THE PRINCIPLE OF LIMITING AMPLITUDE

By

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1. Introduction.

We shall be concerned with the initial value problem

\[
\left[ \frac{\partial^2}{\partial t^2} - \Delta + q(x) \right] u(x, t) = f_2(x) e^{-i\sigma t} \quad (t>0),
\]

\[
u(x, 0) = f_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = f_1(x),
\]

where \( x \) is a point of 3-dimensional Euclidean space \( \mathbb{R}^3 = E \), \( \Delta \) denotes the Laplace operator in \( E \), \( \sigma \) is a positive number, and \( q(x) \) is a real-valued function defined on \( E \). The principle of limiting amplitude for the problem (1.1) consists in the following: The solution \( u(x, t) \) of the problem (1.1) is such that at any point \( x \in E \) we have

\[
\lim_{t \to \infty} u(x, t) e^{it\sigma} = u_+(x, \sigma) \quad (\sigma > 0),
\]

where \( u_+(x, \sigma) \) is the solution of the equation \((-\Delta + q)u = \sigma u + f_2\) with the Sommerfeld radiation condition:

\[
u_+ = 0 (|x|^{-1}), \quad \frac{\partial \nu_+}{\partial |x|} = 0 (|x|^{-2}) \quad \text{at infinity.}
\]

In [1] O. A. Ladyzhenskaya gave this proof for the case that \( q \) and \( f_2 \) have compact supports, \( f_0 = f_1 = 0 \), and the operator \( A \) has no eigenvalue (For the definition see §2). But in this case it should be assumed, in addition, that there exists no solution \( \omega \in L^2(E) \) of the equation \((-\Delta + q)\omega = 0\) satisfying the conditions \( \omega = 0 (|x|^{-1}) \), \( \omega_x = 0 (|x|^{-2}) \) at infinity, although it is not assumed there. In [2] D. M. Éldus gave this proof for the case that \( q \in C^1(E) \), \( \alpha > \frac{1}{6} \) at infinity, and \( f_0 \in \mathfrak{D}(A), f_1 \in \mathfrak{D}(A^1), f_2 \in L^2(E), f_j = 0 \)

\(|x|^{-3-r}) \quad (r > 0) \quad (j = 0, 1, 2) \) at infinity, assuming that \( q(x) \geq 0 \) in \( E \), which guarantee the absence of not only eigenvalues of the operator \( A \), but also the above-mentioned functions \( \omega \). In [3] K. Asano and T. Shirota constructed such a function \( \omega \) for the case that \( q \in C^\infty(E) \) and the inequality
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\[-q(x) \leq \frac{1}{4|x|^2}\]

does not hold in $E$. In [5], by using such a function $\omega$ the present authors constructed a solution $u(x, t)$ of (1.1) with $q \in C_0^0(E), f_0 \in C_0^0(E), f_1 \in C_0^0(E)$ and $f_2 = 0$ such that

\[u(x, t) = \omega(x) \quad \text{for} \quad |x| \leq t .\]

The so constructed solution $u(x, t)$ of (1.1) is special but interesting to us, because that it give a relation of the principle of limiting amplitude and characteristics. They also remarked in [5] that, in the case that $q$ and $f_j$ ($j = 0, 1, 2$) have compact supports and the operator $A$ has no eigenvalue, the principle of limiting amplitude for (1.1) is valid if and only if the before-mentioned functions $\omega$ do not exist. Therefore it seems natural for us to conjecture the following: The principle of limiting amplitude for (1.1) with the condition $q = 0(|x|^{-2-\alpha})$ ($\alpha > 0$) ($|x| \to \infty$) is valid if and only if the operator $A$ has no eigenvalue and the above-mentioned functions $\omega$ do not exist.

In the present paper we establish that the above conjecture is true for $q = 0(|x|^{-3-\alpha})$ ($|x| \to \infty$) with some differentiability conditions. In particular the condition $q = 0(|x|^{-3-\alpha})$ ($\alpha > 0$) ($|x| \to \infty$) is used in proving the equality (5.5), Lemma 8, Lemma 10, and the inequality (5.20). The main results of this paper were presented in our note [5].

2. Results.

For simplicity and clearness we restrict ourselves in what follows to the problem

(2.1) \[\left[ \frac{\partial^2}{\partial t^2} - \Delta + q(x) \right] u(x, t) = 0 \quad (t > 0),
\]
\[u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = f(x).\]

Throughout the present paper $q(x)$ and $f(x)$ are assumed to satisfy the following conditions $(C_1), (C_2)$ and $(C_3)$:

$(C_1)$ $q(x)$ is a locally Hölder continuous real-valued function defined on $E$ and behaves like $0(|x|^{-2-\alpha})$ ($\alpha > 0$) at infinity, i.e. there exist positive numbers $\alpha, C_0, R_0$ such that

\[|q(x)| \leq C_0|x|^{-2-\alpha} \quad \text{for} \quad |x| \geq R_0.\]

Under the condition $(C_1)$ the operator $-\Delta + q$ defined on $C_0^\infty(E)$ has the unique self-adjoint extension in $L^2(E)$, which we denote by $A$. It is known that the domain $\mathcal{D}(A)$ of $A$ is the Sobolev space $W^2_2(E)$. 

(C2) A has no eigenvalue.
Under the condition (C2) A is positive definite. By $\mathfrak{D}(A^{\frac{1}{2}})$ we denote the domain of the self-adjoint operator $A^{\frac{1}{2}}$.

(C3) $f$ belongs to $\mathfrak{D}(A^{\frac{1}{2}})$ and behaves like $0(|x|^{-3-\gamma})$ ($\gamma>0$) at infinity.

In what follows we denote $\int_E \varphi(x)\psi(x)dx$ by $\langle \varphi, \psi \rangle$.

We have the following.

**Theorem 1.** Assume that $\langle f, \omega \rangle=0$ for all solutions $\omega \in L^2(E)$ of the equation $(-\Delta + q)\omega=0$ such that $\omega=0 (|x|^{-1})$, $\frac{\partial\omega}{\partial x_j}=0 (|x|^{-2}) (|x|\rightarrow \infty)$. Then the solution $u(t)=u(x, t)$ of the problem (2.1) is such that we have

$$
\lim_{t\rightarrow \infty} (u(t), \varphi)_{L^2(E)} = 0 \quad \text{for all } \varphi \in L^2(E),
$$

and

$$
\lim_{t\rightarrow \infty} \|u(t)\|_{L^2(K)} = 0 \quad \text{for all compact } K \subset E.
$$

Furthermore, suppose that $f \in \mathfrak{D}(A)$ and $\sup_{x \in E} |\nabla q(x)| < \infty$, where $\nabla q(x)$ denotes grad $q(x)$. Then we have

$$
\lim_{t\rightarrow \infty} \sup_{x \in K} |u(x, t)| = 0 \quad \text{for all compact } K \subset E.
$$

The main result is the following.

**Theorem 2.** Assume that $q \in C^2(E)$ and $q=0 (|x|^{-3-\delta})$, $D^\beta q=0 (|x|^{-\beta-\delta})$ ($|x|\rightarrow \infty$) ($\beta=1, 2$). Then the solution of (2.1) is such that for any $\varphi \in L^2(E)$ satisfying $\varphi=0 (|x|^{-1-\delta}) \left( \delta > \frac{1}{2} \right) (|x|\rightarrow \infty)$ we have

$$
\lim_{t\rightarrow \infty} \langle u(t), \varphi \rangle = 4\pi \langle \varphi, \omega \rangle \cdot \langle f, \omega \rangle \cdot \langle q, \omega \rangle^{-2},
$$

where $\omega$ is the one mentioned in Theorem 1.

By virtue of Theorem 2 and theorem 6 in [2] we have

**Corollary.** Let $q(x)$ be as in Theorem 2. Then the solution $u(x, t)$ of (2.1) is such that we have

$$
\lim_{t\rightarrow \infty} \sup_{x \in K} |u(x, t)| = 0 \quad \text{for all compact } K \subset E
$$

if and only if such functions $\omega$ as in Theorem 1 do not exist.

3. **Proof of Theorem 1.**

Since we can't find the Theorem in the literatures, for the sake of the
completeness of description we shall give the proof.

Let us define an operator $T$ for functions in $L^6(E)$ by

$$T\varphi(x) = -\frac{1}{4\pi} \int_E \frac{q(y) \varphi(y)}{|x-y|} dy \quad \varphi \in L^6(E).$$

**Lemma 1.** 1) $T$ is a compact operator on $L^6(E)$ to $L^6(E)$ and the adjoint operator $T^*$ of $T$ with respect to the bilinear form $\langle \cdot, \cdot \rangle$ is a compact operator on $L^{\frac{6}{2}}(E)$ to $L^{\frac{6}{2}}(E)$ given as follows:

$$T^*\phi(x) = -\frac{1}{4\pi} q(x) \int_E \frac{\phi(y)}{|x-y|} dy \quad \phi \in L^{\frac{6}{2}}(E).$$

2) By $M, M'$ we denote the subspaces $\{\omega \in L^6(E); (I-T)\omega = 0\}, \{\omega' \in L^{\frac{6}{2}}(E); (I-T^*)\omega' = 0\}$ of $L^6(E), L^{\frac{6}{2}}(E)$ respectively, where $I$ denotes the identity operator. Then we have

$$\omega = 0 (|x|^{-1}), \quad \frac{\partial \omega}{\partial x_j} = 0 (|x|^{-2}) \quad (|x| \to \infty) \quad \text{for } \omega \in M,$$

and

$$\omega' = 0 (|x|^{-3-\alpha}) \quad (|x| \to \infty) \quad \text{for } \omega' \in M'.$$

**Proof.** First we shall show that $T$ is a bounded operator on $L^6(E)$ to $L^6(E)$. In what follows we denote by $C$ constants independent of $x \in E$.

we have

$$\int_E \frac{|q(y)\varphi(y)|}{|x-y|} dy \leq \|\varphi\|_{L^6} \left[ \int_E \left( \frac{|q(y)|}{|x-y|} \right)^{\frac{6}{2}} dy \right]^{\frac{2}{3}}.$$

$$\int_E \left( \frac{|q(y)|}{|x-y|} \right)^{\frac{6}{2}} dy \text{ can be estimated as follows:}$$

$$\int_E \left( \frac{|q(y)|}{|x-y|} \right)^{\frac{6}{2}} dy \leq \sup_{|y| \leq R_0} |q(y)|^\frac{6}{2} \int_{|y| \leq R_0} \frac{1}{|x-y|^{\frac{6}{2}}} dy + C_0 \int_{|y| > R_0} \left( \frac{1}{|x-y| |y|^{\frac{3}{2}+\alpha}} \right)^{\frac{6}{2}} dy$$

$$\leq C(1 + |x|)^{-\frac{6}{2} + \rho_\alpha(x)},$$

where

$$|\rho_\alpha(x)| \leq C(1 + |x|)^{-\frac{6}{2} - \frac{3}{2} \alpha} \quad \text{for } 0 < \alpha < \frac{1}{2},$$

$$\leq C(1 + |x|)^{-\frac{6}{2} + \alpha} \quad \text{for } \alpha = \frac{1}{2},$$

$$\leq C(1 + |x|)^{-\frac{6}{2}} \quad \text{for } \alpha > \frac{1}{2},$$
and \( \varepsilon \) is an arbitrary positive number, so that for any \( \alpha > 0 \) \([\rho_{\alpha}(x)]^{\frac{5}{6}} \in L^6(E)\). These estimates imply that

\[(3.1) \quad |T\varphi(x)| \leq C\|\varphi\|_{L^6} \left[(1 + |x|)^{-1} + (\rho_{\alpha}(x))^{\frac{5}{6}}\right],\]

and

\[(3.2) \quad \|T\varphi\|_{L^6} \leq C\|\varphi\|_{L^6},\]

which shows the boundedness of \( T \).

Next we shall show that \( T \) is a compact operator on \( L^6(E) \) to \( L^6(E) \). The inequality (3.2) implies that \( \|T\varphi\|_{L^6} \) are uniformly bounded with respect to \( \varphi \) satisfying \( \|\varphi\|_{L^6} \leq 1 \). On the other hand an argument similar to the one used in deriving (3.2) gives

\[
\left\| \frac{\partial}{\partial x_j} T\varphi \right\|_{L^6} \leq C\|\varphi\|_{L^6}, \quad (j=1,2,3),
\]

which imply that \( T\varphi \) are uniformly equi-continuous in \( L^6(E) \) with respect to \( \varphi \in L^6(E) \) satisfying \( \|\varphi\|_{L^6} \leq 1 \). Thus \( T \) is compact. From this it follows by the Riesz-Schauder theory that \( T^* \) is also a compact operator on \( L^\frac{6}{5}(E) \) to \( L^\frac{6}{5}(E) \) and that \( \text{dim } M = \text{dim } M' < \infty \).

For \( \varphi \in L^6(E) \) and \( \psi \in L^\frac{6}{5}(E) \), since

\[
\int_{E \times E} \frac{|q(y)\varphi(y)\phi(x)|}{|x-y|} \, dx \, dy \leq \|\varphi\|_{L^6} \int_{E} |\phi(x)| \left[ \int_{E} \frac{|q(y)|}{|x-y|} \, dy \right]^{\frac{6}{5}} \, dx,
\]

we have

\[
\langle T^*\psi, \varphi \rangle = \langle \psi, T\varphi \rangle = \int_{E} \phi(x) \left[ -\frac{1}{4\pi} \int_{E} \frac{q(y)\varphi(y)}{|x-y|} \, dy \right] \, dx = \int_{E} \varphi(y) \left[ -\frac{1}{4\pi} q(y) \int_{E} \frac{\phi(x)}{|x-y|} \, dx \right] \, dy,
\]

which implies that

\[
T^*\phi(x) = -\frac{1}{4\pi} q(x) \int_{E} \frac{\phi(y)}{|x-y|} \, dy \quad \text{for } \phi \in L^\frac{6}{5}(E).
\]

For \( \omega \in M \) we have
\[(3.3) \quad \omega(x) = -\frac{1}{4\pi} \int_E \frac{q(y)\omega(y)}{|x-y|} dy.\]

It follows from (3.1) and (3.3) that not only \(\omega(x)\) is bounded, but also \(\omega = 0\) \((|x|^{-1})\), \(\frac{\partial \omega}{\partial x_j} = 0\) \((|x|^{-2})\) \((|x|\to\infty)\).

For \(\omega' \in M'\) we have
\[(3.4) \quad \omega'(x) = -\frac{1}{4\pi} q(x) \int_E \frac{\omega'(y)}{|x-y|} dy,\]
which implies that
\[\omega'(x) = \left(\frac{1}{4\pi}\right)^2 q(x) \int_E \frac{q(y)}{|x-y|} dy \int_E \frac{\omega'(z)}{|y-z|} dz.\]

It follows from the condition \((C_1)\) that
\[\int_E \frac{|q(y)|}{|x-y||y-z|} dy \leq C \frac{1}{1+|z|},\]
where \(C\) is a constant independent of \(x, z \in E\). This implies that
\[\int_{E \times E} \frac{|q(y)\omega'(z)|}{|x-y||y-z|} dy dz \leq C \|\omega'\|_{L^6}.\]

Therefore we obtain
\[|\omega'(x)| \leq C \|\omega'\|_{L^6} (1+|x|)^{-2-s}.\]

This and (3.4) imply that \(\omega' = 0\) \((|x|^{-3-s})\) \((|x|\to\infty)\).

Thus we have proved Lemma 1.

**Lemma 2.** Suppose that \(\varphi \in L^2(E), \varphi = 0\) \((|x|^{-2-s})\) \((\delta > \frac{1}{2})\) \((|x|\to\infty)\), and \(\langle \varphi, \omega \rangle = 0\) for all \(\omega \in M\). Then we have \(\varphi \in R(A^{1/2})\), where \(R(A^{1/2})\) denotes the range of \(A^{1/2}\).

**Proof.** Let \(\omega' \in M'\) and set \(\omega(x) = -\frac{1}{4\pi} \int_E \frac{\omega'(y)}{|x-y|} dy\). Then we find by the equalities (3.3) and (3.4) that \(q(x)\omega(x) = \omega'(x)\) and \(\omega \in M\), because that it follows from Lemma 1 that \(\omega'(x)\) is continuous in \(E\) and \(\omega' = 0\) \((|x|^{-3-s})\) \((|x|\to\infty)\). Hence the assumption implies that \(\frac{1}{4\pi} \int_E \frac{\varphi(y)}{|x-y|} dy \in L^6(E)\) and \(\langle \frac{1}{4\pi} \int_E \frac{\varphi(y)}{|x-y|} dy, \omega' \rangle = 0\) for all \(\omega' \in M'\). Therefore by virtue of Lemma
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1 and the Riesz-Schauder theory we find that there exists a solution \( u \in L^6(E) \) of the equation

\[
(I - T)u = \frac{1}{4\pi} \int_E \varphi(y) \frac{dy}{|x-y|}
\]

which implies that \( u(x) \) satisfies the equation \((-\Delta + q)u = \varphi\) with the conditions \( u = 0 \ (|x|^{-\delta}) \), \( \frac{\partial u}{\partial x_j} = 0 \ (|x|^{-1-\delta}) \ (|x| \to \infty) \), so that \( \Delta u \in L^2(E) \) and \( u \in W_{2,\text{loc}}^{2} \).

Set \( u_\epsilon(x) = u(x) e^{-|x|^2} (\epsilon > 0) \). Then we find that \( u_\epsilon \in \mathfrak{D}(A) \), \( \sup_{0 < \epsilon < \epsilon_0} \|A^{1/2} u_\epsilon\|_{L^2} < \infty \), where \( \epsilon_0 \) is a positive number, and \( \lim_{\epsilon \to 0} \|A u_\epsilon - \varphi\|_{L^2} = 0 \). In fact, \( \Delta u \in L^2(E) \) implies that \( u_\epsilon \in L^2(E) \) and \( \Delta u_\epsilon \in L^2(E) \), from which it follows that \( u_\epsilon \in W_{2}^{2}(E) = \mathfrak{D}(A) \). The conditions \( u = 0 \ (|x|^{-\delta}) \), \( \frac{\partial u}{\partial x_j} = 0 \ (|x|^{-1-\delta}) \ (|x| \to \infty) \) imply that

\[
\|\Delta u_\epsilon - \Delta u\|_{L^2} \leq 2 \int_E |\Delta u|^2 (1 - e^{-|x|^2})^2 dx + 0(\epsilon^{\delta+1}) + 0(\epsilon^2) \quad (\epsilon \to 0),
\]

from which it follows that \( \|\Delta u_\epsilon - \Delta u\|_{L^2} \to 0 \) as \( \epsilon \to 0 \). Since \( qu = 0 \ (|x|^{-2-\alpha-\delta}) \ (|x| \to \infty) \), it is clear that \( \|q u_\epsilon - qu\|_{L^2} \to 0 \) as \( \epsilon \to 0 \). These show that \( \|A u_\epsilon - \varphi\|_{L^2} \to 0 \) as \( \epsilon \to 0 \). Noting that \( A^{1/2} \) is self-adjoint and \( \mathfrak{D}(A) \subset \mathfrak{D}(A^{1/2}) \), we have

\[
\|A^{3/2} u_\epsilon\|_{L^2} \leq 2 \int_E |\Delta u|^2 (1 - e^{-|x|^2})^2 dx + \int_E |q||u|^{2}dx + 0(\epsilon^{\delta+1}) + 0(\epsilon^2) \quad (\epsilon \to 0)
\]

which shows that \( \sup_{0 < \epsilon < \epsilon_0} \|A^{3/2} u_\epsilon\|_{L^2} < \infty \). Thus we have shown that \( u_\epsilon \in \mathfrak{D}(A) \), \( \sup_{0 < \epsilon < \epsilon_0} \|A^{3/2} u_\epsilon\|_{L^2} < \infty \) and \( \|A u_\epsilon - \varphi\|_{L^2} \to 0 \) as \( \epsilon \to 0 \). The fact \( \sup_{0 < \epsilon < \epsilon_0} \|A^{3/2} u_\epsilon\|_{L^2} < \infty \) implies that there exists a function \( g \in L^2(E) \) and a sequence \( \{\epsilon_j\}_{j=1}^\infty \) such that \( \lim_{j \to \infty} \epsilon_j = 0 \) and \( A^{3/2} u_{\epsilon_j} \) converges to \( g \) in \( L^2(E) \) weakly. Let \( H \) be the convex hull of \( \{A^{3/2} u_{\epsilon_j}\}_{j=1}^\infty \). Then it follows from the fact \( \lim_{\epsilon \to 0} \|A u_\epsilon - \varphi\|_{L^2} = 0 \) that \( \{g, \varphi\} \) belongs to the weak closure of the graph \( \{H, A^{3/2} H\} \), so that \( \{g, \varphi\} \) belongs to the strong closure of \( \{H, A^{3/2} H\} \). This shows that \( g \in \mathfrak{D}(A^{1/2}) \) and \( \varphi = A^{3/2} g \), because that \( A^{3/2} u_\epsilon \in \mathfrak{D}(A^{1/2}) \). Thus we have proved Lemma 2.

**Proof of Theorem 1.** Let \( E_\epsilon \) be the resolution of the identity generated by the operator \( A \) such that \( E_{\epsilon, \lambda} = E_\epsilon \). Then, since \( f \in \mathfrak{D}(A^{1/2}) \) and \( A \) has no eigenvalue, the solution \( u(t) \) of the initial value problem (2.1) is given by

\[
u(t) = \int_0^\infty \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dE_\epsilon f.
\]
We decompose $u(t)$ as follows:

$$u(t) = \int_{0}^{N} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dE_{\lambda} f + \int_{N}^{\infty} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dE_{\lambda} f$$

$$\equiv u_{1}(t) + u_{2}(t).$$

Then, since

$$||u_{2}(t)||_{L^{2}}^{2} = \int_{N}^{\infty} \frac{\sin^{2} \sqrt{\lambda} t}{\lambda} d||E_{\lambda} f||_{L^{2}}^{2} \leqq \frac{1}{N} ||f||_{L^{2}}^{2},$$

we can choose $N$ so large that $||u_{2}(t)||_{L^{2}}$ becomes sufficiently small uniformly with respect to $t > 0$. Let $N$ be fixed sufficiently large. Since it follows from the assumption and Lemma 1 that $\langle f, \omega \rangle = 0$ for all $\omega \in M$, we find by the condition $(C_{3})$ and Lemma 2 that there exists a function $g \in \mathfrak{D}(A^{\frac{1}{2}})$ such that

$$A^{\frac{1}{2}} g = f.$$

Therefore we have

$$\langle u_{1}(t), \varphi \rangle_{L^{2}} = \int_{0}^{N} \sin \sqrt{\lambda} t d(E_{\lambda} g, \varphi)$$

for all $\varphi \in L^{2}(E)$.

On the other hand it follows from condition $(C_{3})$ and theorem 6 in [4] that $(E_{\lambda} g, \varphi)_{L^{2}}$ is an absolutely continuous function of $\lambda \in [0, N]$, so that $\frac{d}{d\lambda} (E_{\lambda} g, \varphi)_{L^{2}} \in L^{1}(0, N)$. Consequently by virtue of Riemann-Lebesgue's theorem we obtain

$$\lim_{t \rightarrow \infty} \langle u_{1}(t), \varphi \rangle_{L^{2}} = 0$$

for all $\varphi \in L^{2}(E)$.

This and (3.6) prove (2.2).

Now we proceed to prove (2.3). It follows from the condition $(C_{3})$ that the operator $A$ is strictly positive definite, i.e. $A$ satisfies the inequality

$$\langle A \varphi, \varphi \rangle > 0$$

for $0 \neq \varphi \in \mathfrak{D}(A)$.

We define a functional $\langle \varphi, \phi \rangle_{1}$ for functions $\varphi, \phi \in C_{0}^{\infty}(E)$ as follows:

$$\langle \varphi, \phi \rangle_{1} = \int_{E} \left( \nabla \varphi(x) \overline{\nabla \phi(x)} + q(x) \varphi(x) \overline{\phi(x)} \right) dx.$$

Then by virtue of (3.8) we find that $C_{0}^{\infty}(E)$ is a pre-Hilbert space with the inner product $\langle \varphi, \phi \rangle_{1}$ ($\varphi, \phi \in C_{0}^{\infty}(E)$). Let $H_{1}$ be the completion of $C_{0}^{\infty}(E)$ for the norm $\| \cdot \|_{1} = \sqrt{\langle \cdot, \cdot \rangle_{1}}$ and let $\mathfrak{D}$ be the Hilbert space $H_{1} \times L^{2}(E)$ with the inner product
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\[(\varphi, \phi)_{\mathfrak{H}} = (\varphi_{1}, \phi_{1})_{1} + (\varphi_{2}, \phi_{2})_{L^{2}(E)}, \quad (\varphi, \phi \in \mathfrak{H}),\]

where \(\varphi = [\varphi_{1}, \varphi_{2}], \phi = [\phi_{1}, \phi_{2}]\) and \(\varphi_{1}, \phi_{1} \in H_{1}; \varphi_{2}, \phi_{2} \in L^{2}(E)\).

Let us define an operator \(G_{0}\) of \(\mathfrak{H}\) to \(\mathfrak{H}\) as follows:

\[G_{0} = \begin{bmatrix} 0 & L \\ N & 0 \end{bmatrix},\]

where the operators \(L : L^{2}(E) \rightarrow H_{1}\) and \(N : H_{1} \rightarrow L^{2}(E)\) have domains \(\mathfrak{D}(L) = C_{0}^{\infty}(E) = \mathfrak{D}(N)\), and

\[L\varphi = \varphi \quad \text{for} \quad \varphi \in \mathfrak{D}(L), \quad N\psi = (\Delta - q)\psi \quad \text{for} \quad \psi \in \mathfrak{D}(N).\]

By \(G\) we denote the closure of \(G_{0}\) in \(\mathfrak{H}\). Then we find that the operator \(G\) generates a strongly continuous contraction semi-group on \(\mathfrak{H}\). In fact, the equality

\[(G_{0}\varphi, \varphi)_{\mathfrak{H}} + (\varphi, G_{0}\varphi)_{\mathfrak{H}} = 0 \quad \text{for} \quad \varphi \in \mathfrak{D}(G_{0})\]

implies that for \(\lambda > 0\) we have the inequality

\[||(\lambda I - G)\varphi||_{\mathfrak{H}} \geq \lambda ||\varphi||_{\mathfrak{H}} \quad \text{for} \quad \varphi \in \mathfrak{D}(G_{0}),\]

where \(||\varphi||_{\mathfrak{H}} = \sqrt{(\varphi, \varphi)_{\mathfrak{H}}}\) for \(\varphi \in \mathfrak{H}\). Furthermore the condition \((C_{1})\) and \((3.8)\) imply that

\[(3.11) \quad \mathfrak{D}(G) \supset \mathfrak{D}(A) \times \mathfrak{D}(A^{1/2}).\]

By \((3.11)\) we find that for \(\lambda > 0\)

\[(3.12) \quad R(\lambda I - G) \supset \mathfrak{D}(G_{0})\]

where \(R(\lambda I - G)\) denotes the range of the operator \(\lambda I - G\). \((3.10)\) and \((3.12)\) imply that for any \(\lambda > 0\) the operator \(\lambda I - G\) has the inverse \((\lambda I - G)^{-1}\) defined on \(\mathfrak{H}\) such that \(||(\lambda I - G)^{-1}||_{\mathfrak{H}} \leq \lambda^{-1}\). From this it follows by the Hille-Yosida's theorem that \(G\) is the infinitesimal generator of a strongly continuous semi-group \(U(t)\) on \(\mathfrak{H}\) such that

\[(3.13) \quad ||U(t)||_{\mathfrak{H}} \leq 1 \quad (t \geq 0).\]

Since it follows from \((3.11)\) that \([0, f] \in \mathfrak{D}(G),\) we can find that

\[(3.14) \quad U(t)[0, f] = [u(x, t), u_{t}(x, t)] \quad \text{for} \quad t > 0,\]

where \(u_{t}(x, t) = \frac{\partial}{\partial t} u(x, t).\) \((3.13)\) and \((3.14)\) imply that
\[ \left\| u(\cdot, t), u_t(\cdot, t) \right\|_{L^2}^2 \leq \|f\|_{L^2}^2 \quad \text{for } t > 0, \]

i.e.

\[ \int_E \left[ |Vu(x, t)|^2 + q(x)|u(x, t)|^2 + |u_t(x, t)|^2 \right] dx \leq \|f\|_{L^2}^2 \quad \text{for } t > 0, \]

from which it follows by (3.8) and (3.9) that

\[ \|u(\cdot, t)\|_{L^2} \leq \|f\|_{L^2} \quad \text{for } t > 0. \]

On the other hand (3.5) and (3.7) imply that

\[ \|u(\cdot, t)\|_{L^2} \leq \|g\|_{L^2} \quad \text{for } t > 0. \]

From (3.15), (3.16) and (3.17) it follows by the boundedness of \( q(x) \) that

\[ \|Vu(\cdot, t)\|_{L^2} \leq C \quad \text{for } t > 0. \]

Hereafter we denote by \( C \) constants independent of \( t > 0 \). Thus from (2.2), (3.17) and (3.18) we obtain by the Rellich selection theorem that

\[ \lim_{t \to \infty} \|u(\cdot, t)\|_{L^2(K, x)} = 0 \quad \text{for all compact } K \subset E. \]

Next we shall prove (2.4). It follows from (3.11), (3.13), (3.14) and the identity \( GU(t)[0, f] = U(t)G[0, f] \) that

\[ \int_E \left[ |Vu_t(x, t)|^2 + q(x)|u_t(x, t)|^2 + |(\Delta - q(x))u(x, t)|^2 \right] dx \leq \|A^{1/2}f\|_{L^2}^2 \quad \text{for } t > 0. \]

From this, (3.16) and (3.17) we find by the boundedness of \( q(x) \) that we have

\[ \|\Delta u(\cdot, t)\|_{L^2} \leq C \quad \text{for } t > 0, \]

(3.19)

\[ \|Vu(\cdot, t)\|_{L^2} \leq C \quad \text{for } t > 0. \]

Furthermore, suppose that \( f \in \mathcal{D}(A) \) and \( \sup_{x \in E} |Vu(x)| < \infty \). Then by (3.11) \( [f, 0] \in \mathcal{D}(G) \). Hence, setting \( v(x, t) = u_t(x, t) \) we find that

\[ U(t)[f, 0] = [v(x, t), v_t(x, t)]. \]

Therefore an argument similar to the one used in deriving (3.20) gives

\[ \|Vu_t(\cdot, t)\|_{L^2} \leq C \quad \text{for all } t > 0. \]

On the other hand we have

\[ \nabla \Delta u(x, t) = \nabla v_t(x, t) + \nabla \left( q(x) u(x, t) \right). \]

From this, (3.17), (3.18) and (3.21) it follows by the boundedness of \( q(x) \) and \( \nabla q(x) \) that
(3.22) \[ \| P \Delta u(x, t) \|_{L^2} \leq C \quad \text{for all} \quad t > 0. \]

From (2.2), (3.19) and (3.22) we find by the Rellich selection theorem that

(3.23) \[ \lim_{t \to \infty} \| \Delta u(\cdot, t) \|_{L^2(K)} = 0 \quad \text{for all compact} \quad K \subset E. \]

Consequently from (2.3) and (3.23) we obtain by Sobolev's lemma

(2.4) \[ \lim_{t \to \infty} \sup_{x \in K} |u(x, t)| = 0 \quad \text{for all compact} \quad K \subset E. \]

Thus we have proved Theorem 1.

4. Lemmas.

In this section, we shall establish some lemmas required in proving Theorem 2.

The following lemma 3, 4 and 5 can be verified easily.

**Lemma 3.** 1) Let \( a > 0, \ t > 0 \) be fixed. Then the integral

\[ \int_{a-i\infty}^{a+i\infty} \frac{e^{\zeta t}}{\lambda + \zeta^2} \ d\zeta \]

is a bounded continuous function of \( \lambda \in (0, \infty) \).

2) Let \( a' < 0, \ t_0 > 0 \) be fixed. Then the integral

\[ \int_{a'-i\infty}^{a'+i\infty} \frac{e^{\zeta t}}{\lambda + \zeta^2} \ d\zeta \]

converges uniformly with respect to \( t \geq t_0 \). Furthermore we have

(4.1) \[ \int_{a'-i\infty}^{a'+i\infty} \frac{e^{\zeta t}}{\lambda + \zeta^2} \ d\zeta = 0 \quad \text{for all} \quad t > 0. \]

**Lemma 4.** Let \( t > 0, \ a > 0 \) be fixed. If \( l(\lambda) \) is locally bounded in \( (0, \infty) \) and belongs to \( L^1(0, \infty) \), then the integral

\[ \int_0^\infty |l(\lambda)| \, d\lambda \int_{a-i\infty}^{a+i\infty} \frac{e^{\zeta t}}{\lambda + \zeta^2} \ d\zeta \]

converges for any fixed \( N (|N| > 3a) \).

**Lemma 5.** Let \( a > 0 \) be fixed. Then

\[ u(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\zeta t} R(-\zeta^2) f \ d\zeta \]

is the solution of the problem (2.1), where \( R(-\zeta^2) f \) denotes \( (A + \zeta^2)^{-1} f \).

Now we shall study an asymptotic property of functions of the form
\[ \varphi(x) = \int_{E} \frac{v(y)}{|x-y|} dy. \]

We have the following one similar to Lemma 3.2 in [4].

**Lemma 6.** Let \( v(x) \in L^2(E) \) and \( v(x) = 0 \ (|x|^{-3-s}) \ (0 < s < 1) \ (|x| \to \infty). \) If \( \int_{E} v(x) dx = 0, \) then we have

\[ \varphi(x) = 0 \ (|x|^{-1-s}) \ (|x| \to \infty). \quad (4.2) \]

**Proof.** Let \( R_1 \) be fixed so that \( |v(x)| \leq C_1 |x|^{-3-s} \) for \( |x| \geq R_1 \), and let \( |x| \) be so large that \( \frac{1}{2} |x| = R > R_1. \)

Then we have

\[ \varphi(x) = \int_{|y| \leq R} \frac{v(y)}{|x-y|} dy + \int_{|y| > R} \frac{v(y)}{|x-y|} dy \equiv I + I'. \quad (4.3) \]

\( I' \) can be estimated as follows:

\[
|I'| \leq C_1 \int_{|y| > R} \frac{dy}{|x-y| |y|^{3+s}} \leq C \int_{R}^\infty \int_{0}^{\pi} \frac{r^2 \sin \theta}{r^{3+s}(|x|^2 + r^2 - 2|x| r \cos \theta)^{\frac{1}{2}}} d\theta dr
\]

\[
= C \int_{R}^\infty \int_{0}^{\pi} \frac{r^2}{r^{3+s}} \cdot \frac{1}{|x| r} \frac{d}{d\theta} \left((|x|^2 + r^2 - 2|x| r \cos \theta)^{\frac{1}{2}}\right) d\theta dr
\]

\[
= C \frac{1}{|x|} \int_{R}^{|x|} r^{-1-s} dr + C \int_{|x|}^{\infty} r^{-2-s} dr
\]

\[
\leq C |x|^{-1-s}.
\]

Hereafter we denote by \( C \) constants independent of \( x \). We proceed to estimate \( I \). We have

\[
I = \frac{1}{|x|} \int_{|y| \leq R} v(y) dy - \frac{1}{|x|} \int_{|y| \leq R} \frac{|y|^2 v(y)}{|x-y|(|x| + |x-y|)} dy
\]

\[
+ 2 \int_{|y| \leq R} \frac{\left(\frac{x}{|x|}, y\right) v(y)}{|x-y|(|x| + |x-y|)} dy \equiv I_1 + I_2 + I_3.
\]

It follows from the assumption that

\[
I_1 = -|x|^{-1} \int_{|y| \leq R} v(y) dy.
\]

This implies that
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\[ |I_1| \leq C_1 |x|^{-1} \int_{|y| \geq R} \frac{dy}{|y|^{3+\varepsilon}} \leq C |x|^{-1-\varepsilon}. \]

$I_j$ $(j = 2, 3)$ are estimated as follows:

\[ |I_2| \leq C |x|^{-3} \left[ \int_{|y| \leq R_1} |y|^{2} |v(y)| dy + \int_{R_1 \leq |y| \leq R} |y|^{2} |v(y)| dy \right] \leq C |x|^{-3} + C |x|^{-1-\varepsilon}, \]

\[ |I_3| \leq C |x|^{-2} \left[ \int_{|y| \leq R_1} |y| |v(y)| dy + \int_{R_1 \leq |y| \leq R} |y| |v(y)| dy \right] \leq C |x|^{-2} + C(1-\varepsilon)^{-1} |x|^{-1-\varepsilon}. \]

These estimates and (4.3) give (4.2).

**Lemma 7.** Let $\omega \in M$ and $\omega \neq 0$. Then $\langle q, \omega \rangle \neq 0$ and furthermore $\dim M = \dim M' = 1$, where $M$ and $M'$ are the ones defined in Lemma 1.

**Proof.** Suppose that $\omega \in M$ and $\langle q, \omega \rangle = 0$. Then the function $\omega(x)$ satisfies

\[ \omega(x) = -\frac{1}{4\pi} \int_{E} \frac{q(y)\omega(y)}{|x-y|} dy. \]

Since Lemma 1 implies that $\omega = 0$ ($|x|^{-1}$) ($|x| \to \infty$), from (3.3) it follows by Lemma 6 that $\omega = 0$ ($|x|^{-1-\alpha}$). Applying repeatedly Lemma 6 to (3.3) we find that $\omega \in L^2(E)$. Hence $\omega \in \mathcal{D}(A)$ and $A\omega = 0$. Therefore by the condition (C$_2$) it follows that $\omega = 0$. Thus $\langle q, \omega \rangle \neq 0$ for $\omega \in M$ such that $\omega \neq 0$. From this it follows that $\dim M = 1$, which completes the proof of Lemma 7, since Lemma 1 implies that $\dim M = \dim M'$.

**Remark.** Lemma 7 will be established also in the course of the proof of Theorem 2, for the case that $q$ satisfies the assumptions in Theorem 2.

**Lemma 8.** For $\lambda > 0$ we set

\[ \theta(\lambda) = \theta(x, \lambda) = \frac{1}{2\pi i} \left( u_+(x, \lambda) - u_-(x, \lambda) \right), \]

where $u_\pm(x, \lambda) = R(\lambda \pm i0)f(x)$. Let us define the norm for functions $\varphi \in C^2(E)$ by

\[ \|\varphi\|_{C_{3+\alpha}^2} = \sup_{x \in E, |\beta| \leq 2} \left| D^\beta \varphi(x) \right| \cdot (1 + |x|^2)^{\frac{3+\alpha}{2}} \]

and let $C_{3+\alpha}^2$ be the Banach space $\{ \varphi \in C^2(E); \|\varphi\|_{C_{3+\alpha}^2} < \infty \}$. Then the function of $\lambda$ $T_1(\varphi) = \langle \theta(\lambda), \varphi \rangle$ ($\varphi \in C_{3+\alpha}^2$) is a nuclear operator on $C_{3+\alpha}^2$ to $L^1(0, \infty)$, and
\[ \| T_{i} \| (c_{3+a}^{2})^{*} = \| \theta(\lambda) \| (c_{3+a}^{2})^{*} \text{ belongs to } L^{1}(0, \infty). \]

**Proof.** We introduce the finite measure
\[ d\mu = (1 + |x|^2)^{-\frac{3+a}{2}} \, dx \]
in \( E \), where \( dx \) is the Lebesgue measure in \( E \). Then we decompose \( T_{i} \) as follows:

\[
\begin{align*}
& J_{1} \quad \prod_{|\beta| \leq 2} L^{\infty}(\mu) \quad I_{1} \quad \prod_{|\beta| \leq 2} L^{2}(\mu) \quad J_{2} \quad \prod_{|\beta| \leq 2} L^{2}(dx) \\
& J_{3} \quad W_{2}^{2} \quad I_{2} \quad L^{\infty}(dx) = L^{\infty}(\mu) \quad I_{3} \quad L^{2}(\mu) \quad J_{4} \quad L^{2}(dx) \quad J_{5} \quad L^{1}(0, \infty).
\end{align*}
\]

Let \( F \) be the closed subspace of all vector-valued functions \( \{(D^{\beta} \varphi) \cdot (1 + |x|^2)^{\frac{3+a}{4}}\} \) in \( \prod_{|\beta| \leq 2} L^{2}(dx) \) whose components \( (D^{\beta} \varphi) \cdot (1 + |x|^2)^{\frac{3+a}{4}} \) \( \in L^{2}(dx) \) (\( |\beta| \leq 2 \)).

Furthermore let \( P_{F} \) be the projection operator of \( \prod_{|\beta| \leq 2} L^{2}(dx) \) onto \( F \) and \( J_{3} \) be the operator of \( F \) to \( W_{2}^{2} \) which assigns each vector-valued function \( \{(D^{\beta} \varphi) \cdot (1 + |x|^2)^{\frac{3+a}{4}}\} \) in \( F \) to the scalar-valued function \( \varphi \cdot (1 + |x|^2)^{\frac{3+a}{4}} \) in \( W_{2}^{2} \).

Then \( J_{3} \) may be regarded as the composite operator \( J_{3} \cdot P_{F} \). Hence \( J_{3} \) is defined on the whole space \( \prod_{|\beta| \leq 2} L^{2}(dx) \), so that by the closed graph theorem \( J_{3} \) is a bounded operator on \( \prod_{|\beta| \leq 2} L^{2}(dx) \) to \( W_{2}^{2} \).

\( I_{k} \) \( (k = 1, 2, 3) \) are identities. Since \( d\mu \) is a finite measure we find that \( I_{1} \) and \( I_{3} \) are semi-integral (see [8]). And Sobolev’s lemma implies that \( I_{5} \) is bounded. It is obvious that \( J_{k} \) \( (k = 1, 2, 4) \) are bounded. Thus the identity operator on \( C_{3+a}^{2} \) to \( L^{2}(dx) \) is nuclear (see [8], theorem 14).

On the other hand \( J_{5} \) is a bounded operator on \( L^{2}(E) \) to \( L^{1}(0, \infty) \). In fact, setting
\[ \rho(\varphi) = \int_{0}^{\infty} \frac{d}{d\lambda} (E_{\lambda} f, \varphi) \, d\lambda \quad (\varphi \in L^{2}(E)), \]
we find that \( \rho(\varphi) \) is a semi-norm on \( L^{2}(E) \) and satisfies the inequality
\[ \rho(\varphi) \leq \| f \|_{L^{2}}^2 + \| \varphi \|_{L^{2}}^2 \quad \text{for all } \varphi \in L^{2}(E), \]
which implies that
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\[ \rho(\varphi) \leq \|f\|_{L^2}^2 + 1 \quad \text{for} \quad \|\varphi\|_{L^2} \leq 1. \]

This implies that
\[ \rho(\|\varphi\|_{L^2}^{-1} \varphi) \leq \|f\|_{L^2}^2 + 1 \quad \text{for} \quad 0 \neq \varphi \in L^2(E), \]

so that
\[ \rho(\varphi) \leq (\|f\|_{L^2}^2 + 1)\|\varphi\|_{L^2} \quad \text{for all} \quad \varphi \in L^2(E). \]

This shows that \( J_5 \) is a bounded operator on \( L^2(E) \) to \( L^1(0, \infty) \), if we note that \( \frac{d}{d\lambda}(E_\lambda f, \varphi) = (\theta(\lambda), \varphi) \) for all \( \varphi \in L^2(E) \). Thus \( T_\lambda \) is a nuclear operator on \( C^2_{3+a} \) to \( L^1(0, \infty) \), so that \( \|T_\lambda\|_{(C^2_{3+a})^*} \in L^1(0, \infty) \), which completes the proof of Lemma 8.

**5. Proof of Theorem 2.**

Suppose that there exist functions \( \omega \in M \) such that \( \omega \neq 0 \). Then Lemma 7 implies that \( \dim M = 1 \). Therefore, taking \( \omega \in M \) such that \( \langle q, \omega \rangle = 1 \), since \( q = 0(|x|^{-3-\alpha})(|x| \to \infty) \), we see that any \( \varphi \in L^2(E) \) satisfying the condition \( \varphi = 0(|x|^{-2-\delta})(\delta > \frac{1}{2})(|x| \to \infty) \) can be decomposed as follows:

\[ \varphi = \langle \varphi, \omega \rangle q + (\varphi - \langle \varphi, \omega \rangle q) \equiv \varphi_1 + \varphi_2, \]

while \( \langle \varphi_2, \omega \rangle = 0 \). An argument similar to the one used in proving Theorem 1 gives that \( \lim \langle u(t), \varphi_2 \rangle = 0 \). Therefore we have only to prove

\[ \lim_{t \to \infty} \langle u(t), q \rangle = 4\pi \langle f, \omega \rangle. \]

Now it follows from theorem 4 in [2], the condition \((C_2)\) and theorem 6 in [4] that \( \frac{d}{d\lambda} \langle E_\lambda f, q \rangle \) is continuous in \( \lambda \in (0, \infty) \) and belongs to \( L^1(0, \infty) \). Therefore by virtue of Lemma 3, Lemma 4, Lemma 5 and Fubini’s theorem we obtain

\[
\langle u(t), q \rangle = \frac{1}{2\pi i} \int_{0}^{\infty} \frac{d}{d\lambda} \langle E_\lambda f, q \rangle d\lambda \int_{a-i\infty}^{a+i\infty} \frac{e^{\zeta t}}{\lambda + \zeta^2} d\zeta
= \frac{1}{2\pi i} \int_{0}^{\infty} \frac{d}{d\lambda} \langle E_\lambda f, q \rangle d\lambda \int_{a-i\infty}^{a+i\infty} \frac{e^{\zeta t}}{\lambda + \zeta^2} d\zeta + \int_{N}^{\infty} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \frac{d}{d\lambda} \langle E_\lambda f, q \rangle d\lambda
\]

for \( N > 3a \), where \( \Gamma_1 \) and \( \Gamma_2 \) are the curves \( \{s-iN; 0 < s \leq a\} \cup \{a + i\tau; -N < \}

...
\(\tau < N \cup \{ s+iN; 0 < s \leq a \}, \{ s+iN; -a \leq s < 0 \} \cup \{ -a+i\tau; -N < \tau < N \} \cup \{ s-iN; -a \leq s < 0 \}\) taken in the positive direction respectively. Then we can take \(N\) so large that \(\left| \int_{N^2}^{\infty} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} d\lambda \langle E_{\lambda}f, q \rangle d\lambda \right|\) becomes sufficiently small uniformly with respect to \(t > 0\). Let \(N\) be fixed sufficiently large. Since for \(\zeta \in \Gamma_2\), \(\text{Re} \zeta < 0\), Lemma 4 and Lebesgue’s theorem imply that

\[
\lim_{t \to \infty} \int_{0}^{\infty} \frac{d}{d\lambda} \langle E_{\lambda}f, q \rangle d\lambda \int_{\Gamma_1} \frac{e^{\zeta t}}{\lambda + \zeta^2} d\zeta = 0.
\]

Consequently we have only to prove

\[
(5.2) \quad \lim_{t \to \infty} \int_{0}^{\infty} \frac{d}{d\lambda} \langle E_{\lambda}f, q \rangle d\lambda \int_{\Gamma_1} \frac{e^{\zeta t}}{\lambda + \zeta^2} d\zeta = 8\pi^2 i \langle f, \omega \rangle.
\]

Lemma 4 and Fubini’s theorem imply that

\[
(5.3) \quad \int_{0}^{\infty} \frac{d}{d\lambda} \langle E_{\lambda}f, q \rangle d\lambda \int_{\Gamma_1} \frac{e^{\zeta t}}{\lambda + \zeta^2} d\zeta = \int_{\Gamma_1} \langle R(-\zeta^2)f, q \rangle e^{\zeta t} d\zeta.
\]

For \(\text{Re} \zeta > 0\) we set \(u(\zeta) = R(-\zeta^2)f\). Then \(u(\zeta)\) satisfies

\[
(5.4) \quad u(x, \zeta) = \frac{1}{4\pi} \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) dy - \frac{1}{4\pi} \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} q(y) u(y, \zeta) dy
\]

for \(\text{Re} \zeta > 0\). This and \((3.4)\) imply that

\[
(5.5) \quad \langle u(\zeta), q \rangle = -\frac{1}{\zeta} \int_{E \times E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) q(x) dx dy + \zeta \langle u(\zeta), p(\zeta) \rangle
\]

for \(\text{Re} \zeta > 0\), where

\[
(5.6) \quad p(\zeta) = p(x, \zeta) = q(x) \int_{E} q(y) \omega(y) |x-y| dy \int_{0}^{1} d\tau^\prime \int_{0}^{1} \tau e^{-\zeta|x-y|\tau} d\tau.
\]

In fact, setting \(\omega^\prime = q \omega\) we see that \(\omega^\prime\) satisfies \((3.4)\) and \(\int_{E} \omega^\prime(x) dx = 1\). \((5.4)\) implies that

\[
\langle u(\zeta), \omega^\prime \rangle = \frac{1}{4\pi} \int_{E \times E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) \omega^\prime(x) dx dy - \frac{1}{4\pi} \int_{E \times E} \frac{1}{|x-y|} q(y) u(y, \zeta) \times
\]

\[
\times \omega^\prime(x) dx dy - \frac{1}{4\pi} \int_{E \times E} \frac{e^{-\zeta|x-y|}-1}{|x-y|} q(y) u(y, \zeta) \omega^\prime(x) dx dy.
\]

\((3.4)\) shows that the second term in the right-hand side is equal to \(\langle u(\zeta), \omega^\prime \rangle\). This implies that

\[
\int_{E \times E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) \omega^\prime(x) dx dy = -\zeta \int_{E \times E} q(y) u(y, \zeta) \omega^\prime(x) dx dy \int_{0}^{1} e^{-|x-y|\tau} d\tau
\]

\[
\int_{E \times E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) \omega^\prime(x) dx dy = \int_{E \times E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) \omega^\prime(x) dx dy
\]
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\[ -\zeta \int_{E \times E} q(y) u(y, \zeta) \omega(x) dx dy + \zeta^{2} \int_{E \times E} q(y) u(y, \zeta) \omega(x) |x-y| dx dy \times \int_{0}^{1} d\tau \int_{0}^{\tau} e^{-\zeta|x-y|\tau'} d\tau', \]

which gives (5.5). Furthermore (3.3), (5.3), (5.4), (5.5) and (5.6) imply that

\[ (5.7) \quad \int_{0}^{\infty} \frac{d}{d\lambda} \langle E_{\lambda} f, q \rangle d\lambda \int_{\Gamma_{1}} \frac{e^{r_{t}}}{\zeta} d\zeta = 4\pi \langle f, \omega \rangle \int_{\Gamma_{1}} \frac{e^{r_{t}}}{\zeta} d\zeta + \int_{\Gamma} e^{r_{t}} F(\zeta) d\zeta + \int_{\Gamma_{1}} \zeta e^{r_{t}} \langle R(-\zeta^{2}) f, T_{\zeta}^{*} p(\zeta) \rangle d\zeta, \]

where

\[ F(\zeta) = \int_{E \times E} f(y) q(x) \omega(x) dx dy \int_{0}^{1} e^{-\zeta|x-y|} d\tau + \zeta \sum_{j=0}^{2} \langle T_{\zeta}^{j} \psi, p(\zeta) \rangle, \]

\[ \phi_{\zeta}(x) = \frac{1}{4\pi} \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) dy, \]

\[ T_{\zeta} \phi(x) = -\frac{1}{4\pi} \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} q(y) \phi(y) dy, \]

\[ T_{\zeta}^{*} \phi(x) = -\frac{1}{4\pi} \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} \phi(y) dy, \]

\[ T^{0} \phi(x) = \phi(x), \quad T^{j} \phi(x) = T(T^{j-1} \phi)(x) \quad (j=1,2,3). \]

First we shall show that

\[ (5.8) \quad \lim_{t \to \infty} 4\pi \langle f, \omega \rangle \int_{r_{1}} \frac{e^{r_{t}}}{\zeta} d\zeta = 8\pi^{2} i \langle f, \omega \rangle. \]

We have

\[ \int_{r_{1}} \frac{e^{r_{t}}}{\zeta} d\zeta = \text{Res} \frac{e^{r_{t}}}{\zeta} \bigg|_{\zeta=0} - \int_{r_{1}} \frac{e^{r_{t}}}{\zeta} d\zeta. \]

Lebesgue’s theorem implies that \( \lim_{t \to \infty} \int_{r_{1}} \frac{e^{r_{t}}}{\zeta} d\zeta = 0 \), which gives (5.8).

Next we shall show that

\[ (5.9) \quad \lim_{t \to \infty} \int_{r_{1}} e^{r_{t}} F(\zeta) d\zeta = 0. \]

The conditions \( (C_{3}) \) and \( q = 0 (|x|^{-3-a}) (|x| \to \infty) \) imply that \( F(\zeta) \) is holomorphic in \( \{ \zeta; \text{Re} \zeta > 0 \} \) and is continuous in \( \{ \zeta; \text{Re} \zeta \geq 0 \} \). From this it follows by Lebesgue’s theorem that
\[
\int_{\Gamma_{1}} e^{it} F(\zeta) d\zeta = i \int_{-N}^{N} e^{is} F(is) ds ,
\]
from which we obtain (5.9) by Riemann-Lebesgue's theorem.

Thus the proof of Theorem 2 will be complete, if we prove the following

**Lemma 9.** Let \( q(x) \) be as in Theorem 2. Then

(5.10) \[
\lim_{t \to \infty} \int_{\Gamma_{1}} \zeta e^{it} \langle R(-\zeta^{2})f, p_{3}(\zeta) \rangle d\zeta = 0 ,
\]

where \( p_{3}(\zeta) = p_{3}(x, \zeta) = T_{r}^{*3} p(x, \zeta) \).

To prove Lemma 9 we shall establish the following.

**Lemma 10.** 1) For fixed \( \lambda > 0 \) \( \langle \theta(\lambda), p_{3}(\zeta) \rangle \) is a holomorphic function of \( \zeta \in \{ \text{Re}\zeta > 0 \} \), where \( \theta(\lambda) \) is the one introduced in Lemma 8.

2) Let \( q(x) \) be as in Theorem 2. Then we have

\[
\sup_{x \in E, | \zeta | \leq K} |D_{x}^{\beta} p_{3}(x, \zeta)| (1 + |x|^{2})^{\frac{3+\alpha}{2}} \leq C_{K}
\]

for \( \text{Re}\zeta \geq 0 \), \( |\zeta| \leq K \), where \( K \) is a positive number and \( C_{K} \) is a constant independent of \( x \in E \), \( \zeta \in \{ \text{Re}\zeta \geq 0, |\zeta| \leq K \} \).

**Proof of Lemma 10.** 1) It follows from the condition \( (C_{3}) \) and theorem 4 in [2] that for \( \lambda > 0 \) \( \theta(x, \lambda) \) is a continuous function of \( x \in E \) and behaves like \( 0(|x|^{-1}) (|x| \to \infty) \). Therefore, since \( q=0(|x|^{-2-\alpha}) (|x| \to \infty) \), we see easily that for \( \lambda > 0 \) \( \langle \theta(\lambda), p_{3}(\zeta) \rangle \) is a holomorphic function of \( \zeta \in \{ \text{Re}\zeta > 0 \} \), which proves 1).

We proceed to prove 2). The condition \( q=0(|x|^{-3-\alpha}) (|x| \to \infty) \) and (5.6) imply that for \( \text{Re}\zeta \geq 0 \)

(5.11) \[
|p(x, \zeta)| \leq |q(x)| \left[ |x| \int_{E} |q(y)\omega(y)| dy + \int_{E} |y||q(y)\omega(y)| dy \right] \leq C(1 + |x|)^{-2-\alpha} .
\]

Here and in what follows we denote various constants independent of \( \zeta (\text{Re}\zeta \geq 0) \) by \( C \). (5.11) and the equality

\[
T_{\zeta}^{*} p(x, \zeta) = -\frac{1}{4\pi} q(x) \int_{E} \frac{e^{-\zeta|x-y|}}{|x-y|} p(y, \zeta) dy
\]

imply that

(5.12) \[
|T_{\zeta}^{*} p(x, \zeta)| \leq C(1 + |x|)^{-3-\alpha}
\]

for \( \text{Re}\zeta \geq 0 \).

In the same way we obtain

(5.13) \[
|T_{\zeta}^{*j} p(x, \zeta)| \leq C(1 + |x|)^{-4-\alpha}
\]

for \( \text{Re}\zeta \geq 0 \) \( (j=2,3) \).
The principle of limiting amplitude

The condition $D^\beta q = 0 (|x|^{-2-\alpha}) \ (|x| \rightarrow \infty) \ (|\beta| = 1, 2), \ (5.12)$ and $(5.13)$ imply that

$$|D_x^\beta p_3(x, \zeta)| \leq C_K (1 + |x|)^{-3-\alpha} \quad \text{for } \Re \zeta \geq 0, \ |\zeta| \leq K \ (|\beta| \leq 2),$$

which proves 2) of Lemma 10.

**Proof of Lemma 9.** It follows from 2) of Lemma 10 and Lemma 8 that there exists a function $g(\lambda) \in L^1(0, \infty)$ such that

$$|\langle \theta(\lambda), p_3(\zeta) \rangle| \leq g(\lambda) \quad \text{for } \Re \zeta \geq 0, \ |\zeta| \leq K$$

This, Lemma 4, 1) of Lemma 10 and theorem 4 in [2] imply that

$$\int_{\Gamma_1} \zeta e^{\zeta t} \langle R(-\zeta^2) f, p_3(\zeta) \rangle d\zeta = \int_{\Gamma_1} \zeta e^{\zeta t} d\zeta \int_0^\infty \frac{\langle \theta(\lambda), p_3(\zeta) \rangle}{\lambda + \zeta^2} d\lambda \leq \lim_{\epsilon \rightarrow 0} \int_0^\infty d\lambda \int_{\Gamma_1} \zeta e^{\zeta t} \frac{\langle \theta(\lambda), p_3(\zeta) \rangle}{\lambda + \zeta^2} d\zeta,$$

where $\Gamma_1$ is the path obtained replacing $a$ by $\epsilon$ in $\Gamma_1$. Therefore, by virtue of $(5.14)$, Lemma 4, 2) of Lemma 10, theorem 4 in [2], Lebesgue’s theorem and Riemann-Lebesgue’s theorem we find that we have only to prove

$$\lim_{t \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \int_0^N d\lambda \int_{-N}^N e^{(t+is)t} \frac{\langle \lambda \theta(\lambda^2), p_3(\lambda+s) \rangle}{(\lambda-s)^2 + \epsilon^2} ds = 0.$$

Set $\rho = t - (|x-y| + |y-z| + |z-u| + |u-v| \tau \tau^\prime)$. Then by virtue of Fubini’s theorem, for fixed $t > 0$ and fixed $\epsilon > 0$ we obtain

$$\int_{-N}^N e^{(t+is)t} \frac{\langle \lambda \theta(\lambda^2), p_3(\lambda+s) \rangle}{(\lambda-s)^2 + \epsilon^2} ds = \left(\frac{1}{4\pi}\right) e^{t} \int_E \theta(x, \lambda) \varphi_{*,t}(x, \lambda) dx,$$

where

$$\varphi_{*,t}(x, \lambda) = q(x) \int_E q(y) \frac{q(z)}{|x-y|} dy \int_E q(u) \frac{q(z)}{|x-z|} dz \int_E q(v) \frac{q(u)}{|x-v|} du \times \omega(v) \delta \int_0^1 d\tau^\prime \int_0^1 d\tau e^{-i(t-s)} ds \int_{-N}^N \frac{(s-\lambda) e^{i\tau^\prime}}{(s-\lambda)^2 + \epsilon^2} ds.$$

First we shall show that

$$\lim_{t \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \int_0^N d\lambda \int_{-N}^N e^{(t+is)t} \frac{\langle \lambda \theta(\lambda^2), p_3(\lambda+s) \rangle}{(\lambda-s)^2 + \epsilon^2} ds.$$
$= \left(\frac{1}{4\pi}\right)^3 \int_0^N d\lambda \int_E \lambda \theta(x, \lambda^2) \lim_{t \to 0} \varphi_{*, t}(x, \lambda) dx .

Let $t > 0$ be fixed. Then we find that there exists a constant $C$ such that

$$
\sup_{x \in E, |x| \leq \epsilon} |D^\beta \varphi_{*, t}(x, \lambda)| (1 + |x|^2)^{\frac{3+\alpha}{2}} \leq C \left(1 + \log \frac{N + \lambda}{N - \lambda}\right)
$$

for all $\lambda < N$ and all $\epsilon \leq \epsilon_0$, where $\epsilon_0$ is a positive number. In fact, since $s \cos s$ is an odd function, for $\lambda < N$ we have

$$
\int_{-N}^{N} \frac{(s-\lambda)e^{i\rho s}}{(s-\lambda)^2+\epsilon^2} ds = e^{i\rho \lambda} \left[ \int_{(-N-\lambda)\rho}^{(\lambda-N)\rho} \frac{\cos s}{s} ds - \epsilon^2 \rho \int_{(-N-\lambda)\rho}^{(\lambda-N)\rho} \frac{\cos s}{s(s^2+\epsilon^2\rho^2)} ds \right] + i \left[ \int_{(-N-\lambda)\rho}^{(\lambda-N)\rho} \frac{\sin s}{s} ds - \pi \right] \left[ \int_{(-N-\lambda)\rho}^{(\lambda-N)\rho} \frac{\sin s}{s(s^2+\epsilon^2\rho^2)} ds \right]
$$

which implies that

$$
\left| \int_{-N}^{N} \frac{(s-\lambda)e^{i\rho s}}{(s-\lambda)^2+\epsilon^2} ds \right| \leq C' \left(1 + \log \frac{N + \lambda}{N - \lambda}\right),
$$

where $C'$ is a constant independent of $\epsilon, \lambda, \rho$. Since $q = 0 (|x|^{-3-\alpha})$, $D^\beta q = 0 (|x|^{-2-\alpha}) (|x| \to \infty) (|\beta| = 1, 2)$ and $t - \rho \geq 0$, (5.18) and (5.22) imply (5.20). On the other hand, it follows from theorem 4 in [2] that for each function $\varphi$ satisfying $|\varphi(x)| \leq C(1 + |x|^2)^{-\frac{2+\alpha}{2}}$, $\int_E \lambda \theta(x, \lambda^2) \varphi(x) dx$ is a continuous function of $\lambda \in (0, \infty)$. Consequently by virtue of (5.17), (5.20), Lemma 8 and Lebesgue's theorem we obtain (5.19).

Next we shall show that

$$
\lim_{t \to \infty} \int_0^N d\lambda \int_E \lambda \theta(x, \lambda^2) \lim_{t \to 0} \varphi_{*, t}(x, \lambda) d\lambda = 0 .
$$

From (5.18), (5.21) and (5.22) it follows by Lebesgue's theorem that for fixed $\lambda < N$ we have

$$
\lim_{t \to 0} \varphi_{*, t}(x, \lambda) = q(x) \left[ \int_E \frac{q(y)}{|x-y|} dy \int_E \frac{q(z)}{|y-z|} dz \int_E \frac{q(u)}{|z-u|} du \right] |u-v| x \times q(v) \omega(v) dv \int_0^1 \tau e^{i\rho \tau} d\tau \left[ \int_{(-N-\lambda)\rho}^{(\lambda-N)\rho} \frac{\cos s}{s} ds + i\pi \right] \equiv J_{1,t}(x, \lambda) + J_{2,t}(x, \lambda) + J_{3,t}(x, \lambda) .
$$
The principle of limiting amplitude

Set

\[ \varphi_2(x, \lambda) = e^{-it} J_{2,t}(x, \lambda). \]

Then, since \( \rho = t - (|x - y| + |y - z| + |z - u| + |u - v| \tau') \) and \( q = 0 (|x|^{-3-a}) \), \( D^q q = 0 (|x|^{-2-a}) \) \((\beta = 1, 2)\), we find that

\[ \sup_{x \in \mathbb{R}, |\beta| \leq 2} |D^\beta \varphi_2(x, \lambda)| (1 + |x|^2)^{\frac{3+a}{2}} \leq C \]

for all \( \lambda < N \), where \( C \) is a constant independent of \( \lambda \). Therefore by virtue of Lemma 8 and Riemann-Lebesgue’s theorem we obtain

\[ (5.25) \quad \lim_{t \to \infty} \int_0^N d\lambda \int_E \theta(x, \lambda^2) J_{2,t}(x, \lambda) \, dx = \lim_{t \to \infty} \int_0^N e^{it} d\lambda \int_E \theta(x, \lambda^2) \varphi_2(x, \lambda) dx = 0. \]

Let \( \rho - t \) be fixed. Then \( \rho \to \infty \) as \( t \to \infty \). This implies that

\[ \lim_{t \to \infty} \int_{(\rho-N)^p}^{\rho} \frac{\cos s}{s} ds = \lim_{t \to \infty} \int_{(\rho-N)^p}^{\rho} \frac{\sin s}{s} ds - \pi = 0. \]

Consequently an argument similar to the one used in proving (5.19) gives that

\[ (5.26) \quad \lim_{t \to \infty} \int_0^N d\lambda \int_E \theta(x, \lambda^2) J_{k,t}(x, \lambda) dx = 0 \quad (k = 1, 3). \]

Thus (5.24), (5.25) and (5.26) imply (5.23). Therefore (5.19) and (5.23) prove (5.16).

An argument similar to the one used in proving (5.16) gives that

\[ \lim_{t \to \infty} \int_0^{2N} d\lambda \int_{-N}^{N} e^{(\cdot + is)t} \frac{\langle \lambda \theta(\lambda^2), p_3(z+is) \rangle}{(\lambda-s)^2 + \epsilon^2} ds = 0, \]

\[ \lim_{t \to \infty} \int_0^{2N} d\lambda \int_{-N}^{N} e^{(\cdot + is)t} \frac{\epsilon \langle \lambda \theta(\lambda^2), p_3(z+is) \rangle}{(\lambda-s)^2 + \epsilon^2} ds = 0. \]

These and (5.16) imply that

\[ \lim_{t \to \infty} \int_0^{2N} d\lambda \int_{-iN}^{iN} e^{i\xi} \frac{\langle \lambda \theta(\lambda^2), p_3(\xi) \rangle}{\lambda + i\xi} d\xi = 0. \]

In the same way we obtain

\[ \lim_{t \to \infty} \int_0^{2N} d\lambda \int_{-iN}^{iN} e^{-i\xi} \frac{\langle \lambda \theta(\lambda^2), p_3(\xi) \rangle}{\lambda - i\xi} d\xi = 0. \]
Thus we have proved (5.15), which completes the proof of Lemma 9.

Remark. If the condition (C_2) is not assumed, then, as O. A. Ladyzhenskaya pointed out in [1], it is seen that in general the principle of limiting amplitude is not valid. Let $E_i$ be the resolution of the identity generated by the operator $A$. Furthermore let $-\mu_n (\mu_n>0) \ (n=1,2,\cdots,k)$ be the negative eigenvalues of the operator $A$ and $\varphi_n (n=1,2,\cdots,k)$ be the eigenfunctions associated with the eigenvalues $-\mu_n$, and $\psi_n (n\geq 1)$ be the eigenfunctions of $A$ associated with the eigenvalue zero. Then the solution $u(t)$ of (2.1) is represented as follows:

$$u(t) = \int_0^\infty \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dE_i f + t \sum_{n=1}^k (f, \varphi_n)_{L^2} \varphi_n + \sum_{n=1}^k \frac{e^{\sqrt{\mu_n} t} - e^{-\sqrt{\mu_n} t}}{2\sqrt{\mu_n}} (f, \varphi_n)_{L^2} \varphi_n.$$

The first term in the right-hand side behaves like $O(t)$ at infinity. Therefore Theorem 2 implies that our conjecture is true in the case that $q=0(|x|^{-3-a})$ at infinity.

References


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