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LOCALLY FINITE OUTER GALOIS THEORY

By

Yôichi MIYASHITA

Introduction.

This paper is the continuation of the preceding paper [22]. In §1 and §2, locally finite (outer) Galois extensions are treated. The main results are parallel to those of the finite case. In these studies, Nagahara [12] is our guide. Further several results for finite Galois extensions are added (Th. 1.18). In §3, we give a normal basis theorem for a finite Galois extension.

§1. As to notations and terminologies we follow [22]. Let $A$ be a ring with $1$ ($\neq 0$), $C$ the center of $A$, $G$ a (finite or infinite) group of automorphisms of $A$, $B=A^\sigma=\{x\in A; \sigma(x)=x$ for all $\sigma$ in $G\}$, and $\hat{G}$ the group of all $B$-automorphisms of $A$. $\hat{G}$ is then a topological group in finite topology (cf. Jacobson [7]). We denote the closure of $G$ in $\hat{G}$ by $G^*$. $\Delta$ means the trivial crossed product of $A$ with $G$: $\Delta=\sum_{\sigma\in G}\Delta u_{\sigma}$, $u_{\sigma}u_{\tau}=u_{\sigma\tau}$ ($\sigma, \tau\in G$), $u_{\sigma}x=\sigma(x)u_{\sigma}$ ($x\in A$). Then there is a canonical ring homomorphism $j$ from $\Delta$ to End $(A_B)$ defined by $j(\sum_{\sigma}x_{\sigma}u_{\sigma})(y)=\sum_{\sigma}x_{\sigma}\sigma(y)$ ($\sum_{\sigma}x_{\sigma}u_{\sigma}\in \Delta$, $y\in A$). For any intermediate ring $T$ of $A/B$, $G^T=\{\sigma\in G; \sigma|T=1_T\}$ is a subgroup of $G$, where $\sigma|T$ means the restriction of $\sigma$ to $T$. We call it a fixed subgroup of $G$. For any subgroup $H$ of $G$, $A^H=\{x\in A; \sigma(x)=x$ for all $\sigma$ in $H\}$ is an intermediate ring of $A/B$. We call it a fixed subring of $A$ (with respect to $G$). Then, as is well known, the set of all fixed subgroups of $G$ and the set of all fixed subrings of $A$ are anti-order-isomorphic in the usual sense of Galois theory. A subring $T$ of $A$ is called a $G$-invariant subring of $A$ if $\sigma(T)=T$ for all $\sigma$ in $G$ (or equivalently, $\sigma(T)\subseteq T$ for all $\sigma$ in $G$). Let $N$ be a fixed subgroup of $G$. Then, $A^N$ is $G$-invariant if and only if $N$ is a normal subgroup of $G$: $N\lhd G$. Let $T$ be an intermediate ring of $A/B$, and put $H=G^T$. Then, for $\sigma, \tau$ in $G$, $\sigma|T=\tau|T$ if and only if $\sigma H=\tau H$. Let $H$ and $K$ be subgroups of $G$ such that $H\supseteq K$ and $(H:K)<\infty$, and let $H=\sigma_1K\cup\cdots\cup\sigma_rK$ be the left coset decomposition. For any $x$ in $A^K$ we put $t_{H,K}(x)=\sum_{i}t_{H}(\sigma_i(x))$. Then $t_{H,K}$ is an $A^K-A^H$-homomorphism from $A^K$ to $A^H$, and is independent of the choice of $\sigma_1, \cdots, \sigma_r$. If $K=1$, we write simply $t_H$ instead of $t_{H,1}$.

Here we present several fundamental facts, which are essential throughout the present study. Let $rM_r$ and $rN_r$ be $T$-left, $U$-right modules. If $rM_r$ is
isomorphic to a direct summand of $\tau N^r_U$ for some natural number $r$, then we write $\tau M_U|\tau N_U$, where $\tau N^r_U$ means the direct sum of $r$ copies of $\tau N_U$. If $\tau M_U|\tau N_U$ and $\tau N^r_U|\tau M_U$ we write $\tau M_U\sim \tau N_U$ (similar) (cf. Morita [21]). To be easily seen, $\tau M_U|\tau N_U$ if and only if there are $T-U$-homomorphisms $f_1, \cdots, f_r$ in $\text{Hom}(\tau M_U, \tau N_U)$ and $g_1, \cdots, g_r$ in $\text{Hom}(\tau N_U, \tau M_U)$ such that $\sum f_i g_i=$ the identity of $M$, or equivalently, $\text{Hom}(\tau M_U, \tau N_U).\text{Hom}(\tau N_U, \tau M_U)=\text{Hom}(\tau M_U, \tau M_U)$, where homomorphisms act on the right side.

Let $T$ be a ring with $1$, $M$ a unital $T$-left module, and $T^*=\text{End}(\tau M)$.

S. 1. If $\tau T|\tau M$ then $M_T|T^*$, (i.e. $M_T$ is finitely generated and projective) and $T=\text{End}(M_T)$. (Morita)

S. 2. If $\tau M|\tau T$ then $T^*=M_T$. (Morita)

S. 3. Let $T$ be commutative. If $\tau M|\tau T$ and $\tau M$ is faithful, then $\tau T|\tau M$. (Auslander-Buchsbaum-Goldman)

S. 4. Let $\hat{T}$ be an extension ring of $T$. If $\tau T|\tau \hat{T}$ then $\tau T$ is a direct summand of $\tau \hat{T}$ (and conversely). (Müller)

S. 5. Let $\hat{T}$ be an extension ring of $T$. If $\tau T|\tau \hat{T}$ then $\tau T_T$ is a direct summand of $\tau \hat{T}_T$. (The proof is similar to the one of S.4.)

In [22], $A/B$ was called a $G$-Galois extension if $G$ is finite and there are elements $a_1, \cdots, a_n$; $a_1^*, \cdots, a_n^*$ in $A$ such that $\sum a_i \sigma(a_i^*)=\delta_{i,s} (\sigma \in G)$. In this paper, $A/B$ is called a finite $G$-Galois extension if $A/B$ is $G$-Galois and $t_0(c)=1$ for some $c$ in $A$. Then, the following are equivalent:

(a) $A/B$ is finite $G$-Galois.

(b) $G$ is finite, $A_B\sim B_B$ and $j: A\simeq \text{End}(A_B)$.

(c) $G$ is finite and $A\sim \text{End}(A)$.

(Cf. S.1, S.2, [6] and [21]).

$A/B$ is called a locally finite $G$-Galois extension if there are fixed normal subgroups $N_\lambda (\lambda \in A)$ of $G$ which satisfy the following conditions: (1) $(G:N_\lambda)<\infty$, and $A/N_\lambda$ is a finite $G/N_\lambda$-Galois extension. (2) $A=\bigcup\lambda A^{N_\lambda}$, and $\{A^{N_\lambda}: \lambda \in A\}$ is a directed set with respect to the inclusion relation (abbr. $A=\bigcup\lambda A^{N_\lambda}$ is a directed union). Then we call $A=\bigcup\lambda A^{N_\lambda}$ a representation of the locally finite $G$-Galois extension $A/B$. If $V_A(B)=C$, an extension $A/B$ is said to be outer.

Now we shall prove first the following

**Proposition 1.1.** Let $G=G^*$ (i.e. $G$ is closed in $\hat{G}$). Then the following are equivalent:

(i) $\{\sigma(x); \sigma \in G\}$ is finite for any $x$ in $A$.

(ii) $G$ is compact.

(iii) Every directed union of fixed subrings of $A$ with respect to $G$ is also a fixed subring of $A$ with respect to $G$, and $\cap H=1$, where $H$ ranges
over all fixed subgroups of $G$ such that $(G : H) < \infty$.

Proof. (i) $\Rightarrow$ (ii) If we put $\prod_{x \in A} \{\sigma(x) : \sigma \in G\} = D$, then $G \subseteq D$ and $D$ is compact. Therefore it is sufficient to prove that $G$ is closed in $D$. Let $\rho$ be any element of the closure of $G$ in $D$. Then, as is easily seen, $\rho$ is a $B$-ring isomorphism from $A$ into $A$. Let $a$ be in $A$, and put $F = \{\sigma(a) : \sigma \in G\}$. Then, by assumption, $F$ is a finite subset of $A$, so that there is an element $\tau$ in $G$ such that $\rho|F = \tau|F$. Then, in particular, $\rho(\tau^{-1}(a)) = \tau(\tau^{-1}(a)) = a$. Thus $\rho$ is a $B$-automorphism of $A$. Hence the closure of $G$ in $D$ is contained in $\hat{G}$. Since $G$ is closed in $\hat{G}$, $G$ is closed in $D$, as desired. (ii) $\Rightarrow$ (iii) For any $x$ in $G$, we put $H_x = \{\sigma \in G : \sigma(x) = x\}$. Then $H_x$ is open in $G$, and therefore $\sigma H_x$ is open in $G$ for any $\sigma$ in $G$. Then, since $G$ is compact, we have $(G : H_x) < \infty$. Evidently $\bigcap_{x \in A} H_x = 1$. This proves the second assertion. Let $(A \neq T) T = \bigcup_{i \in A} T_i$ be a directed union of fixed subrings of $A$ with respect to $G$, and let $K_i = G^{\tau_i}$. Then each $K_i$ is a closed subgroup of $G$, and $A^{K_i} = T_i$. Let $a$ be an element of $A - T$, and put $U = \{\sigma \in G : \sigma(a) = a\}$. Then $U$ is open in $G$, so that each $K_i - U$ is closed in $G$. Since $a \not\in T_i$ and $A^{K_i} = T_i$, we have $K_i - U \neq \emptyset$. For any finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of $A$, there is an element $\lambda_0$ of $A$ such that $T_{\lambda_0} \supseteq \bigcup_i T_{\lambda_i}$. Then $K_{\lambda_0} \subseteq \bigcap_i K_{\lambda_i}$, and so $0 \neq K_{\lambda_0} - U \subseteq \bigcap_i K_{\lambda_i} - U = \bigcap_i (K_{\lambda_i} - U)$. Thus $\{K_{\lambda_0} - U ; \lambda \in A\}$ has finite intersection property. Since $G$ is compact, we have $\bigcap_i (K_{\lambda_i} - U) \neq \emptyset$. If $\rho$ is in $\bigcap_i (K_{\lambda_i} - U)$ then $\rho \in G^{\tau}$ and $\rho(a) \neq a$. Therefore $a \not\in A^{K_i}$, where $K = G^{\tau}$. Thus $A^{K_i} = T$. Hence $T$ is a fixed subring of $A$ with respect to $G$. (iii) $\Rightarrow$ (i) Let $H$ and $K$ be fixed subgroups of $G$ such that $(G : H) < \infty$ and $(G : K) < \infty$. Then $H \cap K$ is also a fixed subgroup of $G$ with $(G : H \cap K) < \infty$. Therefore $\bigcup A^H$ is a directed union of fixed subrings of $A$, where $H$ ranges over all fixed subgroups of $G$ with $(G : H) < \infty$. Then, by assumption, $\bigcup A^H$ is a fixed subring of $A$ with respect to $G$. Since $\bigcap H = 1$, we have $A = \bigcup A^H$. For any $x$ in $A$, there is an $A^H$ such that $x \in A^H$. Therefore if we put $L = \{\sigma \in G : \sigma(x) = x\}$ then $(G : L) < \infty$. This implies that $\{\sigma(x) : \sigma \in G\}$ is finite.

Remark. For any $x$ in $A$, $\{\sigma(x) : \sigma \in G\} = \{\sigma(x) : \sigma \in G^*\}$.

Proposition 1.2. Let $N$ be a fixed normal subgroup of $G$ such that $(G : N) < \infty$ and $A^N/B$ is finite $G/N$-Galois, and $G_1$ a subgroup of $G^*$ containing $G$. Then $A^N/B$ is finite $G_1/N_1$-Galois, where $N_1 = \{\sigma \in G_1 : \sigma|A^N = 1_{A^N}\}$.

Proof. Put $T = A^N$. Evidently $A^N = T$. Since $G$ is dense in $G_1$ and $T_B$ is finitely generated, there holds $G|T = G_1|T$. Therefore $T$ is $G_1$-invariant, $N_1 \triangleleft G_1$, and $(G_1 : N_1) < \infty$. There are elements $a_1, \cdots, a_n; a_1^*, \cdots, a_n^*$ in $T$ such that $\sum_i a_i \cdot \sigma(a_i^*) = \delta_{N_1, \sigma}$ for all $\sigma$ in $G$. If $\tau$ is in $G_1 - N_1$ then $\tau|T = \rho|T$ for
some $\rho$ in $G-N$, and $\sum a_i^* \tau(a_i^*) = \sum a_i^* \rho(a_i^*) = 0$. Thus $\sum a_i^* \sigma(a_i^*) = \delta_{N,*}$ for $\sigma$ in $G_1$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, and $G_1$ a subgroup of $G^*$ containing $G$. Then $A/B$ is locally finite $G_1$-Galois.

**Proposition 1.3.** Let $H_i (i \in \Lambda)$ be fixed subgroups of $G$ such that $A = \bigcup_{\lambda \in \Lambda} A_{\lambda}$ is a directed union.

1. If $H$ is a subgroup of $G$ such that $(G:H) < \infty$ then $A_H \subseteq A_{\lambda}$ for some $\lambda$ in $\Lambda$.

2. If $K$ is a subgroup of $G$ such that $(K:1) < \infty$ then $K \cap H_\mu = 1$ for some $\mu$ in $\Lambda$.

**Proof.** (1) Let $[H_i \cup H]$ be the subgroup of $G$ generated by $H_i \cup H$. Since $G \supseteq [H_i \cup H] \supseteq H$, we have $(G:[H_i \cup H]) \leq (G:H)$ for all $\lambda$ in $\Lambda$. Let $(G:[H_i \cup H])$ be maximum. We shall prove that $A_{\mu} \subseteq A_{\lambda}$. For any $H_i$ there is an $H_\mu$ such that $A_{\mu} \supseteq A_{\mu} \cap A_{\lambda}$. Then $H_\mu \subseteq H_i \cap H_\mu$, and so $[H_\mu \cup H] \subseteq [H_i \cup H] \cap [H_\mu \cup H]$. Since $(G:[H_i \cup H])$ is maximum, we have $([H_i \cup H] \supseteq [H_\mu \cup H]) = [H_i \cup H]$. Hence $[H_i \cup H] \subseteq [H_i \cup H]$ for all $\lambda$ in $\Lambda$. Then $A_H = \bigcup (A_{\mu} \cap A_{\lambda}) = \bigcup A_{[H_i \cup H]} = A_{[H_i \cup H]} = A_{[H_i \cup H]} \cap A_H$, which means $A_H \subseteq A_{H_\mu}$. (2) Since $A = \bigcup_{\lambda} A_{\lambda}$, we have $1 = G^A = \cap_i H_i$. Let $K = \{\sigma_1 = 1, \sigma_2, \ldots, \sigma_r\}$. Then, for any $\sigma_i (i \neq 1)$, there is an $H_\mu$ such that $\sigma_i H_i$. By assumption there is a $\mu$ such that $H_\mu \subseteq \cap_{i=1, \ldots, r} H_i$. Then $H \cap H_\mu \subseteq H \cap (\cap_{i=1, \ldots, r} H_i) = 1$.

**Remark.** Let $A/B$ be locally finite $G$-Galois, and $A = \bigcup_{\lambda \in \Lambda} A_{\lambda}$ its representation. If $G$ is finite then $A = A_{\lambda}$ for some $\lambda$.

**Proposition 1.4.** Let $T$ be an intermediate ring of $A/B$ such that $G|T$ is finite, and let $H = G^T$, and $G = \sigma_1 H \cup \cdots \cup \sigma_r H$ a left coset decomposition of $G$. If there are elements $t_1, \ldots, t_n; t_1^*, \ldots, t_n^*$ in $T$ such that $\sum t_i \sigma(t_i^*) = \delta_{k,*}$ for all $\sigma$ in $G$, then there hold the following.

1. $T = A_H^\alpha$, and $T_B$ is finitely generated and projective.

2. $j^*: A(\sum_s u_s) T = \sum_s A u_s \simeq Hom(T_B, A_B)$, where $j^*(\sum_s x_s u_s)(t) = \sum_s x_s \sigma(t)$, and this induces the $B-T$-isomorphism $(B_B T \simeq) (\sum_s u_s) T \simeq Hom(T_B, A_B)$.

3. The following are equivalent: (i) $B_B \mid T_B$. (ii) $B_B \mid T_B$. (iii) $t_{\theta:B}(c) = 1$ for some $c$ in $T$.

**Proof.** (1) $t_{\theta:B}$ is a $B-B$-homomorphism from $A_H^\alpha$ to $B$. For any $y$ in $A_H^\alpha$, $T \ni \sum_i t_i \cdot t_{\theta:B} t_i^* y = \sum_i t_i \sum_k \sigma_k(t_i^*) y = \sum_i \sum_k t_i \cdot \sigma_k(t_i^*) \sigma_k(y) = y$. Hence $A_H^\alpha = T$, and $T_B$ is finitely generated and projective (cf. [3]). (2) $j^*^{-1}$ is the mapping such that $j^*^{-1}(f) = \sum_i f(t_i)(\sum_k u_{\sigma_k}) t_i^*$ (cf. Hom($T_B, A_B$)). The second part will be easily seen. (3) The equivalence (i) $\iff$ (iii) is easy from (2).
Therefore (i) and (ii) are equivalent, because the situation is right-left symmetric.

**Proposition 1.5.** Let $A/B$ be locally finite $G$-Galois. Then there hold the following:

1. $G^*$ is compact.
2. By $j$, $\Delta$ is isomorphic to a dense subring of $\text{Hom}(A_B, A_B)$.
3. A subgroup $H$ of $G$ is a closed subgroup of $G$ if and only if $H$ is a fixed subgroup of $G$.

**Proof.** Let $A = \bigcup \lambda A^{N_\lambda}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) If $x$ is in $A$ then $x \in A^{N_\nu}$ for some $\nu$ in $A$. Then $(G:N_\nu)<\infty$ implies that $\{\{\sigma(x); \sigma \in G\} = \}$ $\{\sigma(x); \sigma \in G^*\}$ is finite. Hence, by Prop. 1.1, $G^*$ is compact. (2) By Prop. 1.4 (2), $\text{Im} j$ is dense in $\text{Hom}(A_B, A_B)$. Therefore it suffices to prove that $j$ is $1-1$. Let $\sigma_1, \cdots, \sigma_r$ be different elements in $G$. Then there is a finite subset $F$ of $A$ such that $\sigma_i[F \neq \sigma_j[F$ provided $i \neq k$. From this fact and Prop. 1.4, we can easily see that $j$ is $1-1$. (3) Evidently, a fixed subgroup is a closed subgroup. Let $H$ be any subgroup of $G$, and put $H' = G^*$, where $T = A^H$. Then $T = A^{H'}$. It suffices to prove that $H$ is dense in $H'$. To prove this, we take any finite subset $F$ of $A$. Then $F \subseteq A^{N_\nu}$ for some $N_\nu$. Put $N = N_\nu$. Then, by finite Galois theory, we obtain $(G/N)^{T_1} = HN/N$ and $(A/N)^{T_1} = H'N/N$, where $T_1 = A^{HN}$ and $T_1' = A^{H'N}$ (cf. [22; Prop. 2.2]). Since $A^{HN} = A^H \cap A^N = A^{H'} \cap A^N = A^{H'N}$, we have $HN/N = H'N/N$, that is, $HN = H'N$. Hence $H|A^N = H'|A^N$, and so $H|F = H'|F$. Since $F$ is arbitrary, this implies that $H$ is dense in $H'$. This completes the proof.

**Theorem 1.6.** Let $A/B$ be locally finite $G$-Galois, $G = G^*$, and $H$ a subgroup of $G$, and let $A'$ be an indecomposable extension ring of $B$ such that $V_{A'}(B) = V_{A'}(A')$. Assume that there is a $B$-ring homomorphism $g$ from $A$ to $A'$. Then, for any $B$-ring homomorphism $f$ from $A^H$ to $A'$, there is an element $\sigma$ in $G$ such that $f = g\sigma|A^H$.

**Proof.** Let $A = \bigcup \lambda \lambda A^{N_\lambda}$ be a representation. For each $N_\lambda$, there is an element $\sigma$ in $G$ such that $f|A^{HN_\lambda} = g\sigma|A^{HN_\lambda}$ ([22; Th. 4.1]). For each $\lambda$, we put $K_\lambda = \{\sigma \in G ; f|A^{HN_\lambda} = g\sigma|A^{HN_\lambda}\}$. Then $K_\lambda \neq \emptyset$, and $\{K_\lambda ; \lambda \in A\}$ has finite intersection property. Let $\tau$ be in the closure of $K_\lambda$ in $G$. Since $(A^{N_\lambda})_B$ is finitely generated, $\tau|A^{N_\lambda} = \alpha|A^{N_\lambda}$ for some $\alpha$ in $K_\lambda$. Then $\tau|A^{HN_\lambda} = \alpha|A^{HN_\lambda}$, and so $f|A^{HN_\lambda} = g\tau|A^{HN_\lambda}$. Hence $\tau \in K_\lambda$, and therefore $K_\lambda$ is closed in $G$. Since $G$ is compact (Prop. 1.5), we have $\bigcap_\lambda K_\lambda \neq \emptyset$. If $\rho$ is in $\bigcap_\lambda K_\lambda$, then $f|A^{HN_\lambda} = g\rho|A^{HN_\lambda}$ for all $\lambda$ in $A$. Since $A^H = \bigcup_\lambda A^{HN_\lambda}$, we know $f = g\rho|A^H$.

The following theorem will follow at once from Th. 1.6 and Cor. to Prop. 1.2.

**Theorem 1.7.** Let $A/B$ be locally finite outer $G$-Galois, and $A$ an
indecomposable ring. Then $G^* = \hat{G}$, that is, $G$ is dense in $\hat{G}$.

**Proposition 1.8.** Let $A/B$ be locally finite $G$-Galois, and $G = G^*$ (cf. Cor. to Prop. 1.2). Then there hold the following.

1. For an intermediate ring $T$ of $A/B$ the following are equivalent.
   (i) $T = A^H$ for some subgroup $H$ of $G$. (ii) There are subgroups $H_t$ ($t \in \Gamma$) of $G$ such that $T = \bigcup t_A^H$, $(G: H_t) < \infty$ and $\{A^H_t ; t \in \Gamma\}$ is a directed set with respect to the inclusion relation.

2. If $H$ is a subgroup of $G$ such that $(G : H) < \infty$ then $(A^H)_B$ is finitely generated.

**Proof.** Let $A = \bigcup_{\lambda \in A} A^{N_\lambda}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) (i) $\Rightarrow$ (ii) $T = A^H = \bigcup_t (A^H \cap A^{N_\lambda}) = \bigcup_t A^{N_\lambda}$ is a directed union, and $(G : H N_t) < \infty$. (ii) $\Rightarrow$ (i) follows from Prop. 1.1. (2) By Prop. 1.3, $A^H \subseteq A^{N_\nu}$ for some $\nu$ in $A$. Then, $A^H = A^{H'}$ is a fixed subring of the finite $G/N$, Galois extension $A^{H'}/B$, and therefore $(A^{H'})_B | (A^{N_\nu})_B$ (cf. [22; §2. p. 118]). Since $(A^{N_\lambda})_B$ is finitely generated, $(A^H)_B$ is finitely generated.

Let $T$ be an intermediate ring of $A/B$, and $S$ a subset of $A$. $T$ is called a $G$-separable cover of $S$ if $T$ satisfies the following conditions:

1. $T/B$ is a separable extension, and $T \supseteq S$.
2. $G|T$ is finite.
3. $G|T$ is strongly distinct (i.e. if $\sigma|T \neq \tau|T$ for $\sigma, \tau$ in $G$ then $\sigma|T$ and $\tau|T$ are strongly distinct).

**Theorem 1.9.** Let $A/B$ be locally finite outer $G$-Galois, and $T$ an intermediate ring of $A/B$. Then the following are equivalent:

1. $T = A^H$ for some subgroup $H$ of $G$ such that $(G : H) < \infty$.
2. $T/B$ is a separable extension, $T_B$ is finitely generated, and $G|T$ is strongly distinct.
3. $T$ is a $G$-separable cover of $B$.

**Proof.** Let $A = \bigcup_{\lambda \in A} A^{N_\lambda}$ be a representation. (i) $\Rightarrow$ (ii) By Prop. 1.3, $T = A^H \subseteq A^{N_\nu}$ for some $\nu$ in $A$. Then $T$ is a fixed subring of the finite $G/N$, Galois extension $A^{N_\nu}/B$. Then, by [19; Prop. 3.4], $T/B$ is a separable extension. By Prop. 1.8 (2) (cf. Cor. to Prop. 1.2), $T_B$ is finitely generated. By [22; Th. 2.6], $G|T$ is strongly distinct. (ii) $\Rightarrow$ (iii) This follows from the fact that $\{\sigma(x) ; \sigma \in G\}$ is finite for any $x$ in $A$. (iii) $\Rightarrow$ (i) Let $\{(t_i, t^*_i)$; $i = 1, \cdots, n\}$ be a $(B, T)$-projective coordinate system of $T/B$. Then, by [22; Prop. 1.2], $\sum_i t_i \cdot \sigma(t^*_i) = \delta_{H, \sigma}$ for $\sigma$ in $G$, where $H = G^T$. $(G|T) < \infty$ implies $(G : H) < \infty$. By Prop. 1.4, $A^H = T$.

Combining Th. 1.9 with Prop. 1.8, we obtain the following theorem (cf. [12; Th. 3], [28; Theorem]).
Theorem 1.10. Let \( A/B \) be locally finite outer \( G \)-Galois, and \( G=G^* \). Then, for an intermediate ring \( T \) of \( A/B \), the following are equivalent.

(i) \( T=A^H \) for some subgroup \( H \) of \( G \).

(ii) For any finite subset \( F \) of \( T \) there is an intermediate ring \( T_0 \) of \( T/B \) such that \( T_0 \supseteq F \), \( T_0/B \) is separable, \( T_0/B \) is finitely generated, and \( G|T_0 \) is strongly distinct.

(iii) Any finite subset of \( T \) has a \( G \)-separable cover which is contained in \( T \).

Next we shall proceed to the characterization of locally finite outer Galois extensions.

Proposition 1.11. Let \( V_A(B)=C \), \( T \) a \( G \)-separable cover of \( B \), and \( \{(t_i, t_i^*)\}; i=1,\cdots,n \) a \((B, T)\)-projective coordinate system for \( T/B \), and put \( H=G^r \). Then there hold the following.

1. \( \sum_i t_i^* \sigma(t_i^*)=\delta_{H,\sigma} \) for all \( \sigma \) in \( G \).

2. \( A^H=T \), \( (G:H)<\infty \), and \( T/B \) is a projective Frobenius extension.

3. Let \( K \) be a subgroup of \( G \) containing \( H \). Then, \( \sum_i t_{K:H}(t_i)\sigma(t_i^*)=\delta_{K,\sigma} \) for all \( \sigma \) in \( G \), \( T \) is \((B, A^K)\)-projective, \( T|A^K \) is a projective Frobenius extension, and \( G|A^K \) is strongly distinct. Further the following are equivalent. (a) \((A^K)_K|T_{A^K} \). (b) \((A^K)_{\delta}(A^K)|\Omega_{(A^K)T} \). (c) \( t_{K:H}(c)=1 \) for some \( c \) in \( T \).

Proof. (1) follows from [22; Prop. 1.2], and (2) is obvious by (1) and Prop. 1.4. (3) It will be easily seen that \( \sum_i t_{K:H}(t_i)\sigma(t_i^*)=\delta_{K,\sigma} \) for all \( \sigma \) in \( G \). Since \( \sum_i t_i t_j^*=\sum_i t_i^* t_j^* t \) \((t_i, t_j, t \in T) \) for \( t \) in \( T \), \( \sum_i y \cdot t_{K:H}(t_i)\otimes t_i^* y \) \((y \in A^K) \) for all \( y \) in \( A^K \). Hence the mapping \( x\mapsto\sum_i t_{K:H}(t_i)\otimes t_i^* x \) from \( T \) to \( A^K \otimes_B T \) is an \( A^K \)-homomorphism. Since \( \sum_i t_{K:H}(t_i)\otimes t_i^* x = x \), it follows that \( T \) is \((B, A^K)\)-projective. Let \( \rho|A^K \neq \tau|A^K \) for \( \rho, \tau \) in \( G \). Then \( \tau^{-1}\rho \notin K \), and so \( 0=\tau(\sum_i t_{K:H}(t_i)\rho(t_i^*))=\sum_i t_{K:H}(t_i)|\rho(t_i^*) \) for all \( \rho \) and \( \tau \) in \( G \). Thus, by [22; Prop. 1.1], \( \rho|A^K \) and \( \tau|A^K \) are strongly distinct. If we set \( G=K \) in Prop. 1.4, the remainder follows from Prop. 1.4.

Theorem 1.12. Let \( V_A(B)=C \). Then the following statements are equivalent.

(i) \( A/B \) is locally finite outer \((G)\)-Galois.

(ii) For any finite subset \( F \) of \( A \) there is a \( G \)-invariant \( G \)-separable cover \( T \) of \( F \) such that \( H\times B|_F \).

(iii) For any finite subset \( F \) of \( A \) there is a \( G \)-separable cover \( T \) of \( F \) which satisfies the following: If \( T_0 \) is an intermediate ring of \( T/B \) such that \( \alpha \) \( T \) is \((B, T_0)\)-projective, \( \beta \) \( T/T_0 \) is a projective Frobenius extension, \( \gamma \) \( G|T_0 \) is strongly distinct, then \( \tau_{r_0T_0\tau_0T} \).

(iv) For any finite subset \( F \) of \( A \) there is a \( G \)-separable cover \( T \) of \( F \)
which satisfies the following: If $T_0$ is an intermediate ring of $T/B$ such that $(\alpha)$ $T$ is $(B, T_0)$-projective, $(\beta)$ $T/T_0$ is a projective Frobenius extension, $(\gamma)$ $G|T_0$ is strongly distinct, $(\delta)$ $T_0$ is a $G$-invariant fixed subring (with respect to $G$), then $\tau_0 T_0|\tau_0 T$.

Proof. (i) $\Rightarrow$ (ii), (iii) Let $A = \bigcup_{\mu} A^N_{\mu}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then any finite subset $F$ of $A$ is contained in some $A^N_{\mu}$ ($\mu \in \Lambda$). By [22; Th. 1.5], $A^N_{\mu}$ is a $G$-invariant $G$-separable cover of $F$ such that $p B|p A^N_{\mu}$. Let $T_0$ be an intermediate ring of $A^N_{\mu}/B$ such that $A^N_{\mu}$ is $(B, T_0)$-projective and that $G|T_0$ is strongly distinct. Then, by [22; Th. 2.6], $T_0$ is a fixed subring of the finite outer $G/N_{\mu}$-Galois extension $A^N_{\mu}/B$, whence $\tau_0 T_0|\tau_0 T$ by [22; §2. p. 118]. (ii) $\Rightarrow$ (i) Let $F$ be a finite subset of $A$, and $T$ a $G$-invariant $G$-separable cover of $F$ such that $p B|p T$. If we put $N = G^r$, then $A^N = T, N \triangleleft G$ and $(G:N) < \infty$. By Prop. 1.11, $A^N/B$ is a finite $G/N$-Galois extension. Noting that $(A^N)^F$ is finitely generated, $A/B$ is a locally finite $G$-Galois extension. (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (i) Let $T_1$ be a separable cover of an element $x \in A$. Put $G_T = H_1$. Then $\#(G|T_1) < \infty$ implies $(G:H_1) < \infty$ and $\# \{ \sigma(x); \sigma \in G \} < \infty$. Thus any finite subset of $A$ is contained in a $G$-invariant finite subset of $A$. Let $F$ be a $G$-invariant finite subset of $A$, and $T$ a $G$-separable cover of $F$ as that in (iv), and let $\{(t_i, t_i^*); i = 1, \ldots, n\}$ be a $(B, T)$-projective coordinate system of $T/B$, and $H = G^r$. Then, by Prop. 1.11, $A^N = T, (G:H) < \infty$, and $\sum_i t_i \sigma(t_i^*) = \delta_{N, \sigma}$ for all $\sigma$ in $G$. Set $N = G^r$. Then $H \subset N \triangleleft G$, and $F \subset A^N \subset A^N$. By Prop. 1.11, $T$ is $(B, A^N)$-projective, $T/A^N$ is a projective Frobenius extension, and $G/A^N$ is strongly distinct. Then, by the assumption for $T$, $(A_N^N(A^N)|A^N), T$, so that $t_{N:B}(c) = 1$ for some $c \in T$ (Prop. 1.11 (3)). Put $t_i^* = t_{N:B}(t_i)$ and $t_i^* = t_{N:B}(t_i^*)$. Then, $t_i, t_i^* \in A^N$, and $\sum_i t_i \sigma(t_i^*) = \delta_{N, \sigma}$ for all $\sigma$ in $G$ (Prop. 1.11 (3)). Further, as is easily seen, $\sum_i t_i \sigma(t_i^*) = \delta_{N, \sigma}$ for all $\sigma$ in $G$. Since $p B|p T$ (Prop. 1.11 (3)), we have $p B|p A^N$. Thus $A^N/B$ is a finite $G/N$-Galois extension. Noting that $(A^N)^F$ is finitely generated, we conclude that $A/B$ is a locally finite $G$-Galois extension.

**Proposition 1.13.** Let $A^* \supseteq T \supseteq B^*$ be rings such that $A^*$ is $(B^*, T)$-projective, $A'$ an extension ring of $B^*$ such that $V_{A'}(B^*) = V_{A'}(A')$, and $f_1, \ldots, f_s, B^*$-ring homomorphisms from $A^*$ to $A'$ such that $f_i|T$ and $f_k|T$ $(i \neq k)$ are strongly distinct. If $(B^*)_{B^*} \rightarrow T_{B^*}$ then $(A')_{A'} \rightarrow (A')_{A'}$.

Proof. Let $\{(t_i, a_i^*); i = 1, \ldots, n\}$ be a $(B^*, T)$-projective coordinate system for $A^*$. Then, by [22; Prop. 1.2], $\sum f_{a_i^*}(t_i) f_k(a_i^*) = \delta_{h, k}$ for all $h, k$. Let $\phi$ be an $A'$-right homomorphism from $T \otimes B^* A'$ to $(A')_{A'}$ defined by $\phi(t \otimes a') = (f_i(t)a'), \ldots, f_s(t)a')$. Since $\sum f_{a_i^*}(t_i) f_k(a_i^*) = \delta_{h, k}$, $\phi$ is an epimorphism. $(B^*)_{B^*} \rightarrow T_{B^*}$ implies that $(A')_{A'} \rightarrow T \otimes B^* A'$. Hence we have $(A')_{A'} \rightarrow (A')_{A'}$, as
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desired.

Concerning Prop. 1.13, we consider the following condition.

Condition (F): If \( A^r \rightarrow A^s \) for positive integers \( r, s \), then \( r \geq s \).

Remark. Let \( A^r \rightarrow A^s \) for positive integers \( r, s \). Then, since \( A^s \) is projective, \( A^s \) is isomorphic to an \( A \)-direct summand of \( A^r \).

(1) If \( A \) is finite dimensional, then \( r \cdot \dim A \geq s \cdot \dim A \), and so \( r \geq s \) (cf. [11]).

(2) Assume that there is a proper ideal \( \mathfrak{U} \) of \( A \) such that \( A \mathfrak{U} / \mathfrak{U} \) is finite dimensional. Then, since \( A^r / \mathfrak{U}^r \rightarrow A^s / \mathfrak{U}^s \), the above (1) yields \( r \geq s \), because \( A^r / \mathfrak{U}^r \cong (A / \mathfrak{U})^r \) and \( A^s / \mathfrak{U}^s \cong (A / \mathfrak{U})^s \).

(3) If \( A \) is commutative, then \( r \geq s \) by (2).

**Proposition 1.14.** Let \( V_\mathfrak{A}(B)=C \), and \( A \) an indecomposable ring satisfying (F), and let \( T \) be an intermediate ring of \( A / B \), and \( S \) a subset of \( A \). Then the following are equivalent:

(i) \( T \) is a \( G \)-separable cover of \( S \).

(ii) \( T \supseteq S \), \( T / B \) is a separable extension, and \( T_B \) is finitely generated.

**Proof.** (i) \( \Rightarrow \) (ii) is evident by Prop. 1.11. (ii) \( \Rightarrow \) (i) By [22; Lemma 2.7], \( A \) is \( (B, T) \)-projective. Then, by Prop. 1.13, we have \#(\( G | T \))<\( \infty \), and hence \( T \) is a \( G \)-separable cover of \( S \).

If \( A \) is commutative, then \( A \) satisfies (F). Therefore, by Th. 1.12, S. 3 and Prop. 1.14, we have the following

**Theorem 1.15** (Nagahara [12]). Let \( A \) be an indecomposable commutative ring. Then the following are equivalent.

(i) \( A / B \) is locally finite \( G \)-Galois.

(ii) For any finite subset \( F \) of \( A \) there is an intermediate ring \( T \) of \( A / B \) such that (a) \( T / B \) is a separable extension, and \( T_B \) is finitely generated, (b) \( T \supseteq F \).

**Proposition 1.16.** Let \( A / B \) be locally finite \( G \)-Galois, and \( H \) a subgroup of \( G \). Then \( G \mid A^H \) is strongly distinct.

**Proof.** Let \( \sigma, \tau \) be in \( G \), and \( e \) a central idempotent of \( A \) such that \( \sigma(x)e=\tau(x)e \) for all \( x \) in \( A^\sigma \). Let \( A=\bigcup_{N\lambda} A_{N\lambda} \) be a representation of the locally finite \( G \)-Galois extension \( A / B \). We may assume that \( e \in A_{N\lambda} \) for all \( \lambda \) in \( \Lambda \). Suppose that \( \sigma \mid A^\sigma \neq \tau \mid A^\tau \). Since \( A^\mu=\bigcup_{N\lambda} A_{N\lambda}^\mu, \sigma \mid A_{N\mu}^\mu \neq \tau \mid A_{N\mu}^\mu \) for some \( \mu \) in \( \Lambda \). Then, by [22; Prop. 2.4], \( (G / N_p) \mid A_{Np}^\mu \) is strongly distinct. Therefore we have \( e=0 \). Thus \( G \mid A^\mu \) is strongly distinct.

**Theorem 1.17.** Let \( A / B \) be locally finite outer \( G \)-Galois, and \( T \) an intermediate ring of \( A / B \). Then the following are equivalent.

(i) \( T=A^H \) for some subgroup \( H \) of \( G \), and \( A_T \) is finitely generated.
(ii) $T=A^r$ for some subgroup $H$ of $G$ such that $(H:1)<\infty$.

(iii) $A/T$ is a projective Frobenius extension, Hom$(A_T, A_T) \subseteq \Delta$, and $G|T$ is strongly distinct.

When any of the above conditions is satisfied $A/A^r$ is finite H-Galois.

Proof. Let $A=\bigcup_{n=1}^{\infty} A^N_n$ be a representation of the locally finite outer $G$-Galois extension $A/B$. (i) $\Rightarrow$ (ii) Let $A=x_1T+\cdots+x_rT$. Then $x_1,\ldots,x_r \in A^N_r$ for some $\mu \in A$, so that $A=A^N_r\cdot T=A^N_r\cdot A^r$. Hence $N_r \cap H=1$. Since $(G:N_r)<\infty$ we have $(H:1)<\infty$. (ii) $\Rightarrow$ (iii) By Prop. 1.3, $H \cap N_r=1$ for some $\mu \in A$. There are elements $a_1,\ldots,a_n; a_1,\ldots,a_n$ in $A^N_r$ such that $\sum_i a_i \cdot \sigma(a_i^\sigma)=\delta_{N_r,\sigma}$ for all $\sigma$ in $G$. Then $\sum_i a_i \cdot \sigma(a_i^\sigma)=\delta_{r,\sigma}$ for all $\sigma$ in $H$. Hence $A/A^r$ is $H$-Galois. Therefore $A/A^r$ is a projective Frobenius extension (cf. [22; p. 121]), and Hom$(A_T, A_T)=\sum_{\tau \in H} Au, \subseteq \Delta$. By Prop. 1.16, $G|T$ is strongly distinct. (iii) $\Rightarrow$ (i) Let $h=\sum_{\tau \in H} a_{\mu}u_{\tau}$ be a Frobenius homomorphism of $A/T$, where $H$ is a finite subset of $G$ and $a_{\mu} \neq 0$ for all $\tau$ in $H$. Then, since $th=ht$ for all $t$ in $T$, we have $ta_{\tau}=\sigma(t)$ for all $t$ in $T$, in particular, $ba_{\tau}=ab$ for all $b$ in $B$. Hence $a_{\tau} \in V(A_B)=C$ for all $\tau$ in $H$. There are elements $r_i, l_i$ in $A$ such that $x=\sum_i h(xr_i)l_i=\sum_i r_i h(l_i x)$ for all $x$ in $A$ (cf. [27]). Then $u_1=\sum_i r_i h l_i=\sum_i \tau a_i \cdot \gamma(l_i)u_i=\sum_{\tau \in H} \sum_i \tau a_i \cdot \gamma(l_i)u_i$, and so $=\sum_i r_i a_i l_i=\sum_i \tau a_i l_i$. Thus $a_1$ is an invertible element in $C$, and $a_1^{-1}=\sum_i r_i l_i$. Since $H$ is finite there is an $N_r$ such that $\tau \mid A^N_r \neq \rho \mid A^N_r$ provided $\tau \neq \rho$ ($\tau, \rho \in H$). Since $A^N_r/B$ is finite $G/N_r$-Galois, there are elements $d_k, e_k$ in $A^N_r$ such that $\sum_k d_k \cdot \sigma(e_k)=\delta_{N_r,\sigma}$ for all $\sigma$ in $G$. Put $D_0=\text{Hom}(A_T, A_T)$. Then $D_0=AhA$, and $D_0 \ni \sum_k \tau(d_k)he_k=\sum_{\tau \in H} \sum_k \tau(d_k) a_i \cdot \sigma(e_k)u_i=a_i u_\tau$ for $\tau$ in $H$. Thus $D_0=AhA=\sum_{\tau \in H} \Delta_{\tau} A a_{\mu} u_{\tau}$. Since $A/T$ is a projective Frobenius extension with Frobenius homomorphism $h\cdot A \otimes \tau A_\mu \simeq \Delta_\mu A_\mu$ by the correspondence $x \otimes y \rightarrow xy$. Let $\varphi$ be the $A$-left homomorphism from $A$ to $D_0$ defined by $\varphi(\sum_{\tau} x_{\tau}u_{\tau})=\sum_{\tau \in H} x_{\tau} \cdot \tau(a_{\mu}u_{\tau})$, and $\psi$ be the $A$-left homomorphism from $D_0$ to $A$ defined by $\psi(xhy)=\sum_{\tau \in H} x \cdot \tau(h(yr_i)u_i)$, where $\nu=\sum_{\tau \in H} u_{\tau}$. Then, as $h(tr_i)a_{\tau}=\tau(h(yr_i)u_{\tau})$ ($\tau \in H$), $\varphi=1$. Since $a_{\mu} u_{\tau}=\sum_k \tau(d_k) he_k$, we have $\varphi(a_{\mu} u_{\tau})=\sum_k \tau d_k h(e_k r_i)u_i=\sum_{\tau \in H} \sum_k \tau(d_k) a_i \cdot \sigma(e_k)u_i$, and so $\varphi(a_{\mu} u_{\tau})=\sum_{\tau \in H} \sum_k \tau(d_k) a_i \cdot \sigma(e_k)u_i$. On the other hand, $\varphi(a_{\mu} u_{\tau})=a_{\mu} u_{\tau}$, and hence $a_{\mu} u_{\tau}=a_{\mu} u_{\tau}$. Therefore $A_{\mu} u_{\tau} \in \Delta_\mu A_{\mu}$. Noting that $a_{\mu} u_{\tau}$ is an invertible element of $C$, $Aa_{\mu} a_{\tau} A \cdot \tau(a_{\mu} u_{\tau})=A a_{\tau}$, and so $A=A a_{\tau} + \text{Ann}_A(a_{\mu})$, where $\text{Ann}_A(a_{\mu})=\{x \in A; xa_{\mu}=0\}$. If $xa_{\mu} \in \text{Ann}_A(a_{\mu})$, then $0=xa_{\mu}=xa_{\mu} \cdot \tau(a_{\mu})$, so that $xa_{\mu}=0$. Thus $A=A a_{\tau} + \text{Ann}_A(a_{\mu})$. Therefore $A a_{\tau}$ is written as $Ag$, with a central idempotent $g$. As $Aa_{\mu} u_{\tau} \subseteq D_0$, we have $g u_{\tau} \in D_0$, and so $g \cdot t=g \cdot \tau(t)$ for all $t$ in $T$. Consequently, $D_0=\sum_{\tau \in H} A a_{\mu} u_{\tau}$, and $H=G^T$. Hence $\text{End}(\sigma A) \subseteq (A^r)_{\sigma}$ the right multiplications of elements of $A$. Since $a_{\mu} u_{\tau} \in D_0=\text{End}(A_T)$, we have $a_{\mu} u_{\tau} \in \text{End}(A_T)$. Noting that $a_{\mu}$ is in $C$, we
can easily seen that \( a_u \in \text{Hom}(A^{(A_u)} - A^{(A_u)}) \). Thus \( h = \sum_{\sigma \in G} a_u \in \text{Hom}(A^{(A_u)} - A^{(A_u)}) \). Then, by [27; Cor. 1], \( A^{A_u} \) is also a projective Frobenius extension with a Frobenius homomorphism \( h \). Since \( (H:1) < \infty \), there is an \( N \) such that \( H \cap N = 1 \) (Prop. 1.3 (2)). Then \( A^{HN} \subseteq A^{N} \), and \( H \cong HN/N \) canonically. Therefore there is an element \( c \) in \( A^{N} \) such that \( t_{H}(c) = 1 \) (cf. [22; §2. p. 118]), which implies \( (A^{u})_{(A_u)}|A^{(A_u)} \), because the \( A^{u} \)-right homomorphism \( x \rightarrow t_{H}(cx) \) \( (x \in A) \) from \( A \) to \( A^{u} \) splits. Therefore there is an element \( d \) in \( A \) such that \( h(d) = 1 \). Then, for any \( x \) in \( A^{u} \), \( T \ni h(dx) = h(d)x = x \). Thus we obtain \( T = A^{u} \), as desired.

**Theorem 1.18.** Let \( A/B \) be finite outer \( G \)-Galois, and \( T \) an intermediate ring of \( A/B \). Then the following are equivalent.

(i) \( T = A^{u} \) for some subgroup \( H \) of \( G \).

(ii) \( A/T \) is a projective Frobenius extension, and \( G \mid T \) is strongly distinct.

(iii) \( T/B \) is a separable extension, and \( G \mid T \) is strongly distinct.

**Proof.** (i) \( \iff \) (ii) is evident from Th. 1.17. (i) \( \implies \) (iii) follows from [22; Th. 2.6] and [19; Prop. 3.4]. (iii) \( \implies \) (i) follows from [22; Th. 2.6 and Lemma 2.7].

§2. Heredity of locally finite Galois extensions.

Let \( A_0 \) be a \( G \)-invariant subring of \( A \) such that the mapping \( \sigma \rightarrow \sigma|A_0 \) \( (\sigma \in G) \) is one-to-one and such that \( A_0/A_0^{G} \) is a locally finite \( G \)-Galois extension, and let \( G \) be compact (as an automorphism group of \( A \)). Put \( B_0 = A_0^{G} \), and let \( A_0 = \bigcup_{\lambda \in A} A_0^{N_{\lambda}} \) be a representation of the locally finite \( G \)-Galois extension \( A_0/B_0 \). Then \( G/N_{\lambda} \) may be considered as a finite group of automorphisms of \( A^{N_{\lambda}} \). And, by [22; Th. 5.1 and §2. p. 118], \( A^{N_{\lambda}} = A_0^{N_{\lambda}} \otimes_{B_0} B, A^{N_{\lambda}}/B \) is finite \( G/N_{\lambda}, \text{-Galois} \). Since \( \bigcup_{\lambda \in A} A^{N_{\lambda}} \) is a directed union, the compactness of \( G \) implies that \( \bigcup_{\lambda \in A} A^{N_{\lambda}} \) is a fixed subring of \( A \) with respect to \( G \) (Prop. 1.1), so that \( A = \bigcup_{\lambda \in A} A^{N_{\lambda}} \), because \( \sigma \rightarrow \sigma|A_0 \) \( (\sigma \in G) \) is 1–1. Thus \( A/B \) is locally finite \( G \)-Galois. Let \( H \) be any subgroup of \( G \). Then, \( A^{u} = \bigcup_{\lambda \in A} (A^{H \cap A^{N_{\lambda}}}) = \bigcup_{\lambda \in A} A^{H^{N_{\lambda}}} \). By [22; Th. 5.1], \( A^{HN_{\lambda}} = (A_0^{N_{\lambda}})^{H_{N_{\lambda}}} \otimes_{B_0} B = A_0^{HN_{\lambda}} \otimes_{B_0} B \). Hence \( A^{H} = \bigcup_{\lambda \in A} (A^{H \cap A^{N_{\lambda}}}) = \bigcup_{\lambda \in A} A^{H^{N_{\lambda}}} \). By [22; Th. 5.1], \( A^{H} = \bigcup_{\lambda \in A} (A_0^{H^{N_{\lambda}}}) = \bigcup_{\lambda \in A} A^{H^{N_{\lambda}}} \). Next we consider the set of all \( A_0 \)-left submodules of \( A \) and the set of all \( B_0 \)-left submodules of \( B \). Let \( \overline{X} \) be any \( A_0 \)-left submodule of \( A \). Then \( \overline{X} \cap A^{N_{\lambda}} \) is an \( A_0^{N_{\lambda}} \)-left submodule of \( A^{N_{\lambda}} \). Therefore, by [22; Th. 5.1], we have \( \overline{X} \cap A^{N_{\lambda}} = A_0^{N_{\lambda}}((\overline{X} \cap A^{N_{\lambda}}) \cap B) = A_0^{N_{\lambda}} \otimes_{B_0} (\overline{X} \cap B) \), so that \( \overline{X} = \bigcup_{\lambda \in A} (\overline{X} \cap A^{N_{\lambda}}) = \bigcup_{\lambda \in A} (A_0^{N_{\lambda}}(\overline{X} \cap B)) = A_0(\overline{X} \cap B) \). Since \( A_0^{N_{\lambda}} \otimes_{B_0} (\overline{X} \cap B) \cong A_0^{N_{\lambda}}(\overline{X} \cap B) \), for any \( \overline{X} \cap A^{N_{\lambda}} \).
of of of of [22; Th. 5.1], $A^\mathfrak{x}_\lambda X \cap B = X$ for all $\lambda$ in $A$, so that $A_0 X \cap B = \cup_1 (A_{0}^{\mathfrak{x}_\lambda} X \cap B) = X$. If $\bar{Y}$ is a $G$-invariant intermediate ring of $A/A_0$, then $\bar{Y} \cap B$ is an intermediate ring of $B/B_0$, and $\bar{Y} = A_0(\bar{Y} \cap B)$. Symmetrically we have $\bar{Y} = (\bar{Y} \cap B) A_0$. If $Y$ is an intermediate ring of $B/B_0$ such that $A_0 Y = YA_0$, then $A_0 Y$ is a $G$-invariant intermediate ring of $A/A_0$. Since $A = \cup_1 A_{0}^{\mathfrak{x}_\lambda}$, we have $\bar{Y} = \cup_1 (\bar{Y} \cap A_{0}^{\mathfrak{x}_\lambda}) = \cup_1 \bar{Y}_{\mathfrak{x}_\lambda}$, and $\bar{Y}_{\mathfrak{x}_\lambda}(\bar{Y} \cap B)$ is finite $G/N_a$-Galois ([22; Th. 5.1]. Hence $\bar{Y}(\bar{Y} \cap B)$ is locally finite $G$-Galois. Thus we have obtained the following.

**Theorem 2.1.** Let $A_0$ be a $G^*$-invariant subring of $A$ such that $\sigma \rightarrow \sigma|A_0$ ($\sigma \in G^*$) is $1$–$1$ and such that $A_0/B_0$ is locally finite $G$-Galois where $B_0 = A_0^H$, and let $G^*$ be compact. Then there hold the following:

1. $A/B$ is locally finite $G$-Galois.
2. $A^H = B \otimes_{B_0} A_0^H = A_0^H \otimes_{B_0} B$ for any subgroup $H$ of $G$. In particular, $A = B \otimes_{B_0} A_0 = A_0 \otimes_{B_0} B$.
3. Let $\{\bar{X}\}$ and $\{X\}$ be the set of all $A_0$-$G$-left submodules of $A$ and the set of all $B_0$-$B$-left submodules of $B$, respectively. Then, $\bar{X} \rightarrow \bar{X} \cap B$ and $X \rightarrow A_0 X = A_0 \otimes_{B_0} X$ are mutually converse order isomorphisms between $\{\bar{X}\}$ and $X$.
4. Let $\{\bar{Y}\}$ and $\{Y\}$ be the set of all $G$-invariant intermediate rings of $A/A_0$ and the set of all intermediate rings of $B/B_0$ such that $A_0 Y = YA_0$, respectively. Then $\bar{Y}(\bar{Y} \cap B)$ is locally finite $G$-Galois, and $\bar{Y} \rightarrow \bar{Y} \cap B$ and $Y \rightarrow A_0 Y = YA_0$ are mutually converse order isomorphisms between $\{\bar{Y}\}$ and $\{Y\}$.

Let $A$, $A'$ be $R$-algebras such that $A \otimes_R A' \neq 0$. Assume that $A/B$ is a locally finite $G$-Galois extension such that $R \cdot 1 \subseteq B$, and assume that $A'$ is a locally finite $G'$-Galois extension such that $R \cdot 1 \subseteq B'$. Then each $\sigma \times \tau$ in $G \times G'$ induces an automorphism of $A \otimes_R A'$. Let $A = \bigcup A^{N_{\lambda}}$ and $A' = \bigcup A^{N_{\lambda}}$ be representations of $A/B$ and $A'/B'$ respectively. Then, by [22; Th. 5.2], $(A^{N_{\alpha}} \otimes_{R} A^{N_{\lambda}})(B \otimes B')$ is a finite $(G/N_{\alpha}) \times (G'/N_{\lambda})$-Galois extension. Let $\varphi_{a^{\beta}}$ be the canonical $R$-algebra homomorphism from $A^{N_{\alpha}} \otimes_{R} A^{N_{\lambda}}$ to $A^{N_{\alpha}N_{\beta}} \subseteq A \otimes_R A'$. We put $(A \otimes_R A' \supseteq) A^{N_{\alpha}} \otimes A^{N_{\beta}} = A_{\alpha}^{\beta}$ and $(A \otimes_R A' \subseteq) B \otimes B' = B^*$. To be easily seen, Ker $\varphi_{a^{\beta}}$ is a $(G/N_{\alpha}) \times (G'/N_{\lambda})$-invariant ideal of $A^{N_{\alpha}} \otimes_{R} A^{N_{\lambda}}$. Hence $A_{\alpha}^{\beta}/B^*$ is $(G/N_{\alpha}) \times (G'/N_{\lambda})$-Galois ([22; Th. 5.6]). There are elements $c$ and $c'$ in $A^{N_{\alpha}}$ and $A^{N_{\beta}}$ respectively such that $t_{(\alpha)/N_{\alpha}}(c) = 1$ and $t_{(\alpha)/N_{\alpha}}(c') = 1$. Then $c \otimes c' \in A_{\alpha}^{\beta}$ and $t_{(\alpha)/N_{\alpha}}(c \otimes c') = 1 \otimes 1$. Hence $A_{\alpha}^{\beta}/B^*$ is a finite $(G/N_{\alpha}) \times (G'/N_{\lambda})$-Galois extension, and $\{\sigma \times \tau \in G \times G'; \sigma \times \tau \mid A_{\alpha}^{\beta} = 1_{A_{\alpha}^{\beta}}\} = N_{\alpha} \times N_{\lambda}$. Since $\cap_{\alpha} (N_{\alpha} \times N_{\lambda}) = (\cap_{\alpha} N_{\alpha}) \times (\cap_{\beta} N_{\lambda}) = 1$, $G \times G'$ may be considered
as a group of automorphisms of $A \otimes_R A'$. Let $H$ and $H'$ be subgroups of $G$ and $G'$, respectively. Then, 
$(A \otimes_R A')^{H \times H'} = \cup_{\alpha, \beta} A_{\alpha \beta}^{H \times H'} = \cup_{\alpha, \beta} (A^N_{\alpha} \otimes A'_{\beta N'}) = (\cup_{\alpha} A_{\alpha}^{N_{\alpha}})^{(\cup_{\beta} A_{\beta}^{N_{\beta}})} = A^{H} \otimes A^{H'}$ by [22; Th. 5.2]. In particular $(A \otimes_R A')^{N_{\alpha} \times N_{\beta}} = A_{\alpha \beta}^{N_{\alpha} \otimes A'_{\beta N'}}$, and evidently $(G \times G': N_{\alpha} \times N_{\beta}) < \infty$. Since $A \otimes_R A' = \cup_{\alpha, \beta} A_{\alpha \beta}^{N_{\alpha} \otimes A'_{\beta N'}}$ is a directed union, $A \otimes_R A'/B \otimes B'$ is a locally finite $G \times G'$-Galois extension. Let $a \in A$ and $a' \in A'$. Then it is evident that \{ $\sigma \times \tau \in G \times G'$; $\sigma(a) \otimes \tau(a') = a \otimes a'$ \} $\supseteq \{ \sigma \in G; \sigma(a) = a \} \times \{ \tau \in G'; \tau(a') = a' \}$. Put \{ $\sigma \in G$; $\sigma(a) = a = K$ and \{ $\tau \in G'$; $\tau(a') = a' = K'$ \}. Then $A^K \subseteq A^N_{\alpha}$ and $A^{'K'} \subseteq A^{N_{\beta}}_{\beta}$ for some $\alpha, \beta$ (Prop. 1.3), so that $N_{\alpha} \subseteq K$ and $N_{\beta} \subseteq K'$. By [22; Th. 5.2], $(G/N_{\alpha} \times G'/N_{\beta})^{A_{\alpha}^{K} \otimes A'_{\beta}^{K'}} = K/N_{\alpha} \times K'/N_{\beta}$, and hence $(G \times G')^{A_{\alpha}^{K} \otimes A'_{\beta}^{K'}} = K \times K'$. Since $(A^K)_{B}$ and $(A'_{K'})_{B}$, are finitely generated, $(A^{K} \otimes A'_{K'})_{B \otimes B'}$ is finitely generated. Hence the finite topology of $G \times G'$ with respect to $A \otimes_R A'$ is the product topology of the finite topology of $G$ with respect to $A$ and the finite topology of $G'$ with respect to $A'$. Thus we have proved the following

**Theorem 2.2.** Let $A$ and $A'$ be $R$-algebras such that $A \otimes_R A' \neq 0$. If $A/B$ is a locally finite $G$-Galois extension such that $B \subseteq C$, and $A'$ a $B$-algebra such that $A \otimes_R A' \neq 0$. Then $(A \otimes_R A')/(B \otimes B')$ is a locally finite $G \times G'$-Galois extension, and $(A \otimes_R A')^{H \times H'} = A^{H} \otimes A'^{H'}$ for any subgroup $H$ of $G$ and any subgroup $H'$ of $G'$. The finite topology of $G \times G'$ with respect to $A \otimes_R A'$ is the product topology of the finite topology of $G$ with respect to $A$ and the finite topology of $G'$ with respect to $A'$.

**Corollary.** Let $A/B$ be a locally finite $G$-Galois extension such that $B \subseteq C$, and $A'$ a $B$-algebra such that $A \otimes_R A' \neq 0$. Then $(A \otimes_R A')/(1 \otimes A')$ is a locally finite $G$-Galois extension, and $(A \otimes_R A')^{H} = A^{H} \otimes A'$ for any subgroup $H$ of $G$.

**Proposition 2.3.** Let $A/B$ be locally finite $G$-Galois, and $G = G^*$. If $H$ and $K$ are closed subgroups of $G$, then $A^{H \cap K} = A^{H} \cdot A^{K}$. In particular, if $H \cap K = 1$ then $A = A^{H} \cdot A^{K} = A^{K} \cdot A^{H}$.

**Proof.** Let $A = \cup_{\mu \in \Lambda} A^{N_{\mu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. First we assume that $(G : K) < \infty$. Then, by Prop. 1.3, $A^{K} \subseteq A^{N_{\mu}}$ for some $\mu \in \Lambda$. Since $(A^{N_{\mu}})_{B}$ is finitely generated and $(A^{K})_{A^{K}}$ is a direct summand of $(A^{N_{\mu}})_{A^{K}}$ ([22; § 2. p. 118]), $(A^{K})_{B}$ is finitely generated. Therefore we may assume that $A^{K} \subseteq A^{N_{\lambda}}$ for all $\lambda \in \Lambda$. Then $N_{\lambda} \subseteq K$ for $\lambda \in \Lambda$, and $A^{H} \cdot A^{K} = \cup_{\lambda \in \Lambda} A^{N_{\lambda}} \cup_{\mu \in \Lambda} A^{N_{\mu} \cap K} = \cup_{\lambda} A^{N_{\lambda} \cap K} = \cup_{\lambda} A^{N_{\lambda} K}$ by [22; Prop. 5.3]. Since $N_{\lambda} H \cap K = N_{\lambda} (H \cap K)$ for all $\lambda$, we have $A^{H} \cdot A^{K} = \cup_{\lambda} A^{N_{\lambda} \cap K}$. Next we return to general case. For any finite subset $F$ of $A^{K}$, we put $K_{F} = \{ \sigma \in G; \sigma | F = 1 \}$. Then $(G : K_{F}) < \infty$, $A^{K_{F}} \subseteq A^{K}$, and $(A^{K_{F}})_{B}$ is finitely generated. Therefore $A^{K} = \cup_{F} A^{K_{F}}$ is a directed union, and
hence $A^H.A^K = A^H(\bigcup_{\gamma} A^{|H_{\gamma}}}) = \bigcup_{\gamma}(A^H.A^{|H_{\gamma}})$ is also a directed union. Since each $A^H.A^K (= A^{H\cap K})$ is a fixed subring of $A$, $A^H.A^K$ is a fixed subring of $A$ (Prop. 1.1). Hence, as is easily seen, $A^H.A^K = A^{H\cap K}$. Symmetrically we have $A^{H\cap K} = A^K.A^H$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, $G = G^*$, and $H, (\gamma \in \Gamma)$ be closed subgroups of $G$. Then, $[\bigcup_{\gamma} A^H] = A^\cap$, where $[\bigcup_{\gamma} A^H]$ means the subring of $A$ generated by $\cup_{\gamma} A^\cap$.  

**Proof.** Evidently $[\bigcup_{\gamma} A^H] = \bigcup [A^{H_1} \cup \cdots \cup A^{H_n}]$, where $\{H_1, \cdots, H_n\}$ ranges over all finite subsets of $\Gamma$. By Prop. 2.3, $A^{H_1 \cap \cdots \cap H_n} = A^{H_1} \cdots A^{H_n}$, and therefore $[\bigcup_{\gamma} A^H]$ is a directed union of fixed subrings of $A$. Hence, by Prop. 1.1, $[\bigcup_{\gamma} A^H]$ is a fixed subring. Since $\{\sigma \in G; \sigma [\bigcup_{\gamma} A^H] = 1\} = \cap_{\gamma} H$, we obtain $[\bigcup_{\gamma} A^H] = A^\cap$, as desired.

**Proposition 2.4.** Let $A/B$ be locally finite $G$-Galois, $\mathfrak{A}$ a $G$-invariant proper ideal of $A$, $K$ a closed subgroup of $G$, and $N$ a closed normal subgroup of $G$ such that $(G : N) < \infty$. Then there hold the following:

1. $A^{K\cap N}/A^K$ is finite $K/(K\cap N)$-Galois. In particular, $A^N/B$ is finite $G/N$-Galois.

2. $(A^N + \mathfrak{A})/(B + \mathfrak{A})$ is finite $G/N$-Galois, and $(A^N + \mathfrak{A})/\mathfrak{A} = (A^N + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

**Proof.** Let $A = \bigcup_{\mu \in A} A^{N_\mu}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) By Prop. 1.3, $A^N \subseteq A^{N_\mu}$, for some $\mu \in A$, and then $N_\mu \subseteq N$, $A^N = (A^{N_\mu})^{K\cap N_\mu}$. Therefore, by [22; Prop. 5.7], $A^N/B$ is finite $(G/N)/(N/N_{K\cap N})$-Galois, or equivalently, finite $G/N$-Galois. Accordingly, $A^N/A^{NK} = A^N/A^{NK}$ is finite $NK/N$-Galois, or equivalently, finite $K/(K\cap N)$-Galois. $K/(K\cap N)$ may be considered as a finite group of automorphisms of $A^{K\cap N}$, because $K\cap N \subseteq K$. Then $A^{K\cap N}/A^K$ is finite $K/(K\cap N)$-Galois. (2) By (1), $A^N/B$ is finite $G/N$-Galois. If $t_0/c = 1$ for $c$ in $A^N$, then $t_0/c = 1 + \mathfrak{A}$. Then, by [22; Th. 5.6], $(A^N + \mathfrak{A})/(B + \mathfrak{A})/\mathfrak{A}$ is finite $G/N$-Galois, and $(A^N + \mathfrak{A})/\mathfrak{A} = (A^N + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

Let $A/B$ be locally finite $G$-Galois, $K$ a closed subgroup of $G$, $N$ a closed normal subgroup of $G$, and $\mathfrak{A}$ a $G$-invariant proper ideal of $A$. Let $A = \bigcup_{\mu \in A} A^{N_\mu}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then $A^N = \bigcup_{\mu} (A^N \cap A^{N_\mu}) = \bigcup_{\mu} A^{NN_\mu}$ is a directed union, and each $NN_\mu$ is a closed normal subgroup of $G$, because $(G : N_\mu) < \infty$. Then, by Prop. 2.4 (1), $A^{NN_\mu}/B$ is finite $G/NN_\mu$-Galois. Therefore there are elements $a_1, \cdots, a_m; b_1, \cdots, b_m$ in $A^{NN_\mu}$ such that $\sum_{i} a_i \cdot \sigma(b_i) = b_{NN_\mu}$, for $\sigma$ in $G$. Hence $A^{NN_\mu}/B$ is finite $(G/N)/(NN_\mu/N)$-Galois. Hence $A^N/B$ is locally finite $G/N$-Galois. Next we consider $K$. $A = \bigcup_{\mu} A^{N_\mu}$ is a directed union, and each $N_\mu \cap K$ is a fixed
normal subgroup of $K$ such that $(K : N_i \cap K) < \infty$. By Prop. 2.4 (1), each $A^{\mathcal{N} \cap K}/A^K$ is finite $K/(N_i \cap K)$-Galois. Hence $A/A^K$ is locally finite $K$-Galois. Finally we consider $\mathfrak{A}$. Evidently, $A/\mathfrak{A} = \bigcup_{i}(A^{\mathcal{N}_i} + \mathfrak{A})/\mathfrak{A}$. By Prop. 2.4 (2), $((A^{\mathcal{N}_i} + \mathfrak{A})/\mathfrak{A})/((B + \mathfrak{A})/\mathfrak{A})$ is finite $G/N_i$-Galois, and $((A^{\mathcal{V}_i} + \mathfrak{A})/\mathfrak{A})^\mathfrak{A} = (A^{\mathcal{V}_i} + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$. Therefore $(A/\mathfrak{A})^\mathfrak{A} = \bigcup_i((A^{\mathcal{N}_i} + \mathfrak{A})/\mathfrak{A})^\mathfrak{A} = \bigcup_i(A^{\mathcal{V}_i} + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$. Hence $(A + \mathfrak{A})/((B + \mathfrak{A})/\mathfrak{A})$ is locally finite $G$-Galois. Thus we have proved the following

**Theorem 2.5.** Let $A|B$ be locally finite $G$-Galois, $N$ a closed normal subgroup of $G$, $K$ a closed subgroup of $G$, and $\mathfrak{A}$ a $G$-invariant proper ideal of $A$. Then there hold the following:

1. $A^{\mathcal{N}}/B$ is locally finite $G/N$-Galois.
2. $A/A^{\mathcal{V}}$ is locally finite $K$-Galois.
3. $((A + \mathfrak{A})/\mathfrak{A})/((B + \mathfrak{A})/\mathfrak{A})$ is locally finite $G$-Galois, and $((A + \mathfrak{A})/\mathfrak{A})^\mathfrak{A} = (A^{\mathcal{V}} + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

**Corollary.** Let $A|B$ be locally finite $G$-Galois, and $e$ a non-zero idempotent in $B \cap C$. Then $Ae|Be$ is locally finite $G$-Galois, and $(Ae)^H = A^H \cdot e$ for any subgroup $H$ of $G$.

Let $A|B$ be locally finite $G$-Galois, $n$ a positive integer, and $J$ the ring of rational integers. Then, $(J)_n$ is a $J$-algebra, and $(J)_n \otimes A \simeq (A)_n \neq 0$. If we define $\sigma((a_{ik})) = \sigma(a_{ik})$ for any $\sigma$ in $G$ and any $(a_{ik})$ in $(J)_n$, then $(A)_n/((B)_n$ is locally finite $G$-Galois and $((A)_n)^J = (A^J)_n$ for any subgroup $H$ of $G$ (Th. 2.2). Now, let $\{e_{ik}; i, k = 1, \ldots, m\}$ a system of matrix units contained in $B$, and $A = \bigcup_{\lambda < \lambda}$ a representation of $A|B$. Put $A_0 = V_A(\{e_{ik}\})$ and $B_0 = B \cap A_0$. Then, as is well known, $A = \sum_{i,k} A^{\mathcal{N}_i} e_{ik}$, $A_0 \simeq A_0 e_{ik}$ by the right multiplication of $e_{ik}$. To be easily seen, $A^{\mathcal{N}_i} = \sum_{i,k} A_0^{\mathcal{N}_i} e_{ik}$, and $A^{\mathcal{N}_i}_0 = V_J(\{e_{ik}\})$. There is an element $c$ in $A^{\mathcal{N}_i}$ such that $t_{\mathcal{N}_i,c} = 1$. Let $c = \sum_{i,k} x_{ik} e_{ik}$ $(x_{ik} \in A^{\mathcal{N}_i}_0)$. Then $1 = t_{\mathcal{N}_i,c} = \sum_{i,k} t_{\mathcal{N}_i,c} x_{ik} e_{ik}$, and so $t_{\mathcal{N}_i,c} = 1$. Thus, by [22; Th. 5.8], $A^{\mathcal{N}_i}/B_0$ is finite $G/N_i$-Galois. Since $A_0 = \bigcup_{\lambda < \lambda} A^{\mathcal{N}_i}_0$ is a directed union, $A_0/B_0$ is locally finite $G$-Galois. Therefore, by Th. 2.1, $A/A_0 \otimes B$. Thus we have obtained the following

**Theorem 2.6.** Let $A|B$ be locally finite $G$-Galois.

1. For any positive integer $n$, $(A)_n/((B)_n$ is locally finite $G$-Galois, and $((A)_n)^J = (A^J)_n$ for any subgroup $H$ of $G$.

2. If $\{e_{ik}; i, k = 1, \ldots, m\}$ is a system of matrix units contained in $B$, $A_0 = V_A(\{e_{ik}\})$, and $B_0 = B \cap A_0$, then $A_0/B_0$ is locally finite $G$-Galois, and $A = A_0 \otimes B$. Let $A|B$ be finite $G$-Galois, and $M$ a $A$-left module. For any subgroup $H$ of $G$, we put $M^H = \{m \in M; u \cdot m = m \text{ for all } u \in H\}$, which is an $A^H$-module.
submodule of $M$. Evidently $M^H \supseteq A^H \cdot M^\alpha$, and the mapping $\varphi : A^H \otimes_B M^\alpha \rightarrow M^H$ defined by $a \otimes m \rightarrow am \ (a \in A, m \in M^\alpha)$ is an $A^H$-left homomorphism. By assumption there are elements $a_1, \ldots, a_n; a_1^*, \ldots, a_n^*$ in $A$ such that $\sum t_i a_i \cdot \sigma(a_i^*) = \delta_{i*} \ (\sigma \in G)$, $t_H(d) = 1$. Put $t_i = t_H(a_i)$. Then, $t_i \in A^H$ and $\sum t_i \cdot \sigma(a_i^*) = \delta_{ii} \sigma$ for $\sigma$ in $G$. If $m$ is in $M^\alpha$, then $A^H \cdot M^\alpha \ni t_i \sum_{\sigma \in \Delta} u_\sigma(a_i^* \sigma m) = \sum t_i \cdot \sigma(a_i^* \sigma m)$ $= t_H(d)m = m$. Hence $\varphi$ is an epimorphism. If $a \in A^H$ and $m_0 \in M^\alpha$, then $\sum t_i \otimes m_0 \cdot (a_i^* \sigma m_0) = \sum t_i \otimes \sum_{\sigma \in \Delta} \sigma(a_i^* \sigma m_0) = \sum t_i \cdot \sum_{\sigma \in \Delta} \sigma(a_i^* \sigma m_0) \otimes m_0 = t_H(da) \otimes m_0 = a \otimes m_0$. From this fact, as is easily seen, $\varphi$ is 1-1. Thus we have $M^H = A^H \otimes_B M^\alpha$. Next we proceed to more general case.

Let $A/B$ be locally finite $G$-Galois, $A = \bigcup_{i \in I} A^{N_i}$ its representation, and $M$ a $\Delta$-left module. Let $G = \sigma_1 N_i \cup \cdots \cup \sigma_r N_i$ be the coset decomposition of $G$, and let $A_i$ be the trivial crossed product of $A^{N_i}$ with $G/N_i$: $A_i = \sum t_i A^{N_1} v_{\sigma} = v_{\sigma_1}, v_{\sigma_1} = a_1, a_1 = a_1 \cdot \sigma(a_1) \cdot \sigma(a_1^*) = \delta_{1, \sigma}$, $\sigma \in G$, $a_1 \cdot \sigma(a_1^*) = \delta_{1, \sigma} \sigma$. If $m$ in $M^{N_i}$, then $A^{N_1} \otimes_B M^{e} = A^{N_1} \cdot M^{e}$ is a $A^{N_2}$-module. Since $A^{N_1}/B$ is finite $G/N_1$-Galois, we obtain that $M^{N_1} = A^{N_1} \otimes_B M^\alpha$ and $\Delta_{N_1} = M^{N_1} \supseteq A^{N_1} \cdot M^{\alpha}$ for any subgroup $H$ of $G$. Since $A = \bigcup_{i \in I} A^{N_i}$ is a directed union, so is $\bigcup_{i \in I} M^{N_i}$. For any subgroup $H$ of $G$, $(\bigcup_{i \in I} M^{N_i})^H = \bigcup_{i \in I} M^{N_i} \cdot M^\alpha = A^H \cdot M^\alpha$, and $A^H \otimes_B M^\alpha \simeq A^H \cdot M^\alpha$ canonically. The last isomorphism may be considered as $A^H \otimes_B M^\alpha \supseteq A^H \otimes_B M^\alpha$, and hence we see that $(\bigcup_{i \in I} M^{N_i})^H = A^H \otimes_B M^\alpha$. For any $m$ in $M$ we put $m_H = \{ \sigma \in G; u \cdot m = m \}$, which is a subgroup of $G$. Assume that $(G:H_m) < \infty$ and that $H_m$ is closed in $G$. Then, by Prop. 1.3, $H_m \supseteq N_\nu$ for some $\nu \in \Delta$, so that $m \in M^{N_\nu}$. Conversely, if $m$ is in $\bigcup_{i \in I} M^{N_i}$, then $m \in M^{N_\nu}$ for some $N_\nu$, so that $H_m \supseteq N_\nu$. Then, since $(G:N_\nu) < \infty$ and $N_\nu$ is closed in $G$, $(G:H_m) < \infty$ and $H_m$ is closed in $G$. Thus we have proved the following

**Theorem 2.7.** Let $A/B$ be locally finite $G$-Galois, and $M$ a $\Delta$-left module. Then there hold the following:

1. $A \cdot M^\alpha$ is a $\Delta$-submodule of $M$, and $(A \cdot M^\alpha)^H = A^H \otimes_B M^\alpha$ for any subgroup $H$ of $G$.

2. $A \cdot M^\alpha = \{ m \in M; (G:H_m) < \infty$ and $H_m$ is closed in $G \}$, where $H_m = \{ \sigma \in G; u \cdot m = m \}$.

**Corollary.** Let $A/B$ be finite $G$-Galois, and $M$ a $\Delta$-left module. Then, $M^H = A^H \otimes_B M^\alpha$ for any subgroup $H$ of $G$, in particular, $M = A \otimes_B M^\alpha$ (cf. [4; Th. 1.3] and [22; Th. 5.1 (2)]).

**Proposition 2.8.** Let $A/B$ be finite $G$-Galois. Then the following are equivalent.

1. There are elements $a_1, \ldots, a_n; a_1^*, \ldots, a_n^*$ in $V_A(B)$ such that $\sum t_i \cdot a_i \cdot \sigma(a_i^*) = \delta_{i*, \sigma} \ (\sigma \in G)$ (cf. [22; Cor. to Th. 5.1]).
(ii) \( bA_B|_B B_B \).

Proof. Since \( (A \supseteq (\sum u_i) A \cong \text{Hom}(A_B, B_B) \) by \( j \), it follows that \( (\sum u_i) V_A(B) \cong \text{Hom}(bA_B, bB_B) \), and it is evident that \( V_A(B) \cong \text{Hom}(bB_B, bA_B) \) canonically. To be easily seen, \( bA_B|_B B_B \) if and only if there are elements \( f_1, \cdots, f_n \) in \( \text{Hom}(bA_B, bB_B) \) and \( g_1, \cdots, g_n \) in \( \text{Hom}(bB_B, bA_B) \) such that \( \sum_i g_i f_i(x) = x \) for all \( x \) in \( A \). Consequently (ii) is equivalent to that \( u_i = \sum a_i (\sum u_i a_i^* ) \)

\[ = \sum \sum a_i \sigma(a_i^* ) u_i \] for some \( a_1, \cdots, a_n ; a_1^*, \cdots, a_n^* \) in \( V_A(B) \). Hence (i) and (ii) are equivalent.

Corollary. Let \( G \) be finite. Then the following are equivalent.

(i) \( A/B \) is outer \( G \)-Galois, and \( bA_B|_B B_B \).

(ii) There are elements \( a_1, \cdots, a_n ; a_1^*, \cdots, a_n^* \) in \( C \) such that \( \sum a_i \sigma(a_i^* ) \)

\[ = \delta_{1, \sigma} (\sigma \in G). \]

Proof. This follows from [22; Prop. 6.4 and Prop. 6.5] and Prop. 2.8. \( A/B \) is called a completely outer \( G \)-Galois extension if \( G \) is finite and completely outer (cf. [22]).

**Theorem 2.9.** Let \( B' \) be a ring with identity, \( Z \) its center, and \( G' \) a finite group.

1. If \( A'/B' \) is completely outer \( G'-\text{Galois} \) and \( bA'_{B'}|_{B'_{B'}} \), then \( A' = B' \otimes_{z} C' \), where \( C' \) is the center of \( A' \), and \( C'/Z \) is \( G'-\text{Galois} \).

2. If \( C'/Z \) is \( G'-\text{Galois} \) and \( C' \) is commutative, then \( A' = B' \otimes_{z} C' \) is a completely outer \( G'-\text{Galois} \) extension over \( B', bA'_{B'}|_{B'_{B'}} \) and \( 1 \otimes C' \) is the center of \( A' \).

Proof. (1) By [22; Prop. 6.4], \( A'/B' \) is outer \( G'-\text{Galois} \) and \( V_{A'}(B') = C' \), where \( C' \) is the center of \( A' \). Then, by Cor. to Prop. 2.8 and [22; Th. 5.1], \( C'/Z \) is \( G'-\text{Galois} \) and \( A' = B' \otimes_{z} C' \). (2) By [22; Th. 5.2 and Prop. 6.5], \( A'/B'(\otimes 1) \) is completely outer \( G'-\text{Galois} \). Since \( zZ \) is a direct summand of \( zC' \), \( B' \simeq B' \otimes 1 \) canonically, and \( bA'_{B'}|_{B'_{B'}} \), because \( zC'|z \). Then, by Cor. to Prop. 2.8, \( C'/Z \) is \( C'-\text{Galois} \), where \( C' \) is the center of \( A' \). Since \( C' \supseteq 1 \otimes C' \supseteq Z \) and \( (1 \otimes C')/Z \) is \( G'-\text{Galois} \) ([22; Th. 5.1 or Th. 5.6]), we have \( C' = Z \cdot (1 \otimes C') = 1 \otimes C' \) ([22; Th. 5.1]).

**Lemma 2.10.** Let \( T \) be a ring, and \( U \) a subring of \( T \).

1. Let \( T/U \) be a separable extension. If a \( T \)-left module \( M \) is \( U \)-projective, then \( M \) is \( T \)-projective.

2. If \( \gamma T \otimes_{\gamma} T \gamma|_{\gamma} T \gamma \) and \( \gamma U|_{\gamma} M \) for a \( T \)-left module \( M \), then \( \gamma T|_{\gamma} M \).

3. Let \( T_0 \) be an intermediate ring of \( T/U \). If \( T \) is \( (U, T_0) \)-projective and \( T_0 \) is a \( T_0 \)-\( T_0 \)-direct summand of \( T \), then \( T_0/U \) is a separable extension.

Proof. (1) Since the mapping \( x \otimes y \rightarrow xy \) form \( T \otimes_{\gamma} T \) to \( T \) splits as a \( T-T \)-homomorphism, the mapping \( x \otimes m \rightarrow xm \) from \( T \otimes_{\gamma} M \) to \( M \) splits as
a $T$-left homomorphism. Since $\nu M$ is projective, so is $T \otimes \nu M$. Therefore $M$ is $T$-projective. (2) Since $\nu U|_0 M$, $T \otimes T \otimes V_M$. Since $\tau T \otimes V_T$, we have $\tau T \otimes \nu M|_0 M$. Hence we have $\tau T|_0 M$. (3) Let $\varphi$ be the canonical homomorphism from $T_0 \otimes \sigma T$ to $T$ defined by $\varphi(t_0 \otimes t)=t_0t$, and let $\psi$ be a $T_0-T_0$-homomorphism from $T$ to $T_0 \otimes \sigma T$ such that $\varphi(x)=x$ for all $x$ in $T$. If $\psi(1)=\sum a_i \otimes b_i \ (a_i \in T_0, b_i \in T)$, then $\sum_i a_i b_i = 1$ and $\sum_i y a_i \otimes b_i = \sum_i a_i \otimes b_i y \ (t_0 \otimes T)$ for all $y$ in $T_0$. Let $\pi$ be a $T_0-T_0$-homomorphism from $T$ to $T_0$ such that $\pi(T_0)=1$. Then, since $\sum_i y a_i \otimes b_i = \sum_i a_i \otimes b_i y \ (t_0 \otimes T)$ for all $y$ in $T_0$, we have $\sum_i a_i \otimes \pi(b_i)=1$ and $\sum_i y a_i \otimes \pi(b_i)=\sum_i a_i \otimes \pi(b_i) y \ (t_0 \otimes T)$ for $y$ in $T_0$. Then the mapping $y \mapsto \sum_i a_i \otimes \pi(b_i) y$ from $T_0$ to $T \otimes \sigma T$ is a $T_0-T_0$-homomorphism, and $\sum_i a_i \otimes \pi(b_i) y = y$. Hence $T_0/U$ is a separable extension.

**Proposition 2.11.** Let $A/B$ be finite $G$-Galois, and $Z$ the center of $B$. If $B$ is a separable $Z$-algebra and $Z \subseteq C$, then $V_A(B)/Z$ is finite $G$-Galois.

**Proof.** By [2; Prop. 1.5], $B \otimes B^\circ$ is a central separable $Z$-algebra, where $B^\circ$ is the opposite ring of $B$. Since $\nu A$ and $\nu B$ are finitely generated and projective, so is $\nu A$. Then, by Lemma 2.10 (1), $\nu \otimes B^\circ|_Z A$ is finitely generated and projective. By [2; Th. 2.1], $\nu \otimes B^\circ|_Z B^\circ B$, and hence $\nu A|_B B^\circ B$. Then, by Prop. 2.8, $V_A(B)/Z$ is finite $G$-Galois (cf. S. 3).

**Theorem 2.12.** Let $G$ be finite, $\pi$ the group homomorphism defined by $\pi(a)=a|_C (a \in G)$, $Z$ the center of $B$, and $Z_0=C^0$, and assume that $A$ is indecomposable. Then the following statements are equivalent.

1. $A/Z_0$ is separable, and $\pi$ is 1-1.
2. $V_A(B)=C$, $A/Z$ is separable, and $\nu A_B|_B B^\circ B$.
3. $V_A(B)=C$, and both $B/Z$ and $C/Z$ are separable.
4. Both $B/Z$ and $C/Z_0$ are separable, and $\pi$ is 1-1.
5. $V_A(B)=C$, $A/B$ is separable, $A$ is $(Z, B)$-projective, and $\nu B_B|_B A_B$.
6. $A=B \cdot C$, and $A/Z$ is separable.
7. $A \otimes Z_0 A \otimes Z_0 A^0 \otimes A^0 A^0 \otimes A^0 A^0$, and $\text{Hom}(A \otimes A, A \otimes A)=0$ for any $\sigma$ in $G$ such that $\sigma \neq 1$.

**Proof.** (i)$\implies$(ii) By [2; Th. 2.3], $A/C$ and $C/Z_0$ are separable. Therefore, by [4; Th. 1.3], $C/Z_0$ is $G$-Galois. Then, by [22; Th. 5.1], $A=B \otimes Z_0 C$. Hence $V_A(B)=C$, and $Z=Z_0$. Since $Z$ is finitely generated and projective, $A_B|_B B^\circ B$. (ii)$\implies$(iii) $V_A(B)=C$ implies $Z=Z_0 \subseteq C$. By [22; Lemma 2.7], $A/C$ and $A/B$ are separable, so that $A/B$ is outer $G$-Galois ([22; Th. 1.5]). Then, by Prop. 2.8, $C/Z$ is $G$-Galois, so that $C/Z$ is separable. Since $A/C$ is separable, $B/Z$ is separable ([22; Cor. to Th. 5.1]). (iii)$\implies$(iv) In this case, $Z=Z_0$. By [2; Th. 3.1], $A=B\cdot C$, whence $\pi$ is 1-1. (iv)$\implies$(v) By
[4; Th. 1.3], $C/Z_0$ is $G$-Galois. Hence, by [22; Th. 5.1], $A/B$ is $G$-Galois, and $A=B·C$. Then $A/B$ is separable, $V_A(B)=C$, and $Z=Z_0$. Since $Z$ is commutative, $Z$ is a direct summand of $ZC$ (S. 3), so that $t_0(c)=1$ for some $c$ in $C$. Then $B$ is a $B$-$B$-direct summand of $A$ (cf. [22; § 2. p. 118]). Since $B/Z$ is separable, $A$ is $(Z, B$)-projective ([22; Lemma 2.7]). (v) $\Rightarrow$ (vi) By Lemma 2.10 (3), $B/Z$ is separable. Then, by [2; Th. 3.1], $A=B\otimes_{B}C$. Since both $A/B$ and $B/Z$ are separable, $A/Z$ is separable ([22; Lemma 2.7]).

(vi) $\Rightarrow$ (i) As $A=B·C$, $V_A(B)=C$, $Z=Z_0$, and $\pi$ is 1–1. Thus we know that (i) $\sim$ (vi) are equivalent. (i) $\Rightarrow$ (vii) In this case, $V_A(B)=C$, $Z=Z_0$, and $B/Z$ is separable. Then, by [2; Th. 2.1], $B\otimes_B\otimes B|_{B}$, and then $A\otimes_B A|_{A} A\otimes_B A$. By [22; Prop. 1.3], $A\otimes_B A\simeq A\otimes_B A$. Hence $A\otimes A\otimes A\otimes A|_{A\otimes A\otimes A\otimes A}$. The second assertion follows from [22; Prop. 6.3]. (vii) $\Rightarrow$ (i) By assumption, $End(\otimes_{A\otimes A\otimes A\otimes A}A)\simeq \oplus_{\sigma \in G} End(\otimes_{A\otimes A\otimes A\otimes A}A)\otimes (external\ direct\ sum\ as\ rings)$. To be easily seen, $End(\otimes_{A\otimes A\otimes A\otimes A}A)$ is a commutative ring. Then, by S. 1 and S. 3, $A\otimes A\otimes A\otimes A$ is finitely generated and projective. Hence $A\otimes A\otimes A\otimes A$ is finitely generated and projective, that is, $A/Z_0$ is separable. Let $f$ be the projection from $A$ to $Au_1$ with respect to the decomposition $A=\sum_{\sigma}Au_{\sigma}$. Then, since $End(\otimes_{A\otimes A\otimes A\otimes A}A)$ is commutative, $f$ is in the center of $A\otimes A\otimes A\otimes A$ (cf. S. 1). By [2; Prop. 1.5], the center of $A\otimes A\otimes A\otimes A$ is $C\otimes C$, so that $f$ is written as $f =\sum_{\sigma}a_{\sigma}a_{\sigma}^{*}$ ($a_{\sigma}$, $a_{\sigma}^{*}$ $\in C$). Then, $u_{\sigma}=\sum_{\sigma}a_{\sigma}a_{\sigma}^{*}u_{\sigma}^{*}$ ($=\sum_{\sigma}(\sum_{\sigma}a_{\sigma}^{*})u_{\sigma}$), and hence $\sum_{\sigma}a_{\sigma}^{*}a_{\sigma}^{*}=\delta_{1,\sigma}$. This completes the proof of the theorem.

Premark. The following are also equivalent to (i) $\leftrightarrow$ (iii).

(viii) $A/C$ is separable, and $C/Z_0$ is $G$-Galois (cf. Kanzaki [8]).

(ix) $A/B$ is outer $G$-Galois, and $B/Z$ is separable.

**Proposition 2.13.** Let $A/B$ be locally finite $G$-Galois, and $b$ an element of $B$ which is not a right zero divisor of $B$. Then $b$ is not a right zero divisor of $A$.

**Proof.** Let $a$ be an element of $A$ such that $ab=0$. Then $Aab=0$, and so $\sigma(\Delta)a\cdot b=0$ for all $\sigma$ in $G$. Hence, $(\sum_{\sigma}(\Delta)a\cdot b)\cap B=0$. Then, by assumption, $(\sum_{\sigma}(\Delta)a\cdot b)\cap B=0$. Then, by Th. 2.1 (3), $\sum_{\sigma}(\Delta)a\cdot b=\sum_{\sigma}(\Delta)a\cdot (\sum_{\sigma}(\Delta)a\cdot b)\cap B=0$. Hence $a=0$.

Let $A/B$ be locally finite $G$-Galois, and $S$ a $G$-invariant multiplicative system of regular elements in $A$ such that a left quotient ring $\overline{A}$ of $A$ with respect to $S$ exists. Then $G$ may be regarded as a group of automorphisms of $\overline{A}$. To be easily seen, $\{s(x); s \in G\}$ is finite for any $x$ in $\overline{A}$. Then, by Th. 2.1, $\overline{A}/\overline{B}$ is locally finite $G$-Galois and $\overline{A}=\overline{B}\otimes_B A=\overline{A}\otimes_B \overline{B}$, where $\overline{B}=\overline{A}$. To be easily seen, any element in $B\cap S$ is a unit of $B$. For $b$ in $\overline{B}$, we put
$\mathfrak{L}=\{x \in A; \exists b \in A\}$, which is a $\mathcal{J}$-left submodule of $A$. Then $(\mathfrak{L} \cap B)b \subseteq B$. If $\mathfrak{L} \cap B \cap S \neq \emptyset$, then $sb \in B$ for some $s$ in $B \cap S$. Therefore, if we assume that $\mathcal{J}(s) \cap B \cap S \neq \emptyset$ for all $s \in S$, then $\overline{B}$ is a left quotient ring of $B$ with respect to $B \cap S$. Thus we obtain the following.

**Theorem 2.14.** Let $A/B$ be locally finite $G$-Galois, and $S \ni 1$ a $G$-invariant multiplicative system of regular elements of $A$ such that a left quotient ring $\overline{A}$ of $A$ with respect to $S$ exists. Further, assume that $\mathcal{J}(s) \cap B \cap S \neq \emptyset$ for all $s \in S$. Then there hold the following:

1. $\overline{A}/\overline{B}$ is locally finite $G$-Galois and $\overline{A}=\overline{B} \otimes_{B}A=A \otimes_{B}\overline{B}$, where $\overline{B}=\overline{A}^o$.
2. $\overline{A}$ is a left quotient ring of $A$ with respect to $B \cap S$. $\overline{B}$ is a left quotient ring of $B$ with respect to $B \cap S$.

**Remark.** Let $A/B$ be locally finite $G$-Galois, and $S$ a $G$-invariant multiplicative system of regular elements in $A$ such that $S \subseteq C$ and $S \ni 1$. Then $S$ satisfies the conditions in Th. 2.14. To see this, we put $H=\{\sigma \in G; \sigma(s)=s\}$ for $s$ in $S$. If $G=\sigma_1H \cup \cdots \cup \sigma_rH$ is the left coset decomposition of $G$, then $\cap_{i=1}^{r}\sigma_i(s) \in \Delta(s) \cap B \cap S$.

A non-zero ring $T$ with 1 is called a left Goldie ring if $T$ satisfies the following conditions: (1) $T$ is a semi-prime ring. (2) Any independent set of non-zero left ideals is finite (i.e., $\mathcal{T}$ is finite dimensional). (3) $T$ satisfies the ascending chain condition for annihilator left ideals.

A left Goldie ring has a complete left quotient ring which is a semi-simple ring with minimum condition for left ideals, and conversely (Goldie [17]). (Cf. [7])

**Theorem 2.15.** Let $A/B$ be locally finite $G$-Galois, $A$ a left Goldie ring, $\overline{A}$ a complete left quotient ring of $A$, and $B$ a semi-prime ring. Then there hold the following:

1. $\overline{A}/\overline{B}$ is locally finite $G$-Galois, where $\overline{B}=\overline{A}^o$.
2. $B$ is a left Goldie ring, and $\overline{B}$ is a complete left quotient ring of $B$.

**Proof.** Let $S$ be the set of all regular elements of $A$. First we shall prove that $B$ is a left Goldie ring. Since $\mathcal{A}A$ is finite dimensional, $\mathcal{A}$ is finite dimensional. Then, by Th. 2.1 (3), $\mathcal{A}$ is finite dimensional. Let $I \subseteq I'$ be left ideals of $B$. Then $l_A(r_B(I)) \subseteq l_A(r_B(I'))$, where $r_B(I)=\{y \in B; l_1y=0\}$ and $l_A(r_B(I))=\{x \in A; x \cdot r_B(I)=0\}$. From this fact, $B$ satisfies the ascending chain condition for annihilator left ideals of $B$. Hence $B$ is a left Goldie ring. By Prop. 2.13, $S \cap B$ is the set of all regular elements of $B$. For any $s$ in $S$, $\mathcal{A}As$ is essential in $\mathcal{A}A$, so that $\mathcal{A}(s)$ is essential in $\mathcal{A}A$. Then, by Th. 2.1 (3), $\mathcal{A}(s) \cap B$ is essential in $\mathcal{A}B$, so that $\mathcal{A}(s) \cap B$ contains a regular element.
of $B$ ([17; Th. (3.9)]). Hence $A(s) \cap B \cap S \neq 0$ for any $s$ in $S$. Thus the present theorem follows from Th. 2.14.

Remark. In the following cases, the condition that $B$ is semi-prime is superfluous.

1. $G$ is finite and completely outer (cf. [22; p. 132]).
2. $B$ is contained in the center of $A$.

Let $T$ be a ring. If $T$-left modules $M$ and $N$ have no non-zero isomorphic subquotients, we say that $\tau M$ and $\tau N$ are unrelated (cf. [22]).

Lemma 2.16. Let $T$ be a ring, and let $M$ and $N$ be $T$-left modules, and $W$ a $T$-submodule of $M$. If $\tau(M/W)$ and $\tau N$ are unrelated, and $\tau W$ and $\tau N$ are unrelated, then $\tau M$ and $\tau N$ are unrelated.

Proof. Assume that there are isomorphic subquotients $X/X_0$ and $Y/Y_0$ of $\tau M$ and $\tau N$, respectively. Then, as is easily seen, $X + W \supseteq X_0 + W$ or $X \cap W \supseteq X_0 \cap W$. If $X + W \supseteq X_0 + W$, then $Y/Y_0 \simeq X/X_0 \rightarrow (X + W)/(X_0 + W) \neq 0$, a contradiction. If $X \cap W \supseteq X_0 \cap W$, then $(X \cap W)/(X_0 \cap W) \simeq (X_0 + (X \cap W))/X_0 \subseteq X/X_0 \simeq Y/Y_0$, which is also a contradiction.

Proposition 2.17. Let $\sigma, \tau$ be in $G$, and assume that $A\sigma A$ and $A\tau A$ are unrelated. Then, for any finite subset $\{x_1, \cdots, x_r; y_1, \cdots, y_s\}$ of $A$, there are elements $a_k, b_k \ (k = 1, \cdots, t)$ in $A$ such that $\sum_x a_kx_i \cdot \sigma(b_k) = x_i$ and $\sum_y a_ky_h \cdot \tau(b_k) = 0$ for all $x_i, y_h$.

Proof. By Lemma 2.16, $A\sigma A$ and $A\tau A$ are unrelated. Then, since $A(x_1 u_\sigma, \cdots, x_r u_\sigma, y_1 u_\tau, \cdots, y_s u_\tau)A$ is an $A$-$A$-submodule of $A\sigma A \oplus (A\tau A)^t$, $A(x_1 u_\sigma, \cdots, x_r u_\sigma, 0, \cdots, 0) \in A(x_1 u_\sigma, \cdots, x_r u_\sigma, y_1 u_\tau, \cdots, y_s u_\tau)A$ (cf. [22; Prop. 6.1]). Therefore there are elements $a_k, b_k \ (k = 1, \cdots, t)$ in $A$ such that $\sum_x a_k(x_1 u_\sigma, \cdots, x_r u_\sigma, y_1 u_\tau, \cdots, y_s u_\tau)b_k = (x_1 u_\sigma, \cdots, x_r u_\sigma, 0, \cdots, 0)$. Then, $\sum_x a_kx_i = x_i$ and $\sum_y a_ky_h \cdot \tau(b_k) = 0$ for all $x_i, y_h$.

Combining Prop. 2.17 with [22; Prop. 6.11] we can easily see the following

Proposition 2.18. Let $A$ and $A'$ be $R$-algebras with $A \otimes_R A' \neq 0$, and let $G$ and $G'$ be completely outer finite groups of $R$-automorphisms of $A$ and $A'$, respectively. Then, $G \times G'$ is completely outer as an automorphism group of $A \otimes_R A'$.

§ 3.

Proposition 3.1. Let $A/B$ be locally finite $G$-Galois, and $X$ a $\Delta$-left submodule of $A$. Then $X = A(X \cap B)$.

Proof. This follows from Th. 2.1 (3).

Proposition 3.2. Let $A/B$ be locally finite $G$-Galois, $\{\mathfrak{B}\}$ the set of
all maximal ideals of $A$, and $\{p\}$ the set of all maximal ideals of $B$. Then the following are equivalent:

(i) $\mathfrak{P} \rightarrow \mathfrak{P} \cap B$ is a mapping from $\{\mathfrak{P}\}$ onto $\{p\}$.

(ii) $ApA \neq A$ for all $p \in \{p\}$, and $\cap_{i \in G} \sigma(\mathfrak{P})$ is $\Delta$-$A$-maximal for all $\mathfrak{P} \in \{\mathfrak{P}\}$.

If (i) holds, then the following are true:

1. $pA = Ap \neq A$ for any $p \in \{p\}$.
2. $\{\cap_{i} \sigma(\mathfrak{P}); \mathfrak{P} \in \{\mathfrak{P}\}\}$ is the set of all maximal $\Delta$-$A$-submodules of $A$.
3. $\mathcal{R}(\Delta A_{A}) = \mathcal{R}(\Delta A_{A}) = \mathcal{R}(\Delta B_{B}) A = A \cdot \mathcal{R}(\Delta B_{B})$, and $\mathcal{R}(\Delta A_{A}) \cap B = \mathcal{R}(\Delta B_{B})$.
4. $B$ is $B$-$B$-completely reducible if and only if $\cap_{i} \cap_{\sigma}(\mathfrak{P}_{i}) = 0$ for some $\mathfrak{P}_{i} (i=1, \cdots, n)$ in $\{\mathfrak{P}\}$.

Proof. (i) $\Rightarrow$ (ii) If $\mathfrak{P}$ is in $\{\mathfrak{P}\}$, then $\mathfrak{P} \cap B = \sigma(\mathfrak{P}) \cap B$ for any $\sigma$ in $G$, and so $\mathfrak{P} \cap B = (\cap_{i} \sigma(\mathfrak{P})) \cap B$. By Prop. 3.1, $A ((\cap_{i} \sigma(\mathfrak{P})) \cap B) = (\cap_{i} \sigma(\mathfrak{P})) ((\cap_{i} \sigma(\mathfrak{P})) \cap B)$. Hence $Ap = pA \neq A$ for all $p$ in $\{p\}$. Let $X$ be a $\Delta$-$A$-submodule of $A$ with $A \cap X \supseteq \cap_{i} \sigma(p)$. Then $B \supseteq X \cap B \supseteq (\cap_{i} \sigma(\mathfrak{P})) \cap B = \mathfrak{P} \cap B$, and so $X \cap B = (\cap_{i} \sigma(\mathfrak{P})) \cap B$. Then, by Prop. 3.1, $X = \cap_{\sigma} \mathfrak{P}$. Thus $\cap_{\sigma} \mathfrak{P}$ is $\Delta$-$A$-maximal. Let $Y$ be a maximal $\Delta$-$A$-submodule of $A$. Take a maximal $\mathfrak{P}_{1}$ of $A$ with $\mathfrak{P}_{1} \supseteq Y$. Then $\cap_{\sigma} \mathfrak{P}_{1} \supseteq Y$, and so $\cap_{\sigma} \mathfrak{P}_{1} = Y$. Thus we obtain (2). Therefore $\mathcal{R}(\Delta A_{A}) = \mathcal{R}(\Delta A_{A})$. Since $\mathcal{R}(\Delta A_{A}) \cap B = \mathcal{R}(\Delta B_{B})$, we have $\mathcal{R}(\Delta A_{A}) = A \cdot \mathcal{R}(\Delta B_{B}) = \mathcal{R}(\Delta B_{B}) A$ (Prop. 3.1). $B$ is $B$-$B$-completely reducible if and only if $\cap_{i} p_{i} = 0$ for some $p_{1}, \cdots, p_{n}$ in $\{p\}$. Thus we obtain (4) (cf. Prop. 3.1). (ii) $\Rightarrow$ (i) Let $p \in \{p\}$. Then, as $ApA \neq A$, $p \subseteq \mathfrak{P}$ for some $\mathfrak{P} \in \{\mathfrak{P}\}$, and so $p = \mathfrak{P} \cap B$ by the maximality of $p$. Let $\mathfrak{Q}$ be in $\{\mathfrak{P}\}$. Then $q \supseteq \cap_{\sigma} \mathfrak{P}$ for some $q \in \{q\}$. There is a $\mathfrak{Q} \in \{\mathfrak{P}\}$ with $\mathfrak{Q} \cap B = q$. Then $(\cap_{\sigma} \mathfrak{Q}) \cap B = \mathfrak{Q} \cap B \supseteq \cap_{\sigma} \mathfrak{P}$, and therefore $\cap_{\sigma} \mathfrak{Q} \supseteq \cap_{\sigma} \mathfrak{P}$ by Prop. 3.1. By assumption, $\cap_{\sigma} \mathfrak{Q} = \cap_{\sigma} \mathfrak{P}$. Hence $q = \cap_{\sigma} \mathfrak{Q} \cap B = \cap_{\sigma} \mathfrak{P}$. This completes the proof.

Concerning Prop. 3.2, we state the following

Lemma 3.3. Let $\mathfrak{P}$ be a maximal ideal of $A$ such that $\cap_{\sigma} \sigma(\mathfrak{P}) = \cap_{i} \sigma_{i}(\mathfrak{P})$ for some $\sigma_{1}, \cdots, \sigma_{n}$ in $G$. Then $\cap_{\sigma} \sigma(\mathfrak{P})$ is $\Delta$-$A$-maximal, and $\{\sigma_{i}(\mathfrak{P}); i=1, \cdots , n\}$ is the set of all maximal ideals containing $\cap_{\sigma} \sigma(\mathfrak{P})$.

Proof. Let $\Delta$ be a maximal ideal of $A$ with $\Delta \supseteq \cap_{\sigma} \sigma(\mathfrak{P})$. If $\Delta \neq \sigma_{i}(\mathfrak{P})$ for all $i$, then $\Delta + \sigma_{i}(\mathfrak{P}) = A$ for all $i$. Then we have a contradiction $A = \Delta + \cap_{i} \sigma_{i}(\mathfrak{P}) = \Delta + \cap_{\sigma} \sigma(\mathfrak{P})$.

Remark. In the following cases, the assumption in Lemma 3.3 holds.

1. $G$ is finite. 2. The ring $A/\mathcal{R}(\Delta A_{A})$ satisfies the descending chain condition for ideals. 3. $G^{*}$ is compact, and every maximal ideal of $A$ is $A$-$A$-finitely generated. (Cf. Prop. 1.1).
Proposition 3.4.

(1) Let $A/B$ be locally finite outer $G$-Galois, and $B$ B-B-completely reducible. Assume that, for any maximal ideal $\mathfrak{P}$ of $A$, there are elements $\sigma_1, \cdots, \sigma_n$ in $G$ such that $\cap_i \sigma_i(\mathfrak{P}) = \cap_i \sigma(\mathfrak{P})$. Then $A$ is $A$-A-completely reducible.

(2) Let $G$ be finite and completely outer, and $B|A_B$. Then $A$ is $A$-A-completely reducible if and only if $B$ is $B$-B-completely reducible. If there is a maximal ideal $\mathfrak{P}$ of $A$ such that $\cap_i \sigma(\mathfrak{P}) = 0$, then $B$ is $B$-B-minimal, and conversely.

Proof. (1) Any maximal ideal $\mathfrak{p}$ of $B$ is written as $\mathfrak{p} = Be$ with a central idempotent $e$ of $B$. Then, by assumption, $(1 \neq e) e \in V_A(B) = C$. Therefore, $A\mathfrak{p} = Ae = eA = \mathfrak{p}A \neq A$. Thus, by Prop. 3.2 and Lemma 3.3, $A$ is $A$-A-completely reducible. (2) In this case, $\alpha A = A\alpha \neq A$ for any proper ideal $\alpha$ of $B$ (cf. [22; p. 132]). Then, by Prop. 3.2 and Lemma 3.3, the first assertion is evident (cf. [22; Prop. 6.4]). For any $\mathfrak{P}$ in $\{\mathfrak{P}\}$, $((\cap_i \sigma(\mathfrak{P})) \cap B =) \mathfrak{P} \cap B = 0$ if and only if $\cap_i \sigma(\mathfrak{P}) = 0$ (Prop. 3.1). Thus we know the second assertion.

Theorem 3.5. Let $A/B$ be finite $G$-Galois, $B$ a semi-primary ring, and $A\mathfrak{p}A \neq A$ for any maximal ideal $\mathfrak{p}$ of $B$. Then $B\mathfrak{p}A \simeq B\mathfrak{p}BG$, that is, $A$ has a normal basis. (Cf. [13; Th. 1]).

Proof. By [22; Th. 1.7], it suffices to prove that $B\mathfrak{p}A$ is free. Let $g = (G:1)$. (1) First we assume that $\mathfrak{R}(B) = 0$. Then $B$ is a direct sum of simple rings: $B = a_1 + \cdots + a_n$. Let $1 = \sum_i e_i$, $e_i \in a_i$. Then $\alpha_i = Be_i = e_i B$ and $e_i^2 = e_i$. By assumption we have $(1-e_i)A = A(1-e_i)$ (Prop. 3.2 and Lemma 3.3), so that $e_i$ is a central idempotent of $A$ contained in $B$. Then each $Ae_i/Be_i$ is $G$-Galois ([22; Cor. to Th. 5.6]). Since $Be_i$ is a simple ring, $Be_iAe_i$ is free (cf. [7]). Hence $Ae_i$ has a normal basis, so that $B\mathfrak{p}Ae_i \simeq B\mathfrak{p}(Be_{i})^p$ for all $i$ ([22; Th. 1.7]). Hence $B\mathfrak{p}A \simeq B\mathfrak{p}^p$. (2) Next we proceed to general case. Since $A$ and $B$ are semi-primary ([22; Prop. 7.3]), $\mathfrak{R}(\mathfrak{a}A_A) = \mathfrak{R}(A)$ and $\mathfrak{R}(\mathfrak{b}B_B) =$ $\mathfrak{R}(B)$. Then, by Prop. 3.2 and Lemma 3.3, $\mathfrak{R}(A) = \mathfrak{R}(B) = A = \mathfrak{R}(B)$ and $\mathfrak{R}(A) \cap B = \mathfrak{R}(B)$. By [22; Th. 5.6], $(A/\mathfrak{R}(A))/(B + \mathfrak{R}(A))/\mathfrak{R}(A)$ is $G$-Galois, and satisfies the same conditions as $A/B$, because $(B + \mathfrak{R}(A))/\mathfrak{R}(A) \simeq B/(\mathfrak{R}(A) \cap B) = B/\mathfrak{R}(B)$ canonically. By (1), we have $B\mathfrak{p}A/\mathfrak{R}(A) \simeq B/\mathfrak{p}(B/\mathfrak{R}(B))^p$. Since $\mathfrak{R}(A) = \mathfrak{R}(B)A$ and $B\mathfrak{p}A$ is finitely generated and projective, we have $B\mathfrak{p}A \simeq B\mathfrak{p}$.

Corollary. Let $A/B$ be finite $G$-Galois, $B$ a semi-primary ring, and $Z$ the center of $B$. Assume that $Z \subseteq C$ and that $B$ is a central separable $Z$-algebra. Then $A$ has a normal basis.

Proof. In this case, any proper ideal of $B$ is written as $\alpha B$ with an ideal
α of \(Z\) (cf. [2]). Then, as \(Z \subseteq C\), \((αB)A = αA = Aα = A(Bα) \neq A\) ([22; Lemma 7.1]).

Let \(A/B\) be finite \(G\)-Galois, \(B \subseteq C\), and \(g = (G:1)\). For any prime ideal \(p\) of \(B\), we denote by \(B_p\) the quotient extension of \(B\) with respect to \(p\). Then \(B_p\) is a \(B\)-algebra, canonically. By [22; Cor. to Th. 5.2], \((B_p \otimes_B A)/B_p\) is \(G\)-Galois. Since \(B_p\) is a local ring, \(B_p B_p \otimes_B A \cong B_p(B_p)^g\) (Cor. to Th. 3.5). We denote by \(K_p\) the quotient field of \(B/p\). Then we have \(K_p K_p \otimes_B A \cong K_p(K_p)^g\) similarly. Thus we obtain the following

**Proposition 3.6.** Let \(A/B\) be finite \(G\)-Galois, \(B \subseteq C\), and \(g = (G:1)\). Then, \(b_p B_p \otimes_B A \cong B_p(B_p)^g\) and \(K_p K_p \otimes_B A \cong K_p(K_p)^g\) for any prime ideal \(p\) of \(B\), where \(B_p\) is the quotient extension of \(B\) with respect to \(p\) and \(K_p\) is the quotient field of \(B/p\).

The following lemma is of some interest.

**Lemma 3.7.** Let \(R \supseteq S\) be rings, \(R_s\) is finitely generated and projective, and \(sS\) is a direct summand of \(sR\). If \(sR\) is injective, then \(sS\) is injective.

**Proof.** Let \(I\) be any left ideal of \(S\), and \(f\) any \(S\)-left homomorphism from \(I\) to \(sR\). Since \(R_s\) is finitely generated and projective, we have \(RI = R \otimes S I\). Therefore \(f\) can be extended to an \(R\)-left homomorphism from \(RI\) to \(R\), canonically. Then, by assumption, there is an element \(a\) in \(R\) such that \(r \cdot (s)f = rsa\) for \(r\) in \(R\) and \(s\) in \(I\), so that \((s)f = sa\) for all \(s\) in \(I\). Therefore, as is well known, \(sR\) is injective. Since \(sS\) is a direct summand of \(sR\), \(sS\) is injective.

**Lemma 3.8.** \(\mathfrak{A}(A) \cap B \subseteq \mathfrak{A}(B)\).

**Proof.** Let \(b\) be in \(\mathfrak{A}(R) \cap B\). Then \(1 - b\) has an inverse in \(A\). Since \(B = A^g\), \(1 - b\) has an inverse in \(B\). Hence \(\mathfrak{A}(A) \cap B\) is a quasi-regular ideal of \(B\), that is, \(\mathfrak{A}(A) \cap B \subseteq \mathfrak{A}(B)\).

**Proposition 3.9.** Let \(G\) be finite. If there is an element \(c\) in \(A\) such that \(1 - t_{o}(c) \in \mathfrak{A}(A)\), then there is an element \(d\) in \(A\) such that \(t_{o}(d) = 1\).

**Proof.** By Lemma 3.8, we have \(1 - t_{o}(c) \in \mathfrak{A}(A) \cap B \subseteq \mathfrak{A}(B)\), so that \(t_{o}(A) + \mathfrak{A}(B) = B\). Since \(t_{o}(A)\) is an ideal of \(B\), we have \(t_{o}(A) = B\). Hence \(t_{o}(d) = 1\) for some \(d\) in \(A\).

**Theorem 3.10.** Let \(A/B\) be \(G\)-Galois, \(A\) a commutative ring, \(H\) a subgroup of \(G\), and \(A'\) a \(B\)-algebra. Then, \(A' \otimes_B A^\alpha\) is a direct sum of minimal ideals if and only if \(A'\) is a direct sum of minimal ideals (cf. [7; p. 178. Th. 2]).

**Proof.** In this case, \((A' \otimes_B A)/A'\) is finite \(G\)-Galois, \(G\) is completely outer as an automorphism group of \(A' \otimes_B A\), and \((A' \otimes_B A)^\alpha = A' \otimes_B A^\alpha\) (cf. [22; Th.
5.2 and Prop. 6.5]). Thus the present theorem is an easy consequence from Prop. 3.4 (2).

Concerning [22; Th. 6.9], we note the following

**Lemma 3.11.** Let $A/C$ be separable, and $e$ an idempotent of $A$ such that $eA \subseteq Ae$. Then $e$ is a central idempotent of $A$.

**Proof.** Since $A\Re(A)$ is a semi-prime ring, we have $(eA + \Re(A))/\Re(A) = (Ae + \Re(A))/\Re(A)$, that is, $eA + \Re(A) = Ae + \Re(A)$, and so $Ae = eA + (Ae \cap \Re(A))$. Since $A$ is a central separable $C$-algebra, $\Re(A) = \Re(C)A$ by [2; Cor. 3.2]. Since $\Re(A_{A}) \supseteq \Re(A) \supseteq \Re(C)A$, we have $\Re(A) = \Re(C)A$, and $Ae = eA + \Re(C)Ae$. Hence $Ae = eA$, because $eAe$ is finitely generated. Consequently, $e$ is a central idempotent of $A$.

**Proposition 3.12.** Let $A/B$ be locally finite $G$-Galois, and assume that there is a representation $A = \bigcup_{N_{i}} A^{N_{i}}$ of $A/B$ such that each $\Re(B)A^{N_{i}}$ is an ideal of $A^{N_{i}}$. Then $\Re(A) = \Re(B)A = A \cdot \Re(B)$, and $\Re(A) \cap B = \Re(B)$.

**Proof.** Let $\mathfrak{J}$ be a right ideal of $A$ such that $\Re(B)A + \mathfrak{J} = A$. Then $\Re(B)A^{N_{i}} + (\mathfrak{J} \cap A^{N_{i}}) \ni 1$ for some $\lambda$ in $A$, so that $\Re(B)A^{N_{i}} + (\mathfrak{J} \cap A^{N_{i}}) = A^{N_{i}}$. Since $\Re(B)A^{N_{i}} \subseteq \Re(A^{N_{i}})$, we have $\mathfrak{J} \cap A^{N_{i}} = A^{N_{i}}$, and hence $\mathfrak{J} = A$. Thus we know that $\Re(B)A \subseteq \Re(A)$. Combining this with Lemma 3.8, we have $\Re(A) \cap B = \Re(B)$. Hence $\Re(A) = \Re(B)A = A \cdot \Re(B)$ (Prop. 3.1).

**Theorem 3.13.** Let $A/B$ be locally finite $G$-Galois, $B \subseteq C$, and $A'$ a $B$-algebra such that $A' \simeq A' \otimes 1 (\subseteq A' \otimes_{B} A)$ canonically.

1. $\Re(A' \otimes_{B} A) = \Re(A' \otimes A)$, and $\Re(A' \otimes A) \cap (A' \otimes 1) = \Re(A') \otimes 1$.

2. If $A$ is commutative, then $\Re(A' \otimes A^H) = \Re(A') \otimes A^H$ for any subgroup $H$ of $G$.

**Proof.** Let $A = \bigcup_{N_{i}} A^{N_{i}}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then $A' \otimes_{B} A / (A' \otimes 1)$ is locally finite $G$-Galois extension with representation $A' \otimes_{B} A = \bigcup_{\lambda} A' \otimes A^{N_{i}}$, where $A' \otimes A^{N_{i}} = (A' \otimes_{B} A)^{N_{i}}$ is a finite $G/N_{i}$-Galois extension over $A' \otimes 1$. (1) This will be easily seen by Prop. 3.12. (2) We may assume that $H$ is closed in $G$. Then each $A^H \cap N_{i} / A^H$ is finite $H/(H \cap N_{i})$-Galois, and $H/(H \cap N_{i})$ is completely outer as an automorphism group of $A^H \cap N_{i}$ (22; Th. 6.6). Then $H/(H \cap N_{i})$ is completely outer as an automorphism group $A' \otimes_{B} A^H \cap N_{i}$ (Prop. 2.18), and so $H/(H \cap N_{i})$ is completely outer as an automorphism group $A' \otimes_{B} A^H \cap N_{i}$ (22; Prop. 6.11). Now, $(A' \otimes_{B} A) / (A' \otimes A^H)$ is a locally finite $H$-Galois extension with representation $A' \otimes_{B} A = \bigcup_{\lambda} A' \otimes A^H \cap N_{i}$, where $A' \otimes A^H \cap N_{i} = (A' \otimes_{B} A)^{H \cap N_{i}}$ is a finite $H/(H \cap N_{i})$-Galois extension over $A' \otimes A^H$. Then, by [22; Th. 7.10] and Prop. 3.12, $\Re(A' \otimes_{B} A) = \Re(A' \otimes A^H) (A' \otimes_{B} A)$. On the other hand,
$\mathcal{R}(A' \otimes_B A) = \mathcal{R}(A') \otimes A = (\mathcal{R}(A') \otimes A^H)(A' \otimes_B A)$. Hence $\mathcal{R}(A' \otimes A^B) = \mathcal{R}(A') \otimes A^H$, as desired (cf. [22; Lemma 7.1]).

References

([1]~[14] are found in [22] below.)


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