ON CARTAN-BRAUER-HUA THEOREM*

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Since Cartan-Brauer-Hua theorem was generalized as in Jacobson [3, Th. 7. 13. 1] and Nagahara and Tominaga [6, Lemma 2], this theorem has been extended to simple rings (cf. Kishimoto [4] and Nagahara, Kishimoto and Tominaga [5]). The purpose of the present paper is to extend [4, Th. 2] and [5, Th. 2] to primitive rings.

Throughout our study, we use the following conventions: U will represent a ring with 1, and B a subdirectly irreducible subring[1)] of U such that the unique minimal ideal T of B is not nilpotent. A primitive ring and a completely primitive ring will mean a right primitive ring and the ring of all the linear transformations in a left vector space over a division ring, respectively. Let R be a ring. If for any finite subset F of R there exists a completely primitive subring of R containing F, R is said to be locally completely primitive. Let Λ be an arbitrary non-empty set. By $(R)_\Lambda$ and $R^{(\Lambda)}$ we denote the ring of all row-finite matrices (x_{ij}) $(i, j \in \Lambda)$ and the direct sum of $\#\Lambda$[2)]-copies of R-left module R; thus $(R)_\Lambda$ can be regarded as the ring of all linear transformations in $R^{(\Lambda)}$.

We shall first prove the following that contains [4, Th. 2].

Theorem 1. *Let A be a subring of U, and $\{2t;\ t \in T\} \neq 0$. If B is invariant relative to all the inner derivations effected by elements of A then either T is A-A-admissible or $A \subseteq V_U(B)$*[3)].

Proof. Let a be an arbitrary element of A. If $\{a, 1\}$ is linearly left independent over B then the same argument used in the proof of [4, Th. 2] enables us to conclude that $a \in V_A(B)$. On the other hand, if $\{a, 1\}$ is linearly left dependent over B then the set $\{b \in B;\ ba \in B\} = \{b \in B;\ ab \in B\}$ forms a non-zero ideal of B, so that $Ta = T^2a = T(Ta) \subseteq TB \subseteq T$ and similarly $aT \subseteq T$. We have thus $A = V_A(B) \cup A_0$, where $A_0 = \{a \in A;\ Ta + aT \subseteq T\}$. Since both $V_A(B)$ and A_0 are submodules, it follows that either $V_A(B) = A$ or $A_0 = A$ (cf. [7, Lemma 3.5]).

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1) Cf. [3] P. 219.

2) $\#\Lambda$ denotes the cardinal number of the set Λ.

3) $V_U(B)$ means the centralizer of B in U.

Corollary 1. *Let A be a subring of U such that B is invariant relative to all the inner derivations effected by elements of A, and $\{2t;\ t\in T\}\neq 0$.*

(1) *Let B be completely primitive. If there exists a primitive idempotent $e\in B$ such that $r_U(eB)=0$*[4] *then either $A\subseteq B$ or $A\subseteq V_U(B)$.*

(1′) *If $A=U$, A is primitive and B is completely primitive then either* $B=A$ or $B\subseteq V_A(A)$.

(2) Let U be completely primitive, and $A\supseteq B$. If A is dense[5] in U then B is either dense in A or contained in the center of A.

Proof. Assume that $A\not\subseteq V_U(B)$. By Th. 1, T is then A-A-admissible.

(1) Since T is the socle of B, we have $eT=eB$. Hence, $A_r\subseteq \mathrm{Hom}_{eBe}(eB, eB)=B_r$, where A_r and B_r are the rings of right multiplications in eB by the elements of A and B, respectively. Since $r_U(eB)=0$, it follows $A\subseteq B$.

(1′) If e' is an arbitrary primitive idempotent in B then $e'A\subseteq e'TA=e'T=e'B$, which implies that e' is a primitive idempotent of A. Hence, $r_U(e'B)=r_A(e'A)=0$, and so $A=B$ by the assertion (1).

(2) Obviously, T is an ideal of A. Hence, (T and so) B is dense in A by [3, Th. 2.4.4].

Next, we shall present an extension of [5, Th. 2].

Let A be a completely primitive unital[6] subring of U different from $(GF(2))_2$. Then one may regard $A=(D)_\Lambda$ where D is a division ring. We identify D with the ring of diagonal matrices of which diagonal elements are the same. e_{ij} ($\in A$) represents the matrix with element 1_D[7] in the (i,j)-position and 0's elsewhere. Let S be the socle of A, then we have $S=\sum_{i\in\Lambda}Ae_{ii}$ We set $E=\{e_{ij};\ i,j\in\Lambda\}$ and $E^*=\{\sum_{i,j}d_{ij}e_{ij}(\text{finite sum});\ d_{ij}\in D\}$, then E^* is a dense subring of A. If R is a subring of U such that $R\tilde{A}=R$[8] and $\#\Lambda>1$, then by using the same argument as in [5, Lemma 2] we can prove that if $Re_{pq}\subseteq R$ for some $e_{pq}\in E$ then $RE^*\subseteq R$.

First we need the following Lemma.

Lemma 1. *If $B\tilde{A}=B$ then $TE^*\subseteq T$ or $E^*\subseteq V_U(B)$.*

Proof. Let a be an arbitrary regular element of A. If $\{1, a\}$ is linearly left dependent over B then the set $\{b\in B;\ ba\in B\}$ is a non-zero ideal of B, and so $Ta\subseteq T$. Now, let a be biregular (i.e. a and $1-a$ are regular). If $\{1, a\}$ is linearly left independent over B, then by the same argument used

4) $r_U(eB)$ means the right annihilator of eB in U.

5) We introduce the finite topology in U.

6) A unital subring of U means a subring containing the identity element of U.

7) 1_D denotes the identity element of D.

8) $\tilde{A}$ represents the group of inner automorphisms of U induced by regular elements of A.

in the proof of [7, Lemma 3.5] one can see that $a\in V_U(B)$. Hence, by making use of the same method as in the proof of [5, Th. 2], we can prove our assertion except for the case $A=(GF(2))_\Lambda$ with $\#\Lambda\geq\aleph_0$. In what follows, we shall restrict ourselves to the exceptional case. By [3, Th. 5.3.1] one may regard then $A=(GF(2))_3\otimes_{GF(2)}A$. Choose an arbitrary index $\lambda\in\Lambda$, and consider the elements $a=\begin{bmatrix}011\\100\\010\end{bmatrix}\otimes 1$, $a^*=\begin{bmatrix}110\\010\\000\end{bmatrix}\otimes e_{\lambda\lambda}$ and $h=\begin{bmatrix}010\\000\\000\end{bmatrix}\otimes e_{\lambda\lambda}$ in A. Then, it is easy to see that $(1+h)a(1+h)^{-1}=a+a^*$ and $a^*+a^{*2}\in E$. Since a is biregular, it follows $Ta\subseteq T$ or $a\in V_U(B)$. If $Ta\subseteq T$ then $T\tilde{A}=T$ implies $T\supseteq(1+h)\,Ta(1+h)^{-1}=T(a+a^*)$, so that $T\supseteq Ta^*\supseteq T(a^*+a^{*2})$. Hence, by the remark stated just before our lemma, it follows $TE^*\subseteq T$. On the other hand, if $a\in V_U(B)$ then the same argument yields $E^*\subseteq V_U(B)$.

Now, we are at the position to prove the following that contains [5, Th. 2].

Theorem 2. *If $B\tilde{A}=B$ and $T\cdot S\neq 0$ then either T is A-A-admissible or $A\subseteq V_U(B)$.*

Proof. By Lemma 1, $TE^*\subseteq T$ or $E^*\subseteq V_U(B)$. Let u be an arbitrary regular element of A. Then, $TSTu=TSu\cdot u^{-1}Tu=TST$. Since E^* is dense in A and $e_i'=ue_{ii}u^{-1}$ is of finite rank, there exists $e_i^*\in E^*$ such that $ue_{ii}=e_i'u=e_i'e_i^*$. If $TE^*\subseteq T$ then $Te_i'=uTe_{ii}u^{-1}\subseteq uTu^{-1}=T$, and so $Tue_{ii}\subseteq Te_i^*\subseteq T$. On the other hand, if $E^*\subseteq V_U(B)$ then $V_U(B)\,\tilde{A}=V_U(B)$ implies $ue_{ii}=e_i'e_i^*\in V_U(B)$. Since A is generated by regular elements by Zelinsky's theorem (cf. [8]) and $S=\sum_{i\in\Lambda}Ae_{ii}$, we have proved that $TSTA=TST$ and there holds either $TS\subseteq T$ or $S\subseteq V_U(B)$. Now, assume first that $(0\neq)TS\subseteq T$. Then $TST=T$, and so $TA=TSTA=TST=T$. Next, assume that $S\subseteq V_U(B)$. Then $r_B(S)=0$ because $TS\neq 0$. If u is any regular element of A and b any element of B, then for every $s\in S$ we have $s(b-ubu^{-1})=sb-(su)bu^{-1}=sb-b(su)u^{-1}=0$, which implies $b-ubu^{-1}=0$ and thus $A\subseteq V_U(B)$.

By virtue of Th. 2, the proof of the following proceeds in the same ways as that of Cor. 1.

Corollary 2. *Let A be a completely primitive unital subring of U such that $A\neq(GF(2))_2$ and $B\tilde{A}=B$.*

(1) *Let B be completely primitive. If there exists a primitive idempotent e in B such that $r_U(eB)=0$ then either $A\subseteq B$ or $A\subseteq V_U(B)$.*

(2) *If $A=U$ then B is either dense in A or contained in $V_A(A)$.*

In the remainder of this paper, we shall give an extension of Th. 2.

Theorem 3. *Let A be a locally completely primitive unital subring of U different from $(GF(2))_2$. If $B\tilde{A}=B$ and $r_A(T)=0$ then either T is*

A-A-admissible or $A \subseteq V_U(B)$.

Proof. If $\# A < \aleph_0$ then A is a simple ring (with minimum condition). Hence, by Th. 2, we may assume $\# A \geq \aleph_0$. Let a be any element of A and A' a completely primitive subring different from $(GF(2))_2$ and containing both a and 1. Since $B\tilde{A}' = B$ and $r_A(T) = 0$, it follows by Th. 2 that T is A'-A'-admissible or $A' \subseteq V_U(B)$. This shows that $A = V_A(B) \cup A_0$, where $A_0 = \{a \in A ; Ta + aT \subseteq T\}$, and therefore $A = V_A(B)$ or $A = A_0$.

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