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CLOSED SUBMANIFOLDS WITH CONSTANT $
u$-TH MEAN CURVATURE RELATED WITH A VECTOR FIELD IN A RIEMANNIAN MANIFOLD

By

Yoshie KATSURADA

Introduction. H. Liebmann (1900) [1], proved the following theorem: The only ovaloids with constant mean curvature $H$ in an Euclidean space $E^3$ are the spheres.

Extension of this theorem to a convex hypersurface in an $n$-dimensional Euclidean space $E^n$ has been given by W. Süss (1929) [2], (cf. also [3], p. 118, and [4]). Then H. Hope (1951) [5], and A. D. Alexandrov (1958) [6], have shown the results that the convexity is not necessary for the validity of the Liebmann–Süss theorem.

Recently the analogous problem for closed hypersurfaces in an $n$-dimensional Riemannian manifold $R^n$ has been discussed by the present author [8], [9], [10], A. D. Alexandrov [7], K. Yano [13], T. Ôtsuki [15], M. Tani [16], and K. Nomizu [17], [18]. And also for a submanifold of codimension 2 in an odd dimensional sphere, M. Okumura has treated the analogue [19].

In the previous papers [11], [12], which are common works by T. Nagai, H. Kôjyo and the present author, we have given a certain extension of this problem to an $m$-dimensional closed submanifold $V^m$ ($1 \leq m \leq n-1$) in the $n$-dimensional Riemannian manifold $R^n$ admitting a vector field $\xi$. However we have given there a restriction such that at each point on $V^m$, the vector $\xi$ lies in the vector space spanned by the tangent space of $V^m$ and the Euler–Schouten vector $n$.

The purpose of this paper is to give more general results except this restriction.

§ 1. Some integral formulas for a submanifold. We suppose an $n$-dimensional Riemannian manifold $R^n$ ($n \geq 3$) of class $C^r$ ($r \geq 3$) wich admits an one-parameter continuous group $G$ of transformations generated by an infinitesimal transformation

\[ \tilde{x}^i = x^i + \xi^i(x) \delta \tau \]

1) Numbers in brackets refer to the references at the end of the paper.
(where \(x^i\) are local coordinates in \(R^n\) and \(\xi^i\) are the components of a contravariant vector \(\xi\)). If \(\xi\) is a Killing vector, a homothetic Killing vector, a conformal Killing vector, etc. ([14], p. 32), then the group \(G\) is called isometric, homothetic, conformal, etc.

In \(R^n\), we consider a domain \(M\). If the domain \(M\) is simply covered by the orbits of the transformations generated by \(\xi\), and \(\xi\) is everywhere of class \(C^r\) and \(\neq 0\) in \(M\); then we call \(M\) a regular domain with respect to the vector field \(\xi\).

Let us denote by \(V^m\) an \(m\)-dimensional closed orientable submanifold of class \(C^3\) imbedded in a regular domain \(M\) with respect to the vector field \(\xi\), locally given by

\[
(1.2) \quad x^i = x^i(u^\alpha) \quad i = 1, \ldots, n
\]

\[
\alpha = 1, \ldots, m,
\]

where \(u^\alpha\) are local coordinates of \(V^m\). Throughout the present paper Latin indices run from 1 to \(n\) and Greek indices from 1 to \(m\). We assume that at any point on \(V^m\) the vector \(\xi\) is not on its tangent space.

We shall indicate by \(n^i (p = m+1, \ldots, n)\) the contravariant unit vectors normal \(V^m\) and suppose that they are mutually orthogonal. Let \(n\) be in the vector space spanned by \(m+1\) independent vectors \(\frac{\partial x^i}{\partial u^\alpha} (\alpha = 1, \ldots, m)\) and \(\xi\) and be the unit vector normal \(V^m\). Then, we may consider \(n\) as one of the unit normal vectors of \(V^m\), that is, \(n^i = n^i\).

Let us consider a differential form of \(m-1\) degree at a point of \(V^m\), defined by

\[
((n, n, \cdots, n, \xi, dx, \cdots, dx) = \sqrt{g} (n, \cdots, n, \xi, dx, \cdots, dx)
\]

\[
= \sqrt{g} \left( (n, \cdots, n, \xi, \frac{\partial x^i}{\partial u^\alpha}, \cdots, \frac{\partial x^i}{\partial u^{m-1}}) du^\alpha \wedge \cdots \wedge du^{m-1},
\right)
\]

where the symbol \(\left( \right)\) means a determinant of order \(n\) whose columns are the components of respective vectors, \(dx\) is a displacement along \(V^m\), \(g\) is the determinant of the metric tensor \(g_{ij}\) of \(R^n\). Then the exterior differential of the differential form \(1.3\) divided by \(m!\) becomes as follows

\[
\frac{1}{m!} d ((n, n, \cdots, n, \xi, dx, \cdots, dx) = \frac{1}{m!} \left\{((\delta n, n, \cdots, n, \xi, dx, \cdots, dx))
\right.
\]

\[
+ ((n, \delta n, \cdots, n, \xi, dx, \cdots, dx)) + \cdots + ((n, \cdots, n, \delta \xi, dx, \cdots, dx))
\]
where $\delta v$ means $v_\alpha du^\alpha$ and the symbol ";'" the operation of $D$-symbol due to van der Waerden–Bortolotti ([20] p. 254).

Let $C^i_j$ be $\sum_{p=m+1}^n n^i n^j$ ($n=m$) and $i (\lambda=1, \cdots, m)$ mutually orthogonal unit tangent vectors of $V^m$. Then we have

$$n^i_a = C^i_j n^j \frac{\partial x^k}{\partial u^a} = -\sum_{i-1}^m \left( i^j k n^j \frac{\partial x^k}{\partial u^a} i^i \right).$$

Therefore we may put

$$n^i_a = \gamma^i_a \frac{\partial x^i}{\partial u^a}.$$

Since we have

$$g_{ij} \left( \frac{\partial x^i}{\partial u^a} \right) n^j = -g_{ij} \frac{\partial x^i}{\partial u^a} n^j,$$

we obtain

(1.5) $$n^i_a = -b^i_a \frac{\partial x^i}{\partial u^a} (p=\xi, m+2, \cdots, n)$$

where $b^i_a$ means $g^{ij} b_{ij}$ and $b_{ij} = \left( \frac{\partial x^i}{\partial u^a} \right) n^j$, and $g^{ij}$ is the contravariant metric tensor of $V^m$.

From (1.5) the first term of the right-hand member of (1.4) becomes

(1.6) $$\frac{1}{m!} ((\delta n, n, \cdots, n, \xi, dx, \cdots, dx)) = (-1)^{(n-m)(n-1)} H^i n^i dA,$$

where $dA$ is the area element of $V^m$ and $H^i$ means the first mean curvature of $V^m$ with respect to the normal direction $n^i$. Similarly, for every integer $p$ satisfying $m+2 \leq p \leq n$ we have

(1.7) $$\frac{1}{m!} ((\delta n, n, \cdots, n, \xi, dx, \cdots, dx)) = (-1)^{(n-m)(n-1)} H^i n^i dA.$$

Because $\xi$ lies in the vector space spanned by $m+1$ independent vectors $\frac{\partial x^i}{\partial u^a}$ $(\alpha=1, \cdots, m)$ and $n$.

On the other hand the last term of the right-hand member of 1.4 becomes

(1.8) $$\frac{1}{m!} ((\delta n, n, \delta \xi, dx, \cdots, dx)) = (-1)^{(n-m)(n-1)} \frac{1}{2m} \left( L g_{ij} \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^b} g^{ab} dA, \right)$$
where $L_{\xi}g$ is the Lie derivative of $g$ with respect to $\xi$ ([14], p. 5).

From (1.6), (1.7) and (1.8), (1.4) is rewritten as follows

$$
\frac{1}{m!}d \left( n, \cdots, n, \xi, dx, \cdots, dx \right) = (-1)^{(n-m)(n-1)} \left\{ H_{1}n_{i}\xi^{i}dA + \frac{1}{2m}(L_{\xi}g_{ij})\frac{\partial x^{i}}{\partial u^{a}}\frac{\partial x^{j}}{\partial u^{b}}g^{ab}dA \right\}.
$$

(1.9)

Integrating both members of (1.9) over the whole submanifold and applying Stokes’ theorem, we obtain

$$
\frac{1}{m!}\int_{\partial V} \left( n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx \right) = (-1)^{(n-m)(n-1)} \left\{ \int_{V} H_{1}n_{i}\xi^{i}dA + \frac{1}{2m}\int_{V} g^{*ij}L_{\xi}g_{ij}dA \right\},
$$

where $\partial V^{m}$ means the boundary of $V^{m}$ and $g^{*ij}$ is $\frac{\partial x^{i}}{\partial u^{a}}\frac{\partial x^{j}}{\partial u^{b}}g^{ab}$. Making use of the fact that $V^{m}$ is closed, we have

$$(1')
\int_{V} H_{1}n_{i}\xi^{i}dA + \frac{1}{2m}\int_{V} g^{*ij}L_{\xi}g_{ij}dA = 0.
$$

If the manifold $R^{n}$ assumes of constant Riemann curvature which includes an Euclidean space, then we consider the following differential form of $m-1$ degree

$$
\left( n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx \right)
= \sqrt{g} \left( n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx \right)
$$

(1.10)

for a fixed integer $\nu$ satisfying $m-1 \geq \nu \geq 1$.

As well-known, a submanifold $V^{m}$ in $R^{n}$ has the following property:

$$
\begin{align*}
\gamma_{\alpha\beta}^{\gamma} - \gamma_{\alpha\gamma}^{\beta} &= -R_{ijkl}n^{i} \frac{\partial x^{j}}{\partial u^{a}} \frac{\partial x^{k}}{\partial u^{b}} \frac{\partial x^{l}}{\partial u^{c}} \quad (20), \text{p. 226},
\end{align*}
$$

where $R_{ijkl}$ is the curvature tensor of $R^{n}$. Since $R^{n}$ is of constant Riemann curvature, we have

$$
\gamma_{\alpha\beta}^{\gamma} - \gamma_{\alpha\beta}^{\gamma} = 0.
$$

(1.11)

Consequently differentiating exteriorly the differential form (1.10), we have
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\[
d((n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx))
\]
\[
= ((\delta n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx))
\]
\[
+ ((n, \delta n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx))
\]
\[
+ \cdots + ((n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx))
\]
\[
+ ((n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx))
\]

(1.12)

because 
\[
((n, n, \cdots, n, \xi, \delta \delta n, \delta n, \cdots, \delta n, dx, \cdots, dx)) = 0
\]

from (1.11).

On substituting \( n^i_{;a} = -\epsilon^i_{a} \text{div}^{2} \frac{\partial x^i}{\partial u^\beta} \) into the first term of the right-hand member of (1.12), we get

\[
((\delta n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx))
\]
\[
= m! (-1)^{(n-m)(n-1)-\nu} H_{\nu+1} n^i \xi^i dA
\]

where \( H_{\nu+1} \) denotes the \( \nu+1 \)-th mean curvature of \( V^m \) with respect to the normal direction \( n^i \) and if we indicate by \( k_1, k_2, \cdots, k_m \) the principal curvatures of \( V^m \) for the normal vector \( n \), \( H_{\nu+1} \) is defined to be the \( \nu+1 \)-th elementary symmetric function of \( k_{\alpha} \) (\( \alpha = 1, \cdots, m \)) divided by the number of terms, that is,

\[
\left( \begin{array}{c} m \\ \nu + 1 \end{array} \right) H_{\nu+1} = \frac{\sum_{a_{1} < a_{2} < \cdots < a_{\nu+1}} k_{a_{1}} k_{a_{2}} \cdots k_{a_{\nu+1}}}{\left( \begin{array}{c} m \\ \nu + 1 \end{array} \right)}.
\]

Also, by virtue of (1.5) we can see that the vectors

\[
n \times \delta n \times n \times n \times \cdots \times n \times \delta n \times \cdots \times \delta n \times dx \times \cdots \times dx,
\]
\[
n \times n \times \delta n \times n \times \cdots \times n \times \delta n \times \cdots \times \delta n \times dx \times \cdots \times dx,
\]
\[
\cdots
\]

and

\[
n \times n \times \cdots \times n \times \delta n \times \cdots \times \delta n \times dx \times \cdots \times dx
\]

have the same direction to the covariant vectors \( n, n, \cdots \) and \( n \) respectively. Thus we obtain

\[
((n, \delta n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx)) = 0,
\]
\[ (n, n, \delta n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx) = 0, \]
\( (n, n, \cdots, n, \delta n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx) = 0, \]

because \( \xi \) lies in the vector space spanned by \( m + 1 \) independent vectors \( \frac{\partial x^\xi}{\partial u^\alpha} \) \((\alpha = 1, \cdots, m)\) and \( n \).

From that the vector \( n \times n \times \cdots \times n \times \delta n \times \cdots \times \delta n \times dx \times \cdots \times dx \) is orthogonal to the normal vectors \( n, \cdots, n, \delta n, \cdots, \delta n \) and \( \delta n^i = -b_{\alpha}^\theta \xi \xi \frac{\partial x^{i}}{\partial u^\beta} du^\alpha \), the last term of the right-hand member of (1.12) becomes as follows

\[ (n, n, \cdots, n, \delta \xi, \delta n, \cdots, \delta n, dx, \cdots, dx) \]
\( = m! (-1)^{(n-m)(n-1)-\nu} \frac{1}{2m} H_{\nu}^{a\beta} L g_{a\beta} dA, \]

where \( L g_{a\beta} = (L g_{ij}) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \) and

\[ H_{\nu}^{a\beta} = \frac{1}{(m-1)!} \varepsilon^{a_{1} \cdots a_{m-1}; \xi, \delta n_{1} \cdots \delta n_{m-1}}, \]

and \( \varepsilon^{a_{1} \cdots a_{m-1}} \) denotes the \( \varepsilon \)-symbol of the submanifold \( V^m \). Accordingly we have

\[ \frac{1}{m!} d((n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx)) \]
\( = (-1)^{(n-m)(n-1)-\nu} \left\{ H_{\nu+1} n_{i} \xi^{i} dA + \frac{1}{2m} H_{\nu}^{a\beta} L g_{a\beta} dA \right\}. \]

Integrating both members of (1.16) over the whole submanifold \( V^m \) and applying Stokes' theorem, we have

\[ \frac{1}{m!} \int_{\nu^{m}} ((n, n, \cdots, n, \xi, \delta n, \cdots, \delta n, dx, \cdots, dx)) \]
\( = (-1)^{(n-m)(n-1)-\nu} \left\{ \int_{\nu^{m}} H_{\nu+1} n_{i} \xi^{i} dA + \frac{1}{2m} \int_{\nu^{m}} H_{\nu}^{a\beta} L g_{a\beta} dA \right\}. \]

Thus, for a closed orientable submanifold \( V^m \) we obtain

\[ \left( \Pi' \right) \]
\( \int_{\nu^{m}} H_{\nu+1} n_{i} \xi^{i} dA + \frac{1}{2m} \int_{\nu^{m}} H_{\nu}^{a\beta} L g_{a\beta} dA = 0. \)

If \( m = n-1 \), that is, \( V^m \) is the hypersurface in \( R^n \), the formulas \( (I') \) and
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(II'') are coincide with the formulas (I) and (II) given in the previous paper [8]. Especially if the vector \( n \) is coincide with the Euler-Schouten unit vector \( n \) at each point on \( V^m \), then the formulas (I'') and (II'') become the formulas (I') and (II') given in the previous paper [12].

\[ \text{§ 2. The integral formulas concerning some special transformations.} \]

In this section, we shall discuss the formulas (I'') and (II'') for a special infinitesimal transformation. Let the group \( G \) of transformations be conformal, that is, \( \xi^i \) satisfies an equation: \[ Lg_{ij} = \xi_{i;j} + \xi_{j;i} = 2\phi g_{ij} \] (\([14]\), p. 32). Then we obtain

\[ g^{*\xi\eta \xi} g_{\xi \eta} = 2m\phi, \quad H^{*\xi \eta \xi} Lg_{\xi \eta} = 2m\phi H_{\xi \eta}. \]

Therefore (I'') and (II'') are rewritten in the following forms:

\[(I'')_c \quad \int_{V^m} H_{\nu} n_{i} \xi^{i} dA + c \int_{V^m} dA = 0, \quad 1 \leq \nu \leq m-1 \]

\[(I'')_h \quad \int_{V^m} H_{\nu} n_{i} \xi^{i} dA + \int_{V^m} dA = 0, \quad 1 \leq \nu \leq m-1 \]

and we can see

\[(I'')_c \quad \int_{V^m} H_{\nu} n_{i} \xi^{i} dA + c \int_{V^m} dA = 0, \quad 1 \leq \nu \leq m-1 \]

\[(I'')_h \quad \int_{V^m} H_{\nu} n_{i} \xi^{i} dA + \int_{V^m} dA = 0, \quad 1 \leq \nu \leq m-1 \]

in case of \( \phi = \text{constant} (\equiv c) \) (\( G \) being homothetic), and

\[(I'')_c \quad \int_{V^m} H_{\nu} n_{i} \xi^{i} dA = 0, \quad 1 \leq \nu \leq m-1 \]

\[(I'')_h \quad \int_{V^m} H_{\nu} n_{i} \xi^{i} dA = 0, \quad 1 \leq \nu \leq m-1 \]

in case of \( \phi = 0 \) (\( G \) being isometric).

Especially if our manifold \( R^n \) is an Euclidean space \( E^n \) and if \( \xi \) is the homothetic Killing vector field on \( E^n \) with components \( \xi^i = x^i \), \( x^i \) being rectangular coordinates with a point in the interior of \( V^m \) as origin in the space \( E^n \), then the orbits of the transformations generated by \( \xi \) are the lines through

\[ 2) \quad \text{In this case, } R^n \text{ becomes an Euclidean space, because if } R^n \text{ with constant Riemann curvature admits an one-parameter group } G \text{ of homothetic transformations, then either } R^n \text{ is } E^n \text{ or the group } G \text{ is isometric.} \]
the origin and we have
\[ Lg_{ij} = 2g_{ij}. \]
Consequently, from \((I'')_h\) and \((II'')_h\) we obtain
\[
(I^*) \quad \int_{\gamma^m} H_1 \rho dA + \int_{\gamma^m} dA = 0,
\]
\[
(II^*) \quad \int_{\gamma^m} H_{\nu+1} \rho dA + \int_{\gamma^m} H_\nu dA = 0,
\]
where \(\rho = n_{i}x^i\). This means that the formulas \((I^*)\) and \((II^*)\) are generalization of those formulas given by C. C. Hsiung [4] for a closed orientable hypersurface in an \(n\)-dimensional Euclidean space \(E^n\).

**§ 3. Some properties of a closed orientable submanifold related with a vector field.** In this section we suppose again that the group \(G\) is conformal. Then we shall prove the following four theorems for a closed orientable submanifold \(V^m\) in a Riemannian manifold \(R^n\) with constant Riemann curvature.

**Theorem 3.1.** If in \(R^n\), there exists such a group \(G\) of conformal transformations as \(\rho\) is positive (or negative) at each point of \(V^m\) and if \(H_1\) is constant, then every point of \(V^m\) is umbilic with respect to the normal vector \(n_{i}\), where \(\rho\) denotes \(n_{i}x^i\).

**Proof.** Multiplying the formula \((I'')_c\) by \(H_1=\text{const.}\), we have
\[
\int_{\gamma^m} H_1^2 \rho dA + \int_{\gamma^m} \phi H_1 dA = 0.
\]
On the other hand, from \((II'')_c\) we have
\[
\int_{\gamma^m} H_2 \rho dA + \int_{\gamma^m} \phi H_1 dA = 0.
\]
Consequently it follows that
\[
\int_{\gamma^m} (H_1^2 - H_2) \rho dA = 0.
\]
From our assumption about \(\rho\), this holds if and only if \(H_1^2 - H_2 = 0\), since
\[
H_1^2 - H_2 = \frac{1}{m^2(m-1)} \sum_{a < \beta} (k_a - k_\beta)^2 \geq 0.
\]
Therefore at each point of \(V^m\) we obtain
\[ k_1 = k_2 = \cdots = k_m. \]

Accordingly every point of \( V^m \) is umbilic with respect to \( \bar{n} \).

**Theorem 3.2.** If in \( \mathbb{R}^n \), there exists such a group \( G \) of conformal transformations as \( \rho \) is positive (or negative) at each point of \( V^m \), and if the principal curvatures \( k_1, k_2, \ldots, k_m \) at each point of \( V^m \) are positive and \( H_\nu \) is constant for any \( \nu \) \((1 < \nu \leq m-1)\), then every point of \( V^m \) is umbilic with respect to the normal vector \( \bar{n} \).

**Proof.** Multiplying the formula \((I'')_c\) by \( H_\nu = \text{const} \), we obtain

\[
(3.1) \quad \int_{\nu^m} H_1 H_\nu \rho dA + \int_{\nu^m} \phi H dA = 0.
\]

By value of \((II'')_c\) and \((3.1)\), we have

\[
\int_{\nu^m} (H_1 H_\nu - H_{\nu+1}) \rho dA = 0.
\]

From our assumption, this holds if and only if \( H_1 H_\nu - H_{\nu+1} = 0 \), since

\[
H_1 H_\nu - H_{\nu+1} = \nu! (m-\nu-1)! \sum k_{a_1} \cdots k_{a_{\nu-1}} (k_{a_\nu} - k_{a_{\nu+1}})^2 \geq 0.
\]

Then at each point of \( V^m \), we obtain

\[ k_1 = k_2 = \cdots = k_m. \]

Accordingly every point of \( V^m \) is umbilic with respect to \( \bar{n} \).

**Theorem 3.3.** If in \( \mathbb{R}^n \), there exists such a group \( G \) of conformal transformations as \( \rho \) is positive (or negative) at each point of \( V^m \), for which \( H_1 \rho + \phi \geq 0 \) (or \( \leq 0 \)) at all points of \( V^m \), then every point of \( V^m \) is umbilic with respect to \( \bar{n} \).

**Proof.** If we express the formula \((I'')_c\) as follows

\[
\int_{\nu^m} (H_1 \rho + \phi) dA = 0,
\]

then from our assumption we have the relation:

\[
(3.2) \quad \phi = -H_1 \rho.
\]

Substituting \((3.2)\) into \((II'')_c\) for \( \nu = 1 \), we have
$$\int_{V^{m}} (H_{1}^{2} - H_{2}) \rho dA = 0.$$ 

Thus, we can see the conclusion.

**Theorem 3.4.** If $H_{1}^{\xi}$ is positive (or negative) at all points of $V^{m}$ and if $R^{n}$ admits such a group $G$ of conformal transformations as $\phi$ is positive (or negative), for which either $\rho \geq \frac{-\phi}{H_{1}^{\xi}}$ or $\rho \leq \frac{-\phi}{H_{1}^{\xi}}$ at all points of $V^{m}$, then every point of $V^{m}$ is umbilic with respect to $n$.

**Proof.** The formula $(V'')_{e}$ is rewritten as follows

$$\int_{V^{m}} H_{1} \left( \rho + \frac{\phi}{H_{1}^{\xi}} \right) dA = 0.$$ 

By virtue of our assumptions $H_{1}^{\xi} > 0$ (or $< 0$) and $\rho + \frac{\phi}{H_{1}^{\xi}} \geq 0$ (or $\leq 0$) at all points of $V^{m}$, we have the following relation

$$\rho = -\frac{\phi}{H_{1}^{\xi}}. \quad (3.3)$$ 

Substituting (3.3) into $(II'')_{e}$ for $\nu = 1$, we obtain

$$\int_{V^{m}} \frac{\phi}{H_{1}^{\xi}} (H_{1}^{2} - H_{2}) dA = 0,$$

which holds if and only if $H_{1}^{2} - H_{2} = 0$. Thus we obtain the conclusion.

**Remark I.** If $V^{m}$ is the hypersurface in $R^{n}$, these four theorems are coincide with the theorems given in the previous paper [8]. Especially if the vector $n$ is coincide with the Euler-Schouten unit vector $n^{E}$ at each point of $V^{m}$, then these four theorems become those theorems given in the previous paper [12].

**Remark II.** In all these sections we have treated the normal unit vector $n^{\xi}$ with respect to the vector field $\xi^{\nu}$ and the mean curvature $H_{\nu}^{\xi}$. These are the notions due to R. E. Stong [21].

**References**


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