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LOCALLY FINITE OUTER GALOIS THEORY

By

Yôichi MIYASHITA

Introduction.

This paper is the continuation of the preceding paper [22]. In §1 and §2, locally finite (outer) Galois extensions are treated. The main results are parallel to those of the finite case. In these studies, Nagahara [12] is our guide. Further several results for finite Galois extensions are added (Th. 1.18). In §3, we give a normal basis theorem for a finite Galois extension.

§1. As to notations and terminologies we follow [22]. Let $A$ be a ring with 1 ($\neq 0$), $C$ the center of $A$, $G$ a (finite or infinite) group of automorphisms of $A, B=A^g=\{x\in A; \sigma(x)=x \text{ for all } \sigma \in G\}$, and $\hat{G}$ the group of all $B$-automorphisms of $A$. $\hat{G}$ is then a topological group in finite topology (cf. Jacobson [7]). We denote the closure of $G$ in $\hat{G}$ by $G^*$. $A$ means the trivial crossed product of $A$ with $G$ : $A=\sum_{\sigma\in G}A_{\sigma}, u_{\sigma}u_{\tau}=u_{\sigma\tau}$ ($\sigma, \tau \in G$), $u_{\sigma}x=\sigma(x)u_{\sigma}$ ($x\in A$). Then there is a canonical ring homomorphism $j$ from $A$ to $\text{End } (A_{\sigma})$ defined by $j(\sum_{\sigma} x_{\sigma}u_{\sigma})(y)=\sum_{\sigma} x_{\sigma}\sigma(y) \ (\sum_{\sigma} x_{\sigma}u_{\sigma}\in A, y\in A)$. For any intermediate ring $T$ of $A/B$, $G^T=\{\sigma\in G; \sigma|T=1\}$ is a subgroup of $G$, where $\sigma|T$ means the restriction of $\sigma$ to $T$. We call it a fixed subgroup of $G$. For any subgroup $H$ of $G$, $A^H=\{x\in A; \sigma(x)=x \text{ for all } \sigma \in H\}$ is an intermediate ring of $A/B$. We call it a fixed subring of $A$ (with respect to $G$). Then, as is well known, the set of all fixed subgroups of $G$ and the set of all fixed subrings of $A$ are anti-order-isomorphic in the usual sense of Galois theory. A subring $T$ of $A$ is called a $G$-invariant subring of $A$ if $\sigma(T)=T$ for all $\sigma$ in $G$ (or equivalently, $\sigma(T)\subseteq T$ for all $\sigma$ in $G$). Let $N$ be a fixed subgroup of $G$. Then, $A^N$ is $G$-invariant if and only if $N$ is a normal subgroup of $G$: $N\triangleleft G$. Let $T$ be an intermediate ring of $A/B$, and put $H=G^T$. Then, for $\sigma, \tau$ in $G$, $\sigma|T=\tau|T$ if and only if $\sigma H=\tau H$. Let $H$ and $K$ be subgroups of $G$ such that $H\supseteq K$ and $(H:K)<\infty$, and let $H=\sigma_1K\cup\cdots\cup\sigma_rK$ be the left coset decomposition. For any $x$ in $A^K$ we put $t_{B,K}(x)=\sum_{i} \sigma_i(x)$. Then $t_{B,K}$ is an $A^K-A^H$-homomorphism from $A^K$ to $A^H$, and is independent of the choice of $\sigma_i, \cdots, \sigma_r$. If $K=1$, we write simply $t_B$ instead of $t_{B,1}$.

Here we present several fundamental facts, which are essential throughout the present study. Let $\tau M_{\nu}$ and $\tau N_{\nu}$ be $T$-left, $U$-right modules. If $\tau M_{\nu}$ is
isomorphic to a direct summand of \( r\mathcal{N}_U \) for some natural number \( r \), then we write \( r\mathcal{M}_U|_{T}r\mathcal{N}_U \), where \( r\mathcal{N}_U \) means the direct sum of \( r \) copies of \( r\mathcal{N}_U \). If \( r\mathcal{M}_U|_{T}r\mathcal{N}_U \) and \( r\mathcal{N}_U|_{T}r\mathcal{M}_U \) we write \( r\mathcal{M}_U \sim r\mathcal{N}_U \) (similar) (cf. Morita [21]). To be easily seen, \( r\mathcal{M}_U|_{T}r\mathcal{N}_U \) if and only if there are \( T-U \)-homomorphisms \( f_1, \ldots, f_r \) in \( \mathrm{Hom}(r\mathcal{M}_U, r\mathcal{N}_U) \) and \( g_1, \ldots, g_r \) in \( \mathrm{Hom}(r\mathcal{N}_U, r\mathcal{M}_U) \) such that \( \Sigma_i f_i g_i = \) the identity of \( M \), or equivalently, \( \mathrm{Hom}(r\mathcal{M}_U, r\mathcal{N}_U) \). (Morita) Let \( T \) be a ring with \( 1 \), \( M \) a unital \( T \)-left module, and \( T^* = \mathrm{End}(rM) \). (Morita)

S. 1. If \( rT \mid rM \) then \( M_T \mid T^* \). (i.e. \( M_T \) is finitely generated and projective) and \( T = \mathrm{End}(M_T) \). (Morita)

S. 2. If \( rM \mid rT \) then \( T^* \mid M_T \). (Morita)

S. 3. Let \( T \) be commutative. If \( rM \mid rT \) and \( rM \) is faithful, then \( rT \mid rM \).

(Auslander-Buchsbaum-Goldman)

S. 4. Let \( \overline{T} \) be an extension ring of \( T \). If \( rT \mid r\overline{T} \) then \( rT \) is a direct summand of \( r\overline{T} \) (and conversely). (Müller)

S. 5. Let \( \overline{T} \) be an extension ring of \( T \). If \( rT \mid r\overline{T} \) then \( rT \) is a direct summand of \( r\overline{T} \). (Müller)

In [22], \( A/B \) was called a \( G \)-Galois extension if \( G \) is finite and there are elements \( a_1, \ldots, a_n \); \( a_1^*, \ldots, a_n^* \) in \( A \) such that \( \Sigma_i a_i \sigma(a_i^*) = \delta_{i, \sigma} \) (\( \sigma \in G \)). In this paper, \( A/B \) is called a finite \( G \)-Galois extension if \( A/B \) is \( G \)-Galois and \( t_0(c) = 1 \) for some \( c \in A \). Then, the following are equivalent:

(a) \( A/B \) is finite \( G \)-Galois.
(b) \( G \) is finite, \( A_B \sim B_B \) and \( j: A \cong \mathrm{End}(A_B) \).
(c) \( G \) is finite and \( A^* \cong A \).

(Cf. S. 1, S. 2, [6] and [21]).

\( A/B \) is called a locally finite \( G \)-Galois extension if there are fixed normal subgroups \( N_\lambda (\lambda \in \Lambda) \) of \( G \) which satisfy the following conditions: (1) \( G : N_\lambda < \infty \), and \( A^{N_\lambda} \mid B \) is a finite \( G/N_\lambda \)-Galois extension. (2) \( A = \bigcup \lambda A^{N_\lambda} \), and \( \{A^{N_\lambda} : \lambda \in \Lambda \} \) is a directed set with respect to the inclusion relation (abbr. \( A = \bigcup \lambda A^{N_\lambda} \) is a directed union). Then we call \( A = \bigcup \lambda A^{N_\lambda} \) a representation of the locally finite \( G \)-Galois extension \( A/B \). If \( V_A(B) = C \), an extension \( A/B \) is said to be outer.

Now we shall prove first the following

**Proposition 1.1.** Let \( G = G^* \) (i.e. \( G \) is closed in \( \hat{G} \)). Then the following are equivalent:

(i) \( \{x(x) ; \sigma \in G \} \) is finite for any \( x \) in \( A \).
(ii) \( G \) is compact.
(iii) Every directed union of fixed subrings of \( A \) with respect to \( G \) is also a fixed subring of \( A \) with respect to \( G \), and \( \cap H = 1 \), where \( H \) ranges
over all fixed subgroups of $G$ such that $(G:H)<\infty$.

Proof. (i) $\Rightarrow$ (ii) If we put $\prod_{x\in A} \{ \sigma(x); \sigma \in G \} = D$, then $G \subseteq D$ and $D$ is compact. Therefore it is sufficient to prove that $G$ is closed in $D$. Let $\rho$ be any element of the closure of $G$ in $D$. Then, as is easily seen, $\rho$ is a $B$-ring isomorphism from $A$ into $A$. Let $a$ be in $A$, and put $F = \{ \sigma(a); \sigma \in G \}$. Then, by assumption, $F$ is a finite subset of $A$, so that there is an element $\tau$ in $G$ such that $\rho|F = \tau|F$. Then, in particular, $\rho(\tau^{-1}(a)) = \tau(\tau^{-1}(a)) = a$. Thus $\rho$ is a $B$-automorphism of $A$. Hence the closure of $G$ in $D$ is contained in $\bar{G}$. Since $G$ is closed in $\bar{G}$, $G$ is closed in $D$, as desired. (ii) $\Rightarrow$ (iii) For any $x$ in $G$, we put $H_x = \{ \sigma \in G; \sigma(x) = x \}$. Then $H_x$ is open in $G$, and therefore $aH_x$ is open in $G$ for any $a$ in $G$. Then, since $G$ is compact, we have $(G:H_x) < \infty$. Evidently $\cap_{x \in A} H_x = 1$. This proves the second assertion. Let $(A \neq) T = \cup_{i \in A} T_i$ be a directed union of fixed subrings of $A$ with respect to $G$, and let $K_i = G^T$. Then each $K_i$ is a closed subgroup of $G$, and $A^{K_i} = T_i$. Let $a$ be an element of $A - T$, and put $U = \{ \sigma \in G; \sigma(a) = a \}$. Then $U$ is open in $G$, so that each $K_i - U$ is closed in $G$. Since $a \notin T_i$ and $A^{K_i} = T_i$, we have $K_i - U \neq \emptyset$. For any finite subset $\{ \lambda_1, \cdots, \lambda_n \}$ of $A$, there is an element $\lambda_0$ of $A$ such that $T_{\lambda_0} \supseteq \cup_i T_i$. Then $K_{\lambda_0} \subseteq \cap_i K_i$, and so $\emptyset \neq K_{\lambda_0} - U \subseteq \cap_i K_i - U = \cap_i (K_i - U)$. Thus $\{ K_i - U; \lambda \in A \}$ has finite intersection property. Since $G$ is compact, we have $\cap_i (K_i - U) \neq \emptyset$. If $\rho$ is in $\cap_i (K_i - U)$ then $\rho \in G^T$ and $\rho(a) \neq a$. Therefore $a \notin A^K$, where $K = G^T$. Thus $A^K = T$. Hence $T$ is a fixed subring of $A$ with respect to $G$. (iii) $\Rightarrow$ (i) Let $H$ and $K$ be fixed subgroups of $G$ such that $(G:H)<\infty$ and $(G:K)<\infty$. Then $H \cap K$ is also a fixed subgroup of $G$ with $(G:H \cap K)<\infty$. Therefore $\cup A^H$ is a directed union of fixed subrings of $A$, where $H$ ranges over all fixed subgroups of $G$ with $(G:H)<\infty$. Then, by assumption, $\cup A^H$ is a fixed subring of $A$ with respect to $G$. Since $\cap H = 1$, we have $A = \cup A^H$. For any $x$ in $A$, there is an $A^H$ such that $x \in A^H$. Therefore if we let $L = \{ \sigma \in G; \sigma(x) \neq x \}$ then $(G:L)<\infty$. This implies that $\{ \sigma(x); \sigma \in G \}$ is finite.

Remark. For any $x$ in $A$, $\{ \sigma(x); \sigma \in G \} = \{ \sigma(x); \sigma \in G^* \}$.

**Proposition 1.2.** Let $N$ be a fixed normal subgroup of $G$ such that $(G:N)<\infty$ and $A^N/B$ is finite $G/N$-Galois, and $G_1$ a subgroup of $G^*$ containing $G$. Then $A^N/B$ is finite $G_1/N_1$-Galois, where $N_1 = \{ \sigma \in G_1; \sigma|A^N = 1_{A^N} \}$.

Proof. Put $T = A^N$. Evidently $A^N = T$. Since $G$ is dense in $G_1$ and $T$ is finitely generated, there holds $G/T = G_1/T$. Therefore $T$ is $G_1$-invariant, $N_1 \subseteq G_1$, and $(G_1:N_1)<\infty$. There are elements $a_1, \cdots, a_n; a_1^*, \cdots, a_n^*$ in $T$ such that $\sum_i a_i \cdot \sigma(a_i^*) = \delta_{N, \sigma}$ for all $\sigma$ in $G$. If $\tau$ is in $G_1 - N_1$ then $\tau|T = \rho|T$ for
some $\rho$ in $G-N$, and $\sum \tau(a^*_i)\rho(a^*_i)=\sum \tau(a^*_i)$ for $\sigma$ in $G_1$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, and $G_1$ a subgroup of $G^*$ containing $G$. Then $A/B$ is locally finite $G_1$-Galois.

**Proposition 1.3.** Let $H_{\lambda}(\lambda \in \Lambda)$ be fixed subgroups of $G$ such that $A=\bigcup_{\lambda \in \Lambda} A^{H_{\lambda}}$ is a directed union.

(1) If $H$ is a subgroup of $G$ such that $(G:H)<\infty$ then $A^H \subseteq A^{H_{\lambda}}$ for some $\lambda$ in $\Lambda$.

(2) If $K$ is a subgroup of $G$ such that $(K:1)<\infty$ then $K \cap H_{\rho}=1$ for some $\mu$ in $\Lambda$.

**Proof.** (1) Let $[H_{\lambda} \cup H]$ be the subgroup of $G$ generated by $H_{\lambda} \cup H$. Since $G \supseteq [H_{\lambda} \cup H] \supseteq H$, we have $(G:[H_{\lambda} \cup H]) \leq (G:H)$ for all $\lambda$ in $\Lambda$. Let $(G:[H_{\lambda} \cup H])$ be maximum. We shall prove that $A^H \subseteq A^{H_{\lambda}}$. For any $H$, there is an $H_{\lambda}$ such that $A^H \supseteq A^{H_{\lambda}} \cap A^H$. Then $H_{\lambda} \subseteq H \cap H_{\lambda}$, and so $[H_{\rho} \cup H] \subseteq [H_{\lambda} \cup H] \cap [H_{\lambda} \cup H]$. Since $(G:[H_{\lambda} \cup H])$ is maximum, we have $([H_{\lambda} \cup H]) \supseteq [H_{\rho} \cup H]=[H_{\lambda} \cup H]$. Hence $[H_{\lambda} \cup H] \subseteq [H_{\lambda} \cup H]$ for all $\lambda$ in $\Lambda$. Then $A^H \subseteq \bigcup_{\lambda} (A^H \cap A^H_{\rho})=H_{\lambda} \cap A^H_{\rho}$, which means $A^H \subseteq A^{H_{\lambda}}$. (2) Since $A=\bigcup_{\sigma} A^H_{\rho}$, we have $1=G^H=\cap_{\lambda} H_{\lambda}$. Let $K=\{\sigma_1=1, \sigma_2, \ldots, \sigma_r\}$. Then, for any $\sigma_\lambda$ ($\lambda \neq 1$), there is an $H_{\rho}$ such that $\sigma_\lambda \notin H_{\rho}$. By assumption there is some $\mu$ such that $H_{\rho} \subseteq \cap_{\lambda=1, \ldots, r} H_{\lambda}$. Then $H \cap H_{\rho} \subseteq H \cap (\cap_{\lambda=1, \ldots, r} H_{\lambda})=1$.

**Remark.** Let $A/B$ be locally finite $G$-Galois, and $A=\bigcup_{\lambda \in \Lambda} A^{H_{\lambda}}$ its representation. If $G$ is finite then $A=A^{H_{\lambda}}$ for some $\lambda$.

**Proposition 1.4.** Let $T$ be an intermediate ring of $A/B$ such that $G|T$ is finite, and let $H=H^T$, and $G=\sigma_1 H \cup \cdots \cup \sigma_r H$ a left coset decomposition of $G$. If there are elements $t_1, \ldots, t_n; t^*_1, \ldots, t^*_n$ in $T$ such that $\sum t_i \sigma(t^*_i)=\delta_{\mu}$ for all $\sigma$ in $G$, then there hold the following.

(1) $T=A^H$, and $T_B$ is finitely generated and projective.

(2) $j^*: A(\sum_k u_k)=\sum_k A u_k \simeq \text{Hom}(T_B, A_B)$, where $j^*(\sum_k x_k u_k)(t)=\sum_k x_k \cdot \sigma_k(t)$, and this induces the $B-T$-isomorphism $(_B T_B \simeq) (\sum_k u_k)T \simeq \text{Hom}(T_B, A_B)$.

(3) The following are equivalent: (i) $B_B|T_B$. (ii) $B_B|T_B$. (iii) $t_{\sigma:B_B}(c)=1$ for some $c$ in $T$.

**Proof.** (1) $t_{\sigma:B_B}$ is a $B-B$-homomorphism from $A^H$ to $B$. For any $y$ in $A^H$, $T \ni \sum t_i \cdot t_{\sigma:B_B}(y^*)=\sum t_i \cdot \sum_k \sigma_k(y^*)=\sum \sum t_i \cdot \sigma_k(t^*_i)\sigma_k(y)=y$. Hence $A^H =T$, and $T_B$ is finitely generated and projective (cf. [3]). (2) $f^{\sigma^{-1}}$ is the mapping such that $f^{\sigma^{-1}}(f)=\sum t_i f(t_i)(\sum_k u_k t^*_i)$ (for $f \in \text{Hom}(T_B, A_B)$). The second part will be easily seen. (3) The equivalence (i)$\iff$ (iii) is easy from (2).
Therefore (i) and (ii) are equivalent, because the situation is right-left symmetric.

**Proposition 1.5.** Let $A/B$ be locally finite $G$-Galois. Then there hold the following:

1. $G^*$ is compact.
2. By $j$, $A$ is isomorphic to a dense subring of $\text{Hom}(A_B, A_B).
3. A subgroup $H$ of $G$ is a closed subgroup of $G$ if and only if $H$ is a fixed subgroup of $G$.

**Proof.** Let $A=\bigcup \nu A^{N_{\nu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) If $x$ is in $A$ then $x \in A^{N_{\nu}}$ for some $\nu$ in $\Lambda$. Then $\langle G: N_{\nu} \rangle < \infty$ implies that $\langle \{ \sigma(x); \sigma \in G \} = \{ \sigma(x); \sigma \in G^* \}$ is finite. Hence, by Prop. 1.1, $G^*$ is compact. (2) By Prop. 1.4 (2), $\text{Im} j$ is dense in $\text{Hom}(A_B, A_B)$. Therefore it suffices to prove that $j$ is 1–1. Let $\sigma_1, \ldots, \sigma_r$ be different elements in $G$. Then there is a finite subset $F$ of $A$ such that $\sigma_i | F \neq \sigma_j | F$ provided $i \neq k$. From this fact and Prop. 1.4, we can easily see that $j$ is 1–1. (3) Evidently, a fixed subgroup is a closed subgroup. Let $H$ be any subgroup of $G$, and put $H' = G^*$, where $T = A^H$. Then $T = A^{H'}$. It suffices to prove that $H$ is dense in $H'$. To prove this, we take any finite subset $F$ of $A$. Then $F \subseteq A^{N_{\nu}}$ for some $N_{\nu}$. Put $N = N_{\nu}$. Then, by finite Galois theory, we obtain $(G/N)^{N_{\nu}} = HN/N$ and $(A/N)^{N_{\nu}} = HN/N$, where $T_1 = A^{H_{\nu}}$ and $T_1' = A^{H'_{\nu}}$ (cf. [22; Prop. 2.2]). Since $A^{H_{\nu}} = A^{H} \cap A^{N_{\nu}} = A^{H} \cap A^{N_{\nu}} = A^{H'_{\nu}}$, we have $HN/N = HN/N$, that is, $HN = H'N$. Hence $H|A^{N_{\nu}} = H'|A^{N_{\nu}}$, and so $H|F = H'F$. Since $F$ is arbitrary, this implies that $H$ is dense in $H'$. This completes the proof.

**Theorem 1.6.** Let $A/B$ be locally finite $G$-Galois, $G = G^*$, and $H$ a subgroup of $G$, and let $A'$ be an indecomposable extension ring of $B$ such that $V_{A'}(B) = V_{A'}(A')$. Assume that there is a B-ring homomorphism $g$ from $A$ to $A'$. Then, for any $B$-ring homomorphism $f$ from $A^H$ to $A'$, there is an element $\sigma$ in $G$ such that $f = g\sigma | A^H$.

**Proof.** Let $A = \bigcup \lambda A^{N_{\lambda}}$ be a representation. For each $N_{\lambda}$, there is an element $\sigma$ in $G$ such that $f | A^{H_{N_{\lambda}}} = g\sigma | A^{H_{N_{\lambda}}}$ ([22; Th. 4.1]). For each $\lambda$, we put $K_{\lambda} = \{ \sigma \in G; f | A^{H_{N_{\lambda}}} = g\sigma | A^{H_{N_{\lambda}}} \}$. Then $K_{\lambda} \neq \emptyset$, and $\{ K_{\lambda}; \lambda \in \Lambda \}$ has finite intersection property. Let $\tau$ be in the closure of $K_{\lambda}$ in $G$. Since $(A^{N_{\lambda}})_B$ is finitely generated, $\tau | A^{N_{\lambda}} = \sigma | A^{N_{\lambda}}$ for some $\alpha$ in $K_{\lambda}$. Then $\tau | A^{H_{N_{\lambda}}} = \sigma | A^{H_{N_{\lambda}}}$, and so $f | A^{H_{N_{\lambda}}} = g\alpha | A^{H_{N_{\lambda}}} = g\tau | A^{H_{N_{\lambda}}}$. Hence $\tau \in K_{\lambda}$, and therefore $K_{\lambda}$ is closed in $G$. Since $G$ is compact (Prop. 1.5), we have $\bigcap \lambda K_{\lambda}$, \neq \emptyset. If $\rho$ is in $\bigcap \lambda K_{\lambda}$, then $f | A^{H_{N_{\lambda}}} = g\rho | A^{H_{N_{\lambda}}}$ for all $\lambda$ in $\Lambda$. Since $A^H = \bigcup \lambda A^{H_{N_{\lambda}}}$, we know $f = g\rho | A^H$.

The following theorem will follow at once from Th. 1.6 and Cor. to Prop. 1.2.

**Theorem 1.7.** Let $A/B$ be locally finite outer $G$-Galois, and $A$ an
indecomposable ring. Then \( G^* = \hat{G} \), that is, \( G \) is dense in \( \hat{G} \).

**Proposition 1.8.** Let \( A/B \) be locally finite G-Galois, and \( G = G^* \) (cf. Cor. to Prop. 1.2). Then there hold the following.

(1) For an intermediate ring \( T \) of \( A/B \) the following are equivalent.
(i) \( T = A^H \) for some subgroup \( H \) of \( G \).
(ii) There are subgroups \( H_\gamma \) (\( \gamma \in \Gamma \)) of \( G \) such that \( T = \bigcup \gamma \), \((G : H_\gamma) < \infty \) and \( \{A^{H_\gamma}; \gamma \in \Gamma \} \) is a directed set with respect to the inclusion relation.

(2) If \( H \) is a subgroup of \( G \) such that \((G : H) < \infty \) then \( (A^H)_B \) is finitely generated.

**Proof.** Let \( A = \bigcup_{\nu \in \Lambda} A^{N_\nu} \) be a representation of the locally finite G-Galois extension \( A/B \). (1) \((i) \implies (ii) \) \( T = A^H = \bigcup_\nu (A^H \cap A^{N_\nu}) = \bigcup_\nu A^{HN_\nu} \) is a directed union, and \((G : HN_\nu) < \infty \). \((ii) \implies (i) \) follows from Prop. 1.1. (2) By Prop. 1.3, \( A^H \subseteq A^N \) for some \( \nu \) in \( A \). Then, \( A^H = A^{HN_\nu} \) is a fixed subring of the finite \( G/N_\nu \)-Galois extension \( A^{N_\nu}/B \), and therefore \((A^H)_B | (A^{N_\nu})_B \) (cf. [22; §2. p. 118]). Since \((A^{N_\nu})_B \) is finitely generated, \((A^H)_B \) is finitely generated.

Let \( T \) be an intermediate ring of \( A/B \), and \( S \) a subset of \( A \). \( T \) is called a G-separable cover of \( S \) if \( T \) satisfies the following conditions:

(1) \( T/B \) is a separable extension, and \( T \supseteq S \).
(2) \( G|T \) is finite.
(3) \( G|T \) is strongly distinct (i.e. if \( \sigma|T \neq \tau|T \) for \( \sigma, \tau \) in \( G \) then \( \sigma|T \) and \( \tau|T \) are strongly distinct).

**Theorem 1.9.** Let \( A/B \) be locally finite outer G-Galois, and \( T \) an intermediate ring of \( A/B \). Then the following are equivalent:

(i) \( T = A^H \) for some subgroup \( H \) of \( G \) such that \((G : H) < \infty \).
(ii) \( T/B \) is a separable extension, \( T_B \) is finitely generated, and \( G|T \) is strongly distinct.

(iii) \( T \) is a G-separable cover of \( B \).

**Proof.** Let \( A = \bigcup_{i \in \Lambda} A^{N_i} \) be a representation. (i) \((i) \implies (ii) \) By Prop. 1.3, \( T = A^H \subseteq A^N \) for some \( \nu \) in \( A \). Then \( T \) is a fixed subring of the finite \( G/N_\nu \)-Galois extension \( A^{N_\nu}/B \). Then, by [19; Prop. 3.4], \( T/B \) is a separable extension. By Prop. 1.8 (2) (cf. Cor. to Prop. 1.2), \( T_B \) is finitely generated. By [22; Th. 2.6], \( G|T \) is strongly distinct. (ii) \((ii) \implies (iii) \) This follows from the fact that \( \{\sigma(x); \sigma \in G\} \) is finite for any \( x \) in \( A \). (iii) \((iii) \implies (i) \) Let \( \{(t_i, t_i^*); i=1, \ldots, n\} \) be a \((B, T)\)-projective coordinate system of \( T/B \). Then, by [22; Prop. 1.2], \( \sum_i t_i \cdot \sigma(t_i^*) = \delta_{H, \sigma} \) for \( \sigma \) in \( G \), where \( H = G^T \). \((G|T) < \infty \) implies \((G : H) < \infty \). By Prop. 1.4, \( A^H = T \).

Combining Th. 1.9 with Prop. 1.8, we obtain the following theorem (cf. [12; Th. 3], [28; Theorem]).
Theorem 1.10. Let $A/B$ be locally finite outer $G$-Galois, and $G=G^*$. Then, for an intermediate ring $T$ of $A/B$, the following are equivalent.

(i) $T=A^H$ for some subgroup $H$ of $G$.

(ii) For any finite subset $F$ of $T$ there is an intermediate ring $T_0$ of $T/B$ such that $T_0$ is separable, $T_0/B$ is finitely generated, and $G|T_0$ is strongly distinct.

(iii) Any finite subset of $T$ has a $G$-separable cover which is contained in $T$.

Next we shall proceed to the characterization of locally finite outer Galois extensions.

Proposition 1.11. Let $V_{A}(B)=C$, $T$ a $G$-separable cover of $B$, and $\{t_i, t_i^*\}; i=1, \cdots, n$ a $(B, T)$-projective coordinate system for $T/B$, and put $H=G^*$. Then there hold the following.

1. \[ \sum_i t_i \sigma(t_i^*) = \delta_{H, \sigma} \text{ for all } \sigma \text{ in } G. \]

2. \[ A^H = T, (G:H)<\infty, \text{ and } T/B \text{ is a projective Frobenius extension}. \]

3. Let $K$ be a subgroup of $G$ containing $H$. Then, $\sum_i t_{K:H}(t_i) \sigma(t_i^*) = \delta_{K, \sigma}$ for all $\sigma$ in $G$, $T$ is $(B, A^K)$-projective, $T/A^K$ is a projective Frobenius extension, and $G/A^K$ is strongly distinct. Further the following are equivalent. (a) $(A^K)_{A^K}|T_{A^K}$. (b) $(A^K)(A^K)|_{(A^K)}T$. (c) $t_{K:H}(c)=1$ for some $c$ in $T$.

Proof. (1) follows from [22; Prop. 1.2], and (2) is obvious by (1) and Prop. 1.4. (3) It will be easily seen that $\sum_i t_{K:H}(t_i) \sigma(t_i^*) = \delta_{K, \sigma}$ for all $\sigma$ in $G$. Since $\sum_i t_i \otimes t_i^* = \sum_i t_i \otimes t_i^*$ for $t$ in $T$, $\sum_i y \cdot t_{K:H}(t_i) \otimes t_i^* = \sum_i t_{K:H}(t_i) \otimes t_i^* y$ for all $y$ in $A^K$. Hence the mapping $x \rightarrow \sum_i t_{K:H}(t_i) \otimes t_i^* x$ from $T$ to $A^K \otimes_B T$ is an $A^K$-$A^K$-homomorphism. Since $\sum_i t_{K:H}(t_i) t_i^* x = x$, it follows that $T$ is $(B, A^K)$-projective. Let $\rho|A^K \neq \tau|A^K$ for $\rho$, $\tau$ in $G$. Then $\tau^{-1} \rho \notin K$, and so $0=\tau(\sum_i t_{K:H}(t_i) \tau^{-1} \rho(t_i^*)) = \sum_i \tau(t_{K:H}(t_i)) \rho(t_i^*)$. Thus, by [22; Prop. 1.1], $\rho|A^K$ and $\tau|A^K$ are strongly distinct. If we set $G=K$ in Prop. 1.4, the remainder follows from Prop. 1.4.

Theorem 1.12. Let $V_{A}(B)=C$. Then the following statements are equivalent.

(i) $A/B$ is locally finite (outer) $G$-Galois.

(ii) For any finite subset $F$ of $A$ there is a $G$-invariant $G$-separable cover $T$ of $F$ such that $pB_{|T}$. 

(iii) For any finite subset $F$ of $A$ there is a $G$-separable cover $T$ of $F$ which satisfies the following: If $T_0$ is an intermediate ring of $T/B$ such that (a) $T$ is $(B, T_0)$-projective, (b) $T/T_0$ is a projective Frobenius extension, (c) $G|T_0$ is strongly distinct, then $T_0/T_0$. 

(iv) For any finite subset $F$ of $A$ there is a $G$-separable cover $T$ of $F$
which satisfies the following: If $T_0$ is an intermediate ring of $T/B$ such that (a) $T$ is $(B, T_0)$-projective, (b) $T/T_0$ is a projective Frobenius extension, (c) $G/T_0$ is strongly distinct, (d) $T_0$ is a $G$-invariant fixed subring (with respect to $G$), then $T_0|_{T_0}T$.

Proof. (i) $\Rightarrow$ (ii), (iii) Let $A = \bigcup_{\alpha} A^{N_\alpha}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then any finite subset $F$ of $A$ is contained in some $A^{N_\alpha}$ ($\alpha \in \Lambda$). By [22; Th. 1.5], $A^{N_\alpha}$ is a $G$-invariant $G$-separable cover of $F$ such that $A_B^{N_\alpha}$ and $A_0^{N_\alpha}$. Let $T_0$ be an intermediate ring of $A^{N_\alpha}/B$ such that $A^{N_\alpha}$ is $(B, T_0)$-projective and that $G/T_0$ is strongly distinct. Then, by [22; Th. 2.6], $T_0$ is a fixed subring of the finite outer $G/N_\alpha$-Galois extension $A^{N_\alpha}/B$, whence $T_0|_{T_0}T$ by [22; §2. p. 118]. (ii) $\Rightarrow$ (i) Let $F$ be a finite subset of $A$, and $T$ a $G$-invariant $G$-separable cover of $F$ such that $A_B^{N_\alpha}$. If we put $N = G_\alpha$, then $A^{N_\alpha} = N \cap G$ and $(G : N) < \infty$ (Prop. 1.11). By Prop. 1.11, $A^{N_\alpha}/B$ is a finite $G/N_\alpha$-Galois extension. Noting that $(A^{N_\alpha})_B$ is finitely generated, $A/B$ is a locally finite $G$-Galois extension. (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (i) Let $T_1$ be a separable cover of an element $x$ in $A$. Put $G_1 = H_1$. Then $(G|T_1) < \infty$ implies $(G : H_1) < \infty$ and $(A|T_1) < \infty$. Thus any finite subset of $A$ is contained in a $G$-invariant finite subset of $A$. Let $F$ be a $G$-invariant finite subset of $A$, and $T$ a $G$-separable cover of $F$ as that in (iv), and let $\sum_{i} t_i \cdot \sigma(t_i^*) = \delta_{\sigma, x}$ for all $\sigma$ in $G$. Set $N = G_\alpha$. Then $H_\alpha \cap N \subseteq G$, and $F \subseteq A^{N_\alpha}$. By Prop. 1.11, $T$ is $(B, A^{N_\alpha})$-projective, $T/A^{N_\alpha}$ is a projective Frobenius extension, and $G/A^{N_\alpha}$ is strongly distinct. Then, by the assumption for $T$, $(A^{N_\alpha}|T) = (A^{N_\alpha}|T_0)$, so that $T_{\alpha} : (x) = 1$ for some $x$ in $T$ (Prop. 1.11 (3)). Put $t_i = t_{\alpha} : (x)$ and $t_i^* = t_{\alpha} : (x)$. Then, $t_i, t_i^* \in A^{N_\alpha}$, and $\sum_{i} t_i \cdot \sigma(t_i^*) = \delta_{\sigma, \alpha}$ for all $\sigma$ in $G$ (Prop. 1.11 (3)). Further, as is easily seen, $\sum_{i} t_i \cdot \sigma(t_i^*) = \delta_{\sigma, \alpha}$ for all $\sigma$ in $G$. Since $p_B|_T (\text{Prop. 1.11 (3)})$, we have $p_B|_T A^{N_\alpha}$. Thus $A^{N_\alpha}/B$ is a finite $G/N_\alpha$-Galois extension. Noting that $(A^{N_\alpha})_B$ is finitely generated, we conclude that $A/B$ is a locally finite $G$-Galois extension.

Proposition 1.13. Let $A^* \supseteq T \supseteq B^*$ be rings such that $A^*$ is $(B^*, T^*)$-projective, $A'$ an extension ring of $B^*$ such that $V_{B'}(B^*) = V_{A'}(A')$, and $f_1, \cdots, f_s$ $B^*$-ring homomorphisms from $A^*$ to $A'$ such that $f_i|_T$ and $f_s|_T (i \neq k)$ are strongly distinct. If $(B^*)_{B^*} \supseteq T_{B^*}$, then $(A')_{A'} \supseteq (A')_{A'}$.

Proof. Let $\{(t_i, a_i^*); i = 1, \cdots, n\}$ be a $(B^*, T^*)$-projective coordinate system for $A^*$. Then, by [22; Prop. 1.2], $\sum f_i(t_i^*) a_i^* = \delta_{h, k}$ for all $h, k$. Let $\varphi$ be a $A'$-right homomorphism from $T \otimes_{B'} A'$ to $(A')_{A'}$ defined by $\varphi(t \otimes a') = (f_i(t) a', \cdots, f_s(t) a')$. Since $\sum f_i(t_i^*) a_i^* = \delta_{h, k}$, $\varphi$ is an epimorphism. $(B')_{B^*} \supseteq T_{B^*}$ implies that $(A')_{A'} \supseteq T \otimes_{B'} A'$, hence we have $(A')_{A'} \supseteq (A')_{A'}$, as
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desired.

Concerning Prop. 1.13, we consider the following condition.

Condition (F): If $\mathcal{A}A^r \rightarrow \mathcal{A}A^s$ for positive integers $r, s$, then $r \geq s$.

Remark. Let $\mathcal{A}A^r \rightarrow \mathcal{A}A^s$ for positive integers $r, s$. Then, since $\mathcal{A}A^s$ is projective, $\mathcal{A}A^s$ is isomorphic to an $A$-direct summand of $\mathcal{A}A^r$.

(1) If $\mathcal{A}A$ is finite dimensional, then $r \cdot \dim \mathcal{A}A \geq s \cdot \dim \mathcal{A}A$, and so $r \geq s$ (cf. [11]).

(2) Assume that there is a proper ideal $\mathfrak{A}$ of $A$ such that $\mathcal{A}A/\mathfrak{A}$ is finite dimensional. Then, since $\mathcal{A}A^r/\mathfrak{A}^r \rightarrow \mathcal{A}A^s/\mathfrak{A}^s$, the above (1) yields $r \geq s$, because $\mathcal{A}A^r/\mathfrak{A}^r \simeq \mathcal{A}(A/\mathfrak{A})^r$ and $\mathcal{A}A^s/\mathfrak{A}^s \simeq \mathcal{A}(A/\mathfrak{A})^s$.

(3) If $A$ is commutative, then $r \geq s$ by (2).

**Proposition 1.14.** Let $V_A(B) = C$, and $A$ an indecomposable ring satisfying (F), and let $T$ be an intermediate ring of $A/B$, and $S$ a subset of $A$. Then the following are equivalent:

(i) $T$ is a $G$-separable cover of $S$.

(ii) $T \supseteq S$, $T/B$ is a separable extension, and $T_B$ is finitely generated.

*Proof.* (i) $\Rightarrow$ (ii) is evident by Prop. 1.11. (ii) $\Rightarrow$ (i) By [22; Lemma 2.7], $A$ is $(B, T)$-projective. Then, by Prop. 1.13, we have $\#(G|T) < \infty$, and hence $T$ is a $G$-separable cover of $S$.

If $A$ is commutative, then $A$ satisfies (F). Therefore, by Th. 1.12, S. 3 and Prop. 1.14, we have the following

**Theorem 1.15** (Nagahara [12]). Let $A$ be an indecomposable commutative ring. Then the following are equivalent.

(i) $A/B$ is locally finite $G$-Galois.

(ii) For any finite subset $F$ of $A$ there is an intermediate ring $T$ of $A/B$ such that (a) $T/B$ is a separable extension, and $T_B$ is finitely generated, (b) $T \supseteq F$.

**Proposition 1.16.** Let $A/B$ be locally finite $G$-Galois, and $H$ a subgroup of $G$. Then $G|A^H$ is strongly distinct.

*Proof.* Let $\sigma, \tau$ be in $G$, and $e$ a central idempotent of $A$ such that $\sigma(x)e = \tau(x)e$ for all $x$ in $A^H$. Let $A = \bigcup_{\lambda \in \Lambda} A^{N_\lambda}$ be a representation of the locally finite $G$-Galois extension $A/B$. We may assume that $e \in A^{N_\lambda}$ for all $\lambda$ in $\Lambda$. Suppose that $\sigma|A^H \neq \tau|A^H$. Since $A^H = \bigcup_{\lambda \in \Lambda} A^{N_\mu|H}$, $\sigma|A^{N_\mu|H} = \tau|A^{N_\nu|H}$ for some $\mu$ in $\Lambda$. Then, by [22; Prop. 2.4], $(G/N_\mu)|A^{N_\mu}$ is strongly distinct. Therefore we have $e = 0$. Thus $G|A^H$ is strongly distinct.

**Theorem 1.17.** Let $A/B$ be locally finite outer $G$-Galois, and $T$ an intermediate ring of $A/B$. Then the following are equivalent.

(i) $T = A^H$ for some subgroup $H$ of $G$, and $A_T$ is finitely generated.
(ii) $T=An$ for some subgroup $H$ of $G$ such that $(H:1)<\infty$.
(iii) $A/T$ is a projective Frobenius extension, Hom$(A_T, A_T) \subseteq \Delta$, and $G|T$ is strongly distinct.

When any of the above conditions is satisfied $A/A^n$ is finite $H$-Galois.

Proof. Let $A = \bigcup_{\mu \in A} A^{N_{\mu}}$ be a representation of the locally finite outer $G$-Galois extension $A/B$. (i) $\Rightarrow$ (ii) Let $A = x_1 T + \cdots + x_r T$. Then $x_1, \ldots, x_r \in A^{N_\mu}$ for some $\mu \in A$. There are elements $a_1, \ldots, a_n; a_1^*, \ldots, a_n^*$ in $A^{N_{\mu}}$ such that $\sum_i a_i \sigma(a_i^*) = \delta_{N_\mu}$ for all $\sigma$ in $G$. Then $\sum_i a_i \sigma(a_i^*) = \delta_{N_\mu}$ for all $\sigma$ in $G$. Hence $A/A^n$ is $H$-Galois. Therefore $A/A^n$ is a projective Frobenius extension (cf. [22; p. 121]), and Hom $(A_T, A_T) = \sum_{\epsilon \in H} Au, \subseteq \Delta$. By Prop. 1.16, $G|T$ is strongly distinct. (iii) $\Rightarrow$ (i) Let $h = \sum_{\epsilon \in H} a_u$, be a Frobenius homomorphism of $A/T$, where $H$ is a finite subset of $G$ and $a_u \neq 0$ for all $\tau$ in $H$. Then, since $th = ht$ for all $t$ in $T$, we have $ta_\tau = a_\tau \tau(t)$ for all $t$ in $T$, in particular, $ba_\tau = ab$ for all $b$ in $B$. Hence $a_\tau \in V_\Lambda(A) = C$ for all $\tau$ in $H$. There are elements $r_i, l_i$ in $A$ such that $x = \sum_i h(xr_i)l_i = \sum_i r_i h(l_ix)$ for all $x$ in $A$ (cf. [27]). Then $u_\tau = \sum_i r_i h(l_i) = \sum_i r_i \sum_{\epsilon \in H} a_\tau \sigma(l_i) u_\tau = \sum_{\epsilon \in H} \sum_i r_i a_\tau \tau(l_i) u_\tau$, and so $1 = \sum_i r_i a_\tau l_i = a_1 \sum_i r_i l_i$. Thus $a_1$ is an invertible element in $C$, and $a_1^{-1} = \sum_i r_i l_i$. Since $H$ is finite there is an $N_\mu$ such that $\tau|A^{N_\mu} \neq \rho|A^{N_\mu}$ provided $\tau \neq \rho, \rho \in H$. Since $A^{N_\mu}/B$ is finite $G/N_\mu$-Galois, there are elements $d_k, e_k$ in $A^{N_\mu}$ such that $\sum_k d_k \sigma(e_k) = \delta_{N_\mu}$ for all $\sigma$ in $G$. Put $A_0 = \text{Hom}(A_T, A_T)$. Then $A_0 = AHA$, and $A_0 \ni \sum_k \tau(d_k) h e_k = \sum_{\epsilon \in H} \sum_k \tau(d_k) a_\tau \sigma(e_k) u_\tau = a_\tau u_\tau$ for $\tau$ in $H$. Thus $A_0 = AHA = \sum_{\epsilon \in H} A a_\tau u_\tau$. Since $A/T$ is a projective Frobenius extension with Frobenius homomorphism $h, A \otimes_{\tau} A \simeq A \to \delta \lambda$ by the correspondence $x \otimes y \rightarrow xhy$. Let $\varphi$ be the $A$-left homomorphism from $A$ to $A_0$ defined by $\varphi(\sum u_x x) = \sum_{\epsilon \in H} x a_u u_x$, and $\psi$ be the $A$-left homomorphism from $A_0$ to $A$ defined by $\psi(xhy) = \sum_i x h(yr_i) v l_i$, where $v = \sum_{\epsilon \in H} u_x$. Then, as $h(tr_i) a_\tau = h(yr_i) a_\tau \tau(t) \in H$, $\varphi = 1$. Since $a_\tau u_x = \sum_k \tau(d_k) h e_k$, we have $\varphi(a_\tau u_x) = \sum_k \sum_l \tau(d_k) h(e_l r_i) v l_i = \sum_{\epsilon \in H} \sum_k \sum_l \tau(d_k) a_\tau \sigma(e_l) \rho(r_i) u_i l_i$, and so $\varphi(a_\tau u_x) = \sum_{\epsilon \in H} \sum_k a_\tau \sigma(e_l) \rho(r_i) \nu(l_i) u_i$. On the other hand, $\varphi(a_\tau u_x) = a_\tau u_x$, and hence $a_\tau^2 \tau(\sum_i r_i l_i) = a_\tau$ for all $\tau$ in $H$. Since $a_1 = \sum_i r_i l_i$, we have $a_\tau^2 = a_\tau \tau(a_1)$. Noting that $\tau(a_1)$ is an invertible element of $C$, $Aa_\tau A a_\tau = Aa_\tau A a_\tau$, so $A = Aa_\tau + \text{Ann}_A(a_\tau)$, where $\text{Ann}_A(a_\tau) = \{x \in A; xa_\tau = 0\}$. If $xa_\tau \in \text{Ann}_A(a_\tau)$, then $0 = xa_\tau^2 = xa_\tau \tau(a_\tau)$, so that $xa_\tau = 0$. Thus $A = Aa_\tau + \text{Ann}_A(a_\tau)$. Therefore $a_\tau$ is written as $A_g$, with a central idempotent $g$, of $A$. Since $Aa_\tau u_x \subseteq A_0$, we have $g u_x \in A_0$, and so $g \cdot t = g \tau(t)$ for all $t$ in $T$. Consequently, $A_0 = \sum_{\epsilon \in H} A u_x$, and $H = G^T$. Hence End$(A_A(A) = A^\mu)$, the right multiplications of elements of $A$. Since $a_\tau u_x \in A_0 = \text{End}(A_T)$, we have $a_\tau u_x \in \text{End}(A_A(A))^\mu$. Noting that $a_\tau$ is in $C$, we
can easily seen that \(a, u \in \text{Hom}(\langle A \rangle A_{\langle A \rangle}, \langle A \rangle A_{\langle A \rangle})\). Thus \(h = \sum_{a, u} a, u \in \text{Hom}(\langle A \rangle A_{\langle A \rangle}, \langle A \rangle A_{\langle A \rangle})\). Then, by [27; Cor. 1], \(A/A^h\) is also a projective Frobenius extension with a Frobenius homomorphism \(h\). Since \((H:1)<\infty\), there is an \(N_i\) such that \(H \cap N_i = 1\) (Prop. 1.3 (2)). Then \(A^{HN_i} \subseteq A^{N_i}\), and \(H \approx HN_i / N_i\) canonically. Therefore there is an element \(c\) in \(A^{N_i}\) such that \(t_H(c) = 1\) (cf. [22; §2. p. 118]), which implies \((A^h_{\langle A \rangle})|A_{\langle A \rangle}\), because the \(A^h\)-right homomorphism \(x \rightarrow t_H(cx)\) \((x \in A)\) from \(A\) to \(A^h\) splits. Therefore there is an element \(d\) in \(A\) such that \(h(d) = 1\). Then, for any \(x\) in \(A^h\), \(T \ni h(dx) = h(d) x = x\). Thus we obtain \(T = A^h\), as desired.

**Theorem 1.18.** Let \(A/B\) be finite outer \(G\)-Galois, and \(T\) an intermediate ring of \(A/B\). Then the following are equivalent.

(i) \(T = A^h\) for some subgroup \(H\) of \(G\).

(ii) \(A/T\) is a projective Frobenius extension, and \(G|T\) is strongly distinct.

(iii) \(T/B\) is a separable extension, and \(G|T\) is strongly distinct.

**Proof.** (i) \(\Leftrightarrow\) (ii) is evident from Th. 1.17. (i) \(\Rightarrow\) (iii) follows from [22; Th. 2.6] and [19; Prop. 3.4]. (iii) \(\Rightarrow\) (i) follows from [22; Th. 2.6 and Lemma 2.7].

**§ 2. Heredity of locally finite Galois extensions.**

Let \(A_0\) be a \(G^*\)-invariant subring of \(A\) such that the mapping \(\sigma \rightarrow \sigma|A_0\) \((\sigma \in G^*)\) is one-to-one and such that \(A_0/A_0^G\) is a locally finite \(G\)-Galois extension, and let \(G^*\) be compact (as an automorphism group of \(A\)). Put \(B_0 = A_0^G\), and let \(A_0 = \bigcup_{\lambda} A_{\lambda}\) be a representation of the locally finite \(G\)-Galois extension \(A_0/B_0\). Then \(G/N_i\) may be considered as a finite group of automorphisms of \(A^{N_i}\). And, by [22; Th. 5.1 and §2. p. 118], \(A^{N_i} = A_0^{N_i} \otimes_{B_0} B\), \(A^{N_i}/B\) is finite \(G/N_i\)-Galois. Since \(\bigcup_{\lambda} A_{\lambda}^{N_{\lambda}}\) is a directed union, the compactness of \(G^*\) implies that \(\bigcup_{\lambda} A_{\lambda}^{N_{\lambda}} (\subseteq A_0)\) is a fixed subring of \(A\) with respect to \(G^*\) (Prop. 1.1), so that \(A = \bigcup_{\lambda} A_{\lambda}^{N_{\lambda}}\), because \(\sigma \rightarrow \sigma|A_0\) \((\sigma \in G^*)\) is 1–1. Thus \(A/B\) is locally finite \(G\)-Galois. Let \(H\) be any subgroup of \(G\). Then, \(A^H = \bigcup_{\lambda} (A^H \cap A_{\lambda}^{N_{\lambda}}) = \bigcup_{\lambda} A_{\lambda}^{H_{\lambda}}\). By [22; Th. 5.1], \(A^{H_{\lambda}} = (A_{\lambda}^{N_{\lambda}})^{H_{\lambda}} \otimes_{B_0} B\). Hence \(A^H = \bigcup_{\lambda} (A_{\lambda}^{N_{\lambda}} \otimes_{B_0} B) = A_0^{N_{\lambda}} \otimes_{B_0} B\). By [22; Th. 5.1], \(A^{H_{\lambda}} = (A_{\lambda}^{N_{\lambda}})^{H_{\lambda}} \otimes_{B_0} B\). Hence \(A^H = \bigcup_{\lambda} (A_{\lambda}^{N_{\lambda}} \otimes_{B_0} B) = A_0^{N_{\lambda}} \otimes_{B_0} B\). Therefore, \(A^H = \bigcup_{\lambda} A_{\lambda}^{H_{\lambda}}\). Next we consider the set of all \(A_0\)-left submodules of \(A\) and the set of all \(B_0\)-left submodules of \(B\). Let \(X\) be any \(A_0\)-left submodule of \(A\). Then \(\overline{X} \cap A^{N}\) is an \(A_0^{N}\)(\(G/N_i\))-left submodule of \(A^{N}\). Therefore, by [22; Th. 5.1], we have \(\overline{X} \cap A^{N} = A_0^{N}((\overline{X} \cap A^{N}) \cap B) = A_0^{N} \otimes_{B_0} (\overline{X} \cap B)\), so that \(\overline{X} = \bigcup_{\lambda} (\overline{X} \cap A_{\lambda}^{N}) = \bigcup_{\lambda} (A_{\lambda}^{N} ((\overline{X} \cap B)) = A_0^{N} (\overline{X} \cap B)\). Since \(A_0^{N} \otimes_{B_0} (\overline{X} \cap B) \simeq A_0^{N} (\overline{X} \cap B)\), etc.
\[\cap B \subseteq \overline{X}\] for all \(\lambda\), we have \(\overline{X} = A_0 \otimes_{B_0} (\overline{X} \cap B)\). Evidently \(\overline{X} \cap B\) is a \(B_0\)-left submodule of \(B\). Let \(X\) be any \(B_0\)-left submodule of \(B\). Then, as is easily seen, \(A_0 X\) is an \(A_0\)-\(G\)-left submodule of \(A_1\), and \(A_0 X = \cup_1 A_0^{N_1} X\). By [22; Th. 5.1], \(A_0^{N_1} X \cap B = X\) for all \(\lambda\) in \(A_1\), so that \(A_0 X \cap B = \cup_1 (A_0^{N_1} X \cap B) = X\). If \(\overline{Y}\) is a \(G\)-invariant intermediate ring of \(A/A_0\), then \(\overline{Y} \cap B\) is an intermediate ring of \(B/B_0\), and \(\overline{Y} = A_0 (\overline{Y} \cap B)\). Symmetrically we have \(\overline{Y} = (\overline{Y} \cap B) A_0\). If \(Y\) is an intermediate ring of \(B/B_0\) such that \(A_0 Y = YA_0\), then \(A_0 Y\) is a \(G\)-invariant intermediate ring of \(A/A_0\). Since \(A = \cup_1 A_0^{N_1}\), we have \(\overline{Y} = \cup_1 (\overline{Y} \cap A_0^{N_1}) = \cup_1 \overline{Y}^{N_1}\), and \(\overline{Y}/(\overline{Y} \cap B)\) is finite \(G/N_{\alpha}\)-Galois ([22; Th. 5.1]. Hence \(\overline{Y}/(\overline{Y} \cap B)\) is locally finite \(G\)-Galois. Thus we have obtained the following.

**Theorem 2.1.** Let \(A_0\) be a \(G\)-invariant subring of \(A\) such that \(\sigma \mapsto \sigma | A_0\) (\(\sigma \in G\)) is 1-1 and such that \(A_0/B_0\) is locally finite \(G\)-Galois where \(B_0 = A_0^0\), and let \(G\) be compact. Then there hold the following:

1. \(A/B\) is locally finite \(G\)-Galois.
2. \(A^H = A \otimes_{B_0} A_0^H = A_0^H \otimes_{B_0} B\) for any subgroup \(H\) of \(G\). In particular, \(A = B \otimes_{B_0} A_0 = A_0 \otimes_{B_0} B\).
3. Let \(\overline{X}\) and \(\{X\}\) be the set of all \(A_0\)-\(G\)-left submodules of \(A\) and the set of all \(B_0\)-left submodules of \(B\), respectively. Then, \(\overline{X} \rightarrow \overline{X} \cap B\) and \(X \rightarrow A_0 X = A_0 \otimes_{B_0} X\) are mutually converse order isomorphisms between \(\overline{X}\) and \(X\).
4. Let \(\overline{Y}\) and \(\{Y\}\) be the set of all \(G\)-invariant intermediate rings of \(A/A_0\) and the set of all intermediate rings of \(B/B_0\) such that \(A_0 Y = YA_0\), respectively. Then \(\overline{Y}/(\overline{Y} \cap B)\) is locally finite \(G\)-Galois, and \(\overline{Y} \rightarrow \overline{Y} \cap B\) and \(Y \rightarrow A_0 Y = YA_0\) are mutually converse order isomorphisms between \(\overline{Y}\) and \(\{Y\}\).

Let \(A, A'\) be \(R\)-algebras such that \(A \otimes_R A' \neq 0\). Assume that \(A/B\) is a locally finite \(G\)-Galois extension such that \(R \cdot 1 \subseteq B\), and assume that \(A'\) is a locally finite \(G'\)-Galois extension such that \(R \cdot 1 \subseteq B'\). Then each \(\sigma \times \tau\) in \(G \times G'\) induces an automorphism of \(A \otimes_R A'\). Let \(A = \cup_a A_a^{N_a}\) and \(A' = \cup_{a \in B_0} A_a^{N_{a\beta}}\) be representations of \(A/B\) and \(A'/B'\), respectively. Then, by [22; Th. 5.2], \((A_0 \otimes_R A_0^{N_{a\beta}})/(B \otimes B')\) is a finite \((G/N_a) \times (G'/N_{a\beta})\)-Galois extension. Let \(\varphi_{a\beta}\) be the canonical \(R\)-algebra homomorphism from \(A_a^{N_a} \otimes_R A_a^{N_{a\beta}}\) to \(A_a \otimes R A_a^{N_{a\beta}}\) \((\subseteq A \otimes R A')\). We put \(A = \cup_1 A_a^{N_a} \otimes R A_a^{N_{a\beta}}\) and \(A' = \cup_1 A_a^{N_{a\beta}}\) \((\subseteq A \otimes R A')\). To be easily seen, \(\ker \varphi_{a\beta}\) is a \((G/N_a) \times (G'/N_{a\beta})\)-invariant ideal of \(A_0 \otimes_R A_0^{N_{a\beta}}\). Hence \(A_0/B_a^{*}\) is \((G/N_a) \times (G'/N_{a\beta})\)-Galois ([22; Th. 5.6]). There are elements \(c\) and \(c'\) in \(A_a^{N_a}\) and \(A_a^{N_{a\beta}}\) respectively such that \(t_{G/NA_a}(c) = 1\) and \(t_{G'/NA_{a\beta}}(c') = 1\). Then \(c \otimes c' \in A\) and \(t_{G \times G'}(c \otimes c') = 1 \otimes 1\). Hence \(A_b^{*} = \{G/N_a\} \times (G'/N_{a\beta})\)-Galois extension, and \(\{\sigma \times \tau \in G \times G' ; \sigma \times \tau | A_0 = 1\} = N_a \times N_{a\beta}\). Since \(G \times G'\) may be considered
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as a group of automorphisms of $A \otimes_{R} A'$. Let $H$ and $H'$ be subgroups of $G$ and $G'$, respectively. Then, $(A \otimes_{R} A')^{H \times H'} = \cup_{a, b} A_{a, b}^{H \times H'} = \cup_{a, b} (A^{N_{a}H} \otimes A^{N_{b}H'}) = (\cup_{a} A^{N_{a}H}) \otimes (\cup_{b} A^{N_{b}H'}) = A^{H} \otimes A^{H'}$ by [22; Th. 5.2]. In particular $(A \otimes_{R} A')^{N_{a} \times N_{b}'} = A^{N_{a} \otimes N_{b}'} = A_{a, b}$, and evidently $(G \times G'): N_{a} \times N_{b}' < \infty$. Since $A \otimes_{R} A' = \cup_{a, b} A^{N_{a} \otimes N_{b}'}$ is a directed union, $A \otimes_{R} A'/B \otimes B'$ is a locally finite $G \times G'$-Galois extension. Let $a \in A$ and $a' \in A'$. Then it is evident that $\{\sigma \times \tau \in G \times G'; \sigma(a) \otimes \tau(a') = a \otimes a'\} \supseteq \{\sigma \in G; \sigma(a) = a\} \times \{\tau \in G'; \tau(a') = a'\}$. Put $\{\sigma \in G; \sigma(a) = a\} = K$ and $\{\tau \in G'; \tau(a') = a'\} = K'$. Then $A^{K} \subseteq A^{N_{a}}$ and $A^{K'} \subseteq A^{N_{b}'}$ for some $\alpha, \beta$ (Prop. 1.3), so that $N_{a} \subseteq K$ and $N_{b}' \subseteq K'$. By [22; Th. 5.2], $(G/N_{a} \times G'/N_{b}')^{H \otimes K'} = K/N_{a} \times K'/N_{b}'$, and hence $(G \times G')^{H \otimes K'} = K \times K'$. Since $(A^{K})_{B}$ and $(A^{K'})_{B'}$ are finitely generated, $(A^{K} \otimes A^{K'})_{B \otimes B'}$ is finitely generated. Hence the finite topology of $G \times G'$ with respect to $A \otimes_{R} A'$ is the product topology of the finite topology of $G$ with respect to $A$ and the finite topology of $G'$ with respect to $A'$. Thus we have proved the following

**Theorem 2.2.** Let $A$ and $A'$ be $R$-algebras such that $A \otimes_{R} A' \neq 0$. If $A/B$ is a locally finite $G$-Galois extension such that $R \cdot 1 \subseteq B$, and $A'/B'$ is a locally finite $G'$-Galois extension such that $R \cdot 1 \subseteq B'$, then $(A \otimes_{R} A')/(B \otimes B')$ is a locally finite $G \times G'$-Galois extension, and $(A \otimes_{R} A')^{H \times H'} = A^{H} \otimes A^{H'}$ for any subgroup $H$ of $G$ and any subgroup $H'$ of $G'$. The finite topology of $G \times G'$ with respect to $A \otimes_{R} A'$ is the product topology of the finite topology of $G$ with respect to $A$ and the finite topology of $G'$ with respect to $A'$.

**Corollary.** Let $A/B$ be a locally finite $G$-Galois extension such that $B \subseteq C$, and $A'$ a $B$-algebra such that $A \otimes_{R} A' \neq 0$. Then $(A \otimes_{R} A')/(1 \otimes A')$ is a locally finite $G$-Galois extension, and $(A_{B} \otimes A')^{H} = A^{H} \otimes A'$ for any subgroup $H$ of $G$.

**Proposition 2.3.** Let $A/B$ be locally finite $G$-Galois, and $G = G^{*}$. If $H$ and $K$ are closed subgroups of $G$, then $A^{H \cap K} = A^{H} \cdot A^{K} = A^{K} \cdot A^{H}$. In particular, if $H \cap K = 1$ then $A = A^{H}. A^{K} = A^{K}. A^{H}$.

**Proof.** Let $A = \cup_{\lambda \in A} A^{N_{\lambda}}$ be a representation of the locally finite $G$-Galois extension $A/B$. First we assume that $(G : K) < \infty$. Then, by Prop. 1.3, $A^{K} \subseteq A^{N_{\mu}}$ for some $\mu \in A$. Since $(A^{N_{\mu}})_{B}$ is finitely generated, and $(A^{K})_{B}$ is a direct summand of $(A^{N_{\mu}})_{A^{K}}$ ([22; § 2. p. 118]), $(A^{K})_{B}$ is finitely generated. Therefore we may assume that $A^{K} \subseteq A^{N_{\lambda}}$ for all $\lambda \in A$. Then $N_{\lambda} \subseteq K$ for $\lambda \in A$, and $A^{H}. A^{K} = (\cup_{\lambda \in \lambda} A^{N_{\lambda}}) (\cup_{\mu \in \mu} A^{N_{\mu}}) = A_{\lambda} A^{N_{\lambda} \cap N_{\mu}} A^{N_{\mu}} = A \cup A^{N_{\lambda} \cap N_{\mu}}$ by [22; Prop. 5.3]. Since $N_{\lambda} H \cap K = N_{\lambda} (H \cap K)$ for all $\lambda$, we have $A^{H}. A^{K} = A_{\lambda} A^{N_{\lambda} \cap N_{\mu}} A^{N_{\mu}}$. Next we return to general case. For any finite subset $F$ of $A^{K}$, we put $K_{F} = \{\sigma \in G; \sigma | F = 1_{F}\}$. Then $(G : K_{F}) < \infty$, $A^{K_{F}} \subseteq A^{K}$, and $(A^{K_{F}})_{B}$ is finitely generated. Therefore $A^{K} = \cup_{F} A^{K_{F}}$ is a directed union, and
hence $A^\mu A^K = A^\mu (\cup_i A^\mu K_i) = \cup_i (A^\mu A^\mu K_i)$ is also a directed union. Since each $A^\mu A^\mu K_i$ is a fixed subring of $A$, $A^\mu A^K$ is a fixed subring of $A$ (Prop. 1.1). Hence, as is easily seen, $A^\mu A^K = A^\mu \cap K$. Symmetrically we have $A^\mu \cap K = A^K A^\mu$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, $G = G^*$, and $H\cap \Gamma_i \neq 0$ be closed subgroups of $G$. Then, $[\cup_i A^\mu r_i] = A^\cap r_i$, where $[\cup_i A^\mu r_i]$ means the subring of $A$ generated by $\cup_i A^\mu r_i$.

**Proof.** Evidently $[\cup_i A^\mu r_i] = \cup_i [A^{r_i} \cup \cdots \cup A^{r_n}]$, where $\{r_1, \cdots, r_n\}$ ranges over all finite subsets of $\Gamma$. By Prop. 2.3, $A^{r_i \cap \cdots \cap r_n} = A^{r_i} \cdots A^{r_n} = [A^{r_i} \cdots \cup A^{r_n}]$, and therefore $[\cup_i A^\mu r_i]$ is a directed union of fixed subrings of $A$. Hence, by Prop. 1.1, $[\cup_i A^\mu r_i]$ is a fixed subring. Since $\{\sigma \in G; \sigma [\cup_i A^\mu r_i] = 1\} = \cap_i H_i$, we obtain $[\cup_i A^\mu r_i] = A^\cap H_i$, as desired.

**Proposition 2.4.** Let $A/B$ be locally finite $G$-Galois, $\mathfrak{A}$ a $G$-invariant proper ideal of $A$, $K$ a closed subgroup of $G$, and $N$ a closed normal subgroup of $G$ such that $(G : N) < \infty$. Then there hold the following:

1. $A^{K \cap N}/A^K$ is finite $K/(K \cap N)$-Galois. In particular, $A^N/B$ is finite $G/N$-Galois.

2. $(\langle A^N + \mathfrak{A}\rangle/\mathfrak{A})/(B + \mathfrak{A})/\mathfrak{A})$ is finite $G/N$-Galois, and $((A^N + \mathfrak{A})/\mathfrak{A})^n = (A^{NH} + \mathfrak{U})/\mathfrak{A}$ for any subgroup $H$ of $G$.

**Proof.** Let $A = \cup_{\lambda \epsilon A} A^{N_{\lambda}}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) By Prop. 1.3, $A^N \subseteq A^{N_{\mu}}$ for some $\mu \in A$, and then $N_{\mu} \subseteq N$, $A^N = (A^{N_{\mu}})^{N/N_{\mu}}$. Therefore, by [22; Prop. 5.7], $A^N/B$ is finite $(G/N_{\mu})/(N/N_{\mu})$-Galois, or equivalently finite $G/N$-Galois. Accordingly, $A^N/A^K$ is finite $NK/N$-Galois, or equivalently, finite $K/(H \cap N)$-Galois. $K/(H \cap N)$ may be considered as a finite group of automorphisms of $A^{K \cap N}$, because $K \cap N \subseteq K$. Then $A^{K \cap N}/A^K$ is finite $K/(K \cap N)$-Galois. (2) By (1), $A^N/B$ is finite $G/N$-Galois. If $t_{\sigma/c} = 1$ for $c$ in $A^N$, then $t_{\sigma/c}(c + \mathfrak{A}) = 1 + \mathfrak{A}$. Then, by [22; Th. 5.6], $((A^N + \mathfrak{A})/\mathfrak{A})/(B + \mathfrak{A})/\mathfrak{A})$ is finite $G/N$-Galois, and $((A^N + \mathfrak{A})/\mathfrak{A})^n = (A^{NH} + \mathfrak{U})/\mathfrak{A}$ for any subgroup $H$ of $G$.

Let $A/B$ be locally finite $G$-Galois, $K$ a closed subgroup of $G$, $N$ a closed normal subgroup of $G$, and $\mathfrak{A}$ a $G$-invariant proper ideal of $A$. Let $A = \cup_{\lambda \epsilon A} A^{N_{\lambda}}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then $A^N = \cup_i (A^N \cap A^{N_i}) = \cup_i A^{N_{N_i}}$ is a directed union, and each $NN_i$ is a closed normal subgroup of $G$, because $(G : N_i) < \infty$. Then, by Prop. 2.4 (1), $A^{NN_i}/B$ is finite $G/NN_i$-Galois. Therefore there are elements $a_1, \cdots, a_m; b_1, \cdots, b_m$ in $A^{NN_i}$ such that $\Sigma_i a_i \sigma(b_i) = \delta_{NN_i, \sigma}$ for $\sigma$ in $G$. Hence $A^{NN_i}/B$ is finite $(G/N)/((NN_i)/N)$-Galois. Hence $A^N/B$ is locally finite $G/N$-Galois. Next we consider $K$. $A = \cup_i A^{N_i \cap K}$ is a directed union, and each $N_i \cap K$ is a fixed
normal subgroup of $K$ such that $(K : N_i \cap K) < \infty$. By Prop. 2.4 (1), each $A^{N_i \cap K}/A^K$ is finite $K/(N_i \cap K)$-Galois. Hence $A/A^K$ is locally finite $K$-Galois. Finally we consider $\mathfrak{U}$. Evidently, $A/\mathfrak{U} = \bigcup_i ((A^{N_i} + \mathfrak{U})/\mathfrak{U})$. By Prop. 2.4 (2), $((A^{N_i} + \mathfrak{U})/\mathfrak{U})/((B + \mathfrak{U})/\mathfrak{U})$ is finite $G/N_i$-Galois, and $((A^{N_i} + \mathfrak{U})/\mathfrak{U})^H = (A^{N_i \cap K} + \mathfrak{U})/\mathfrak{U}$ for any subgroup $H$ of $G$. Therefore $(A/\mathfrak{U})^H = \bigcup_i ((A^{N_i} + \mathfrak{U})/\mathfrak{U})^H = \bigcup_i (A^{N_i \cap K} + \mathfrak{U})/\mathfrak{U}$ for any subgroup $H$ of $G$. Hence $((A + \mathfrak{U})/\mathfrak{U})/((B + \mathfrak{U})/\mathfrak{U})$ is locally finite $G$-Galois. Thus we have proved the following

**Theorem 2.5.** Let $A/B$ be locally finite $G$-Galois, $N$ a closed normal subgroup of $G$, $K$ a closed subgroup of $G$, and $\mathfrak{U}$ a $G$-invariant proper ideal of $A$. Then there hold the following:

1. $A^N/B$ is locally finite $G/N$-Galois.
2. $A/A^K$ is locally finite $K$-Galois.
3. $((A + \mathfrak{U})/\mathfrak{U})/((B + \mathfrak{U})/\mathfrak{U})$ is locally finite $G$-Galois, and $((A + \mathfrak{U})/\mathfrak{U})^H = (A^H + \mathfrak{U})/\mathfrak{U}$ for any subgroup $H$ of $G$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, and $e$ a non-zero idempotent in $B \cap C$. Then $Ae/Be$ is locally finite $G$-Galois, and $(Ae)^H = A^He$ for any subgroup $H$ of $G$.

Let $A/B$ be locally finite $G$-Galois, $n$ a positive integer, and $J$ the ring of rational integers. Then, $(J)_n$ is a $J$-algebra, and $(J)_n \otimes_J A \simeq (A)_n \neq 0$. If we define $\sigma((a_{ik})) = (\sigma(a_{ik}))$ for any $\sigma$ in $G$ and any $(a_{ik})$ in $(A)_n$, then $(A)_n/(B)_n$ is locally finite $G$-Galois and $(A)_n^H = (A^H)_n$ for any subgroup $H$ of $G$ (Th. 2.2). Now, let $\{e_{ik}; i, k = 1, \ldots, m\}$ a system of matrix units contained in $B$, and $A = \bigcup_{i \in \lambda} A^{N_i}$ a representation of $A/B$. Put $A_0 = V_A(\{e_{ik}\})$ and $B_0 = B \cap A_0$. Then, as is well known, $A = \sum_{i, k} A_0 e_{ik}$, $A_0 \simeq A_0 e_{ik}$ by the right multiplication of $e_{ik}$. To be easily seen, $A^{N_i} = \sum_{i, k} A_0^{N_i} e_{ik}$, and $A_0^{N_i} = V_{A^{N_i}}(\{e_{ik}\})$. There is an element $c$ in $A^{N_i}$ such that $t_{\Theta, N_i}(c) = 1$. Let $c = \sum_{i, k} x_{ik} e_{ik}$, then $c \in A_0^{N_i}$. Then $1 = t_{\Theta, X_i}(c) = \sum_{i, k} t_{\Theta, X_i}(x_{ik}) e_{ik}$, and so $t_{\Theta, X_i}(x_{ik}) = 1$. Thus, by [22; Th. 5.8], $A_0^{N_i}/B_0$ is finite $G/N_i$-Galois. Since $A_0 = \bigcup_i A_0^{N_i}$ is a directed union, $A_0/B_0$ is locally finite $G$-Galois. Therefore, by Th. 2.1, $A = A_0 \otimes_{B_0} B$. Thus we have obtained the following

**Theorem 2.6.** Let $A/B$ be locally finite $G$-Galois.

1. For any positive integer $n$, $(A)_n/(B)_n$ is locally finite $G$-Galois, and $(A)_n^H = (A^H)_n$ for any subgroup $H$ of $G$.
2. If $\{e_{ik}; i, k = 1, \ldots, m\}$ is a system of matrix units contained in $B$, $A_0 = V_A(\{e_{ik}\})$, and $B_0 = B \cap A_0$, then $A_0/B_0$ is locally finite $G$-Galois, and $A = A_0 \otimes_{B_0} B$.

Let $A/B$ be finite $G$-Galois, and $M$ a $\Delta$-left module. For any subgroup $H$ of $G$, we put $M^H = \{m \in M; u \cdot m = m \text{ for all } \tau \in H\}$, which is an $A^H$. 

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submodule of $M$. Evidently $M^H \supseteq A^H \cdot M^\theta$, and the mapping $\varphi : A^H \otimes_B M^\theta \rightarrow M^\theta$ defined by $a \otimes m \rightarrow am$ ($a \in A$, $m \in M^\theta$) is an $A^H$-left homomorphism. By assumption there are elements $a_1, \ldots, a_n; a_1^*, \ldots, a_n^*$, $d$ in $A$ such that $\sum \iota \sigma (a_\iota^*) = \delta_{i,\iota}$ ($\sigma \in G$), $t_H(d) = 1$. Put $t_i = t_H(a_i)$. Then, $t_i \in A^H$ and $\sum \iota \sigma (a_\iota^*) = \delta_{i,\iota}$ for $\sigma$ in $G$. If $m$ is in $M^\theta$, then $A^H : M^\theta \ni t_i \sum \iota \sigma (a_\iota^* dm) = \sum \iota t_i \sum \iota \sigma (a_\iota^* d)m = t_H(d)m=m$. Hence $\varphi$ is an epimorphism. If $a \in A^H$ and $m_0 \in M^\theta$, then $\sum \iota \otimes \sum \iota \sigma (a_\iota^* dm_0) = \sum \iota \otimes \sum \iota \sigma (a_\iota^* da)m_0 = \sum \iota t_i \sum \iota \sigma (a_\iota^* da) \otimes m_0 = t_H(da) \otimes m_0 = a \otimes m_0$. From this fact, as easily seen, $\varphi$ is 1–1. Thus we have $M^H = A^H \otimes_B M^\theta$. Next we proceed to more general case.

Let $A/B$ be locally finite $G$-Galois, $A = \bigcup_{\iota} A^\iota$ its representation, and $M$ a $\Delta$-left module. Let $G = \sigma_1 N_1 \cup \cdots \cup \sigma_r N_r$ be the coset decomposition of $G$, and let $\Delta_i$ be the trivial crossed product of $A^\iota$ with $G/N_i$: $\Delta_i = \sum \iota \otimes A^\iota v_{\iota_\iota^*}$, $v_{\iota_\iota^*} = v_{\iota_\iota^*}^1$, $v_{\iota_\iota^*} \sigma = \sigma_\iota v_{\iota_\iota^*}$. For any $m$ in $M^\iota$, and any $\sum \iota \sigma (a_\iota^* v_{\iota_\iota^*})$ in $\Delta_i$, we define $\sum \iota \sigma (a_\iota^* v_{\iota_\iota^*}) (m) = \sum \iota \sigma (a_\iota^* m)$. Then, as is easily seen, $M^\iota$ is a $\Delta_i$-left module. Since $A^\iota$ is a $\Delta_i$-left module, we obtain that $M^\iota = A^\iota \otimes_B M^\theta$ and $M^\iota^H = A^\iota^H \otimes_B M^\theta$ for any subgroup $H$ of $G$. Since $A = \bigcup_i A^\iota$ is a directed union, so is $\bigcup_i M^\iota$. For any subgroup $H$ of $G$, $\bigcup_i M^\iota^H = \bigcup_i A^\iota^H . M^\theta = A^H : M^\theta$, and $A^\iota^H \otimes_B M^\theta \simeq A^\iota^H \cdot M^\theta$ canonically. The last isomorphism may be considered as $A^\iota^H \otimes_B M^\theta \rightarrow A^H \otimes_B M^\theta = A^H : M^\theta$, and hence we see that $\bigcup_i M^\iota^H = A^H \otimes_B M^\theta$. For any $m$ in $M$ we put $H_m = \{ \sigma \in G; u_m = m \}$, which is a subgroup of $G$. Assume that $(G : H_m) < \infty$ and that $H_m$ is closed in $G$. Then, by Prop. 1.3, $H_m \supseteq N_\nu$ for some $\nu \in A$, so that $m \in M^N$. Conversely, if $m$ is in $\bigcup_i M^i$, then $m \in M^\iota$, for some $N_\iota$, so that $H_m \supseteq N_\iota$. Then, since $(G : N_\iota) < \infty$ and $N_\iota$ is closed in $G$, $(G : H_m) < \infty$ and $H_m$ is closed in $G$. Thus we have proved the following.

**Theorem 2.7.** Let $A/B$ be locally finite $G$-Galois, and $M$ a $\Delta$-left module. Then there hold the following:

1. $A \cdot M^\theta$ is a $\Delta$-submodule of $M$, and $(A \cdot M^\theta)^H = A^H \otimes_B M^\theta$ for any subgroup $H$ of $G$.

2. $A \cdot M^\theta = \{ m \in M; (G : H_m) < \infty$ and $H_m$ is closed in $G \}$, where $H_m = \{ \sigma \in G; u_m = m \}$.

**Corollary.** Let $A/B$ be finite $G$-Galois, and $M$ a $\Delta$-left module. Then, $M^H = A^H \otimes_B M^\theta$ for any subgroup $H$ of $G$, in particular, $M = A \otimes_B M^\theta$ (cf. [4; Th. 1.3] and [22; Th. 5.1 (2)]).

**Proposition 2.8.** Let $A/B$ be finite $G$-Galois. Then the following are equivalent.

1. There are elements $a_1, \cdots, a_n; a_1^*, \cdots, a_n^*$ in $V_A(B)$ such that $\sum \iota \sigma (a_\iota^*) = \delta_{i,\iota}$ ($\sigma \in G$) (cf. [22; Cor. to Th. 5.1]).
(ii) \( bA_B |_B B_B \).

Proof. Since \((A \supseteq (\sum_{\sigma} u_{\sigma})A \simeq \text{Hom}(A_B, B_B)\) by \( j \), it follows that \((\sum_{\sigma} u_{\sigma})V_A(B) \simeq \text{Hom}(bA_B, B_B)\), and it is evident that \(V_A(B) \simeq \text{Hom}(bB_B, bA_B)\) canonically. To be easily seen, \( bA_B |_B B_B \) if and only if there are elements \(f_1, \ldots, f_n\) in \(\text{Hom}(bA_B, B_B)\) and \(g_1, \ldots, g_n\) in \(\text{Hom}(bB_B, bA_B)\) such that \(\sum_i g_if_i(x) = x\) for all \(x\) in \(A\). Consequently (ii) is equivalent to that \(u_i = \sum_i a_i \cdot \sigma(a_i^{*})u_{\sigma}\) for some \(a_1, \ldots, a_n; a_1^{*}, \ldots, a_n^{*}\) in \(V_A(B)\). Hence (i) and (ii) are equivalent.

Corollary. Let \(G\) be finite. Then the following are equivalent.

(i) \(A/B\) is outer \(G\)-Galois, and \(bA_B |_B B_B \).

(ii) There are elements \(a_1, \ldots, a_n; a_1^{*}, \ldots, a_n^{*}\) in \(C\) such that \(\sum_i a_i \cdot \sigma(a_i^{*}) = \delta_{1,\sigma}\) (\(\sigma \in G\)).

Proof. This follows from [22; Prop. 6.4 and Prop. 6.5] and Prop. 2.8. \(A/B\) is called a completely outer \(G\)-Galois extension if \(G\) is finite and completely outer (cf. [22]).

Theorem 2.9. Let \(B'\) be a ring with identity, \(Z\) its center, and \(G'\) a finite group.

(1) If \(A'/B'\) is completely outer \(G'\)-Galois and \((bA'_B |_B B'_B)\), then \(A' = B' \otimes_{z} C'\), where \(C'\) is the center of \(A'\), and \(C'/Z\) is \(G'\)-Galois.

(2) If \(C'/Z\) is \(G'\)-Galois and \(C'\) is commutative, then \(A' = B' \otimes_{z} C'\) is a completely outer \(G'\)-Galois extension over \(B'\), \((bA'_B |_B B'_B)\), and \(1 \otimes C'\) is the center of \(A'\).

Proof. (1) By [22; Prop. 6.4], \(A'/B'\) is outer \(G'\)-Galois and \(V_{A'}(B') = C'\), where \(C'\) is the center of \(A'\). Then, by Cor. to Prop. 2.8 and [22; Th. 5.1], \(C'/Z\) is \(G'\)-Galois and \(A' = B' \otimes_{z} C'\). (2) By [22; Th. 5.2 and Prop. 6.5], \(A'/B' \otimes 1\) is completely outer \(G'\)-Galois. Since \(zZ\) is a direct summand of \(zC'\), \(B' \simeq B' \otimes 1\) canonically, and \((bA'_B |_B B'_B)\), because \(zC | z\). Then, by Cor. to Prop. 2.8, \(C^*/Z\) is \(C'\)-Galois, where \(C^*\) is the center of \(A'\). Since \(C^* \supseteq 1 \otimes C' \supseteq Z\) and \((1 \otimes C')/Z\) is \(G'\)-Galois ([22; Th. 5.1 or Th. 5.6]), we have \(C^* = Z(1 \otimes C') = 1 \otimes C'\) ([22; Th. 5.1]).

Lemma 2.10. Let \(T\) be a ring, and \(U\) a subring of \(T\).

(1) Let \(T/U\) be a separable extension. If a \(T\)-left module \(M\) is \(U\)-projective, then \(M\) is \(T\)-projective.

(2) If \(\tau T \otimes_{\tau} T_T |_{\tau} T_T\) and \(U|_{\nu} M\) for a \(T\)-left module \(M\), then \(\tau T|_{\tau} M\).

(3) Let \(T_0\) be an intermediate ring of \(T/U\). If \(T\) is \((U, T_0)\)-projective and \(T_0\) is a \(T_0 \otimes T_0\)-direct summand of \(T\), then \(T_0/U\) is a separable extension.

Proof. (1) Since the mapping \(x \otimes y \rightarrow xy\) form \(T \otimes_{\nu} T\) to \(T\) splits as a \(T-T\)-homomorphism, the mapping \(x \otimes m \rightarrow xm\) from \(T \otimes_{\nu} M\) to \(M\) splits as
a $T$-left homomorphism. Since $_\tau M$ is projective, so is $\tau T \otimes \nu M$. Therefore $M$ is $T$-projective. (2) Since $_\nu U|_\nu M$, $\tau T|_\tau T \otimes \nu M$. Since $\tau T \otimes \nu T|_\tau T$, we have $\tau T \otimes \nu M|_\tau M$. Hence we have $\tau T|_\tau M$. (3) Let $\varphi$ be the canonical homomorphism from $T_0 \otimes \nu T$ to $T$ defined by $\varphi(t_0 \otimes t) = t_0t$, and let $\phi$ be a $T_0$-$T_0$-homomorphism from $T$ to $T_0 \otimes \nu T$ such that $\varphi \phi(x) = x$ for all $x$ in $T$. If $\phi(1) = \sum_i a_i \otimes b_i$ ($a_i \in T_0$, $b_i \in T$), then $\sum_i a_i b_i = 1$ and $\sum_i y a_i \otimes b_i = \sum_i a_i \otimes b_i y$ ($\in T_0 \otimes \nu T$) for all $y$ in $T_0$. Let $\pi$ be a $T_0$-$T_0$-homomorphism from $T$ to $T_0$ such that $\pi|T_0 = 1_{\tau T}$. Then, since $\sum_i y a_i \otimes b_i = \sum_i a_i \otimes b_i y$ ($\in T_0 \otimes \nu T_0$) for all $y$ in $T_0$, we have $\sum_i a_i \cdot \pi(b_i) = 1$ and $\sum_i y a_i \otimes \pi(b_i) = \sum_i a_i \otimes \pi(b_i) y$ ($\in T_0 \otimes \nu T_0$) for $y$ in $T_0$. Then the mapping $y \rightarrow \sum_i a_i \otimes \pi(b_i) y$ from $T_0$ to $T_0 \otimes \nu T_0$ is a $T_0$-$T_0$-homomorphism, and $\sum_i a_i \cdot \pi(b_i) y = y$. Hence $T_0/U$ is a separable extension.

**Proposition 2.11.** Let $A/B$ be finite $G$-Galois, and $Z$ the center of $B$. If $B$ is a separable $Z$-algebra and $Z \subseteq C$, then $V_A(B)/Z$ is finite $G$-Galois.

**Proof.** By [2; Prop. 1.5], $B \otimes \nu B'$ is a central separable $Z$-algebra, where $B'$ is the opposite ring of $B$. Since $\nu A$ and $\nu B$ are finitely generated and projective, so is $\nu A$. Then, by Lemma 2.10 (1), $\nu \otimes \nu A$ is finitely generated and projective. By [2; Th. 2.1], $\nu \otimes \nu B \otimes \nu B'$, and hence $\nu A_{\nu B}|_{\nu B}$. Then, by Prop. 2.8, $V_A(B)/Z$ is finite $G$-Galois (cf. S. 3).

**Theorem 2.12.** Let $G$ be finite, $\pi$ the group homomorphism defined by $\pi(a) = a|C$ ($a \in G$), $Z$ the center of $B$, and $Z = C^\sigma$, and assume that $A$ is indecomposable. Then the following statements are equivalent.

(i) $A/Z_0$ is separable, and $\pi$ is 1-1.
(ii) $V_A(B) = C$, $A/Z$ is separable, and $\nu A_B|_{\nu B}$.
(iii) $V_A(B) = C$, and both $B/Z$ and $C/Z$ are separable.
(iv) Both $B/Z$ and $C/Z_0$ are separable, and $\pi$ is 1-1.
(v) $V_A(B) = C$, $A/B$ is separable, $A$ is $(Z, B)$-projective, and $\nu A_B|_{\nu B}$.
(vi) $A = B_C$, and $A/Z$ is separable.
(vii) $A \otimes_{Z_0} A = A \otimes_{Z_0} A^\sigma$, $A \otimes_{Z_0} A^\sigma$, and $\text{Hom}(A_{\nu A}, A_{\nu A'}) = 0$ for any $\sigma$ in $G$ such that $\sigma \neq 1$.

**Proof.** (i) $\Rightarrow$ (ii) By [2; Th. 2.3], $A/C$ and $C/Z_0$ are separable. Therefore, by [4; Th. 1.3], $C/Z_0$ is $G$-Galois. Then, by [22; Th. 5.1], $A = B \otimes_{Z_0} C$. Hence $V_A(B) = C$, and $Z = Z_0$. Since $Z$ is finitely generated and projective, $\nu A_B|_{\nu B}$. (ii) $\Rightarrow$ (iii) $V_A(B) = C$ implies $Z = Z_0$ (cf. C). By [22; Lemma 2.7], $A/C$ and $A/B$ are separable, so that $A/B$ is outer $G$-Galois ([22; Th. 1.5]). Then, by Prop. 2.8, $C/Z$ is $G$-Galois, so that $C/Z$ is separable. Since $A/C$ is separable, $B/Z$ is separable ([22; Cor. to Th. 5.1]). (iii) $\Rightarrow$ (iv) In this case, $Z = Z_0$. By [2; Th. 3.1], $A = B \cdot C$, whence $\pi$ is 1-1. (iv) $\Rightarrow$ (v) By
[4; Th. 1.3], $C/Z_0$ is $G$-Galois. Hence, by [22; Th. 5.1], $A/B$ is $G$-Galois, and $A=B\cdot C$. Then $A/B$ is separable, $V_A(B)=C$, and $Z=Z_0$. Since $Z$ is commutative, $\pi Z$ is a direct summand of $\pi C$ (S. 3), so that $\pi\sigma(c)=1$ for some $c$ in $C$. Then $B$ is a $B$-$B$-direct summand of $A$ (cf. [22; § 2. p. 118]). Since $B/Z$ is separable, $A$ is $(Z, B)$-projective ([22; Lemma 2.7]). \( (v) \Rightarrow (vi) \) By Lemma 2.10 (3), $B/Z$ is separable. Then, by [2; Th. 3.1], $A=B\otimes_B C$. Since both $A/B$ and $B/Z$ are separable, $A/Z$ is separable ([22; Lemma 2.7]).

\( (vi) \Rightarrow (i) \) As $A=B\cdot C$, $V_A(B)=C$, $Z=Z_0$, and $\pi$ is 1-1. Thus we know that (i)$\sim$(vi) are equivalent. (i) \( \Rightarrow \) (vi) In this case, $V_A(B)=C$, $Z=Z_0$, and $B/Z$ is separable. Then, by [2; Th. 2.1], $B=B\otimes_B B^0$. Therefore $A\otimes_B\overline{B^0}B, and then $A\otimes_B\overline{B^0}$ $A\otimes_B A$. By [22; Prop. 1.3], $\mathcal{A}$ $A\otimes_B A$. Hence $A\otimes_B A$. The second assertion follows from [22; Prop. 6.3]. \( (vi) \Rightarrow (i) \) By assumption, $End((\otimes_B A)\simeq \oplus_{\sigma\epsilon G} End((\otimes_B A\otimes_B B) (\text{external direct sum as rings}). To be easily seen, $End((\otimes_B A)\simeq C$, which is commutative. Hence $End((\otimes_B A)\simeq C$.

\textbf{Remark.} The following are also equivalent to (i) \( \iff \) (iii).

(viii) $A/C$ is separable, and $C/Z_0$ is $G$-Galois (cf. Kanzaki [8]).

(ix) $A/B$ is outer $G$-Galois, and $B/Z$ is separable.

\textbf{Proposition 2.13.} Let $A/B$ be locally finite $G$-Galois, and $b$ an element of $B$ which is not a right zero divisor of $B$. Then $b$ is not a right zero divisor of $A$.

\textbf{Proof.} Let $a$ be an element of $A$ such that $ab=0$. Then $Aab=0$, and so $\sigma(Aa)b=0$ for all $\sigma$ in $G$. Hence, $((\sum\sigma(Aa))\cap B)b=0$. Then, by assumption, $(\sum\sigma(Aa))\cap B=0$. Then, by Th. 2.1 (3), $\sum\sigma(Aa)=A((\sum\sigma(Aa))\cap B) =0$. Hence $a=0$.

Let $A/B$ be locally finite $G$-Galois, and $S\ni 1$ a $G$-invariant multiplicative system of regular elements in $A$ such that a left quotient ring $\overline{A}$ of $A$ with respect to $S$ exists. Then $G$ may be regarded as a group of automorphisms of $\overline{A}$. To be easily seen, $\{\sigma(x); \sigma\in G\}$ is finite for any $x$ in $\overline{A}$. Then, by Th. 2.1, $\overline{A}/\overline{B}$ is locally finite $G$-Galois and $\overline{A}=\overline{B}\otimes_B A=A\otimes_B B$, where $\overline{B}=\overline{A}^0$. To be easily seen, any element in $B\cap S$ is a unit of $B$. For $b$ in $\overline{B}$, we put \( =0 \).
$\mathcal{U} = \{x \in A; \exists b \in A\}$, which is a $\mathcal{A}$-left submodule of $A$. Then $(\mathcal{U} \cap B)b \subseteq B$. If $\mathcal{U} \cap B \cap S \neq \emptyset$, then $sb \in B$ for some $s$ in $B \cap S$. Therefore, if we assume that $\mathcal{A}(s) \cap B \cap S \neq \emptyset$ for all $s \in S$, then $\overline{B}$ is a left quotient ring of $B$ with respect to $B \cap S$. Thus we obtain the following.

**Theorem 2.14.** Let $A/B$ be locally finite $G$-Galois, and $S \ni 1$ a $G$-invariant multiplicative system of regular elements of $A$ such that a left quotient ring $\overline{A}$ of $A$ with respect to $S$ exists. Further, assume that $\mathcal{A}(s) \cap B \cap S \neq \emptyset$ for all $s \in S$. Then there hold the following:

1. $\overline{A}/\overline{B}$ is locally finite $G$-Galois and $\overline{A} = \overline{B} \otimes_{B} A = A \otimes_{B} \overline{B}$, where $\overline{B} = \overline{A}^{\circ}$.

2. $\overline{A}$ is a left quotient ring of $A$ with respect to $B \cap S$. $\overline{B}$ is a left quotient ring of $B$ with respect to $B \cap S$.

**Remark.** Let $A/B$ be locally finite $G$-Galois, and $S$ a $G$-invariant multiplicative system of regular elements in $A$ such that $S \subseteq C$ and $S \ni 1$. Then $S$ satisfies the conditions in Th. 2.14. To see this, we put $H = \{s \in G; s(s) = s\}$ for $s$ in $S$. If $G = \sigma_{1}H \cup \cdots \cup \sigma_{r}H$, the left coset decomposition of $G$, then $\{1, \sigma_{1}(s) \in \mathcal{A}(s) \cap B \cap S\}$.

A non-zero ring $T$ with 1 is called a left Goldie ring if $T$ satisfies the following conditions: (1) $T$ is a semi-prime ring. (2) Any independent set of non-zero left ideals is finite (i.e., $T$ is finite dimensional). (3) $T$ satisfies the ascending chain condition for annihilator left ideals.

A left Goldie ring has a complete left quotient ring which is a semi-simple ring with minimum condition for left ideals, and conversely (Goldie [17]). (Cf. [7])

**Theorem 2.15.** Let $A/B$ be locally finite $G$-Galois, $A$ a left Goldie ring, $\overline{A}$ a complete left quotient ring of $A$, and $B$ a semi-prime ring. Then there hold the following:

1. $\overline{A}/\overline{B}$ is locally finite $G$-Galois, where $\overline{B} = \overline{A}^{\circ}$.

2. $B$ is a left Goldie ring, and $\overline{B}$ is a complete left quotient ring of $B$.

**Proof.** Let $S$ be the set of all regular elements of $A$. First we shall prove that $B$ is a left Goldie ring. Since $\mathcal{A}A$ is finite dimensional, $\mathcal{A}A$ is finite dimensional. Then, by Th. 2.1 (3), $\mathcal{A}B$ is finite dimensional. Let $I \subseteq I'$ be left ideals of $B$. Then $l_{A}(r_{B}(I)) \subseteq l_{A}(r_{B}(I'))$, where $r_{B}(I) = \{y \in B; 1y = 0\}$ and $l_{A}(r_{B}(I)) = \{x \in A; x \cdot r_{B}(I) = 0\}$. From this fact, $B$ satisfies the ascending chain condition for annihilator left ideals of $B$. Hence $B$ is a left Goldie ring. By Prop. 2.13, $S \cap B$ is the set of all regular elements of $B$. For any $s$ in $S$, $\mathcal{A}s$ is essential in $\mathcal{A}A$, so that $\mathcal{A}(s)$ is essential in $\mathcal{A}A$. Then, by Th. 2.1 (3), $s(\mathcal{A}(s) \cap B)$ is essential in $\mathcal{A}B$, so that $\mathcal{A}(s) \cap B$ contains a regular element.
of $B$ ([17; Th. (3.9)]). Hence $\Delta(s) \cap B \cap S \neq \emptyset$ for any $s$ in $S$. Thus the present theorem follows from Th. 2.14.

Remark. In the following cases, the condition that $B$ is semi-prime is superfluous.

(1) $G$ is finite and completely outer (cf. [22; p. 132]).
(2) $B$ is contained in the center of $A$.

Let $T$ be a ring. If $T$-left modules $M$ and $N$ have no non-zero isomorphic subquotients, we say that $\tau M$ and $\tau N$ are unrelated (cf. [22]).

Lemma 2.16. Let $T$ be a ring, and let $M$ and $N$ be $T$-left modules, and $W$ a $T$-submodule of $M$. If $\tau(M/W)$ and $\tau N$ are unrelated, and $\tau W$ and $\tau N$ are unrelated, then $\tau M$ and $\tau N$ are unrelated.

Proof. Assume that there are isomorphic subquotients $X/X_0$ and $Y/Y_0$ of $\tau M$ and $\tau N$, respectively. Then, as is easily seen, $X + W \nsubseteq X_0 + W$ or $X \cap W \supsetneq X_0 \cap W$. If $X + W \supsetneq X_0 + W$, then $Y/Y_0 \simeq X/X_0 \rightarrow (X + W)/(X_0 + W) \neq 0$, a contradiction. If $X \cap W \nsubseteq X_0 \cap W$, then $(X \cap W)/(X_0 \cap W) \simeq (X_0 + (X \cap W))/X_0 \subseteq X/X_0 \simeq Y/Y_0$, which is also a contradiction.

Proposition 2.17. Let $\sigma, \tau$ be in $G$, and assume that $\tau A A \sigma$ and $\tau A A \sigma$ are unrelated. Then, for any finite subset \{\(x_1, \cdots, x_r; y_1, \cdots, y_s\)\} of $A$, there are elements $a_k, b_k$ (\(k = 1, \cdots, t\)) in $A$ such that $\Sigma_k a_kx_i \cdot \sigma(b_k) = x_i$ and $\Sigma_k a_k y_h \cdot \tau(b_k) = 0$ for all $x_i, y_h$.

Proof. By Lemma 2.16, $\tau(A A \sigma)$ and $\tau(A A \sigma)$ are unrelated. Then, since $A(x_1 u_\sigma, \cdots, x_r u_\sigma, y_1 u_\sigma, \cdots, y_s u_\sigma)A$ is an $A$-$A$-submodule of $A(x_1 u_\sigma) \oplus (A u_\sigma)_A$, $\{x_1 u_\sigma, \cdots, x_r u_\sigma, 0, \cdots, 0\} \in A(x_1 u_\sigma, \cdots, x_r u_\sigma, y_1 u_\sigma, \cdots, y_s u_\sigma)A$ (cf. [22; Prop. 6.1]). Therefore there are elements $a_k, b_k$ (\(k = 1, \cdots, t\)) in $A$ such that $\Sigma_k a_k(x_1 u_\sigma, \cdots, x_r u_\sigma, y_1 u_\sigma, \cdots, y_s u_\sigma) b_k = (x_1 u_\sigma, \cdots, x_r u_\sigma, 0, \cdots, 0)$. Then, $\Sigma_k a_k x_i \cdot \sigma(b_k) = x_i$ and $\Sigma_k a_k y_h \cdot \tau(b_k) = 0$ for all $x_i, y_h$.

Combining Prop. 2.17 with [22; Prop. 6.11] we can easily see the following.

Proposition 2.18. Let $A$ and $A'$ be $R$-algebras with $A \otimes_R A' \neq 0$, and let $G$ and $G'$ be completely outer finite groups of $R$-automorphisms of $A$ and $A'$, respectively. Then, $G \times G'$ is completely outer as an automorphism group of $A \otimes_R A'$.

§ 3.

Proposition 3.1. Let $A/B$ be locally finite $G$-Galois, and $X$ a $\Delta$-left submodule of $A$. Then $X = A(X \cap B)$.

Proof. This follows from Th. 2.1 (3).

Proposition 3.2. Let $A/B$ be locally finite $G$-Galois, $\{B\}$ the set of
all maximal ideals of $A$, and $\{p\}$ the set of all maximal ideals of $B$. Then the following are equivalent:

(i) $\mathfrak{P} \rightarrow B \cap B$ is a mapping from $\{\mathfrak{P}\}$ onto $\{p\}$.

(ii) $ApA \neq A$ for all $p \in \{p\}$, and $\cap_{\sigma \in \sigma(\mathfrak{P})}$ is $A$-$A$-maximal for all $\mathfrak{P} \in \{\mathfrak{P}\}$.

If (i) holds, then the following are true:

1. $pA = Ap \neq A$ for any $p \in \{p\}$.

2. $\{\cap_{\sigma}(\mathfrak{P}); \mathfrak{P} \in \{\mathfrak{P}\}\}$ is the set of all maximal $A$-$A$-submodules of $A$.

3. $\mathfrak{M}(4A_\mathfrak{A}) = \mathfrak{M}(4A_\mathfrak{B}) = \mathfrak{M}(bB_\mathfrak{B})A = A \cdot \mathfrak{M}(bB_\mathfrak{B})$, and $\mathfrak{M}(4A_\mathfrak{A}) \cap B = \mathfrak{M}(bB_\mathfrak{B})$.

4. $B$ is $B$-$B$-completely reducible if and only if $\cap_{i} \cap_{\sigma}(\mathfrak{P}_{i}) = 0$ for some $\mathfrak{P}_{i}$ $(i=1, \cdots, n)$ in $\{\mathfrak{P}\}$.

Proof. (i) $\Rightarrow$ (ii) If $\mathfrak{P}$ is in $\{\mathfrak{P}\}$, then $\mathfrak{P} \cap B = \sigma(\mathfrak{P}) \cap B$ for any $\sigma$ in $G$, and so $\mathfrak{P} \cap B = (\cap_{\sigma}(\mathfrak{P})) \cap B$. By Prop. 3.1, $A((\cap_{\sigma}(\mathfrak{P})) \cap B) = \cap_{\sigma}(\mathfrak{P}) = (\cap_{\sigma}(\mathfrak{P})) \cap B$. Hence $Ap = pA \neq A$ for all $p$ in $\{p\}$. Let $X$ be a $A$-$A$-submodule of $A$ with $A \supseteq X \supseteq \cap_{\sigma}(\mathfrak{P})$. Then $B \supseteq X \cap B \supseteq (\cap_{\sigma}(\mathfrak{P})) \cap B = \mathfrak{P} \cap B$, and so $X \cap B = (\cap_{\sigma}(\mathfrak{P})) \cap B$. Then, by Prop. 3.1, $X = \cap_{\sigma}(\mathfrak{P})$. Thus $\cap_{\sigma}(\mathfrak{P})$ is $A$-$A$-maximal. Let $Y$ be a maximal $A$-$A$-submodule of $A$. Take a maximal ideal $\mathfrak{P}_{1}$ of $A$ with $\mathfrak{P}_{1} \supseteq Y$. Then $\cap_{\sigma}(\mathfrak{P}_{1}) \supseteq Y$, and so $\cap_{\sigma}(\mathfrak{P}_{1}) = Y$. Thus we obtain (2). Therefore $\mathfrak{M}(4A_\mathfrak{A}) = \mathfrak{M}(4A_\mathfrak{B})$. Since $\mathfrak{M}(4A_\mathfrak{A}) \cap B = \mathfrak{M}(bB_\mathfrak{B})$, we have $\mathfrak{M}(4A_\mathfrak{A}) = A \cdot \mathfrak{M}(bB_\mathfrak{B}) = \mathfrak{M}(bB_\mathfrak{B})A$ (Prop. 3.1). $B$ is $B$-$B$-completely reducible if and only if $\cap_{i} p_{i} = 0$ for some $p_{1}, \cdots, p_{n}$ in $\{p\}$. Thus we obtain (4) (cf. Prop. 3.1). (ii) $\Rightarrow$ (i). Let $p \in \{p\}$. Then, as $ApA \neq A$, $p \subseteq \mathfrak{P}$ for some $\mathfrak{P} \in \{\mathfrak{P}\}$, and so $p = \mathfrak{P} \cap B$ by the maximality of $p$. Let $\mathfrak{Q}$ be in $\{\mathfrak{P}\}$. Then $q \supseteq \mathfrak{Q} \cap B$ for some $q \in \{p\}$. There is a $\mathfrak{Q} \in \{\mathfrak{P}\}$ with $\mathfrak{Q} \cap B = q$. Then $(\cap_{i} \sigma(\mathfrak{Q}) \cap B = \mathfrak{Q} \cap B \supseteq \cap_{i} \sigma(\mathfrak{Q}) \cap B = (\cap_{\sigma}(\mathfrak{P})) \cap B$, and therefore $\cap_{\sigma}(\mathfrak{Q}) \supseteq \cap_{\sigma}(\mathfrak{Q})$ by Prop. 3.1. By assumption, $\cap_{\sigma}(\mathfrak{Q}) = \cap_{\sigma}(\mathfrak{Q})$. Hence $q = \mathfrak{Q} \cap B = \mathfrak{Q} \cap B$. This completes the proof.

Concerning Prop. 3.2, we state the following

**Lemma 3.3.** Let $\mathfrak{P}$ be a maximal ideal of $A$ such that $\cap_{\sigma} \sigma(\mathfrak{P}) = \cap_{i} \sigma(\mathfrak{P})$ for some $\sigma_{1}, \cdots, \sigma_{n}$ in $G$. Then $\cap_{\sigma}(\mathfrak{P})$ is $A$-$A$-maximal, and $\{\sigma_{i}(\mathfrak{P}); i=1, \cdots, n\}$ is the set of all maximal ideals containing $\cap_{\sigma}(\mathfrak{P})$.

**Proof.** Let $\mathfrak{Q}$ be a maximal ideal of $A$ with $\mathfrak{Q} \supseteq \cap_{\sigma}(\mathfrak{P})$. If $\mathfrak{Q} \neq \sigma_{i}(\mathfrak{P})$ for all $i$, then $\mathfrak{Q} + \sigma_{i}(\mathfrak{P}) = A$ for all $i$. Then we have a contradiction $A = \mathfrak{Q} + \cap_{\sigma} \sigma_{i}(\mathfrak{P}) = \mathfrak{Q} + \cap_{\sigma}(\mathfrak{P})$.

**Remark.** In the following cases, the assumption in Lemma 3.3 holds. (1) $G$ is finite. (2) The ring $A/\mathfrak{M}(A_{\mathfrak{A}})$ satisfies the descending chain condition for ideals. (3) $G^{*}$ is compact, and every maximal ideal of $A$ is $A$-$A$-finitely generated. (Cf. Prop. 1.1).
Proposition 3.4.
(1) Let \( A/B \) be locally finite outer \( G \)-Galois, and \( B \) \( B-B \)-completely reducible. Assume that, for any maximal ideal \( \mathfrak{P} \) of \( A \), there are elements \( \sigma_1, \ldots, \sigma_n \) in \( G \) such that \( \cap_i \sigma_i(\mathfrak{P}) = \cap_i \sigma(\mathfrak{P}) \). Then \( A \) is \( A-A \)-completely reducible.

(2) Let \( G \) be finite and completely outer, and \( B_B \mid A_B \). Then \( A \) is \( A-A \)-completely reducible if and only if \( B \) is \( B-B \)-completely reducible. If there is a maximal ideal \( \mathfrak{P} \) of \( A \) such that \( \cap_i \sigma(\mathfrak{P}) = 0 \), then \( B \) is \( B-B \)-minimal, and conversely.

Proof. (1) Any maximal ideal \( \mathfrak{p} \) of \( B \) is written as \( \mathfrak{p} = Be \) with a central idempotent \( e \) of \( B \). Then, by assumption, \( (1 \neq) e \in V_{A}(B) = C \). Therefore, \( A\mathfrak{p} = Ae = eA = \mathfrak{p} \neq \mathfrak{a} \). Thus, by Prop. 3.2 and Lemma 3.3, \( A \) is \( A-A \)-completely reducible. (2) In this case, \( \alpha A = A \alpha \neq A \) for any proper ideal \( \alpha \) of \( B \) (cf. [22; p. 132]). Then, by Prop. 3.2 and Lemma 3.3, the first assertion is evident (cf. [22; Prop. 6.4]). For any \( \mathfrak{P} \) in \( \{ \mathfrak{P} \} \), \( \mathfrak{P} \cap B = 0 \) if and only if \( \cap_i \sigma(\mathfrak{P}) = 0 \) (Prop. 3.1). Thus we know the second assertion.

Theorem 3.5. Let \( A/B \) be finite \( G \)-Galois, \( B \) a semi-primary ring, and \( A \alpha A \neq A \) for any maximal ideal \( \alpha \) of \( B \). Then \( _{BG}A \simeq _{BG}B \), that is, \( A \) has a normal basis. (Cf. [13; Th. 1]).

Proof. By [22; Th. 1.7], it suffices to prove that \( _{B}A \) is free. Let \( g = (G:1) \). (1) First we assume that \( \mathfrak{R}(B) = 0 \). Then \( B \) is a direct sum of simple rings: \( B = \alpha_1 + \cdots + \alpha_n \). Let \( 1 = \sum_i e_i \), \( e_i \in \alpha_i \). Then \( \alpha_i = Be_i \) and \( e_i^2 = e_i \). By assumption we have \( (1-e_i)A = A(1-e_i) \) (Prop. 3.2 and Lemma 3.3), so that \( e_i \) is a central idempotent of \( A \) contained in \( B \). Then each \( A\alpha_i/Be_i \) is \( G \)-Galois ([22; Cor. to Th. 5.6]). Since \( Be_i \) is a simple ring, \( _{Be_i}A \) is free (cf. [7]). Hence \( A\alpha_i \) has a normal basis, so that \( _{Be_i}A \simeq _{Be_i}(Be_i)^{\alpha_i} \) for all \( i \) ([22; Th. 1.7]). Hence \( _{B}A \simeq _{B}B \). (2) Next we proceed to general case. Since \( A \) and \( B \) are semi-primary ([22; Prop. 7.3]), \( \mathfrak{R}(A) = \mathfrak{R}(A)B = \mathfrak{R}(A) \cap B = \mathfrak{R}(B) \). Then, by Prop. 3.2 and Lemma 3.3, \( \mathfrak{R}(A) = \mathfrak{R}(B)A = A \cdot \mathfrak{R}(B) \) and \( \mathfrak{R}(A) \cap B = \mathfrak{R}(B) \). By [22; Th. 5.6], \( (A/\mathfrak{R}(A))/((B+\mathfrak{R}(A))/\mathfrak{R}(A)) \) is \( G \)-Galois, and satisfies the same conditions as \( A/B \), because \( (B+\mathfrak{R}(A))/\mathfrak{R}(A) \simeq B/(\mathfrak{R}(A) \cap B) = B/\mathfrak{R}(B) \) canonically. By (1), we have \( _{B}A/\mathfrak{R}(A) \simeq _{B}B/\mathfrak{R}(B) \) and \( _{B}A \) is finitely generated and projective, we have \( _{B}A \simeq _{B}B \) by the uniqueness of projective cover. This completes the proof.

Corollary. Let \( A/B \) be finite \( G \)-Galois, \( B \) a semi-primary ring, and \( Z \) the center of \( B \). Assume that \( Z \subseteq C \) and that \( B \) is a central separable \( Z \)-algebra. Then \( A \) has a normal basis.

Proof. In this case, any proper ideal of \( B \) is written as \( \alpha B \) with an ideal
α of $Z$ (cf. [2]). Then, as $Z \subseteq C$, $(aB)A = aA = Aα = A(Bα) \neq A$ ([22; Lemma 7.1]).

Let $A/B$ be finite $G$-Galois, $B \subseteq C$, and $g = (G : 1)$. For any prime ideal $\mathfrak{p}$ of $B$, we denote by $B_\mathfrak{p}$ the quotient extension of $B$ with respect to $\mathfrak{p}$. Then $B_\mathfrak{p}$ is a $B$-algebra, canonically. By [22; Cor. to Th. 5.2], $(B_\mathfrak{p} \otimes_B A)/B_\mathfrak{p}$ is $G$-Galois. Since $B_\mathfrak{p}$ is a local ring, $B_\mathfrak{p} B_\mathfrak{p} \otimes_B A \simeq_{B_\mathfrak{p}} B_\mathfrak{p}^\mathfrak{p}$ (Cor. to Th. 3.5). We denote by $K_\mathfrak{p}$ the quotient field of $B/\mathfrak{p}$. Then we have $K_\mathfrak{p} K_\mathfrak{p} \otimes_B A \simeq_{K_\mathfrak{p}} (K_\mathfrak{p})^\mathfrak{p}$ similarly. Thus we obtain the following

**Proposition 3.6.** Let $A/B$ be finite $G$-Galois, $B \subseteq C$, and $g = (G : 1)$. Then, $B_\mathfrak{p} B_\mathfrak{p} \otimes_B A \simeq_{B_\mathfrak{p}} (B_\mathfrak{p})^\mathfrak{p}$ and $K_\mathfrak{p} K_\mathfrak{p} \otimes_B A \simeq_{K_\mathfrak{p}} (K_\mathfrak{p})^\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of $B$, where $B_\mathfrak{p}$ is the quotient extension of $B$ with respect to $\mathfrak{p}$ and $K_\mathfrak{p}$ is the quotient field of $B/\mathfrak{p}$.

The following lemma is of some interest.

**Lemma 3.7.** Let $R \supseteq S$ be rings, $R_S$ is finitely generated and projective, and $sS$ is a direct summand of $sR$. If $sR$ is injective, then $sS$ is injective.

**Proof.** Let $I$ be any left ideal of $S$, and $f$ any $S$-left homomorphism from $I$ to $sR$. Since $R_S$ is finitely generated and projective, we have $RI = R \otimes_S I$. Therefore $f$ can be extended to an $R$-left homomorphism from $RI$ to $R$, canonically. Then, by assumption, there is an element $a$ in $R$ such that $r \cdot (s)f = rsa$ for $r$ in $R$ and $s$ in $I$, so that $(s)f = sa$ for all $s$ in $I$. Therefore, as is well known, $sR$ is injective. Since $sS$ is a direct summand of $sR$, $sS$ is injective.

**Lemma 3.8.** $\Re(A) \cap B \subseteq \Re(B)$.

**Proof.** Let $b$ be in $\Re(R) \cap B$. Then $1 - b$ has an inverse in $A$. Since $B = A^\#$, $1 - b$ has an inverse in $B$. Hence $\Re(A) \cap B$ is a quasi-regular ideal of $B$, that is, $\Re(A) \cap B \subseteq \Re(B)$.

**Proposition 3.9.** Let $G$ be finite. If there is an element $c$ in $A$ such that $1 - t_\alpha(c) \in \Re(A)$, then there is an element $d$ in $A$ such that $t_\alpha(d) = 1$.

**Proof.** By Lemma 3.8, we have $1 - t_\alpha(c) \in \Re(A) \cap B \subseteq \Re(B)$, so that $t_\alpha(A) + \Re(B) = B$. Since $t_\alpha(A)$ is an ideal of $B$, we have $t_\alpha(A) = B$. Hence $t_\alpha(d) = 1$ for some $d$ in $A$.

**Theorem 3.10.** Let $A/B$ be $G$-Galois, $A$ a commutative ring, $H$ a subgroup of $G$, and $A'$ a $B$-algebra. Then, $A' \otimes_B A^\#$ is a direct sum of minimal ideals if and only if $A'$ is a direct sum of minimal ideals (cf. [7; p. 178. Th. 2]).

**Proof.** In this case, $(A' \otimes_B A)/A'$ is finite $G$-Galois, $G$ is completely outer as an automorphism group of $A' \otimes_B A$, and $(A' \otimes_B A)^\# = A' \otimes_B A^\#$ (cf. [22; Th.
Thus the present theorem is an easy consequence from Prop. 3.4 (2).

Concerning [22; Th. 6.9], we note the following

**Lemma 3.11.** Let $A/C$ be separable, and $e$ an idempotent of $A$ such that $eA \subseteq Ae$. Then $e$ is a central idempotent of $A$.

**Proof.** Since $A/\Re(A)$ is a semi-prime ring, we have $(eA + \Re(A))/\Re(A)$

$= (eA + \Re(A))/\Re(A)$, that is, $eA + \Re(A) = Ae + \Re(A)$, and so $Ae = eA + (Ae \cap \Re(A))= 2A + \Re(A)e$. Since $A$ is a central separable C-algebra, $\Re(aA_a) = \Re(C)A$

by [2; Cor. 3.2]. Since $\Re(aA) \supseteq \Re(A) \supseteq \Re(C)A$, we have $\Re(A) = \Re(C)A$, and $Ae = eA + \Re(C)Ae$. Hence $Ae = eA$, because $cAe$ is finitely generated. Consequently, $e$ is a central idempotent of $A$.

**Proposition 3.12.** Let $A/B$ be locally finite G-Galois, and assume that there is a representation $A = \bigcup_{x \in A} A^{x}$ of $A/B$ such that each $\Re(B)A^{x}$ is an ideal of $A^{x}$. Then $\Re(A) = \Re(B)A = A \cdot \Re(B)$, and $\Re(A) \cap B = \Re(B)$.

**Proof.** Let $\mathfrak{F}$ be a right ideal of $A$ such that $\Re(B)A + \mathfrak{F} = A$. Then $\Re(B)A^{x} + (\mathfrak{F} \cap A^{x}) \ni 1$ for some $\lambda$ in $A$, so that $\Re(B)A^{\lambda} + (\mathfrak{F} \cap A^{\lambda}) = A^{\lambda}$. Since $\Re(B)A^{\lambda} \subseteq \Re(A^{\lambda})$, we have $\mathfrak{F} \cap A^{\lambda} = A^{\lambda}$, and hence $\mathfrak{F} = A$. Thus we know that $\Re(B)A \subseteq \Re(A)$. Combining this with Lemma 3.8, we have $\Re(A) \cap B = \Re(B)$. Hence $\Re(A) = \Re(B)A = A \cdot \Re(B)$ (Prop. 3.1).}

**Theorem 3.13.** Let $A/B$ be locally finite G-Galois, $B \subseteq C$, and $A'$ a $B$-algebra such that $A' \cong A' \otimes_{B} 1$ (\subseteq A' \otimes_{B} A) canonically.

1. $\Re(A' \otimes_{B} A) = \Re(A' \otimes A)$, and $\Re(A' \otimes A) \cap (A' \otimes 1) = \Re(A') \otimes 1$.

2. If $A$ is commutative, then $\Re(A' \otimes A') = \Re(A') \otimes A'$ for any subgroup $H$ of $G$.

**Proof.** Let $A = \bigcup_{x \in A} A^{x}$ be a representation of the locally finite G-Galois extension $A/B$. Then $(A' \otimes_{B} A)/(A' \otimes 1)$ is a locally finite G-Galois extension with representation $A' \otimes_{B} A = \bigcup_{\lambda} A' \otimes A^{\lambda}$, where $A' \otimes A^{\lambda} = (A' \otimes_{B} A)^{\lambda}$ is a finite $G/N_{\lambda}$-Galois extension over $A' \otimes 1$. (1) This will be easily seen by Prop. 3.12. (2) We may assume that $H$ is closed in $G$. Then each $A' \otimes A'/A'$ is finite $H/(H \cap N_{\lambda})$-Galois, and $H/(H \cap N_{\lambda})$ is completely outer as an automorphism group of $A' \otimes A'$ ([22; Th. 6.6]). Then $H/(H \cap N_{\lambda})$ is completely outer as an automorphism group of $A' \otimes A'/A'$ (Prop. 2.18), and so $H/(H \cap N_{\lambda})$ is completely outer as an automorphism group of $A' \otimes A'$ (Prop. 6.11). Now, $(A' \otimes_{B} A)/(A' \otimes A')$ is a locally finite $H$-Galois extension with representation $A' \otimes_{B} A = \bigcup_{\lambda} A' \otimes A' \otimes N_{\lambda}$, where $A' \otimes A' \otimes N_{\lambda} = (A' \otimes_{B} A)^{H \cap N_{\lambda}}$ is a finite $H/(H \cap N_{\lambda})$-Galois extension over $A' \otimes A'$. Then, by [22; Th. 7.10] and Prop. 3.12, $\Re(A' \otimes_{B} A) = \Re(A' \otimes A') (A' \otimes_{B} A)$. On the other hand,
$\Re(A' \otimes_B A) = \Re(A') \otimes A = (\Re(A') \otimes A^H)(A' \otimes_B A)$. Hence $\Re(A' \otimes A^B) = \Re(A') \otimes A^H$, as desired (cf. [22; Lemma 7.1]).

References

([1]~[14] are found in [22] below.)


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