LOCAL FINITE OUTER GALOIS THEORY

By

Yôichi MIYASHITA

Introduction.

This paper is the continuation of the preceding paper [22]. In §1 and §2, locally finite (outer) Galois extensions are treated. The main results are parallel to those of the finite case. In these studies, Nagahara [12] is our guide. Further several results for finite Galois extensions are added (Th. 1.18). In §3, we give a normal basis theorem for a finite Galois extension.

§1. As to notations and terminologies we follow [22]. Let $A$ be a ring with $1$ ($\neq 0$), $C$ the center of $A$, $G$ a (finite or infinite) group of automorphisms of $A$, $B=A^g=\{x\in A; \sigma(x)=x \text{ for all } \sigma \text{ in } G\}$, and $\hat{G}$ the group of all $B$-automorphisms of $A$. $\hat{G}$ is then a topological group in finite topology (cf. Jacobson [7]). We denote the closure of $G$ in $\hat{G}$ by $G^*$. $A$ means the trivial crossed product of $A$ with $G$: $A=\sum_{\sigma \in G} A \sigma$, $u_{\sigma}u_{\tau}=u_{\sigma \tau}$ ($\sigma, \tau \in G$), $u_{\sigma}x=\sigma(x)u_{\sigma}$ ($x\in A$). Then there is a canonical ring homomorphism $j$ from $A$ to End ($A_{g}$) defined by $j(\sum_{\sigma} x_{\sigma}u_{\sigma})(y)=\sum_{\sigma} x_{\sigma} \sigma(y)$ ($\sum_{\sigma} x_{\sigma}u_{\sigma} \in A$, $y \in A$). For any intermediate ring $T$ of $A/B$, $G^r=\{\sigma \in G; \sigma|T=1_{T}\}$ is a subgroup of $G$, where $\sigma|T$ means the restriction of $\sigma$ to $T$. We call it a fixed subgroup of $G$. For any subgroup $H$ of $G$, $A^{H}=\{x\in A; \sigma(x)=x \text{ for all } \sigma \in H\}$ is an intermediate ring of $A/B$. We call it a fixed subring of $A$ (with respect to $G$). Then, as is well known, the set of all fixed subgroups of $G$ and the set of all fixed subrings of $A$ are anti-order-isomorphic in the usual sense of Galois theory.

A subring $T$ of $A$ is called a $G$-invariant subring of $A$ if $\sigma(T)=T$ for all $\sigma$ in $G$ (or equivalently, $\sigma(T)\subseteq T$ for all $\sigma$ in $G$). Let $N$ be a fixed subgroup of $G$. Then, $A^{N}$ is $G$-invariant if and only if $N$ is a normal subgroup of $G$: $N \triangleleft G$. Let $T$ be an intermediate ring of $A/B$, and put $H=G^r$. Then, for $\sigma, \tau$ in $G$, $\sigma|T=\tau|T$ if and only if $\sigma H=\tau H$. Let $H$ and $K$ be subgroups of $G$ such that $H \subseteq K$ and $(H:K)<\infty$, and let $H=\sigma_{1}K \cup \cdots \cup \sigma_{r}K$ be the left coset decomposition. For any $x$ in $A^{K}$ we put $t_{H,K}(x)=\sum_{i} \sigma_{i}(x)$. Then $t_{H,K}$ is an $A^{K}-A^{H}$-homomorphism from $A^{K}$ to $A^{H}$, and is independent of the choice of $\sigma_{1},\cdots,\sigma_{r}$. If $K=1$, we write simply $t_{H}$ instead of $t_{H,1}$.

Here we present several fundamental facts, which are essential throughout the present study. Let $\tau M_{\nu}$ and $\tau N_{\nu}$ be $T$-left, $U$-right modules. If $\tau M_{\nu}$ is
isomorphic to a direct summand of \( rN_{V} \) for some natural number \( r \), then we write \( _{T}M_{U}\mid_{T}N_{U} \), where \( rN_{V} \) means the direct sum of \( r \) copies of \( N_{U} \). If \( _{T}M_{U}\mid_{T}N_{U} \) and \( _{T}N_{V}\mid_{T}M_{U} \) we write \( _{T}M_{U}\sim_{T}N_{V} \) (similar) (cf. Morita [21]). To be easily seen, \( _{T}M_{U}\mid_{T}N_{U} \) if and only if there are \( T-U \)-homomorphisms \( f_{i}, \cdots, f_{r} \) in \( \text{Hom}_{T}(M_{U}, N_{U}) \) and \( g_{i}, \cdots, g_{r} \) in \( \text{Hom}_{T}(N_{U}, M_{U}) \) such that \( \Sigma_{i}f_{i}g_{i}=\text{the identity of } M_{i} \), or equivalently, \( \text{Hom}_{T}(M_{U}, N_{U})\cdot \text{Hom}_{T}(N_{U}, M_{U})=\text{Hom}_{T}(M_{U}, N_{U}) \). Then, \( _{T}M_{U}\mid_{T}N_{U} \), where homomorphisms act on the right side.

Let \( T \) be a ring with \( 1 \), \( M \) a unital \( T \)-left module, and \( T^{*}=\text{End}_{T}(M) \).

S. 1. If \( _{T}T\mid_{T}M \) then \( M_{T}\mid_{T}T^{*} \). (i.e. \( M_{T} \) is finitely generated and projective) and \( T=\text{End}_{T}(M) \). \( \text{(Morita)} \)

S. 2. If \( _{T}M\mid_{T}T \) then \( T^{*}\mid_{T}M_{T} \). \( \text{(Morita)} \)

S. 3. Let \( T \) be commutative. If \( _{T}M\mid_{T}T \) and \( _{T}M \) is faithful, then \( _{T}T\mid_{T}M \). \( \text{(Auslander-Buchsbaum-Goldman)} \)

S. 4. Let \( \bar{T} \) be an extension ring of \( T \). If \( _{T}T\mid_{T}\bar{T} \) then \( _{T}T \) is a direct summand of \( _{T}\bar{T} \) (and conversely). \( \text{(Müller)} \)

S. 5. Let \( \bar{T} \) be an extension ring of \( T \). If \( _{T}\bar{T}\mid_{T}T_{T} \) then \( _{T}\bar{T} \) is a direct summand of \( _{T}T_{T} \). \( \text{(The proof is similar to the one of S.4.)} \)

In [22], \( A/B \) was called a \( G \)-Galos extension if \( G \) is finite and there are elements \( a_{1}, \cdots, a_{n} \); \( a_{1}^{*}, \cdots, a_{n}^{*} \) in \( A \) such that \( \Sigma_{i}a_{i}\sigma(a_{i}^{*})=\delta_{1,\sigma} \) (\( \sigma \in G \)). In this paper, \( A/B \) is called a finite \( G \)-Galos extension if \( A/B \) is \( G \)-Galos and \( t_{\sigma}(c)=1 \) for some \( c \) in \( A \). Then, the following are equivalent:

(a) \( A/B \) is finite \( G \)-Galos.
(b) \( G \) is finite, \( A_{B}\sim B_{B} \) and \( j: A=\text{End}(A_{B}) \).
(c) \( G \) is finite and \( A_{B}\sim B_{B} \).
(Cf. S. 1, S. 2, [6] and [21]).

\( A/B \) is called a locally finite \( G \)-Galos extension if there are fixed normal subgroups \( N_{i} \) (\( \lambda \in \Lambda \)) of \( G \) which satisfy the following conditions: (1) \( (G: N_{i})<\infty \), and \( A/N_{i}|B \) is a finite \( G/N_{i} \)-Galos extension. (2) \( A=\bigcup_{i}A^{N_{i}} \), and \( \{A^{N_{i}} ; \lambda \in \Lambda \} \) is a directed set with respect to the inclusion relation (abbre. \( A=\bigcup_{i}A^{N_{i}} \) is a directed union). Then we call \( A=\bigcup_{i}A^{N_{i}} \) a representation of the locally finite \( G \)-Galos extension \( A/B \). If \( V_{A}(B)=C \), an extension \( A/B \) is said to be outer.

Now we shall prove first the following

**Proposition 1.1.** Let \( G=G^{*} \) (i.e. \( G \) is closed in \( \hat{G} \)). Then the following are equivalent:

(i) \( \{\sigma(x) ; \sigma \in G \} \) is finite for any \( x \) in \( A \).

(ii) \( G \) is compact.

(iii) Every directed union of fixed subrings of \( A \) with respect to \( G \) is also a fixed subring of \( A \) with respect to \( G \), and \( \bigcap H=1 \), where \( H \) ranges
over all fixed subgroups of $G$ such that $(G:H)<\infty$.

Proof. (i) $\Rightarrow$ (ii) If we put $\prod_{x\in A} \{\sigma(x); \sigma \in G\} = D$, then $G \subseteq D$ and $D$ is compact. Therefore it is sufficient to prove that $G$ is closed in $D$. Let $\rho$ be any element of the closure of $G$ in $D$. Then, as is easily seen, $\rho$ is a $B$-ring isomorphism from $A$ into $A$. Let $a$ be in $A$, and put $F = \{\sigma(a); \sigma \in G\}$. Then, by assumption, $F$ is a finite subset of $A$, so that there is an element $\tau$ in $G$ such that $\rho|F = \tau|F$. Then, in particular, $\rho(\tau^{-1}(a)) = \tau(\tau^{-1}(a)) = a$. Thus $\rho$ is a $B$-automorphism of $A$.

Hence the closure of $G$ in $D$ is contained in $\hat{G}$. Since $G$ is closed in $\hat{G}$, $G$ is closed in $D$, as desired. (ii) $\Rightarrow$ (iii) For any $x$ in $G$, we put $H_x = \{\sigma \in G; \sigma(x) = x\}$. Then $H_x$ is open in $G$, and therefore $\sigma H_x$ is open in $G$ for any $\sigma$ in $G$. Then, since $G$ is compact, we have $(G:H_x)<\infty$. Evidently $\cap_{x \in A} H_x = 1$. This proves the second assertion.

Let $(A \neq) T = \cup_{i \in A} T_i$ be a directed union of fixed subrings of $A$ with respect to $G$, and let $K_i = G^{T_i}$. Then each $K_i$ is a closed subgroup of $G$, and $A^{K_i} = T_i$. Let $a$ be an element of $A - T$, and put $U = \{\sigma \in G; \sigma(a) = a\}$. Then $U$ is open in $G$, so that each $K_i - U$ is closed in $G$. Since $a \notin T_i$ and $A^{K_i} = T_i$, we have $K_i - U \neq \emptyset$. For any finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of $A$, there is an element $\lambda_0$ of $A$ such that $T_i \supseteq \cup_{i \in A} T_i$. Then $K_\lambda \subseteq \cap_{i} K_{\lambda_i}$, and so $0 \neq K_\lambda - U \subseteq \cap_{i} K_{\lambda_i} - U = \cap_{i} (K_{\lambda_i} - U)$. Thus $\{K_i - U; i \in A\}$ has finite intersection property. Since $G$ is compact, we have $\cap_{i} (K_i - U) \neq \emptyset$. If $\rho$ is in $\cap_{i} (K_i - U)$ then $\rho \in G^{T_i}$ and $\rho(a) \neq a$. Therefore $a \neq A^{\rho}$, where $K = G^{T_i}$. Thus $A^{\rho} = T$. Hence $T$ is a fixed subgroup of $A$ with respect to $G$. (iii) $\Rightarrow$ (i) Let $H$ and $K$ be fixed subgroups of $G$ such that $(G:H)<\infty$ and $(G:K)<\infty$. Then $H \cap K$ is also a fixed subgroup of $G$ with $(G:H \cap K)<\infty$. Therefore $\cup A^H$ is a directed union of fixed subrings of $A$, where $H$ ranges over all fixed subgroups of $G$ with $(G:H)<\infty$. Then, by assumption, $\cup A^H$ is a fixed subgroup of $A$ with respect to $G$. Since $\cap H = 1$, we have $A = \cup A^H$. For any $x$ in $A$, there is an $A^H$ such that $x \in A^H$. Therefore if we put $L = \{\sigma \in G; \sigma(x) = x\}$ then $(G:L)<\infty$. This implies that $\{\sigma(x); \sigma \in G\}$ is finite.

Remark. For any $x$ in $A$, $\{\sigma(x); \sigma \in G\} = \{\sigma(x); \sigma \in G^n\}$.

**Proposition 1.2.** Let $N$ be a fixed normal subgroup of $G$ such that $(G:N)<\infty$ and $A^N/B$ is finite $G/N$-Galois, and $G_1$ a subgroup of $G^*$ containing $G$. Then $A^N/B$ is finite $G_1/N_1$-Galois, where $N_1 = \{\sigma \in G_1; \sigma|A^N = 1_{A^N}\}$.

Proof. Put $T = A^N$. Evidently $A^N = T$. Since $G$ is dense in $G_1$ and $T_B$ is finitely generated, there holds $G|T = G_1|T$. Therefore $T$ is $G_1$-invariant, $N_1 \lhd G_1$, and $(G_1 : N_1)<\infty$. There are elements $a_1, \cdots, a_n; a_1^*, \cdots, a_n^*$ in $T$ such that $\sum a_i \cdot \sigma(a_i^*) = \delta_{N, \sigma}$ for all $\sigma$ in $G$. If $\tau$ is in $G_1 - N_1$ then $\tau|T = \rho|T$ for
some \( \rho \) in \( G - N \), and \( \sum_i a_i \cdot \tau(a_i^*) = \sum_i a_i \cdot \rho(a_i^*) = 0 \). Thus \( \sum_i a_i \cdot \sigma(a_i^*) = \delta_{N, \sigma} \) for \( \sigma \) in \( G_1 \).

**Corollary.** Let \( A/B \) be locally finite \( G \)-Galois, and \( G_1 \) a subgroup of \( G^* \) containing \( G \). Then \( A/B \) is locally finite \( G_1 \)-Galois.

**Proposition 1.3.** Let \( H_\lambda (\lambda \in \Lambda) \) be fixed subgroups of \( G \) such that \( A = \bigcup_{\lambda \in \Lambda} A^{H_\lambda} \) is a directed union.

1. If \( H \) is a subgroup of \( G \) such that \( (G:H) < \infty \) then \( A^H \subseteq A^{H_\lambda} \) for some \( \lambda \) in \( \Lambda \).

2. If \( K \) is a subgroup of \( G \) such that \( (K:1) \) < \( \infty \) then \( K \cap H_\mu = 1 \) for some \( \mu \) in \( \Lambda \).

**Proof.** (1) Let \([H_\lambda \cup H]\) be the subgroup of \( G \) generated by \( H_\lambda \cup H \). Since \( G \supseteq [H_\lambda \cup H] \supseteq H \), we have \( (G:[H_\lambda \cup H]) \subseteq (G:H) \) for all \( \lambda \) in \( \Lambda \). Let \( (G:[H_\lambda \cup H]) \) be maximum. We shall prove that \( A^H \subseteq A^{H_\lambda} \). For any \( H_\lambda \) there is an \( H_\mu \) such that \( A^{H_\mu} \supseteq A^{H_\lambda} \cup A^{H_\lambda_0} \). Then \( H_\mu \subseteq H_\lambda \cap H_\lambda_0 \), and so \( [H_\mu \cup H] \subseteq [H_\lambda \cup H] \cap [H_\lambda_0 \cup H] \). Since \( (G:[H_\lambda \cup H]) \) is maximum, we have \( (G_\lambda \cup H) \supseteq [H_\lambda \cup H] \cap [H_\lambda_0 \cup H] \). Hence \( [H_\lambda \cup H] \subseteq [H_\lambda \cup H] \) for all \( \lambda \) in \( \Lambda \). Then \( A^H = \bigcup_{\lambda} (A^{H_\lambda} \cap A^H) = \bigcup_{\lambda} A^{[H_\lambda \cup H]} = A^{[H_\lambda \cup H]} \cap A^H \), which means \( A^H \subseteq A^{H_\lambda} \). (2) Since \( A = \bigcup_{\lambda} A^{H_\lambda} \), we have \( 1 = G^A = \bigcap_{\lambda} H_\lambda \). Let \( K = \{ \sigma_1 = 1, \sigma_2, \ldots, \sigma_r \} \). Then, for any \( \sigma_\mu (i \neq 1) \), there is an \( H_\mu \) such that \( \sigma_\mu \in H_\mu \). By assumption there is a \( \mu \) such that \( H_\mu \subseteq \bigcap_{\lambda \in 1, \ldots, r} H_\mu \). Then \( H \cap H_\mu \subseteq H \cap (\bigcap_{\lambda \in 1, \ldots, r} H_\mu) = 1 \).

**Remark.** Let \( A/B \) be locally finite \( G \)-Galois, and \( A = \bigcup_{\lambda \in \Lambda} A^{N_\lambda} \) its representation. If \( G \) is finite then \( A = A^{N_\lambda} \) for some \( \lambda \).

**Proposition 1.4.** Let \( T \) be an intermediate ring of \( A/B \) such that \( G|T \) is finite, and \( H = G^T \), and \( G = \sigma_1 H \cup \cdots \cup \sigma_r H \) a left coset decomposition of \( G \). If there are elements \( t_1, \ldots, t_n ; t^*_1, \ldots, t^*_n \) in \( T \) such that \( \sum_i t_i \cdot \sigma(t^*_i) = \delta_{\mu, \sigma} \) for all \( \sigma \) in \( G \), then there hold the following.

1. \( T = A^H \), and \( T_\mu \) is finitely generated and projective.

2. \( j^*: A(\sum_k u_{k_\mu})T = \sum_k A u_{k_\mu} \cong \text{Hom}(T_\mu, A_\mu) \), where \( j^*(\sum_k u_{k_\mu})(t) = \sum_k x_k \cdot s_k(t) \), and this induces the \( B - T \)-isomorphism \( (\sum_k u_{k_\mu})T \cong \text{Hom}(T_\mu, A_\mu) \).

3. The following are equivalent: (i) \( B_\mu | T_\mu \). (ii) \( B_\mu | T_\mu \). (iii) \( \delta_{\mu, \sigma}(c) = 1 \) for some \( c \) in \( T \).

**Proof.** (1) \( \delta_{\mu, \sigma} \) is a \( B - B \)-homomorphism from \( A^H \) to \( B \). For any \( y \) in \( A^H \), \( T \), \( \sum_i t_i \cdot \delta_{\mu, \sigma}(t^*_i) = \sum_i t_i \sum_k s_k(t^*_i) = \sum_i \sum_k t_i \cdot s_k(t^*_i) = \delta_{\mu, \sigma}(y) = y \). Hence \( A^H = T \), and \( T_\mu \) is finitely generated and projective (cf. [3]). (2) \( j^* \) is the mapping such that \( j^*(f) = \sum_i f(t_i)(\sum_k u_{k_\mu})t_i^* \) (\( f \in \text{Hom}(T_\mu, A_\mu) \)). The second part will be easily seen. (3) The equivalence (i) \( \iff \) (iii) is easy from (2).
Therefore (i) and (ii) are equivalent, because the situation is right-left symmetric.

**Proposition 1.5.** Let $A/B$ be locally finite $G$-Galois. Then there hold the following:

1. $G^*$ is compact.
2. By $j$, $\Delta$ is isomorphic to a dense subring of $\text{Hom}(A_B, A_B)$.
3. A subgroup $H$ of $G$ is a closed subgroup of $G$ if and only if $H$ is a fixed subgroup of $G$.

**Proof.** Let $A=\bigcup_{\lambda} A_{\lambda}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) If $x$ is in $A$ then $x\in A_{\nu}$ for some $\nu$ in $\Lambda$. Then $(G:N_{\nu})<\infty$ implies that $(\{\sigma(x); \sigma \in G\} = \{\sigma(x); \sigma \in G^*\}$ is finite. Hence, by Prop. 1.1, $G^*$ is compact. (2) By Prop. 1.4 (2), $\text{Im} j$ is dense in $\text{Hom}(A_B, A_B)$. Therefore it suffices to prove that $j$ is 1–1. Let $\sigma_1, \cdots, \sigma_r$ be different elements in $G$. Then there is a finite subset $F$ of $A$ such that $\sigma_i|F \neq \sigma_j|F$ provided $i \neq k$. From this fact and Prop. 1.4, we can easily see that $j$ is 1–1. (3) Evidently, a fixed subgroup is a closed subgroup. Let $H$ be any subgroup of $G$, and put $H'=G^*$, where $T=A^H$. Then $T=A^H$. It suffices to prove that $H$ is dense in $H'$. To prove this, we take any finite subset $F$ of $A$. Then $F \subseteq A_{\nu}$ for some $N_{\nu}$. Put $N=N_{\nu}$. Then, by finite Galois theory, we obtain $(G/N)^{T}=HN/N$ and $(A/N)^{T}=HN/N$, where $T=H^N$ and $T'=A^H$. (cf. [22; Prop. 2.2]). Since $A^H=H \cap A^N=A^H \cap A^N=A^W$, we have $HN/N=HN/N$, that is, $HN=H'N$. Hence $H|A^N=H'|A^N$, and so $H|F=H'|F$. Since $F$ is arbitrary, this implies that $H$ is dense in $H'$. This completes the proof.

**Theorem 1.6.** Let $A/B$ be locally finite $G$-Galois, $G=G^*$, and $H$ a subgroup of $G$, and let $A'$ be an indecomposable extension ring of $B$ such that $V_{A'}(B)=V_{A'}(A')$. Assume that there is a $B$-ring homomorphism $g$ from $A$ to $A'$. Then, for any $B$-ring homomorphism $f$ from $A^H$ to $A'$, there is an element $\sigma$ in $G$ such that $f=g\sigma|A^H$.

**Proof.** Let $A=\bigcup_{\lambda} A_{\lambda}$ be a representation. For each $N_{\lambda}$, there is an element $\sigma$ in $G$ such that $f|A_{\lambda}^{HN_{\lambda}}=g\sigma|A_{\lambda}^{HN_{\lambda}}$. (cf. [22; Th. 4.1]). For each $\lambda$, we put $K_{\lambda}=\{\sigma \in G; f|A_{\lambda}^{HN_{\lambda}}=g\sigma|A_{\lambda}^{HN_{\lambda}}\}$. Then $K_{\lambda} \neq \emptyset$, and $\{K_{\lambda} ; \lambda \in \Lambda\}$ has finite intersection property. Let $\tau$ be in the closure of $K_{\lambda}$ in $G$. Since $(A_{\lambda}^{N_{\lambda}})_{\lambda}$ is finitely generated, $\tau|A_{\lambda}^{N_{\lambda}}=\alpha|A_{\lambda}^{N_{\lambda}}$ for some $\alpha$ in $K_{\lambda}$. Then $f|A_{\lambda}^{HN_{\lambda}}=g\alpha|A_{\lambda}^{HN_{\lambda}}$. Hence $\tau \in K_{\lambda}$, and therefore $K_{\lambda}$ is closed in $G$. Since $G$ is compact (Prop. 1.5), we have $\cap_{\lambda} K_{\lambda} \neq \emptyset$. If $\rho$ is in $\cap_{\lambda} K_{\lambda}$, then $f|A_{\lambda}^{HN_{\lambda}}=g\rho|A_{\lambda}^{HN_{\lambda}}$ for all $\lambda$. Since $A^H=\bigcup_{\lambda} A_{\lambda}^{HN_{\lambda}}$, we know $f=g\rho|A^H$.

The following theorem will follow at once from Th. 1.6 and Cor. to Prop. 1.2.

**Theorem 1.7.** Let $A/B$ be locally finite outer $G$-Galois, and $A$ an
indecomposable ring. Then $G^* = \hat{G}$, that is, $G$ is dense in $\hat{G}$.

**Proposition 1.8.** Let $A/B$ be locally finite $G$-Galois, and $G=G^*$ (cf. Cor. to Prop. 1.2). Then there hold the following.

(1) For an intermediate ring $T$ of $A/B$ the following are equivalent.
   (i) $T=A^H$ for some subgroup $H$ of $G$. (ii) There are subgroups $H_\nu (\nu \in \Gamma)$ of $G$ such that $T=\bigcup_{\nu} A^{H}_\nu$, $(G:H)_\nu<\infty$ and $\{A^{H}_\nu; \nu \in \Gamma\}$ is a directed set with respect to the inclusion relation.

(2) If $H$ is a subgroup of $G$ such that $(G:H)<\infty$ then $(A^H)_B$ is finitely generated.

**Proof.** Let $A=\bigcup_{i \in I_A} A^{N_i}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) (i) $\iff$ (ii) $T=A^H=\bigcup_{i} (A^H \cap A^{N_i})=\bigcup_{i} A^{H+N_i}$ is a directed union, and $(G:HN_\nu)<\infty$. (ii) $\iff$ (i) follows from Prop. 1.1. (2) By Prop. 1.3, $A^H \subseteq A^{N_\nu}$ for some $\nu$ in $A$. Then, $A^H=\bigcup_{\nu} A^{H_\nu}$ is a fixed subring of the finite $G/N_\nu$-Galois extension $A^{N_\nu}/B$, and therefore $(A^{N_\nu})_B | (A^H)_B$ (cf. [22; §2. p. 118]). Since $(A^{N_\nu})_B$ is finitely generated, $(A^H)_B$ is finitely generated.

Let $T$ be an intermediate ring of $A/B$, and $S$ a subset of $A$. $T$ is called a $G$-separable cover of $S$ if $T$ satisfies the following conditions:

(1) $T/B$ is a separable extension, and $T \supseteq S$.
(2) $G|T$ is finite.
(3) $G|T$ is strongly distinct (i.e. if $\sigma|T \neq \tau|T$ for $\sigma, \tau$ in $G$ then $\sigma|T$ and $\tau|T$ are strongly distinct).

**Theorem 1.9.** Let $A/B$ be locally finite outer $G$-Galois, and $T$ an intermediate ring of $A/B$. Then the following are equivalent:

(i) $T=A^H$ for some subgroup $H$ of $G$ such that $(G:H)<\infty$.
(ii) $T/B$ is a separable extension, $T_B$ is finitely generated, and $G|T$ is strongly distinct.
(iii) $T$ is a $G$-separable cover of $B$.

**Proof.** Let $A=\bigcup_{i \in I_A} A^{N_i}$ be a representation. (i) $\iff$ (ii) By Prop. 1.3, $T=A^H \subseteq A^{N_\nu}$ for some $\nu$ in $A$. Then $T$ is a fixed subring of the finite $G/N_\nu$-Galois extension $A^{N_\nu}/B$. Then, by [19; Prop. 3.4], $T/B$ is a separable extension. By Prop. 1.8 (2) (cf. Cor. to Prop. 1.2), $T_B$ is finitely generated. By [22; Th. 2.6], $G|T$ is strongly distinct. (ii) $\iff$ (iii) This follows from the fact that $\{\sigma(x); \sigma \in G\}$ is finite for any $x$ in $A$. (iii) $\iff$ (i) Let $\{(t_i, t_i^*); i=1, \ldots, n\}$ be a $(B, T)$-projective coordinate system of $T/B$. Then, by [22; Prop. 1.2], $\sum_i t_i \cdot \sigma(t_i^*)=\delta_{H, \sigma}$ for $\sigma$ in $G$, where $H=G^T$. $(G:H)<\infty$ implies $(G:H)<\infty$. By Prop. 1.4, $A^H=T$.

Combining Th. 1.9 with Prop. 1.8, we obtain the following theorem (cf. [12; Th. 3], [28; Theorem]).
Theorem 1.10. Let $A/B$ be locally finite outer $G$-Galois, and $G=G^*$. Then, for an intermediate ring $T$ of $A/B$, the following are equivalent.

(i) $T=A^H$ for some subgroup $H$ of $G$.

(ii) For any finite subset $F$ of $T$ there is an intermediate ring $T_0$ of $T/B$ such that $T_0\supseteq F$, $T_0/B$ is separable, $T_0$ is finitely generated, and $G\mid T_0$ is strongly distinct.

(iii) Any finite subset of $T$ has a $G$-separable cover which is contained in $T$.

Next we shall proceed to the characterization of locally finite outer Galois extensions.

Proposition 1.11. Let $V_A(B)=C$, $T$ a $G$-separable cover of $B$, and $(t_i, t_i^*)$ a $(B, T)$-projective coordinate system for $T/B$, and put $H=G^H$. Then there hold the following.

(1) $\sum_i t_i \cdot \sigma(t_i^*) = \delta_{H, \sigma}$ for all $\sigma$ in $G$.

(2) $A^H=T$, $(G:H)<\infty$, and $T/B$ is a projective Frobenius extension.

(3) Let $K$ be a subgroup of $G$ containing $H$. Then, $\sum_i t_{K:H}(t_i)\sigma(t_i^*) = \delta_{K, \sigma}$ for all $\sigma$ in $G$, $T$ is $(B, A^K)$-projective, $T\mid A^K$ is a projective Frobenius extension, and $G\mid A^K$ is strongly distinct. Further the following are equivalent. $(\alpha) (A^K)_{A^K}|T_{A^K}$. $(\beta) (A^K)_{A^K}|\langle A^K \rangle T$. $(\gamma) t_{K:H}(c)=1$ for some $c$ in $T$.

Proof. (1) follows from [22; Prop. 1.2], and (2) is obvious by (1) and Prop. 1.4. (3) It will be easily seen that $\sum_i t_{K:H}(t_i)\sigma(t_i^*) = \delta_{K, \sigma}$ for all $\sigma$ in $G$. Since $\sum_i t_i \otimes t_i^* = \sum_i t_i \otimes t_i^* t \in T \otimes_B T$ for $t$ in $T$, $\sum_i t_{K:H}(t_i)\otimes t_i^* = \sum_i t_{K:H}(t_i) \otimes t_i^* \in A^K \otimes_B T$ for all $y$ in $A^K$. Hence the mapping $x \rightarrow \sum_i t_{K:H}(t_i)\otimes t_i^* x$ from $T$ to $A^K \otimes_B T$ is an $A^K$-homomorphism. Since $\sum_i t_{K:H}(t_i)\otimes t_i^* x = x$, it follows that $T$ is $(B, A^K)$-projective. Let $\rho|A^K \neq \tau|A^K$ for $\rho, \tau$ in $G$. Then $\tau^{-1}\rho \notin K$, and so $0 = \tau(\sum_i t_{K:H}(t_i)\tau^{-1}\rho(t_i^*)) = \sum_i \tau(t_{K:H}(t_i))\rho(t_i^*)$. Thus, by [22; Prop. 1.1], $\rho|A^K$ and $\tau|A^K$ are strongly distinct. If we set $G=K$ in Prop. 1.4, the remainder follows from Prop. 1.4.

Theorem 1.12. Let $V_A(B)=C$. Then the following statements are equivalent.

(i) $A/B$ is locally finite (outer) $G$-Galois.

(ii) For any finite subset $F$ of $A$ there is a $G$-invariant $G$-separable cover $T$ of $F$ such that $F B_{|T} T$.

(iii) For any finite subset $F$ of $A$ there is a $G$-separable cover $T$ of $F$ which satisfies the following: If $T_0$ is an intermediate ring of $T/B$ such that $(\alpha) T$ is $(B, T_0)$-projective, $(\beta) T/T_0$ is a projective Frobenius extension, $(\gamma) G\mid T_0$ is strongly distinct, then $T_0 \cap T_0 \subseteq T_0 \cap T$.

(iv) For any finite subset $F$ of $A$ there is a $G$-separable cover $T$ of $F$
which satisfies the following: If $T_0$ is an intermediate ring of $T/B$ such that (α) $T$ is $(B, T_0)$-projective, (β) $T/T_0$ is a projective Frobenius extension, (γ) $G|T_0$ is strongly distinct, (δ) $T_0$ is a $G$-invariant fixed subring (with respect to $G$), then $\tau_0T_0|\tau_0T$.

Proof. (i) $\Rightarrow$ (ii), (iii) Let $A=\bigcup_{\mu\in\Lambda}A^{N_{\mu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then any finite subset $F$ of $A$ is contained in some $A^{N_{\mu}} (\mu \in \Lambda)$. By [22; Th. 1.5], $A^{N_{\mu}}$ is a $G$-invariant $G$-separable cover of $F$ such that $\rho_B|_{B}A^{N_{\mu}}$. Let $T_0$ be an intermediate ring of $A^{N_{\mu}}/B$ such that $A^{N_{\mu}}$ is $(B, T_0)$-projective and that $G|T_0$ is strongly distinct. Then, by [22; Th. 2.6], $T_0$ is a fixed subring of the finite outer $G/N_{\mu}$-Galois $A^{N_{\mu}}/B$, whence $\tau_0T_0|\tau_0T$ by [22; §2. p. 118]. (ii) $\Rightarrow$ (i) Let $F$ be a finite subset of $A$, and $T$ a $G$-invariant $G$-separable cover of $F$ such that $\rho_B|_{B}T$. If we put $N=G^r$, then $A^{N}=T, N \triangleleft G$ and $(G:N)<\infty$ (Prop. 1.11). By Prop. 1.11, $A^{N}/B$ is a finite $G/N$-Galois extension. Noting that $(A^{N})_{B}$ is finitely generated, $A/B$ is a locally finite $G$-Galois extension. (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (i) Let $T_1$ be a separable cover of an element $x$ in $A$. Put $G^x=H_1$. Then $\#(G|T_1)<\infty$ implies $(G:H_1)<\infty$ and $\# \{\sigma(x) ; \sigma \in G\}<\infty$. Thus any finite subset of $A$ is contained in a $G$-invariant finite subset of $A$. Let $F$ be a $G$-invariant finite subset of $A$, and $T$ a $G$-separable cover of $F$ as that in (iv), and let $\{(t_i, t^*_i) ; i=1, \cdots, n\}$ be a $(B, T)$-projective coordinate system of $T/B$, and $H=G^r$. Then, by Prop. 1.11, $A^{r}=T$, $(G:H)<\infty$, and $\sum_{i}t_i \sigma(t^*_i)=\delta_{\sigma,x}$ for all $\sigma \in G$. Set $N=G^r$. Then $H \triangleleft N \triangleleft G$, and $F \subseteq A^{N} \subseteq A^{r}=T$. By Prop. 1.11, $T$ is $(B, A^{N})$-projective, $T/A^{N}$ is a projective Frobenius extension, and $G|A^{N}$ is strongly distinct. Then, by the assumption for $T$, $(A^{N})|_{(A^{N})∩(N_1)}T$, so that $t_{N:B}(c)=1$ for some $c \in T$ (Prop. 1.11 (3)). Put $t^*_i=t_{N:B}(t_i)$ and $t^*_i\in A^{N}$. Then, $t_i, t^*_i \in A^{N}$, and $\sum t_i \sigma(t^*_i)=\delta_{\sigma,x}$ for all $\sigma \in G$ (Prop. 1.11 (3)). Further, as is easily seen, $\sum t_i \sigma(t^*_i)=\delta_{\sigma,x}$ for all $\sigma \in G$. Since $\rho_B|_{B}T$ (Prop. 1.11 (3)), we have $\rho_B|_{B}A^{N}$. Thus $A^{N}/B$ is a finite $G/N$-Galois extension. Noting that $(A^{N})_{B}$ is finitely generated, we conclude that $A/B$ is a locally finite $G$-Galois extension.

Proposition 1.13. Let $A^{*} \supseteq T \supseteq B^{*}$ be rings such that $A^{*}$ is $(B^{*}, T)$-projective, $A'$ an extension ring of $B^{*}$ such that $V_{A'}(B^{*})=V_{A'}(A')$, and $f_{1}, \cdots, f_{s}$ $B^{*}$-ring homomorphisms from $A^{*}$ to $A'$ such that $f_{i}|T$ and $f_{k}|T$ ($i \neq k$) are strongly distinct. If $(B^{*})^{s}_{B^{*}} \rightarrow T^{s}_{B^{*}}$, then $(A'^{*})^{s}_{B^{*}} \rightarrow (A')^{s}_{A'^{*}}$.

Proof. Let $\{(t_i, a^*_i) ; i=1, \cdots, n\}$ be a $(B^{*}, T)$-projective coordinate system for $A^{*}$. Then, by [22; Prop. 1.2], $\sum f_{h}(t_i) f_{k}(a^*_i)=\delta_{h,k}$ for all $h, k$. Let $\varphi$ be a $A'$-right homomorphism from $T \otimes_{B^{*}}A'$ to $(A')_{A'^{*}}$ defined by $\varphi(t \otimes a')= (f_{h}(t)a', \cdots, f_{s}(t)a')$. Since $\sum f_{h}(t_i) f_{k}(a^*_i)=\delta_{h,k}$, $\varphi$ is an epimorphism. $(B^{*})^{s}_{B^{*}} \rightarrow T^{s}_{B^{*}}$ implies that $(A')^{s}_{A'^{*}} \rightarrow T \otimes_{B^{*}}A'$ Hence we have $(A')^{s}_{A'^{*}} \rightarrow (A')^{s}_{A'^{*}}$, as
desired.

Concerning Prop. 1.13, we consider the following condition.

Condition (F): If $\mathcal{A}A^r \rightarrow \mathcal{A}A^s$ for positive integers $r, s$, then $r \geq s$.

Remark. Let $\mathcal{A}A^r \rightarrow \mathcal{A}A^s$ for positive integers $r, s$. Then, since $\mathcal{A}A^s$ is projective, $\mathcal{A}A^r$ is isomorphic to an $A$-direct summand of $\mathcal{A}A^r$.

1. If $\mathcal{A}A$ is finite dimensional, then $r \cdot \text{dim} \mathcal{A}A \geq s \cdot \text{dim} \mathcal{A}A$, and so $r \geq s$ (cf. [11]).

2. Assume that there is a proper ideal $\mathfrak{A}$ of $A$ such that $\mathcal{A}A/\mathfrak{A}$ is finite dimensional. Then, since $\mathcal{A}A^r/\mathfrak{A}^r \rightarrow \mathcal{A}A^s/\mathfrak{A}^s$, the above (1) yields $r \geq s$, because $\mathcal{A}A^r/\mathfrak{A}^r \cong (A/\mathfrak{A})^r$ and $\mathcal{A}A^s/\mathfrak{A}^s \cong (A/\mathfrak{A})^s$.

3. If $A$ is commutative, then $r \geq s$ by (2).

**Proposition 1.14.** Let $V_\mathcal{A}(B)=C$, and $A$ an indecomposable ring satisfying (F), and let $T$ be an intermediate ring of $A/B$, and $S$ a subset of $A$. Then the following are equivalent:

(i) $T$ is a $G$-separable cover of $S$.

(ii) $T \supseteq S$, $T/B$ is a separable extension, and $T_B$ is finitely generated.

**Proof.** (i) $\Rightarrow$ (ii) is evident by Prop. 1.11. (ii) $\Rightarrow$ (i) By [22; Lemma 2.7], $A$ is $(B, T)$-projective. Then, by Prop. 1.13, we have $\#(G|T) < \infty$, and hence $T$ is a $G$-separable cover of $S$.

If $A$ is commutative, then $A$ satisfies (F). Therefore, by Th. 1.12, S. 3 and Prop. 1.14, we have the following

**Theorem 1.15** (Nagahara [12]). Let $A$ be an indecomposable commutative ring. Then the following are equivalent.

(i) $A/B$ is locally finite $G$-Galois.

(ii) For any finite subset $F$ of $A$ there is an intermediate ring $T$ of $A/B$ such that

(a) $T/B$ is a separable extension, and $T_B$ is finitely generated,

(b) $T \supseteq F$.

**Proposition 1.16.** Let $A/B$ be locally finite $G$-Galois, and $H$ a subgroup of $G$. Then $G|A^H$ is strongly distinct.

**Proof.** Let $\sigma, \tau$ be in $G$, and $e$ a central idempotent of $A$ such that $\sigma(x)e = \tau(x)e$ for all $x$ in $A^H$. Let $A = \bigcup_{\lambda \in \Lambda} A^{N_\lambda}$ be a representation of the locally finite $G$-Galois extension $A/B$. We may assume that $e \in A^{N_\lambda}$ for all $\lambda \in \Lambda$. Suppose that $\sigma|A^H \neq \tau|A^H$. Since $A^H = \bigcup_{\mu \in \Lambda} A^{N_\mu}$, $\sigma|A^{N_\mu} \neq \tau|A^{N_\mu}$ for some $\mu$ in $\Lambda$. Then, by [22; Prop. 2.4], $(G/N_\mu)|A^{N_{\mu}}$ is strongly distinct. Therefore we have $e = 0$. Thus $G|A^H$ is strongly distinct.

**Theorem 1.17.** Let $A/B$ be locally finite outer $G$-Galois, and $T$ an intermediate ring of $A/B$. Then the following are equivalent.

(i) $T = A^H$ for some subgroup $H$ of $G$, and $A_T$ is finitely generated.
(ii) \( T = A^\sigma \) for some subgroup \( H \) of \( G \) such that \( (H:1) < \infty \).

(iii) \( A|T \) is a projective Frobenius extension, \( \Hom(A_T, A_T) \subseteq \Delta \), and \( G|T \) is strongly distinct.

When any of the above conditions is satisfied \( A|A^H \) is finite \(-\)Galois.

Proof. Let \( A = \bigcup_{\mu \in \Lambda} A^\mu \) be a representation of the locally finite outer \(-\)Galois extension \( A/B \). (i) \( \Rightarrow \) (ii) Let \( A = x_T T + \cdots + x_T T \). Then \( x_T, \ldots, x_T \in A^{N_T} \) for some \( \mu \in \Lambda \), so that \( A = A^{N_T} \cdot T = A^{N_T} \cdot A^H \). Hence \( N_T \cap H = 1 \). Since \( (G : N_T) < \infty \) we have \( (H : 1) < \infty \). (ii) \( \Rightarrow \) (iii) By Prop. 1.3, \( H \cap N_T = 1 \) for some \( \mu \in \Lambda \). There are elements \( a_1, \ldots, a_n \), \( a_1^*, \ldots, a_n^* \in A^{N_T} \) such that \( \sum_i a_i \cdot \sigma(a_i^*) = \delta_{N_T} \) for all \( \sigma \) in \( G \). Then \( \sum_i a_i \cdot \sigma(a_i^*) = \delta_{N_T} \), for all \( \sigma \) in \( H \). Hence \( A/A^H \) is \( H \)-Galois. Therefore \( A/A^H \) is a projective Frobenius extension (cf. [22; p. 121]), and \( \Hom(A_T, A_T) = \bigcap_{\sigma \in H} A u, \subseteq \Delta \). By Prop. 1.16, \( G|T \) is strongly distinct. (iii) \( \Rightarrow \) (i) Let \( h = \sum_{\tau \in H} a_i u_i \) be a Frobenius homomorphism of \( A/T \), where \( H \) is a finite subset of \( G \) and \( a_i \neq 0 \) for all \( \tau \) in \( H \). Then, since \( th = ht \) for all \( t \) in \( T \), we have \( ta_i = a_i \cdot \tau(t) \) for all \( t \) in \( T \), in particular, \( ba_i = a_b \) for all \( b \) in \( B \). Hence \( a_i \in V(A)_B = C \) for all \( t \) in \( H \). There are elements \( r_i, l_i \) in \( A \) such that \( x = \sum_i x \cdot h(x r_i) l_i = \sum_i r_i h(l_i x) \) for all \( x \in A \) (cf. [27]). Then \( u_i = \sum_i r_i h(l_i) = \sum_i r_i \sum_{\tau \in H} a_i \cdot \tau(l_i) u_i = \sum_{\tau \in H} \sum_i r_i a_i \cdot \tau(l_i) u_i \), and so \( 1 = \sum_i r_i a_i l_i = a_i \sum_i r_i l_i \). Thus \( a_i \) is an invertible element in \( C \), and \( a_i^{-1} = \sum_i r_i l_i \). Since \( H \) is finite there is an \( N_T \) such that \( \tau \cdot A^{N_T} \neq \rho \cdot A^{N_T} \) provided \( \tau \neq \rho \). \( \tau, \rho \in H \). Since \( A^{N_T}/B \) is finite \( G/N_T \)-Galois, there are elements \( d_k, e_k \in A^{N_T} \) such that \( \sum_k d_k \cdot \sigma(e_k) = \delta_{N_T} \) for all \( \sigma \) in \( G \). Put \( D_0 = \Hom(A_T, A_T) \). Then \( D_0 = AhA \), and \( D_0 \ni \sum_k \tau(d_k) h e_k = \sum_{\tau \in H} \sum_k \tau(d_k) a_i \cdot \sigma(e_k) u_i = a_i u_i \) for \( \tau \) in \( H \). Thus \( D_0 = AhA = \sum_{\tau \in H} \oplus A u_i \). Since \( A/T \) is a projective Frobenius extension with Frobenius homomorphism \( h, A \otimes_T A \simeq_A b_A \) by the correspondence \( x \otimes y \to x y h \). Let \( \varphi \) be the \( A \)-left homomorphism from \( A \) to \( D_0 \) defined by \( \varphi(\sum x_i u_i) = \sum_{\tau \in H} x_i h(y r_i) v l_i \), for all \( \mu \in \Lambda \). and \( \psi \) the \( A \)-left homomorphism from \( D_0 \) to \( D \) defined by \( \psi(x y h) = \sum_i x \cdot h(y r_i) v l_i \), for all \( \mu \in \Lambda \). Then, \( h(\tau r_i) a_i = \tau h(y r_i) a_i \). \( \tau \in H \), so \( \varphi = 1 \). Since \( a_i u_i = \sum_k \tau(d_k) h e_k \), we have \( \psi(a_i u_i) = \sum_k \sum \tau(d_k) h(e_k r_i) v l_i = \sum_{\tau \in H} \sum_k \tau(d_k) a_i \cdot \tau(r_i) u_i l_i \), and so \( \varphi(a_i u_i) = \psi(\sum \tau(d_k) a_i \cdot \tau(r_i) u_i l_i \). On the other hand, \( \varphi(a_i u_i) = a_i u_i \), and hence \( \sum \tau(d_k) a_i \cdot \tau(r_i) = a_i \cdot \tau(a_i) \), for all \( \tau \) in \( H \). Since \( a_i^{-1} = \sum_i r_i l_i \), we have \( a_i^* \cdot \tau(a_i) \). Noting that \( \tau(a_i) \) is an invertible element of \( C \), \( A a_i a_i = A a_i \cdot \tau(a_i) = A a_i \), and so \( A = A a_i + \Ann_A(a_i) \), where \( \Ann_A(a_i) = \{ x \in A ; x a_i = 0 \} \). If \( x a_i \in \Ann_A(a_i) \), then \( 0 = x a_i^* = x a_i \cdot \tau(a_i) \), so that \( x a_i = 0 \). Therefore \( A a_i \) is written as \( A \cdot \tau \), with a central idempotent \( g \) of \( A \). Since \( A a_i u_i \subseteq D_0 \), we have \( g u_i \in D_0 \), and so \( g \cdot t = g \cdot \tau(t) \) for all \( t \) in \( T \). Consequently, \( D_0 = \sum_{\tau \in H} \oplus A u_i \), and \( H = G \cdot T \). Hence \( \End(A) = (A^H)_T \) the right multiplications of elements of \( A \). Since \( a_i u_i \in D_0 = \End(A_T) \), we have \( a_i u_i \in \End(A_{(\mu)}) \). Noting that \( a_i \) is in \( C \), we
can easily seen that \(a,u \in \text{Hom}(A_{(A^{H})}, A_{(A^{H})})\). Thus \(h = \sum_{i \in H} a, u \in \text{Hom}(A_{(A^{H})}A_{(A^{H})}, A_{(A^{H})})\). Then, by [27; Cor. 1], \(A/A^H\) is also a projective Frobenius extension with a Frobenius homomorphism \(h\). Since \((H:1) < \infty\), there is an \(N_i\) such that \(H \cap N_i = 1\) (Prop. 1.3 (2)). Then \(A^{H\cap N_i} \subseteq A^{N_i}\), and \(H \simeq HN_i/N_i\) canonically. Therefore there is an element \(c \in A^{N_i}\) such that \(t_{H}(c) = 1\) (cf. [22; §2. p. 118]), which implies \((A^{H})_{(A^{H})}|A_{(A^{H})}\), because the \(A^H\)-right homomorphism \(x \rightarrow t_{H}(cx)\) \((x \in A)\) from \(A\) to \(A^H\) splits. Therefore there is an element \(d\) in \(A\) such that \(h(d) = 1\). Then, for any \(x\) in \(A^H\), \(T \ni h(dx) = h(d)x = x\). Thus we obtain \(T = A^H\), as desired.

**Theorem 1.18.** Let \(A/B\) be finite outer \(G\)-Galois, and \(T\) an intermediate ring of \(A/B\). Then the following are equivalent.

(i) \(T = A^H\) for some subgroup \(H\) of \(G\).

(ii) \(A/T\) is a projective Frobenius extension, and \(G|T\) is strongly distinct.

(iii) \(T/B\) is a separable extension, and \(G|T\) is strongly distinct.

**Proof.** (i) \(\Leftrightarrow\) (ii) is evident from Th. 1.17. (i) \(\Rightarrow\) (iii) follows from [22; Th. 2.6] and [19; Prop. 3.4]. (iii) \(\Rightarrow\) (i) follows from [22; Th. 2.6 and Lemma 2.7].

**§2. Heredity of locally finite Galois extensions.**

Let \(A_0\) be a \(G^*\)-invariant subring of \(A\) such that the mapping \(\sigma \rightarrow \sigma|A_0\) \((\sigma \in G^*)\) is one-to-one and such that \(A_0/A_0^{\sigma}\) is a locally finite \(G\)-Galois extension, and let \(G^*\) be compact (as an automorphism group of \(A\)). Put \(B_0 = A_0^{\sigma}\), and let \(A_0 = \bigcup_{\lambda \in A} A_0^{N_i}\) be a representation of the locally finite \(G\)-Galois extension \(A_0/B_0\). Then \(G/N_i\) may be considered as a finite group of automorphisms of \(A^{N_i}\). And, by [22; Th. 5.1 and §2. p. 118], \(A^{N_i} = A_0^{N_i} \otimes_{B_0} B\), \(A^{N_i}/B\) is finite \(G/N_i\)-Galois. Since \(\cup_i A^{N_i}\) is a directed union, the compactness of \(G^*\) implies that \(\cup_i A^{N_i}(\subseteq A_0)\) is a fixed subring of \(A\) with respect to \(G^*\) (Prop. 1.1), so that \(A = \cup_i A^{N_i}\), because \(\sigma \rightarrow \sigma|A_0\) \((\sigma \in G^*)\) is 1–1. Thus \(A/B\) is locally finite \(G\)-Galois. Let \(H\) be any subgroup of \(G\). Then, \(A^H = \bigcup_{i} (A^H \cap A^{N_i}) = \cup_i A^{HN_i}\). By [22; Th. 5.1], \(A^{HN_i} = (A_0^{N_i})^{HN_i}/N_i \otimes_{B_0} B = A_0^{HN_i} \otimes_{B_0} B\). Hence \(A^H = \bigcup_i(A_0^{HN_i} \otimes_{B_0} B) = A_0^H \otimes_{B_0} B\), and \(A_0^H \otimes_{B_0} B \rightarrow A_0^H = A^H\) canonically. Since the isomorphism \(A_0^{HN_i} \otimes_{B_0} B \simeq A_0^{HN_i}/N_i \otimes_{B_0} B\) may be considered as \(A_0^{HN_i} \otimes_{B_0} B \rightarrow A_0^{H \cap N_i} \otimes_{B_0} B \rightarrow A^H\), we know \(A^H = A_0^H \otimes_{B_0} B\). Symmetrically we obtain \(A^H = B \otimes_{B_0} A_0^H\). Next we consider the set of all \(A_0^G\)-left submodules of \(A\) and the set of all \(B_0\)-left submodules of \(B\). Let \(\overline{X}\) be any \(A_0^G\)-left submodule of \(A\). Then \(\overline{X} \cap A^{N_i}\) is an \(A_0^{N_i}(G/N_i)\)-left submodule of \(A^{N_i}\). Therefore, by [22; Th. 5.1], we have \(\overline{X} \cap A^{N_i} = A_0^{N_i}(\overline{X} \cap A^{N_i}) \cap B = A_0^{N_i} \otimes_{B_0}(\overline{X} \cap B)\), so that \(\overline{X} = \cup_i(\overline{X} \cap A^{N_i}) = \cup_i(A_0^{N_i}(\overline{X} \cap B)) = A_0(\overline{X} \cap B)\). Since \(A_0^{N_i} \otimes_{B_0}(\overline{X} \cap B) \simeq A_0^{N_i}(\overline{X}\)
\( \bigcap B \subseteq X \) for all \( \lambda \), we have \( X = A_0 \otimes_{B_0} (X \cap B) \). Evidently \( X \cap B \) is a \( B_0 \)-left submodule of \( B \). Let \( X \) be any \( B_0 \)-left submodule of \( B \). Then, as is easily seen, \( A_0 X \) is an \( A_0 \)-\( G \)-left submodule of \( A \), and \( A_0 X = \bigcup_\lambda A_0 X_\lambda \). By [22; Th. 5.1], \( A_0 X \cap B = X \) for all \( \lambda \) in \( A \), so that \( A_0 X \cap B = \bigcup_\lambda (A_0 X_\lambda \cap B) \). If \( \overline{Y} \) is a \( G \)-invariant intermediate ring of \( A/A_0 \), then \( \overline{Y} \cap B \) is an intermediate ring of \( B/B_0 \), and \( \overline{Y} = A_0 (\overline{Y} \cap B) \). Symmetrically we have \( \overline{Y} = (\overline{Y} \cap B) A_0 \). If \( Y \) is an intermediate ring of \( B/B_0 \) such that \( A_0 Y = Y A_0 \), then \( A_0 Y \) is a \( G \)-invariant intermediate ring of \( A/A_0 \). Since \( A = \bigcup \lambda A_\lambda X_\lambda \), we have \( \overline{Y} = \bigcup \lambda (\overline{Y} \cap A_\lambda X_\lambda \cap B) \), and \( \overline{Y} / (\overline{Y} \cap B) \) is finite \( G/N_{\alpha} \)-Galois (22; Th. 5.1). Hence \( \overline{Y} / (\overline{Y} \cap B) \) is locally finite \( G \)-Galois. Thus we have obtained the following.

**Theorem 2.1.** Let \( A_0 \) be a \( G^* \)-invariant subring of \( A \) such that \( \sigma \rightarrow \sigma|A_0 \ (\sigma \in G^*) \) is 1–1 and such that \( A_0 / B_0 \) is locally finite \( G \)-Galois where \( B_0 = A_0^\theta \), and let \( G^* \) be compact. Then there hold the following:

1. \( A/B \) is locally finite \( G \)-Galois.
2. \( A^H = B \otimes_{B_0} A_0^H = A_0^H \otimes_{B_0} B \) for any subgroup \( H \) of \( G \). In particular, \( A = B \otimes_{B_0} A_0 = A_0 \otimes_{B_0} B \).
3. Let \( \{X\} \) and \( \{X\} \) be the set of all \( A_0 \)-\( G \)-left submodules of \( A \) and the set of all \( B_0 \)-left submodules of \( B \), respectively. Then, \( X \rightarrow X \cap B \) and \( X \rightarrow A_0 X = A_0 X \cap B \) are mutually converse order isomorphisms between \( \{X\} \) and \( \{X\} \).
4. Let \( \{\overline{Y}\} \) and \( \{Y\} \) be the set of all \( G \)-invariant intermediate rings of \( A/A_0 \) and the set of all intermediate rings of \( B/B_0 \) such that \( A_0 Y = Y A_0 \), respectively. Then \( \overline{Y} / (\overline{Y} \cap B) \) is locally finite \( G \)-Galois, and \( \overline{Y} \rightarrow \overline{Y} \cap B \) and \( Y \rightarrow A_0 Y = Y A_0 \) are mutually converse order isomorphisms between \( \{\overline{Y}\} \) and \( \{Y\} \).

Let \( A, A' \) be \( R \)-algebras such that \( A \otimes_R A' \neq 0 \). Assume that \( A/B \) is a locally finite \( G \)-Galois extension such that \( R \cdot 1 \subseteq B \), and assume that \( A' \) is a locally finite \( G' \)-Galois extension such that \( R \cdot 1 \subseteq B' \). Then each \( \sigma \times \tau \) in \( G \times G' \) induces an automorphism of \( A \otimes \tau \). Let \( A = \bigcup \alpha A^\alpha \) and \( A' = \bigcup \beta A'^\beta \) be representations of \( A/B \) and \( A'/B' \) respectively. Then, by [22; Th. 5.2], \( (A^\alpha \otimes_R A'^\beta)/(B \otimes B') \) is a finite \( (G/N_\alpha) \times (G'/N'_\beta) \)-Galois extension. Let \( \varphi_{\alpha \beta} \) be the canonical \( R \)-algebra homomorphism from \( A^\alpha \otimes_R A'^\beta \) to \( A^\alpha \otimes A'^\beta \) \( \subseteq A \otimes_R A' \). We put \( A \otimes_R A' \cong A^\alpha \otimes A'^\beta \) and \( A \otimes_R A' \cong B \otimes B' = B^* \). To be easily seen, \( \text{Ker } \varphi_{\alpha \beta} \) is a \( (G/N_\alpha) \times (G'/N'_{\beta}) \)-invariant ideal of \( A^\alpha \otimes_R A'^\beta \). Hence \( A_{\alpha \beta} / B^* \) is \( (G/N_\alpha) \times (G'/N'_{\beta}) \)-Galois ([22; Th. 5.6]). There are elements \( c \) and \( c' \) in \( A^\alpha \) and \( A'^\beta \) respectively such that \( t_{\alpha / N_{\alpha}}(c) = 1 \) and \( t_{\beta / N_{\beta}}(c') = 1 \). Then \( c \otimes c' \in A_{\alpha \beta} \) and \( t_{(\alpha / N_{\alpha}) \times (\beta / N_{\beta})}(c \otimes c') = 1 \otimes 1 \). Hence \( A_{\alpha \beta} / B^* \) is a finite \( (G/N_\alpha) \times (G'/N'_{\beta}) \)-Galois extension, and \( \{\sigma \times \tau \in G \times G'; \ \sigma \times \tau | A_{\alpha \beta} = 1_{A_{\alpha \beta}}\} = N_\alpha \times N'_\beta \). Since \( A_{\alpha \beta}(N_\alpha \times N'_\beta) = (N_\alpha N_\beta) \times (N_{\alpha \beta} N'_\beta) = 1 \), \( G \times G' \) may be considered
as a group of automorphisms of $A \otimes_R A'$. Let $H$ and $H'$ be subgroups of $G$ and $G'$, respectively. Then, $(A \otimes_R A')^{H \times H'} = \bigcup_{a, b} A_{a, b}^{H \times H'} = \bigcup_{a, b} (A_{a}^{H} \otimes A_{b}^{H'}) = (\bigcup_{a} A_{a}^{X_{a}^{H}})(\bigcup_{b} A_{b}^{X_{b}^{H'}}) = A^{H} \otimes A^{H'}$ by [22; Th. 5.2]. In particular, $(A \otimes_R A')^{N_{a} \times N_{b}'} = A_{a}^{N_{a}} \otimes A_{b}'^{N_{b}'} = A_{a} = A_{b}'$, and evidently $(G \times G' : N_{a} \times N_{b}') \leq \infty$. Since $A \otimes_R A' = \bigcup_{a, b} A_{a}^{X_{a} \times X_{b}'}$ is a directed union, $A \otimes_R A'/B \otimes B'$ is a locally finite $G \times G'$-Galois extension. Let $a \in A$ and $a' \in A'$. Then it is evident that $\{\sigma \times \tau \in G \times G'; \sigma(a) \otimes \tau(a') = a \otimes a'\} \supseteq \{a \in G; \sigma(a) = a\} \times \{\tau \in G'; \tau(a') = a'\}$. Put $\{a \in G; \sigma(a) = a\} = K$ and $\{\tau \in G'; \tau(a') = a'\} = K'$. Then $A^{K} \subseteq A_{a}^{X}$ and $A^{K'} \subseteq A_{a}'^{N_{b}'}$ for some $a, b \in \Lambda$ (Prop. 1.3), so that $N_{a} \subseteq K$ and $N_{b}' \subseteq K'$. By [22; Th. 5.2], $(G/N_{a} \times G'/N_{b}')(A_{a} \otimes A_{b}') = (K/N_{a} \times K'/N_{b}')$, and hence $(G \times G')(A_{a} \otimes A_{b}') = K \times K'$. Since $(A^{K})_{B}$ and $(A^{K'})_{B}$ are finitely generated, $(A^{K} \otimes_{B} A^{K'})_{B}$ is finitely generated. Hence the finite topology of $G \times G'$ with respect to $A \otimes R A'$ is the product topology of the finite topology of $G$ with respect to $A$ and the finite topology of $G'$ with respect to $A'$. Thus we have proved the following

**Theorem 2.2.** Let $A$ and $A'$ be $R$-algebras such that $A \otimes_R A' \neq 0$. If $A/B$ is a locally finite $G$-Galois extension such that $R \cdot 1 \subseteq B$, and $A'/B'$ is a locally finite $G'$-Galois extension such that $R \cdot 1 \subseteq B'$, then $(A \otimes_R A')/(B \otimes B')$ is a locally finite $G \times G'$-Galois extension, and $(A \otimes_R A')^{H \times H'} = A^{H} \otimes A^{H'}$ for any subgroup $H$ of $G$ and any subgroup $H'$ of $G'$. The finite topology of $G \times G'$ with respect to $A \otimes R A'$ is the product topology of the finite topology of $G$ with respect to $A$ and the finite topology of $G'$ with respect to $A'$.

**Corollary.** Let $A/B$ be a locally finite $G$-Galois extension such that $B \subseteq C$, and $A'$ a $B$-algebra such that $A \otimes_R A' \neq 0$. Then $(A \otimes_R A')/(1 \otimes A')$ is a locally finite $G$-Galois extension, and $(A \otimes_{B} A')(A \otimes_{B} A') = A^{H} \otimes A'$ for any subgroup $H$ of $G$.

**Proposition 2.3.** Let $A/B$ be locally finite $G$-Galois, and $G = G^*$. If $H$ and $K$ are closed subgroups of $G$, then $A^{H \cap K} = A^{H} \cdot A^{K} = A^{K} \cdot A^{H}$. In particular, if $H \cap K = 1$ then $A = A^{H} \cdot A^{K} = A^{K} \cdot A^{H}$.

**Proof.** Let $A = \bigcup_{\mu \in \Lambda} A^{N_{\mu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. First we assume that $(G : K) < \infty$. Then, by Prop. 1.3, $A^{K} \subseteq A^{N_{\mu}}$ for some $\mu \in \Lambda$. Since $(A^{N_{\mu}})_{B}$ is finitely generated and $(A^{K})_{B}$ is a direct summand of $(A^{N_{\mu}})_{A^{K}}$ ([22; § 2. p. 118]), $(A^{K})_{B}$ is finitely generated. Therefore we may assume that $A^{K} \subseteq A^{N_{\mu}}$ for all $\mu \in \Lambda$. Then $N_{\mu} \subseteq K$ for $\lambda \in \Lambda$, and $A_{\mu}^{H} \cdot A^{K} = (\bigcup_{\mu \in \Lambda} A^{N_{\mu}})(\bigcup_{\mu \in \Lambda} A^{N_{\mu}}) = \bigcup_{\mu \in \Lambda} (A^{N_{\mu}} \cdot A^{N_{\mu}}) = \bigcup_{\lambda} A^{N_{\mu} \cap N_{\mu}', K}$ by [22; Prop. 5.3]. Since $N_{\mu}H \cap K = N_{\mu}(H \cap K)$ for all $\lambda$, we have $A^{H} \cdot A^{K} = \bigcup_{\lambda} A^{N_{\mu} \cap N_{\mu}'; K} = A^{H} \cdot A^{K}$. Next we return to general case. For any finite subset $F$ of $A^{K}$, we put $K_{F} = \{\sigma \in G; \sigma[F = 1_{F}]\}$. Then $(G : K_{F}) < \infty$, $A^{K_{F}} \subseteq A^{K}$, and $(A^{K_{F}})_{B}$ is finitely generated. Therefore $A^{K} = \bigcup_{F} A^{K_{F}}$ is a directed union, and
hence $A^H,A^K=A^H(\bigcup_{\lambda}A^{K_{\lambda}})=\bigcup_{\lambda}(A^H\cdot A^{K_{\lambda}})$ is also a directed union. Since each $A^H\cdot A^{K_{\lambda}} (=A^H\cap K_{\lambda})$ is a fixed subring of $A$, $A^H\cdot A^K$ is a fixed subring of $A$ (Prop. 1.1). Hence, as is easily seen, $A^H\cdot A^K=A^{H\cap K}$. Symmetrically we have $A^{H\cap K}=A^K\cdot A^H$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, $G=G^*$, and $H_i (i \in \Gamma)$ be closed subgroups of $G$. Then, $[\bigcup_i A^{H_i}]=A^{\cap H_i}$, where $[\bigcup_i A^{H_i}]$ means the subring of $A$ generated by $\bigcup_i A^{H_i}$.

**Proof.** Evidently $[\bigcup_i A^{H_i}]=\bigcup [A^{H_{i_1}} \cup \cdots \cup A^{H_{i_n}}]$, where $\{i_1, \ldots, i_n\}$ ranges over all finite subsets of $\Gamma$. By Prop. 2.3, $A^{H_{i_1}\cap \cdots \cap H_{i_n}}=A^{H_{i_1}} \cdots A^{H_{i_n}}=[A^{H_{i_1}} \cup \cdots \cup A^{H_{i_n}}]$, and therefore $[\bigcup_i A^{H_i}]$ is a directed union of fixed subrings of $A$. Hence, by Prop. 1.1, $[\bigcup_i A^{H_i}]$ is a fixed subring. Since $\{\sigma \in G ; \sigma [\bigcup_i A^{H_i}]=1\}=\bigcap_i H_i$, we obtain $[\bigcup_i A^{H_i}]=A^{\cap H_i}$, as desired.

**Proposition 2.4.** Let $A/B$ be locally finite $G$-Galois, $\mathfrak{N}$ a $G$-invariant proper ideal of $A$, $K$ a closed subgroup of $G$, and $N$ a closed normal subgroup of $G$ such that $(G:N)<\infty$. Then there hold the following:

1. $A^{K\cap N}/A^K$ is finite $K/(K\cap N)$-Galois. In particular, $A^N/B$ is finite $G/N$-Galois.

2. $(A^N+\mathfrak{A})/\mathfrak{A}/((B+\mathfrak{A})/\mathfrak{A})$ is finite $G/N$-Galois, and $((A^N+\mathfrak{A})/\mathfrak{A})^{H}= (A^{NH}+\mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

**Proof.** Let $A=\bigcup_{\mu \in A} A^{N_{\mu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) By Prop. 1.3, $A^N \subseteq A^{N_{\mu}}$ for some $\mu \in A$, and then $N_{\mu} \subseteq N$, $A^N=(A^{N_{\mu}})^{N\cap N_{\mu}}$. Therefore, by [22; Prop. 5.7], $A^N/B$ is finite $(G/N)\times (N/N_{\mu})$-Galois, or equivalently, finite $G/N$-Galois. Accordingly, $A^N/A^{NK}$ is finite $NK/N$-Galois, or equivalently, finite $K/(K\cap N)$-Galois. $K/(K\cap N)$ may be considered as a finite group of automorphisms of $A^{K\cap N}$, because $K\cap N \triangleleft K$. Then $A^{K\cap N}/A^K$ is finite $K/(K\cap N)$-Galois. (2) By (1), $A^N/B$ is finite $G/N$-Galois. If $t_{\theta/N}(c)=1$ for $c$ in $A^N$, then $t_{\theta/N}(c+\mathfrak{A})=1+\mathfrak{A}$. Then, by [22; Th. 5.6], $((A^N+\mathfrak{A})/\mathfrak{A})/(B+\mathfrak{A})/\mathfrak{A})$ is finite $G/N$-Galois, and $((A^N+\mathfrak{A})/\mathfrak{A})^{H}= (A^{NH}+\mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

Let $A/B$ be locally finite $G$-Galois, $K$ a closed subgroup of $G$, and $N$ a $G$-invariant proper ideal of $A$. Let $A=\bigcup_{\mu \in A} A^{N_{\mu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then $A^N=\bigcup_{\mu} (A^N \cap A^{N_{\mu}})=\bigcup_{\mu} A^{NN_{\mu}}$ is a directed union, and each $NN_{\mu}$ is a closed normal subgroup of $G$, because $(G:N)<\infty$. Then, by Prop. 2.4 (1), $A^{NN_{\mu}}/B$ is finite $G/NN_{\mu}$-Galois. Therefore there are elements $a_1, \ldots, a_m ; b_1, \ldots, b_m$ in $A^{NN_{\mu}}$ such that $\sum a_i \cdot \sigma(b_i)=\delta_{NN_{\mu},\sigma}$ for $\sigma$ in $G$. Hence $A^{NN_{\mu}}/B$ is finite $(G/N)/(NN_{\mu})/N$-Galois. Hence $A^N/B$ is locally finite $G/N$-Galois. Next we consider $K$. $A=\bigcup_{\mu} A^{N_{\mu}\cap K}$ is a directed union, and each $N_{\mu}\cap K$ is a fixed
normal subgroup of $K$ such that $(K : N_{i} \cap K) < \infty$. By Prop. 2.4 (1), each $A_{0}^{N_{i}/K}/A^{K}$ is finite $K/(N_{i} \cap K)$-Galois. Hence $A/A^{K}$ is locally finite $K$-Galois. Finally we consider $\mathfrak{A}$. Evidently, $A/\mathfrak{A} = \bigcup_{\lambda}((A^{N_{\lambda}}+\mathfrak{U})/\mathfrak{A})$. By Prop. 2.4 (2), $((A^{N_{\lambda}}+\mathfrak{U})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is finite $G/N_{i}$-Galois, and $((A^{N_{\lambda}}+\mathfrak{U})/\mathfrak{A})^{H} = (A^{N_{\lambda}H} + \mathfrak{U})/\mathfrak{A}$ for any subgroup $H$ of $G$. Therefore $(A/\mathfrak{A})^{H} = \bigcup_{\lambda}((A^{N_{\lambda}}+\mathfrak{U})/\mathfrak{A})^{H} = \bigcup_{\lambda}(A^{N_{\lambda}H} + \mathfrak{U})/\mathfrak{A}$ for any subgroup $H$ of $G$. Hence $((A+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is locally finite $G$-Galois. Thus we have proved the following

**Theorem 2.5.** Let $A/B$ be locally finite $G$-Galois, $N$ a closed normal subgroup of $G$, $K$ a closed subgroup of $G$, and $\mathfrak{A}$ a $G$-invariant proper ideal of $A$. Then there hold the following:

1. $A^{N}/B$ is locally finite $G/N$-Galois.
2. $A/A^{K}$ is locally finite $K$-Galois.
3. $((A+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is locally finite $G$-Galois, and $(A+\mathfrak{A})/\mathfrak{A}^{H} = (A^{H} + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, and $e$ a non-zero idempotent in $B \cap C$. Then $Ae/Be$ is locally finite $G$-Galois, and $(Ae)^{H} = A^{H}e$ for any subgroup $H$ of $G$.

Let $A/B$ be locally finite $G$-Galois, $n$ a positive integer, and $J$ the ring of rational integers. Then, $(J)n$ is a $J$-algebra, and $(J)n \otimes_{J} A \simeq (A)n \neq 0$. If we define $\sigma(a_{et}) = (\sigma(a_{et}))$ for any $\sigma$ in $G$ and any $(a_{et})$ in $(A)n$, then $(A)n/(B)n$ is locally finite $G$-Galois and $(A)m = (A^{H})m$ for any subgroup $H$ of $G$ (Th. 2.2). Now, let $\{e_{ik} ; i, k = 1, \cdots, m\}$ a system of matrix units contained in $B$, and $A = \bigcup_{\lambda}A^{N_{\lambda}}$ a representation of $A/B$. Put $A_{0} = V_{A} \{e_{ik}\}$ and $B_{0} = B \cap A_{0}$. Then, as is well known, $A = \sum_{i, k} A_{0}e_{ik}$, $A_{0} \simeq A_{0}e_{ik}$ by the right multiplication of $e_{ik}$. To be easily seen, $A^{N_{\lambda}} = \sum_{i, k} A_{0}^{N_{\lambda}} e_{ik}$, and $A^{N_{\lambda}H} = V_{A^{H}} \{e_{ik}\}$. There is an element $c$ in $A^{N_{\lambda}}$ such that $t_{\lambda;N_{\lambda}}(c) = 1$. Let $c = \sum_{i, k} x_{ik} e_{ik}$ ($x_{ik} \in A^{N_{\lambda}H}$). Then $1 = t_{\lambda;N_{\lambda}}(c) = \sum_{i, k} t_{\lambda;N_{\lambda}}(x_{ik}) e_{ik}$, and so $t_{\lambda;N_{\lambda}}(x_{ik}) = 1$. Thus, by [22; 5.8], $A_{0}^{N_{\lambda}}/B_{0}$ is finite $G/N_{i}$-Galois. Since $A_{0} = \bigcup_{\lambda}, A_{0}^{N_{\lambda}}$ is a directed union, $A_{0}/B_{0}$ is locally finite $G$-Galois. Therefore, by Th. 2.1, $A_{0} = A_{0} \otimes_{B_{0}} B$. Thus we have obtained the following

**Theorem 2.6.** Let $A/B$ be locally finite $G$-Galois.

1. For any positive integer $n$, $(A)n/(B)n$ is locally finite $G$-Galois, and $(A)n^{H} = (A^{H})n$ for any subgroup $H$ of $G$.
2. If $\{e_{ik} ; i, k = 1, \cdots, m\}$ is a system of matrix units contained in $B$, $A_{0} = V_{A} \{e_{ik}\}$, and $B_{0} = B \cap A_{0}$, then $A_{0}/B_{0}$ is locally finite $G$-Galois, and $A = A_{0} \otimes_{B_{0}} B$.

Let $A/B$ be finite $G$-Galois, and $M$ a $\Delta$-left module. For any subgroup $H$ of $G$, we put $M^{H} = \{m \in M ; u, m = m$ for all $\tau \in H\}$, which is an $A^{H}$.\[1\]
submodule of $M$. Evidently $M^H \supseteq A^H \cdot M^\sigma$, and the mapping $\varphi : A^H \otimes_B M^\sigma \rightarrow M^H$ defined by $a \otimes m \mapsto am$ ($a \in A$, $m \in M^\sigma$) is an $A^H$-left homomorphism. By assumption there are elements $a_1, \ldots, a_n; a_1^*, \ldots, a_n^*$ in $A$ such that $\sum_i a_i \cdot \sigma(a_i^*) = \delta_{i, \sigma}$ ($\sigma \in G$), $t_H(d) = 1$. Put $t_i = t_H(a_i)$. Then, $t_i \in A^H$ and $\sum_i t_i \cdot \sigma(a_i^*) = \delta_{i, \sigma}$ for $\sigma$ in $G$. If $m$ is in $M^\sigma$, then $A^H \cdot M^\sigma \ni t_i \sum_{\sigma \in G} u_{\sigma}(a_i^* dm) = \sum_i t_i \sum_{\sigma \in G} (a_i^* d) u_m = t_H(d) m = \sum_i t_i \sum_{\sigma \in G} (a_i^* da) \otimes m_0 = t_H(da) \otimes m_0 = a \otimes m_0$. From this fact, as is easily seen, $\varphi$ is 1–1. Thus we have $M^H = A^H \otimes_B M^\sigma$. Next we proceed to more general case.

Let $A/B$ be locally finite $G$-Galois, $A = \bigcup_{i=1}^r A^N_i$ its representation, and $M$ a $A$-left module. Let $G = \sigma_1 N_1 \cup \cdots \cup \sigma_r N_r$ be the coset decomposition of $G$, and let $A_i$ be the trivial crossed product of $A^N_i$ with $G/N_i$: $A_i = \bigoplus \sigma \in G/N_i A^N_i v_{\overline{\sigma}}$, $v_{\overline{\sigma}} v_{\overline{\sigma'}} = v_{\overline{\sigma \sigma'}}$, $v_{\overline{\sigma}} a = \sigma(a) v_{\overline{\sigma}}$ ($\sigma = \sigma_i N_i, \sigma_k = \sigma_k N \in G/N_i, a \in A^N_i$). For any $m$ in $M^N_i$ and any $\sum_i x_i v_{\overline{\sigma}} (m) = \sum_i x_i u_{\sigma_i} m$. Then, as is easily seen, $M^N_i$ is a $A_i$-left module. Since $A^N_i/B$ is finite $G/N_i$-Galois, we obtain that $M^N_i = A^N_i \otimes_B M^\sigma$ and $M^N_i H = A^N_i H \otimes_B M^\sigma$ for any subgroup $H$ of $G$. Since $A = \bigcup_i A^N_i$ is a directed union, so is $\bigcup_i M^N_i$. For any subgroup $H$ of $G$, $(\bigcup_i M^N_i)^H = \bigcup_i M^N_i H = \bigcup_i A^N_i H \cdot M^\sigma = A^H \cdot M^\sigma$, and $A^N_i H \otimes_B M^\sigma \simeq A^N_i H \cdot M^\sigma$ canonically. The last isomorphism may be considered as $A^N_i H \otimes_B M^\sigma \rightarrow A^H \otimes_B M^\sigma \rightarrow A^H \cdot M^\sigma$, and hence we see that $(\bigcup_i M^N_i)^H = A^H \otimes_B M^\sigma$. For any $m$ in $M$ we put $H_m = \{ \sigma \in G ; \sigma m = m \}$, which is a subgroup of $G$. Assume that $(G : H_m) < \infty$ and that $H_m$ is closed in $G$. Then, by Prop. 1.3, $H_m \supseteq N_\nu$ for some $\nu \in A$, so that $m \in M^N_\nu$. Conversely, if $m$ is in $\bigcup_\nu M^N_\nu$, then $m \in M^N_\nu$ for some $N_\nu$, so that $H_m \supseteq N_\nu$. Then, since $(G : N_\nu) < \infty$ and $N_\nu$ is closed in $G$, $(G : H_m) < \infty$ and $H_m$ is closed in $G$. Thus we have proved the following:

**Theorem 2.7.** Let $A/B$ be locally finite $G$-Galois, and $M$ a $A$-left module. Then there hold the following:

1. $A \cdot M^\sigma$ is a $A$-submodule of $M$, and $(A \cdot M^\sigma)^H = A^H \otimes_B M^\sigma$ for any subgroup $H$ of $G$.

2. $A \cdot M^\sigma = \{ m \in M ; (G : H_m) < \infty$ and $H_m$ is closed in $G \}$, where $H_m = \{ \sigma \in G ; \sigma m = m \}$.

**Corollary.** Let $A/B$ be finite $G$-Galois, and $M$ a $A$-left module. Then, $M^H = A^H \otimes_B M^\sigma$ for any subgroup $H$ of $G$, in particular, $M = A \otimes_B M^\sigma$ (cf. [4; Th. 1.3] and [22; Th. 5.1 (2)]).

**Proposition 2.8.** Let $A/B$ be finite $G$-Galois. Then the following are equivalent.

1. There are elements $a_1, \ldots, a_n; a_1^*, \ldots, a_n^*$ in $V_A(B)$ such that $\sum_i a_i \cdot \sigma(a_i^*) = \delta_{i, \sigma}$ ($\sigma \in G$) (cf. [22; Cor. to Th. 5.1]).
(ii) \( \text{If } \ A|B \text{ is outer } G\text{-Galois, and } \ A|B \text{.} \)

Proof. Since \( \text{(A)} \ (\sum_{\sigma} u_{\sigma}) A \cong \text{Hom}(A, B) \) by \( j \), it follows that \( \text{(A)} \ V_{A}(B) \cong \text{Hom}(B, B) \) and it is evident that \( V_{A}(B) \cong \text{Hom}(B, B) \) canonically. To be easily seen, \( A|B \) if and only if there are elements \( f_{1}, \cdots, f_{n} \) in \( \text{Hom}(B, B) \) such that \( \sum_{i} g_{i} f_{i}(x) = x \) for all \( x \) in \( A \). Consequently \( (ii) \) is equivalent to that \( u_{1} = \sum_{i} a_{i}(\sum_{\sigma} u_{\sigma}) a_{i}^{*} \) (= \( \sum_{i} a_{i} \sigma(a_{i}^{*})u_{\sigma} \)) for some \( a_{1}, \cdots, a_{n} \; a_{1}^{*}, \cdots, a_{n}^{*} \) in \( V_{A}(B) \). Hence \( (i) \) and \( (ii) \) are equivalent.

**Corollary.** Let \( G \) be finite. Then the following are equivalent.

(i) \( \ A|B \) is outer \( G\text{-Galois, and } \ A|B \).

(ii) There are elements \( a_{1}, \cdots, a_{n} \; a_{1}^{*}, \cdots, a_{n}^{*} \) in \( C \) such that \( \sum_{i} a_{i} \sigma(a_{i}^{*}) = \delta_{1,\sigma} \) \( (\sigma \in G) \).

Proof. This follows from [22; Prop. 6.4 and Prop. 6.5] and Prop. 2.8. \( A|B \) is called a completely outer \( G\text{-Galois extension if } G \) is finite and completely outer (cf. [22]).

**Theorem 2.9.** Let \( B' \) be a ring with identity, \( Z \) its center, and \( G' \) a finite group.

1. If \( A'\not|B' \) is completely outer \( G'\text{-Galois and } \ A'|B', \text{ then } A' = B' \otimes_{Z} C' \), where \( C' \) is the center of \( A' \), and \( C'|Z \) is \( G'\text{-Galois.} \)

2. If \( C'|Z \) is \( G'\text{-Galois and } C' \text{ is commutative, then } A' = B' \otimes_{Z} C' \) is a completely outer \( G'\text{-Galois extension over } B', \ A'|B', \text{ and } 1\otimes C' \text{ is the center of } A'. \)

Proof. (1) By [22; Prop. 6.4], \( A'|B' \) is outer \( G'\text{-Galois and } V_{A'}(B') = C' \), where \( C' \) is the center of \( A' \). Then, by Cor. to Prop. 2.8 and [22; Th. 5.1], \( C'|Z \) is \( G'\text{-Galois and } A' = B' \otimes_{Z} C' \). (2) By [22; Th. 5.2 and Prop. 6.5], \( A'|B(\otimes 1) \) is completely outer \( G'\text{-Galois. Since } Z \text{ is a direct summand of }_{Z} C', B' \cong B' \otimes 1 \text{ canonically, and } \ A'|B', \text{ because }_{Z} C'|Z. \text{ Then, by Cor. to Prop. 2.8, } C'|Z \text{ is } C'\text{-Galois, where } C'^{*} \text{ is the center of } A'. \text{ Since } C'^{*} \supseteq 1 \otimes C' \supseteq Z \text{ and } (1 \otimes C')/Z \text{ is } G'\text{-Galois ([22; Th. 5.1 or Th. 5.6])}, \text{ we have } C'^{*} = Z(1 \otimes C') = 1 \otimes C' ([22; Th. 5.1]).

**Lemma 2.10.** Let \( T \) be a ring, and \( U \) a subring of \( T \).

1. Let \( T|U \) be a separable extension. If a \( T\)-left module \( M \) is \( U\)-projective, then \( M \) is \( T\)-projective.

2. If \( \sigma T \otimes_{U} T|_{T} \) and \( \sigma U|_{U} M \) for a \( T\)-left module \( M \), then \( \sigma T|_{T} M \).

3. Let \( T_{0} \) be an intermediate ring of \( T|U \). If \( T \) is \( (U, T_{0})\)-projective and \( T_{0} \) is a \( T_{0}\)-\( T_{0}\)-direct summand of \( T \), then \( T_{0}|U \) is a separable extension.

Proof. (1) Since the mapping \( x \otimes y \to xy \text{ form } T \otimes_{U} T \text{ to } T \) splits as a \( T\)-\( T\)-homomorphism, the mapping \( x \otimes m \to xm \text{ from } T \otimes_{U} M \text{ to } M \) splits as
a $T$-left homomorphism. Since $_\rho M$ is projective, so is $T \otimes _\rho M$. Therefore $M$ is $T$-projective. (2) Since $_\rho U|_\rho M$, $T|_\rho T \otimes _\rho M$. Since $T \otimes _\rho T|_\rho T$, we have $T \otimes _\rho M|_\rho M$. Hence we have $T|_\rho M$. (3) Let $\phi$ be the canonical homomorphism from $T_0 \otimes _\rho T$ to $T$ defined by $\phi(t \otimes m) = t \cdot m$, and let $\phi$ be a $T_0 \otimes _\rho T$-homomorphism from $T$ to $T_0 \otimes _\rho T$ such that $\phi(x) = x$ for all $x$ in $T$. If $\phi(1) = \sum a_i \otimes b_i$ ($a_i \in T_0$, $b_i \in T$), then $\sum_i a_i b_i = 1$ and $\sum_i y a_i \otimes b_i = \sum_i a_i \otimes b_i y$ ($\in T_0 \otimes _\rho T$) for all $y$ in $T_0$. Let $\pi$ be a $T_0 \otimes _\rho T$-homomorphism from $T$ to $T_0$ such that $\pi|T_0 = 1_{T_0}$. Then, since $\sum_i y a_i \otimes b_i = \sum_i a_i \otimes b_i y$ ($\in T_0 \otimes _\rho T$) for all $y$ in $T_0$, we have $\sum_i a_i \cdot \pi(b_i) = 1$ and $\sum_i y a_i \otimes \pi(b_i) = \sum_i a_i \otimes \pi(b_i) y$ ($\in T_0 \otimes _\rho T_0$) for $y$ in $T_0$. Then the mapping $y \mapsto \sum_i a_i \otimes \pi(b_i) y$ from $T_0$ to $T_0 \otimes _\rho T_0$ is a $T_0 \otimes _\rho T$-homomorphism, and $\sum_i a_i \cdot \pi(b_i) y = y$. Hence $T_0/U$ is a separable extension.

Proposition 2.11. Let $A/B$ be finite $G$-Galois, and $Z$ the center of $B$. If $B$ is a separable $Z$-algebra and $Z \subseteq C$, then $V_A(B)/Z$ is finite $G$-Galois.

Proof. By [2; Prop. 1.5], $B \otimes _Z B^0$ is a central separable $Z$-algebra, where $B^0$ is the opposite ring of $B$. Since $_B A$ and $_B B$ are finitely generated and projective, so is $A$. Then, by Lemma 2.10 (1), $A \otimes _Z B$ is finitely generated and projective. By [2; Th. 2.1], $A \otimes _Z B \otimes _Z B^0 |_{A \otimes _Z B}$, and hence $V_B(B)/B \otimes B$. Then, by Prop. 2.8, $V_A(B)/Z$ is finite $G$-Galois (cf. S. 3).

Theorem 2.12. Let $G$ be finite, $\pi$ the group homomorphism defined by $\pi(a) = \sigma|C$ ($\sigma \in G$), $Z$ the center of $B$, and $Z_0 = C^0$, and assume that $A$ is indecomposable. Then the following statements are equivalent.

(i) $A/Z_0$ is separable, and $\pi$ is 1-1.
(ii) $V_A(B) = C$, $A/Z$ is separable, and $B \otimes _B B$.
(iii) $V_A(B) = C$, and both $B/Z$ and $C/Z$ are separable.
(iv) Both $B/Z$ and $C/Z_0$ are separable, and $\pi$ is 1-1.
(v) $V_A(B) = C$, $A/B$ is separable, $A$ is ($Z$, $B$)-projective, and $B \otimes _B A$.
(vi) $A = B \cdot C$, and $A/Z$ is separable.
(vii) $A \otimes _Z A^0 \cdot A^0 |_{A \otimes _Z A^0}$, and $\operatorname{Hom}(A \otimes _Z A^0, A \otimes _Z A^0) = 0$ for any $\sigma$ in $G$ such that $\sigma \neq 1$.

Proof. (i) $\Rightarrow$ (ii) By [2; Th. 2.3], $A/C$ and $C/Z_0$ are separable. Therefore, by [4; Th. 1.3], $C/Z_0$ is $G$-Galois. Then, by [22; Th. 5.1], $A = B \otimes _Z C$. Hence $V_A(B) = C$, and $Z = Z_0$. Since $Z$ is finitely generated and projective, $B \otimes _B B$. (ii) $\Rightarrow$ (iii) $V_A(B) = C$ implies $Z = Z_0$ ($\subseteq C$). By [22; Lemma 2.7], $A/C$ and $A/B$ are separable, so that $A/B$ is outer $G$-Galois ([22; Th. 1.5]). Then, by Prop. 2.8, $C/Z$ is $G$-Galois, so that $C/Z$ is separable. Since $A/C$ is separable, $B/Z$ is separable ([22; Cor. to Th. 5.1]). (iii) $\Rightarrow$ (iv) In this case, $Z = Z_0$. By [2; Th. 3.1], $A = B \cdot C$, whence $\pi$ is 1-1. (iv) $\Rightarrow$ (v) By
[4; Th. 1.3], $C/Z_0$ is $G$-Galois. Hence, by [22; Th. 5.1], $A/B$ is $G$-Galois, and $A=B\cdot C$. Then $A/B$ is separable, $V_A(B)=C$, and $Z=Z_0$. Since $Z$ is commutative, $\tau Z$ is a direct summand of $\tau C$ (S. 3), so that $\tau_0(c)=1$ for some $c$ in $C$. Then $B$ is a $B$-direct summand of $A$ (cf. [22; § 2. p. 118]). Since $B/Z$ is separable, $A$ is $(Z, B)$-projective ([22; Lemma 2.7]). $(v) \implies (vi)$ By Lemma 2.10 (3), $B/Z$ is separable. Then, by [2; Th. 3.1], $A=B\otimes_\tau C$. Since both $A/B$ and $B/Z$ are separable, $A/Z$ is separable ([22; Lemma 2.7]).

$(vi) \implies (i)$ As $A=B\cdot C$, $V_A(B)=C$, $Z=Z_0$, and $\pi$ is 1-1. Thus we know that $(i) \sim (vi)$ are equivalent. $(i) \implies (vii)$ In this case, $V_A(B)=C$, $Z=Z_0$, and $B/Z$ is separable. Then, by [2; Th. 2.1], $\B_{\otimes_\tau B}\otimes_\tau B|_{\B\otimes_\tau B}$. Therefore $\B\otimes_\tau B|_{\B\otimes_\tau B}$. Hence $\B\otimes_\tau B|_{\B\otimes_\tau B}$. By [22; Prop. 1.3], $\B\otimes_\tau B|_{\B\otimes_\tau B}$. The second assertion follows from [22; Prop. 6.3]. $(vii) \implies (i)$ By assumption, $\End(\otimes_\tau B|_{\B\otimes_\tau B})\simeq \oplus_{\sigma\in G} \End(\otimes_\tau B|_{\B\otimes_\tau B})$ (external direct sum as rings). To be easily seen, $\End(\otimes_\tau B|_{\B\otimes_\tau B})\simeq C$, which is commutative. Hence $\End(\otimes_\tau B|_{\B\otimes_\tau B})$ is a commutative ring.

Proposition 2.13. Let $A/B$ be locally finite $G$-Galois, and $b$ an element of $B$ which is not a right zero divisor of $B$. Then $b$ is not a right zero divisor of $A$.

Proof. Let $a$ be an element of $A$ such that $ab=0$. Then $Aab=0$, and $\sigma(Aa)b=0$ for all $\sigma$ in $G$. Hence, $(\sum_\sigma \sigma(Aa)) \cap B=0$. Then, by assumption, $\sum_\sigma \sigma(Aa) \cap B=0$. Then, by Th. 2.1 (3), $\sum_\sigma \sigma(Aa)=A(\sum_\sigma \sigma(Aa)) \cap B=0$. Hence $a=0$.

Let $A/B$ be locally finite $G$-Galois, and $S \ni 1$ a $G$-invariant multiplicative system of regular elements in $A$ such that a left quotient ring $\overline{A}$ of $A$ with respect to $S$ exists. Then $G$ may be regarded as a group of automorphisms of $\overline{A}$. To be easily seen, $\{\sigma(x); \sigma \in G\}$ is finite for any $x$ in $\overline{A}$. Then, by Th. 2.1, $\overline{A}/\overline{B}$ is locally finite $G$-Galois and $\overline{A}=\overline{B}\otimes_\tau A=\overline{A}\otimes_\tau \overline{B}$, where $\overline{B}=\overline{A}^G$. To be easily seen, any element in $B \cap S$ is a unit of $\overline{B}$. For $b$ in $\overline{B}$, we put
$\mathfrak{L} = \{x \in A \mid xb \in A\}$, which is a $\mathfrak{J}$-left submodule of $A$. Then $(\mathfrak{S} \cap B)b \subseteq B$. If $\mathfrak{S} \cap B \cap S \neq 0$, then $sb \in B$ for some $s$ in $B \cap S$. Therefore, if we assume that $\mathfrak{J}(s) \cap B \cap S \neq 0$ for all $s \in S$, then $\overline{B}$ is a left quotient ring of $B$ with respect to $B \cap S$. Thus we obtain the following

**Theorem 2.14.** Let $A/B$ be locally finite $G$-Galois, and $S \ni 1$ a $G$-invariant multiplicative system of regular elements of $A$ such that a left quotient ring $\overline{A}$ of $A$ with respect to $S$ exists. Further, assume that $\mathfrak{J}(s) \cap B \cap S \neq 0$ for all $s \in S$. Then there hold the following:

1. $\overline{A}/\overline{B}$ is locally finite $G$-Galois and $\overline{A} = \overline{B} \otimes_B A = A \otimes_B \overline{B}$, where $\overline{B} = \overline{A}_0$.
2. $\overline{A}$ is a left quotient ring of $A$ with respect to $B \cap S$. $\overline{B}$ is a left quotient ring of $B$ with respect to $B \cap S$.

**Remark.** Let $A/B$ be locally finite $G$-Galois, and $S$ a $G$-invariant multiplicative system of regular elements in $A$ such that $S \subseteq C$ and $S \ni 1$. Then $S$ satisfies the conditions in Th. 2.14. To see this, we put $H = \{\sigma \in G; \sigma(s) = s\}$ for $s \in S$. If $G = \sigma_1 H \cup \cdots \cup \sigma_r H$ is the left coset decomposition of $G$, then $\mathfrak{L} = \mathfrak{L} \cap B \cap S$ for $s \in S$. If $G = \sigma_1 H \cup \cdots \cup \sigma_r H$ is the left coset decomposition of $G$, then

A non-zero ring $T$ with 1 is called a left Goldie ring if $T$ satisfies the following conditions: (1) $T$ is a semi-prime ring. (2) Any independent set of non-zero left ideals is finite (i.e., $T$ is finite dimensional). (3) $T$ satisfies the ascending chain condition for annihilator left ideals.

A left Goldie ring has a complete left quotient ring which is a semi-simple ring with minimum condition for left ideals, and conversely (Goldie [17]). (Cf. [7])

**Theorem 2.15.** Let $A/B$ be locally finite $G$-Galois, $A$ a left Goldie ring, $\overline{A}$ a complete left quotient ring of $A$, and $B$ a semi-prime ring. Then there hold the following:

1. $\overline{A}/\overline{B}$ is locally finite $G$-Galois, where $\overline{B} = \overline{A}_0$.
2. $B$ is a left Goldie ring, and $\overline{B}$ is a complete left quotient ring of $B$.

**Proof.** Let $S$ be the set of all regular elements of $A$. First we shall prove that $B$ is a left Goldie ring. Since $\mathfrak{A} A$ is finite dimensional, $\mathfrak{A} A$ is finite dimensional. Then, by Th. 2.1 (3), $\mathfrak{A} B$ is finite dimensional. Let $I \subseteq I'$ be left ideals of $B$. Then $l_A(r_B(I)) \subseteq l_A(r_B(I'))$, where $r_B(I) = \{y \in B; l_B = 0\}$ and $l_A(r_B(I)) = \{x \in A; x \cdot r_B(I) = 0\}$. From this fact, $B$ satisfies the ascending chain condition for annihilator left ideals of $B$. Hence $B$ is a left Goldie ring. By Prop. 2.13, $S \cap B$ is the set of all regular elements of $B$. For any $s$ in $S$, $\mathfrak{A} A s$ is essential in $\mathfrak{A} A$, so that $\mathfrak{A} s$ is essential in $\mathfrak{A} A$. Then, by Th. 2.1 (3), $\mathfrak{A} s(\mathfrak{A} s \cap B)$ is essential in $\mathfrak{A} B$, so that $\mathfrak{A} s \cap B$ contains a regular element.
of $B$ ([17; Th. (3.9)]). Hence $A(s) \cap B \cap S \neq 0$ for any $s$ in $S$. Thus the present theorem follows from Th. 2.14.

**Remark.** In the following cases, the condition that $B$ is semi-prime is superfluous.

(1) $G$ is finite and completely outer (cf. [22; p. 132]).

(2) $B$ is contained in the center of $A$.

Let $T$ be a ring. If $T$-left modules $M$ and $N$ have no non-zero isomorphic subquotients, we say that $\tau M$ and $\tau N$ are unrelated (cf. [22]).

**Lemma 2.16.** Let $T$ be a ring, and let $M$ and $N$ be $T$-left modules, and $W$ a $T$-submodule of $M$. If $\tau(M/W)$ and $\tau N$ are unrelated, and $\tau W$ and $\tau N$ are unrelated, then $\tau M$ and $\tau N$ are unrelated.

**Proof.** Assume that there are isomorphic subquotients $X/X_0$ and $Y/Y_0$ of $\tau M$ and $\tau N$, respectively. Then, as is easily seen, $X + W \supset X_0 + W$ or $X \cap W \supset X_0 \cap W$. If $X + W \supset X_0 + W$, then $Y/X_0 \simeq (X + W)/(X_0 + W) \neq 0$, a contradiction. If $X \cap W \supset X_0 \cap W$, then $(X \cap W)/(X_0 \cap W) \simeq (X_0 + (X \cap W))/X_0 \simeq Y/Y_0$, which is also a contradiction.

**Proposition 2.17.** Let $\sigma, \tau$ be in $G$, and assume that $\tau Au_{\sigma}$ and $\tau Au_{\tau}$ are unrelated. Then, for any finite subset $\{x_1, \cdots, x_r; y_1, \cdots, y_s\}$ of $A$, there are elements $a_k, b_k \ (k=1, \cdots, t)$ in $A$ such that $\sum_k a_k x_i \cdot \sigma(b_k) = x_i$ and $\sum_k a_k y_h \cdot \tau(b_k) = 0$ for all $x_i, y_h$.

**Proof.** By Lemma 2.16, $\tau (Au_{\sigma})_A$ and $\tau (Au_{\tau})_A$ are unrelated. Then, since $A(x_1 u_{\sigma}, \cdots, x_r u_{\sigma}, y_1 u_{\tau}, \cdots, y_s u_{\tau})A$ is an $A$-$A$-submodule of $\tau (Au_{\sigma})_A \oplus (Au_{\tau})_A$, the set $\{(x_1 u_{\sigma}, \cdots, x_r u_{\sigma}, 0, \cdots, 0) \in A(x_1 u_{\sigma}, \cdots, x_r u_{\sigma}, y_1 u_{\tau}, \cdots, y_s u_{\tau})A \ (cf. [22; Prop. 6.1])$. Therefore there are elements $a_k, b_k \ (k=1, \cdots, t)$ in $A$ such that $\sum_k a_k(x_1 u_{\sigma}, \cdots, x_r u_{\sigma}, y_1 u_{\tau}, \cdots, y_s u_{\tau})b_k = (x_1 u_{\sigma}, \cdots, x_r u_{\sigma}, 0, \cdots, 0)$. Then, $\sum_k a_k x_i \cdot \sigma(b_k) = x_i$ and $\sum_k a_k y_h \cdot \tau(b_k) = 0$ for all $x_i, y_h$.

Combining Prop. 2.17 with [22; Prop. 6.11] we can easily see the following

**Proposition 2.18.** Let $A$ and $A'$ be $R$-algebras with $A \otimes_R A' \neq 0$, and let $G$ and $G'$ be completely outer finite groups of $R$-automorphisms of $A$ and $A'$, respectively. Then, $G \times G'$ is completely outer as an automorphism group of $A \otimes_R A'$.

§ 3.

**Proposition 3.1.** Let $A/B$ be locally finite $G$-Galois, and $X$ a $\Delta$-left submodule of $A$. Then $X = A(X \cap B)$.

**Proof.** This follows from Th. 2.1 (3).

**Proposition 3.2.** Let $A/B$ be locally finite $G$-Galois, $\{\mathbb{P}\}$ the set of
all maximal ideals of $A$, and $\{\mathfrak{p}\}$ the set of all maximal ideals of $B$. Then the following are equivalent:

(i) $\mathfrak{P} \to \mathfrak{P} \cap B$ is a mapping from $\{\mathfrak{P}\}$ onto $\{\mathfrak{p}\}$.

(ii) $A\mathfrak{p}A \neq A$ for all $\mathfrak{p} \in \{\mathfrak{p}\}$, and $\cap_{\mathfrak{p} \in \mathfrak{P}} \sigma(\mathfrak{P})$ is $\Delta$-$A$-maximal for all $\mathfrak{P} \in \{\mathfrak{P}\}$.

If (i) holds, then the following are true:

1. $\mathfrak{p}A = A\mathfrak{p} \neq A$ for any $\mathfrak{p} \in \{\mathfrak{p}\}$.

2. $\{\cap_{\sigma}(\mathfrak{P}); \mathfrak{P} \in \{\mathfrak{P}\}\}$ is the set of all maximal $\Delta$-$A$-submodules of $A$.

3. $\mathfrak{R}(\Delta_{A}) = \mathfrak{R}(\Delta_{A}) = \mathfrak{R}(\Delta_{B}) = A = \mathfrak{R}(\Delta_{B})$, and $\mathfrak{R}(\Delta_{A}) \cap B = \mathfrak{R}(\Delta_{B})$.

4. $B$ is $B$-$B$-completely reducible if and only if $\cap_{i} \cap_{\sigma}(\mathfrak{P}) = 0$ for some $\mathfrak{P}_{i} (i = 1, \ldots, n)$ in $\{\mathfrak{P}\}$.

Proof. (i) $\implies$ (ii) If $\mathfrak{P}$ is in $\{\mathfrak{P}\}$, then $\mathfrak{P} \cap B = \sigma(\mathfrak{P}) \cap B$ for any $\sigma$ in $G$, and so $\mathfrak{P} \cap B = (\cap_{\sigma}(\mathfrak{P})) \cap B$. By Prop. 3.1, $A((\cap_{\sigma}(\mathfrak{P})) \cap B) = \cap_{\sigma}(\mathfrak{P}) = (\cap_{\sigma}(\mathfrak{P}) \cap B)A$. Hence $A\mathfrak{p} = A\mathfrak{p} \neq A$ for all $\mathfrak{p}$ in $\{\mathfrak{p}\}$. Let $X$ be a $\Delta$-$A$-submodule of $A$ with $A X = X \cap_{\sigma}(\mathfrak{P})$. Then $B X \cap B Y = (\cap_{\sigma}(\mathfrak{P})) \cap B = \mathfrak{P} \cap B$, and so $X \cap B = (\cap_{\sigma}(\mathfrak{P})) \cap B$. Then, by Prop. 3.1, $X = \cap_{\sigma}(\mathfrak{P})$, so $\cap_{\sigma}(\mathfrak{P})$ is $\Delta$-$A$-maximal. Let $Y$ be a maximal $\Delta$-$A$-submodule of $A$. Take a maximal ideal $\mathfrak{P}_{1}$ of $A$ with $\mathfrak{P}_{1} \supseteq Y$. Then $\cap_{\sigma}(\mathfrak{P}_{1}) \supseteq Y$, and so $\cap_{\sigma}(\mathfrak{P}_{1}) = Y$. Thus we obtain (2). Therefore $\mathfrak{R}(\Delta_{A}) = \mathfrak{R}(\Delta_{A})$. Since $\mathfrak{R}(\Delta_{A}) \cap B = \mathfrak{R}(\Delta_{B})$, we have $\mathfrak{R}(\Delta_{A}) = A \cdot \mathfrak{R}(\Delta_{B}) = \mathfrak{R}(\Delta_{B})$ (Prop. 3.1). B is $B$-$B$-completely reducible if and only if $\cap_{i} \mathfrak{p}_{i} = 0$ for some $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{n}$ in $\{\mathfrak{P}\}$. Thus we obtain (4) (cf. Prop. 3.1). (ii) $\implies$ (i) Let $\mathfrak{p} \in \{\mathfrak{p}\}$. Then, as $A\mathfrak{p}A \neq A$, $\mathfrak{p} \not\subseteq \mathfrak{P}$ for some $\mathfrak{P} \in \{\mathfrak{P}\}$, and so $\mathfrak{p} = \mathfrak{P} \cap B$ by the maximality of $\mathfrak{p}$. Let $\mathfrak{Q}$ be in $\{\mathfrak{P}\}$. Then $\mathfrak{Q} \supseteq \mathfrak{Q} \cap B$ for some $\mathfrak{q} \in \{\mathfrak{p}\}$. There is a $\Sigma \in \{\mathfrak{P}\}$ with $\mathfrak{Q} \cap B = \mathfrak{q}$. Then $(\cap_{\sigma}(\Sigma)) \cap B = \Sigma \cap B \supseteq \Sigma \cap B = (\cap_{\sigma}(\Sigma)) \cap B$, and therefore $\cap_{\sigma}(\Sigma) \supseteq \cap_{\sigma}(\mathfrak{Q})$ by Prop. 3.1. By assumption, $\cap_{\sigma}(\Sigma) = \cap_{\sigma}(\mathfrak{P})$. Hence $\mathfrak{q} = \mathfrak{Q} \cap B = \mathfrak{Q} \cap B$. This completes the proof.

Concerning Prop. 3.2, we state the following

**Lemma 3.3.** Let $\mathfrak{P}$ be a maximal ideal of $A$ such that $\cap_{\sigma}(\mathfrak{P}) = \cap_{\sigma}(\mathfrak{P})$ for some $\sigma_{1}, \ldots, \sigma_{n}$ in $G$. Then $\cap_{\sigma}(\mathfrak{P})$ is $\Delta$-$A$-maximal, and $\{\cap_{\sigma}(\mathfrak{P}); \cap_{\sigma}(\mathfrak{P}); i = 1, \ldots, n\}$ is the set of all maximal ideals containing $\cap_{\sigma}(\mathfrak{P})$.

Proof. Let $\mathfrak{Q}$ be a maximal ideal of $A$ with $\mathfrak{Q} \supseteq \cap_{\sigma}(\mathfrak{P})$. If $\mathfrak{Q} \neq \cap_{\sigma}(\mathfrak{P})$ for all $i$, then $\mathfrak{Q} + \cap_{\sigma}(\mathfrak{P}) = A$ for all $i$. Then we have a contradiction $A = \mathfrak{Q} + \cap_{\sigma}(\mathfrak{P}) = \mathfrak{Q} + \cap_{\sigma}(\mathfrak{P})$.

Remark. In the following cases, the assumption in Lemma 3.3 holds.

1. $G$ is finite.
2. The ring $A/\mathfrak{R}(\Delta_{A})$ satisfies the descending chain condition for ideals.
Proposition 3.4.

(1) Let $A/B$ be locally finite outer $G$-Galois, and $B$ $B$-B-completely reducible. Assume that, for any maximal ideal $\mathfrak{P}$ of $A$, there are elements $\sigma_1, \ldots, \sigma_n$ in $G$ such that $\cap_i \sigma_i(\mathfrak{P}) = \cap_e \sigma(\mathfrak{P})$. Then $A$ is $A$-$A$-completely reducible.

(2) Let $G$ be finite and completely outer, and $B \mid A_B$. Then $A$ is $A$-$A$-completely reducible if and only if $B$ is $B$-B-completely reducible. If there is a maximal ideal $\mathfrak{P}$ of $A$ such that $\cap_e \sigma(\mathfrak{P}) = 0$, then $B$ is $B$-B-minimal, and conversely.

Proof. (1) Any maximal ideal $\mathfrak{p}$ of $B$ is written as $\mathfrak{p} = Be$ with a central idempotent $e$ of $B$. Then, by assumption, $(1 \neq) \ e \in V_A(B) = C$. Therefore, $A\mathfrak{p} = Ae = eA = \mathfrak{p}A \neq A$. Thus, by Prop. 3.2 and Lemma 3.3, $A$ is $A$-$A$-completely reducible. (2) In this case, $\alpha A = A\alpha \neq A$ for any proper ideal $\alpha$ of $B$ (cf. [22; p. 132]). Then, by Prop. 3.2 and Lemma 3.3, the first assertion is evident (cf. [22; Prop. 6.4]). For any $\mathfrak{P}$ in $\{\mathfrak{P}\}$, $(\cap_e \sigma(\mathfrak{P}) \cap B) = \mathfrak{P} \cap B = 0$ if and only if $\cap_e \sigma(\mathfrak{P}) = 0$ (Prop. 3.1). Thus we know the second assertion.

Theorem 3.5. Let $A/B$ be finite $G$-Galois, $B$ a semi-primary ring, and $A\mathfrak{p}A \neq A$ for any maximal ideal $\mathfrak{p}$ of $B$. Then $A_B \simeq A_B^g$, that is, $A$ has a normal basis. (Cf. [13; Th. 1]).

Proof. By [22; Th. 1.7], it suffices to prove that $A_B$ is free. Let $g = (G : 1)$. (1) First we assume that $R(B) = 0$. Then $B$ is a direct sum of simple rings: $B = a_1 + \cdots + a_n$. Let $1 = \sum_i e_i$, $e_i \in a_i$. Then $a_i = Be_i = e_i B$ and $e_i^2 = e_i$. By assumption we have $(1 - e_i)A = A(1 - e_i)$ (Prop. 3.2 and Lemma 3.3), so that $e_i$ is a central idempotent of $A$ contained in $B$. Then each $Ae_i/Be_i$ is $G$-Galois ([22; Cor. to Th. 5.6]). Since $Be_i$ is a simple ring, $B$ has a normal basis (cf. [7]). Hence $Ae_i$ has a normal basis, so that $Ae_i \simeq B(\mathfrak{b} \mathfrak{b})$ for all $i$ ([22; Th. 1.7]). Hence $A_B \simeq A_B^g$. (2) Next we proceed to general case. Since $A$ and $B$ are semi-primary ([22; Prop. 7.3]), $R(A_A) = R(A)$ and $R(B_B) = R(B)$. Then, by Prop. 3.2 and Lemma 3.3, $R(A) = R(B_A) = A \cdot R(B)$ and $R(A) \cap B = R(B)$. By [22; Th. 5.6], $(A/R(A))/(B/R(B)) \cong R(B)$ is $G$-Galois, and satisfies the same conditions as $A/B$, because $(B + R(A))/R(A) \simeq B/(R(A) \cap B) = B/R(B)$ canonically. By (1), we have $A_B \\ R(A) \simeq B_B/R(B)$ for all $i$. Since $R(B) \simeq R(B)A$ and $A_B$ is finitely generated and projective, we have $A_B \simeq B_B^g$. This completes the proof.

Corollary. Let $A/B$ be finite $G$-Galois, $B$ a semi-primary ring, and $Z$ the center of $B$. Assume that $Z \subseteq C$ and that $B$ is a central separable $Z$-algebra. Then $A$ has a normal basis.

Proof. In this case, any proper ideal of $B$ is written as $\alpha A$ with an ideal
a of $Z$ (cf. [2]). Then, as $Z \subseteq C$, $(aB)A = aA = Aa = A(Ba) \neq A$ ([22; Lemma 7.1]).

Let $A/B$ be finite $G$-Galois, $B \subseteq C$, and $g = (G:1)$. For any prime ideal $\mathfrak{p}$ of $B$, we denote by $B_\mathfrak{p}$ the quotient extension of $B$ with respect to $\mathfrak{p}$. Then $B_\mathfrak{p}$ is a $B$-algebra, canonically. By [22; Cor. to Th. 5.2], $(B_\mathfrak{p} \otimes_B A)/B_\mathfrak{p}$ is $G$-Galois. Since $B_\mathfrak{p}$ is a local ring, $b_{\mathfrak{p}} B_\mathfrak{p} \otimes_B A \simeq_{b_{\mathfrak{p}}}(B_\mathfrak{p})^g$ (Cor. to Th. 3.5). We denote by $K_\mathfrak{p}$ the quotient field of $B/\mathfrak{p}$. Then we have $\mathfrak{k}_\mathfrak{p} K_\mathfrak{p} \otimes_B A \simeq_{\mathfrak{k}_\mathfrak{p}}(K_\mathfrak{p})^g$ similarly. Thus we obtain the following

**Proposition 3.6.** Let $A/B$ be finite $G$-Galois, $B \subseteq C$, and $g = (G:1)$. Then, $b_{\mathfrak{p}} B_\mathfrak{p} \otimes_B A \simeq_{b_{\mathfrak{p}}}(B_\mathfrak{p})^g$ and $\mathfrak{k}_\mathfrak{p} K_\mathfrak{p} \otimes_B A \simeq_{\mathfrak{k}_\mathfrak{p}}(K_\mathfrak{p})^g$ for any prime ideal $\mathfrak{p}$ of $B$, where $B_\mathfrak{p}$ is the quotient extension of $B$ with respect to $\mathfrak{p}$ and $K_\mathfrak{p}$ is the quotient field of $B/\mathfrak{p}$.

The following lemma is of some interest.

**Lemma 3.7.** Let $R \supseteq S$ be rings, $R_s$ is finitely generated and projective, and $sS$ is a direct summand of $sR$. If $rS$ is injective, then $sS$ is injective.

**Proof.** Let I be any left ideal of $S$, and $f$ any $S$-left homomorphism from $I$ to $sR$. Since $R_s$ is finitely generated and projective, we have $R|I = R \otimes S$. Therefore $f$ can be extended to an $R$-left homomorphism from $R|I$ to $R$, canonically. Then, by assumption, there is an element $a$ in $R$ such that $r \cdot (s)f = rsa$ for $r$ in $R$ and $s$ in $I$, so that $(s)f = sa$ for all $s$ in $I$. Therefore, as is well known, $sR$ is injective. Since $sS$ is a direct summand of $sR$, $sS$ is injective.

**Lemma 3.8.** $\Re(A) \cap B \subseteq \Re(B)$.

**Proof.** Let $b$ be in $\Re(R) \cap B$. Then $1 - b$ has an inverse in $A$. Since $B = A^g$, $1 - b$ has an inverse in $B$. Hence $\Re(A) \cap B$ is a quasi-regular ideal of $B$, that is, $\Re(A) \cap B \subseteq \Re(B)$.

**Proposition 3.9.** Let $G$ be finite. If there is an element $c$ in $A$ such that $1 - t_\alpha(c) \in \Re(A)$, then there is an element $d$ in $A$ such that $t_\alpha(d) = 1$.

**Proof.** By Lemma 3.8, we have $1 - t_\alpha(c) \in \Re(A) \cap B \subseteq \Re(B)$, so that $t_\alpha(A) + \Re(B) = B$. Since $t_\alpha(A)$ is an ideal of $B$, we have $t_\alpha(A) = B$. Hence $t_\alpha(d) = 1$ for some $d$ in $A$.

**Theorem 3.10.** Let $A/B$ be $G$-Galois, $A$ a commutative ring, $H$ a subgroup of $G$, and $A'$ a $B$-algebra. Then, $A' \otimes_B A''$ is a direct sum of minimal ideals if and only if $A'$ is a direct sum of minimal ideals (cf. [7; p. 178. Th. 2]).

**Proof.** In this case, $(A' \otimes_B A)/A'$ is finite $G$-Galois, $G$ is completely outer as an automorphism group of $A' \otimes_B A$, and $(A' \otimes_B A)'' = A' \otimes_B A''$ (cf. [22; Th.
5.2 and Prop. 6.5]). Thus the present theorem is an easy consequence from Prop. 3.4 (2).

Concerning [22; Th. 6.9], we note the following

Lemma 3.11. Let $A/C$ be separable, and $e$ an idempotent of $A$ such that $eA \subseteq Ae$. Then $e$ is a central idempotent of $A$.

Proof. Since $A/\mathfrak{M}(A)$ is a semi-prime ring, we have $(eA+\mathfrak{M}(A))/\mathfrak{M}(A)\cong (Ae+\mathfrak{M}(A))/\mathfrak{M}(A)$, that is, $eA+\mathfrak{M}(A)=Ae+\mathfrak{M}(A)$, and so $Ae=eA+(Ae \cap \mathfrak{M}(A))=2A+\mathfrak{M}(A)e$. Since $A$ is a central separable C-algebra, $\mathfrak{M}(A)\subseteq \mathfrak{M}(C)A$ by [2; Cor. 3.2]. Since $\mathfrak{M}(A)<\mathfrak{M}(A)\subseteq \mathfrak{M}(C)A$, we have $\mathfrak{M}(A)=\mathfrak{M}(C)A$, and $Ae=eA+\mathfrak{M}(C)e$. Hence $Ae=eA$, because $eA$ is finitely generated. Consequently, $e$ is a central idempotent of $A$.

Proposition 3.12. Let $A/B$ be locally finite $G$-Galois, and assume that there is a representation $A=\bigcup_{\lambda}A^{N_{\lambda}}$ of $A/B$ such that each $\mathfrak{M}(B)A^{N_{\lambda}}$ is an ideal of $A^{N_{\lambda}}$. Then $\mathfrak{M}(A)=\mathfrak{M}(B)A=A \cdot \mathfrak{M}(B)$, and $\mathfrak{M}(A)\cap B=\mathfrak{M}(B)$.

Proof. Let $\mathfrak{N}$ be a right ideal of $A$ such that $\mathfrak{M}(B)A+\mathfrak{N}=A$. Then $\mathfrak{M}(B)A^{N_{\lambda}}+(\mathfrak{N} \cap A^{N_{\lambda}}) \ni 1$ for some $\lambda$ in $A$, so that $\mathfrak{M}(B)A^{N_{\lambda}}+(\mathfrak{N} \cap A^{N_{\lambda}})=A^{N_{\lambda}}$. Since $\mathfrak{M}(B)A^{N_{\lambda}} \subseteq \mathfrak{M}(A^{N_{\lambda}})$, we have $\mathfrak{N} \cap A^{N_{\lambda}}=A^{N_{\lambda}}$, and hence $\mathfrak{N}=A$. Thus we know that $\mathfrak{M}(B)A \subseteq \mathfrak{M}(A)$. Combining this with Lemma 3.8, we have $\mathfrak{M}(A)\cap B=\mathfrak{M}(B)$. Hence $\mathfrak{M}(A)=\mathfrak{M}(B)A=A \cdot \mathfrak{M}(B)$ (Prop. 3.1).

Theorem 3.13. Let $A/B$ be locally finite $G$-Galois, $B \subseteq C$, and $A'$ a $B$-algebra such that $A'\simeq A' \otimes 1$ ($\subseteq A' \otimes B$) canonically.

1. $\mathfrak{M}(A' \otimes B A)=\mathfrak{M}(A' \otimes A)$, and $\mathfrak{M}(A' \otimes A) \cap (A' \otimes 1)=\mathfrak{M}(A') \otimes 1$.

2. If $A$ is commutative, then $\mathfrak{M}(A' \otimes A^H)=\mathfrak{M}(A') \otimes A^H$ for any subgroup $H$ of $G$.

Proof. Let $A=\bigcup_{\lambda}A^{N_{\lambda}}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then $(A' \otimes B A)/(A' \otimes 1)$ is a locally finite $G$-Galois extension with representation $A' \otimes B A=\bigcup_{\lambda}A' \otimes A^{N_{\lambda}}$, where $A' \otimes A^{N_{\lambda}}=(A' \otimes B A)^{N_{\lambda}}$ is a finite $G/N_{\lambda}$-Galois extension over $A' \otimes 1$. (1) This will be easily seen by Prop. 3.12. (2) We may assume that $H$ is closed in $G$. Then each $A^H/A^{N_{\lambda}} \subseteq A^H$ is finite $H/(H \cap N_{\lambda})$-Galois, and $H/(H \cap N_{\lambda})$ is completely outer as an automorphism group of $A^H/N_{\lambda}$ ([22; Th. 6.6]). Then $H/(H \cap N_{\lambda})$ is completely outer as an automorphism group $A' \otimes B A^{H/N_{\lambda}}$ (Prop. 2.18), and so $H/(H \cap N_{\lambda})$ is completely outer as an automorphism group of $A' \otimes A^{H/N_{\lambda}}$ (Prop. 6.11). Now, $(A' \otimes B A)/(A' \otimes A^H)$ is a locally finite $H$-Galois extension with representation $A' \otimes B A=\bigcup_{\lambda}A' \otimes A^{H/N_{\lambda}}$, where $A' \otimes A^{H/N_{\lambda}}=(A' \otimes B A)^{H/N_{\lambda}}$ is a finite $H/(H \cap N_{\lambda})$-Galois extension over $A' \otimes A^H$. Then, by [22; Th. 7.10] and Prop. 3.12, $\mathfrak{M}(A' \otimes B A)=\mathfrak{M}(A' \otimes A^H)(A' \otimes B A)$. On the other hand,
\( \Re(A' \otimes_B A) = \Re(A') \otimes A = (\Re(A') \otimes A^H)(A' \otimes_B A) \). Hence \( \Re(A' \otimes A^B) = \Re(A') \otimes A^H \), as desired (cf. [22; Lemma 7.1]).

References

([1]–[14] are found in [22] below.)


Department of Mathematics, Hokkaido University

(Received June 10, 1967)