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HOKKAIDO UNIVERSITY
LOCALLY FINITE OUTER GALOIS THEORY

By

Yōichi MIYASHITA

Introduction.

This paper is the continuation of the preceding paper [22]. In §1 and §2, locally finite (outer) Galois extensions are treated. The main results are parallel to those of the finite case. In these studies, Nagahara [12] is our guide. Further several results for finite Galois extensions are added (Th. 1.18). In §3, we give a normal basis theorem for a finite Galois extension.

§1. As to notations and terminologies we follow [22]. Let $A$ be a ring with $1$ $(\neq 0)$, $C$ the center of $A$, $G$ a (finite or infinite) group of automorphisms of $A$, $B=A^\sigma=\{x\in A; \sigma(x)=x$ for all $\sigma$ in $G\}$, and $\hat{G}$ the group of all $B$-automorphisms of $A$. $\hat{G}$ is then a topological group in finite topology (cf. Jacobson [7]). We denote the closure of $G$ in $\hat{G}$ by $G^*$. $\Delta$ means the trivial crossed product of $A$ with $G$: $\Delta=\sum_{\sigma\in \Theta} A\sigma_{u_{\sigma}}, \ u_{\sigma}u_{\tau}=\sigma_{\tau}u_{\sigma_{\tau}} (\sigma, \tau\in G)$. $A\Delta_{x} = \sigma(x)u_{\sigma_{\tau}} (x\in A)$. Then there is a canonical ring homomorphism $j$ from $\Delta$ to End $(A_{\Theta})$ defined by $j(\sum_{\sigma\in \Theta} x_{\sigma}u_{\sigma})(y)=\sum_{\sigma\in \Theta} x_{\sigma}\cdot \sigma(y)$ $(\sum_{\sigma\in \Theta} x_{\sigma}u_{\sigma}\in A, y\in A)$. For any intermediate ring $T$ of $A/B$, $G^T=\{\sigma\in \Theta; \sigma|T=1_{T}\}$ is a subgroup of $G$, where $\sigma|T$ means the restriction of $\sigma$ to $T$. We call it a fixed subgroup of $G$. For any subgroup $H$ of $G$, $A^H=\{x\in A; \sigma(x)=x$ for all $\sigma$ in $H\}$ is an intermediate ring of $A/B$. We call it a fixed subring of $A$ (with respect to $G$). Then, as is well known, the set of all fixed subgroups of $G$ and the set of all fixed subrings of $A$ are anti-order-isomorphic in the usual sense of Galois theory. A subring $T$ of $A$ is called a $G$-invariant subring of $A$ if $\sigma(T)=T$ for all $\sigma$ in $G$ (or equivalently, $\sigma(T)\subseteq T$ for all $\sigma$ in $G$). Let $N$ be a fixed subgroup of $G$. Then, $A^N$ is $G$-invariant if and only if $N$ is a normal subgroup of $G$: $N\triangleleft G$. Let $T$ be an intermediate ring of $A/B$, and put $H=G^T$. Then, for $\sigma, \tau$ in $G$, $\sigma|T=\tau|T$ if and only if $\sigma H=\tau H$. Let $H$ and $K$ be subgroups of $G$ such that $H \supseteq K$ and $(H:K)<\infty$, and let $H=\sigma_{1}K\cup \cdots \cup \sigma_{r}K$ be the left coset decomposition. For any $x$ in $A^\kappa$ we put $t_{B,K}(x)=\sum_{i} \sigma_{i}(x)$. Then $t_{B,K}$ is an $A^\kappa-A^\nu$-homomorphism from $A^\kappa$ to $A^\nu$, and is independent of the choice of $\sigma_{1}, \cdots, \sigma_{r}$. If $K=1$, we write simply $t_B$ instead of $t_{B,1}$.

Here we present several fundamental facts, which are essential throughout the present study. Let $rM_{\nu}$ and $rN_{\nu}$ be $T$-left, $U$-right modules. If $rM_{\nu}$ is
isomorphic to a direct summand of \( rN_{U} \) for some natural number \( r \), then we write \( rM_{U}|_{T}rN_{U} \), where \( rN_{U} \) means the direct sum of \( r \) copies of \( rN_{U} \). If \( rM_{U}|_{T}rN_{U} \) and \( rN_{U}|_{T}rM_{U} \) we write \( rM_{U} \sim rN_{U} \) (similar) (cf. Morita [21]). To be easily seen, \( rM_{U}|_{T}N_{U} \) if and only if there are \( T-U \)-homomorphisms \( f_{1}, \ldots, f_{r} \) in \( \text{Hom}(rM_{U}, rN_{U}) \) and \( g_{1}, \ldots, g_{r} \) in \( \text{Hom}(rN_{U}, rM_{U}) \) such that \( \sum_{i=1}^{r} f_{i}g_{i} \) is the identity of \( M \), or equivalently, \( \text{Hom}(rM_{U}, rN_{U}) \cdot \text{Hom}(rN_{U}, rM_{U})=\text{Hom}(rM_{U}, rM_{U}) \), where homomorphisms act on the right side.

Let \( T \) be a ring with \( 1 \), \( M \) a unital \( T \)-left module, and \( T^*=\text{End}(rM) \).

S.1. If \( rT|_{T}M \) then \( M_{T}|^{*}T_{T} \). \( \text{(i.e. } M_{T} \text{ is finitely generated and projective) and } T=\text{End}(M_{T}). \) (Morita)

S.2. If \( rM|_{T}T \) then \( T^{*}_{T}|_{M_{T}} \). \( \text{(Morita) \quad (Auslander-Buchsbaum-Goldman) \quad \text{(Müller)}} \)

S.3. Let \( T \) be commutative. If \( rM|_{T}T \) and \( rM \) is faithful, then \( rT|_{T}M \).

S.4. Let \( \bar{T} \) be an extension ring of \( T \). If \( rT|_{T}\bar{T} \) then \( rT \) is a direct summand of \( r\bar{T} \) (and conversely). \( \text{(Müller}} \)

S.5. Let \( \bar{T} \) be an extension ring of \( T \). If \( rT_{T}|_{T}\bar{T} \) then \( rT_{T} \) is a direct summand of \( r\bar{T}_{T} \). \( \text{(The proof is similar to the one of S.4.)} \)

In [22], \( A/B \) was called a \( G \)-Galois extension if \( G \) is finite and there are elements \( a_{1}, \ldots, a_{n} \); \( a_{1}^{*}, \ldots, a_{n}^{*} \) in \( A \) such that \( \sum_{i=1}^{n} a_{i}^{*} \sigma(a_{i})=\delta_{1,\sigma} \) (\( \sigma \in G \)). In this paper, \( A/B \) is called a finite \( G \)-Galois extension if \( A/B \) is \( G \)-Galois and \( t_{0}(c)=1 \) for some \( c \) in \( A \). Then, the following are equivalent:

(a) \( A/B \) is finite \( G \)-Galois.
(b) \( G \) is finite, \( A_{B} \sim B_{B} \) and \( j: A\simeq \text{End}(A_{B}). \)
(c) \( G \) is finite and \( A_{A} \sim A \).

(Cf. S.1, S.2, [6] and [21]).

\( A/B \) is called a locally finite \( G \)-Galois extension if there are fixed normal subgroups \( N_{i} (\lambda \in \Lambda) \) of \( G \) which satisfy the following conditions: \( (1) \ (G:N_{i})<\infty \), and \( A^{N_{i}}|B \) is a finite \( G/N_{i} \)-Galois extension. \( (2) \ A=\bigcup_{\lambda} A^{N_{\lambda}}, \text{ and } \{A^{N_{\lambda}} ; \lambda \in \Lambda\} \) is a directed set with respect to the inclusion relation (abbre. \( A=\bigcup_{\lambda} A^{N_{\lambda}} \) is a directed union). Then we call \( A=\bigcup_{\lambda} A^{N_{\lambda}} \) a representation of the locally finite \( G \)-Galois extension \( A/B \). If \( V_{A}(B)=C \), an extension \( A/B \) is said to be outer.

Now we shall prove first the following

**Proposition 1.1.** Let \( G=G* \) (i.e. \( G \) is closed in \( \hat{G} \)). Then the following are equivalent:

(i) \( \{\sigma(x); \sigma \in G\} \) is finite for any \( x \) in \( A \).
(ii) \( G \) is compact.
(iii) Every directed union of fixed subrings of \( A \) with respect to \( G \) is also a fixed subring of \( A \) with respect to \( G \), and \( \cap H=1 \), where \( H \) ranges
over all fixed subgroups of $G$ such that $(G:H)<\infty$.

Proof. (i) $\implies$ (ii) If we put $\Pi_{x\in A}\{\sigma(x); \sigma\in G\}=D$, then $G\subseteq D$ and $D$ is compact. Therefore it is sufficient to prove that $G$ is closed in $D$. Let $\rho$ be any element of the closure of $G$ in $D$. Then, as is easily seen, $\rho$ is a $B$-ring isomorphism from $A$ into $A$. Let $a$ be in $A$, and put $F=\{\sigma(a); \sigma\in G\}$. Then, by assumption, $F$ is a finite subset of $A$, so that there is an element $\tau$ in $G$ such that $\rho|F=\tau|F$. Then, in particular, $\rho(\tau^{-1}(a))=\tau(\tau^{-1}(a))=a$. Thus $\rho$ is a $B$-automorphism of $A$. Hence the closure of $G$ in $D$ is contained in $\hat{G}$.

Since $G$ is closed in $\hat{G}$, $G$ is closed in $D$, as desired. (ii) $\implies$ (iii) For any $x$ in $G$, we put $H_x=\{\sigma\in G; \sigma(x)=x\}$. Then $H_x$ is open in $G$, and therefore $\sigma H_x$ is open in $G$ for any $\sigma$ in $G$. Then, since $G$ is compact, we have $(G:H_x)<\infty$. Evidently $\cap_{x\in A}H_x=1$. This proves the second assertion.

Let $(\Lambda\neq) T=\bigcup_{i\in A}T_i$ be a directed union of fixed subrings of $A$ with respect to $G$, and let $K_i=G^{T_i}$. Then each $K_i$ is a closed subgroup of $G$, and $A^{K_i}=T_i$. Let $a$ be an element of $A-T$, and put $U=\{\sigma\in G; \sigma(a)=a\}$. Then $U$ is open in $G$, so that each $K_i-U$ is closed in $G$. Since $a\not\in T_i$ and $A^{K_i}=T_i$, we have $K_i-U\neq\emptyset$. For any finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of $\Lambda$, there is an element $\lambda_0$ of $\Lambda$ such that $T_{\lambda_0}\supseteq \bigcup_i T_{\lambda_i}$. Then $K_{\lambda_0}\subseteq \cap_i K_{\lambda_i}$, and so $0\neq K_{\lambda_0}-U\subseteq \cap_i K_{\lambda_i}-U=\cap_i (K_{\lambda_i}-U)$. Thus $\{K_i-U; \lambda\in \Lambda\}$ has finite intersection property. Since $G$ is compact, we have $\cap_{\lambda}(K_{\lambda}-U)\neq\emptyset$. If $\rho$ is in $\cap_{\lambda}(K_{\lambda}-U)$ then $\rho\in G^{T}$ and $\rho(a)\neq a$. Therefore $a\neq A^{K}$, where $K=G^{T}$. Thus $A^{K}=T$. Hence $T$ is a fixed subring of $A$ with respect to $G$. (iii) $\implies$ (i) Let $H$ and $K$ be fixed subgroups of $G$ such that $(G:H)<\infty$ and $(G:K)<\infty$. Then $H\cap K$ is a fixed subgroup of $G$ with $(G:H\cap K)<\infty$. Therefore $\bigcup A^{H}$ is a directed union of fixed subrings of $A$, where $H$ ranges over all fixed subgroups of $G$ with $(G:H)<\infty$. Then, by assumption, $\bigcup A^{H}$ is a fixed subring of $A$ with respect to $G$. Since $\cap H=1$, we have $A=\bigcup A^{H}$. For any $x$ in $A$, there is an $A^{H}$ such that $x\in A^{H}$. Therefore if we put $L=\{\sigma\in G; \sigma(x)\neq x\}$ then $(G:L)<\infty$. This implies that $\{\sigma(x); \sigma\in G\}$ is finite.

Remark. For any $x$ in $A$, $\{\sigma(x); \sigma\in G\}=\{\sigma(x); \sigma\in G^{*}\}$.

Proposition 1.2. Let $N$ be a fixed normal subgroup of $G$ such that $(G:N)<\infty$ and $A^{N}/B$ is finite $G/N$-Galois, and $G_{1}$ a subgroup of $G^{*}$ containing $G$. Then $A^{N}/B$ is finite $G_{1}/N_{1}$-Galois, where $N_{1}=\{\sigma\in G_{1}; \sigma A^{N}=A^{N}\}$.

Proof. Put $T=A^{N}$. Evidently $A^{N}=T$. Since $G$ is dense in $G_{1}$ and $T_{B}$ is finitely generated, there holds $G|T=G_{1}|T$. Therefore $T$ is $G_{1}$-invariant, $N_{1}\lhd G_{1}$, and $(G_{1}:N_{1})<\infty$. There are elements $a_{1}, \cdots, a_{n}; a_{1}^{*}, \cdots, a_{n}^{*}$ in $T$ such that $\sum_{i} a_{i}\cdot \sigma(a_{i}^{*})=\delta_{N_{1}}$ for all $\sigma$ in $G$. If $\tau$ is in $G_{1}-N_{1}$ then $\tau|T=\rho|T$ for
some \( \rho \) in \( G-N \), and \( \sum_i a_i \cdot \tau(a_i^*) = \sum_i a_i \cdot \rho(a_i^*) = 0 \). Thus \( \sum_i a_i \cdot \sigma(a_i^*) = \delta_{\lambda, \sigma} \) for \( \sigma \) in \( G_1 \).

Corollary. Let \( A|B \) be locally finite \( G \)-Galois, and \( G_1 \) a subgroup of \( G^* \) containing \( G \). Then \( A|B \) is locally finite \( G_1 \)-Galois.

Proposition 1.3. Let \( H_\lambda (\lambda \in \Lambda) \) be fixed subgroups of \( G \) such that \( A = \bigcup_{\lambda \in \Lambda} A^{H_\lambda} \) is a directed union.

1. If \( H \) is a subgroup of \( G \) such that \( (G:H) < \infty \) then \( A^H \subseteq A^{H_\lambda} \) for some \( \lambda \) in \( \Lambda \).

2. If \( K \) is a subgroup of \( G \) such that \( (K:1) < \infty \) then \( K \cap H = 1 \) for some \( \mu \) in \( \Lambda \).

Proof. 1. Let \( [H_\lambda \cup H] \) be the subgroup of \( G \) generated by \( H_\lambda \cup H \). Since \( G \supseteq [H_\lambda \cup H] \supseteq H_\lambda \), we have \( (G:[H_\lambda \cup H]) \leq (G:H) \) for all \( \lambda \) in \( \Lambda \). Let \( (G:[H_\lambda \cup H]) \) be maximum. We shall prove that \( A^H \subseteq A^{H_\lambda} \). For any \( H_\lambda \) there is an \( H_\lambda \) such that \( A^H \supseteq A^{H_\lambda} \cup A^{H_{\lambda_0}} \). Then \( H_\lambda \subseteq H_\lambda \cap H_{\lambda_0} \), and so \( [H_\lambda \cup H] \subseteq [H_\lambda \cup H] \cap [H_{\lambda_0} \cup H] \). Since \( (G:[H_\lambda \cup H]) \) is maximum, we have \( ([H_\lambda \cup H] \supseteq [H_\lambda \cup H] = [H_{\lambda_0} \cup H] \). Hence \( [H_\lambda \cup H] \subseteq [H_\lambda \cup H] \) for all \( \lambda \) in \( \Lambda \). Then \( A^H = \bigcup_\lambda (A^H \cap A^{H_{\lambda_0}}) \cup \bigcup_\lambda A^{H_{\lambda_0} \cup H_{\lambda_0}} \). Since \( (G:[H_\lambda \cup H]) \) is maximum, we have \( ([H_\lambda \cup H] \supseteq [H_\lambda \cup H] = [H_{\lambda_0} \cup H] \). Hence \( [H_\lambda \cup H] \subseteq [H_\lambda \cup H] \) for all \( \lambda \) in \( \Lambda \). Then \( A^H = \bigcup_\lambda (A^H \cap A^{H_{\lambda_0}}) \cup \bigcup_\lambda A^{H_{\lambda_0} \cup H_{\lambda_0}} \). By assumption there is a \( \mu \) such that \( H_\mu \subseteq \bigcap_\lambda H_\lambda \). Then \( H_\mu \subseteq H \cap (\bigcap_\lambda H_\lambda) \).

Remark. Let \( A|B \) be locally finite \( G \)-Galois, and \( A = \bigcup_{\lambda \in \Lambda} A^{N_{\lambda}} \) its representation. If \( G \) is finite then \( A = A^{N_1} \) for some \( \lambda \).

Proposition 1.4. Let \( T \) be an intermediate ring of \( A|B \) such that \( G|T \) is finite, and let \( H = G^T \), and \( G = \sum_{i \in I} H \cup \cdots \cup H \) a left coset decomposition of \( G \). If there are elements \( t_1, \ldots, t_r; t^*_1, \ldots, t^*_r \) in \( T \) such that \( \sum t_i \cdot \sigma(t^*_i) = \delta_{\mu, \sigma} \) for all \( \sigma \) in \( G \), then there hold the following.

1. \( T = A^H \), and \( T_B \) is finitely generated and projective.

2. \( \sum t_i \cdot \sigma(t^*_i) = \delta_{\mu, \sigma} \) for all \( \sigma \) in \( G \), then there hold the following.

3. The following are equivalent: (i) \( B|T \). (ii) \( B|T \). (iii) \( (B) = 1 \) for some \( c \) in \( T \).

Proof. 1. \( t_{\sigma:B} \) is a \( B-B \)-homomorphism from \( A^H \) to \( B \). For any \( y \) in \( A^H \), \( T \cap \sum t_i \cdot t_{\sigma:B}(t^*_i) = \sum t_i \cdot \sigma(t^*_i) = \sum t_i \cdot \sigma(t^*_i) = y \). Hence \( A^H = T \), and \( T_B \) is finitely generated and projective (cf. [3]). (2) \( \sigma^{-1} \) is the mapping such that \( \sum_{f \in \Hom(T_B, A_B)}(f) = \sum t_i \cdot f(t_i)(t^*_i) \). (\( f \in \Hom(T_B, A_B) \)). The second part will be easily seen. (3) The equivalence (i) \( \iff \) (iii) is easy from (2).
Therefore (i) and (ii) are equivalent, because the situation is right-left symmetric.

**Proposition 1.5.** Let $A/B$ be locally finite $G$-Galois. Then there hold the following:

1. $G^*$ is compact.
2. By $j$, $\Delta$ is isomorphic to a dense subring of $\text{Hom}(A_B, A_B)$.
3. A subgroup $H$ of $G$ is a closed subgroup of $G$ if and only if $H$ is a fixed subgroup of $G$.

**Proof.** Let $A = \bigcup_{\nu} A^{N_{\nu}}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) If $x$ is in $A$ then $x \in A^{N_{\nu}}$ for some $\nu$ in $\Lambda$. Then $(G: N_{\nu}) < \infty$ implies that $\{\sigma(x); \sigma \in G\} = \{\sigma(x); \sigma \in G^*\}$ is finite. Hence, by Prop. 1.1, $G^*$ is compact. (2) By Prop. 1.4 (2), $\text{Im} j$ is dense in $\text{Hom}(A_B, A_B)$. Therefore it suffices to prove that $j$ is $1 - 1$. Let $\sigma_1, \ldots, \sigma_r$ be different elements in $G$. Then there is a finite subset $F$ of $A$ such that $\sigma_i[F \neq \sigma_j[F$ provided $i \neq k$. From this fact and Prop. 1.4, we can easily see that $j$ is $1 - 1$. (3) Evidently, a fixed subgroup is a closed subgroup. Let $H$ be any subgroup of $G$, and put $H' = G^*$, where $T = A^H$. Then $T = A^{H'}$. It suffices to prove that $H$ is dense in $H'$. To prove this, we take any finite subset $F$ of $A$. Then $F \subseteq A^{N_{\nu}}$ for some $N_{\nu}$. Put $N = N_{\nu}$. Then, by finite Galois theory, we obtain $(G/N)^{\nu}_i = HN/N$ and $(A/N)^{\nu}_i = H^N/N$, where $T_i = A^{HN}$ and $T_i = A^{H^N}$ (cf. [22; Prop. 2.2]). Since $A^{HN} = A^H \cap A^N = A^{N} \cap A^N = A^{N^N}$, we have $HN/N = H^N/N$, that is, $HN = H'N$. Hence $H|A^N = H'|A^N$, and so $H|F = H'F$. Since $F$ is arbitrary, this implies that $H$ is dense in $H'$. This completes the proof.

**Theorem 1.6.** Let $A/B$ be locally finite $G$-Galois, $G = G^*$, and $H$ a subgroup of $G$, and let $A'$ be an indecomposable extension ring of $B$ such that $\text{V}_{A'}(B) = \text{V}_{A'}(A')$. Assume that there is a $B$-ring homomorphism $g$ from $A$ to $A'$. Then, for any $B$-ring homomorphism $f$ from $A^H$ to $A'$, there is an element $\sigma$ in $G$ such that $f = g\sigma|A^H$.

**Proof.** Let $A = \bigcup_{\nu \in A} A^{N_{\nu}}$ be a representation. For each $N_{\nu}$, there is an element $\sigma$ in $G$ such that $f|A^{HN_{\nu}} = g\sigma|A^{HN_{\nu}}$ ([22; Th. 4.1]). For each $\lambda$, we put $K_{\lambda} = \{\sigma \in G; f|A^{HN_{\nu}} = g\sigma|A^{HN_{\nu}}\}$. Then $K_{\lambda} \neq \emptyset$, and $\{K_{\lambda}; \lambda \in A\}$ has finite intersection property. Let $\tau$ be in the closure of $K_{\lambda}$ in $G$. Since $(A^{N_{\nu}})_B$ is finitely generated, $\tau|A^{N_{\nu}} = \alpha|A^{N_{\nu}}$ for some $\alpha$ in $K_{\lambda}$. Then $\tau|A^{HN_{\nu}} = \alpha|A^{HN_{\nu}}$, and so $f|A^{HN_{\nu}} = g\alpha|A^{HN_{\nu}} = g\tau|A^{HN_{\nu}}$. Hence $\tau \in K_{\lambda}$, and therefore $K_{\lambda}$ is closed in $G$. Since $G$ is compact (Prop. 1.5), we have $\bigcap_{\lambda} K_{\lambda} \neq \emptyset$. If $\rho$ is in $\bigcap_{\lambda} K_{\lambda}$, then $f|A^{HN_{\nu}} = g\rho|A^{HN_{\nu}}$ for all $\lambda$ in $A$. Since $A^H = \bigcup_{\nu} A^{HN_{\nu}}$, we know $f = g\rho|A^H$.

The following theorem will follow at once from Th. 1.6 and Cor. to Prop. 1.2.

**Theorem 1.7.** Let $A/B$ be locally finite outer $G$-Galois, and $A$ an
indecomposable ring. Then $G^* = \hat{G}$, that is, $G$ is dense in $\hat{G}$.

**Proposition 1.8.** Let $A/B$ be locally finite $G$-Galois, and $G = G^*$ (cf. Cor. to Prop. 1.2). Then there hold the following.

(1) For an intermediate ring $T$ of $A/B$ the following are equivalent.

(i) $T = A^H$ for some subgroup $H$ of $G$. (ii) There are subgroups $H_i$ ($i \in \Gamma$) of $G$ such that $T = \bigcup_i A^{H_i}$, $G : H_i < \infty$ and $\{A^{H_i} ; i \in \Gamma\}$ is a directed set with respect to the inclusion relation.

(2) If $H$ is a subgroup of $G$ such that $(G : H) < \infty$ then $(A^H)_B$ is finitely generated.

**Proof.** Let $A = \bigcup_{\lambda \in \Lambda} A^{N_{\lambda}}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) $(i) \implies (ii)$ $T = A^H = \bigcup_i (A^H \cap A^{N_{\lambda}}) = \bigcup_i A^{N_{\lambda}}$ is a directed union, and $(G : H N_{\lambda}) < \infty$. (ii) $(\implies)$ (i) follows from Prop. 1.1. (2) By Prop. 1.3, $A^H \subseteq A^\nu$ for some $\nu$ in $A$. Then, $A^H = A^{H N_{\nu}}$ is a fixed subring of the finite $G/N_{\nu}$-Galois extension $A^{H N_{\nu}}/B$, and therefore $(A^{H N_{\nu}})_B = (A^{N_{\nu}})_B$ (cf. [22; §2, p. 118]). Since $(A^{N_{\nu}})_B$ is finitely generated, $(A^H)_B$ is finitely generated.

Let $T$ be an intermediate ring of $A/B$, and $S$ a subset of $A$. $T$ is called a $G$-separable cover of $S$ if $T$ satisfies the following conditions:

(1) $T/B$ is a separable extension, and $T \supseteq S$.

(2) $G|T$ is finite.

(3) $G|T$ is strongly distinct (i.e. if $\sigma|T \neq \tau|T$ for $\sigma, \tau$ in $G$ then $\sigma|T$ and $\tau|T$ are strongly distinct).

**Theorem 1.9.** Let $A/B$ be locally finite outer $G$-Galois, and $T$ an intermediate ring of $A/B$. Then the following are equivalent:

(i) $T = A^H$ for some subgroup $H$ of $G$ such that $(G : H) < \infty$.

(ii) $T|B$ is a separable extension, $T_B$ is finitely generated, and $G|T$ is strongly distinct.

(iii) $T$ is a $G$-separable cover of $B$.

**Proof.** Let $A = \bigcup_{\lambda \in \Lambda} A^{N_{\lambda}}$ be a representation. (i) $(\implies)$ (ii) By Prop. 1.3, $T = A^H \subseteq A^\nu$ for some $\nu$ in $A$. Then $T$ is a fixed subring of the finite $G/N_{\nu}$-Galois extension $A^{H N_{\nu}}/B$. Then, by [19; Prop. 3.4], $T/B$ is a separable extension. By Prop. 1.8 (2) (cf. Cor. to Prop. 1.2), $T_B$ is finitely generated. By [22; Th. 2.6], $G|T$ is strongly distinct. (ii) $(\implies)$ (iii) This follows from the fact that $\{\sigma(x) ; \sigma \in G\}$ is finite for any $x$ in $A$. (iii) $(\implies)$ (i) Let $\{(t_i, t_i^*) ; i = 1, \ldots, n\}$ be a $(B, T)$-projective coordinate system of $T/B$. Then, by [22; Prop. 1.2], $\sum_i t_i \cdot \sigma(t_i^*) = \delta_{H, \sigma}$ for $\sigma$ in $G$, where $H = G^T$. \#($G|T$)$ < \infty$ implies $(G : H) < \infty$. By Prop. 1.4, $A^H = T$.

Combining Th. 1.9 with Prop. 1.8, we obtain the following theorem (cf. [12; Th. 3], [28; Theorem]).
Theorem 1.10. Let \( A/B \) be locally finite outer \( G \)-Galois, and \( G = G^* \). Then, for an intermediate ring \( T \) of \( A/B \), the following are equivalent.

(i) \( T = A^H \) for some subgroup \( H \) of \( G \).

(ii) For any finite subset \( F \) of \( T \) there is an intermediate ring \( T_0 \) of \( T/B \) such that \( T_0 \supseteq F \), \( T_0/B \) is separable, \( T_0/\rho \) is finitely generated, and \( G|T_0 \) is strongly distinct.

(iii) Any finite subset of \( T \) has a \( G \)-separable cover which is contained in \( T \).

Next we shall proceed to the characterization of locally finite outer Galois extensions.

Proposition 1.11. Let \( V_A(B) = C \), \( T \) a \( G \)-separable cover of \( B \), and \( \{ (t_i, t_i^*) ; i = 1, \cdots, n \} \) a \( (B, T) \)-projective coordinate system for \( T/B \), and put \( H = G^* \). Then there hold the following.

1. \( \sum t_i \cdot \sigma(t_i^*) = \delta_{H}, \) for all \( \sigma \) in \( G \).

2. \( A^H = T, \) \( (G : H) < \infty \), and \( T/B \) is a projective Frobenius extension.

3. Let \( K \) be a subgroup of \( G \) containing \( H \). Then, \( \sum t_{K,H}(t_i) \cdot \sigma(t_i^*) = \delta_{K,H} \) for all \( \sigma \) in \( G \), \( T \) is \( (B, A^K) \)-projective, \( T/A^K \) is a projective Frobenius extension, and \( G|A^K \) is strongly distinct. Further the following are equivalent. (a) \( (A^K)_K \times T_{A^K} \). (b) \( (A^K)_K \times \tau A^K \). (c) \( t_{K,H}(c) = 1 \) for some \( c \) in \( T \).

Proof. (1) follows from [22; Prop. 1.2], and (2) is obvious by (1) and Prop. 1.4. (3) It will be easily seen that \( \sum t_{K,H}(t_i) \cdot \sigma(t_i^*) = \delta_{K,H} \) for all \( \sigma \) in \( G \). Since \( \sum t_i \otimes t_i^* = \sum t_i \otimes t_i^* \otimes t \) \( t \in T \otimes_B T \), \( \sum t \cdot t_{K,H}(t_i) \otimes t_i^* \otimes t \) \( t \in T \) to \( A^K \otimes_B T \) is an \( A^K \)-homomorphism. Since \( \sum t_{K,H}(t_i) t_i^* x = x \), it follows that \( T \) is \( (B, A^K) \)-projective. Let \( \rho|A^K \neq \tau |A^K \) for \( \rho, \tau \) in \( G \). Then \( t^{-1} \rho \notin K \), and so \( 0 = \tau(\sum t_{K,H}(t_i) t^{-1} \rho(t_i^*)) = \sum t_{K,H}(t_i) \rho(t_i^*) \). Thus, by [22; Prop. 1.11], \( \rho|A^K \) and \( \tau|A^K \) are strongly distinct. If we set \( G = K \) in Prop. 1.4, the remainder follows from Prop. 1.4.

Theorem 1.12. Let \( V_A(B) = C \). Then the following statements are equivalent.

(i) \( A/B \) is locally finite outer \( G \)-Galois.

(ii) For any finite subset \( F \) of \( A \) there is a \( G \)-invariant \( G \)-separable cover \( T \) of \( F \) such that \( pB|p|T \).

(iii) For any finite subset \( F \) of \( A \) there is a \( G \)-separable cover \( T \) of \( F \) which satisfies the following: If \( T_0 \) is an intermediate ring of \( T/B \) such that \( \alpha T \) is \( (B, T_0) \)-projective, \( \beta \) \( T/T_0 \) is a projective Frobenius extension, \( \gamma \) \( G|T_0 \) is strongly distinct, then \( \tau_{T_0} T \).

(iv) For any finite subset \( F \) of \( A \) there is a \( G \)-separable cover \( T \) of \( F \)
which satisfies the following: If $T_0$ is an intermediate ring of $T|B$ such that (a) $T$ is $(B, T_0)$-projective, (b) $T|T_0$ is a projective Frobenius extension, (c) $G|T_0$ is strongly distinct, (d) $T_0$ is a $G$-invariant fixed subring (with respect to $G$), then $\tau_0T_0|\tau_0T$.

**Proof.** (i) $\Rightarrow$ (ii), (iii) Let $A = \bigcup_{x \in A} A^x$ be a representation of the locally finite $G$-Galois extension $A/B$. Then any finite subset $F$ of $A$ is contained in some $A^{x_\mu}$ ($\mu \in \Lambda$). By [22; Th. 1.5], $A^{x_\mu}$ is a $G$-invariant $G$-separable cover of $F$ such that $B|A^{x_\mu}$. Let $T_0$ be an intermediate ring of $A^{x_\mu}/B$ such that $A^{x_\mu}$ is $(B, T_0)$-projective and that $G|T_0$ is strongly distinct. Then, by [22; Th. 2.6], $T_0$ is a fixed subring of the finite outer $G|N_x$-Galois extension $A^{x_\mu}/B$, whence $\tau_0T_0|\tau_0T$ by [22; §2. p. 118]. (ii) $\Rightarrow$ (i) Let $F$ be a finite subset of $A$, and $T$ a $G$-invariant $G$-separable cover of $F$ such that $B|B$. If we put $N = G^x$, then $A^x = T, N \triangleleft G$ and $(G : N) < \infty$ (Prop. 1.11). By Prop. 1.11, $A^x/B$ is a finite $G/N$-Galois extension. Noting that $(A^x)^B$ is finitely generated, $A/B$ is a locally finite $G$-Galois extension. (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (i) Let $T_1$ be a separable cover of an element $x$ in $A$. Put $G^x = H_1$. Then $\#(G' \cap T_1) < \infty$ implies $(G : H_1) < \infty$ and $\# \{\sigma(x) ; \sigma \in G\} < \infty$. Thus any finite subset of $A$ is contained in a $G$-invariant finite subset of $A$. Let $F$ be a $G$-invariant finite subset of $A$, and $T$ a $G$-separable cover of $F$ as that in (iv), and let $\{(t_i, t_i^*) ; i = 1, \cdots, n\}$ be a $(B, T)$-projective coordinate system of $T/B$, and $H = G^x$. Then, by Prop. 1.11, $A^x = T, (G : H) < \infty$, and $\sum t_i^* \sigma(t_i^*) = \delta_{\sigma, \tau}$ for all $\sigma$ in $G$. Set $N = G^x$. Then $H \triangleleft N \triangleleft G$, and $F \subseteq A^x \subseteq A^x$. By Prop. 1.1.1, $T$ is $(B, A^x)$-projective, $T/A^x$ is a projective Frobenius extension, and $G/A^x$ is strongly distinct. Then, by the assumption for $T$, $(A^x)_T(T)T$, so that $t_{H\cap c} = 1$ for some $c$ in $T$ (Prop. 1.11 (3)). Put $t_i = t_{N:B}(t_i)$ and $t_i^* = t_{N:B}(t_i^*)$. Then, $t_i^*, t_i^* \subseteq A^x$, and $\sum t_i^* \sigma(t_i^*) = \delta_{\sigma, \tau}$ for all $\sigma$ in $G$ (Prop. 1.11 (3)). Further, as is easily seen, $\sum t_i^* \sigma(t_i^*) = \delta_{\sigma, \tau}$ for all $\sigma$ in $G$. Since $B|B$ (Prop. 1.11 (3)), we have $B|B$. Thus $A^x/B$ is a finite $G/N$-Galois extension. Noting that $(A^x)^B$ is finitely generated, we conclude that $A/B$ is a locally finite $G$-Galois extension.

**Proposition 1.13.** Let $A^* \supseteq T \supseteq B^*$ be rings such that $A^*$ is $(B^*, T)$-projective, $A'$ an extension ring of $B^*$ such that $V_{A'}(B^*) = V_{A'}(A')$, and $f_1, \cdots, f_s$ $B^*$-ring homomorphisms from $A^*$ to $A'$ such that $f_i|T$ and $f_s \supseteq T (i \neq k)$ are strongly distinct. If $(B^*)_{B^*} \rightarrow T_{B^*}$, then $(A')_{A'} \rightarrow (A')_{A'}$.

**Proof.** Let $\{(t_i, a_i^*); i = 1, \cdots, n\}$ be a $(B^*, T)$-projective coordinate system for $A^*$. Then, by [22; Prop. 1.2], $\sum f_i^*(t_i) a_i^* = \delta_{h, k}$ for all $h, k$. Let $\varphi$ be a $A'$-right homomorphism from $T \otimes_{B^*} A'$ to $(A')_{A'}$ defined by $\varphi(t \otimes a') = (f_i(t)a', \cdots, f_s(t)a')$. Since $\sum f_i \otimes a_i^* = \delta_{h, k}, \varphi$ is an epimorphism. $(B^*)_{B^*} \rightarrow T_{B^*}$ implies that $(A')_{A'} \rightarrow T \otimes_{B^*} A'_{A'}$. Hence we have $(A')_{A'} \rightarrow (A')_{A'}$, as
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desired.

Concerning Prop. 1.13, we consider the following condition.

Condition (F): If $\mathcal{A}A^r \rightarrow \mathcal{A}A^s$ for positive integers $r$, $s$, then $r \geq s$.

Remark. Let $\mathcal{A}A^r \rightarrow \mathcal{A}A^s$ for positive integers $r$, $s$. Then, since $\mathcal{A}A^r$ is projective, $\mathcal{A}A^s$ is isomorphic to an $A$-direct summand of $\mathcal{A}A^r$.

(1) If $\mathcal{A}A$ is finite dimensional, then $r \cdot \dim \mathcal{A}A \geq s \cdot \dim \mathcal{A}A$, and so $r \geq s$ (cf. [11]).

(2) Assume that there is a proper ideal $\mathfrak{A}$ of $A$ such that $\mathcal{A}A/\mathfrak{A}$ is finite dimensional. Then, since $\mathcal{A}A^r/\mathfrak{A}^r \rightarrow \mathcal{A}A^s/\mathfrak{A}^s$, the above (1) yields $r \geq s$, because $\mathcal{A}A^r/\mathfrak{A}^r \simeq (A/\mathfrak{A})^r$ and $\mathcal{A}A^s/\mathfrak{A}^s \simeq (A/\mathfrak{A})^s$.

(3) If $A$ is commutative, then $r \geq s$ by (2).

**Proposition 1.14.** Let $V_A(B)=C$, and $A$ an indecomposable ring satisfying (F), and let $T$ be an intermediate ring of $A/B$, and $S$ a subset of $A$. Then the following are equivalent:

(i) $T$ is a $G$-separable cover of $S$.

(ii) $T \supseteq S$, $T/B$ is a separable extension, and $T_B$ is finitely generated.

Proof. (i) $\Rightarrow$ (ii) is evident by Prop. 1.11. (ii) $\Rightarrow$ (i) By [22; Lemma 2.7], $A$ is $(B, T)$-projective. Then, by Prop. 1.13, we have $\#(G|T)<\infty$, and hence $T$ is a $G$-separable cover of $S$.

If $A$ is commutative, then $A$ satisfies (F). Therefore, by Th. 1.12, S. 3 and Prop. 1.14, we have the following

**Theorem 1.15** (Nagahara [12]). Let $A$ be an indecomposable commutative ring. Then the following are equivalent.

(i) $A/B$ is locally finite $G$-Galois.

(ii) For any finite subset $F$ of $A$ there is an intermediate ring $T$ of $A/B$ such that (a) $T/B$ is a separable extension, and $T_B$ is finitely generated, (b) $T \supseteq F$.

**Proposition 1.16.** Let $A/B$ be locally finite $G$-Galois, and $H$ a subgroup of $G$. Then $G|A^H$ is strongly distinct.

Proof. Let $\sigma$, $\tau$ be in $G$, and $e$ a central idempotent of $A$ such that $\sigma(x)e=\tau(x)e$ for all $x$ in $A^H$. Let $A=\bigcup_{\lambda \in \Lambda}A^{N_{\lambda}}$ be a representation of the locally finite $G$-Galois extension $A/B$. We may assume that $e \in A^{N_{\lambda}}$ for all $\lambda$ in $\Lambda$. Suppose that $\sigma|A^H \neq \tau|A^H$. Since $A^H=\bigcup_{\lambda \in \Lambda}A^{N_{\lambda}H}$, $\sigma|A^{N_{\mu}H} \neq \tau|A^{N_{\mu}H}$ for some $\mu$ in $\Lambda$. Then, by [22; Prop. 2.4], $(G/N_H)|A^{N_{\mu}H}$ is strongly distinct. Therefore we have $e=0$. Thus $G|A^H$ is strongly distinct.

**Theorem 1.17.** Let $A/B$ be locally finite outer $G$-Galois, and $T$ an intermediate ring of $A/B$. Then the following are equivalent.

(i) $T=A^H$ for some subgroup $H$ of $G$, and $A_T$ is finitely generated.
(ii) \( T = A^\mu \) for some subgroup \( H \) of \( G \) such that \((H:1)<\infty\).

(iii) \( A/T \) is a projective Frobenius extension, \( \text{Hom}(A_T, A_T)\subseteq\Delta \), and \( G|T \) is strongly distinct.

When any of the above conditions is satisfied \( A/A^\mu \) is finite \( H \)-Galois.

Proof. Let \( A = \bigcup_{\lambda \in A} A_{N_{\mu}} \) be a representation of the locally finite outer \( G \)-Galois extension \( A/B \). (i) \( \Rightarrow \) (ii) Let \( A = x_1T + \cdots + x_rT \). Then \( x_1, \ldots, x_r \in A^{N_{\mu}} \) for some \( \mu \in A \), so that \( A = A^{N_{\mu}} \cdot T = A^{N_{\mu}} \cdot A^\mu \). Hence \( N_{\mu} \cap H = 1 \). Since \( (G:N_{\mu}) < \infty \) we have \((H:1)<\infty\). (ii) \( \Rightarrow \) (iii) By Prop. 1.3, \( H \cap N_{\mu} = 1 \) for some \( \mu \in A \). There are elements \( a_1, \ldots, a_n; a_1^*, \ldots, a_n^* \) in \( A^{N_{\mu}} \) such that \( \Sigma_i a_i^* \sigma(a_i^*) = \delta_{N_{\mu},*} \) for all \( \sigma \) in \( G \). Then \( \Sigma_i a_i^* \sigma(a_i^*) = \delta_{1,*} \) for all \( \sigma \) in \( H \). Hence \( A/A^\mu \) is \( H \)-Galois. Therefore \( A/A^\mu \) is a projective Frobenius extension (cf. [22; p. 121]), and \( \text{Hom}(A_T, A_T) = \sum_{x \in H} Au_x \subseteq \Delta \). By Prop. 1.16, \( G|T \) is strongly distinct. (iii) \( \Rightarrow \) (i) Let \( h = \sum_{x \in H} a_x u_x \) be a Frobenius homomorphism of \( A/T \), where \( H \) is a finite subset of \( G \) and \( a_x \neq 0 \) for all \( x \) in \( H \). Then, since \( th = ht \) for all \( t \) in \( T \), we have \( ta_t = a_\tau \cdot \tau(t) \) for all \( t \) in \( T \), in particular, \( ba_a = ab \) for all \( b \) in \( B \). Hence \( a_x \in V(A) = C \) for all \( x \) in \( H \). There are elements \( r_i, l_i \) in \( A \) such that \( x = \sum_i r_i h(xr_i)l_i = \sum_i r_i h(l_i x) \) for all \( x \) in \( A \) (cf. [27]). Then \( u_i = \sum_i r_i h(l_i) = \sum_i r_i \sum_{x \in H} a_x \tau(l_i)u_x = \sum_{x \in H} \sum_i r_i a_x \tau(l_i)u_x \), and so \( 1 = \sum_i r_i a_i l_i = a_1 \sum_i r_i l_i \). Thus \( a_1 \) is an invertible element in \( C \), and \( a_1 \cdot \cdot = \sum_i r_i l_i \). Since \( H \) is finite there is an \( N_{\mu} \) such that \( \tau | A^{N_{\mu}} \neq \rho | A^{N_{\mu}} \) provided \( \tau \neq \rho \) (\( \tau, \rho \in H \)). Since \( A^{N_{\mu}}/B \) is finite \( G/N_{\mu} \)-Galois, there are elements \( d_k, e_k \) in \( A^{N_{\mu}} \) such that \( \sum_k d_k \tau(e_k) = \delta_{N_{\mu},*} \) for all \( \sigma \) in \( G \). Put \( \Delta_0 = \text{Hom}(A_T, A_T) \). Then \( \Delta_0 = AhA \), and \( \Delta_0 \ni \sum_{x \in H} \tau(d_k)he_k = \sum_{x \in H} \sum_k \tau(d_k)a_x e_k u_x \subseteq \Delta \), for \( \tau \) in \( H \). Thus \( \Delta_0 = AhA = \sum_{x \in H} \Delta_0 \oplus Aa_x u_x \). Since \( A/T \) is a projective Frobenius extension with Frobenius homomorphism \( h_\tau A \otimes \tau A_\tau \cong A \otimes A_\tau \) by the correspondence \( x \otimes y \rightarrow xyh. \) Let \( \varphi \) be the \( A \)-left homomorphism from \( A \) to \( \Delta_0 \) defined by \( \varphi(\sum x_{i}u_{i}) = \sum_{x \in H} x_{i}a_{x_{i}}u_{x_{i}} \), and \( \psi \) the \( A \)-left homomorphism from \( \Delta_0 \) to \( \Delta \) defined by \( \psi(xyh) = \sum_{x \in H} x_{i}h(yr_{i})vl_{i} \), where \( v = \sum_{x \in H} u_{x_{i}} \). Then, as \( h(tr_{i})a_{\tau} = \tau(h(yr_{i})a_{\tau} \in H) \), \( \varphi \psi = 1 \). Since \( a_{x_{i}}u_{x_{i}} = \sum_{x \in H} \tau(d_k)he_k \), we have \( \varphi(a_{x_{i}}u_{x_{i}}) = \sum_{x \in H} \tau(d_k)he_k = \sum_{x \in H} \sum_{\tau} e_{k} \rho(e_k) \rho(r_{i})u_{x_{i}}l_{i} \), and so \( \varphi \psi(a_{x_{i}}u_{x_{i}}) = \sum_{x \in H} \tau(d_k)he_k \varphi \psi (a_{x_{i}}u_{x_{i}}) \subseteq \Delta \), for \( \tau \) in \( H \). Since \( a_1 \cdot \cdot = \sum_i r_i l_i \), we have \( a_{x_{i}} \cdot a_{x_{i}} = a_{x_{i}} \cdot a_{x_{i}} \). Noting that \( \tau(a_{x_{i}}) \) is an invertible element of \( C \), \( a_{x_{i}}a_{x_{i}} = a_{x_{i}} \cdot a_{x_{i}} = Aa_{x_{i}} \), and so \( A = Aa_{x_{i}} + \text{Ann}_{A}(a_{x_{i}}) \), where \( \text{Ann}_{A}(a_{x_{i}}) = \{ x \in A ; xa_{x_{i}} = 0 \} \). If \( xa_{x_{i}} \in \text{Ann}_{A}(a_{x_{i}}) \), then \( 0 = xa_{x_{i}} = x \cdot a_{x_{i}} \cdot a_{x_{i}} \), so that \( xa_{x_{i}} = 0 \). Thus \( A = Aa_{x_{i}} \oplus \text{Ann}_{A}(a_{x_{i}}) \). Therefore \( Aa_{x_{i}} \) is written as \( A_{g} \), with a central idempotent \( g_{x_{i}} \) of \( A \). Since \( Aa_{x_{i}}, u_{x_{i}} \subseteq \Delta_0 \), we have \( g_{x_{i}} u_{x_{i}} \subseteq \Delta_0 \), and so \( g_{x_{i}} t = g_{x_{i}} \cdot \tau(t) \) for all \( t \) in \( T \). Consequently, \( \Delta_0 = \sum_{x \in H} \Delta_0 \oplus Aa_{x_{i}} \) and \( H = G^{T} \). Hence End \( (A,T) = (A^{N_{\mu}}) \), the right multiplications of elements of \( A \). Since \( a_{x_{i}} \in \Delta_0 = \text{End}(A_T) \), we have \( a_{x_{i}} u_{x_{i}} \in \text{End}(A_{(A^{N_{\mu}})}) \). Noting that \( a_{x_{i}} \in C \), we
can easily seen that $a,u_i \in \text{Hom}(\langle a \rangle \mathcal{H}, \langle a \rangle \mathcal{H})$. Thus $h = \sum_{i \in \mathcal{H}} a_0 u_i \in \text{Hom}(\langle a \rangle \mathcal{H}, \langle a \rangle \mathcal{H})$. Then, by [27; Cor. 1], $A/\mathcal{H}$ is also a projective Frobenius extension with a Frobenius homomorphism $h$. Since $(H:1)<\infty$, there is an $N_i$ such that $H \cap N_i = 1$ (Prop. 1.3 (2)). Then $A_N^H \subseteq A_N$, and $H \simeq HN_i/N_i$ canonically. Therefore there is an element $c$ in $A_N$ such that $t_H(c) = 1$ (cf. [22; §2. p. 118]), which implies $(A_H)^{(A_H)N_i}|A_H$, because the $A_H$-right homomorphism $x \mapsto t_H(cx) \ (x \in A)$ from $A$ to $A_H$ splits. Therefore there is an element $d$ in $A$ such that $h(d) = 1$. Then, for any $x$ in $A_H$, $T \ni h(dx) = h(d)x = x$. Thus we obtain $T = A_H$, as desired.

**Theorem 1.18.** Let $A/B$ be finite outer $G$-Galois, and $T$ an intermediate ring of $A/B$. Then the following are equivalent.

(i) $T = A_H$ for some subgroup $H$ of $G$.

(ii) $A/T$ is a projective Frobenius extension, and $G|T$ is strongly distinct.

(iii) $T/B$ is a separable extension, and $G|T$ is strongly distinct.

**Proof.** (i) $\iff$ (ii) is evident from Th. 1.17. (i) $\implies$ (iii) follows from [22; Th. 2.6] and [19; Prop. 3.4]. (iii) $\implies$ (i) follows from [22; Th. 2.6 and Lemma 2.7].

**§ 2. Heredity of locally finite Galois extensions.**

Let $A_0$ be a $G$*-invariant subring of $A$ such that the mapping $\sigma \mapsto \sigma|A_0$ ($\sigma \in G^*$) is one-to-one and such that $A_0/A_0^G$ is a locally finite $G$-Galois extension, and let $G^*$ be compact (as an automorphism group of $A$). Put $B_0 = A_0^G$, and let $A_0 = \bigcup_{\lambda \in A} A_0^{N_\lambda}$ be a representation of the locally finite $G$-Galois extension $A_0/B_0$. Then $G/N_\lambda$ may be considered as a finite group of automorphisms of $A_0^{N_\lambda}$. And, by [22; Th. 5.1 and §2. p. 118], $A_0^{N_\lambda} = A_0^{N_\lambda} \otimes_{B_0} B$, $A_0^{N_\lambda}/B$ is finite $G/N_\lambda$-Galois. Since $\bigcup A_0^{N_\lambda}$ is a directed union, the compactness of $G^*$ implies that $\bigcup A_0^{N_\lambda}(\subseteq A_0)$ is a fixed subring of $A$ with respect to $G^*$ (Prop. 1.1), so that $A = \bigcup A_0^{N_\lambda}$, because $\sigma \mapsto \sigma|A_0$ ($\sigma \in G^*$) is $1-1$. Thus $A/B$ is locally finite $G$-Galois. Let $H$ be any subgroup of $G$. Then, $A^H = \bigcup_i (A^H \cap A^{N_\lambda}) = \bigcup_i A^{H_{N_\lambda}}$. By [22; Th. 5.1], $A^H = (A_0^{N_\lambda})^{H/N_\lambda} \otimes_{B_0} B = A_0^{H_{N_\lambda}} \otimes_{B_0} B$. Hence

\[A^H = \bigcup_i (A_0^{H_{N_\lambda}} \otimes_{B_0} B) = A_0^H \cdot B, \text{ and } A_0^H \otimes_{B_0} B \rightarrow A_0^H \cdot B = A^H \text{ canonically.} \]

Since the isomorphism $A_0^{H_{N_\lambda}} \otimes_{B_0} B \simeq A_0^{H_{N_\lambda}} \cdot B (\subseteq A^H)$ may be considered as $A_0^{H_{N_\lambda}} \otimes_{B_0} B \rightarrow A_0^H \otimes_{B_0} B = A^H$, we know $A^H = A_0^H \otimes_{B_0} B$. Symmetrically we obtain $A^H = B \otimes_{B_0} A_0^H$. Next we consider the set of all $A_0$-$G$-left submodules of $A$ and the set of all $B_0$-$G$-left submodules of $B$. Let $\overline{X}$ be any $A_0$-$G$-left submodule of $A$. Then $\overline{X} \cap A^{N_\lambda}$ is an $A_0^{N_\lambda}(G/N_\lambda)$-left submodule of $A^{N_\lambda}$. Therefore, by [22; Th. 5.1], we have $\overline{X} \cap A^{N_\lambda} = A_0^{N_\lambda}((\overline{X} \cap A^{N_\lambda}) \cap B) = A_0^{N_\lambda} \otimes_{B_0} (\overline{X} \cap B)$, so that $\overline{X} = \bigcup_i (\overline{X} \cap A^{N_\lambda}) = \bigcup_i (A_0^{N_\lambda} \overline{X} \cap B) = A_0(\overline{X} \cap B)$. Since $A_0^{N_\lambda} \otimes_{B_0} (\overline{X} \cap B) \simeq A_0^{N_\lambda}(\overline{X}$
$\cap B) \subseteq \overline{X}$ for all $\lambda$, we have $\overline{X} = A_0 \otimes_{B_0}(\overline{X} \cap B)$. Evidently $\overline{X} \cap B$ is a $B_0$-left submodule of $B$. Let $X$ be any $B_0$-left submodule of $B$. Then, as is easily seen, $A_0 X$ is an $A_0 G$-left submodule of $A$, and $A_0 X = \cup_\lambda A_0^{N_\lambda} X$. By [22; Th. 5.1], $A_0^{N_\lambda} X \cap B = X$ for all $\lambda$ in $\Lambda$, so that $A_0 X \cap B = \cup_\lambda (A_0^{N_\lambda} X \cap B) = X$. If $\overline{Y}$ is a $G$-invariant intermediate ring of $A/A_0$, then $\overline{Y} \cap B$ is an intermediate ring of $B/B_0$, and $\overline{Y} = A_0(\overline{Y} \cap B)$. Symmetrically we have $\overline{Y} = (\overline{Y} \cap B)A_0$. If $Y$ is an intermediate ring of $B/B_0$ such that $A_0 Y = Y A_0$, then $A_0 Y$ is a $G$-invariant intermediate ring of $A/A_0$. Since $A = \cup_\lambda A^{N_\lambda}$, we have $\overline{Y} = \cup_\lambda (\overline{Y} \cap A^{N_\lambda}) = \cup_\lambda \overline{Y}^{N_\lambda}$, and $\overline{Y}^{N_\lambda}(\overline{Y} \cap B)$ is finite $G/N_\lambda$-Galois ([22; Th. 5.1]. Hence $\overline{Y}/(\overline{Y} \cap B)$ is locally finite $G$-Galois. Thus we have obtained the following.

**Theorem 2.1.** Let $A_0$ be a $G^*$-invariant subring of $A$ such that $\sigma \rightarrow \sigma|A_0$ ($\sigma \in G^*$) is 1-1 and such that $A_0/B_0$ is locally finite $G$-Galois where $B_0 = A_0^G$, and let $G^*$ be compact. Then there hold the following:

1. $A/B$ is locally finite $G$-Galois.

2. $A^H = B \otimes_{B_0} A_0^H = A_0^H \otimes_{B_0} B$ for any subgroup $H$ of $G$. In particular, $A = B \otimes_{B_0} A_0 = A_0 \otimes_{B_0} B$.

3. Let $\{\overline{X}\}$ and $\{X\}$ be the set of all $A_0 G$-left submodules of $A$ and the set of all $B_0$-left submodules of $B$, respectively. Then, $\overline{X} \rightarrow \overline{X} \cap B$ and $X \rightarrow \overline{X} = A_0 \otimes_{B_0} X$ are mutually converse order isomorphisms between $\{\overline{X}\}$ and $X$.

4. Let $\{\overline{Y}\}$ and $\{Y\}$ be the set of all $G$-invariant intermediate rings of $A/A_0$ and the set of all intermediate rings of $B/B_0$ such that $A_0 Y = Y A_0$, respectively. Then $\overline{Y}((\overline{Y} \cap B)$ is locally finite $G$-Galois, and $Y \rightarrow \overline{Y} \cap B$ and $\overline{Y} / (\overline{Y} \cap B)$ are mutually converse order isomorphisms between $\{\overline{Y}\}$ and $\{Y\}$.

Let $A, A'$ be $R$-algebras such that $A \otimes_R A' \neq 0$. Assume that $A/B$ is a locally finite $G$-Galois extension such that $R \cdot 1 \subseteq B$, and assume that $A'$ is a locally finite $G'$-Galois extension such that $R \cdot 1 \subseteq B'$. Then each $\sigma \times \tau$ in $G \times G'$ induces an automorphism of $A \otimes_R A'$. Let $A = A^{N_\sigma} \subseteq A_0 \otimes_{B_0} B$ and $A' = A^{N_\tau} \subseteq A_0 \otimes_{B_0} B$. For any subgroup $H$ of $G$, let $A^H = A \otimes_{B_0} A_0^H = A_0^H \otimes_{B_0} B$.

Hence $A_0/B_0$ is a $(G/N_\sigma) \times (G'/N_\tau)$-Galois extension. Then, by [22; Th. 5.2], $(A^{N_\sigma} \otimes_R A^{N_\tau})/B \otimes B'$ is a finite $(G/N_\sigma) \times (G'/N_\tau)$-Galois extension. Let $\varphi_{a,b}$ be the canonical $R$-algebra homomorphism from $A^{N_\sigma} \otimes_R A^{N_\tau}$ to $A^{N_\sigma} \otimes A^{N_\tau}$, where $\varphi_{a,b}$ is the $(G/N_\sigma) \times (G'/N_\tau)$-Galois extension. Hence $A_{a,b}/B^*$ is a finite $(G/N_\sigma) \times (G'/N_\tau)$-Galois extension, and $\{\sigma \times \tau \in G \times G'; \sigma \times \tau | A_{a,b} = 1_{A_{a,b}}\} = N_\sigma \times N_\tau$. Since $\cap_{a,b}(N_\sigma \times N_\tau) = (\cap_{a,b} N_\sigma) \times (\cap_{b} N_\tau) = 1$, $G \times G'$ may be considered...
as a group of automorphisms of $A \otimes_R A'$. Let $H$ and $H'$ be subgroups of $G$ and $G'$, respectively. Then, $(A \otimes_R A')^{H \times H'} = \bigcup_{\alpha, \beta} A_{\alpha, \beta}^{H \times H'} = \bigcup_{\alpha, \beta} (A_{\alpha}^N \otimes A_{\beta}^{N'}) = (\bigcup_{\alpha} A_{\alpha}^N \otimes (\bigcup_{\beta} A_{\beta}^{N'}) = A^H \otimes A'^{H'}$ by [22; Th. 5.2]. In particular $(A \otimes_R A')^{N_a \times N_{\beta'}} = (A_{\alpha}^N \otimes A_{\beta'}^{N'}) = A_{\alpha}^N \otimes A_{\beta'}^{N'}$, and evidently $(G \times G' : N_a \times N_{\beta'}) < \infty$. Since $A \otimes_R A' = \bigcup_{\alpha, \beta} A_{\alpha}^N \otimes A_{\beta}^{N'}$ is a directed union, $A \otimes_R A'/B \otimes B'$ is a locally finite $G \times G'$-Galois extension. Let $a \in A$ and $a' \in A'$. Then it is evident that $\{ \sigma \times \tau \in G \times G'; \sigma(a) \otimes \tau(a') = a \otimes a' \} \supseteq \{ \sigma \in G; \sigma(a) = a \} \times \{ \tau \in G'; \tau(a') = a' \}$. Put $\{ \sigma \in G; \sigma(a) = a \} = K$ and $\{ \tau \in G'; \tau(a') = a' \} = K'$. Then $A \times K \subseteq A^N$ and $A \times K' \subseteq A^{N'}$ for some $\alpha, \beta$ (Prop. 1.3), so that $N_a \subseteq K$ and $N_{\beta'} \subseteq K'$. By [22; Th. 5.2], $(G/N_a \times G'/N_{\beta'})^{K \otimes K'} = K/N_a \times K'/N_{\beta'}$, and hence $(G \times G')^{K \otimes K'} = K \times K'$. Since $(A^K)_B$ and $(A^{K'})_{B'}$ are finitely generated, $(A^K \otimes A^{K'})_{B \otimes B'}$ is finitely generated. Hence the finite topology of $G \times G'$ with respect to $A \otimes_R A'$ is the product topology of the finite topology of $G$ with respect to $A$ and the finite topology of $G'$ with respect to $A'$. Thus we have proved the following

**Theorem 2.2.** Let $A$ and $A'$ be $R$-algebras such that $A \otimes_R A' \neq 0$. If $A/B$ is a locally finite $G$-Galois extension such that $R \cdot 1 \subseteq B$, and $A'/B'$ is a locally finite $G'$-Galois extension such that $R \cdot 1 \subseteq B'$, then $(A \otimes_R A')/(B \otimes B')$ is a locally finite $G \times G'$-Galois extension, and $(A \otimes_R A')^{H \times H'} = A^H \otimes A'^{H'}$ for any subgroup $H$ of $G$ and any subgroup $H'$ of $G'$. The finite topology of $G \times G'$ with respect to $A \otimes_R A'$ is the product topology of the finite topology of $G$ with respect to $A$ and the finite topology of $G'$ with respect to $A'$.

**Corollary.** Let $A/B$ be a locally finite $G$-Galois extension such that $B \subseteq C$, and $A'$ a $B$-algebra such that $A \otimes_R A' \neq 0$. Then $(A \otimes_R A')/(1 \otimes A')$ is a locally finite $G'$-Galois extension, and $(A_B \otimes A')^{H} = A^H \otimes A'$. For any subgroup $H$ of $G$.

**Proposition 2.3.** Let $A/B$ be locally finite $G$-Galois, and $G = G^*$. If $H$ and $K$ are closed subgroups of $G$, then $A^{H \cap K} = A^H \cdot A^K = A^H \cdot A^K$. In particular, if $H \cap K = 1$ then $A = A^H \cdot A^K = A^H \cdot A^K$.

**Proof.** Let $A = \bigcup_{\mu \in \Lambda} A_N^\mu$ be a representation of the locally finite $G$-Galois extension $A/B$. First we assume that $(G : K) < \infty$. Then, by Prop. 1.3, $A^\mu \subseteq A_N^\mu$ for some $\mu \in \Lambda$. Since $(A_N^\mu)_B$ is finitely generated and $(A^\mu)_B$ is a direct summand of $(A_N^\mu)_B$ ([22; §2. p. 118]), $(A^\mu)_B$ is finitely generated. Therefore we may assume that $A^\mu \subseteq A_N^{\mu_1}$ for all $\mu_1 \in \Lambda$. Then $N_{\mu \subseteq K}$ for all $\mu \in \Lambda$, and $A^H \cdot A^K = (\bigcup_{\mu \in \Lambda} A_N^\mu) \cdot (\bigcup_{\mu' \in \Lambda} A_N^\mu') = (\bigcup_{\mu \in \Lambda} A_N^\mu \cdot A_N^{\mu'}) = (\bigcup_{\mu \in \Lambda} A_N^{H \cap K}) = \bigcup_{\lambda \in \Lambda} A_N^{\mu \in H \cap K}$ by [22; Prop. 5.3]. Since $N_{\mu \cap H \cap K} = N_{\mu \cap (H \cap K)}$ for all $\mu$, we have $A^{H \cap K} = \bigcup_{\lambda \in \Lambda} A_N^{H \cap K} = A^{H \cap K}$. Next we return to general case. For any finite subset $F$ of $A^\mu$, we put $K_F = \{ \sigma \in G; \sigma |F = 1_F \}$. Then $(G : K_F) < \infty$, $A^{K_F} \subseteq A^K$, and $(A^{K_F})_B$ is finitely generated. Therefore $A^K = \bigcup_{F} A^{K_F}$ is a directed union, and
hence $A^{H}.A^{K} = A^{H}(\bigcup_{\gamma} A^{K_{\gamma}}) = \bigcup_{\gamma} (A^{H_{\gamma}}.A^{K_{\gamma}})$ is also a directed union. Since each $A^{H}.A^{K_{\gamma}} (= A^{H_{\gamma}}.A^{K_{\gamma}})$ is a fixed subring of $A$, $A^{H}.A^{K}$ is a fixed subring of $A$ (Prop. 1.1). Hence, as is easily seen, $A^{H}.A^{K} = A^{H \cap K}$. Symmetrically we have $A^{H_{\gamma}}.A^{K_{\gamma}} = A^{K_{\gamma}}.A^{H_{\gamma}}$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, $G = G^{*}$, and $H, (\gamma \in \Gamma)$ be closed subgroups of $G$. Then, $[\cup_{\gamma} A^{H_{\gamma}}] = A^{\cap H_{\gamma}}$, where $[\cup_{\gamma} A^{H_{\gamma}}]$ means the subring of $A$ generated by $\cup_{\gamma} A^{H_{\gamma}}$.

**Proof.** Evidently $[\cup_{\gamma} A^{H_{\gamma}}] = \bigcup \{ A^{B_{1}} \cup \cdots \cup A^{B_{n}} \}$, where $\{ B_{1}, \cdots, B_{n} \}$ ranges over all finite subsets of $\Gamma$. By Prop. 2.3, $A^{B_{1}} \cup \cdots \cup A^{B_{n}} = A^{B_{1}} A^{B_{2}} \cdots A^{B_{n}} = \bigcup_{\gamma \in \Gamma_{i}} (A^{H_{\gamma_{1}}} \cdots A^{H_{\gamma_{n}}})_{\gamma_{i}}$, and therefore $[\cup_{\gamma} A^{H_{\gamma}}]$ is a directed union of fixed subrings of $A$. Hence, by Prop. 1.1, $[\cup_{\gamma} A^{H_{\gamma}}]$ is a fixed subring. Since $\{ \sigma \in G ; \sigma[\cup_{\gamma} A^{H_{\gamma}}] = 1 \} = \bigcap_{\gamma \in \Gamma_{i}} H_{\gamma}$, we obtain $[\cup_{\gamma} A^{H_{\gamma}}]$ as desired.

**Proposition 2.4.** Let $A/B$ be locally finite $G$-Galois, $\mathfrak{A}$ a $G$-invariant proper ideal of $A$, $K$ a closed subgroup of $G$, and $N$ a closed normal subgroup of $G$ such that $(G:N) < \infty$. Then there hold the following:

1. $A^{N}/A^{K}$ is finite $G/N$-Galois. In particular, $A^{N}/B$ is finite $G/N$-Galois.

2. $(A^{N} + \mathfrak{A})/(B + \mathfrak{A})$ is finite $G/N$-Galois, and $((A^{N} + \mathfrak{A})/\mathfrak{A})^{H} = (A^{NH} + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

**Proof.** Let $A = \bigcup_{\gamma \in \Lambda} A^{N_{\gamma}}$ be a representation of the locally finite $G$-Galois extension $A/B$. (1) By Prop. 1.3, $A^{N} \subseteq A^{N_{\mu}}$ for some $\mu \in \Lambda$, and then $N_{\mu} \subseteq N$, $A^{N} = (A^{N_{\mu}})^{N_{\mu}}$. Therefore, by [22; Prop. 5.7], $A^{N}/B$ is finite $(G/N_{\mu})(N_{\mu}N/N_{\mu})$-Galois, or equivalently, finite $G/N$-Galois. Accordingly, $A^{N}/A^{N_{\mu}}$ is finite $NK/N$-Galois, or equivalently, finite $K/(K \cap N)$-Galois. $K/(K \cap N)$ may be considered as a finite group of automorphisms of $A^{N}$, because $K \cap N \subseteq K$. Then $A^{K}/A^{N}$ is finite $K/(K \cap N)$-Galois. (2) By (1), $A^{N}/B$ is finite $G/N$-Galois. If $t_{0,N}(c) = 1$ for $c$ in $A^{N}$, then $t_{0,N}(c + \mathfrak{A}) = 1 + \mathfrak{A}$. Then, by [22; Th. 5.6], $((A^{N} + \mathfrak{A})/\mathfrak{A})/(B + \mathfrak{A})/\mathfrak{A}$ is finite $G/N$-Galois, and $((A^{N} + \mathfrak{A})/\mathfrak{A})^{H} = (A^{NH} + \mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

Let $A/B$ be locally finite $G$-Galois, $K$ a closed subgroup of $G$, $N$ a closed normal subgroup of $G$, and $\mathfrak{A}$ a $G$-invariant proper ideal of $A$. Let $A = \bigcup_{\gamma \in \Lambda} A^{N_{\gamma}}$ be a representation of the locally finite $G$-Galois extension $A/B$. Then $A^{N} = \bigcup_{\gamma} (A^{N} \cap A^{N_{\gamma}}) = \bigcup_{\gamma} A^{N_{\gamma}}$ is a directed union, and each $NN_{\gamma}$ is a closed normal subgroup of $G$, because $(G:N) < \infty$. Then, by Prop. 2.4 (1), $A^{NN_{\gamma}}/B$ is finite $G/NN_{\gamma}$-Galois. Therefore there are elements $a_{1}, \cdots, a_{m}; b_{1}, \cdots, b_{m}$ in $A^{NN_{\gamma}}$ such that $\sum_{i} a_{i} \sigma(b_{i}) = 0_{NN_{\gamma}}$ for $\sigma$ in $G$. Hence $A^{NN_{\gamma}}/B$ is finite $(G/N)($(NN_{\gamma})/N$)-Galois. Hence $A^{N}/B$ is locally finite $G/N$-Galois. Next we consider $K$. $A = \bigcup_{\gamma} A^{N_{\gamma},K}$ is a directed union, and each $N_{\gamma} \cap K$ is a fixed
normal subgroup of $K$ such that $(K:N_i\cap K)<\infty$. By Prop. 2.4 (1), each $A^{N_i\cap K}/A^K$ is finite $K/(N_i\cap K)$-Galois. Hence $A/A^K$ is locally finite $K$-Galois. Finally we consider $\mathfrak{A}$. Evidently, $A/\mathfrak{A}=\bigcup_i((A^{N_i}+\mathfrak{A})/\mathfrak{A})$. By Prop. 2.4 (2), $((A^{N_i}+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is finite $G/N_i$-Galois, and $((A^{N_i}+\mathfrak{A})/\mathfrak{A})^H=(A^{N_iH}+\mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$. Therefore $(A/\mathfrak{A})^H=\bigcup_i((A^{N_i}+\mathfrak{A})/\mathfrak{A})^H=\bigcup_i(A^{N_iH}+\mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$. Hence $((A+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is locally finite $G$-Galois. Thus we have proved the following

**Theorem 2.5.** Let $A/B$ be locally finite $G$-Galois, $N$ a closed normal subgroup of $G$, $K$ a closed subgroup of $G$, and $\mathfrak{A}$ a $G$-invariant proper ideal of $A$. Then there hold the following:

1. $A^N/B$ is locally finite $G/N$-Galois.
2. $A/A^K$ is locally finite $K$-Galois.
3. $((A+\mathfrak{A})/\mathfrak{A})/((B+\mathfrak{A})/\mathfrak{A})$ is locally finite $G$-Galois, and $((A+\mathfrak{A})/\mathfrak{A})^H=(A^H+\mathfrak{A})/\mathfrak{A}$ for any subgroup $H$ of $G$.

**Corollary.** Let $A/B$ be locally finite $G$-Galois, and $e$ a non-zero idempotent in $B \cap C$. Then $Ae/Be$ is locally finite $G$-Galois, and $(Ae)^H=A^He$ for any subgroup $H$ of $G$.

Let $A/B$ be locally finite $G$-Galois, $n$ a positive integer, and $J$ the ring of rational integers. Then, $(J)_n$ is a $J$-algebra, and $(J)_n\otimes_AA\simeq(A)_n\neq 0$. If we define $\sigma((a_{ik}))=(\sigma(a_{ik}))$ for any $\sigma$ in $G$ and any $(a_{ik})$ in $(J)_n$, then $(A)_n/(B)_n$ is locally finite $G$-Galois and $((A)_n)^H=(A^H)_n$ for any subgroup $H$ of $G$ (Th. 2.2). Now, let $\{e_{ik}; i, k=1, \cdots, m\}$ a system of matrix units contained in $B$, and

$A=\bigcup_{x\in A}A_{N_i}$ a representation of $A/B$. Put $A_0=V_A(\{e_{ik}\})$ and $B_0=B\cap A_0$. Then, as is well known, $A=\sum_{x_i}A_0e_{ik}$, $A_0\simeq A_0e_{ik}$ by the right multiplication of $e_{ik}$. To be easily seen, $A_{N_i}=\sum_{x_i}A_0^{N_i}e_{ik}$, and $A_0^{N_i}=V_{x_i}(\{e_{ik}\})$. There is an element $c$ in $A_{N_i}$ such that $t_{\tau;N_i}(c)=1$. Let $c=\sum_{x_i}x_ie_{ik}$ $(x_i\in A_{N_i})$. Then $1=t_{\tau;N_i}(c)=\sum_{x_i}t_{\tau;N_i}(x_i)e_{ik}$, and so $t_{\tau;N_i}(x_i)=1$. Thus, by [22; Th. 5.8], $A_0^{N_i}/B_0$ is finite $G/N_i$-Galois. Since $A_0=\bigcup_iA_0^{N_i}$ is a directed union, $A_0/B_0$ is locally finite $G$-Galois. Therefore, by Th. 2.1, $A=A_0\otimes_{B_0}B$. Thus we have obtained the following

**Theorem 2.6.** Let $A/B$ be locally finite $G$-Galois.

1. For any positive integer $n$, $(A)_n/(B)_n$ is locally finite $G$-Galois, and $((A)_n)^H=(A^H)_n$ for any subgroup $H$ of $G$.
2. If $\{e_{ik}; i, k=1, \cdots, m\}$ is a system of matrix units contained in $B$, $A_0=V_A(\{e_{ik}\})$, and $B_0=B\cap A_0$, then $A_0/B_0$ is locally finite $G$-Galois, and $A=A_0\otimes_{B_0}B$.

Let $A/B$ be finite $G$-Galois, and $M$ a $A$-left module. For any subgroup $H$ of $G$, we put $M^\tau=\{m\in M; u\cdot m=m$ for all $\tau\in H\}$, which is an $A^\tau$.
submodule of $M$. Evidently $M^H \supseteq A^H \cdot M^\theta$, and the mapping $\varphi : A^H \otimes_B M^\theta \rightarrow M^H$ defined by $a \otimes m \rightarrow am$ ($a \in A$, $m \in M^\theta$) is an $A^H$-left homomorphism. By assumption there are elements $a_1, \ldots, a_n; a_1^*, \ldots, a_n^*$ in $A$ such that $\sum_i a_i \cdot \sigma(a_i^*) = \delta_{i,e}$ ($\sigma \in G$), $t_H(d)=1$. Put $t_i = t_H(a_i)$. Then, $t_i \in A^H$ and $\sum_i t_i \cdot \sigma(a_i^*) = \delta_{i,e}$ for $\sigma$ in $G$. If $m$ is in $M^\theta$, then $A^H \cdot M^\theta \ni t_i \sum_{\sigma \in \theta} u_{\sigma}(a_i^* dm) = \sum_i t_i \sum_{\sigma \in \theta} \sigma(a_i^* dm) u_{\theta} m = t_H(dm) m = m$. Hence $\varphi$ is an epimorphism. If $a \in A^H$ and $m_0 \in M^\theta$, then $\sum_i t_i \sum_{\sigma \in \theta} u_{\sigma}(a_i^* dm_0) = \sum_i t_i \sum_{\sigma \in \theta} \sigma(a_i^* dm_0) m_0 = \sum_i t_i \sum_{\sigma \in \theta} \sigma(a_i^* dm_0) \otimes m_0 = t_H(\Delta) \otimes m_0 = a \otimes m_0$. From this, as is easily seen, $\varphi$ is 1-1. Thus we have $M^H = A^H \otimes_B M^\theta$. Next we proceed to more general case.

Let $A/B$ be locally finite $G$-Galois, $A = \bigcup_{\lambda \in A} A_{N_{\lambda}}$ its representation, and $M$ a $A$-left module. Let $G = \sigma_1 N_1 \cup \cdots \cup \sigma_r N_r$ be the coset decomposition of $G$, and let $A_1$ be the trivial crossed product of $A_{N_{\lambda}}$ with $G/N_i$: $A_1 = \sum_i A_{N_{\lambda}} \otimes_{B} \mathbb{C}$, $v_{e \sigma} = v_{\sigma} = v_{e \sigma_1} = v_{\sigma_1} \sigma_1 = \sigma_{i} a \otimes \sigma_{i} N$. For any $m \in M_{N_{\lambda}}$ and any $\sum_i a \cdot v_{e \sigma}$ in $A$, we define $\sum_i a \cdot v_{e \sigma}(m) = \sum_i a \cdot u_{\lambda \sigma} m$. Then, as is easily seen, $M_{N_{\lambda}}$ is a $A_1$-left module. Since $M_{N_{\lambda}}$ is finite $G/N_i$-Galois, we obtain that $M_{N_{\lambda}} = A_{N_{\lambda}} \otimes_B M^\theta$ and $M_{N_{\lambda}}^H = A_{N_{\lambda}}^H \otimes_B M^\theta$ for any subgroup $H$ of $G$. Since $A = \bigcup_i A_{N_{\lambda}}$ is a directed union, so is $\bigcup_i M_{N_{\lambda}}$. For any subgroup $H$ of $G$, $(\bigcup_i M_{N_{\lambda}})^H = \bigcup_i A_{N_{\lambda}}^H \otimes_B M^\theta = A^H \otimes_B M^\theta$, and $A^H \otimes_B M^\theta \simeq A^{H_{N_{\lambda}}} \otimes_B M^\theta$ canonically. The last isomorphism may be considered as $A^{H_{N_{\lambda}}} \otimes_B M^\theta \rightarrow A^H \otimes_B M^\theta$, and hence we see that $(\bigcup_i M_{N_{\lambda}})^H = A^H \otimes_B M^\theta$. For any $m$ in $M$ we put $H_m = \{ \sigma \in G; \varphi \sigma m = m \}$, which is a subgroup of $G$. Assume that $(G : H_m) < \infty$ and that $H_m$ is closed in $G$. Then, by Prop. 1.3, $H_m \supseteq N_\nu$ for some $\nu \in A$, so that $m \in M_{N_\nu}$. Conversely, if $m$ is in $\bigcup_i M_{N_{\lambda}}$, then $m \in M_{N_{\lambda}}$ for some $N_{\lambda}$, so that $H_m \supseteq N_\nu$. Then, since $(G : N_{\lambda}) < \infty$ and $N_{\lambda}$ is closed in $G$, $(G : H_m) < \infty$ and $H_m$ is closed in $G$. Thus we have proved the following.

**Theorem 2.7.** Let $A/B$ be locally finite $G$-Galois, and $M$ a $A$-left module. Then there hold the following:

1. $A \cdot M^\theta$ is a $A$-submodule of $M$, and $(A \cdot M^\theta)^H = A^H \otimes_B M^\theta$ for any subgroup $H$ of $G$.

2. $A \cdot M^\theta = \{ m \in M; (G : H_m) < \infty$ and $H_m$ is closed in $G \}$, where $H_m = \{ \sigma \in G; u_{\sigma} m = m \}$.

**Corollary.** Let $A/B$ be finite $G$-Galois, and $M$ a $A$-left module. Then, $M^H = A^H \otimes_B M^\theta$ for any subgroup $H$ of $G$, in particular, $M = A \otimes_B M^\theta$ (cf. [4; Th. 1.3] and [22; Th. 5.1 (2)]).

**Proposition 2.8.** Let $A/B$ be finite $G$-Galois. Then the following are equivalent.

1. There are elements $a_{1, \ldots, a_n}; a_1^*, \ldots, a_n^*$ in $V_A(B)$ such that $\sum_i a_i \cdot \sigma(a_i^*) = \delta_{i,e}$ ($\sigma \in G$) (cf. [22; Cor. to Th. 5.1]).
(ii) \( _bA_B|_B B_B. \)

Proof. Since \((A \supseteq \Sigma u_i)A \simeq \text{Hom}(A_B, B_B)\) by \(j\), it follows that \((\Sigma u_i) V_A(B) \simeq \text{Hom}(\_bA_B, \_bB_B)\), and it is evident that \(V_A(B) \simeq \text{Hom}(\_bB_B, _bA_B)\) canonically. To be easily seen, \(_bA_B|_B B_B\) if and only if there are elements \(f_1, \ldots, f_n\) in \(\text{Hom}(\_bA_B, \_bB_B)\) and \(g_1, \ldots, g_n\) in \(\text{Hom}(\_bB_B, _bA_B)\) such that \(\sum g_i f_i(x) = x\) for all \(x\) in \(A\). Consequently (ii) is equivalent to that \(u_i = \sum a_i (\Sigma \ast u_i) a_i^\ast\)

\(= \sum_i a_i^\ast \sigma(a_i^\ast) u_i\) for some \(a_1, \ldots, a_n, a_1^\ast, \ldots, a_n^\ast\) in \(V_A(B)\). Hence (i) and (ii) are equivalent.

Corollary. Let \(G\) be finite. Then the following are equivalent.

(i) \(A/B\) is outer \(G\)-Galois, and \(_bA_B|_B B_B.\)

(ii) There are elements \(a_1, \ldots, a_n, a_1^\ast, \ldots, a_n^\ast\) in \(C\) such that \(\sum_i a_i^\ast \sigma(a_i^\ast) = \delta_{1, \sigma}\) (\(\sigma \in G\)).

Proof. This follows from [22; Prop. 6.4 and Prop. 6.5] and Prop. 2.8. \(A/B\) is called a completely outer \(G\)-Galois extension if \(G\) is finite and completely outer (cf. [22]).

Theorem 2.9. Let \(B'\) be a ring with identity, \(Z\) its center, and \(G'\) a finite group.

(1) If \(A'/B'\) is completely outer \(G'\)-Galois and \(_bA'_B|_B B'_B\), then \(A' = B' \otimes_2 C'\), where \(C'\) is the center of \(A'\), and \(C'/Z\) is \(G'\)-Galois.

(2) If \(C'/Z\) is \(G'\)-Galois and \(C'\) is commutative, then \(A' = B' \otimes_2 C'\) is a completely outer \(G'\)-Galois extension over \(B', _bA'_B|_B B'_B\). and \(1 \otimes C'\) is the center of \(A'\).

Proof. (1) By [22; Prop. 6.4], \(A'/B'\) is outer \(G'\)-Galois and \(V_{A'}(B') = C'\), where \(C'\) is the center of \(A'\). Then, by Cor. to Prop. 2.8 and [22; Th. 5.1], \(C'/Z\) is \(G'\)-Galois and \(A' = B' \otimes_2 C'\) is completely outer \(G'\)-Galois. Since \(Z\) is a direct summand of \(2C'\), \(B' \simeq B' \otimes 1\) canonically, and \(_bA'_B|_B B'_B\), because \(2C'|Z\). Then, by Cor. to Prop. 2.8, \(C'/Z\) is \(G'\)-Galois, where \(C'\) is the center of \(A'\). Since \(C' \supseteq 1 \otimes C' \supseteq Z\) and \((1 \otimes C')/Z\) is \(G'\)-Galois ([22; Th. 5.1 or Th. 5.6]), we have \(C' = Z(1 \otimes C') = 1 \otimes C'\) ([22; Th. 5.1]).

Lemma 2.10. Let \(T\) be a ring, and \(U\) a subring of \(T\).

(1) Let \(T|U\) be a separable extension. If a \(T\)-left module \(M\) is \(U\)-projective, then \(M\) is \(T\)-projective.

(2) If \(xT \otimes_0 T_U|_U T_U \otimes_0 T_T\) and \(xU|_U M\) for a \(T\)-left module \(M\), then \(xT|_U M\).

(3) Let \(T_0\) be an intermediate ring of \(T|U\). If \(T\) is \((U, T_0)\)-projective and \(T_0\) is a \(T_0\)-\(T_0\)-direct summand of \(T\), then \(T/U\) is a separable extension.

Proof. (1) Since the mapping \(x \otimes y \rightarrow xy\) form \(T \otimes_0 T\) to \(T\) splits as a \(T-T\)-homomorphism, the mapping \(x \otimes m \rightarrow xm\) from \(T \otimes_0 M\) to \(M\) splits as
a $T$-left homomorphism. Since $rM$ is projective, so is $rT\otimes TM$. Therefore $M$ is $T$-projective. (2) Since $vu|vM$, $\tau T|\tau T\otimes vM$. Since $\tau T\otimes vT|\tau T$, we have $\tau T\otimes vM|\tau M$. Hence we have $\tau T|\tau M$. (3) Let $\varphi$ be the canonical homomorphism from $T_0\otimes_\sigma T$ to $T$ defined by $\varphi(t_0\otimes t)=t_0t$, and let $\phi$ be a $T_0$-$T_0$-homomorphism from $T$ to $T_0\otimes_\sigma T$ such that $\varphi\phi(x)=x$ for all $x$ in $T$. If $\phi(1)=\sum_i a_i\otimes b_i$ ($a_i\in T_0$, $b_i\in T$), then $\sum_i a_ib_i=1$ and $\sum_i y a_i\otimes b_i=\sum_i a_i\otimes b_i y$ ($\in T_0\otimes_\sigma T$) for all $y$ in $T_0$. Let $\pi$ be a $T_0$-$T_0$-homomorphism from $T$ to $T_0$ such that $\pi T_0=1$. Then, since $\sum_i y a_i\otimes b_i=\sum_i a_i\otimes b_i y$ ($\in T_0\otimes_\sigma T$) for all $y$ in $T_0$, we have $\sum_i a_i\cdot \pi(b_i)=1$ and $\sum_i y a_i\otimes \pi(b_i)=\sum_i a_i\otimes \pi(b_i) y$ ($\in T_0\otimes_\sigma T$) for $y$ in $T_0$. Then the mapping $y\rightarrow \sum_i a_i\otimes \pi(b_i) y$ from $T_0$ to $T_0\otimes_\sigma T_0$ is a $T_0$-$T_0$-homomorphism, and $\sum_i a_i\cdot \pi(b_i) y=y$. Hence $T_0/U$ is a separable extension.

**Proposition 2.11.** Let $A/B$ be finite $G$-Galois, and $Z$ the center of $B$. If $B$ is a separable $Z$-algebra and $Z\subseteq C$, then $V_A(B)/Z$ is finite $G$-Galois.

**Proof.** By [2; Prop. 1.5], $B\otimes Z B$ is a central separable $Z$-algebra, where $B^0$ is the opposite ring of $B$. Since $\pi A$ and $\pi B$ are finitely generated and projective, so is $\pi A$. Then, by Lemma 2.10 (1), $\pi A$ is finitely generated and projective. By [2; Th. 2.1], $\pi B\otimes Z B^0|_{\pi B^0} B$, and hence $\pi A|B B$. Then, by Prop. 2.8, $V_A(B)/Z$ is finite $G$-Galois (cf. S. 3).

**Theorem 2.12.** Let $G$ be finite, $\pi$ the group homomorphism defined by $\pi(\sigma)=\sigma|C$ ($\sigma\in G$), $Z$ the center of $B$, and $Z_0=C^0$, and assume that $A$ is indecomposable. Then the following statements are equivalent.

(i) $A/Z_0$ is separable, and $\pi$ is 1-1.
(ii) $\pi A(B)=C$, $A/Z$ is separable, and $\pi A|B B$.
(iii) $\pi A(B)=C$, and both $B/Z$ and $C/Z$ are separable.
(iv) Both $B/Z$ and $C/Z_0$ are separable, and $\pi$ is 1-1.
(v) $\pi A(B)=C$, $A/B$ is separable, $A$ is $(Z, B)$-projective, and $\pi B|B A_B$.
(vi) $A=B\cdot C$, and $A/Z$ is separable.
(vii) $\pi A|B=\pi A|Z_0$, and $\pi A|B=\pi A|Z_0$, and $\pi A|B=0$ for any $\sigma$ in $G$ such that $\sigma \neq 1$.

**Proof.** (i) $\Rightarrow$ (ii) By [2; Th. 2.3], $A/C$ and $C/Z_0$ are separable. Therefore, by [4; Th. 1.3], $C/Z_0$ is $G$-Galois. Then, by [22; Th. 5.1], $A=B\otimes Z_C$. Hence $V_A(B)=C$, and $Z=Z_0$. Since $\pi C$ is finitely generated and projective, $\pi A|B B$. (ii) $\Rightarrow$ (iii) $\pi A(B)=C$ implies $Z=Z_0$ ($\subseteq C$). By [22; Lemma 2.7], $A/C$ and $A/B$ are separable, so that $A/B$ is outer $G$-Galois ([22; Th. 1.5]). Then, by Prop. 2.8, $A/C$ is $G$-Galois, so that $C/Z$ is separable. Since $A/C$ is separable, $B/Z$ is separable ([22; Cor. to Th. 5.1]). (iii) $\Rightarrow$ (iv) In this case, $Z=Z_0$. By [2; Th. 3.1], $A=B\cdot C$, whence $\pi$ is 1-1. (iv) $\Rightarrow$ (v) By
[4; Th. 1.3], $C/Z_0$ is $G$-Galois. Hence, by [22; Th. 5.1], $A/B$ is $G$-Galois, and $A = B \cdot C$. Then $A/B$ is separable, $V_A(B) = C$, and $Z = Z_0$. Since $Z$ is commutative, $zZ$ is a direct summand of $zC$ (S. 3), so that $t_0(c) = 1$ for some $c$ in $C$. Then $B$ is a $B \cdot B$-direct summand of $A$ (cf. [22; § 2. p. 118]). Since $B/Z$ is separable, $A$ is $(Z, B)$-projective (cf. [22; Lemma 2.7]). (v) $\Rightarrow$ (vi) By Lemma 2.10 (3), $B/Z$ is separable. Then, by [2; Th. 3.1], $A = B \otimes zC$. Since both $A/B$ and $B/Z$ are separable, $A/Z$ is separable (cf. [22; Lemma 2.7]). (vi) $\Rightarrow$ (i) As $A = B \cdot C$, $V_A(B) = C$, $Z = Z_0$, and $\pi$ is 1–1. Thus we know that (i) $\sim$ (vi) are equivalent. (i) $\Rightarrow$ (vii) In this case, $V_A(B) = C$, $Z = Z_0$, and $B/Z$ is separable. Then, by [2; Th. 2.1], $B \otimes zB \otimes zB^0 | B \otimes zB$. Therefore $B \otimes zB | B$, and thus $A = B \otimes zA | A \otimes B A$. By [22; Prop. 1.3], $A \approx z A \otimes z A$. Hence, $A \otimes zA | A \otimes zA$. The second assertion follows from [22; Prop. 6.3]. (vii) $\Rightarrow$ (i) By assumption, $\text{End}(A \otimes zA \otimes zA^0) \simeq \oplus_{z \in \theta} \text{End}(A \otimes zA \otimes zA)$. (vi) $\Rightarrow$ (i) By assumption, $\text{End}(A \otimes zA \otimes zA^0) \simeq C$, which is commutative. Hence $\text{End}(A \otimes zA \otimes zA)$ is a commutative ring. Then, by S. 1 and S. 3, $A \otimes zA \otimes zA$ is finitely generated and projective. Hence $A \otimes zA \otimes zA$ is finitely generated and projective, that is, $A/Z_0$ is separable. Let $f$ be the projection from $A$ to $A u_1$ with respect to the decomposition $A = \sum_{i} \oplus A u_i$. Then, since $\text{End}(A \otimes zA \otimes zA^0)$ is commutative, $f$ is in the center of $A \otimes zA^0$ (cf. S. 1). By [2; Prop. 1.5], the center of $A \otimes zA^0$ is $C \otimes C$, so that $f$ is written as $f = \sum_{i} a_i \otimes a_i^* (a_i, a_i^* \in C)$. Then, $u_i = \sum_{i} a_i (\sum_{u} u_{\sigma}) a_i^* (= \sum_{i} (\sum_{\sigma} a_i \sigma(a_i^*)) u_{\sigma})$, and hence $\sum_{i} a_i \sigma(a_i^*) = \delta_{1, u}$. This completes the proof of the theorem.

**Proposition 2.13.** Let $A/B$ be locally finite $G$-Galois, and $b$ an element of $B$ which is not a right zero divisor of $B$. Then $b$ is not a right zero divisor of $\overline{A}$.

**Proof.** Let $a$ be an element of $A$ such that $ab = 0$. Then $Aab = 0$, and so $\sigma(Aa)b = 0$ for all $\sigma$ in $G$. Hence, $((\sum_{\sigma} \sigma(Aa)) \cap B)b = 0$. Then, by assumption, $(\sum_{\sigma} \sigma(Aa)) \cap B = 0$. Then, by Th. 2.1 (3), $\sum_{\sigma} \sigma(Aa) = A((\sum_{\sigma} \sigma(Aa)) \cap B) = 0$. Hence $a = 0$.

Let $A/B$ be locally finite $G$-Galois, and $S \ni 1$ a $G$-invariant multiplicative system of regular elements in $A$ such that a left quotient ring $\overline{A}$ of $A$ with respect to $S$ exists. Then $G$ may be regarded as a group of automorphisms of $\overline{A}$. To be easily seen, $\{ \sigma(x); \sigma \in G \}$ is finite for any $x$ in $\overline{A}$. Then, by Th. 2.1, $\overline{A} \cup \overline{B}$ is locally finite $G$-Galois and $\overline{A} = \overline{B} \otimes zA = A \otimes zB$, where $\overline{B} = \overline{A}^0$. To be easily seen, any element in $B \cap S$ is a unit of $\overline{B}$. For $b$ in $\overline{B}$, we put
\( \mathfrak{L} = \{ x \in A \mid xb \in A \} \), which is a \( \mathcal{D} \)-left submodule of \( A \). Then \( (\mathfrak{L} \cap B)b \subseteq B \). If \( \mathfrak{L} \cap B \cap S \neq \emptyset \), then \( sb \in B \) for some \( s \) in \( B \cap S \). Therefore, if we assume that \( \mathcal{D}(s) \cap B \cap S \neq \emptyset \) for all \( s \in S \), then \( \overline{B} \) is a left quotient ring of \( B \) with respect to \( B \cap S \). Thus we obtain the following

**Theorem 2.14.** Let \( A/B \) be locally finite \( G \)-Galois, and \( S \ni 1 \) a \( G \)-invariant multiplicative system of regular elements of \( A \) such that a left quotient ring \( \overline{A} \) of \( A \) with respect to \( S \) exists. Further, assume that \( \mathcal{D}(s) \cap B \cap S \neq \emptyset \) for all \( s \in S \). Then there hold the following:

1. \( \overline{A}/\overline{B} \) is locally finite \( G \)-Galois and \( \overline{A} = \overline{B} \otimes_{B} A = A \otimes_{B} \overline{B} \), where \( \overline{B} = \overline{A}^{o} \).
2. \( \overline{A} \) is a left quotient ring of \( A \) with respect to \( B \cap S \). \( \overline{B} \) is a left quotient ring of \( B \) with respect to \( B \cap S \).

**Remark.** Let \( A/B \) be locally finite \( G \)-Galois, and \( S \) a \( G \)-invariant multiplicative system of regular elements in \( A \) such that \( S \subseteq C \) and \( S \ni 1 \). Then \( S \) satisfies the conditions in Th. 2.14. To see this, we put \( H = \{ \sigma \in G \mid \sigma(s) = s \} \) for \( s \) in \( S \). If \( G = \sigma_{1}H \cup \cdots \cup \sigma_{r}H \) is the left coset decomposition of \( G \), then \( \mathfrak{L} = \sigma_{1}H \cap \cdots \cap \sigma_{r}H \) is the left quotient ring which is a semi-simple ring with minimum condition for left ideals, and conversely (Goldie [17]). (Cf. [7])

**Theorem 2.15.** Let \( A/B \) be locally finite \( G \)-Galois, \( A \) a left Goldie ring, \( \overline{A} \) a complete left quotient ring of \( A \), and \( B \) a semi-prime ring. Then there hold the following:

1. \( \overline{A}/\overline{B} \) is locally finite \( G \)-Galois, where \( \overline{B} = \overline{A}^{o} \).
2. \( \overline{B} \) is a left Goldie ring, and \( \overline{B} \) is a complete left quotient ring of \( B \).

**Proof.** Let \( S \) be the set of all regular elements of \( A \). First we shall prove that \( B \) is a left Goldie ring. Since \( A \) is finite dimensional, \( S \) is finite dimensional. Then, by Th. 2.1 (3), \( \mathfrak{p}B \) is finite dimensional. Let \( I \subseteq I' \) be left ideals of \( B \). Then \( \mathfrak{l}_{A}(r_{B}(I)) \subseteq \mathfrak{l}_{A}(r_{B}(I')) \), where \( r_{B}(I) = \{ y \in B \mid Iy = 0 \} \) and \( r_{B}(I') = \{ x \in A \mid x \cdot r_{B}(I') = 0 \} \). From this fact, \( B \) satisfies the ascending chain condition for annihilator left ideals of \( B \). Hence \( B \) is a left Goldie ring. By Prop. 2.13, \( S \cap B \) is the set of all regular elements of \( B \). For any \( s \) in \( S \), \( \mathfrak{A}s \) is essential in \( \mathfrak{A}A \), so that \( \mathfrak{A}(s) \) is essential in \( \mathfrak{A}A \). Then, by Th. 2.1 (3), \( \mathfrak{p}(\mathfrak{A}(s) \cap B) \) is essential in \( \mathfrak{p}B \), so that \( \mathfrak{A}(s) \cap B \) contains a regular element
of $B$ ([17; Th. (3.9)]). Hence $A(s) \cap B \cap S \neq \emptyset$ for any $s$ in $S$. Thus the present theorem follows from Th. 2.14.

**Remark.** In the following cases, the condition that $B$ is semi-prime is superfluous.

1. $G$ is finite and completely outer (cf. [22; p. 132]).
2. $B$ is contained in the center of $A$.

Let $T$ be a ring. If $T$-left modules $M$ and $N$ have no non-zero isomorphic subquotients, we say that $\tau M$ and $\tau N$ are unrelated (cf. [22]).

**Lemma 2.16.** Let $T$ be a ring, and let $M$ and $N$ be $T$-left modules, and $W$ a $T$-submodule of $M$. If $\tau(M/W)$ and $\tau N$ are unrelated, and $\tau W$ and $\tau N$ are unrelated, then $\tau M$ and $\tau N$ are unrelated.

**Proof.** Assume that there are isomorphic subquotients $X/X_0$ and $Y/Y_0$ of $\tau M$ and $\tau N$, respectively. Then, as is easily seen, $X + W \supseteq X_0 + W$ or $X \cap W \supseteq X_0 \cap W$. If $X + W \supseteq X_0 + W$, then $Y/Y_0 \simeq X/X_0 \rightarrow (X + W)/(X_0 + W) \neq 0$, a contradiction. If $X \cap W \supseteq X_0 \cap W$, then $(X \cap W)/(X_0 \cap W) \simeq (X_0 + (X \cap W))/X_0 \subseteq X/X_0 \simeq Y/Y_0$, which is also a contradiction.

**Proposition 2.17.** Let $\sigma$, $\tau$ be in $G$, and assume that $A\sigma A$ and $A\tau A$ are unrelated. Then, for any finite subset $\{x_1, \cdots, x_r ; y_1, \cdots, y_s\}$ of $A$, there are elements $a_k$, $b_k$ ($k=1, \cdots, t$) in $A$ such that $\sum a_k x_i \cdot \sigma(b_k) = x_i$ and $\sum a_k y_h \cdot \tau(b_k) = 0$ for all $x_i$, $y_h$.

**Proof.** By Lemma 2.16, $A\sigma A$ and $A\tau A$ are unrelated. Then, since $A(x_1 u_1, \cdots, x_r u_r, y_1 u_1, \cdots, y_s u_s)A$ is an $A$-$A$-submodule of $A\sigma A \oplus A\tau A$, $(x_1 u_1, \cdots, x_r u_r, 0, \cdots, 0) \in A(x_1 u_1, \cdots, x_r u_r, y_1 u_1, \cdots, y_s u_s)A$ (cf. [22; Prop. 6.1]). Therefore there are elements $a_k$, $b_k$ ($k=1, \cdots, t$) in $A$ such that $\sum a_k x_i u_1, \cdots, x_r u_r, y_1 u_1, \cdots, y_s u_s)A = (x_i u_1, \cdots, x_r u_r, 0, \cdots, 0)$. Then, $\sum a_k x_i \cdot \sigma(b_k) = x_i$ and $\sum a_k y_h \cdot \tau(b_k) = 0$ for all $x_i$, $y_h$.

Combining Prop. 2.17 with [22; Prop. 6.11] we can easily see the following

**Proposition 2.18.** Let $A$ and $A'$ be $R$-algebras with $A \otimes_R A' \neq 0$, and let $G$ and $G'$ be completely outer finite groups of $R$-automorphisms of $A$ and $A'$, respectively. Then, $G \times G'$ is completely outer as an automorphism group of $A \otimes_R A'$.

§ 3.

**Proposition 3.1.** Let $A/B$ be locally finite $G$-Galois, and $X$ a $A$-left submodule of $A$. Then $X = A(X \cap B)$.

**Proof.** This follows from Th. 2.1 (3).

**Proposition 3.2.** Let $A/B$ be locally finite $G$-Galois, $\{\mathfrak{P}\}$ the set of
all maximal ideals of $A$, and $\{\mathfrak{p}\}$ the set of all maximal ideals of $B$. Then the following are equivalent:

(i) $\mathfrak{P} \rightarrow \mathfrak{P} \cap B$ is a mapping from $\{\mathfrak{P}\}$ onto $\{\mathfrak{p}\}$.

(ii) $A \mathfrak{p} A \neq A$ for all $\mathfrak{p} \in \{\mathfrak{p}\}$, and $\cap_{i \in G} \sigma(\mathfrak{P})$ is $A$-$A$-maximal for all $\mathfrak{P} \in \{\mathfrak{P}\}$.

If (i) holds, then the following are true:

(1) $\mathfrak{p} A = A \mathfrak{p} \neq A$ for any $\mathfrak{p} \in \{\mathfrak{p}\}$.

(2) $\{\cap_{\sigma} \sigma(\mathfrak{P}); \mathfrak{P} \in \{\mathfrak{P}\}\}$ is the set of all maximal $A$-$A$-submodules of $A$.

(3) $\mathfrak{R}(A A) = \mathfrak{R}(A A) = \mathfrak{R}(b B B) A = A \cdot \mathfrak{R}(b B B)$, and $\mathfrak{R}(A A) \cap B = \mathfrak{R}(b B B)$.

(4) $B$ is $B$-$B$-completely reducible if and only if $\cap_{i} \cap_{\sigma} \sigma(\mathfrak{P}) = 0$ for some $\mathfrak{P}_{i} (i = 1, \cdots, n)$ in $\{\mathfrak{P}\}$.

Proof. (i) \(\Rightarrow\) (ii) If $\mathfrak{P}$ is in $\{\mathfrak{P}\}$, then $\mathfrak{P} \cap B = \sigma(\mathfrak{P}) \cap B$ for any $\sigma$ in $G$, and so $\mathfrak{P} \cap B = (\cap_{\sigma} \sigma(\mathfrak{P})) \cap B$. By Prop. 3.1, $A (\cap_{\sigma} \sigma(\mathfrak{P})) \cap B = \cap_{\sigma} \sigma(\mathfrak{P}) = (\cap_{\sigma} \sigma(\mathfrak{P})) \cap B)$. Hence $A \mathfrak{p} \neq A \mathfrak{p}$ for all $\mathfrak{p}$ in $\{\mathfrak{p}\}$. Let $X$ be a $A$-$A$-submodule of $A$ with $A \supseteq X \supseteq \cap_{\sigma} \sigma(\mathfrak{P})$. Then $B \supseteq X \cap B \supseteq (\cap_{\sigma} \sigma(\mathfrak{P})) \cap B = \mathfrak{P} \cap B$, and so $X \cap B = (\cap_{\sigma} \sigma(\mathfrak{P})) \cap B$. Then, by Prop. 3.1, $X = \cap_{\sigma} \sigma(\mathfrak{P})$. Thus $\cap_{\sigma} \sigma(\mathfrak{P})$ is $A$-$A$-maximal. Let $Y$ be a maximal $A$-$A$-submodule of $A$. Take a maximal ideal $\mathfrak{P}_{1}$ of $A$ with $\mathfrak{P}_{1} \supseteq Y$. Then $\cap_{\sigma} \sigma(\mathfrak{P}_{1}) \supseteq Y$, and so $\cap_{\sigma} \sigma(\mathfrak{P}_{1}) = Y$. Thus we obtain (2). Therefore $\mathfrak{R}(A A) = \mathfrak{R}(A A)$. Since $\mathfrak{R}(A A) \cap B = \mathfrak{R}(b B B)$, we have $\mathfrak{R}(A A) = A \cdot \mathfrak{R}(b B B) = \mathfrak{R}(b B B) A$ (Prop. 3.1). $B$ is $B$-$B$-completely reducible if and only if $\cap_{i} \cap_{\sigma} \sigma(\mathfrak{P}_{i}) = 0$ for some $\mathfrak{P}_{i}, \cdots, \mathfrak{P}_{n}$ in $\{\mathfrak{P}\}$. Thus we obtain (4) (cf. Prop. 3.1). (ii) \(\Rightarrow\) (i). Let $\mathfrak{p} \in \{\mathfrak{p}\}$. Then, as $A \mathfrak{p} A \neq A$, $\mathfrak{p} \supseteq \mathfrak{P}$ for some $\mathfrak{P} \in \{\mathfrak{P}\}$, and so $\mathfrak{p} = \mathfrak{P} \cap B$ by the maximality of $\mathfrak{p}$. Let $\mathfrak{Q}$ be in $\{\mathfrak{P}\}$. Then $\mathfrak{Q} \cap B$ for some $\mathfrak{q} \in \{\mathfrak{p}\}$. There is a $\mathfrak{Q}' \in \{\mathfrak{Q}\}$ with $\mathfrak{Q}' \cap B = q$. Then $(\cap_{\sigma} \sigma(\mathfrak{Q}')) \cap B = \mathfrak{Q}' \cap B \supseteq \mathfrak{Q} \cap B = (\cap_{\sigma} \sigma(\mathfrak{Q})) \cap B$, and therefore $\cap_{\sigma} \sigma(\mathfrak{Q}') \supseteq \cap_{\sigma} \sigma(\mathfrak{Q})$ by Prop. 3.1. By assumption, $\cap_{\sigma} \sigma(\mathfrak{Q}') = \cap_{\sigma} \sigma(\mathfrak{Q})$. Hence $q = \mathfrak{Q}' \cap B = \mathfrak{Q} \cap B$. This completes the proof.

Concerning Prop. 3.2, we state the following

**Lemma 3.3.** Let $\mathfrak{P}$ be a maximal ideal of $A$ such that $\cap_{\sigma} \sigma(\mathfrak{P}) = \cap_{\sigma} \sigma(\mathfrak{P})$ for some $\sigma_{1}, \cdots, \sigma_{n}$ in $G$. Then $\cap_{\sigma} \sigma(\mathfrak{P})$ is $A$-$A$-maximal, and $\cap_{\sigma} \sigma(\mathfrak{P}); i = 1, \cdots, n$ is the set of all maximal ideals containing $\cap_{\sigma} \sigma(\mathfrak{P})$.

Proof. Let $\Omega$ be a maximal ideal of $A$ with $\Omega \supseteq \cap_{\sigma} \sigma(\mathfrak{P})$. If $\Omega \neq \cap_{\sigma} \sigma(\mathfrak{P})$ for all $i$, then $\Omega + \cap_{\sigma} \sigma(\mathfrak{P}) = A$ for all $i$. Then we have a contradiction $A = \Omega + \cap_{\sigma} \sigma(\mathfrak{P}) = \Omega + \cap_{\sigma} \sigma(\mathfrak{P})$.

Remark. In the following cases, the assumption in Lemma 3.3 holds.

(1) $G$ is finite. (2) The ring $A/\mathfrak{R}(A A)$ satisfies the descending chain condition for ideals. (3) $G'$ is compact, and every maximal ideal of $A$ is $A$-$A$-finitely generated. (Cf. Prop. 1.1).
**Proposition 3.4.**

(1) Let $A/B$ be locally finite outer $G$-Galois, and $B$ a $B$-completely reducible. Assume that, for any maximal ideal $\mathfrak{P}$ of $A$, there are elements $\sigma_1, \cdots, \sigma_n$ in $G$ such that $\cap_i \sigma_i(\mathfrak{P}) = \cap_i \sigma(\mathfrak{P})$. Then $A$ is $A$-completely reducible.

(2) Let $G$ be finite and completely outer, and $B/\mathfrak{p}|A/\mathfrak{p}$. Then $A$ is $A$-completely reducible if and only if $B$ is $B$-completely reducible. If there is a maximal ideal $\mathfrak{P}$ of $A$ such that $\cap_i \sigma(\mathfrak{P}) = 0$, then $B$ is $B$-minimal, and conversely.

**Proof.** (1) Any maximal ideal $\mathfrak{p}$ of $B$ is written as $\mathfrak{p} = Be$ with a central idempotent $e$ of $B$. Then, by assumption, $(1 \neq) e \in V_e(A) = C$. Therefore, $A\mathfrak{p} = Ae = eA = \mathfrak{p}A \neq A$. Thus, by Prop. 3.2 and Lemma 3.3, $A$ is $A$-completely reducible. (2) In this case, $\mathfrak{a}A = A\mathfrak{a} \neq A$ for any proper ideal $\mathfrak{a}$ of $B$ (cf. [22; p. 132]). Then, by Prop. 3.2 and Lemma 3.3, the first assertion is evident (cf. [22; Prop. 6.4]). For any $\mathfrak{P}$ in $\{\mathfrak{P}\}$, $((\cap_i \sigma(\mathfrak{P})) \cap B) \mathfrak{P} \cap B = 0$ if and only if $\mathfrak{p} = \sigma(\mathfrak{P}) = 0$ (Prop. 3.1). Thus we know the second assertion.

**Theorem 3.5.** Let $A/B$ be finite $G$-Galois, $B$ a semi-primary ring, and $A\mathfrak{p}A \neq A$ for any maximal ideal $\mathfrak{p}$ of $B$. Then $\mathfrak{p}A \simeq \mathfrak{p}B$, that is, $A$ has a normal basis. (Cf. [13; Th. 1]).

**Proof.** By [22; Th. 1.7], it suffices to prove that $\mathfrak{p}A$ is free. Let $g = (G : 1)$. (1) First we assume that $\mathfrak{g}(B) = 0$. Then $B$ is a direct sum of simple rings: $B = a_1 + \cdots + a_n$. Let $1 = \sum_i e_i$, $e_i \in a_i$. Then $Ae_i = Be_i = e_iB$ and $e_i^2 = e_i$. By assumption we have $(1 - e_i)A = A(1 - e_i)$ (Prop. 3.2 and Lemma 3.3), so that $e_i$ is a central idempotent of $A$ contained in $B$. Then each $Ae_i = Be_i$ is $G$-Galois ([22; Cor. to Th. 5.6]). Since $Be_i$ is a simple ring, $Be_i$ is free (cf. [7]). Hence $Ae_i$ has a normal basis, so that $\mathfrak{p}A \simeq \mathfrak{p}B$ for all $i$ ([22; Th. 1.7]). Hence $\mathfrak{p}A \simeq \mathfrak{p}B$.

(2) Next we proceed to general case. Since $A$ and $B$ are semi-primary ([22; Prop. 7.3]), $\mathfrak{R}(A) = \mathfrak{R}(A)$ and $\mathfrak{R}(B) = \mathfrak{R}(B)$, $\mathfrak{R}(A) \cap B = \mathfrak{R}(B)$. Then, by Prop. 3.2 and Lemma 3.3, $\mathfrak{R}(A) = \mathfrak{R}(A) = \mathfrak{R}(A) \cap B = \mathfrak{R}(B)$. By [22; Th. 5.6], $\mathfrak{R}(A) = \mathfrak{R}(A) \cap B = \mathfrak{R}(A) \cap B = \mathfrak{R}(B)$, and satisfies the same conditions as $A/B$, because $(B + \mathfrak{R}(A)) / \mathfrak{R}(A) \simeq B / \mathfrak{R}(A) \cap B = B / \mathfrak{R}(B)$ canonically. By (1), we have $\mathfrak{p}A / \mathfrak{p}B \simeq \mathfrak{p}B / \mathfrak{p}B$ and $\mathfrak{p}A$ is finitely generated and projective, we have $\mathfrak{p}A \simeq \mathfrak{p}B$.

**Corollary.** Let $A/B$ be finite $G$-Galois, $B$ a semi-primary ring, and $Z$ the center of $B$. Assume that $Z \subseteq C$ and that $B$ is a central separable $Z$-algebra. Then $A$ has a normal basis.

**Proof.** In this case, any proper ideal of $B$ is written as $\mathfrak{a}B$ with an ideal
a of \( Z \) (cf. [2]). Then, as \( Z \subseteq C \), \( (aB)A = aA = Aa = A(Ba) \neq A \) ([22; Lemma 7.1]).

Let \( A/B \) be finite \( G \)-Galois, \( B \subseteq C \), and \( g = (G : 1) \). For any prime ideal \( \mathfrak{p} \) of \( B \), we denote by \( B_\mathfrak{p} \) the quotient extension of \( B \) with respect to \( \mathfrak{p} \). Then \( B_\mathfrak{p} \) is a \( B \)-algebra, canonically. By [22; Cor. to Th. 5.2], \( (B_\mathfrak{p} \otimes_B A)/B_\mathfrak{p} \) is \( G \)-Galois. Since \( B_\mathfrak{p} \) is a local ring, \( B_\mathfrak{p} B_\mathfrak{p} \otimes_B A \simeq_{B_\mathfrak{p}} \Re(B_\mathfrak{p}) \) (Cor. to Th. 3.5). We denote by \( K_\mathfrak{p} \) the quotient field of \( B/\mathfrak{p} \). Then we have \( \Re(K_\mathfrak{p}) \simeq \Re(K_\mathfrak{p}) \) similarly. Thus we obtain the following

**Proposition 3.6.** Let \( A/B \) be finite \( G \)-Galois, \( B \subseteq C \), and \( g = (G : 1) \). Then, \( b_\mathfrak{p} B_\mathfrak{p} \otimes_B A \simeq B_\mathfrak{p} \Re(B_\mathfrak{p}) \) and \( \Re(K_\mathfrak{p}) \Re(B_\mathfrak{p}) \) for any prime ideal \( \mathfrak{p} \) of \( B \), where \( B_\mathfrak{p} \) is the quotient extension of \( B \) with respect to \( \mathfrak{p} \) and \( K_\mathfrak{p} \) is the quotient field of \( B/\mathfrak{p} \).

The following lemma is of some interest.

**Lemma 3.7.** Let \( R \supseteq S \) be rings, \( R_S \) is finitely generated and projective, and \( _S S \) is a direct summand of \( _S R \). If \( _R R \) is injective, then \( _S S \) is injective.

**Proof.** Let \( I \) be any left ideal of \( S \), and \( f \) any \( S \)-left homomorphism from \( I \) to \( _S R \). Since \( R_S \) is finitely generated and projective, we have \( RI = R \otimes_S I \). Therefore \( f \) can be extended to an \( R \)-left homomorphism from \( RI \) to \( R \), canonically. Then, by assumption, there is an element \( a \in R \) such that \( r \cdot (s)f = rsa \) for \( r \in R \) and \( s \in I \), so that \( (s)f = sa \) for all \( s \in I \). Therefore, as is well known, \( _S R \) is injective. Since \( _S S \) is a direct summand of \( _S R \), \( _S S \) is injective.

**Lemma 3.8.** \( \Re(A) \cap B \subseteq \Re(B) \).

**Proof.** Let \( b \) be in \( \Re(R) \cap B \). Then \( 1 - b \) has an inverse in \( A \). Since \( B = A^\#, 1 - b \) has an inverse in \( B \). Hence \( \Re(A) \cap B \) is a quasi-regular ideal of \( B \), that is, \( \Re(A) \cap B \subseteq \Re(B) \).

**Proposition 3.9.** Let \( G \) be finite. If there is an element \( c \) in \( A \) such that \( 1 - t_\theta(c) \in \Re(A) \), then there is an element \( d \) in \( A \) such that \( t_\theta(d) = 1 \).

**Proof.** By Lemma 3.8, we have \( 1 - t_\theta(c) \in \Re(A) \cap B \subseteq \Re(B) \), so that \( t_\theta(A) + \Re(B) = B \). Since \( t_\theta(A) \) is an ideal of \( B \), we have \( t_\theta(A) = B \). Hence \( t_\theta(d) = 1 \) for some \( d \) in \( A \).

**Theorem 3.10.** Let \( A/B \) be \( G \)-Galois, \( A \) a commutative ring, \( H \) a subgroup of \( G \), and \( A' \) a \( B \)-algebra. Then, \( A' \otimes_B A^\# \) is a direct sum of minimal ideals if and only if \( A' \) is a direct sum of minimal ideals (cf. [7; p. 178. Th. 2]).

**Proof.** In this case, \( (A' \otimes_B A)/A' \) is finite \( G \)-Galois, \( G \) is completely outer as an automorphism group of \( A' \otimes_B A \), and \( (A' \otimes_B A)^\# = A' \otimes_B A^\# \) (cf. [22; Th.
5.2 and Prop. 6.5]). Thus the present theorem is an easy consequence from Prop. 3.4 (2).

Concerning [22; Th. 6.9], we note the following

**Lemma 3.11.** Let \( A/C \) be separable, and \( e \) an idempotent of \( A \) such that \( eA \subseteq Ae \). Then \( e \) is a central idempotent of \( A \).

**Proof.** Since \( A/\Re(A) \) is a semi-prime ring, we have \( (eA + \Re(A))/\Re(A) = (eA + \Re(A))/\Re(A) \), that is, \( eA + \Re(A) = Ae + \Re(A) \), and so \( Ae = eA + (eA \cap \Re(A))e \). Since \( A \) is a central separable \( \mathcal{C} \)-algebra, \( \Re(A) = \Re(C)A \) and \( Ae = eA + \Re(C)Ae \). Hence \( Ae = eA \), because \( eA \) is finitely generated. Consequently, \( e \) is a central idempotent of \( A \).

**Proposition 3.12.** Let \( A/B \) be locally finite \( G \)-Galois, and assume that there is a representation \( A = \bigcup_{\lambda \in \Lambda} A^{N_{\lambda}} \) of \( A/B \) such that each \( \Re(B)A^{N_{\lambda}} \) is an ideal of \( A^{N_{\lambda}} \). Then \( \Re(A) = \Re(B)A = A \cdot \Re(B) \), and \( \Re(A) \cap B = \Re(B) \).

**Proof.** Let \( \mathfrak{Z} \) be a right ideal of \( A \) such that \( \Re(B)A + \mathfrak{Z} = A \). Then \( \Re(B)A^{N_{\lambda}} \cap \Re(B)A^{N_{\lambda}} \) is an ideal of \( A \) for some \( \lambda \) in \( A \), so that \( \Re(B)A^{N_{\lambda}} + (\mathfrak{Z} \cap A^{N_{\lambda}}) = A^{N_{\lambda}} \). Since \( \Re(B)A^{N_{\lambda}} \subseteq \Re(A^{N_{\lambda}}) \), we have \( \mathfrak{Z} \cap A^{N_{\lambda}} = A^{N_{\lambda}} \), and hence \( \mathfrak{Z} = A \). Thus we know that \( \Re(B)A \subseteq \Re(A) \). Combining this with Lemma 3.8, we have \( \Re(A) \cap B = \Re(B) \). Hence \( \Re(A) = \Re(B)A = A \cdot \Re(B) \) (Prop. 3.1).

**Theorem 3.13.** Let \( A/B \) be locally finite \( G \)-Galois, \( B \subseteq C \), and \( A' \) a \( B \)-algebra such that \( A' \simeq A' \otimes 1 \) (\( \subseteq A' \otimes_{B} A \)) canonically.

1. \( \Re(A' \otimes_{B} A) = \Re(A' \otimes A) \), and \( \Re(A' \otimes A) \cap (A' \otimes 1) = \Re(A') \otimes 1 \).

2. If \( A \) is commutative, then \( \Re(A' \otimes A^{H}) = \Re(A') \otimes A^{H} \) for any subgroup \( H \) of \( G \).

**Proof.** Let \( A = \bigcup_{\lambda \in \Lambda} A^{N_{\lambda}} \) be a representation of the locally finite \( G \)-Galois extension \( A/B \). Then \( (A' \otimes_{B} A)/(A' \otimes 1) \) is a locally finite \( G \)-Galois extension with representation \( A' \otimes_{B} A = \bigcup_{\lambda} A' \otimes A^{N_{\lambda}} \), where \( A' \otimes A^{N_{\lambda}} = (A' \otimes_{B} A)^{N_{\lambda}} \) is a finite \( G/N_{\lambda} \)-Galois extension over \( A' \otimes 1 \). (1) This will be easily seen by Prop. 3.12. (2) We may assume that \( H \) is closed in \( G \). Then each \( A^{H \cap N_{\lambda}}/A^{H} \) is finite \( H/(H \cap N_{\lambda}) \)-Galois, and \( H/(H \cap N_{\lambda}) \) is completely outer as an automorphism group of \( A^{H \cap N_{\lambda}} \) ([22; Th. 6.6]). Then \( H/(H \cap N_{\lambda}) \) is completely outer as an automorphism group \( A' \otimes A^{H \cap N_{\lambda}} \) (Prop. 2.18), and so \( H/(H \cap N_{\lambda}) \) is completely outer as an automorphism group of \( A' \otimes A^{H \cap N_{\lambda}} \) ([22; Prop. 6.11]). Now, \( (A' \otimes_{B} A)/(A' \otimes A^{H}) \) is a locally finite \( H \)-Galois extension with representation \( A' \otimes_{B} A = \bigcup_{\lambda} A' \otimes A^{H \cap N_{\lambda}} \), where \( A' \otimes A^{H \cap N_{\lambda}} = (A' \otimes_{B} A)^{H \cap N_{\lambda}} \) is a finite \( H/(H \cap N_{\lambda}) \)-Galois extension over \( A' \otimes A^{H} \). Then, by [22; Th. 7.10] and Prop. 3.12, \( \Re(A' \otimes_{B} A) = \Re(A' \otimes A^{H})(A' \otimes_{B} A) \). On the other hand,
\[ \mathcal{R}(A' \otimes_B A) = \mathcal{R}(A') \otimes A = (\mathcal{R}(A') \otimes A^H)(A' \otimes_B A). \]

Hence \( \mathcal{R}(A' \otimes A^B) = \mathcal{R}(A') \otimes A^H \), as desired (cf. [22; Lemma 7.1]).

References

([1]~[14] are found in [22] below.)


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