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<td>Author(s)</td>
<td>Miyashita, Yôichi</td>
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<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 20(1-2), 001-026</td>
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<td>Issue Date</td>
<td>1967</td>
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<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56082">http://hdl.handle.net/2115/56082</a></td>
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<td>File Information</td>
<td>JFSHIU_20_N1-2_001-026.pdf</td>
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<td>Hokkaido University Collection of Scholarly and Academic Papers</td>
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LOCALLY FINITE OUTER GALOIS THEORY

By

Yôichi MIYASHITA

Introduction.

This paper is the continuation of the preceding paper [22]. In §1 and §2, locally finite (outer) Galois extensions are treated. The main results are parallel to those of the finite case. In these studies, Nagahara [12] is our guide. Further several results for finite Galois extensions are added (Th. 1.18). In §3, we give a normal basis theorem for a finite Galois extension.

§1. As to notations and terminologies we follow [22]. Let A be a ring with 1 (≠ 0), C the center of A, G a (finite or infinite) group of automorphisms of A, \( B = A^g = \{ x \in A ; \sigma(x) = x \text{ for all } \sigma \in G \} \), and \( \hat{G} \) the group of all B-automorphisms of A. \( \hat{G} \) is then a topological group in finite topology (cf. Jacobson [7]). We denote the closure of G in \( \hat{G} \) by \( G^* \). \( \Delta \) means the trivial crossed product of A with G: \( \Delta = \sum_{\sigma \in G} A \sigma \), \( u_r u_r = u_r \sigma, \) (\( \sigma, \tau \in G \)), \( u_r x = \sigma(x) u_r \) (\( x \in A \)). Then there is a canonical ring homomorphism j from \( \Delta \) to End \( (A_B) \) defined by \( j(\sum_{\sigma} x_{\sigma}u_{\sigma})(y) = \sum_{\sigma} x_{\sigma}(y) \) \( (\sum_{\sigma} x_{\sigma}u_{\sigma} \in \Delta, y \in A) \). For any intermediate ring T of A/B, \( G^T = \{ \sigma \in G ; \sigma|T = 1 \} \) is a subgroup of G, where \( \sigma|T \) means the restriction of \( \sigma \) to T. We call it a fixed subgroup of G. For any subgroup H of G, \( A^H = \{ x \in A ; \sigma(x) = x \text{ for all } \sigma \in H \} \) is an intermediate ring of A/B. We call it a fixed subring of A (with respect to G). Then, as is well known, the set of all fixed subgroups of G and the set of all fixed subrings of A are anti-order-isomorphic in the usual sense of Galois theory. A subring T of A is called a G-invariant subring of A if \( \sigma(T) = T \text{ for all } \sigma \in G \) (or equivalently, \( \sigma(T) \subseteq T \) for all \( \sigma \in G \)). Let N be a fixed subgroup of G. Then, \( A^N \) is G-invariant if and only if N is a normal subgroup of G: \( N \triangleleft G \). Let T be an intermediate ring of A/B, and put \( H = G^T \). Then, for \( \sigma, \tau \in G, \sigma|T = \tau|T \text{ if and only if } \sigma H = \tau H \). Let H and K be subgroups of G such that \( H \supseteq K \text{ and } (H:K) < \infty \), and let \( H = \sigma_1 K \cup \cdots \cup \sigma_r K \) be the left coset decomposition. For any \( x \in A^K \) we put \( t_{B;K}(x) = \sum_{\sigma} \sigma(x) \). Then \( t_{B;K} \) is an \( A^K - A^K \)-homomorphism from \( A^K \) to \( A^K \), and is independent of the choice of \( \sigma_1, \cdots, \sigma_r \). If \( K = 1 \), we write simply \( t_B \) instead of \( t_{B;1} \).

Here we present several fundamental facts, which are essential throughout the present study. Let \( \tau M_B \) and \( \tau N_B \) be T-left, U-right modules. If \( \tau M_B \) is
isomorphic to a direct summand of $\tau N^r_U$ for some natural number $r$, then we write $\tau M_U|_{\tau N^r_U}$, where $\tau N^r_U$ means the direct sum of $r$ copies of $\tau N_U$. If $\tau M_U|_{\tau N_U}$ and $\tau N_U|\tau M_U$ we write $\tau M_U \sim \tau N_U$ (similar) (cf. Morita [21]). To be easily seen, $\tau M_U|_{\tau N^r_U}$ if and only if there are $T-U$-homomorphisms $f_1, \cdots, f_r$ in Hom $(\tau M_U, \tau N_U)$ and $g_1, \cdots, g_r$ in Hom $(\tau N_U, \tau M_U)$ such that $\Sigma_i f_i g_i=$ the identity of $M$, or equivalently, Hom $(\tau M_U, \tau N_U)$Hom $(\tau N_U, \tau M_U)$Hom $(\tau M_U, \tau M_U)$, where homomorphisms act on the right side.

Let $T$ be a ring with 1, $M$ a unital $T$-left module, and $T^* = \text{End}(\tau M)$. 

S.1. If $\tau T |_{\tau M}$ then $M_T |_{T^*}$ (i.e. $M_T$ is finitely generated and projective) and $T = \text{End} (M_T)$. (Morita)

S.2. If $\tau M |_{\tau T}$ then $T^* |_{T^*}$. (Morita)

S.3. Let $T$ be commutative. If $\tau M |_{\tau T}$ and $\tau M$ is faithful, then $\tau T |_{\tau M}$. (Auslander-Buchsbaum-Goldman)

S.4. Let $T$ be an extension ring of $T$. If $\tau T |_{\tau T}$ then $\tau T$ is a direct summand of $\tau T^*$ (and conversely). (Müller)

S.5. Let $T$ be an extension ring of $T$. If $\tau T_T |_{\tau T}$ then $\tau T_T$ is a direct summand of $\tau T_T$. (The proof is similar to the one of S.4.)

In [22], $A/B$ was called a G-Galois extension if $G$ is finite and there are elements $a_1, \cdots, a_n; a_1^*, \cdots, a_n^*$ in $A$ such that $\Sigma_i a_i \sigma(a_i^*) = \delta_{1, \sigma}$ ($\sigma \in G$). In this paper, $A/B$ is called a finite G-Galois extension if $A/B$ is G-Galois and $t_\sigma (c) = 1$ for some $c$ in $A$. Then, the following are equivalent:

(a) $A/B$ is finite G-Galois.
(b) $G$ is finite, $A_{\sigma} \sim B_{\sigma}$ and $j: A \simeq \text{End} (A_{\sigma})$.
(c) $G$ is finite and $\Delta \sim_A A$.

(Cf. S.1, S.2, [6] and [21]).

$A/B$ is called a locally finite G-Galois extension if there are fixed normal subgroups $N_\lambda$ ($\lambda \in \Delta$) of $G$ which satisfy the following conditions: (1) $(G: N_\lambda) < \infty$, and $A^N/B$ is a finite $G/N_\lambda$-Galois extension. (2) $A = \bigcup_{\lambda} A^{N_\lambda}$, and $\{A^{N_\lambda}; \lambda \in \Delta\}$ is a directed set with respect to the inclusion relation (abbre. $A = \bigcup_{\lambda} A^{N_\lambda}$ is a directed union). Then we call $A = \bigcup_{\lambda} A^{N_\lambda}$ a representation of the locally finite G-Galois extension $A/B$. If $V_{A}(B)=C$, an extension $A/B$ is said to be outer.

Now we shall prove first the following

**Proposition 1.1.** Let $G = G^*$ (i.e. $G$ is closed in $\hat{G}$). Then the following are equivalent:

(i) $\{\sigma(x); \sigma \in G\}$ is finite for any $x$ in $A$.
(ii) $G$ is compact.
(iii) Every directed union of fixed subrings of $A$ with respect to $G$ is also a fixed subring of $A$ with respect to $G$, and $\bigcap H = 1$, where $H$ ranges
over all fixed subgroups of $G$ such that $(G:H)<\infty$.

Proof. (i) $\Rightarrow$ (ii) If we put $\prod_{x\in A}\{\sigma(x); \sigma \in G\}=D$, then $G \subseteq D$ and $D$ is compact. Therefore it is sufficient to prove that $G$ is closed in $D$. Let $\rho$ be any element of the closure of $G$ in $D$. Then, as is easily seen, $\rho$ is a $B$-ring isomorphism from $A$ into $A$. Let $a$ be in $A$, and put $F=\{\sigma(a); \sigma \in G\}$. Then, by assumption, $F$ is a finite subset of $A$, so that there is an element $\tau$ in $G$ such that $\rho|F=\tau|F$. Then, in particular, $\rho(\tau^{-1}(a))=\tau(\tau^{-1}(a))=a$. Thus $\rho$ is a $B$-automorphism of $A$. Hence the closure of $G$ in $D$ is contained in $\hat{G}$. Since $G$ is closed in $\hat{G}$, $G$ is closed in $D$, as desired. (ii) $\Rightarrow$ (iii) For any $x$ in $G$, we put $H_x=\{\sigma \in G; \sigma(x)=x\}$. Then $H_x$ is open in $G$, and therefore $\sigma H_x$ is open in $G$ for any $\sigma$ in $G$. Then, since $G$ is compact, we have $(G:H_x)<\infty$. Evidently $\cap_{x\in A}H_x=1$. This proves the second assertion. Let $(A\neq) T=\bigcup_{i\in A}T_i$ be a directed union of fixed subrings of $A$ with respect to $G$, and let $K_i=G^{T_i}$. Then each $K_i$ is a closed subgroup of $G$, and $A^{K_i}=T_i$. Let $a$ be an element of $A-T$, and put $U=\{\sigma \in G; \sigma(a)=a\}$. Then $U$ is open in $G$, so that each $K_i-U$ is closed in $G$. Since $a \notin T_i$ and $A^{K_i}=T_i$, we have $K_i-U \neq \emptyset$. For any finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of $A$, there is an element $\lambda_0$ of $A$ such that $T_{\lambda_0} \supseteq \bigcup_i T_{\lambda_i}$. Then $K_{\lambda_0} \subseteq \bigcap_i K_{\lambda_i}$, and so $\emptyset \neq K_{\lambda_0}-U \subseteq \bigcap_i K_{\lambda_i}-U=\bigcap_i (K_{\lambda_i}-U)$. Thus $\{K_{\lambda_i}-U; \lambda \in A\}$ has finite intersection property. Since $G$ is compact, we have $\bigcap_i (K_{\lambda_i}-U) \neq \emptyset$. If $\rho$ is in $\bigcap_i (K_{\lambda_i}-U)$ then $\rho \in G^T$ and $\rho(a) \neq a$. Therefore $a \neq A^\rho$, where $K=G^T$. Thus $A^\rho=T$. Hence $T$ is a fixed subring of $A$ with respect to $G$. (iii) $\Rightarrow$ (i) Let $H$ and $K$ be fixed subgroups of $G$ such that $(G:H)<\infty$ and $(G:K)<\infty$. Then $H \cap K$ is also a fixed subgroup of $G$ with $(G:H \cap K)<\infty$. Therefore $\cup A^U$ is a directed union of fixed subrings of $A$, where $H$ ranges over all fixed subgroups of $G$ with $(G:H)<\infty$. Then, by assumption, $\cup A^U$ is a fixed subring of $A$ with respect to $G$. Since $\bigcap H=1$, we have $A=\cup A^U$. For any $x$ in $A$, there is an $A^U$ such that $x \in A^U$. Therefore if we put $L=\{\sigma \in G; \sigma(x)=x\}$ then $(G:L)<\infty$. This implies that $\{\sigma(x); \sigma \in G\}$ is finite.

Remark. For any $x$ in $A$, $\{\sigma(x); \sigma \in G\}=\{\sigma(x); \sigma \in G^*\}$.

Proposition 1.2. Let $N$ be a fixed normal subgroup of $G$ such that $(G:N)<\infty$ and $A^N/B$ is finite $G/N$-Galois, and $G_1$ a subgroup of $G^*$ containing $G$. Then $A^N/B$ is finite $G_1/N_1$-Galois, where $N_1=\{\sigma \in G_1; \sigma|A^N=1_{A^N}\}$.

Proof. Put $T=A^N$. Evidently $A^N=T$. Since $G$ is dense in $G_1$ and $T$ is finitely generated, there holds $G|T=G_1|T$. Therefore $T$ is $G_1$-invariant, $N_1 \subseteq G_1$, and $(G_1:N_1)<\infty$. There are elements $a_1, \cdots, a_n; a_1^*, \cdots, a_n^*$ in $T$ such that $\sum_ia_i\cdot \sigma(a_i^*)=\delta_{N, \sigma}$ for all $\sigma$ in $G$. If $\tau$ is in $G_1-N_1$ then $\tau|T=\rho|T$ for
some $\rho$ in $G-N$, and $\sum_{i}a_{i}\cdot\rho(a_{i}t)=\sum_{i}a_{i}\cdot\rho(a_{i}t)\cdot\delta_{N_{i},t}$ for $\sigma$ in $G_{1}$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, and $G_{1}$ a subgroup of $G^{*}$ containing $G$. Then $A/B$ is locally finite $G_{1}$-Galois.

**Proposition 1.3.** Let $H_{i}(\lambda \in \Lambda)$ be fixed subgroups of $G$ such that $A=\bigcup_{\lambda \in \Lambda}A^{H_{i}}$ is a directed union.

1. If $H$ is a subgroup of $G$ such that $(G:H)<\infty$ then $A^{H}\subseteq A^{H_{i}}$ for some $\lambda \in \Lambda$.

2. If $K$ is a subgroup of $G$ such that $(K:1)<\infty$ then $K\cap H_{i}=1$ for some $\mu \in \Lambda$.

**Proof.** (1) Let $[H_{i}\cup H]$ be the subgroup of $G$ generated by $H_{i}\cup H$. Since $G\supseteq[H_{i}\cup H]\supseteq H$, we have $(G:[H_{i}\cup H])<(G:H)$ for all $\lambda \in \Lambda$. Let $(G:[H_{i}\cup H])$ be maximum. We shall prove that $A^{H}\subseteq A^{H_{i}}$. For any $H_{i}$ there is an $H_{i}$ such that $A^{H_{i}}\supseteq A^{H_{i}}$. Then $H_{i}\subseteq H_{i}\cap H_{i}$, and so $[H_{i}\cup H]\subseteq[H_{i}\cup H]$. Hence $[H_{i}\cup H]\subseteq[H_{i}\cup H]$ for all $\lambda \in \Lambda$. Then $A^{H}=\bigcup_{\lambda}(A^{H}\cap A^{H})=\bigcup_{\lambda}A^{[H\cup H]}=A^{[H_{i}\cup H]}A^{H_{i}}\cap A^{H}$, which means $A^{H}\subseteq A^{H_{i}}$. (2) Since $A=\bigcup_{\lambda}A^{H_{i}}$, we have $1=A=A^{\bigcap_{\lambda}H_{i}}$. Let $K=\{\sigma_{i}=1, \sigma_{2}, \cdots, \sigma_{r}\}$. Then, for any $\sigma_{i} (i \neq 1)$, there is an $H_{i}$ such that $\sigma_{i} \notin H_{i}$. By assumption there is a $\mu$ such that $H_{\mu}\subseteq \bigcap_{i=1, \cdots, r}H_{i}$. Then $H\cap H_{\mu}\subseteq H\cap (\bigcap_{i=1, \cdots, r}H_{i})=1$.

**Remark.** Let $A/B$ be locally finite $G$-Galois, and $A=\bigcup_{\lambda \in \Lambda}A^{N_{i}}$ its representation. If $G$ is finite then $A=A^{N_{i}}$ for some $\lambda$.

**Proposition 1.4.** Let $T$ be an intermediate ring of $A/B$ such that $G|T$ is finite, and let $H=\sigma_{1}H_{1}\cup\cdots\cup\sigma_{r}H_{r}$ a left coset decomposition of $G$. If there are elements $t_{1}, \cdots, t_{n}; t_{1}^{*}, \cdots, t_{n}^{*}$ in $T$ such that $\sum_{i}t_{i}^{*}\sigma(t_{i}^{*})=\delta_{u_{i}}$ for all $\sigma$ in $G$, then there hold the following.

1. $T=A^{u}$, and $T_{n}$ is finitely generated and projective.

2. $j^{*}: A(\Sigma_{k}u_{k})T=\Sigma_{k}Au_{k}\simeq\text{Hom}(T_{n}, A_B)$, where $j^{*}(\Sigma_{k}x_{k}u_{k})T=(\Sigma_{k}x_{k}\cdot\sigma(t_{k}^{*}))T$, and this induces the $B-T$-isomorphism $(\Sigma_{k}u_{k})T\simeq\text{Hom}(T_{n}, A_B)$.

3. The following are equivalent: (i) $B_{n}|T_{n}$. (ii) $B|T$. (iii) $t_{n}\cdot T(c)=1$ for some $c$ in $T$.

**Proof.** (1) $t_{n}: B\to B$ is a $B-B$-homomorphism from $A^{u}$ to $B$. For any $y$ in $A^{u}$, $T\subseteq\Sigma_{k}t_{k}^{*}t_{n}(y)=\Sigma_{k}t_{k}^{*}\Sigma_{k}\sigma_{k}(y)=\Sigma_{k}\Sigma_{i}t_{i}^{*}\sigma_{k}(t_{k}^{*})\sigma_{k}(y)=y$. Hence $A^{u}=T$, and $T_{n}$ is finitely generated and projective (cf. [3]). (2) $j^{*^{-1}}$ is the mapping such that $j^{*^{-1}}(f)=\Sigma_{i}f(t_{i}(\Sigma_{k}u_{k}))t_{n}^{*}$ ($f \in \text{Hom}(T_{n}, A_B)$). The second part will be easily seen. (3) The equivalence (i)$\iff$ (iii) is easy from (2).
Therefore (i) and (ii) are equivalent, because the situation is right-left symmetric.

**Proposition 1.5.** Let $A/B$ be locally finite $G$-Galois. Then there hold the following:

1. $G^*$ is compact.
2. By $j$, $A$ is isomorphic to a dense subring of $\text{Hom}(A_B, A_B)$.
3. A subgroup $H$ of $G$ is a closed subgroup of $G$ if and only if $H$ is a fixed subgroup of $G$.

**Proof.** Let $A = \bigcup_{\nu} A^{N_{\nu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) If $x$ is in $A$ then $x \in A^{N_{\nu}}$ for some $\nu$ in $A$. Then $(G : N_{\nu}) < \infty$ implies that $(\{\sigma(x); \sigma \in G\} = \{\sigma(x); \sigma \in G^*\}$ is finite. Hence, by Prop. 1.1, $G^*$ is compact. (2) By Prop. 1.4 (2), $\text{Im} j$ is dense in $\text{Hom}(A_B, A_B)$. Therefore it suffices to prove that $j$ is 1–1. Let $\sigma_1, \ldots, \sigma_r$ be different elements in $G$. Then there is a finite subset $F$ of $A$ such that $\sigma_i|F \neq \sigma_j|F$ provided $i \neq k$. From this fact and Prop. 1.4, we can easily see that $j$ is 1–1. (3) Evidently, a fixed subgroup is a closed subgroup. Let $H$ be any subgroup of $G$, and put $H' = G^*$, where $T = A^H$. Then $T = A^H$. It suffices to prove that $H$ is dense in $H'$. To prove this, we take any finite subset $F$ of $A$. Then $F \subseteq A^{N_{\nu}}$ for some $N_{\nu}$. Put $N = N_{\nu}$. Then, by finite Galois theory, we obtain $(G/N)^{T_{1}} = HN/N$ and $(A/N)^{T_{1}} = H'N/N$, where $T_1 = A^{HN}$ and $T_1 = A^{H'N}$ (cf. [22; Prop. 2.2]). Since $A^{HN} = A^H \cap A^N = A^W \cap A^N = A^{H'N}$, we have $HN/N = H'N/N$, that is, $HN = H'N$. Hence $H|A^N = H'|A^N$, and so $H|F = H'|F$. Since $F$ is arbitrary, this implies that $H$ is dense in $H'$. This completes the proof.

**Theorem 1.6.** Let $A/B$ be locally finite $G$-Galois, $G = G^*$, and $H$ a subgroup of $G$, and let $A'$ be an indecomposable extension ring of $B$ such that $V_{A'}(B) = V_{A'}(A')$. Assume that there is a $B$-ring homomorphism $g$ from $A$ to $A'$. Then, for any $B$-ring homomorphism $f$ from $A^H$ to $A'$, there is an element $\sigma$ in $G$ such that $f = g\sigma|A^H$.

**Proof.** Let $A = \bigcup_{\nu} A^{N_{\nu}}$ be a representation. For each $N_{\nu}$, there is an element $\sigma$ in $G$ such that $f|A^{N_{\nu}} = g\sigma|A^{N_{\nu}}$ ([22; Th. 4.1]). For each $\lambda$, we put $K_{\lambda} = \{\sigma \in G; f|A^{N_{\nu}} = g\sigma|A^{N_{\nu}}\}$. Then $K_{\lambda} \neq \emptyset$, and $\{K_{\lambda}; \lambda \in \Lambda\}$ has finite intersection property. Let $\tau$ be in the closure of $K_{\lambda}$ in $G$. Since $(A^{N_{\lambda}})_B$ is finitely generated, $\tau|A^{N_{\lambda}} = \alpha|A^{N_{\lambda}}$ for some $\alpha$ in $K_{\lambda}$. Then $\tau|A^{N_{\lambda}} = \alpha|A^{N_{\lambda}}$, and so $f|A^{N_{\lambda}} = g\alpha|A^{N_{\lambda}} = g\tau|A^{N_{\lambda}}$. Hence $\tau \in K_{\lambda}$, and therefore $K_{\lambda}$ is closed in $G$. Since $G$ is compact (Prop. 1.5), we have $\bigcap K_{\lambda} \neq \emptyset$. If $\rho$ is in $\bigcap K_{\lambda}$, then $f|A^{N_{\lambda}} = g\rho|A^{N_{\lambda}}$ for all $\lambda$ in $\Lambda$. Since $A^H = \bigcup A^{N_{\nu}}$, we know $f = g\rho|A^H$.

The following theorem will follow at once from Th. 1.6 and Cor. to Prop. 1.2.

**Theorem 1.7.** Let $A/B$ be locally finite outer $G$-Galois, and $A$ an
indecomposable ring. Then \(G^* = \hat{G}\), that is, \(G\) is dense in \(\hat{G}\).

**Proposition 1.8.** Let \(A/B\) be locally finite \(G\)-Galois, and \(G = G^*\) (cf. Cor. to Prop. 1.2). Then there hold the following.

(1) For an intermediate ring \(T\) of \(A/B\) the following are equivalent.

\((i)\) \(T = A^H\) for some subgroup \(H\) of \(G\). \((ii)\) There are subgroups \(H_t\) (\(t \in \Gamma\)) of \(G\) such that \(T = \bigcup_t A^{H_t}\), \((G:H_t) < \infty\) and \(\{A^{H_t}; t \in \Gamma\}\) is a directed set with respect to the inclusion relation.

(2) If \(H\) is a subgroup of \(G\) such that \((G:H) < \infty\) then \((A^H)_B\) is finitely generated.

**Proof.** Let \(A = \bigcup_{i \epsilon I} A^{N_i}\) be a representation of the locally finite \(G\)-Galois extension \(A/B\). (1) \((i) \Rightarrow (ii)\) \(T = A^H = \bigcup_i (A^H \cap A^{N_i}) = \bigcup_i A^{H\cap N_i}\) is a directed union, and \((G:HN_i) < \infty\). \((ii) \Rightarrow (i)\) follows from Prop. 1.1. (2) By Prop. 1.3, \(A^H \subseteq A^\nu\) for some \(\nu\) in \(A\). Then, \(A^H = A^{H\nu}\) is a fixed subring of the finite \(G/N_\nu\)-Galois extension \(A^\nu/B\), and therefore \((A^\nu)_B/(A^\nu)_B\) (cf. [22; §2. p. 118]). Since \((A^\nu)_B\) is finitely generated, \((A^H)_B\) is finitely generated.

Let \(T\) be an intermediate ring of \(A/B\), and \(S\) a subset of \(A\). \(T\) is called a \(G\)-separable cover of \(S\) if \(T\) satisfies the following conditions:

(1) \(T/B\) is a separable extension, and \(T \supseteq S\).

(2) \(G/T\) is finite.

(3) \(G/T\) is strongly distinct (i.e. if \(\sigma|T \neq \tau|T\) for \(\sigma, \tau\) in \(G\) then \(\sigma|T\) and \(\tau|T\) are strongly distinct).

**Theorem 1.9.** Let \(A/B\) be locally finite outer \(G\)-Galois, and \(T\) an intermediate ring of \(A/B\). Then the following are equivalent:

(i) \(T = A^H\) for some subgroup \(H\) of \(G\) such that \((G:H) < \infty\).

(ii) \(T/B\) is a separable extension, \(T_B\) is finitely generated, and \(G/T\) is strongly distinct.

(iii) \(T\) is a \(G\)-separable cover of \(B\).

**Proof.** Let \(A = \bigcup_{i \epsilon I} A^{N_i}\) be a representation. (i) \(\Rightarrow (ii)\) By Prop. 1.3, \(T = A^H \subseteq A^\nu\) for some \(\nu\) in \(A\). Then \(T\) is a fixed subring of the finite \(G/N_\nu\)-Galois extension \(A^\nu/B\). Then, by [19; Prop. 3.4], \(T/B\) is a separable extension. By Prop. 1.8 (2) (cf. Cor. to Prop. 1.2), \(T_B\) is finitely generated. By [22; Th. 2.6], \(G/T\) is strongly distinct. (ii) \(\Rightarrow (iii)\) This follows from the fact that \(\{\sigma(x); \sigma \in G\}\) is finite for any \(x\) in \(A\). (iii) \(\Rightarrow (i)\) Let \(\{(t_i, r_i); i = 1, \ldots, n\}\) be a \((B, T)\)-projective coordinate system of \(T/B\). Then, by [22; Prop. 1.2], \(\sum_i t_i \cdot \sigma(t_i) = \delta_{H, \sigma}\) for \(\sigma\) in \(G\), where \(H = G^T\). \(#(G/T) < \infty\) implies \((G:H) < \infty\). By Prop. 1.4, \(A^H = T\).

Combining Th. 1.9 with Prop. 1.8, we obtain the following theorem (cf. [12; Th. 3], [28; Theorem]).
Theorem 1.10. Let $A/B$ be locally finite outer $G$-Galois, and $G=G^*$. Then, for an intermediate ring $T$ of $A/B$, the following are equivalent.

(i) $T=A^H$ for some subgroup $H$ of $G$.

(ii) For any finite subset $F$ of $T$ there is an intermediate ring $T_0$ of $T/B$ such that $T_0\supseteq F$, $T_0/B$ is separable, $T_0$ is finitely generated, and $G|T_0$ is strongly distinct.

(iii) Any finite subset of $T$ has a $G$-separable cover which is contained in $T$.

Next we shall proceed to the characterization of locally finite outer Galois extensions.

Proposition 1.11. Let $V_A(B)=C$, $T$ a $G$-separable cover of $B$, and $\{ (t_i, t_i^*) ; i=1, \ldots, n \}$ a $(B, T)$-projective coordinate system for $T/B$, and put $H=G^T$. Then there hold the following.

1. $\sum_i t_i \cdot \sigma(t_i^*) = \delta_{H,a}$ for all $\sigma$ in $G$.

2. $T^H = T$, $(G:H) < \infty$, and $T/B$ is a projective Frobenius extension.

3. Let $K$ be a subgroup of $G$ containing $H$. Then, $\sum t_{K,H}(t_i) \sigma(t_i^*) = \delta_{K,a}$ for all $\sigma$ in $G$, $T$ is $(B, A^K)$-projective, $T/A^K$ is a projective Frobenius extension, and $G|A^K$ is strongly distinct. Further the following are equivalent. (a) $(A^K|T_A^K)$. (b) $(A^K\cap \{ (A^K)_A^K\})|\{ (A^K)_A^K\}$. (c) $t_{K,H}(c)=1$ for some $c$ in $T$.

Proof. (1) follows from [22; Prop. 1.2], and (2) is obvious by (1) and Prop. 1.4. (3) It will be easily seen that $\sum_t t_{K,H}(t_i) \sigma(t_i^*) = \delta_{K,a}$ for all $\sigma$ in $G$. Since $\sum_t y \cdot t_{K,H}(t_i) \otimes t_i^* = \sum_t t_{K,H}(t_i) \otimes t_i^* t$ $(t \in T \otimes B T)$ for $t$ in $T$, $\sum_t y \cdot t_{K,H}(t_i) \otimes t_i^* = \sum t_{K,H}(t_i) \otimes t_i^* y$ $(y \in A^K \otimes B T)$ for all $y$ in $A^K$. Hence the mapping $x \rightarrow \sum t_{K,H}(t_i) \otimes t_i^* x$ from $T$ to $A^K \otimes B T$ is an $A^K$-homomorphism. Since $\sum t_{K,H}(t_i) \otimes t_i^* x = x$, it follows that $T$ is $(B, A^K)$-projective. Let $\rho|A^K \neq \tau|A^K$ for $\rho$, $\tau$ in $G$. Then $\tau^{-1} \rho \notin K$, and so $0 = \tau(\sum t_{K,H}(t_i) \tau^{-1} \rho(t_i^*)) = \sum \tau(t_{K,H}(t_i)) \rho(t_i^*)$. Thus, by [22; Prop. 1.1], $\rho|A^K$ and $\tau|A^K$ are strongly distinct. If we set $G=K$ in Prop. 1.4, the remainder follows from Prop. 1.4.

Theorem 1.12. Let $V_A(B)=C$. Then the following statements are equivalent.

(i) $A/B$ is locally finite (outer) $G$-Galois.

(ii) For any finite subset $F$ of $A$ there is a $G$-invariant $G$-separable cover $T$ of $F$ such that $B|T$.

(iii) For any finite subset $F$ of $A$ there is a $G$-separable cover $T$ of $F$ which satisfies the following: If $T_0$ is an intermediate ring of $T/B$ such that (a) $T$ is $(B, T_0)$-projective, (b) $T/T_0$ is a projective Frobenius extension, (c) $G|T_0$ is strongly distinct, then $T_0 \supseteq T$. (d) $G|T$.

(iv) For any finite subset $F$ of $A$ there is a $G$-separable cover $T$ of $F$
which satisfies the following: If $T_0$ is an intermediate ring of $T/B$ such that (α) $T$ is $(B, T_0)$-projective, (β) $T/T_0$ is a projective Frobenius extension, (γ) $G|T_0$ is strongly distinct, (δ) $T_0$ is a $G$-invariant fixed subring (with respect to $G$), then $T_0|T_0$.

Proof. (i) $\Rightarrow$ (ii), (iii) Let $A = \bigcup_{\mu \in A} A^\mu$ be a representation of the locally finite $G$-Galois extension $A/B$. Then any finite subset $F$ of $A$ is contained in some $A^\mu$ ($\mu \in A$). By [22; Th. 1.5], $A^\mu$ is a $G$-invariant $G$-separable cover of $F$ such that $p_B|_{\mu} A^\mu$. Let $T_0$ be an intermediate ring of $A^\mu/B$ such that $A^\mu$ is $(B, T_0)$-projective and that $G|T_0$ is strongly distinct. Then, by [22; Th. 2.6], $T_0$ is a fixed subring of the finite outer $G/N$-Galois extension $A^\mu/B$, whence $T_0|T_0$ by [22; §2. p. 118]. (ii) $\Rightarrow$ (i) Let $F$ be a finite subset of $A$, and $T$ a $G$-invariant $G$-separable cover of $F$ such that $p_B|_{\mu} T$. If we put $N = G^\mu$, then $A^\mu = T$, $N \cap G$ and $(G:N) < \infty$ (Prop. 1.11). By Prop. 1.11, $A^\mu/B$ is a finite $G/N$-Galois extension. Noting that $(A^\mu)_B$ is finitely generated, $A/B$ is a locally finite $G$-Galois extension. (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (i) Let $T_1$ be a separable cover of an element $x$ in $A$. Put $G^\mu = H_1$. Then $\#(G|T_1) < \infty$ implies $(G:H_1) < \infty$ and $\# \{x(a); \sigma \in G\} < \infty$. Thus any finite subset of $A$ is contained in a $G$-invariant finite subset of $A$. Let $F$ be a $G$-invariant finite subset of $A$, and $T$ a $G$-separable cover of $F$ as that in (iv), and let $\{(t, t^*_\sigma); i = 1, \ldots, n\}$ be a $(B, T)$-projective coordinate system of $T/B$, and $H = G^\mu$. Then, by Prop. 1.11, $A^n = T$, $(G:H) < \infty$, and $\sum_i t_i^* \sigma = \delta_{\mu,\sigma}$ for all $\sigma$ in $G$. Set $N = G^\mu$. Then $H \subseteq N \cap G$, and $F \subseteq A^\mu \subseteq A^n = T$. By Prop. 1.11, $T$ is $(B, A^\mu)$-projective, $T/A^\mu$ is a projective Frobenius extension, and $T/A^\mu$ is strongly distinct. Then, by the assumption for $F$, $(A^\mu)_B \subseteq T$, so that $t_{N:B}(c) = 1$ for some $c$ in $T$ (Prop. 1.11 (3)). Put $t^*_\sigma = t_{N:B}(t^*_\sigma)$ and $t^*_n = t_{N:B}(t^*_n \sigma)$. Then, $t^*_\sigma$, $t^*_n$ is $A^\mu$, and $\sum_i t^*_i \sigma = \delta_{\mu,\sigma}$ for all $\sigma$ in $G$ (Prop. 1.11 (3)). Further, as is easily seen, $\sum_i t^*_i \sigma = \delta_{\mu,\sigma}$ for all $\sigma$ in $G$. Since $p_B|_{\mu} T$ (Prop. 1.11 (3)), we have $p_B|_{\mu} A^\mu$. Thus $A^\mu/B$ is a finite $G/N$-Galois extension. Noting that $(A^\mu)_B$ is finitely generated, we conclude that $A/B$ is a locally finite $G$-Galois extension.

Proposition 1.13. Let $A^* \supseteq T \supseteq B^*$ be rings such that $A^*$ is $(B^*, T^*)$-projective, $A'$ an extension ring of $B^*$ such that $V_{A'}(B^*) = V_{A'}(A')$, and $f_1, \ldots, f_s$ $B^*$-ring homomorphisms from $A^*$ to $A'$ such that $f_i|T$ and $f_k|T$ (i ≠ k) are strongly distinct. If $(B^*)_{B^*} \rightarrow T_{B^*}$, then $(A')_{A'} \rightarrow T_{A'}$.

Proof. Let $\{(t, a^*_i); i = 1, \ldots, n\}$ be a $(B^*, T)$-projective coordinate system for $A^*$. Then, by [22; Prop. 1.2], $\sum_i f_h(t_i) f_k(a^*_i) = \delta_{h,k}$ for all $h, k$. Let $\phi$ be a $A'$-right homomorphism from $T \otimes_B A'$ to $(A')_{A'}$ defined by $\phi(t \otimes a') = (f(t) a', \ldots, f(t) a')$. Since $\sum_i f_h(t_i) f_k(a^*_i) = \delta_{h,k}$, $\phi$ is an epimorphism. $(B^*)_{B^*} \rightarrow T_{B^*}$ implies that $(A')_{A'} \rightarrow T \otimes_B A'. Hence we have $(A')_{A'} \rightarrow (A')_{A'}$.
desired.

Concerning Prop. 1.13, we consider the following condition.
Condition (F): If $\mathcal{A}A^r \rightarrow \mathcal{A}A^s$ for positive integers $r, s$, then $r \geq s$.

Remark. Let $\mathcal{A}A^r \rightarrow \mathcal{A}A^s$ for positive integers $r, s$. Then, since $\mathcal{A}A^s$ is projective, $\mathcal{A}A^s$ is isomorphic to an $A$-direct summand of $\mathcal{A}A^r$.

(1) If $\mathcal{A}A$ is finite dimensional, then $r \cdot \dim \mathcal{A}A \geq s \cdot \dim \mathcal{A}A$, and so $r \geq s$ (cf. [11]).

(2) Assume that there is a proper ideal $\mathfrak{U}$ of $A$ such that $\mathcal{A}A/\mathfrak{U}$ is finite dimensional. Then, since $\mathcal{A}A^r/\mathfrak{U}^r \rightarrow \mathcal{A}A^s/\mathfrak{U}^s$, the above (1) yields $r \geq s$, because $\mathcal{A}A^r/\mathfrak{U}^r \simeq \mathcal{A}(A/\mathfrak{U})^r$ and $\mathcal{A}A^s/\mathfrak{U}^s \simeq \mathcal{A}(A/\mathfrak{U})^s$.

(3) If $A$ is commutative, then $r \geq s$ by (2).

**Proposition 1.14.** Let $\mathcal{V}_A(B) = C$, and $A$ an indecomposable ring satisfying (F), and let $T$ be an intermediate ring of $A/B$, and $S$ a subset of $A$. Then the following are equivalent:

(i) $T$ is a $G$-separable cover of $S$.

(ii) $T \supseteq S$, $T/B$ is a separable extension, and $T_B$ is finitely generated.

**Proof.** (i) $\Rightarrow$ (ii) is evident by Prop. 1.11. (ii) $\Rightarrow$ (i) By [22; Lemma 2.7], $A$ is $(B, T)$-projective. Then, by Prop. 1.13, we have $\#(G|T) < \infty$, and hence $T$ is a $G$-separable cover of $S$.

If $A$ is commutative, then $A$ satisfies (F). Therefore, by Th. 1.12, S. 3 and Prop. 1.14, we have the following

**Theorem 1.15** (Nagahara [12]). Let $A$ be an indecomposable commutative ring. Then the following are equivalent.

(i) $A/B$ is locally finite $G$-Galois.

(ii) For any finite subset $F$ of $A$ there is an intermediate ring $T$ of $A/B$ such that (a) $T/B$ is a separable extension, and $T_B$ is finitely generated, (b) $T \supseteq F$.

**Proposition 1.16.** Let $A/B$ be locally finite $G$-Galois, and $H$ a subgroup of $G$. Then $G|A^H$ is strongly distinct.

**Proof.** Let $\sigma, \tau$ be in $G$, and $e$ a central idempotent of $A$ such that $\sigma(x)e = \tau(x)e$ for all $x$ in $A^H$. Let $A = \bigcup_{\lambda \in \Lambda} A^{N_{\lambda}}$ be a representation of the locally finite $G$-Galois extension $A/B$. We may assume that $e \in A^{N_{\lambda}}$ for all $\lambda$ in $\Lambda$. Suppose that $\sigma|A^H \neq \tau|A^H$. Since $A^H = \bigcup_{\lambda \in \Lambda} A^{N_{\lambda}H}$, $\sigma|A^{N_{\mu}H} \neq \tau|A^{N_{\mu}H}$ for some $\mu$ in $\Lambda$. Then, by [22; Prop. 2.4], $(G/N_{\mu})|A^{N_{\mu}H}$ is strongly distinct. Therefore we have $e = 0$. Thus $G|A^H$ is strongly distinct.

**Theorem 1.17.** Let $A/B$ be locally finite outer $G$-Galois, and $T$ an intermediate ring of $A/B$. Then the following are equivalent.

(i) $T = A^H$ for some subgroup $H$ of $G$, and $A_T$ is finitely generated.
(ii) $T=A^H$ for some subgroup $H$ of $G$ such that $(H:1)<\infty$.

(iii) $A/T$ is a projective Frobenius extension, $	ext{Hom}(A_T,A_T)\subseteq\Delta$, and $G|T$ is strongly distinct.

When any of the above conditions is satisfied $A/A^H$ is finite $H$-Galois.

Proof. Let $A=\bigcup_{\mu\in A}A^{N_{\mu}}$ be a representation of the locally finite outer $G$-Galois extension $A/B$. (i) $\implies$ (ii) Let $A=x_1T+\cdots+x_rT$. Then $x_1,\ldots,x_r\in A^{N_{\mu}}$ for some $\mu\in A$, so that $A=A^{N_{\mu}}T=A^{N_{\mu}A^H}$. Hence $N_{\mu}\cap H=1$. Since $(G:N_{\mu})<\infty$ we have $(H:1)<\infty$. (ii) $\implies$ (iii) By Prop. 1.3, $H\cap N_{\mu}=1$ for some $\mu \in A$. There are elements $a_1,\ldots,a_n, a_1\cdots,a_n\in A^{N_{\mu}}$ such that $\Sigma_i a_i\cdot (a_i^{\ast}) = \delta_{N_{\mu},\tau}$ for all $\tau$ in $G$. Then $\Sigma_i a_i\cdot (a_i^{\ast}) = \delta_{N_{\mu},\tau}$ for all $\tau$ in $G$. Hence $A/A^{H}$ is $H$-Galois. Therefore $A/A^H$ is a projective Frobenius extension (cf. [22; p. 121]), and $\text{Hom}(A_T,A_T)\subseteq\Delta_0\subseteq\Delta$. By Prop. 1.16, $G|T$ is strongly distinct. (iii) $\implies$ (i) Let $h=\Sigma_{e\in H}a_{ue}u$ be a Frobenius homomorphism of $A/T$, where $H$ is a finite subset of $G$ and $u \ne 0$ for all $\tau$ in $H$. Then, since $th=ht$ for all $t$ in $T$, we have $ta_t=a_\tau t(t)$ for all $t$ in $T$, in particular, $ba_t=a_t b$ for all $b$ in $B$. Hence $a_{\tau}e_{V}(A_{B})=C$ for all $\tau$ in $H$. There are elements $r_i, l_i$ in $A$ such that $x=\Sigma_i h(xr_i)l_i=\Sigma_i r_i h(l_i x)$ for all $x$ in $A$ (cf. [27]). Then $u_{e}=\Sigma_i r_i h(\sum r_i)\tau(l_i)\tau(u_{e})=\Sigma_{e\in H}\sum_i r_i a_{e}\tau(l_i)\tau(u_{e})$, and so $1=\Sigma_i r_i a_{e}l_i= a_{1}\Sigma_i r_i l_i$. Thus $a_{e}$ is an invertible element in $C$, and $a_{1}^{-1}=\Sigma_i r_i l_i$. Since $H$ is finite there is an $N_{\mu}$ such that $\tau|A^{N_{\mu}}\neq\rho|A^{N_{\mu}}$ for all $\tau$ in $H$. Since $A^{N_{\mu}}/B$ is finite $G|N_{\mu}$-Galois, there are elements $d_{k}, e_{k} \in A^{N_{\mu}}$ such that $\Sigma_k d_{k}\cdot e_{k}=\delta_{N_{\mu},\tau}$ for all $\tau$ in $G$. Put $\Delta=\text{Hom}(A_T,A_T)$. Then $\Delta=A^{H}A$, and $\Delta=A_{\tau}u_{e}\subseteq\Delta_0=A_{\tau}u_{e}$ for all $\tau$ in $H$. Thus $\Delta=A^{H}A=\Sigma_{e\in H}A\tau u_{e}$. Since $A/T$ is a projective Frobenius extension with Frobenius homomorphism $h_{\tau}: A\otimes_{\tau}A\simeq A\otimes_{\tau}A$ by the correspondence $x\otimes y \mapsto xhy$. Let $\varphi$ be the $A$-left homomorphism from $A$ to $\Delta$ defined by $\varphi(\Sigma_{e\in H}x_{e}u_{e})=\Sigma_{e\in H}x_{e}a_{e}u_{e}$, and $\varphi$ be the $A$-left homomorphism from $\Delta$ to $\Delta$ defined by $\varphi(xy)=\Sigma_{e\in H}x_{e}h(yr_{e})u_{e}$, where $u=\Sigma_{e\in H}u_{e}$. Then, as $h(tr_{e})a_{e}=h(yr_{e})a_{e}$ ($\tau\in H$), $\varphi=1$. Since $a_{e}u_{e}=\Sigma_{e\in H}r_{e}a_{e}u_{e}$, we have $\varphi(a_{e}u_{e})=\Sigma_{e\in H}\Sigma_{e\in H}\tau(d_{e})h_{\tau}(e_{e}r_{e})u_{e}=\Sigma_{e\in H}a_{e}\tau(l_{e})u_{e}$, and so $\varphi(a_{e}u_{e})=\Sigma_{e\in H}a_{e}\tau(l_{e})u_{e}\tau(l_{e})u_{e}$. On the other hand, $\varphi(a_{e}u_{e})=a_{e}u_{e}$, and hence $a_{e}^{\ast}a_{e}u_{e}=a_{e}a_{e}u_{e}$, for all $\tau$ in $H$. Since $a_{1}^{-1}=\Sigma_i r_i l_i$, we have $a_{e}^{\ast}=a_{e}a_{e}$. Noting that $a_{e}$ is an invertible element of $C$, $Aa_{e}=Aa_{e}a_{e}=Aa_{e}$, and so $A=Aa_{e}+\text{Ann}_{A}(a_{e})$, where $\text{Ann}_{A}(a_{e})=\{x\in A; xa_{e}=0\}$. If $xa_{e}\in \text{Ann}_{A}(a_{e})$, then $0=xa_{e}^{\ast}=xa_{e}a_{e}^{\ast}a_{e}$, so that $xa_{e}=0$. Thus $A=Aa_{e}+\text{Ann}_{A}(a_{e})$. Therefore $Aa_{e}$ is written as $A_{g}$, with a central idempotent $g$. Since $Aa_{e}, u_{e}\subseteq\Delta$, we have $g_{u_{e}}u_{e}\in\Delta_{0}$, and so $g_{u_{e}}u_{e}=g_{u_{e}}u_{e}t$ for all $t$ in $T$. Consequently, $\Delta_{0}=\Sigma_{e\in H}Aa_{e}$, and $H=Ga_{e}$. Hence $\text{End}_{A}(A_{T})=A^{H}A$, the right multiplications of elements of $A$. Since $a_{e}u_{e}\in \Delta_{0}=\text{End}(A_{T})$, we have $a_{e}u_{e}\in \text{End}(A_{(H,u_{e})})$. Noting that $a_{e}$ is in $C$, we
can easily seen that \( a_\tau u_\tau \in \text{Hom}(A^{(\lambda H)}, A^{(\lambda H)}) \). Thus \( h = \sum_{\tau \in H} a_\tau u_\tau \in \text{Hom}(A^{(\lambda H)}A^{(\lambda H)}, A^{(\lambda H)}A^{(\lambda H)}) \). Then, by [27; Cor. 1], \( A/A^H \) is also a projective Frobenius extension with a Frobenius homomorphism \( h \). Since \( (H:1) < \infty \), there is an \( N_i \) such that \( H \cap N_i = 1 \) (Prop. 1.3 (2)). Then \( A^{H i} \subseteq A^{N_i} \), and \( H \cong H N_i / N_i \) canonically. Therefore there is an element \( c \) in \( A^{N_i} \) such that \( t_H(c) = 1 \) (cf. [22; §2. p. 118]), which implies \( (A^{H})^{(\lambda H_i)} / A^{(\lambda H)} \), because the \( A^H \)-right homomorphism \( x \rightarrow t_H(cx) \) (\( x \in A \)) from \( A \) to \( A^H \) splits. Therefore there is an element \( d \) in \( A \) such that \( h(d) = 1 \). Then, for any \( x \) in \( A^H \), \( T \ni h(dx) = h(d)x = x \). Thus we obtain \( T = A^H \), as desired.

**Theorem 1.18.** Let \( A/B \) be finite outer \( G \)-Galois, and \( T \) an intermediate ring of \( A/B \). Then the following are equivalent.

(i) \( T = A^H \) for some subgroup \( H \) of \( G \).

(ii) \( A/T \) is a projective Frobenius extension, and \( G|T \) is strongly distinct.

(iii) \( T/B \) is a separable extension, and \( G|T \) is strongly distinct.

**Proof.** (i) \( \iff \) (ii) is evident from Th. 1.17. (i) \( \implies \) (iii) follows from [22; Th. 2.6] and [19; Prop. 3.4]. (iii) \( \implies \) (i) follows from [22; Th. 2.6 and Lemma 2.7].

**§ 2. Heredity of locally finite Galois extensions.**

Let \( A_0 \) be a \( G^* \)-invariant subring of \( A \) such that the mapping \( \sigma \rightarrow \sigma|A_0 \) (\( \sigma \in G^* \)) is one-to-one and such that \( A_0/A_0^\sigma \) is a locally finite \( G \)-Galois extension, and let \( G^* \) be compact (as an automorphism group of \( A \)). Put \( B_0 = A_0^\sigma \), and let \( A_0 = \bigcup_{\lambda \in A} A_0^N \) be a representation of the locally finite \( G \)-Galois extension \( A_0/B_0 \). Then \( G/N_\lambda \) may be considered as a finite group of automorphisms of \( A^{N_\lambda} \). And, by [22; Th. 5.1 and §2. p. 118], \( A^{N_\lambda} = A_0^{N_\lambda} \otimes_{B_0} B, A^{N_\lambda}/B \) is finite \( G/\bigcup_{\lambda \in A} A^{N_\lambda} \). Since \( \bigcup_{\lambda \in A} A^{N_\lambda} \) is a directed union, the compactness of \( G^* \) implies that \( \bigcup_{\lambda \in A} A^{N_\lambda}(\bigcup_{\lambda \in A} A_0) \) is a fixed subring of \( A \) with respect to \( G^* \) (Prop. 1.1), so that \( A = \bigcup_{\lambda \in A} A^{N_\lambda} \), because \( \sigma \rightarrow \sigma|A_0 \) (\( \sigma \in G^* \)) is 1–1. Thus \( A/B \) is locally finite \( G \)-Galois. Let \( H \) be any subgroup of \( G \). Then, \( A^H = \bigcup_{\lambda \in A} (A^{H \cap A^{N_\lambda}}) = \bigcup_{\lambda \in A} A^{H \cap A^{N_\lambda}} \). By [22; Th. 5.1], \( A^{N_\lambda} = (A^N)^{H \cap A^{N_\lambda}} \otimes_{B_0} B = A_0^{H \cap A^{N_\lambda}} \otimes_{B_0} B \). Hence \( A^H = \bigcup_{\lambda \in A} (A_0^{H \cap A^{N_\lambda}}) \cdot B = A_0^{H \cap A^{N_\lambda}} \otimes_{B_0} B \). And \( A^H \otimes_{B_0} B \rightarrow A^H \otimes_{B_0} B \rightarrow A^H \), we know \( A^H = A_0^H \otimes_{B_0} B \). Symmetrically we obtain \( A^H = B \otimes_{B_0} A_0^H \). Next we consider the set of all \( A_0 \)-left submodules of \( A \) and the set of all \( B_0 \)-left submodules of \( B \). Let \( \overline{X} \) be any \( A_0 \)-left submodule of \( A \). Then \( \overline{X} \cap A^{N_\lambda} \) is an \( A_0^{N_\lambda}(G/\bigcup_{\lambda \in A} A^{N_\lambda}) \)-left submodule of \( A^{N_\lambda} \). Therefore, by [22; Th. 5.1], we have \( \overline{X} \cap A^{N_\lambda} = A_0^{N_\lambda}(\overline{X} \cap A^{N_\lambda}) \cap B = A_0^{N_\lambda} \otimes_{B_0} (\overline{X} \cap B) \), so that \( \overline{X} = \bigcup_{\lambda \in A} (\overline{X} \cap A^{N_\lambda}) = \bigcup_{\lambda \in A} (A_0^{N_\lambda}(\overline{X} \cap B)) = A_0(\overline{X} \cap B) \). Since \( A_0^{N_\lambda} \otimes_{B_0} (\overline{X} \cap B) \cong A_0^{N_\lambda}(\overline{X} \cap B) \).
of $\exists = A_0 \otimes_{B_0} (X \cap B)$ for all $\lambda$, we have $\exists = A_0 \otimes_{B_0} (X \cap B)$. Evidently $X \cap B$ is a $B_0$-left submodule of $B$. Let $X$ be any $B_0$-left submodule of $B$. Then, as is easily seen, $A_0 X$ is an $A_0 G$-left submodule of $A$, and $A_0 X = \cup_1 A_0^{N \lambda} X$. By [22; Th. 5.1], $A_0^{N \lambda} X \cap B = X$ for all $\lambda$ in $\Lambda$, so that $A_0 X \cap B = \cup_1 (A_0^{N \lambda} X \cap B) = X$. If $\overline{Y}$ is a $G$-invariant intermediate ring of $A/A_0$, then $\overline{Y} \cap B$ is an intermediate ring of $B/B_0$, and $\overline{Y} = A_0 (\overline{Y} \cap B)$. Symmetrically we have $\overline{Y} = (\overline{Y} \cap B) A_0$. If $Y$ is an intermediate ring of $B/B_0$ such that $A_0 Y = Y A_0$, then $A_0 Y$ is a $G$-invariant intermediate ring of $A/A_0$. Since $A = \cup_1 A^{N \lambda}$, we have $\overline{Y} = \cup_1 (\overline{Y} \cap A^{N \lambda}) = \cup_1 \overline{Y}^{N \lambda}$, and $\overline{Y}^{N \lambda} (\overline{Y} \cap B)$ is finite $G/N_{\lambda}$-Galois ([22; Th. 5.1]. Hence $\overline{Y} (\overline{Y} \cap B)$ is locally finite $G$-Galois. Thus we have obtained the following.

**Theorem 2.1.** Let $A_0$ be a $G^*$-invariant subring of $A$ such that $\sigma \rightarrow \sigma | A_0$ ($\sigma \in G^*$) is 1-1 and such that $A_0/B_0$ is locally finite $G$-Galois where $B_0 = A_0^G$, and let $G^*$ be compact. Then there hold the following:

1. $A/B$ is locally finite $G$-Galois.
2. $A^H = B \otimes_{B_0} A_0^H = A_0^H \otimes_{B_0} B$ for any subgroup $H$ of $G$. In particular, $A = B \otimes_{B_0} A_0 = A_0 \otimes_{B_0} B$.
3. Let $\{X\}$ and $\{X\}$ be the set of all $A_0 G$-left submodules of $A$ and the set of all $B_0$-left submodules of $B$, respectively. Then, $X \rightarrow X \cap B$ and $X \rightarrow A_0 X = A_0 \otimes_{B_0} X$ are mutually converse order isomorphisms between $\{X\}$ and $X$.
4. Let $\{\overline{Y}\}$ and $\{Y\}$ be the set of all $G$-invariant intermediate rings of $A/A_0$ and the set of all intermediate rings of $B/B_0$ such that $A_0 Y = Y A_0$, respectively. Then $\overline{Y} (\overline{Y} \cap B)$ is locally finite $G$-Galois, and $\overline{Y} \rightarrow \overline{Y} \cap B$ and $Y \rightarrow A_0 Y = Y A_0$ are mutually converse order isomorphisms between $\{\overline{Y}\}$ and $\{Y\}$.

Let $A', A'$ be $R$-algebras such that $A \otimes_{R} A' \neq 0$. Assume that $A/B$ is a locally finite $G$-Galois extension such that $R \cdot 1 \subseteq B$, and assume that $A'$ is a locally finite $G'$-Galois extension such that $R \cdot 1 \subseteq B'$. Then each $\sigma \times \tau$ in $G \times G'$ induces an automorphism of $A \otimes_{R} A'$. Let $A = \cup_\sigma A^{N \sigma}$ and $A' = \cup_\beta A'^{N \beta}$ be representations of $A/B$ and $A'/B'$ respectively. Then, by [22; Th. 5.2], $A^{N \sigma} \otimes_{R} A'^{N \beta} (B \otimes B')$ is a finite $(G/N_{\sigma}) \times (G'/N_{\beta})$-Galois extension. Let $\varphi_{\alpha \beta}$ be the canonical $R$-algebra homomorphism from $A^{N \sigma} \otimes_{R} A'^{N \beta}$ to $A^{N_{\alpha \beta}} \otimes_{R} A'^{N_{\beta \alpha}}$ ($\subseteq A \otimes_{R} A'$). We put $(A \otimes_{R} A') \supseteq A^{N \sigma} \otimes_{R} A'^{N \beta} = A_{\alpha \beta}$ and $(A \otimes_{R} A') \supseteq B \otimes B' = B^*$. To be easily seen, Ker $\varphi_{\alpha \beta}$ is a $(G/N_{\alpha}) \times (G'/N'_{\beta})$-invariant ideal of $A^{N \sigma} \otimes_{R} A'^{N \beta}$. Hence $A_{\alpha \beta}/B^*$ is $(G/N_{\alpha}) \times (G'/N'_{\beta})$-Galois ([22; Th. 5.6]). There are elements $c$ and $c'$ in $A^{N \sigma}$ and $A'^{N \beta}$ respectively such that $t_{\alpha \beta}(c) = 1$ and $t_{\beta \alpha}(c') = 1$. Then $c \otimes c' \in A_{\alpha \beta}$ and $t_{(\alpha /N_{\alpha}) \times (\beta /N'_{\beta})} (c \otimes c') = 1 \otimes 1$. Hence $A_{\alpha \beta}/B^*$ is a finite $(G/N_{\alpha}) \times (G'/N'_{\beta})$-Galois extension, and $\{\sigma \times \tau \in G \times G'; \sigma \times \tau | A_{\alpha \beta} = 1_{A_{\alpha \beta}}\} = N_{\alpha} \times N'_{\beta}$. Since $\cap_{\alpha \beta} (N_{\alpha} \times N'_{\beta}) = (\cap_{\alpha} N_{\alpha}) \times (\cap_{\beta} N'_{\beta}) = 1$, $G \times G'$ may be considered
as a group of automorphisms of \(A \otimes_{R} A'\). Let \(H\) and \(H'\) be subgroups of \(G\) and \(G'\), respectively. Then, \((A \otimes_{R} A')^{H \times H'} = \bigcup_{\alpha, \beta} A_{\alpha \beta}^{H \times H'} = \bigcup_{\alpha} A_{\alpha}^{H} \otimes A_{\beta}^{H'} = (\bigcup_{\alpha} A_{\alpha}^{H'}) \otimes (\bigcup_{\beta} A_{\beta}^{H'}) = A^{H} \otimes A^{H'}\) by [22; Th. 5.2]. In particular, \((A \otimes_{R} A')^{N_{\alpha} \times N_{\beta}} = A_{\alpha \beta}^{N_{\alpha} \times N_{\beta}}\) and we have \((G \times G')^{N_{\alpha} \times N_{\beta}} < \infty\). Since \(A \otimes_{R} A' = \bigcup_{\alpha, \beta} A_{\alpha \beta}^{N_{\alpha} \times N_{\beta}}\) is a directed union, \(A \otimes_{R} A'/B \otimes B'\) is a locally finite \(G \times G'\)-Galois extension. Let \(a \in A\) and \(a' \in A'\). Then it is evident that \(\{\sigma \times \tau \in G \times G' \mid \sigma(a) \otimes \tau(a') = a \otimes a'\} \supseteq \{\sigma \in G \mid \sigma(a) = a\} \times \{\tau \in G' \mid \tau(a') = a'\}\). Put \(\{\sigma \in G \mid \sigma(a) = a\} = K\) and \(\{\tau \in G' \mid \tau(a') = a'\} = K'\). Then \(A^{K} \subseteq A_{\alpha}^{N_{\alpha}}\) and \(A^{K'} \subseteq A_{\alpha}^{N_{\beta}}\) for some \(\alpha, \beta\) (Prop. 1.3), so that \(N_{\alpha} \subseteq K\) and \(N_{\beta} \subseteq K'\). By [22; Th. 5.2], \((G/N_{\alpha} \times G'/N_{\beta})^{A^{K} \otimes A^{K'}} = K/(N_{\alpha} \times K'/N_{\beta})\), and hence \((G \times G')^{A^{K} \otimes A^{K'}} = K \times K'\). Since \((A^{K})_{B}\) and \((A^{K'})_{B}\) are finitely generated, \((A^{K} \otimes A^{K'})_{B} = A^{K} \otimes A^{K'}\) is finitely generated. Hence the finite topology of \(G \times G'\) with respect to \(A \otimes_{R} A'\) is the product topology of the finite topology of \(G\) with respect to \(A\) and the finite topology of \(G'\) with respect to \(A'\). Thus we have proved the following

**Theorem 2.2.** Let \(A\) and \(A'\) be \(R\)-algebras such that \(A \otimes_{R} A' \neq 0\). If \(A/B\) is a locally finite \(G\)-Galois extension such that \(R \cdot 1 \subseteq B\), and \(A'/B'\) is a locally finite \(G'\)-Galois extension such that \(R \cdot 1 \subseteq B'\), then \((A \otimes_{R} A')/(B \otimes B')\) is a locally finite \(G \times G'\)-Galois extension, and \((A \otimes_{R} A')^{H \times H'} = A^{H} \otimes A^{H'}\) for any subgroup \(H\) of \(G\) and any subgroup \(H'\) of \(G'\). The finite topology of \(G \times G'\) with respect to \(A \otimes_{R} A'\) is the product topology of the finite topology of \(G\) with respect to \(A\) and the finite topology of \(G'\) with respect to \(A'\).

**Corollary.** Let \(A/B\) be a locally finite \(G\)-Galois extension such that \(B \subseteq C\), and \(A'\) a \(B\)-algebra such that \(A \otimes_{B} A' \neq 0\). Then \((A \otimes_{B} A')/(1 \otimes A')\) is a locally finite \(G\)-Galois extension, and \((A \otimes_{B} A')^{H} = A^{H} \otimes A'\) for any subgroup \(H\) of \(G\).

**Proposition 2.3.** Let \(A/B\) be locally finite \(G\)-Galois, and \(G = G^{*}\). If \(H\) and \(K\) are closed subgroups of \(G\), then \(A^{H} \cap K = A^{H} \cdot A^{K} = A^{K} \cdot A^{H}\) in particular, if \(H \cap K = 1\) then \(A = A^{H} \cdot A^{K} = A^{K} \cdot A^{H}\).

**Proof.** Let \(A = \bigcup_{\mu} A_{\mu}^{N_{\mu}}\) be a representation of the locally finite \(G\)-Galois extension \(A/B\). First we assume that \((G : K) < \infty\). Then, by Prop. 1.3, \(A^{K} \subseteq A_{\mu}^{N_{\mu}}\) for some \(\mu \in \Lambda\). Since \((A^{N_{\mu}})_{B}\) is finitely generated and \((A^{K})_{A_{\mu}}^{K}\) is a direct summand of \((A^{N_{\mu}})_{A_{\mu}}^{K}\) ([22; §2. p. 118]), \((A^{K})_{B}\) is finitely generated. Therefore we may assume that \(A^{K} \subseteq A_{\mu}^{N_{\mu}}\) for all \(\lambda \in \Lambda\). Then \(N_{\mu} \subseteq K\) for \(\lambda \in \Lambda\), and \(A^{H} \cdot A^{K} = (\bigcup_{\mu} A_{\mu}^{N_{\mu}})(\bigcup_{\mu} A_{\mu}^{N_{\mu} K}) = \bigcup_{\mu} A_{\mu}^{N_{\mu} H \cap N_{\mu} K} = \bigcup_{\mu} A_{\mu}^{N_{\mu} H \cap K}\) by [22; Prop. 5.3]. Since \(N_{\mu} \cap H = N_{\mu} \cap K\) for all \(\lambda\), we have \(A^{H} \cdot A^{K} = \bigcup_{\mu} A_{\mu}^{N_{\mu} H \cap K} = A^{H} \cap K\). Next we return to general case. For any finite subset \(F\) of \(A^{K}\), we put \(K_{F} = \{\sigma \in G \mid \sigma[F = 1_{F}]\}\). Then \((G : K_{F}) < \infty\), \(A^{K_{F}} \subseteq A^{K}\), and \((A^{K_{F}})_{B}\) is finitely generated. Therefore \(A^{K} = \bigcup_{F} A^{K_{F}}\) is a directed union, and
hence $A^{H}A^{K} = A^{H}(\cup_{\gamma}A^{K_{\gamma}}) = \cup_{\gamma}A^{H}A^{K_{\gamma}}$ is also a directed union. Since each $A^{H}A^{K_{\gamma}} = A^{H}(\cup_{\gamma}A^{K_{\gamma}})$ is a fixed subring of $A$, $A^{H}A^{K}$ is a fixed subring of $A$ (Prop. 1.1). Hence, as is easily seen, $A^{H}A^{K} = A^{H\cap K}$. Symmetrically we have $A^{H\cap K} = A^{K}A^{H}$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, $G = G^{*}$, and $H, (r \in \Gamma)$ be closed subgroups of $G$. Then, $[\cup_{r}A^{H_{r}}] = A^{H}$, where $[\cup_{r}A^{H_{r}}]$ means the subring of $A$ generated by $\cup_{r}A^{H_{r}}$.

**Proof.** Evidently $[\cup_{r}A^{H_{r}}] = \cup[A^{H_{r_{1}}} \cup \cdots \cup A^{H_{r_{n}}}]$, where $\{r_{1}, \ldots, r_{n}\}$ ranges over all finite subsets of $\Gamma$. By Prop. 2.3, $A^{H_{r_{1}\cap\cdots\cap H_{r_{n}}} = A^{H_{r_{1}}} \cdots A^{H_{r_{n}}} = [A^{H_{1}} \cdots \cup A^{H_{n}}]}$, and therefore $[\cup_{r}A^{H_{r}}]$ is a directed union of fixed subrings of $A$. Hence, by Prop. 1.1, $[\cup_{r}A^{H_{r}}]$ is a fixed subring. Since $\{\sigma \in G; \sigma|[\cup_{r}A^{H_{r}}] = 1\} = \cap_{r}H_{r}$, we obtain $[\cup_{r}A^{H_{r}}] = A^{H_{r}}$, as desired.

**Proposition 2.4.** Let $A/B$ be locally finite $G$-Galois, $\mathfrak{A}$ a $G$-invariant proper ideal of $A$, $K$ a closed subgroup of $G$, and $N$ a closed normal subgroup of $G$ such that $(G:N) < \infty$. Then there hold the following:

1. $A^{N\cap K}/A^{K}$ is finite $K/(K\cap N)$-Galois. In particular, $A^{N}/B$ is finite $G/N$-Galois.

2. $(A^{N} + \mathfrak{A})/(B + \mathfrak{A})/\mathfrak{A}$ is finite $G/N$-Galois, and $((A^{N} + \mathfrak{A})/\mathfrak{A})^{H} = (A^{N}/B + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

**Proof.** Let $A = \cup_{\mu}A^{N_{\mu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) By Prop. 1.3, $A^{N} \subseteq A^{N_{\mu}}$ for some $\mu \in \Lambda$, and then $N_{\mu} \subseteq N$, $A^{N} = (A^{N_{\mu}})^{N/N_{\mu}}$. Therefore, by [22; Prop. 5.7], $A^{N}/B$ is finite $(G/N)/(N/N_{\mu})$-Galois, or equivalently, finite $G/N$-Galois. Accordingly, $A^{N}/A^{NK}$ is finite $N/(KN)$-Galois, or equivalently, finite $K/(K\cap N)$-Galois. $K/(K\cap N)$ may be considered as a finite group of automorphisms of $A^{N}$, because $K\cap N < K$. Then $A^{N\cap K}/A^{K}$ is finite $K/(K\cap N)$-Galois. (2) By (1), $A^{N}/B$ is finite $G/N$-Galois. If $t_{\lambda}/N(c) = 1$ for $c$ in $A^{N}$, then $t_{\lambda}/N(c + \mathfrak{A}) = 1 + \mathfrak{A}$. Then, by [22; Th. 5.6], $((A^{N} + \mathfrak{A})/\mathfrak{A})/(B + \mathfrak{A})/\mathfrak{A}$ is finite $G/N$-Galois, and $((A^{N} + \mathfrak{A})/\mathfrak{A})^{H} = (A^{N}/B + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

Let $A/B$ be locally finite $G$-Galois, $K$ a closed subgroup of $G$, $N$ a closed normal subgroup of $G$, and $\mathfrak{A}$ a $G$-invariant proper ideal of $A$. Let $A = \cup_{\mu}A^{N_{\mu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then $A^{N} = \cup_{\mu}(A^{N} \cap A^{N_{\mu}}) = \cup_{\mu}A^{NN_{\mu}}$ is a directed union, and each $NN_{\mu}$ is a closed normal subgroup of $G$, because $(G:N) < \infty$. Then, by Prop. 2.4 (1), $A^{NN}$ is finite $G/NN$-Galois. Therefore there are elements $a_{1}, \ldots, a_{m}; b_{1}, \ldots, b_{m}$ in $A^{NN}$ such that $\sum_{i=1}^{m}a_{i}\sigma(b_{i}) = 0_{NN}$ for $\sigma$ in $G$. Hence $A^{NN}/B$ is finite $(G/N)/(NN)/N$-Galois. Hence $A^{N}/B$ is locally finite $G/N$-Galois. Next we consider $K$. $A = \cup_{\mu}A^{N_{\mu}\cap K}$ is a directed union, and each $N_{\mu}\cap K$ is a fixed
normal subgroup of $K$ such that $(K : N_{i} \cap K) < \infty$. By Prop. 2.4 (1), each $A^{N_{i} \cap K}/A^{K}$ is finite $K/(N_{i} \cap K)$-Galois. Hence $A/A^{K}$ is locally finite $K$-Galois. Finally we consider $\mathfrak{U}$. Evidently, $A/\mathfrak{U} = \bigcup_{\lambda}((A^{N_{i}} + \mathfrak{U})/\mathfrak{U})$. By Prop. 2.4 (2), $((A^{N_{i}} + \mathfrak{U})/\mathfrak{U})/((B + \mathfrak{U})/\mathfrak{U})$ is finite $G/N_{i}$-Galois, and $((A^{N_{i}} + \mathfrak{U})/\mathfrak{U})^{H} = (A^{N_{i} \cap K}/\mathfrak{U})/((B + \mathfrak{U})/\mathfrak{U})$ for any subgroup $H$ of $G$. Therefore $(A/\mathfrak{U})^{H} = \bigcup_{\lambda}((A^{N_{i}} + \mathfrak{U})/\mathfrak{U})^{H} = \bigcup_{\lambda}((A^{N_{i} \cap K}/\mathfrak{U})/((B + \mathfrak{U})/\mathfrak{U})$ is locally finite $G$-Galois. Thus we have proved the following

**Theorem 2.5.** Let $A/B$ be locally finite $G$-Galois, $N$ a closed normal subgroup of $G$, $K$ a closed subgroup of $G$, and $\mathfrak{U}$ a $G$-invariant proper ideal of $A$. Then there hold the following:

1. $A^{N}/B$ is locally finite $G/N$-Galois.
2. $A/A^{K}$ is locally finite $K$-Galois.
3. $((A + \mathfrak{U})/\mathfrak{U})/((B + \mathfrak{U})/\mathfrak{U})$ is locally finite $G$-Galois, and $((A + \mathfrak{U})/\mathfrak{U})^{H} = (A^{H} + \mathfrak{U})/\mathfrak{U}$ for any subgroup $H$ of $G$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, and $e$ a non-zero idempotent in $B \cap C$. Then $Ae/Be$ is locally finite $G$-Galois, and $(Ae)^{H} = A^{H}e$ for any subgroup $H$ of $G$.

Let $A/B$ be locally finite $G$-Galois, $n$ a positive integer, and $J$ the ring of rational integers. Then, $(J)_{n}$ is a $J$-algebra, and $(J)_{n} \otimes_{J}A \simeq (A)_{n}$. If we define $\sigma((a_{ik})) = (\sigma(a_{ik}))$ for any $\sigma$ in $G$ and any $(a_{ik})$ in $(J)_{n}$, then $(J)_{n}/((B)_{n}$ is locally finite $G$-Galois. $((A)_{n})^{H} = (A^{H})_{n}$ for any subgroup $H$ of $G$ (Th. 2.2). Now, let $\{e_{ik} ; i, k = 1, \cdots, m\}$ a system of matrix units contained in $B$, and $A = \bigcup_{i \in I} A^{N_{i}}$ a representation of $A/B$. Put $A_{0} = V_{\mathfrak{A}}(\{e_{ik}\})$ and $B_{0} = B \cap A_{0}$. Then, as is well known, $A = \bigcup_{i \in I} A^{N_{i}}$, $A_{0} \simeq A_{0}e_{ik}$ by the right multiplication of $e_{ik}$. To be easily seen, $A^{N_{i}} = \bigcup_{i \in I} A_{0}e_{ik}^{i}$, and $A_{0}^{N_{i}} = V(A^{N_{i}}(\{e_{ik}\})$. There is an element $c$ in $A^{N_{i}}$ such that $t_{\theta;N_{i}}(x_{i}) = 1$. Let $c = \sum_{i \in I} x_{i}e_{ik}$ ($x_{i} \in A_{0}^{N_{i}}$). Then $1 = t_{\theta;N_{i}}(x_{i}) = \sum_{i \in I} x_{i}e_{ik} \in A_{0}^{N_{i}}$. For any positive integer $n$, $(A)_{n}/((B)_{n}$ is locally finite $G$-Galois, and $(A)_{n}^{H} = (A^{H})_{n}$ for any subgroup $H$ of $G$.

**Theorem 2.6.** Let $A/B$ be locally finite $G$-Galois.

1. For any positive integer $n$, $(A)_{n}/((B)_{n}$ is locally finite $G$-Galois, and $(A)_{n}^{H} = (A^{H})_{n}$ for any subgroup $H$ of $G$.
2. If $\{e_{ik}; i, k = 1, \cdots, m\}$ is a system of matrix units contained in $B$, $A_{0} = V_{\mathfrak{A}}(\{e_{ik}\})$, and $B_{0} = B \cap A_{0}$, then $A_{0}/B_{0}$ is locally finite $G$-Galois, and $A = A_{0} \otimes_{B_{0}}B$. Thus we have obtained the following

Let $A/B$ be finite $G$-Galois, and $M$ a $\Delta$-left module. For any subgroup $H$ of $G$, we put $M^{H} = \{m \in M; u_{m}m = m \text{ for all } \tau \in H\}$, which is an $A^{H}$. 
submodule of $M$. Evidently $M^H \supseteq A^H \cdot M^g$, and the mapping $\varphi : A^H \otimes_B M^g \to M^H$ defined by $a \otimes m \to am$ ($a \in A$, $m \in M^g$) is an $A^H$-left homomorphism. By assumption there are elements $a_1, \ldots, a_n; a_1^*, \ldots, a_n^*$ in $A$ such that $\sum_{i} a_i^* \cdot \sigma (a_i^*) = \delta_{i, \sigma}$ ($\sigma \in G$), $t_H(d) = 1$. Put $t_\iota = t_H(a_\iota)$. Then, $t_\iota \in A^H$ and $\sum_{i} t_\iota \cdot \sigma (a_i^*) = \delta_{H, \iota}$ for $\iota$ in $G$. If $m$ is in $M^g$, then $A^H : \cdot M^g \ni t_\iota \sum_{\iota \in G} u_\iota (a_\iota^* dm) = \sum_{\iota} t_\iota \sum_{\iota \in G} \sigma (a_\iota^* d) u_m = t_H(d) m = m$. Hence $\varphi$ is an epimorphism. If $a \in A^H$ and $m_0 \in M^g$, then $\sum_{\iota} t_\iota \otimes \sum_{\iota \in G} u_\iota (a_\iota^* dm_0) = \sum_{\iota} t_\iota \otimes \sum_{\iota \in G} \sigma (a_\iota^* da) m_0 = \sum_{\iota} t_\iota \sum_{\iota \in G} \sigma (a_\iota^* da) \otimes m_0 = a \otimes m_0$. From this fact, as is easily seen, $\varphi$ is 1-1. Thus we have $M^H = A^H \otimes_B M^g$. Next we proceed to more general case.

Let $A/B$ be locally finite $G$-Galois, $A = \bigcup_{\iota \in I} A^{N_\iota}$ its representation, and $M$ a $\Delta$-left module. Let $G = \sigma_1 N_1 \cup \cdots \cup \sigma_r N_r$ be the coset decomposition of $G$, and let $A_\iota$ be the trivial crossed product of $A^{N_\iota}$ with $G/N_\iota$: $A_\iota = \Delta_{\iota} \subseteq A^{N_\iota}$, $\Delta_{\iota} \subseteq \Delta_{\iota} = \{a \in G; u_{\sigma} m = m \}$, for any $m$ in $M^{N_\iota}$, and any $\sum a_i v_{\iota a_i} = \sum a_i v_{\iota a_i}$ in $A_\iota$. Since $A^{N_\iota}$ is a $\Delta$-left module, we have $M^{N_\iota} = A^{N_\iota} \otimes_B M^g$ and $M^{N_\iota, H} = A^{N_\iota H} \otimes_B M^g$ for any subgroup $H$ of $G$. Since $A = \bigcup_{\iota \in I} A^{N_\iota}$ is a directed union, so is $\bigcup_{\iota \in I} M^{N_\iota}$. For any subgroup $H$ of $G$, $(\bigcup_{\iota \in I} M^{N_\iota})^g = \bigcup_{\iota \in I} M^{N_\iota, H} = \bigcup_{\iota \in I} A^{N_\iota H} \cdot M^g = A^H \cdot M^g$, and $A^{N_\iota H} \otimes_B M^g \cong A^{N_\iota H} \cdot M^g$ canonically. The last isomorphism may be considered as $A^{N_\iota H} \otimes_B M^g \to A^H \otimes_B M^g \to A^H : \cdot M^g$, and hence we see that $(\bigcup_{\iota \in I} M^{N_\iota})^g = A^H \otimes_B M^g$. For any $m$ in $M$ we put $H_m = \{a \in G; u_m m = m \}$, which is a subgroup of $G$. Assume that $(G: H_m) < \infty$ and that $H_m$ is closed in $G$. Then, by Prop. 1.3, $H_m \supseteq N_\iota$, for some $\iota \in I$, so that $m \in M^{N_\iota}$. Conversely, if $m$ is in $\bigcup_{\iota \in I} M^{N_\iota}$, then $m \in M^{N_\iota}$ for some $N_\iota$, so that $H_m \supseteq N_\iota$. Then, since $(G: N_\iota) < \infty$ and $N_\iota$ is closed in $G$, $(G: H_m) < \infty$ and $H_m$ is closed in $G$. Thus we have proved the following

**Theorem 2.7.** Let $A/B$ be locally finite $G$-Galois, and $M$ a $\Delta$-left module. Then there hold the following:

1. $A \cdot M^g$ is a $\Delta$-submodule of $M$, and $(A \cdot M^g)^g = A^H \otimes_B M^g$ for any subgroup $H$ of $G$.
2. $A \cdot M^g = \{m \in M; (G: H_m) < \infty \text{ and } H_m \text{ is closed in } G \}$, where $H_m = \{a \in G; u_m m = m \}$.

**Corollary.** Let $A/B$ be finite $G$-Galois, and $M$ a $\Delta$-left module. Then, $M^H = A^H \otimes_B M^g$ for any subgroup $H$ of $G$, in particular, $M = A \otimes_B M^g$ (cf. [4; Th. 1.3] and [22; Th. 5.1 (2)]).

**Proposition 2.8.** Let $A/B$ be finite $G$-Galois. Then the following are equivalent.

1. There are elements $a_1, \ldots, a_n; a_1^*, \ldots, a_n^*$ in $V_A(B)$ such that $\sum a_i^* \cdot \sigma (a_i^*) = \delta_{i, \sigma}$ ($\sigma \in G$) (cf. [22; Cor. to Th. 5.1]).
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(ii) $\nu A_B\|\nu B_B$.

Proof. Since $(A \supseteq) (\sum_\sigma u_\sigma) A \simeq \text{Hom}(A_B, B_B)$ by $j$, it follows that $(\sum_\sigma u_\sigma) V_A(B) \simeq \text{Hom}(\nu A_B, \nu B_B)$, and it is evident that $V_A(B) \simeq \text{Hom}(\nu B_B, \nu A_B)$ canonically. To be easily seen, $\nu A_B\|\nu B_B$ if and only if there are elements $f_1, \ldots, f_n$ in $\text{Hom}(\nu A_B, \nu B_B)$ and $g_1, \ldots, g_n$ in $\text{Hom}(\nu B_B, \nu A_B)$ such that $\sum g_i f_i(x) = x$ for all $x$ in $A$. Consequently (ii) is equivalent to that $u_i = \sum a_i (\sum_\sigma u_\sigma) a_i^*$ ($= \sum a_i^* \sigma(a_i^*) u_\sigma$) for some $a_1, \ldots, a_n$; $a_1^*, \ldots, a_n^*$ in $V_A(B)$. Hence (i) and (ii) are equivalent.

Corollary. Let $G$ be finite. Then the following are equivalent.

(i) $A/B$ is outer $G$-Galois, and $\nu A_B\|\nu B_B$.

(ii) There are elements $a_1, \ldots, a_n$; $a_1^*, \ldots, a_n^*$ in $C$ such that $\sum a_i^* \sigma(a_i^*) = \delta_{1,\sigma}$ ($\sigma \in G$).

Proof. This follows from [22; Prop. 6.4 and Prop. 6.5] and Prop. 2.8. $A/B$ is called a completely outer $G$-Galois extension if $G$ is finite and completely outer (cf. [22]).

Theorem 2.9. Let $B'$ be a ring with identity, $Z$ its center, and $G'$ a finite group.

(i) If $A'/B'$ is completely outer $G'$-Galois and $\nu A'/B'|\nu B '|$ then $A' = B' \otimes_{Z} C'$, where $C'$ is the center of $A'$, and $C'/Z$ is $G'$-Galois.

(ii) If $C'/Z$ is $G'$-Galois and $C'$ is commutative, then $A' = B' \otimes_{Z} C'$ is a completely outer $G'$-Galois extension over $B'$, $\nu A'/B'|\nu B'$. and $1 \otimes C'$ is the center of $A'$.

Proof. (1) By [22; Prop. 6.4], $A'/B'$ is outer $G'$-Galois and $V_{A'}(B') = C'$, where $C'$ is the center of $A'$. Then, by Cor. to Prop. 2.8 and [22; Th. 5.1], $C'/Z$ is $G'$-Galois and $A' = B' \otimes_{Z} C'$. (2) By [22; Th. 5.2 and Prop. 6.5], $A'/(B' \otimes 1)$ is completely outer $G'$-Galois. Since $Z$ is a direct summand of $\otimes C'$, $B' \simeq B' \otimes 1$ canonically, and $\nu A'/B'|\nu B'$, because $Z$ is $G'$-Galois. Then, by Cor. to Prop. 2.8, $C'/Z$ is $G'$-Galois, where $C'$ is the center of $A'$. Since $C' \supseteq 1 \otimes C' \supseteq Z$ and $(1 \otimes C')/Z$ is $G'$-Galois ([22; Th. 5.1 or Th. 5.6]), we have $C' = Z \cdot (1 \otimes C') = 1 \otimes C'$ ([22; Th. 5.1]).

Lemma 2.10. Let $T$ be a ring, and $U$ a subring of $T$.

(1) Let $T/U$ be a separable extension. If a $T$-left module $M$ is $U$-projective, then $M$ is $T$-projective.

(2) If $\tau T \otimes_{\nu} T' \tau T$ and $\tau U \nu M$ for a $T$-left module $M$, then $\tau T \nu M$.

(3) Let $T_0$ be an intermediate ring of $T/U$. If $T$ is $(U, T_0)$-projective and $T_0$ is a $T_0\otimes_{U} T_0$-direct summand of $T$, then $T_0/U$ is a separable extension.

Proof. (1) Since the mapping $x \otimes y \rightarrow xy$ form $T \otimes_{\nu} T$ to $T$ splits as a $T-T$-homomorphism, the mapping $x \otimes m \rightarrow xm$ from $T \otimes_{\nu} M$ to $M$ splits as
a $T$-left homomorphism. Since $_{r}M$ is projective, so is $_{r}T\otimes_{U}M$. Therefore $M$ is $T$-projective. (2) Since $_{U}U|_{U}M$, $_{r}T|_{r}T\otimes_{U}M$. Since $_{r}T\otimes_{U}T|_{r}T$, we have $_{r}T\otimes_{U}M|_{r}M$. Hence we have $_{r}T|_{r}M$. (3) Let $\varphi$ be the canonical homomorphism from $T_{0}\otimes_{U}T$ to $T$ defined by $\varphi(t_{0}\otimes t)=t_{0}$, and let $\phi$ be a $T_{0}$-$T$-homomorphism from $T$ to $T_{0}\otimes_{U}T$ such that $\varphi\phi(x)=x$ for all $x$ in $T$. If $\phi(1)=\sum a_{i}\otimes b_{i}$ ($a_{i}\in T_{0}$, $b_{i}\in T$), then $\sum a_{i}b_{i}=1$ and $\sum ya_{i}\otimes b_{i}=$ $\sum a_{i}\otimes b_{i}y$ ($\in T_{0}\otimes_{U}T$) for all $y$ in $T_{0}$. Let $\pi$ be a $T_{0}$-$T$-homomorphism from $T$ to $T_{0}$ such that $\pi|T_{0}=1_{T_{0}}$. Then, since $\sum ya_{i}\otimes b_{i}=$ $\sum a_{i}\otimes b_{i}y$ ($\in T_{0}\otimes_{U}T$) for all $y$ in $T_{0}$, we have $\sum a_{i}\pi(b_{i})=1$ and $\sum ya_{i}\otimes \pi(b_{i})=$ $\sum a_{i}\otimes \pi(b_{i})y$ ($\in T_{0}\otimes_{U}T_{0}$) for $y$ in $T_{0}$. Then the mapping $y\rightarrow \sum a_{i}\otimes \pi(b_{i})y$ from $T_{0}$ to $T_{0}\otimes_{U}T_{0}$ is a $T_{0}$-$T$-homomorphism, and $\sum a_{i}\pi(b_{i})y=y$. Hence $T_{0}/U$ is a separable extension.

**Proposition 2.11.** Let $A/B$ be finite $G$-Galois, and $Z$ the center of $B$. If $B$ is a separable $Z$-algebra and $Z\subseteq C$, then $V_{A}(B)|Z$ is finite $G$-Galois.

**Proof.** By [2; Prop. 1.5], $B\otimes_{Z}B^{0}$ is a central separable $Z$-algebra, where $B^{0}$ is the opposite ring of $B$. Since $A$ and $ZB$ are finite generated and projective, so is $A$. Then, by Lemma 2.10 (1), $n\otimes_{Z}Z^{0}$ is a finite generated and projective. By [2; Th. 2.1], $nZB\otimes_{Z}B^{0}$ is a $Z$-Galois extension, and hence $nA_{B}|_{B}B^{0}$. Then, by Prop. 2.8, $V_{A}(B)|Z$ is finite $G$-Galois (cf. S. 3).

**Theorem 2.12.** Let $G$ be finite, $\pi$ the group homomorphism defined by $\pi(\sigma)=\sigma|C$ ($\sigma\in G$), $Z$ the center of $B$, and $Z=Z^{0}$, and assume that $A$ is indecomposable. Then the following statements are equivalent.

(i) $A/Z_{0}$ is separable, and $\pi$ is 1–1.
(ii) $V_{A}(B)=C$, $A/Z$ is separable, and $nA_{B}|_{B}B^{0}$.
(iii) $V_{A}(B)=C$, and both $B/Z$ and $C/Z$ are separable.
(iv) Both $B/Z$ and $C/Z_{0}$ are separable, and $\pi$ is 1–1.
(v) $V_{A}(B)=C$, $A/B$ is separable, $A$ is $(Z, B)$-projective, and $nB_{B}|_{B}A_{B}^{0}$.
(vi) $A=B-C$, and $A/Z$ is separable.
(vii) $\alpha\otimes_{Z_{0}}A\otimes_{Z_{0}}A^{0}$, $\alpha\otimes_{Z_{0}}, A^{0}, \alpha_{n}\alpha_{A}^{0}$, and $Hom(\alpha A_{n}A_{A}=0$ for any $\sigma$ in $G$ such that $\sigma\neq 1$.

**Proof.** (i) $\Rightarrow$ (ii) By [2; Th. 2.3], $A/C$ and $C/Z_{0}$ are separable. Therefore, by [4; Th. 1.3], $C/Z_{0}$ is $G$-Galois. Then, by [22; Th. 5.1], $A=B\otimes_{Z_{0}}C$. Hence $V_{A}(B)=C$, and $Z=Z_{0}$. Since $A$ is finitely generated and projective, $nA_{B}|_{B}B^{0}$. (ii) $\Rightarrow$ (iii) $V_{A}(B)=C$ implies $Z=Z_{0}$. By [22; Lemma 2.7], $A/C$ and $A/B$ are separable, so that $A/B$ is outer $G$-Galois ([22; Th. 1.5]). Then, by Prop. 2.8, $C/Z$ is $G$-Galois, so that $C/Z$ is separable. Since $A/C$ is separable, $B/Z$ is separable ([22; Cor. to Th. 5.1]). (iii) $\Rightarrow$ (iv) In this case, $Z=Z_{0}$. By [2; Th. 3.1], $A=B-C$, whence $\pi$ is 1–1. (iv) $\Rightarrow$ (v) By
[4; Th. 1.3], $C/Z_0$ is $G$-Galois. Hence, by [22; Th. 5.1], $A/B$ is $G$-Galois, and $A=BC$. Then $A/B$ is separable, $V_A(B)=C$, and $Z=Z_0$. Since $Z$ is commutative, $Z$ is a direct summand of $ZC$ (S. 3), so that $t_0(c)=1$ for some $c$ in $C$. Then $B$ is a $B$-direct summand of $A$ (cf. [22; § 2. p. 118]). Since $B/Z$ is separable, $A$ is $(G, B)$-projective ([22; Lemma 2.7]). (v) $\Rightarrow$ (vi) By Lemma 2.10 (3), $B/Z$ is separable. Then, by [2; Th. 3.1], $A=Z\otimes_B C$. Since both $A/B$ and $B/Z$ are separable, $A/Z$ is separable ([22; Lemma 2.7]).

(vi) $\Rightarrow$ (i) As $A=BC$, $V_A(B)=C$, $Z=Z_0$, and $\pi$ is 1-1. Thus we know that (i) $\sim$ (vi) are equivalent. (i) $\Rightarrow$ (vi) In this case, $V_A(B)=C$, $Z=Z_0$, and $B/Z$ is separable. Then, by [2; Th. 2.1], $B\otimes_Z B|_B B$. Therefore $B\otimes_Z B|_B B$, and then $A\otimes_Z A|_A A\otimes_B A$. By [22; Prop. 1.3], $A\simeq A\otimes_B A$. Hence $A\otimes_Z A|_A A\otimes_B A$. The second assertion follows from [22; Prop. 6.3].

(vii) $\Rightarrow$ (i) By assumption, $\text{End}(A\otimes_Z A|_A A\otimes_B A)$ is a commutative ring. Then, by S. 1 and S. 3, $A\otimes_Z A$ is finitely generated and projective. Hence $A\otimes_Z A$ is finitely generated and projective, that is, $A/Z_0$ is separable. Let $f$ be the projection from $A$ to $A_1$ with respect to the decomposition $A=\sum_\sigma A$. Then, since $\text{End}(A\otimes Z_A)$ is commutative, $f$ is in the center of $A\otimes A$. (cf. S. 1). By [2; Prop. 1.5], the center of $A\otimes A$ is $C\otimes C$, so that $f$ is written as $f=\sum_\sigma a_\sigma d_\sigma^* (a_\sigma, d_\sigma^* \in C)$. Then, $u_1=\sum_\sigma a_\sigma (\sum_\sigma u_\sigma) a_\sigma^* \left(=\sum_\sigma (\sum_\sigma a_\sigma \sigma(a_\sigma^*)) u_\sigma\right)$, and hence $\sum_\sigma a_\sigma \sigma(a_\sigma^*)=d_1$. This completes the proof of the theorem.

\textbf{Remark.} The following are also equivalent to (i) $\Leftrightarrow$ (iii).

(viii) $A/C$ is separable, and $C/Z_0$ is $G$-Galois (cf. Kanzaki [8]).

(ix) $A/B$ is outer $G$-Galois, and $B/Z$ is separable.

\textbf{Proposition 2.13.} Let $A/B$ be locally finite $G$-Galois, and $b$ an element of $B$ which is not a right zero divisor of $B$. Then $b$ is not a right zero divisor of $A$.

\textbf{Proof.} Let $a$ be an element of $A$ such that $ab=0$. Then $AaB=0$, and so $\sigma(Aa)b=0$ for all $\sigma$ in $G$. Hence, $(\sum_\sigma \sigma(Aa))\cap B=0$. Then, by assumption, $(\sum_\sigma \sigma(Aa))\cap B=0$. Then, by Th. 2.1 (3), $\sum_\sigma \sigma(Aa)=A((\sum_\sigma \sigma(Aa))\cap B)=0$. Hence $a=0$.

Let $A/B$ be locally finite $G$-Galois, and $S \ni 1$ a $G$-invariant multiplicative system of regular elements in $A$ such that a left quotient ring $A$ of $A$ with respect to $S$ exists. Then $G$ may be regarded as a group of automorphisms of $A$. To be easily seen, $\{\sigma(x); \sigma \in G\}$ is finite for any $x$ in $A$. Then, by Th. 2.1, $A/B$ is locally finite $G$-Galois and $A=\overline{B}\otimes_B A=\overline{A}\otimes_B B$, where $\overline{B}=A^{\sigma}$. To be easily seen, any element in $B\cap S$ is a unit of $B$. For $b$ in $B$, we put
$\mathcal{L} = \{ x \in A ; xb \in A \}$, which is a $\mathcal{D}$-left submodule of $A$. Then $(\mathcal{L} \cap B)b \subseteq B$. If $\mathcal{L} \cap B \cap S \neq \emptyset$, then $sb \in B$ for some $s$ in $B \cap S$. Therefore, if we assume that $\mathcal{D}(s) \cap B \cap S \neq \emptyset$ for all $s \in S$, then $\overline{B}$ is a left quotient ring of $B$ with respect to $B \cap S$. Thus we obtain the following

**Theorem 2.14.** Let $A/B$ be locally finite $G$-Galois, and $S \ni 1$ a $G$-invariant multiplicative system of regular elements of $A$ such that a left quotient ring $\overline{A}$ of $A$ with respect to $S$ exists. Further, assume that $\Delta(s) \cap B \cap S \neq \emptyset$ for all $s \in S$. Then there hold the following:

1. $\overline{A}/\overline{B}$ is locally finite $G$-Galois and $\overline{A} = \overline{B} \otimes_{B} A = A \otimes_{B} \overline{B}$, where $\overline{B} = \overline{A^{o}}$.

2. $\overline{A}$ is a left quotient ring of $A$ with respect to $B \cap S$. $\overline{B}$ is a left quotient ring of $B$ with respect to $B \cap S$.

**Remark.** Let $A/B$ be locally finite $G$-Galois, and $S$ a $G$-invariant multiplicative system of regular elements in $A$ such that $S \subseteq C$ and $S \ni 1$. Then $S$ satisfies the conditions in Th. 2.14. To see this, we put $H = \{ \sigma \in G ; \sigma(s) = s \}$ for $s$ in $S$. If $G = \sigma_{1}H \cup \cdots \cup \sigma_{r}H$ is the left coset decomposition of $G$, then $\mathfrak{L}, \mathfrak{L}(s) \subseteq \Delta(s) \cap B \cap S$.

A non-zero ring $T$ with 1 is called a left Goldie ring if $T$ satisfies the following conditions: (1) $T$ is a semi-prime ring. (2) Any independent set of non-zero left ideals is finite (i.e., $T$ is finite dimensional). (3) $T$ satisfies the ascending chain condition for annihilator left ideals.

A left Goldie ring has a complete left quotient ring which is a semi-simple ring with minimum condition for left ideals, and conversely (Goldie [17]). (Cf. [7])

**Theorem 2.15.** Let $A/B$ be locally finite $G$-Galois, $A$ a left Goldie ring, $\overline{A}$ a complete left quotient ring of $A$, and $B$ a semi-prime ring. Then there hold the following:

1. $\overline{A}/\overline{B}$ is locally finite $G$-Galois, where $\overline{B} = \overline{A^{o}}$.

2. $B$ is a left Goldie ring, and $\overline{B}$ is a complete left quotient ring of $B$.

**Proof.** Let $S$ be the set of all regular elements of $A$. First we shall prove that $B$ is a left Goldie ring. Since $\mathcal{A}A$ is finite dimensional, $\mathcal{A}A$ is finite dimensional. Then, by Th. 2.1 (3), $\mathcal{A}B$ is finite dimensional. Let $I \subseteq I'$ be left ideals of $B$. Then $l_{A}(r_{B}(I)) \subseteq l_{A}(r_{B}(I'))$, where $r_{B}(I) = \{ y \in B ; Iy = 0 \}$ and $l_{A}(r_{B}(I)) = \{ x \in A ; x \cdot r_{B}(I) = 0 \}$. From this fact, $B$ satisfies the ascending chain condition for annihilator left ideals of $B$. Hence $B$ is a left Goldie ring. By Prop. 2.13, $S \cap B$ is the set of all regular elements of $B$. For any $s$ in $S$, $\mathcal{A}s$ is essential in $\mathcal{A}A$, so that $\mathcal{A}(s)$ is essential in $\mathcal{A}A$. Then, by Th. 2.1 (3), $s(\mathcal{A}(s) \cap B)$ is essential in $\mathcal{A}B$, so that $\mathcal{A}(s) \cap B$ contains a regular element.
of $B$ ([17; Th. (3.9)]). Hence $A(s) \cap B \cap S \neq 0$ for any $s$ in $S$. Thus the present theorem follows from Th. 2.14.

Remark. In the following cases, the condition that $B$ is semi-prime is superfluous.

1. $G$ is finite and completely outer (cf. [22; p. 132]).
2. $B$ is contained in the center of $A$.

Let $T$ be a ring. If $T$-left modules $M$ and $N$ have no non-zero isomorphic subquotients, we say that $_TN$ and $_TM$ are unrelated (cf. [22]).

Lemma 2.16. Let $T$ be a ring, and let $M$ and $N$ be $T$-left modules, and $W$ a $T$-submodule of $M$. If $_T(M/W)$ and $_TN$ are unrelated, and $_TW$ and $_TM$ are unrelated, then $_TM$ and $_TN$ are unrelated.

Proof. Assume that there are isomorphic subquotients $X/X_0$ and $Y/Y_0$ of $_TM$ and $_TN$, respectively. Then, as is easily seen, $X + W \supseteq X_0 + W$ or $X \cap W \supsetneq X_0 \cap W$. If $X + W \supsetneq X_0 + W$, then $Y/Y_0 \approx X/X_0 \implies (X + W)/(X_0 + W) \neq 0$, a contradiction. If $X \cap W \supsetneq X_0 \cap W$, then $(X \cap W)/(X_0 \cap W) \approx (X_0 + (X \cap W))/X_0 \subseteq X/X_0 \approx Y/Y_0$, which is also a contradiction.

Proposition 2.17. Let $\sigma, \tau$ be in $G$, and assume that $_A\left[A\right]$ and $_A\left[A\right]$ are unrelated. Then, for any finite subset $\{x_1, \cdots, x_r, y_1, \cdots, y_s\}$ of $A$, there are elements $a_k, b_k$ $(k = 1, \cdots, t)$ in $A$ such that $\sum_k a_kx_i \cdot \sigma(b_k) = x_i$ and $\sum_k a_ky_h \cdot \tau(b_k) = 0$ for all $x_i, y_h$.

Proof. By Lemma 2.16, $_A\left[A\right]$ and $_A\left[A\right]$ are unrelated. Then, since $A(x_1 u_\sigma, \cdots, x_r u_\sigma, y_1 u_\tau, \cdots, y_s u_\tau)A$ is an $A$-$A$-submodule of $_A\left[A\right] \oplus (_A\left[A\right]$ $A(x_1 u_\sigma, \cdots, x_r u_\sigma, 0, \cdots, 0) \in A(x_1 u_\sigma, \cdots, x_r u_\sigma, y_1 u_\tau, \cdots, y_s u_\tau)A$ (cf. [22; Prop. 6.1]). Therefore there are elements $a_k, b_k$ $(k = 1, \cdots, t)$ in $A$ such that $\sum_k a_k(x_1 u_\sigma, \cdots, x_r u_\sigma, y_1 u_\tau, \cdots, y_s u_\tau)b_k = (x_1 u_\sigma, \cdots, x_r u_\sigma, 0, \cdots, 0)$. Then, $\sum_k a_kx_i \cdot \sigma(b_k) = x_i$ and $\sum_k a_ky_h \cdot \tau(b_k) = 0$ for all $x_i, y_h$.

Combining Prop. 2.17 with [22; Prop. 6.11] we can easily see the following:

Proposition 2.18. Let $A$ and $A'$ be $R$-algebras with $A \otimes_R A' \neq 0$, and let $G$ and $G'$ be completely outer finite groups of $R$-automorphisms of $A$ and $A'$, respectively. Then, $G \times G'$ is completely outer as an automorphism group of $A \otimes_R A'$.

§ 3.

Proposition 3.1. Let $A/B$ be locally finite $G$-Galois, and $X$ a $A$-left submodule of $A$. Then $X = A(X \cap B)$.

Proof. This follows from Th. 2.1 (3).

Proposition 3.2. Let $A/B$ be locally finite $G$-Galois, $\{B\}$ the set of
all maximal ideals of $A$, and $\{p\}$ the set of all maximal ideals of $B$. Then the following are equivalent:

(i) $\mathfrak{P} \rightarrow \mathfrak{B} \cap B$ is a mapping from $\{\mathfrak{P}\}$ onto $\{p\}$.

(ii) $A \mathfrak{p} A \neq A$ for all $\mathfrak{p} \in \{p\}$, and $\cap_{\sigma \in G} \mathfrak{P}$ is $A$-$A$-maximal for all $\mathfrak{P} \in \{\mathfrak{P}\}$.

If (i) holds, then the following are true:

(1) $p A = A \mathfrak{p} \neq A$ for any $p \in \{p\}$.

(2) $\{\cap_{\sigma} \mathfrak{P}; \mathfrak{P} \in \{\mathfrak{P}\}\}$ is the set of all maximal $A$-$A$-submodules of $A$.

(3) $\mathfrak{R}(A A_\mathfrak{p}) = \mathfrak{R}(A A_\mathfrak{b}) = \mathfrak{R}(B B_\mathfrak{b}) A = A \cdot \mathfrak{R}(B B_\mathfrak{b})$, and $\mathfrak{R}(A A_\mathfrak{p}) \cap B = \mathfrak{R}(B B_\mathfrak{b})$.

(4) $B$ is $B$-$B$-completely reducible if and only if $\cap_{i} \cap_{\sigma} \sigma(\mathfrak{P}) = 0$ for some $\mathfrak{P}_i (i = 1, \cdots, n)$ in $\{\mathfrak{P}\}$.

Proof. (i) $\Rightarrow$ (ii) If $\mathfrak{P}$ is in $\{\mathfrak{P}\}$, then $\mathfrak{P} \cap B = \sigma(\mathfrak{P}) \cap B$ for any $\sigma$ in $G$, and so $\mathfrak{P} \cap B = (\cap_{\sigma} \sigma(\mathfrak{P})) \cap B$. By Prop. 3.1, $A((\cap_{\sigma} \sigma(\mathfrak{P})) \cap B) \cap B = \cap_{\sigma} \sigma(\mathfrak{P}) \cap B$. Hence $A \mathfrak{p} = \mathfrak{p} A \neq A$ for all $p \in \{p\}$. Let $X$ be a $A$-$A$-submodule of $A$ with $A \supseteq X \supseteq \sigma(\mathfrak{P})$. Then $B \supseteq X \cap B \supseteq (\cap_{\sigma} \sigma(\mathfrak{P})) \cap B = \mathfrak{P} \cap B$, and so $X \cap B = (\cap_{\sigma} \sigma(\mathfrak{P})) \cap B$. Then, by Prop. 3.1, $X = \cap_{\sigma} \sigma(\mathfrak{P})$ is $A$-$A$-maximal. Let $Y$ be a maximal $A$-$A$-submodule of $A$. Take a maximal ideal $\mathfrak{P}_1$ of $A$ with $\mathfrak{P}_1 \supseteq Y$. Then $\cap_{\sigma} \sigma(\mathfrak{P}) \supseteq Y$, and so $\cap_{\sigma} \sigma(\mathfrak{P}) = Y$. Thus we obtain (2). Therefore $\mathfrak{R}(A A_\mathfrak{p}) = \mathfrak{R}(A A_\mathfrak{b})$. Since $\mathfrak{R}(A A_\mathfrak{p}) \cap B = \mathfrak{R}(B B_\mathfrak{b})$, we have $\mathfrak{R}(A A_\mathfrak{p}) = A \cdot \mathfrak{R}(B B_\mathfrak{b}) = \mathfrak{R}(B B_\mathfrak{b}) A$ (Prop. 3.1). $B$ is $B$-$B$-completely reducible if and only if $\cap_{i} \cap_{\sigma} \sigma(\mathfrak{P}) = 0$ for some $\mathfrak{P}_i (i = 1, \cdots, n)$ in $\{\mathfrak{P}\}$. Thus we obtain (4) (cf. Prop. 3.1). (ii) $\Rightarrow$ (i). Let $p \in \{p\}$. Then, as $A \mathfrak{p} A \neq A$, $\mathfrak{p} \subseteq \{\mathfrak{P}\}$ for some $\mathfrak{P} \in \{\mathfrak{P}\}$, and so $\mathfrak{p} = \mathfrak{P} \cap B$ by the maximality of $\mathfrak{p}$. Let $\mathfrak{O}$ be in $\{\mathfrak{P}\}$. Then $q \subseteq \mathfrak{O} \cap B$ for some $q \in \{q\}$. There is a $\mathfrak{O}' \in \{\mathfrak{P}\}$ with $\mathfrak{O}' \cap B = q$. Then $\cap_{\sigma} \sigma(\mathfrak{O}') \cap B = \mathfrak{O}' \cap B \supseteq \mathfrak{O} \cap B = (\cap_{\sigma} \sigma(\mathfrak{O})) \cap B$, and therefore $\cap_{\sigma} \sigma(\mathfrak{O}') \supseteq \cap_{\sigma} \sigma(\mathfrak{O})$ by Prop. 3.1. By assumption, $\cap_{\sigma} \sigma(\mathfrak{O}') = \cap_{\sigma} \sigma(\mathfrak{O})$. Hence $q = \mathfrak{O}' \cap B = \mathfrak{O} \cap B$. This completes the proof.

Concerning Prop. 3.2, we state the following

**Lemma 3.3.** Let $\mathfrak{P}$ be a maximal ideal of $A$ such that $\cap_{\sigma \in G} \sigma(\mathfrak{P}) = \cap_{i} \sigma_i (\mathfrak{P})$ for some $\sigma_1, \cdots, \sigma_n$ in $G$. Then $\cap_{\sigma} (\mathfrak{P})$ is $A$-$A$-maximal, and $\cap_{i} \sigma_i (\mathfrak{P}); i = 1, \cdots, n$ is the set of all maximal ideals containing $\cap_{\sigma} (\mathfrak{P})$.

Proof. Let $\mathfrak{O}$ be a maximal ideal of $A$ with $\mathfrak{O} \supseteq \cap_{\sigma} (\mathfrak{P})$. If $\mathfrak{O} \neq \sigma_i (\mathfrak{P})$ for all $i$, then $\mathfrak{O} + \mathfrak{O}_i (\mathfrak{P}) = A$ for all $i$. Then we have a contradiction $A = \mathfrak{O} + \cap_{i} \sigma_i (\mathfrak{P}) = \mathfrak{O} + \cap_{\sigma} (\mathfrak{P})$.

Remark. In the following cases, the assumption in Lemma 3.3 holds.

(1) $G$ is finite. (2) The ring $A/\mathfrak{R}(A A_\mathfrak{p})$ satisfies the descending chain condition for ideals. (3) $G^*$ is compact, and every maximal ideal of $A$ is $A$-$A$-finitely generated. (Cf. Prop. 1.1).
Proposition 3.4.

(1) Let $A/B$ be locally finite outer $G$-Galois, and $B$ $B$-$B$-completely reducible. Assume that, for any maximal ideal $\mathfrak{P}$ of $A$, there are elements $\sigma_1, \ldots, \sigma_n$ in $G$ such that $\cap_i \sigma_i(\mathfrak{P}) = \cap \sigma(\mathfrak{P})$. Then $A$ is $A$-$A$-completely reducible.

(2) Let $G$ be finite and completely outer, and $B_B|A_B$. Then $A$ is $A$-$A$-completely reducible if and only if $B$ is $B$-$B$-completely reducible. If there is a maximal ideal $\mathfrak{P}$ of $A$ such that $\cap \sigma(\mathfrak{P}) = 0$, then $B$ is $B$-$B$-minimal, and conversely.

Proof. (1) Any maximal ideal $\mathfrak{p}$ of $B$ is written as $\mathfrak{p} = \mathfrak{p}Be$ with a central idempotent $e$ of $B$. Then, by assumption, $(1 \neq e) \in V_e(B) = C$. Therefore, $A\mathfrak{p} = Ae = eA = \mathfrak{p}A \neq A$. Thus, by Prop. 3.2 and Lemma 3.3, $A$ is $A$-$A$-completely reducible. (2) In this case, $\mathfrak{a}A = A\mathfrak{a} \neq A$ for any proper ideal $\mathfrak{a}$ of $B$ (cf. [22; p. 132]). Then, by Prop. 3.2 and Lemma 3.3, the first assertion is evident (cf. [22; Prop. 6.4]). For any $\mathfrak{P}$ in $\{\mathfrak{P}\}$, $((\cap \sigma(\mathfrak{P})) \cap B = \mathfrak{P} \cap B = 0$ if and only if $\cap \sigma(\mathfrak{P}) = 0$ (Prop. 3.1). Thus we know the second assertion.

Theorem 3.5. Let $A/B$ be finite $G$-Galois, $B$ a semi-primary ring, and $A\mathfrak{p}A \neq A$ for any maximal ideal $\mathfrak{p}$ of $B$. Then $A \simeq B_B$, that is, $A$ has a normal basis. (cf. [13; Th. 1]).

Proof. By [22; Th. 1.7], it suffices to prove that $B_B$ is free. Let $g = (G : 1)$. (1) First we assume that $\mathfrak{N}(B) = 0$. Then $B$ is a direct sum of simple rings: $B = a_1 + \cdots + a_n$. Let $1 = \sum e_i, e_i \in a_i$. Then $a_i = Be_i = e_i B$ and $e_i^2 = e_i$. By assumption we have $(1 - e_i)A = A(1 - e_i)$ (Prop. 3.2 and Lemma 3.3), so that $e_i$ is a central idempotent of $A$ contained in $B$. Then each $Ae_i/Be_i$ is $G$-Galois ([22; Cor. to Th. 5.6]). Since $Be_i$ is a simple ring, $Be_i A e_i$ is free (cf. [7]). Hence $Ae_i$ has a normal basis, so that $B_Be_i A e_i \simeq B_B e_i$ for all $i$ ([22; Th. 1.7]). Hence $B_B \simeq B_B$. (2) Next we proceed to general case. Since $A$ and $B$ are semi-primary ([22; Prop. 7.3]), $\mathfrak{N}(A) = \mathfrak{N}(A)$ and $\mathfrak{N}(B) = \mathfrak{N}(B)$. Then, by Prop. 3.2 and Lemma 3.3, $\mathfrak{N}(A) = \mathfrak{N}(B) A = A \mathfrak{N}(B)$ and $\mathfrak{N}(A) \cap B = \mathfrak{N}(B)$. By [22; Th. 5.6], $(A/\mathfrak{N}(A))/(B + \mathfrak{N}(A)) \simeq (A/\mathfrak{N}(A))/\mathfrak{N}(A)$ is $G$-Galois, and satisfies the same conditions as $A/B$, because $(B + \mathfrak{N}(A))/\mathfrak{N}(A) \simeq B/(\mathfrak{N}(A) \cap B) = B/\mathfrak{N}(B)$ canonically. By (1), we have $B_B/\mathfrak{N}(B) \times B_B/\mathfrak{N}(B)$ and $B_B$ is finitely generated and projective, we have $B_B \simeq B_B$. Since $\mathfrak{N}(A) = \mathfrak{N}(B) A$ and $B_B$ is finitely generated and projective, we have $B_B \simeq B_B$. This completes the proof.

Corollary. Let $A/B$ be finite $G$-Galois, $B$ a semi-primary ring, and $Z$ the center of $B$. Assume that $Z \subseteq C$ and that $B$ is a central separable $Z$-algebra. Then $A$ has a normal basis.

Proof. In this case, any proper ideal of $B$ is written as $\mathfrak{a}B$ with an ideal
a of $Z$ (cf. [2]). Then, as $Z \subseteq C$, $(aB)A = \alpha A = A\alpha = A(B\alpha) \neq A$ ([22; Lemma 7.1]).

Let $A/B$ be finite $G$-Galois, $B \subseteq C$, and $g=(G:1)$. For any prime ideal $\mathfrak{p}$ of $B$, we denote by $B_{\mathfrak{p}}$ the quotient extension of $B$ with respect to $\mathfrak{p}$. Then $B_{\mathfrak{p}}$ is a $B$-algebra, canonically. By [22; Cor. to Th. 5.2], $(B_{\mathfrak{p}} \otimes_{B} A)/B_{\mathfrak{p}}$ is $G$-Galois. Since $B_{\mathfrak{p}}$ is a local ring, $R_{\mathfrak{p}}B_{\mathfrak{p}} \otimes_{B} A \simeq B_{\mathfrak{p}}(B_{\mathfrak{p}})^{g}$ (Cor. to Th. 3.5). We denote by $K_{\mathfrak{p}}$ the quotient field of $B/\mathfrak{p}$. Then we have $K_{\mathfrak{p}}K_{\mathfrak{p}} \otimes_{B} A \simeq K_{\mathfrak{p}}(K_{\mathfrak{p}})^{g}$ similarly. Thus we obtain the following

**Proposition 3.6.** Let $A/B$ be finite $G$-Galois, $B \subseteq C$, and $g=(G:1)$. Then, $B_{\mathfrak{p}}B_{\mathfrak{p}} \otimes_{B} A \simeq B_{\mathfrak{p}}(B_{\mathfrak{p}})^{g}$ and $K_{\mathfrak{p}}K_{\mathfrak{p}} \otimes_{B} A \simeq K_{\mathfrak{p}}(K_{\mathfrak{p}})^{g}$ for any prime ideal $\mathfrak{p}$ of $B$, where $B_{\mathfrak{p}}$ is the quotient extension of $B$ with respect to $\mathfrak{p}$ and $K_{\mathfrak{p}}$ is the quotient field of $B/\mathfrak{p}$.

The following lemma is of some interest.

**Lemma 3.7.** Let $R \supseteq S$ be rings, $R_{S}$ is finitely generated and projective, and $S$ is a direct summand of $R$. If $R$ is injective, then $S$ is injective.

**Proof.** Let $I$ be any left ideal of $S$, and $f$ any $S$-left homomorphism from $I$ to $S$. Since $R_{S}$ is finitely generated and projective, we have $RI=R \otimes_{S} I$. Therefore $f$ can be extended to an $R$-left homomorphism from $R$ to $R$, canonically. Then, by assumption, there is an element $a$ in $R$ such that $r \cdot (s)f = rsa$ for $r$ in $R$ and $s$ in $I$, so that $(s)f = sa$ for all $s$ in $I$. Therefore, as is well known, $S$ is injective. Since $S$ is a direct summand of $R$, $S$ is injective.

**Lemma 3.8.** $\Re(A) \cap B \subseteq \Re(B)$.

**Proof.** Let $b$ be in $\Re(R) \cap B$. Then $1-b$ has an inverse in $A$. Since $B=A^{\sigma}$, $1-b$ has an inverse in $B$. Hence $\Re(A) \cap B$ is a quasi-regular ideal of $B$, that is, $\Re(A) \cap B \subseteq \Re(B)$.

**Proposition 3.9.** Let $G$ be finite. If there is an element $c$ in $A$ such that $1-\sigma(c) \in \Re(A)$, then there is an element $d$ in $A$ such that $t_{\sigma}(d)=1$.

**Proof.** By Lemma 3.8, we have $1-\sigma(c) \in \Re(A) \cap B \subseteq \Re(B)$, so that $t_{\sigma}(A)+\Re(B)=B$. Since $t_{\sigma}(A)$ is an ideal of $B$, we have $t_{\sigma}(A)=B$. Hence $t_{\sigma}(d)=1$ for some $d$ in $A$.

**Theorem 3.10.** Let $A/B$ be $G$-Galois, $A$ a commutative ring, $H$ a subgroup of $G$, and $A'$ a $B$-algebra. Then, $A' \otimes_{B} A^{\sigma}$ is a direct sum of minimal ideals if and only if $A'$ is a direct sum of minimal ideals (cf. [7; p. 178. Th. 2]).

**Proof.** In this case, $(A' \otimes_{B} A)/A'$ is finite $G$-Galois, $G$ is completely outer as an automorphism group of $A' \otimes_{B} A$, and $(A' \otimes_{B} A)^{\sigma}=A' \otimes_{B} A^{\sigma}$ (cf. [22; Th.
5.2 and Prop. 6.5]). Thus the present theorem is an easy consequence from Prop. 3.4 (2).

Concerning [22; Th. 6.9], we note the following

**Lemma 3.11.** Let $A/C$ be separable, and $e$ an idempotent of $A$ such that $eA \subseteq Ae$. Then $e$ is a central idempotent of $A$.

**Proof.** Since $A/\mathfrak{R}(A)$ is a semi-prime ring, we have $(eA+\mathfrak{R}(A))/\mathfrak{R}(A) = (eA+\mathfrak{R}(A))/\mathfrak{R}(A)$, that is, $eA+\mathfrak{R}(A) = Ae+\mathfrak{R}(A)$, and so $Ae = eA + (Ae \cap \mathfrak{R}(A)) = eA + \mathfrak{R}(A)e$. Since $A$ is a central separable $C$-algebra, $\mathfrak{R}(A)A = \mathfrak{R}(C)A$ by [2; Cor. 3.2]. Since $\mathfrak{R}(A)A \supseteq \mathfrak{R}(A) \supseteq \mathfrak{R}(C)A$, we have $\mathfrak{R}(A) = \mathfrak{R}(C)A$, and $Ae = eA + \mathfrak{R}(C)Ae$. Hence $Ae = eA$, because $eAe$ is finitely generated. Consequently, $e$ is a central idempotent of $A$.

**Proposition 3.12.** Let $A/B$ be locally finite $G$-Galois, and assume that there is a representation $A = \bigcup_{i \in I} A^{N_i}$ of $A/B$ such that each $\mathfrak{R}(A)A^{N_i}$ is an ideal of $A^{N_i}$. Then $\mathfrak{R}(A) = \mathfrak{R}(B)A = A \cdot \mathfrak{R}(B)$, and $\mathfrak{R}(A) \cap B = \mathfrak{R}(B)$.

**Proof.** Let $\mathfrak{J}$ be a right ideal of $A$ such that $\mathfrak{R}(B)A + \mathfrak{J} = A$. Then $\mathfrak{R}(B)A^{N_i} + (\mathfrak{J} \cap A^{N_i}) \ni 1$ for some $\lambda$ in $A$, so that $\mathfrak{R}(B)A^{N_i} + (\mathfrak{J} \cap A^{N_i}) = A^{N_i}$. Since $\mathfrak{R}(B)A^{N_i} \subseteq \mathfrak{R}(A^{N_i})$, we have $\mathfrak{J} \cap A^{N_i} = A^{N_i}$, and hence $\mathfrak{J} = A$. Thus we know that $\mathfrak{R}(B)A \subseteq \mathfrak{R}(A)$. Combining this with Lemma 3.8, we have $\mathfrak{R}(A) \cap B = \mathfrak{R}(B)$. Hence $\mathfrak{R}(A) = \mathfrak{R}(B)A = A \cdot \mathfrak{R}(B)$ (Prop. 3.1).

**Theorem 3.13.** Let $A/B$ be locally finite $G$-Galois, $B \subseteq C$, and $A'$ a $B$-algebra such that $A' \approx A' \otimes 1$ ($\subseteq A' \otimes B$) canonically.

1. $\mathfrak{R}(A' \otimes B) = \mathfrak{R}(A' \otimes A)$, and $\mathfrak{R}(A' \otimes A) \cap (A' \otimes 1) = \mathfrak{R}(A') \otimes 1$.

2. If $A$ is commutative, then $\mathfrak{R}(A' \otimes A') = \mathfrak{R}(A') \otimes A'$ for any subgroup $H$ of $G$.

**Proof.** Let $A = \bigcup_{i \in I} A^{N_i}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then $(A' \otimes B)A/(A' \otimes 1)$ is a locally finite $G$-Galois extension with representation $A' \otimes B = \bigcup_{i \in I} A' \otimes A^{N_i}$, where $A' \otimes A^{N_i} = (A' \otimes B)^{N_i}$ is a finite $G/N_i$-Galois extension over $A' \otimes 1$. (1) This will be easily seen by Prop. 3.12. (2) We may assume that $H$ is closed in $G$. Then each $A'/N_i = A'H$ is finite $H/(H \cap N_i)$-Galois, and $H/(H \cap N_i)$ is completely outer as an automorphism group of $A'/N_i$ ([22; Th. 6.6]). Then $H/(H \cap N_i)$ is completely outer as an automorphism group $A' \otimes B A'^{H \cap N_i}$ (Prop. 2.18), and so $H/(H \cap N_i)$ is completely outer as an automorphism group of $A' \otimes B A'^{H \cap N_i}$ (Prop. 2.11). Now, $(A' \otimes B)A/(A' \otimes A')$ is a locally finite $H$-Galois extension with representation $A' \otimes B = \bigcup_{i \in I} A' \otimes A'^{H \cap N_i}$, where $A' \otimes A'^{H \cap N_i} = (A' \otimes B)A'^{H \cap N_i}$ is a finite $H/(H \cap N_i)$-Galois extension over $A' \otimes A'$. Then, by [22; Th. 7.10] and Prop. 3.12, $\mathfrak{R}(A' \otimes B) = \mathfrak{R}(A' \otimes A')(A' \otimes B)$. On the other hand,
$\Re(A' \otimes_B A) = \Re(A') \otimes A = (\Re(A') \otimes A^H) (A' \otimes_B A)$. Hence $\Re(A' \otimes A^B) = \Re(A') \otimes A^H$, as desired (cf. [22; Lemma 7.1]).

References

([1]~[14] are found in [22] below.)


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(Received June 10, 1967)