for } \omega' \in M'.$$

**Proof.** First we shall show that $T$ is a bounded operator on $L^6(E)$ to $L^6(E)$. In what follows we denote by $C$ constants independent of $x \in E$. we have

$$\int_E \frac{|q(y)\varphi(y)|}{|x-y|} dy \leq \|\varphi\|_L \left[ \int_E \left( \frac{|q(y)|}{|x-y|} \right)^\frac{6}{\alpha} dy \right]^\frac{\alpha}{6}. $$

$$\int_E \left( \frac{|q(y)|}{|x-y|} \right)^\frac{6}{\alpha} dy \text{ can be estimated as follows:}$$

$$\int_E \left( \frac{|q(y)|}{|x-y|} \right)^\frac{6}{\alpha} dy \leq \sup_{|y| \leq R_0} |q(y)| \int_{|y| \leq R_0} \frac{1}{|x-y|^\alpha} dy + C_0 \int_{|y| > R_0} \left( \frac{1}{|x-y||y|^{2+\alpha}} \right)^\alpha dy$$

$$\leq C(1 + |x|)^{-\frac{6}{\alpha}} + \rho_\alpha(x),$$

where

$$|\rho_\alpha(x)| \leq C(1 + |x|)^{-\frac{3-\alpha}{\alpha}} \quad \text{for } 0 < \alpha < \frac{1}{2},$$

$$\leq C(1 + |x|)^{-\frac{1}{2}} \quad \text{for } \alpha = \frac{1}{2},$$

$$\leq C(1 + |x|)^{-\frac{3}{2}} \quad \text{for } \alpha > \frac{1}{2},$$

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and $\epsilon$ is an arbitrary positive number, so that for any $\alpha > 0 [\rho_\alpha(x)]^\beta \in L^6(E)$. These estimates imply that

\begin{equation}
|T\varphi(x)| \leq C\|\varphi\|_{L^6} \left[(1+|x|)^{-1} + (\rho_\alpha(x))^\beta \right],
\end{equation}

and

\begin{equation}
\|T\varphi\|_{L^6} \leq C\|\varphi\|_{L^6},
\end{equation}

which shows the boundedness of $T$.

Next we shall show that $T$ is a compact operator on $L^6(E)$ to $L^6(E)$. The inequality (3.2) implies that $\|T\varphi\|_{L^6}$ are uniformly bounded with respect to $\varphi$ satisfying $\|\varphi\|_{L^6} \leq 1$. On the other hand an argument similar to the one used in deriving (3.2) gives

\[ \left\| \frac{\partial}{\partial x_j} T\varphi \right\|_{L^6} \leq C\|\varphi\|_{L^6} \quad (j=1,2,3), \]

which imply that $T\varphi$ are uniformly equi-continuous in $L^6(E)$ with respect to $\varphi \in L^6(E)$ satisfying $\|\varphi\|_{L^6} \leq 1$. Thus $T$ is compact. From this it follows by the Riesz-Schauder theory that $T^*$ is also a compact operator on $L^6(E)$ to $L^6(E)$ and that $\dim M = \dim M' < \infty$.

For $\varphi \in L^6(E)$ and $\psi \in L^6(E)$, since

\[ \int_{E \times E} \frac{|q(y)\varphi(y)\psi(x)|}{|x-y|} dxdy \leq \|\varphi\|_{L^6} \int_{E} |\psi(x)| \left[ \int_{E} \left( \frac{|q(y)|}{|x-y|} \right)^6 dy \right]^\frac{6}{\beta} dx \]

\[ \leq C\|\varphi\|_{L^6}\|\psi\|_{L^6}^\frac{6}{\beta}, \]

we have

\[ \langle T^*\psi, \varphi \rangle = \langle \psi, T\varphi \rangle \]

\[ = \int_{E} \varphi(x) \left[ -\frac{1}{4\pi} \int_{E} \frac{q(y)\varphi(y)}{|x-y|} dy \right] dx \]

\[ = \int_{E} \varphi(y) \left[ -\frac{1}{4\pi} q(y) \int_{E} \frac{\varphi(x)}{|x-y|} dx \right] dy, \]

which implies that

\[ T^*\varphi(x) = -\frac{1}{4\pi} q(x) \int_{E} \frac{\varphi(y)}{|x-y|} dy \quad \text{for} \quad \varphi \in L^6(E). \]

For $\omega \in M$ we have
It follows from (3.1) and (3.3) that not only $\omega(x)$ is bounded, but also $\omega=0$ ($|x|^{-1}), \frac{\partial \omega}{\partial x_j} = 0$ ($|x|^{-2})$ ($|x|\to\infty$).

For $\omega' \in M'$ we have
\begin{equation}
(3.4) \quad \omega'(x) = -\frac{1}{4\pi} q(x) \int_E \omega'(y) \frac{dy}{|x-y|},
\end{equation}
which implies that
\begin{equation*}
\omega'(x) = \left( \frac{1}{4\pi} \right)^2 q(x) \int_E \frac{q(y)}{|x-y|} \, dy \int_E \frac{\omega'(z)}{|y-z|} \, dz.
\end{equation*}
It follows from the condition $(C_1)$ that
\begin{equation*}
\int_E \frac{|q(y)|}{|x-y||y-z|} \, dy \leq C \frac{1}{1+|z|},
\end{equation*}
where $C$ is a constant independent of $x, z \in E$. This implies that
\begin{equation*}
\int_{E \times E} \frac{|q(y)\omega'(z)|}{|x-y||y-z|} \, dy \, dz \leq C \|\omega'\|_{L^6(E)}.
\end{equation*}
Therefore we obtain
\begin{equation*}
|\omega'(x)| \leq C \|\omega'\|_{L^6} (1+|x|)^{-2-a}.
\end{equation*}
This and (3.4) imply that $\omega'=0$ ($|x|^{-3-a})$ ($|x|\to\infty$).

Thus we have proved Lemma 1.

**Lemma 2.** Suppose that $\varphi \in L^2(E), \varphi=0$ ($|x|^{-2-\delta})$ ($|x|\to\infty), and$
\langle \varphi, \omega \rangle = 0$ for all $\omega \in M$. Then we have $\varphi \in R(A^{\frac{1}{2}})$, where $R(A^{\frac{1}{2}})$ denotes the range of $A^{\frac{1}{2}}$.

**Proof.** Let $\omega' \in M'$ and set $\omega(x) = -\frac{1}{4\pi} \int_E \frac{\omega'(y)}{|x-y|} \, dy$. Then we find by the equalities (3.3) and (3.4) that $q(x) \omega(x) = \omega'(x)$ and $\omega \in M$, because that it follows from Lemma 1 that $\omega'(x)$ is continuous in $E$ and $\omega'=0$ ($|x|^{-3-\delta})$ ($|x|\to\infty$). Hence the assumption implies that $\frac{1}{4\pi} \int_E \varphi(y) \frac{dy}{|x-y|} \in L^6(E)$ and
\begin{equation*}
\langle \varphi, \omega' \rangle = 0 \text{ for all } \omega' \in M'.
\end{equation*}
Therefore by virtue of Lemma
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1 and the Riesz-Schauder theory we find that there exists a solution $u \in L^6(E)$ of the equation

$$(I-T)u = \frac{1}{4\pi} \int_E \frac{\varphi(y)}{|x-y|} dy$$

which implies that $u(x)$ satisfies the equation $(-\Delta + q)u = \varphi$ with the conditions $u = 0$ ($|x|^{-\delta}$), $\frac{\partial u}{\partial x} = 0$ ($|x|^{-1-\delta}$) ($|x| \to \infty$), so that $\Delta u \in L^6(E)$ and $u \in W^2_{2, loc}$. Set $u(x) = u(x)e^{-|x|^2}$ ($\epsilon > 0$). Then we find that $u \in \mathfrak{D}(A)$, sup $\|A^{\frac{1}{2}}u\|_{L^2} < \infty$, where $\epsilon_0$ is a positive number, and $\lim_{\epsilon \to 0} \|Au_\epsilon - \varphi\|_{L^2} = 0$. In fact, $\Delta u \in L^2(E)$ implies that $u_\epsilon \in L^2(E)$ and $\Delta u_\epsilon \in L^2(E)$, from which it follows that $u_\epsilon \in W^2_{2}(E) = \mathfrak{D}(A)$. The conditions $u = 0$ ($|x|^{-\delta}$), $\frac{\partial u}{\partial x} = 0$ ($|x|^{-1-\delta}$) ($|x| \to \infty$) imply that

$$\|\Delta u_\epsilon - \Delta u\|_{L^2} \leq 2\int_E |\Delta u|^2 (1-e^{-|x|^2}) dx + 0(\epsilon^{\delta+1}) + 0(\epsilon^2)$$

($\epsilon \to 0$),

from which it follows that $\|\Delta u_\epsilon - \Delta u\|_{L^2} \to 0$ as $\epsilon \to 0$. Since $qu = 0$ ($|x|^{-2-\alpha-\delta}$) ($|x| \to \infty$), it is clear that $\|qu - qu\|_{L^2} \to 0$ as $\epsilon \to 0$. These show that $\sup_{0<\epsilon \leq \epsilon_0} \|A^{\frac{1}{2}}u\|_{L^2} < \infty$. Noting that $A^{\frac{1}{2}}$ is self-adjoint and $\mathfrak{D}(A) \subset \mathfrak{D}(A^{\frac{1}{2}})$, we have

$$\|A^{\frac{1}{2}}u_\epsilon\|_{L^2} \leq 2\sum_{j=1}^{3} \int_E \left| \frac{\partial u}{\partial x_j} \right|^2 dx + \int_E |q||u|^2 dx + 0(\epsilon^{\delta+1}) + 0(\epsilon^2)$$

($\epsilon \to 0$)

which shows that $\sup_{0<\epsilon \leq \epsilon_0} \|A^{\frac{1}{2}}u_\epsilon\|_{L^2} < \infty$. Thus we have shown that $u_\epsilon \in \mathfrak{D}(A)$, sup $\|A^{\frac{1}{2}}u_\epsilon\|_{L^2} < \infty$ and $\lim_{\epsilon \to 0} \|Au_\epsilon - \varphi\|_{L^2} = 0$ as $\epsilon \to 0$. The fact sup $\|A^{\frac{1}{2}}u_\epsilon\|_{L^2} < \infty$ implies that there exists a function $g \in L^2(E)$ and a sequence $\{\epsilon_j\}_{j=1}^\infty$ such that $\lim_{j \to \infty} \epsilon_j = 0$ and $A^{\frac{1}{2}}u_{\epsilon_j}$ converges to $g$ in $L^2(E)$ weakly. Let $H$ be the convex hull of $\{A^{\frac{1}{2}}u_{\epsilon_j}\}_{j=1}^\infty$. Then it follows from the fact $\lim_{\epsilon \to 0} \|Au_\epsilon - \varphi\|_{L^2} = 0$ that

$$\{g, \varphi\} \text{ belongs to the weak closure of the graph } \{H, A^{\frac{1}{2}}H\}, \text{ so that } \{g, \varphi\} \text{ belongs to the strong closure of } \{H, A^{\frac{1}{2}}H\}. \text{ This shows that } g \in \mathfrak{D}(A^{\frac{1}{2}}) \text{ and } \varphi = A^{\frac{1}{2}}g, \text{ because that } A^{\frac{1}{2}}u \in \mathfrak{D}(A^{\frac{1}{2}}). \text{ Thus we have proved Lemma 2.}

Proof of Theorem 1. Let $E_\epsilon$ be the resolution of the identity generated by the operator $A$ such that $E_{\epsilon_1} = E_\epsilon$. Then, since $f \in \mathfrak{D}(A^{\frac{1}{2}})$ and $A$ has no eigenvalue, the solution $u(t)$ of the initial value problem (2.1) is given by

$$u(t) = \int_0^\infty \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dE_\epsilon f.$$
We decompose \( u(t) \) as follows:

\[
  u(t) = \int_{0}^{N} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dE_{\lambda} f + \int_{N}^{\infty} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dE_{\lambda} f
  \equiv u_{1}(t) + u_{2}(t).
\]

Then, since

\[
  \|u_{2}(t)\|_{L^2}^{2} = \int_{N}^{\infty} \frac{\sin^{2} \sqrt{\lambda} t}{\lambda} d\|E_{\lambda} f\|_{L^2}^{2} 
  \leqq \frac{1}{N} \|f\|_{L^2}^{2}
\]

we can choose \( N \) so large that \( \|u_{2}(t)\|_{L^2} \) becomes sufficiently small uniformly with respect to \( t>0 \). Let \( N \) be fixed sufficiently large. Since it follows from the assumption and Lemma 1 that \( \langle f, \omega \rangle = 0 \) for all \( \omega \in M \), we find by the condition \((C_{3})\) and Lemma 2 that there exists a function \( g \in \mathfrak{D}(A^{1/2}) \) such that

\[
  \langle f, \omega \rangle = 0 \quad \text{for all} \quad \omega \in M.
\]

Therefore we have

\[
  \langle u_{1}(t), \varphi \rangle_{L^2} = \int_{0}^{N} \sin \sqrt{\lambda} t d(E_{\lambda} g, \varphi)
  \quad \text{for all} \quad \varphi \in L^2(E).
\]

On the other hand it follows from condition \((C_{3})\) and theorem 6 in [4] that \( (E_{\lambda} g, \varphi)_{L^2} \) is an absolutely continuous function of \( \lambda \in [0, N] \), so that \( \frac{d}{d\lambda} (E_{\lambda} g, \varphi)_{L^2} \in L^1(0, N) \). Consequently by virtue of Riemann-Lebesgue's theorem we obtain

\[
  \lim_{t \to \infty} \langle u_{1}(t), \varphi \rangle_{L^2} = 0 \quad \text{for all} \quad \varphi \in L^2(E).
\]

This and (3.6) prove (2.2).

Now we proceed to prove (2.3). It follows from the condition \((C_{3})\) that the operator \( A \) is strictly positive definite, i.e. \( A \) satisfies the inequality

\[
  (A \varphi, \varphi) > 0 \quad \text{for} \quad 0 \neq \varphi \in \mathfrak{D}(A).
\]

We define a functional \( (\varphi, \phi)_{1} \) for functions \( \varphi, \phi \in C_{0}^{\infty}(E) \) as follows:

\[
  (\varphi, \phi)_{1} = \int_{E} \left( \nabla \varphi(x) \nabla \phi(x) + q(x) \varphi(x) \phi(x) \right) dx.
\]

Then by virtue of (3.8) we find that \( C_{0}^{\infty}(E) \) is a pre-Hilbert space with the inner product \( (\varphi, \phi)_{1} \), \( (\varphi, \phi)_{C_{0}^{\infty}(E)} \). Let \( H_{1} \) be the completion of \( C_{0}^{\infty}(E) \) for the norm \( \| \cdot \|_{1} = \sqrt{(\cdot, \cdot)_{1}} \) and let \( \mathfrak{H} \) be the Hilbert space \( H_{1} \times L^{2}(E) \) with the inner product
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\[ (\varphi, \psi)_\mathfrak{H} = (\varphi_1, \psi_1) + (\varphi_2, \psi_2)_{L^2(E)} \]

where \( \varphi = [\varphi_1, \varphi_2], \psi = [\psi_1, \psi_2] \) and \( \varphi_1, \psi_1 \in H_1; \varphi_2, \psi_2 \in L^2(E) \).

Let us define an operator \( G_0 \) of \( \mathfrak{H} \) to \( \mathfrak{H} \) as follows:

\[ G_0 = \begin{bmatrix} 0 & L \\ N & 0 \end{bmatrix}, \]

where the operators \( L : L^2(E) \rightarrow H_1 \) and \( N : H_1 \rightarrow L^2(E) \) have domains \( \mathfrak{D}(L) = C^\infty_0(E) = \mathfrak{D}(N) \), and

\[ L\varphi = \varphi \quad \text{for} \quad \varphi \in \mathfrak{D}(L), \]

\[ N\psi = (\Delta - q)\psi \quad \text{for} \quad \psi \in \mathfrak{D}(N). \]

By \( G \) we denote the closure of \( G_0 \) in \( \mathfrak{H} \). Then we find that the operator \( G \) generates a strongly continuous contraction semi-group on \( \mathfrak{H} \). In fact, the equality

\[ (G_0 \varphi, \varphi)_\mathfrak{H} + (\varphi, G_0 \varphi)_\mathfrak{H} = 0 \quad \text{for} \quad \varphi \in \mathfrak{D}(G_0) \]

implies that for \( \lambda > 0 \) we have the inequality

\[ \| (\lambda I - G) \varphi \|_\mathfrak{H} \geq \lambda \| \varphi \|_\mathfrak{H} \quad \text{for} \quad \varphi \in \mathfrak{D}(G_0), \]

where \( \| \varphi \|_\mathfrak{H} = \sqrt{(\varphi, \varphi)_\mathfrak{H}} \) for \( \varphi \in \mathfrak{H} \). Furthermore the condition \( (C_1) \) and (3.8) imply that

\[ \mathfrak{D}(G) \supset \mathfrak{D}(A) \times \mathfrak{D}(A^{1/2}). \]

By (3.11) we find that for \( \lambda > 0 \)

\[ R(\lambda I - G) \supset \mathfrak{D}(G_0) \]

where \( R(\lambda I - G) \) denotes the range of the operator \( \lambda I - G \). (3.10) and (3.12) imply that for any \( \lambda > 0 \) the operator \( \lambda I - G \) has the inverse \( (\lambda I - G)^{-1} \) defined on \( \mathfrak{H} \) such that \( \| (\lambda I - G)^{-1} \|_\mathfrak{H} \leq \lambda^{-1} \). From this it follows by the Hille-Yosida's theorem that \( G \) is the infinitesimal generator of a strongly continuous semi-group \( U(t) \) on \( \mathfrak{H} \) such that

\[ \| U(t) \|_\mathfrak{H} \leq 1 \quad (t \geq 0). \]

Since it follows from (3.11) that \([0, f] \in \mathfrak{D}(G)\), we can find that

\[ U(t)[0, f] = [u(x, t), u_t(x, t)] \quad \text{for} \quad t > 0, \]

where \( u_t(x, t) = \frac{\partial}{\partial t} u(x, t) \). (3.13) and (3.14) imply that
\[ \Vert[u(\cdot, t), u_t(\cdot, t)]\Vert_{\mathfrak{H}}^{2} \leqq ||f||_{L^{2}}^{2} \quad \text{for } t>0, \]

i.e.

\[ \int_{E} \left[ |\nabla u(x, t)|^{2} + q(x)|u(x, t)|^{2} + |u_t(x, t)|^{2} \right] dx \leqq ||f||_{L^{2}}^{2} \quad \text{for } t>0, \]

from which it follows by (3.8) and (3.9) that

\[ ||u_t(\cdot, t)||_{L^{2}} \leqq ||f||_{L^{2}} \quad \text{for } t>0. \]

On the other hand (3.5) and (3.7) imply that

\[ ||u(\cdot, t)||_{L^{2}} \leqq ||g||_{L^{2}} \quad \text{for } t>0. \]

From (3.15), (3.16) and (3.17) it follows by the boundedness of \( q(x) \) that

\[ \Vert\nabla u(\cdot, t)\Vert_{L^{2}} \leqq C \quad \text{for } t>0. \]

Hereafter we denote by \( C \) constants independent of \( t>0 \). Thus from (2.2), (3.17) and (3.18) we obtain by the Rellich selection theorem that

\[ \lim_{t \to \infty} ||u(\cdot, t)||_{L^{2}(X)} = 0 \quad \text{for all compact } K \subset E. \]

Next we shall prove (2.4). It follows from (3.11), (3.13), (3.14) and the identity \( G U(t)[0, f] = U(t) G[0, f] \) that

\[ \int_{E} \left[ |\nabla u_t(x, t)|^{2} + q(x)|u_t(x, t)|^{2} + \left| \left( \Delta - q(x) \right) u(x, t) \right|^{2} \right] dx \leqq \|A^{1/2} f\|_{L^{2}}^{2} \quad \text{for } t>0. \]

From this, (3.16) and (3.17) we find by the boundedness of \( q(x) \) that we have

\[ \|\Delta u(\cdot, t)\|_{L^{2}} \leqq C \quad \text{for } t>0, \]

\[ \|\nabla u_t(\cdot, t)\|_{L^{2}} \leqq C \quad \text{for } t>0. \]

Furthermore, suppose that \( f \in \mathfrak{D}(A) \) and \( \sup_{x \in E} |\nabla q(x)| < \infty \). Then by (3.11) \([f, 0] \in \mathfrak{D}(G)\). Hence, setting \( v(x, t) = u_t(x, t) \) we find that

\[ U(t)[f, 0] = [v(x, t), v_t(x, t)]. \]

Therefore an argument similar to the one used in deriving (3.20) gives

\[ \|\nabla v_t(\cdot, t)\|_{L^{2}} \leqq C \quad \text{for all } t>0. \]

On the other hand we have

\[ \nabla \Delta u(x, t) = \nabla v_t(x, t) + \nabla \left( q(x) u(x, t) \right). \]

From this, (3.17), (3.18) and (3.21) it follows by the boundedness of \( q(x) \) and \( \nabla q(x) \) that
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\[ (3.22) \quad ||\nabla \Delta u(x, t)||_{L^2} \leq C \quad \text{for all} \quad t > 0. \]

From (2.2), (3.19) and (3.22) we find by the Rellich selection theorem that

\[ (3.23) \quad \lim_{t \to \infty} ||\Delta u(\cdot, t)||_{L^2(K)} = 0 \quad \text{for all compact} \quad K \subset E. \]

Consequently from (2.3) and (3.23) we obtain by Sobolev’s lemma

\[ (2.4) \quad \lim_{t \to \infty} \sup_{x \in K} |u(x, t)| = 0 \quad \text{for all compact} \quad K \subset E. \]

Thus we have proved Theorem 1.

4. Lemmas.

In this section, we shall establish some lemmas required in proving Theorem 2.

The following lemma 3, 4 and 5 can be verified easily.

**Lemma 3.** 1) Let \( a > 0, \, t > 0 \) be fixed. Then the integral

\[ \int_{a-i\infty}^{a+i\infty} \left| \frac{e^{\zeta t}}{\lambda + \zeta^2} \right| d\zeta \]

is a bounded continuous function of \( \lambda \in (0, \infty) \).

2) Let \( a' < 0, \, t_0 > 0 \) be fixed. Then the integral

\[ \int_{a'-i\infty}^{a'+i\infty} \left| \frac{e^{\zeta t}}{\lambda + \zeta^2} \right| d\zeta \]

converges uniformly with respect to \( t \geq t_0 \). Furthermore we have

\[ (4.1) \quad \int_{a'-i\infty}^{a'+i\infty} \frac{e^{\zeta t}}{\lambda + \zeta^2} d\zeta = 0 \quad \text{for all} \quad t > 0. \]

**Lemma 4.** Let \( t > 0, \, a > 0 \) be fixed. If \( l(\lambda) \) is locally bounded in \( (0, \infty) \) and belongs to \( L^1(0, \infty) \), then the integral

\[ \int_0^\infty |l(\lambda)| d\lambda \int_{a-iN}^{a+iN} \left| \frac{e^{\zeta t}}{\lambda + \zeta^2} \right| d\zeta \]

converges for any fixed \( N \) (\( |N| > 3a \)).

**Lemma 5.** Let \( a > 0 \) be fixed. Then

\[ u(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\zeta t} R(-\zeta^2) f d\zeta \]

is the solution of the problem (2.1), where \( R(-\zeta^2) f \) denotes \( (A - \zeta^2)^{-1} f \).

Now we shall study an asymptotic property of functions of the form
\[ \varphi(x) = \int_E \frac{v(y)}{|x-y|} \, dy. \]

We have the following one similar to Lemma 3.2 in [4].

**Lemma 6.** Let \( v(x) \in L^2(E) \) and \( v(x) = 0 \) (\(|x|^{- \varepsilon} \)) \( (0 < \varepsilon < 1) \) \( (|x| \to \infty) \).

If \( \int_E v(x) \, dx = 0 \), then we have

\[ (4.2) \quad \varphi(x) = 0 \, (|x|^{-1-\varepsilon}) \, (|x| \to \infty). \]

**Proof.** Let \( R_1 \) be fixed so that \( |v(x)| \leq C_1 \, |x|^{- \varepsilon} \) for \( |x| \geq R_1 \) and let \( |x| \) be so large that \( \frac{1}{2} |x| = R > R_1 \).

Then we have

\[ (4.3) \quad \varphi(x) = \int_{|y| \leq R} \frac{v(y)}{|x-y|} \, dy + \int_{|y| \geq R} \frac{v(y)}{|x-y|} \, dy \equiv I + I'. \]

\( I' \) can be estimated as follows:

\[
|I'| \leq C_1 \int_{|y| \geq R} \frac{dy}{|x-y|||y|^{3+\varepsilon}}
\]
\[
\leq C \int_{R}^\infty \int_{0}^{\pi} \frac{r^2 \sin \theta}{r^{3+\varepsilon}(|x|^2 + r^2 - 2|x|r \cos \theta)^{\frac{1}{2}}} \, d\theta \, dr
\]
\[
= C \int_{R}^\infty \int_{0}^{\pi} \frac{1}{|x|} \frac{1}{r^{3+\varepsilon}} \, (|x|^2 + r^2 - 2|x|r \cos \theta)^{\frac{1}{2}} \, d\theta \, dr
\]
\[
= C \frac{1}{|x|} \int_{R}^{|x|} r^{-1-\varepsilon} \, dr + C \int_{|x|}^\infty r^{-2-\varepsilon} \, dr
\]
\[
\leq C \frac{1}{|x|} \int_{|x|}^{|x|} r^{-1-\varepsilon} \, dr + C \int_{|x|}^\infty r^{-2-\varepsilon} \, dr
\]

Hereafter we denote by \( C \) constants independent of \( x \). We proceed to estimate \( I \). We have

\[
I = \frac{1}{|x|} \int_{|y| \leq R} v(y) \, dy - \frac{1}{|x|} \int_{|y| \leq R} \frac{|y|^2 v(y)}{|x-y|(|x| + |x-y|)} \, dy
\]
\[
+ 2 \int_{|y| \leq R} \frac{(\frac{x}{|x|}, y) v(y)}{|x-y|(|x| + |x-y|)} \, dy \equiv I_1 + I_2 + I_3.
\]

It follows from the assumption that

\[
I_1 = -|x|^{-1} \int_{|y| \geq R} v(y) \, dy.
\]

This implies that
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\[ |I_1| \leq C_1 |x|^{-1} \int_{|y| \geq R} \frac{dy}{|y|^{3+\varepsilon}} \leq C |x|^{-1-\varepsilon}. \]

$I_j$ ($j = 2, 3$) are estimated as follows:

\[ |I_2| \leq C |x|^{-3} \left[ \int_{|y| \leq R_1} |y|^2 |v(y)| dy + \int_{R_1 \leq |y| \leq R} |y|^2 |v(y)| dy \right] \leq C |x|^{-3} + C |x|^{-1-\varepsilon}. \]

\[ |I_3| \leq C |x|^{-2} \left[ \int_{|y| \leq R_1} |y||v(y)| dy + \int_{R_1 \leq |y| \leq R} |y||v(y)| dy \right] \leq C |x|^{-2} + C (1-\varepsilon)^{-1} |x|^{-1-\varepsilon}. \]

These estimates and (4.3) give (4.2).

**Lemma 7.** Let $\omega \in M$ and $\omega \neq 0$. Then $\langle q, \omega \rangle \neq 0$ and furthermore $\dim M = \dim M' = 1$, where $M$ and $M'$ are the ones defined in Lemma 1.

**Proof.** Suppose that $\omega \in M$ and $\langle q, \omega \rangle = 0$. Then the function $\omega(x)$ satisfies

\[ (3.3) \quad \omega(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{q(y) \omega(y)}{|x-y|} dy. \]

Since Lemma 1 implies that $\omega = 0 (|x|^{-1})$ ($|x| \to \infty$), from (3.3) it follows by Lemma 6 that $\omega = 0 (|x|^{-1-\varepsilon})$. Applying repeatedly Lemma 6 to (3.3) we find that $\omega \in \mathcal{D}(A)$. Hence $\omega \in \mathcal{D}(A)$ and $A \omega = 0$. Therefore by the condition $(C_2)$ it follows that $\omega = 0$. Thus $\langle q, \omega \rangle \neq 0$ for $\omega \in M$ such that $\omega \neq 0$. From this it follows that $\dim M = 1$, which completes the proof of Lemma 7, since Lemma 1 implies that $\dim M = \dim M'$.

**Remark.** Lemma 7 will be established also in the course of the proof of Theorem 2, for the case that $q$ satisfies the assumptions in Theorem 2.

**Lemma 8.** For $\lambda > 0$ we set

\[ \theta(\lambda) = \theta(x, \lambda) = \frac{1}{2\pi i} \left( u_+(x, \lambda) - u_-(x, \lambda) \right), \]

where $u_{\pm}(x, \lambda) = R(\lambda \pm i0) f(x)$. Let us define the norm for functions $\varphi \in C^2(E)$ by

\[ \|\varphi\|_{C^2_{\lambda+\alpha}}^2 = \sup_{x \in \mathbb{R}^3, |\beta| \leq 2} |D^\beta \varphi(x)| \cdot (1 + |x|^3)^{3+\alpha} \]

and let $C^2_{\lambda+\alpha}$ be the Banach space \( \{ \varphi \in C^2(E); \|\varphi\|_{C^2_{\lambda+\alpha}} < \infty \} \). Then the function of $\lambda T_1(\varphi) = \langle \theta(\lambda), \varphi \rangle$ ($\varphi \in C^2_{\lambda+\alpha}$) is a nuclear operator on $C^2_{\lambda+\alpha}$ to $L^1(0, \infty)$, and
$$\|T_{\lambda}\|_{(c^{2+a}_{3})^*} = \|\theta(\lambda)\|_{(c^{2+a}_{3})^*}$$ belongs to $L^1(0, \infty)$.

Proof. We introduce the finite measure

$$d\mu = (1 + |x|^2)^{-\frac{3+a}{2}} \, dx$$

in $E$, where $dx$ is the Lebesgue measure in $E$. Then we decompose $T_{\lambda}$ as follows:

$$C^{2+a}_{3} \xrightarrow{\psi} \prod_{\beta \leq 2} L^\infty(\mu) \xrightarrow{I_1} \prod_{\beta \leq 2} L^2(\mu) \xrightarrow{J_2} \prod_{\beta \leq 2} L^2(dx)$$

$$\varphi \xrightarrow{J_3} W^2_2 \xrightarrow{I_2} L^\infty(dx) = L^\infty(\mu) \xrightarrow{I_3} L^2(\mu) \xrightarrow{J_4} L^1(0, \infty)$$

Let $F$ be the closed subspace of all vector-valued functions \{(D^\beta \varphi) \cdot (1 + |x|^2)^{\frac{3+a}{4}} \} \subset \prod_{\beta \leq 2} L^2(dx)$ whose components $(D^\beta \varphi) \cdot (1 + |x|^2)^{\frac{3+a}{4}} \in L^2(dx) (|\beta| \leq 2)$.

Furthermore let $P_F$ be the projection operator of $\prod_{\beta} L^2(dx)$ onto $F$ and $J_3'$ be the operator of $F$ to $W^2_2$ which assigns each vector-valued function \{(D^\beta \varphi) \cdot (1 + |x|^2)^{\frac{3+a}{4}} \} in $F$ to the scalar-valued function $\varphi \cdot (1 + |x|^2)^{\frac{3+a}{4}}$ in $W^2_2$. Then $J_3$ may be regarded as the composite operator $J_3' \cdot P_F$. Hence $J_3$ is defined on the whole space $\prod_{\beta} L^2(dx)$, so that by the closed graph theorem $J_3$ is a bounded operator on $\prod_{\beta} L^2(dx)$ to $W^2_2$.

$I_k (k=1, 2, 3)$ are identities. Since $d\mu$ is a finite measure we find that $I_1$ and $I_3$ are semi-integral (see [8]). And Sobolev's lemma implies that $I_5$ is bounded. It is obvious that $J_k (k=1, 2, 4)$ are bounded. Thus the identity operator on $C^{2+a}_{3}$ to $L^2(dx)$ is nuclear (see [8], theorem 14).

On the other hand $J_5$ is a bounded operator on $L^2(E)$ to $L^1(0, \infty)$. In fact, setting

$$\rho(\varphi) = \int_0^\infty \left| \frac{d}{d\lambda} (E_{\lambda} f, \varphi) \right| \, d\lambda \quad (\varphi \in L^2(E)),$$

we find that $\rho(\varphi)$ is a semi-norm on $L^2(E)$ and satisfies the inequality

$$\rho(\varphi) \leq \|f\|^2_{L^2} + \|\varphi\|^2_{L^2}$$

for all $\varphi \in L^2(E)$, which implies that
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\[ \rho(\varphi) \leq ||f||_{L^2}^2 + 1 \quad \text{for} \quad ||\varphi||_{L^2} \leq 1. \]

This implies that

\[ \rho(||\varphi||_{L^2}^{-1} \varphi) \leq ||f||_{L^2}^2 + 1 \quad \text{for} \quad 0 \neq \varphi \in L^2(E), \]

so that

\[ \rho(\varphi) \leq (||f||_{L^2}^2 + 1) ||\varphi||_{L^2} \quad \text{for all} \quad \varphi \in L^2(E). \]

This shows that \( J_5 \) is a bounded operator on \( L^2(E) \) to \( L^1(0, \infty) \), if we note that \( \frac{d}{d\lambda}(E_\lambda f, \varphi) = (\theta(\lambda), \varphi) \) for all \( \varphi \in L^2(E) \). Thus \( T_1 \) is a nuclear operator on \( C_{3+a}^2 \) to \( L^1(0, \infty) \), so that \( ||T_1||_{(C_{3+a}^2)^*} \in L^1(0, \infty) \), which completes the proof of Lemma 8.

5. Proof of Theorem 2.

Suppose that there exist functions \( \omega \in M \) such that \( \omega \neq 0 \). Then Lemma 7 implies that \( \dim M = 1 \). Therefore, taking \( \omega \in M \) such that \( \langle q, \omega \rangle = 1 \), since \( q = 0 (|x|^{-3-\alpha}) (|x| \to \infty) \), we see that any \( \varphi \in L^2(E) \) satisfying the condition \( \varphi = 0 (|x|^{-2-\delta}) (\delta > \frac{1}{2}) (|x| \to \infty) \) can be decomposed as follows:

\[ \varphi = \langle \varphi, \omega \rangle q + (\varphi - \langle \varphi, \omega \rangle q) \]

\[ = \varphi_1 + \varphi_2, \]

while \( \langle \varphi_2, \omega \rangle = 0 \). An argument similar to the one used in proving Theorem 1 gives that \( \lim_{t \to \infty} \langle u(t), \varphi_2 \rangle = 0 \). Therefore we have only to prove

\[ \lim_{t \to \infty} \langle u(t), q \rangle = 4\pi \langle f, \omega \rangle. \]  

Now it follows from theorem 4 in [2], the condition \( (C_2) \) and theorem 6 in [4] that \( \frac{d}{d\lambda} \langle E_\lambda f, q \rangle \) is continuous in \( \lambda \in (0, \infty) \) and belongs to \( L^1(0, \infty) \). Therefore by virtue of Lemma 3, Lemma 4, Lemma 5 and Fubini's theorem we obtain

\[ \langle u(t), q \rangle = \frac{1}{2\pi i} \int_0^\infty \frac{d}{d\lambda} \langle E_\lambda f, q \rangle d\lambda \int_{a-iN}^{a+i\infty} \frac{e^{\zeta t}}{\lambda + \zeta^2} d\zeta \]

\[ = \frac{1}{2\pi i} \int_0^\infty \frac{d}{d\lambda} \langle E_\lambda f, q \rangle d\lambda \int_{t_1}^{t_2} \frac{e^{\zeta t}}{\lambda + \zeta^2} d\zeta + \int_0^\infty \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \frac{d}{d\lambda} \langle E_\lambda f, q \rangle d\lambda \]

for \( N > 3a \), where \( t_1 \) and \( t_2 \) are the curves \( \{s-iN; 0 < s \leq a\} \cup \{a+it; -N < \} \).
\[
\tau < N \cup \{ s + iN; 0 < s \leq a \}, \{s + iN; -a \leq s < 0\} \cup \{-a + i\tau; -N < \tau < N\} \cup \{s - iN; -a \leq s < 0\} \text{ taken in the positive direction respectively. Then we can take} \\
N \text{ so large that} \left| \int_{-\infty}^{0} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \frac{d}{d\lambda} \langle E_{\lambda} f, q \rangle d\lambda \right| \text{ becomes sufficiently small uniformly} \\
\text{with respect to} \ t > 0. \text{ Let} \ N \text{ be fixed sufficiently large. Since for} \ \zeta \in \Gamma_{2} \ \text{Re} \zeta < 0, \text{ Lemma 4 and Lebesgue’s theorem imply that} \\
\lim_{t \to \infty} \int_{0}^{\infty} \frac{d}{d\lambda} \langle E_{\lambda} f, q \rangle d\lambda \int_{\Gamma_{2}} \frac{e^{\zeta t}}{\lambda + \zeta^{2}} d\zeta = 0.
\]

Consequently we have only to prove

\[
\left(5.2\right) \quad \lim_{t \to \infty} \int_{0}^{\infty} \frac{d}{d\lambda} \langle E_{\lambda} f, q \rangle d\lambda \int_{\Gamma_{2}} \frac{e^{\zeta t}}{\lambda + \zeta^{2}} d\zeta = 8\pi^{2}i \langle f, \omega \rangle.
\]

Lemma 4 and Fubini’s theorem imply that

\[
\left(5.3\right) \quad \int_{0}^{\infty} \frac{d}{d\lambda} \langle E_{\lambda} f, q \rangle d\lambda \int_{\Gamma_{2}} \frac{e^{\zeta t}}{\lambda + \zeta^{2}} d\zeta = \int_{\Gamma_{1}} \langle R(-\zeta^{2}) f, q \rangle e^{\zeta t} d\zeta.
\]

For Re \( \zeta > 0 \) we set \( u(\zeta) = R(-\zeta^{2}) f \). Then \( u(\zeta) \) satisfies

\[
\left(5.4\right) \quad u(x, \zeta) = \frac{1}{4\pi} \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) dy - \frac{1}{4\pi} \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} q(y) u(y, \zeta) dy
\]

for Re \( \zeta > 0 \). This and (3.4) imply that

\[
\left(5.5\right) \quad \langle u(\zeta), q \rangle = -\frac{1}{\zeta} \int_{E \times E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) q(x) dx dy + \zeta \langle u(\zeta), p(\zeta) \rangle
\]

for Re \( \zeta > 0 \), where

\[
\left(5.6\right) \quad p(\zeta) = p(x, \zeta) = q(x) \int_{E} q(y) \omega(y)|x-y|dy \int_{0}^{1} dy \int_{0}^{1} \tau e^{-\zeta|x-y|\tau} d\tau.
\]

In fact, setting \( \omega' = q\omega \) we see that \( \omega' \) satisfies (3.4) and \( \int_{E} \omega'(x) dx = 1 \). (5.4) implies that

\[
\langle u(\zeta), \omega' \rangle = \frac{1}{4\pi} \int_{E \times E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) \omega'(x) dx dy - \frac{1}{4\pi} \int_{E \times E} \frac{1}{|x-y|} q(y) u(y, \zeta) \times
\]

\[
\times \omega'(x) dx dy - \frac{1}{4\pi} \int_{E \times E} \frac{e^{-\zeta|x-y|} - 1}{|x-y|} q(y) u(y, \zeta) \omega'(x) dx dy.
\]

(3.4) shows that the second term in the right-hand side is equal to \( \langle u(\zeta), \omega' \rangle \). This implies that

\[
\int_{E \times E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) \omega'(x) dx dy = -\zeta \int_{E \times E} q(y) u(y, \zeta) \omega'(x) dx dy \int_{0}^{1} e^{-|x-y|\tau} d\tau
\]
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\[ -\zeta \int_{E \times E} q(y) u(y, \zeta) \omega'(x) dx \, dy + \zeta^2 \int_{E \times E} q(y) u(y, \zeta) \omega'(x) |x - y| dx \, dy \times \int_0^1 d\tau^1 \int_0^1 \tau e^{-\zeta|x-y| \tau'} d\tau, \]

which gives (5.5). Furthermore (3.3), (5.3), (5.4), (5.5) and (5.6) imply that

\[ \int_0^\infty \frac{d}{d\lambda} \langle E, f, q \rangle d\lambda \int_{\Gamma_1} \frac{e^{\zeta t}}{\lambda + \zeta^2} d\zeta = 4\pi \langle f, \omega \rangle \int_{\Gamma_1} \frac{e^{\zeta t}}{\zeta} d\zeta + \int_{\Gamma_1} \zeta e^{\zeta t} \langle R(-\zeta^2) f, T_{\zeta}^* p(\zeta) \rangle d\zeta, \]

where

\[ F(\zeta) = \int_{E \times E} f(y) q(x) \omega(x) dx \, dy \int_0^1 e^{-\zeta|x-y|} d\tau + \zeta \sum_{j=0}^2 \langle T_{\zeta}^j \psi, p(\zeta) \rangle, \]

\[ \phi(x) = \frac{1}{4\pi} \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) dy, \]

\[ T_{\zeta} \phi(x) = -\frac{1}{4\pi} \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} q(y) \phi(y) dy, \]

\[ T_{\zeta}^* \phi(x) = -\frac{1}{4\pi} q(x) \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} \phi(y) dy, \]

\[ T^0 \phi(x) = \phi(x), \quad T^j \phi(x) = T(T^{j-1} \phi)(x) \quad (j=1,2,3). \]

First we shall show that

\[ \lim_{t \to \infty} 4\pi \langle f, \omega \rangle \int_{r_1} \frac{e^{\zeta t}}{\zeta} d\zeta = 8\pi^2 i \langle f, \omega \rangle. \]

We have

\[ \int_{r_1} \frac{e^{\zeta t}}{\zeta} d\zeta = \mathrm{Res} \frac{e^{\zeta t}}{\zeta} \bigg|_{\zeta=0} - \int_{r_1} \frac{e^{\zeta t}}{\zeta} d\zeta. \]

Lebesgue's theorem implies that \( \lim_{t \to \infty} \int_{r_1} \frac{e^{\zeta t}}{\zeta} d\zeta = 0 \), which gives (5.8).

Next we shall show that

\[ \lim_{t \to \infty} \int_{r_1} e^{\zeta t} F(\zeta) d\zeta = 0. \]

The conditions \( (C_3) \) and \( q = 0 \ (|x|^{-3-a}) \ (|x| \to \infty) \) imply that \( F(\zeta) \) is holomorphic in \( \{\zeta; \ \mathrm{Re} \ \zeta > 0\} \) and is continuous in \( \{\zeta; \ \mathrm{Re} \ \zeta \geq 0\} \). From this it follows by Lebesgue's theorem that
\[
\int_{\Gamma_{1}} e^{\zeta t} F(\zeta) d\zeta = i \int_{-N}^{N} e^{\sigma jst} F(is) ds,
\]
from which we obtain (5.9) by Riemann-Lebesgue's theorem.

Thus the proof of Theorem 2 will be complete, if we prove the following

**Lemma 9.** Let \( q(x) \) be as in Theorem 2. Then

\[(5.10) \quad \lim_{t \to \infty} \int_{\Gamma_{1}} \zeta e^{\zeta t} \langle R(-\zeta^{2}) f, p_{3}(\zeta) \rangle d\zeta = 0,
\]
where \( p_{3}(\zeta) = p_{3}(x, \zeta) = T_{\zeta}^{*3} p(x, \zeta). \)

To prove Lemma 9 we shall establish the following.

**Lemma 10.** 1) For fixed \( \lambda > 0 \langle \theta(\lambda), p_{3}(\zeta) \rangle \) is a holomorphic function of \( \zeta \in \{\text{Re}\zeta > 0\} \), where \( \theta(\lambda) \) is the one introduced in Lemma 8.

2) Let \( q(x) \) be as in Theorem 2. Then we have

\[
\sup_{x \in E, |\beta| \leq 2} |D_{x}^{\beta} p_{3}(x, \zeta)|(1 + |x|^{2})^{\frac{3+\alpha}{2}} \leq C_{K}
\]
for \( \text{Re}\zeta \geq 0, |\zeta| \leq K \), where \( K \) is a positive number and \( C_{K} \) is a constant independent of \( x \in E, \zeta \in \{\text{Re}\zeta \geq 0, |\zeta| \leq K\} \).

**Proof of Lemma 10.** 1) It follows from the condition \((C_{3})\) and theorem 4 in [2] that for \( \lambda > 0 \theta(x, \lambda) \) is a continuous function of \( x \in E \) and behaves like \( 0(|x|^{-1}) (|x| \to \infty) \). Therefore, since \( q = 0(|x|^{-2-\alpha})(|x| \to \infty) \), we see easily that for \( \lambda > 0 \langle \theta(\lambda), p_{3}(\zeta) \rangle \) is a holomorphic function of \( \zeta \in \{\text{Re}\zeta > 0\} \), which proves 1).

We proceed to prove 2). The condition \( q = 0(|x|^{-3-\alpha})(|x| \to \infty) \) and (5.6) imply that for \( \text{Re}\zeta \geq 0 \)

\[(5.11) \quad |p(x, \zeta)| \leq |q(x)| \left[ |x| \int_{E} |q(y)\omega(y)| dy + \int_{E} |y| |q(y)\omega(y)| dy \right] \leq C(1 + |x|)^{-2-\alpha}.
\]
Here and in what follows we denote various constants independent of \( \zeta (\text{Re}\zeta \geq 0) \) by \( C \). (5.11) and the equality

\[
T_{\zeta}^{*} p(x, \zeta) = -\frac{1}{4\pi} q(x) \int_{E} e^{-\zeta|x-y|} p(y, \zeta) dy
\]

imply that

\[(5.12) \quad |T_{\zeta}^{*} p(x, \zeta)| \leq C(1 + |x|)^{-3-\alpha} \quad \text{for} \quad \text{Re}\zeta \geq 0
\]

In the same way we obtain

\[(5.13) \quad |T_{\zeta}^{*j} p(x, \zeta)| \leq C(1 + |x|)^{-4-\alpha} \quad \text{for} \quad \text{Re}\zeta \geq 0 \quad (j=2,3).
\]
The principle of limiting amplitude

The condition \( D^\beta q = 0 (|x|^{-2-alpha}) (|x|->\infty) \) (\( |\beta| = 1, 2 \)), (5.12) and (5.13) imply that

\[
|D_x^9 p_3(x, \zeta)| \leq C_X (1 + |x|)^{-3-alpha}
\]

for \( Re \zeta \geq 0, \ |\zeta| \leq K (|\beta| \leq 2) \), which proves 2) of Lemma 10.

**Proof of Lemma 9.** It follows from 2) of Lemma 10 and Lemma 8 that there exists a function \( g(h) \in L^1(0, \infty) \) such that

(5.14)

\[ |\langle \theta(h), p_3(\zeta) \rangle| \leq g(h) \]

for \( Re \zeta \geq 0, \ |\zeta| \leq K \).

This, Lemma 4, 1) of Lemma 10 and theorem 4 in [2] imply that

\[
\int_{\Gamma_1} \zeta e^{\zeta t} \langle R(-\zeta^2)f, p_3(\zeta) \rangle d\zeta = \int_{\Gamma_1} \zeta e^{\zeta t} \langle \theta(h), p_3(\zeta) \rangle d\zeta
\]

\[ = \lim_{t \rightarrow 0} \int_{0}^{\infty} d\lambda \int_{\Gamma_1} \zeta e^{\zeta t} \langle \theta(h), p_3(\zeta) \rangle d\zeta,
\]

where \( \Gamma_1 \) is the path obtained replacing \( a \) by \( \epsilon \) in \( \Gamma_1 \). Therefore, by virtue of (5.14), Lemma 4, 2) of Lemma 10, theorem 4 in [2], Lebesgue's theorem and Riemann-Lebesgue's theorem we find that we have only to prove

(5.15)

\[ \lim_{t \rightarrow 0} \int_{0}^{\infty} d\lambda \int_{-iN}^{iN} \frac{\langle \theta(h), p_3(\zeta) \rangle}{\lambda + \zeta^2} d\zeta = 0. \]

Now we shall prove

(5.16)

\[ \lim_{t \rightarrow 0} \int_{0}^{N} d\lambda \int_{-N}^{N} e^{(\epsilon+is)t} \frac{\langle \lambda \theta(\lambda^2), p_3(\epsilon+is) \rangle}{(\lambda-s)^2 + \epsilon^2} ds = 0. \]

Set \( \rho = t - (|x-y| + |y-z| + |z-u| + |u-v|/\tau) \). Then by virtue of Fubini's theorem, for fixed \( t > 0 \) and fixed \( \epsilon > 0 \) we obtain

(5.17)

\[ \int_{-N}^{N} e^{(\epsilon+is)t} \frac{(\lambda-s)\langle \lambda \theta(\lambda^2), p_3(\epsilon+is) \rangle}{(\lambda-s)^2 + \epsilon^2} ds = \left( \frac{1}{4\pi} \right)^3 e^{\epsilon t} \int_{E} \theta(x, \lambda^2) \varphi_{*,t}(x, \lambda) dx,
\]

where

(5.18)

\[ \varphi_{*,t}(x, \lambda) = q(x) \int_{E} \frac{q(y)}{|x-y|} dy \int_{E} \frac{q(z)}{|x-z|} dz \int_{E} \frac{q(u)}{|x-u|} du \int_{E} q(u) \times \omega(v) dv \int_{0}^{1} d\tau \int_{0}^{1} \tau e^{-\epsilon(t-\rho)} d\tau \int_{-N}^{N} \frac{(s-\lambda) e^{\epsilon \tau}}{(s-\lambda)^2 + \epsilon^2} ds.
\]

First we shall show that

(5.19)

\[ \lim_{t \rightarrow 0} \int_{0}^{N} d\lambda \int_{-N}^{N} e^{(\epsilon+is)t} \frac{(\lambda-s)\langle \lambda \theta(\lambda^2), p_3(\epsilon+is) \rangle}{(\lambda-s)^2 + \epsilon^2} ds
\]
Let $t>0$ be fixed. Then we find that there exists a constant $C$ such that

\[(5.20) \quad \sup_{x \in E, |\beta| \leq 2} |D^\beta \varphi_{*,t}(x, \lambda)| |1 + |x|^2|^{\alpha/2} \leq C \left(1 + \log \frac{N+\lambda}{N-\lambda}\right)\]

for all $\lambda < N$ and all $\epsilon \leq \epsilon_0$, where $\epsilon_0$ is a positive number. In fact, since $s \cos s$ is an odd function, for $\lambda < N$ we have

\[(5.21) \quad \int_{-N}^{N} \frac{(s-\lambda)e^{i\rho s}}{(s-\lambda)^2 + \epsilon^2} ds = e^{i\rho \lambda} \left[ \int_{(-N-\lambda)\rho}^{(N-\lambda)\rho} \frac{\cos s}{s} ds + \epsilon^2 \int_{(-N-\lambda)\rho}^{(N-\lambda)\rho} \frac{1}{s^2 + \epsilon^2} ds \right] + i \left[ \int_{(-N-\lambda)\rho}^{(N-\lambda)\rho} \frac{\sin s}{s} ds + \epsilon^2 \int_{(-N-\lambda)\rho}^{(N-\lambda)\rho} \frac{1}{s^2 + \epsilon^2} ds \right]

which implies that

\[(5.22) \quad \left| \int_{-N}^{N} \frac{(s-\lambda)e^{i\rho s}}{(s-\lambda)^2 + \epsilon^2} ds \right| \leq C \left(1 + \log \frac{N+\lambda}{N-\lambda}\right),\]

where $C'$ is a constant independent of $\epsilon, \lambda, \rho$. Since $q=0(|x|^{-3-\alpha}), D^\beta q=0 (|x|^{-2-\alpha}) (|x| \to \infty) (|\beta|=1,2)$ and $t-\rho \geq 0$, (5.18) and (5.22) imply (5.20). On the other hand it follows from theorem 4 in [2] that for each function $\varphi$ satisfying $|\varphi(x)| \leq C(1 + |x|^2)^{-\alpha/2}$, the integral of $\theta(x, \lambda^2) \varphi(x) dx$ is a continuous function of $\lambda \in (0, \infty)$. Consequently by virtue of (5.17), (5.20), Lemma 8 and Lebesgue's theorem we obtain (5.19).

Next we shall show that

\[(5.23) \quad \lim_{t \to \infty} \int_{0}^{N} d\lambda \int_{E} \theta(x, \lambda^2) \lim_{\epsilon \to 0} \varphi_{*,\epsilon}(x, \lambda) d\lambda = 0 .\]

From (5.18), (5.21) and (5.22) it follows by Lebesgue's theorem that for fixed $\lambda < N$ we have

\[(5.24) \quad \lim_{\epsilon \to 0} \varphi_{*,\epsilon}(x, \lambda) = q(x) \int_{E} \frac{q(y)}{|x-y|} dy \int_{E} \frac{q(z)}{|y-z|} dz \int_{E} \frac{q(u)}{|z-u|} du \int_{E} |u-v| \times

\times q(v) \omega(v) dv \int_{0}^{1} \tau e^{i\rho \lambda} d\tau \left[ \int_{(-N-\lambda)\rho}^{(N-\lambda)\rho} \frac{\cos s}{s} ds + i\pi \right] \equiv J_{1,\epsilon}(x, \lambda) + J_{2,\epsilon}(x, \lambda) + J_{3,\epsilon}(x, \lambda) .\]
The principle of limiting amplitude

Set
\[ \varphi_2(x, \lambda) = e^{-i\lambda t} J_{2,t}(x, \lambda). \]

Then, since \[ \rho = t - (|x - y| + |y - z| + |z - u| + |u - v| \tau') \] and \[ q = 0 (|x|^{-3-a}) \]
\[ D^\beta q = 0 (|x|^{-2-a}) (|x|\to \infty) (|\beta| = 1, 2), \]
we find that
\[ \sup_{x \in E, |\beta| \leq 2} |D^\beta \varphi_2(x, \lambda)| (1 + |x|^2)^{3+a/2} \leq C \]
for all \( \lambda < N \), where \( C \) is a constant independent of \( \lambda \). Therefore by virtue of Lemma 8 and Riemann-Lebesgue’s theorem we obtain

\[
\lim_{t \to \infty} \int_0^N d\lambda \int_E \lambda \theta(x, \lambda) J_{2,t}(x, \lambda) dx = 0. 
\]

Let \( \rho - t \) be fixed. Then \( \rho \to \infty \) as \( t \to \infty \). This implies that for fixed \( \lambda < N \) and fixed \( \rho - t \)
\[
\lim_{t \to \infty} \int_{(-N-N)\rho}^{(N-N)\rho} \frac{\cos s}{s} ds = \lim_{t \to \infty} \int_{(-N-N)\rho}^{(N-N)\rho} \frac{\sin s}{s} ds - \pi = 0. 
\]

Consequently an argument similar to the one used in proving (5.19) gives that

\[
\lim_{t \to \infty} \int_0^N d\lambda \int_E \lambda \theta(x, \lambda) J_{k,t}(x, \lambda) dx = 0 \quad (k = 1, 3). 
\]

Thus (5.24), (5.25) and (5.26) imply (5.23). Therefore (5.19) and (5.23) prove (5.16).

An argument similar to the one used in proving (5.16) gives that
\[
\lim_{t \to \infty} \lim_{\epsilon \to 0} \int_0^{2N} d\lambda \int_{-iN}^{iN} e^{it} \frac{\lambda \theta(\lambda^2) \langle p_3(\lambda + i\epsilon), p_3(\lambda^2) \rangle}{\lambda + i\zeta} d\lambda = 0. 
\]

These and (5.16) imply that
\[
\lim_{t \to \infty} \lim_{\epsilon \to 0} \int_0^{2N} d\lambda \int_{-iN}^{iN} e^{it} \frac{\lambda \theta(\lambda^2) \langle p_3(\lambda^2), p_3(\zeta) \rangle}{\lambda - i\zeta} d\lambda = 0. 
\]

In the same way we obtain
\[
\lim_{t \to \infty} \lim_{\epsilon \to 0} \int_0^{2N} d\lambda \int_{-iN}^{iN} e^{it} \frac{\lambda \theta(\lambda^2) \langle p_3(\lambda^2), p_3(\zeta) \rangle}{\lambda + i\zeta} d\lambda = 0. 
\]
Thus we have proved (5.15), which completes the proof of Lemma 9.

Remark. If the condition $(C_2)$ is not assumed, then, as O. A. Ladyzhenskaya pointed out in [1], it is seen that in general the principle of limiting amplitude is not valid. Let $E_i$ be the resolution of the identity generated by the operator $A$. Furthermore let $-\mu_n (\mu_n > 0) \ (n=1, 2, \cdots, k)$ be the negative eigenvalues of the operator $A$ and $\varphi_n (n=1, 2, \cdots, k)$ be the eigenfunctions associated with the eigenvalues $-\mu_n$, and $\phi_n (n \geq 1)$ be the eigenfunctions of $A$ associated with the eigenvalue zero. Then the solution $u(t)$ of (2.1) is represented as follows:

$$u(t) = \int_{0+}^{\infty} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dE_{\lambda} f + t \sum_{n=1}^{\infty} (f, \varphi_n)_{L^2} \varphi_n + \sum_{n=1}^{k} \frac{e^{\sqrt{\mu_n} t} - e^{-\sqrt{\mu_n} t}}{2 \sqrt{\mu_n}} (f, \varphi_n)_{L^2} \varphi_n.$$ 

The first term in the right-hand side behaves like $0(t)$ at infinity. Therefore Theorem 2 implies that our conjecture is true in the case that $q = 0(|x|^{-3-\gamma})$ at infinity.

References


