ON CARTAN-BRAUER-HUA THEOREM*

By

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Since Cartan-Brauer-Hua theorem was generalized as in Jacobson [3, Th. 7. 13.1] and Nagahara and Tominaga [6, Lemma 2], this theorem has been extended to simple rings (cf. Kishimoto [4] and Nagahara, Kishimoto and Tominaga [5]). The purpose of the present paper is to extend [4, Th. 2] and [5, Th. 2] to primitive rings.

Throughout our study, we use the following conventions: $U$ will represent a ring with 1, and $B$ a subdirectly irreducible subring1) of $U$ such that the unique minimal ideal $T$ of $B$ is not nilpotent. A primitive ring and a completely primitive ring will mean a right primitive ring and the ring of all the linear transformations in a left vector space over a division ring, respectively. Let $R$ be a ring. If for any finite subset $F$ of $R$ there exists a completely primitive subring of $R$ containing $F$, $R$ is said to be locally completely primitive. Let $A$ be an arbitrary non-empty set. By $\Lambda$ be an arbitrary non-empty set.

Let $R$ be a ring. If for any finite subset $F$ of $R$ there exists a completely primitive subring of $R$ containing $F$, $R$ is said to be locally completely primitive. Let $A$ be an arbitrary non-empty set. By $(R)_{A}$ and $R^{(A)}$ we denote the ring of all row-finite matrices $(x_{ij})(i,j \in \Lambda)$ and the direct sum of $\#\Lambda^{2}$-copies of $R$-left module $R$; thus $(R)_{A}$ can be regarded as the ring of all linear transformations in $R^{(A)}$.

We shall first prove the following that contains [4, Th. 2].

**Theorem 1.** Let $A$ be a subring of $U$, and $\{2t; t \in T\} \neq 0$. If $B$ is invariant relative to all the inner derivations effected by elements of $A$ then either $T$ is $A-A$-admissible or $A \subseteq V_{A}(B)^{3})$.

**Proof.** Let $a$ be an arbitrary element of $A$. If $\{a, 1\}$ is linearly left independent over $B$ then the same argument used in the proof of [4, Th. 2] enables us to conclude that $a \in V_{A}(B)$. On the other hand, if $\{a, 1\}$ is linearly left dependent over $B$ then the set $\{b \in B; ba \in B\} = \{b \in B; ab \in B\}$ forms a non-zero ideal of $B$, so that $Ta = T^2a = T(Ta) \subseteq TB \subseteq T$ and similarly $aT \subseteq T$. We have thus $A = V_{A}(B)^{\cup}A_{0}$, where $A_{0} = \{a \in A; Ta + aT \subseteq T\}$. Since both $V_{A}(B)$ and $A_{0}$ are submodules, it follows that either $V_{A}(B) = A$ or $A_{0} = A$ (cf. [7, Lemma 3.5]).

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2) $\#A$ denotes the cardinal number of the set $A$.

3) $V_{A}(B)$ means the centralizer of $B$ in $U$. 

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Corollary 1. Let $A$ be a subring of $U$ such that $B$ is invariant relative to all the inner derivations effected by elements of $A$, and $\{2t; t \in T\} \neq \emptyset$.

1 Let $B$ be completely primitive. If there exists a primitive idempotent $e \in B$ such that $\text{r}_{\nu}(eB)=0^{4)}$ then either $A \subseteq B$ or $A \subseteq V_{U}(B)$.

1' If $A=U$, $A$ is primitive and $B$ is completely primitive then either $B=A$ or $B \subseteq V_{A}(A)$.

(2) Let $U$ be completely primitive, and $A \supseteq B$. If $A$ is dense$^{5)}$ in $U$ then $B$ is either dense in $A$ or contained in the center of $A$.

Proof. Assume that $A \not\subseteq V_{U}(B)$. By Th. 1, $T$ is then $A$-$A$-admissible.

1 Since $T$ is the socle of $B$, we have $eT=eB$. Hence, $A_{r} \subseteq \text{Hom}_{eBe}$ $(eB, eB)=B_{r}$, where $A_{r}$ and $B_{r}$ are the rings of right multiplications in $eB$ by the elements of $A$ and $B$, respectively. Since $\text{r}_{\nu}(eB)=0$, it follows $A \subseteq B$.

1' If $e'$ is an arbitrary primitive idempotent in $B$ then $e'A \subseteq e'TA=e'T=e'B$, which implies that $e'$ is a primitive idempotent of $A$. Hence, $\text{r}_{\nu}(e'B)=\text{r}_{\nu}(e'A)=0$, and so $A=B$ by the assertion (1).

2 Obviously, $T$ is an ideal of $A$. Hence, $(T$ and so) $B$ is dense in $A$ by [3, Th. 2.4.4].

Next, we shall present an extension of [5, Th. 2].

Let $A$ be a completely primitive unital$^{6)}$ subring of $U$ different from $(GF(2))_{2}$. Then one may regard $A=(D)_{A}$ where $D$ is a division ring. We identify $D$ with the ring of diagonal matrices of which diagonal elements are the same. $e_{ij} \in A$ represents the matrix with element $1_{D}$ in the $(i, j)$-position and 0's elsewhere. Let $S$ be the socle of $A$, then we have $S=\sum_{i \epsilon A} A_{e_{ii}}$. We set $E=\{e_{ij}; i, j \epsilon A\}$ and $E^{*}=\{\sum_{i \epsilon A} d_{ij} e_{ij}; d_{ij} \epsilon D\}$. Then $E^{*}$ is a dense subring of $A$. If $R$ is a subring of $U$ such that $R\tilde{A}=R^{7)}$ and $\# A \geq 1$, then by using the same argument as in [5, Lemma 2] we can prove that if $Re_{pq} \subseteq R$ for some $e_{pq} \epsilon E$ then $RE^{*} \subseteq R$.

First we need the following Lemma.

Lemma 1. If $B\tilde{A}=B$ then $TE^{*} \subseteq T$ or $E^{*} \subseteq V_{U}(B)$.

Proof. Let $a$ be an arbitrary regular element of $A$. If $\{1, a\}$ is linearly left dependent over $B$ then the set $\{b \epsilon B; ba \epsilon B\}$ is a non-zero ideal of $B$, and so $Ta \subseteq T$. Now, let $a$ be biregular (i.e. $a$ and $1-a$ are regular). If $\{1, a\}$ is linearly left independent over $B$, then by the same argument used

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4) $\text{r}_{\nu}(eB)$ means the right annihilator of $eB$ in $U$.

5) We introduce the finite topology in $U$.

6) A unital subring of $U$ means a subring containing the identity element of $U$.

7) $1_{D}$ denotes the identity element of $D$.

8) $\tilde{A}$ represents the group of inner automorphisms of $U$ induced by regular elements of $A$.
in the proof of [7, Lemma 3.5] one can see that \( a \in V_U(B) \). Hence, by making use of the same method as in the proof of [5, Th. 2], we can prove our assertion except for the case \( A=(GF(2))_A \) with \( \#A \geq \aleph_0 \). In what follows, we shall restrict ourselves to the exceptional case. By [3, Th. 5.3.1] one may regard then \( A=(GF(2))_B \otimes_{aGF(2)} A \). Choose an arbitrary index \( \lambda \in A \), and consider the elements \( a= \begin{bmatrix} 011 & 000 \\ 100 & 010 \\ 100 & 000 \end{bmatrix} \) and \( h= \begin{bmatrix} 010 & 000 \\ 010 & 000 \end{bmatrix} \) in \( A \).

Then, it is easy to see that \((1+h)a(1+h)^{-1}=a+a^* \) and \( a^*+a^*2 \in E \). Since \( a \) is biregular, it follows \( Ta \subseteq T \) or \( a \in V_U(B) \). If \( Ta \subseteq T \) then \( T \bar{A}=T \) implies \( T \supseteq (1+h)Ta(1+h)^{-1}=T(a+a^*) \). Hence, by the remark stated just before our lemma, it follows \( TE^* \subseteq T \). On the other hand, if \( a \in V_U(B) \) then the same argument yields \( E^* \subseteq V_U(B) \).

Now, we are at the position to prove the following that contains [5, Th. 2].

**Theorem 2.** If \( B\bar{A}=B \) and \( T \cdot S \neq 0 \) then either \( T \) is \( A-A \)-admissible or \( A \subseteq V_U(B) \).

**Proof.** By Lemma 1, \( TE^* \subseteq T \) or \( E^* \subseteq V_U(B) \). Let \( u \) be an arbitrary regular element of \( A \). Then, \( TSTu=TSu \cdot u^{-1}Tu=TST \). Since \( E^* \) is dense in \( A \) and \( \epsilon \cdot u = \epsilon u u^{-1} \) is of finite rank, there exists \( \epsilon^* \in E^* \) such that \( u \epsilon u = \epsilon^* u = \epsilon \epsilon^* \). If \( TE^* \subseteq T \) then \( Tu \epsilon^* = uTu \epsilon^* \subseteq uTu^{-1} = T \), and so \( Tu \epsilon u \subseteq Tu \epsilon^* \subseteq T \). On the other hand, if \( E^* \subseteq V_U(B) \) then \( V_U(B) \bar{A}=V_U(B) \) implies \( u \epsilon u = \epsilon \epsilon^* \in V_U(B) \). Since \( A \) is generated by regular elements by Zelinsky's theorem (cf. [8]) and \( S=\sum_{\epsilon \in \Lambda} A \epsilon u \), we have proved that \( TSTA=TST \) and there holds either \( TS \subseteq T \) or \( S \subseteq V_U(B) \). Now, assume first that \((0 \neq) TS \subseteq T \). Then \( TST=T \), and so \( TA=TSTA=TST=T \). Next, assume that \( S \subseteq V_U(B) \). Then \( r_B(S) =0 \) because \( TS \neq 0 \). If \( u \) is any regular element of \( A \) and \( b \) any element of \( B \), then for every \( s \in S \) we have \( s(b-ubu^{-1})=sb-(su)bu^{-1}=sb-b(su)u^{-1}=0 \), which implies \( b-ubu^{-1}=0 \) and thus \( A \subseteq V_U(B) \).

By virtue of Th. 2, the proof of the following proceeds in the same ways as that of Cor. 1.

**Corollary 2.** Let \( A \) be a completely primitive unital subring of \( U \) such that \( A \neq (GF(2))_A \) and \( B\bar{A}=B \).

1. Let \( B \) be completely primitive. If there exists a primitive idempotent \( e \) in \( B \) such that \( r_B(eB)=0 \) then either \( A \subseteq B \) or \( A \subseteq V_U(B) \).

2. If \( A=U \) then \( B \) is either dense in \( A \) or contained in \( V_A(A) \).

In the remainder of this paper, we shall give an extension of Th. 2.

**Theorem 3.** Let \( A \) be a locally completely primitive unital subring of \( U \) different from \( (GF(2))_A \). If \( B\bar{A}=B \) and \( r_A(T)=0 \) then either \( T \) is
$A$-$A$-admissible or $A \subseteq V_U(B)$.

*Proof.* If $\#A < \aleph_0$ then $A$ is a simple ring (with minimum condition). Hence, by Th. 2, we may assume $\#A \geq \aleph_0$. Let $a$ be any element of $A$ and $A'$ a completely primitive subring different from $(GF(2))_2$ and containing both $a$ and 1. Since $B A' = B$ and $r'_A(T) = 0$, it follows by Th. 2 that $T$ is $A'$-$A'$-admissible or $A' \subseteq V_U(B)$. This shows that $A = V_A(B) \cup A_0$, where $A_0 = \{a \in A; Ta + aT \subseteq T\}$, and therefore $A = V_A(B)$ or $A = A_0$.

**References**


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