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ON CERTAIN CONDITIONS FOR A SUBMANIFOLD IN A RIEMANN SPACE TO BE ISOMETRIC TO A SPHERE

By

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Introduction. In an Euclidean space of three dimensions E^3 a sphere is characterized by certain special properties of a closed surface. It has been proved by H. Liebmann [1][1)] that the only ovaloid with constant mean curvature H in E^3 is a sphere. W. Süss [2] generalized this result for a closed convex hypersurface in an n-dimensional Euclidean space E^n. The analogous problem for a closed orientable hypersurface in E^n has been investigated by T. Bonnesen and W. Fenchel [3], H. Hopf [4], C. C. Hsiung [5] and A. D. Alexandrov [6]. The characterization of a sphere has been studied by many investigators and it is one of the interesting problem within the differential geometry in the large. There are also investigations about generalizing the condition $H=const.$ in the Liebmann-Süss theorem. The interesting results of this problem were given by A. D. Alexandrov [7] and S. S. Chern [9]. In the field of these investigations the integral formulas of Minkowski type has played one of the important role.

Let F be an ovaloid in E^3 and H and K the mean curvature and the Gauss curvature at a point P of F respectively. Then the integral formula of Minkowski is

$$\iint_F (Kp+H)dA=0\,,$$

where p denotes the oriented distance from a fixed point O in E^3 to the tangent space of F at P and dA is the area element of F at P. C. C. Hsiung [5] derived the generalization of this formula for a closed orientable hypersurface in E^n and gave certain characterizations of hyperspheres in E^n. Afterward Y. Katsurada [10], [11] derived the integral formulas of Minkowski type which are valid for a closed orientable hypersurface V^{n-1} in an n-dimensional Riemann space R^n and proved the following two theorems:

Theorem 0.1. (Y. Katsurada) *Let R^n be an Einstein space which ad-*

1) Numbers in brackets refer to the references at the end of the paper.

mits a vector field ξ^i *generating a continuous one-parameter group of conformal transformations in* R^n *and* V^{n-1} *a closed orientable hypersurface in* R^n *such that*

(i) $H_1 = const.$,

(ii) $n_i\xi^i$ *has fixed sign on* V^{n-1}.

Then every point of V^{n-1} *is umbilic, where* H_1 *and* n_i *denote the first mean curvature of* V^{n-1} *and covariant component of a unit normal vector of* V^{n-1} *respectively.*

Theorem 0. 2. (Y. Katsurada) *Let* R^n *be a constant Riemann curvature space which admits a vector field* ξ^i *generating a continuous one-parameter group of conformal transformations in* R^n *and* V^{n-1} *a closed orientable hypersurface in* R^n *such that*

(i) $k_1, k_2, \cdots, k_{n-1}>0$ *on* V^{n-1} *and* $H_\nu = const.$ *for any* ν $(1<\nu\leqq n-2)$,

(ii) $n_i\xi^i$ *has fixed sign on* V^{n-1}.

Then every point of V^{n-1} *is umbilic, where* k_p $(p=1,2,\cdots,n-1)$ *and* H_ν $(1<\nu\leqq n-2)$ *denote principal curvature of* V^{n-1} *and the* ν*-th mean curvature of* V^{n-1} *respectively.*

The analogous problems for a closed orientable hypersurface V^{n-1} in R^n have been discussed by A. D. Alexandrov [8], K. Yano [14], T. Ôtsuki [19], M. Tani [20] and T. Koyanagi [21]. Most of these investigations are related to the characterization of an umbilical hypersurface in R^n.

Certain generalization of Theorem 0. 1 and Theorem 0. 2 for an m-dimensional closed orientable submanifold V^m in R^n have been studied by Y. Katsurada, H. Kôjyô and the present author [12], [13] and the following three theorems were proved:

Theorem 0. 3. (Y. Katsurada and H. Kôjyô) *Let* R^n *be a constant Riemann curvature space which admits a vector field* ξ^i *generating a continuous one-parameter group of conformal transformations in* R^n *and* V^m *a closed orientable submanifold in* R^n *such that*

(i) $H_1 = const.$,

(ii) $\underset{E}{n_i}\xi^i$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* m *independent tangent vectors and* $\underset{E}{n^i}$ *at each point on* V^m.

Then every point of V^m *is umbilic with respect to Euler-Schouten unit vector* $\underset{E}{n^i}$, *where* H_1 *and* $\underset{E}{n^i}$ *denote the first mean curvature of* V^m *and covariant component of a unit normal vector which has the same direction with Euler-*

Schouten vector of V^m *respectively*[2].

Theorem 0.4. (Y. Katsurada and H. Kôjyô) *Let* R^n *be a constant Riemann curvature space which admits a vector field* ξ^i *generating a continuous one-parameter group of conformal transformations in* R^n *and* V^m *a closed orientable submanifold in* R^n *such that*

(i) $\underset{E}{k_1}, \underset{E}{k_2}, \cdots, \underset{E}{k_m} > 0$ *on* V^m *and* $\underset{E}{H_\nu} = const.$ *for any* ν $(1 < \nu \leqq m-1)$,

(ii) $\underset{E}{n_i}\xi^i$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* m *independent tangent vectors and* $\underset{E}{n^i}$ *at each point on* V^m.

Then every point of V^m *is umbilic with respect to Euler-Schouten unit vector* $\underset{E}{n^i}$, *where* $\underset{E}{k_p}$ $(p=1, 2, \cdots, m)$ *and* $\underset{E}{H_\nu}$ $(1 < \nu \leqq m-1)$ *denote principal curvature of* V^m *for the normal vector* $\underset{E}{n^i}$ *and the* ν*-th mean curvature of* V^m *for the normal vector* $\underset{E}{n^i}$ *respectively.*

Theorem 0.5. (Y. Katsurada and T. Nagai) *Let* R^n *be a Riemann space which admits a vector field* ξ^i *generating a continuous one-parameter group of homothetic transformations in* R^n *and* V^m *a closed orientable submanifold in* R^n *such that*

(i) $H_1 = const.$,

(ii) $\underset{E}{n_i}\xi^i$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* m *independent tangent vectors and* $\underset{E}{n^i}$ *at each point on* V^m,

(iv) $R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}g^{\alpha\beta}B^j_\alpha B^k_\beta \geqq 0$ *at each point on* V^m[3].

Then every point of V^m *is umbilic with respect to Euler-Schouten unit vector* $\underset{E}{n^i}$.

Because of the existence of cartesian coordinate system in E^n we can prove that if every point of a closed orientable hypersurface in E^n is umbilic, then the hypersurface is isometric to a sphere. However, in a general Riemann space we can not expect the validity of the same result even if every point of a closed orientable hypersurface is umbilic. It is one of the interesting problem to find certain conditions for an umbilical hypersurface in a Riemann space to

2) With respect to these objects we shall find again in §1 of the present paper.

3) Throughout the present paper the Latin indices run from 1 to n and the Greek indices from 1 to m $(m \leqq n-1)$. R_{ijhk}, $g^{\alpha\beta}$ and B^i_α denote curvature tensor of R^n, contravariant component of induced metric tensor in V^m and $\dfrac{\partial x^i}{\partial u^\alpha}$ respectively, where x^i are local coordinates in R^n and u^α are local coordinates on V^m. (Cf. §1 of the present paper.)

be isometric to a sphere. Recently one result on this problem was given by K. Yano [15], that is,

Theorem 0.6. (K. Yano) *Let R^n be an n-dimensional Riemann space admitting a non-constant scalar field v such that*

$$v_{;i;j}=f(v)g_{ij}\ ^{4)}$$

and V^{n-1} a closed orientable hypersurface in R^n such that

(i) $H_1 = const.$,

(ii) $v_{;i}n^i$ *has fixed sign on* V^{n-1},

(iii) $\left\{(n-1)f'(v)g_{ij}+R_{ij}\right\}n^in^j \geqq 0$ *at each point on* V^{n-1} [5].

Then every point of V^{n-1} is umbilic. If $f(v)=kv$ or $f(v)=k$ ($k=const.$) and $v\neq const.$ on V^{n-1}, the hypersurface V^{n-1} is isometric to a sphere.

To prove that the hypersurface under consideration is isometric to a sphere, the following theorem due to M. Obata [22] has been used:

Theorem (M. Obata) *If R^n ($n\geqq 2$) is complete and admits a non-null function φ such that $\varphi_{;i;j}=-c^2\varphi g_{ij}$ ($c=const.$), then R^n is isometric to a sphere of radius $\frac{1}{c}$.*

The purpose of the present paper is to investigate certain conditions for a closed orientable submanifold V^m in R^n to be isometric to a sphere. **§1** is devoted to give notations and fundamental formulas in the theory of submanifolds in a general Riemann space. In **§2** we give certain generalization of Theorem 0.6 due to K. Yano. The method of proof of our theorem is learned much from the paper of K. Yano [15].

Let us denote by M^n an n-dimensional Riemann space which admits a vector field ξ^i generating a continuous one-parameter group G of conformal transformations in M^n such that there exist a family of hypersurfaces of M^n orthogonal to the trajectories of the group G. Analytic representations of characteristic properties of M^n are shown in §3. In **§4** we derive the integral formulas which are valid for a closed orientable submanifold V^m in M^n. In **§5** we apply the integral formulas obtained in **§4** to a closed orientable submanifold V^m whose first mean curvature H_1 is constant and give some conditions for every point of V^m to be umbilic with respect to Euler-Schouten unit vector $\underset{E}{n}{}^i$. Making use of results in **§2** and **§5**, we give in the last

4) " ; " denotes the covariant differentiation with respect to the Christoffel symbol formed with metric tensor g_{ij} of R^n.

5) $f'(v)$ denotes $\frac{df}{dv}$ and R_{ij} is the Ricci tensor.

section §6 certain conditions for a closed orientable submanifold V^m in M^n to be isometric to a sphere. Especially when M^n is a constant Riemann curvature space or admits a homothetic Killing vector field, making use of Theorem 0. 3, Theorem 0. 4 and Theorem 0. 5 we can arrive at a conclusion such that V^m is isometric to a sphere.

The present author wishes to express his very sincere thanks to Professor Y. Katsurada for her many valuable advices and constant guidances.

§1. Notations and fundamental formulas in the theory of submanifolds. Let R^n $(n\geqq 3)$ be an n-dimensional Riemann space of class C^r $(r\geqq 3)$ and x^i, g_{ij} and R_{ijhk} be local coordinates, positive definite metric tensor and curvature tensor of R^n respectively. We now consider an m-dimensional closed orientable submanifold V^m $(m\leqq n-1)$ in R^n whose local expression is

$$x^i = x^i(u^\alpha),$$

where u^α denotes local coordinate on V^m. If we put

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha},$$

then B^i_α $(\alpha=1, 2, \cdots, m)$ are m independent vectors tangent to V^m and an induced metric tensor $g_{\alpha\beta}$ is given by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta .$$

We indicate by $\underset{P}{n}{}^i$ $(P=m+1,\ m+2, \cdots, n)$[6] the contravariant components of $n-m$ unit vectors which are normal to V^m and mutually orthogonal. Hence they satisfy the following relations:

$$g_{ij}B^i_\alpha \underset{P}{n}{}^j = 0\,, \qquad g_{ij}\underset{P}{n}{}^i \underset{Q}{n}{}^j = \delta_{PQ}\,,$$

where δ_{PQ} means the Kronecker delta. In this case a set of n independent vectors

$$(B^i_1, B^i_2, \cdots, B^i_m, \underset{m+1}{n}{}^i, \underset{m+2}{n}{}^i, \cdots, \underset{n}{n}{}^i) \tag{1.1}$$

determines an ennuple at each point on V^m. We put

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta\,, \qquad \underset{P}{n}{}_i = g_{ij}\underset{P}{n}{}^j\,,$$

where $g^{\alpha\beta}$ are defined by the equations $g^{\alpha\beta}g_{\beta\gamma}=\delta^\alpha_\gamma$. Then we have

$$\begin{aligned} g^{ij} &= g^{\alpha\beta}B^i_\alpha B^j_\beta + \sum_{P=m+1}^{n} \underset{P}{n}{}^i \underset{P}{n}{}^j\,, \\ g_{ij} &= g_{\alpha\beta}B^\alpha_i B^\beta_j + \sum_{P=m+1}^{n} \underset{P}{n}{}_i \underset{P}{n}{}_j\,, \end{aligned} \tag{1.2}$$

6) Throughout the present paper the capital Latin indices run from $m+1$ to n.

$$(1.3)\qquad B_j^i + C_j^i = \delta_j^i ,$$

where we put

$$B_j^i = B_\alpha^i B_j^\alpha , \qquad C_j^i = \sum_{P=m+1}^{n} \underset{P}{n}^i \underset{P}{n}_j .$$

Denoting by " ; " the operation of D-symbol due to van der Waerden-Bortolotti [23], from the definition we have

$$(1.4)\qquad B^i_{\alpha;\beta} = B^i_{j;k} B^j_\alpha B^k_\beta ,$$

$$(1.5)\qquad \underset{P}{n}^i{}_{;\alpha} = C^i_{j;k} \underset{P}{n}^j B^k_\alpha .$$

Putting $H_{\alpha\beta}^{\ i} = B^i_{\alpha;\beta}$, we call $H_{\alpha\beta}^{\ i}$ the Euler-Schouten curvature tensor. By means of (1.3) and (1.4) it follows that

$$\begin{aligned} H_{\alpha\beta}^{\ i} &= (\delta_j^i - C_j^i)_{;k} B^j_\alpha B^k_\beta \\ &= -\sum_{P=m+1}^{n} (\underset{P}{n}_{j;k} B^j_\alpha B^k_\beta) \underset{P}{n}^i . \end{aligned}$$

Therefore if we put $\underset{P}{b}_{\alpha\beta} = H_{\alpha\beta}^{\ i} \underset{P}{n}_i$, we have

$$(1.6)\qquad H_{\alpha\beta}^{\ i} = \sum_{P=m+1}^{n} \underset{P}{b}_{\alpha\beta} \underset{P}{n}^i .$$

Multiplying (1.6) by $g^{\alpha\beta}$ and summing for α and β, we get

$$(1.7)\qquad g^{\alpha\beta} H_{\alpha\beta}^{\ i} = \sum_{P=m+1}^{n} m \underset{P}{H}_1 \underset{P}{n}^i ,$$

where we put $\underset{P}{H}_1 = \dfrac{1}{m} g^{\alpha\beta} \underset{P}{b}_{\alpha\beta}$. $\underset{P}{H}_1$ is called the first mean curvature of V^m for the normal vector $\underset{P}{n}^i$.

Let $\underset{E}{n}^i$ be a unit vector which has the same direction to the vector $g^{\alpha\beta} H_{\alpha\beta}^{\ i}$ and we call it the Euler-Schouten unit vector. Then the components of the vector $\underset{E}{n}^i$ are independent of a change of parameters u^α on V^m, that is, the vector $\underset{E}{n}^i$ is determined uniquely at each point on V^m. By means of (1.6) $\underset{E}{n}^i$ is a unit normal vector of V^m. In the set of n independent vectors (1.1), $n-m$ unit normal vectors $\underset{P}{n}^i$ may be chosen in a multiply infinite number of ways. Hence we may consider $\underset{E}{n}^i$ as one of $\underset{P}{n}^i$ in (1.1). Consequently, putting $\underset{m+1}{n}^i = \underset{E}{n}^i$ we take a set of n independent vectors

$$(1.8)\qquad (B_1^i, B_2^i, \cdots, B_m^i, \underset{E}{n}^i, \underset{m+2}{n}^i, \cdots, \underset{n}{n}^i)$$

as an ennuple at each point on V^m.

The first mean curvature of V^m for normal vector $\underset{E}{n}^i$ is the socalled first mean curvature of V^m. Hence we denote it by H_1 without subscript $_E$. In this case, with respect to the ennuple (1. 8) we get from (1. 7)

$$g^{\alpha\beta}H^i_{\alpha\beta}=mH_1\underset{E}{n}^i. \tag{1.9}$$

If we denote by $\underset{E}{k}_1, \underset{E}{k}_2, \cdots, \underset{E}{k}_m$ the principal curvatures of V^m for the normal vector $\underset{E}{n}^i$, that is, the roots of the characteristic equation

$$\text{det. }(\underset{E}{b}_{\alpha\beta}-kg_{\alpha\beta})=0\,,$$

the ν-th mean curvature $\underset{E}{H}_\nu$ of V^m for the normal vector $\underset{E}{n}^i$ is defined to be the ν-th elementary symmetric function of $\underset{E}{k}_1, \underset{E}{k}_2, \cdots, \underset{E}{k}_m$ divided by the number of terms, i. e.,

$$\binom{m}{\nu}\underset{E}{H}_\nu=\sum_{\alpha_1<\alpha_2<\cdots<\alpha_\nu}\underset{E}{k}_{\alpha_1}\underset{E}{k}_{\alpha_2}\cdots\underset{E}{k}_{\alpha_\nu}\qquad(1\leqq\nu\leqq m)\,.$$

From the above definition, especially H_1 and $\underset{E}{H}_2$ satisfy the following relations:

$$mH_1=\sum_\alpha\underset{E}{k}_\alpha=\underset{E}{b}^\alpha_\alpha\,, \tag{1.10}$$

$$\binom{m}{2}\underset{E}{H}_2=\sum_{\alpha<\beta}\underset{E}{k}_\alpha\underset{E}{k}_\beta=\frac{1}{2}(\underset{E}{b}^\alpha_\alpha\underset{E}{b}^\beta_\beta-\underset{E}{b}^\alpha_\beta\underset{E}{b}^\beta_\alpha)\,, \tag{1.11}$$

where $\underset{E}{b}^\beta_\alpha=g^{\beta\gamma}\underset{E}{b}_{\alpha\gamma}$. By virtue of (1. 10) and (1. 11) it follows that

$$\underset{E}{b}^\alpha_\beta\underset{E}{b}^\beta_\alpha=m\left\{mH_1^2-(m-1)\underset{E}{H}_2\right\}. \tag{1.12}$$

On the other hand, by means of (1. 3) and (1. 5) it follows that

$$\begin{aligned}\underset{P}{n}^i{}_{;\alpha}&=(\delta^i_j-B^i_j)_{;k}\underset{P}{n}^jB^k_\alpha\\&=-(B^\beta_{j;k}\underset{P}{n}^jB^k_\alpha)B^i_\beta\,.\end{aligned}$$

Therefore we may put as follows:

$$\underset{P}{n}^i{}_{;\alpha}=\underset{P}{\gamma}^\beta_\alpha B^i_\beta\,. \tag{1.13}$$

Multiplying (1. 13) by $g_{ij}B^j_\gamma$ and contracting, we have

$$g_{ij}B^j_\gamma\underset{P}{n}^i{}_{;\alpha}=\underset{P}{\gamma}^\beta_\alpha g_{\beta\gamma}\,. \tag{1.14}$$

Since we have

$$\underset{P}{b}_{\gamma\alpha}=g_{ij}B^j_{\gamma;\alpha}\underset{P}{n}^i=-g_{ij}B^j_\gamma\underset{P}{n}^i{}_{;\alpha}\,,$$

by means of (1. 14) we get

$$\underset{P}{\gamma}{}^{\beta}_{\alpha} g_{\beta\gamma} = -\underset{P}{b}_{\alpha\gamma}\,.$$

Consequently we obtain

$$\underset{P}{n}{}^{i}{}_{;\alpha} = -\underset{P}{b}{}^{\gamma}_{\alpha} B^{i}_{\gamma}\,. \tag{1. 15}$$

By virtue of (1. 4) and (1. 5), after some calculations we get

$$\begin{aligned} B^{i}_{\alpha;\beta} &= \frac{\partial B^{i}_{\alpha}}{\partial u^{\beta}} + \Gamma^{i}_{hj} B^{h}_{\alpha} B^{j}_{\beta} - \Gamma'{}^{\gamma}_{\alpha\beta} B^{i}_{\gamma}\,, \\ \underset{P}{n}{}^{i}{}_{;\alpha} &= \frac{\partial \underset{P}{n}{}^{i}}{\partial x^{j}} B^{j}_{\alpha} + \Gamma^{i}_{hj} \underset{P}{n}{}^{h} B^{j}_{\alpha} - \Gamma''{}^{Q}_{P\alpha} \underset{Q}{n}{}^{i}\,, \end{aligned} \tag{1. 16}$$

where Γ^{i}_{hj} are the Christoffel symbols formed with g_{ij} and

$$\Gamma'{}^{\gamma}_{\alpha\beta} = \Gamma^{i}_{hj} B^{h}_{\alpha} B^{j}_{\beta} B^{\gamma}_{i} + \frac{\partial B^{i}_{\alpha}}{\partial u^{\beta}} B^{\gamma}_{i}\,,$$

$$\Gamma''{}^{Q}_{P\alpha} = \frac{\partial \underset{P}{n}{}^{i}}{\partial x^{j}} B^{j}_{\alpha} \underset{Q}{n}_{i} + \Gamma^{i}_{hj} \underset{P}{n}{}^{h} B^{j}_{\alpha} \underset{Q}{n}_{i}\,.$$

Since $\underset{P}{n}{}^{i} \underset{Q}{n}_{i} = \delta_{PQ}$, from the last relation we can easily find

$$\Gamma''{}^{Q}_{P\alpha} + \Gamma''{}^{P}_{Q\alpha} = 0\,. \tag{1. 17}$$

By virtue of (1. 6) and (1. 15) we have

$$\underset{P}{b}_{\alpha\delta;\beta} - \underset{P}{b}_{\alpha\beta;\delta} = (H_{\alpha\delta}{}^{i}{}_{;\beta} - H_{\alpha\beta}{}^{i}{}_{;\delta}) \underset{P}{n}_{i}\,.$$

Consequently we obtain the equation of Codazzi in the form

$$\underset{P}{b}_{\alpha\delta;\beta} - \underset{P}{b}_{\alpha\beta;\delta} = -R_{ikjl} \underset{P}{n}{}^{i} B^{k}_{\alpha} B^{j}_{\delta} B^{l}_{\beta} \qquad \text{([23], p. 266).} \tag{1. 18}$$

§ 2. Some properties of V^m in R^n admitting a scalar field v such that $v_{;i;j}=f(v)g_{ij}$. Let us consider that R^n admit a scalar field v such that

$$v_{;i;j} = f(v) g_{ij} \tag{2. 1}$$

and V^m be a closed orientable submanifold in R^n. We have

$$v_{;\alpha} = v_{;i} B^{i}_{\alpha}\,.$$

Covariantly differentiating[7] the vector $v_{;\alpha}$ we have

$$v_{;\alpha;\beta} = v_{;i;j} B^{i}_{\alpha} B^{j}_{\beta} + v_{;i} H_{\alpha\beta}{}^{i}\,. \tag{2. 2}$$

7) In the present paper, covariant differentiation means always the operation of D-symbol.

Multiplying (2. 2) by $g^{\alpha\beta}$ and contracting, by means of (1. 9) and (2. 1) we get

$$v^{\alpha}{}_{;\alpha} = mf(v) + mH_1 v_{;i} \underset{E}{n}{}^{i} . \qquad (v^{\alpha} = g^{\alpha\gamma} v_{;\gamma})$$

Hence by virtue of Green's theorem ([16], p. 31) we obtain the following integral formula:

$$\int_{V^m} f(v)dA + \int_{V^m} H_1 v_{;i} \underset{E}{n}{}^{i} dA = 0 , \tag{I}$$

where dA is the area element of V^m.

Now we put

$$w_{\alpha} = \left(\frac{\partial \underset{E}{n}{}^{i}}{\partial x^{j}} B^{j}_{\alpha} + \Gamma^{i}_{hj} \underset{E}{n}{}^{h} B^{j}_{\alpha} \right) v_{;i} .$$

By means of (1. 16) we have

$$w_{\alpha} = (\underset{E}{n}{}^{i}{}_{;\alpha} + \Gamma''{}_{E}{}^{P}{}_{\alpha} \underset{P}{n}{}^{i}) v_{;i} .$$

We assume that at each point of V^m the vector $v^{i} (= g^{ij} v_{;j})$ is contained in the vector space spanned by $m+1$ vectors B^{i}_{α} $(\alpha = 1, 2, \cdots, m)$ and $\underset{E}{n}{}^{i}$. Then we put as follows:

$$v^{i} = v^{\alpha} B^{i}_{\alpha} + \phi \underset{E}{n}{}^{i} . \tag{2. 3}$$

Then by means of (1. 15), (1. 17) and (2. 3) we have

$$w_{\alpha} = - \underset{E}{b}{}^{\gamma}_{\alpha} B^{i}_{\gamma} v_{;i} .$$

From the definition of w_{α}, it is remarkable that the subscript E in the last relation does not mean the vector component. Accordingly, covariantly differentiating the covariant vector w_{α}, we have

$$w_{\alpha;\beta} = -(\underset{E}{b}{}^{\gamma}_{\alpha;\beta} B^{i}_{\gamma} v_{;i} + \Gamma''{}_{E}{}^{P}{}_{\beta} \underset{P}{b}{}^{\gamma}_{\alpha} B^{i}_{\gamma} v_{;i} + \underset{E}{b}{}^{\gamma}_{\alpha} H^{i}_{\gamma\beta} v_{;i} + \underset{E}{b}{}^{\gamma}_{\alpha} B^{i}_{\gamma} B^{j}_{\beta} v_{;i;j}) . \tag{2. 4}$$

Making use of (1. 15), (1. 17), (2. 1) and (2. 3) we can easily find

$$\Gamma''{}_{E}{}^{P}{}_{\beta} \underset{P}{b}{}^{\gamma}_{\alpha} B^{i}_{\gamma} v_{;i} = 0$$

and

$$\underset{E}{b}{}^{\gamma}_{\alpha} H^{i}_{\gamma\beta} v_{;i} = \underset{E}{b}{}^{\gamma}_{\alpha} \underset{E}{b}_{\gamma\beta} \underset{E}{n}{}^{i} v_{;i} , \qquad \underset{E}{b}{}^{\gamma}_{\alpha} B^{i}_{\gamma} B^{j}_{\beta} v_{;i;j} = f(v) \underset{E}{b}_{\alpha\beta} .$$

Then, multiplying (2. 4) by $g^{\alpha\beta}$ and contracting we have

$$w^{\alpha}{}_{;\alpha} = -g^{\alpha\beta} \underset{E}{b}{}^{\gamma}_{\alpha;\beta} B^{i}_{\gamma} v_{;i} - \underset{E}{b}{}^{\gamma}_{\alpha} \underset{E}{b}{}^{\alpha}_{\gamma} \underset{E}{n}{}^{i} v_{;i} - f(v) \underset{E}{b}{}^{\alpha}_{\alpha} . \tag{2. 5}$$

By means of (1. 10) and the equations of Codazzi, the first term of the right hand side of (2. 5) becomes

$$g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha;\beta}B^{i}_{\gamma}v_{;i}=(mH_{1;\delta}g^{\gamma\delta}-g^{\alpha\beta}g^{\gamma\delta}R_{ikjl}\underset{E}{n}{}^{i}B^{k}_{\alpha}B^{j}_{\delta}B^{l}_{\beta})B^{h}_{\gamma}v_{;h}\,.$$

By virtue of (1. 2) the last term of the right hand side in the preceding equation is rewritten as follows:

$$g^{\alpha\beta}g^{\gamma\delta}R_{ikjl}\underset{E}{n}{}^{i}B^{k}_{\alpha}B^{j}_{\delta}B^{l}_{\beta}B^{h}_{\gamma}v_{;h}=R_{ikjl}\underset{E}{n}{}^{i}v^{j}g^{*kl}-R_{ikjl}\underset{E}{n}{}^{i}\underset{E}{n}{}^{j}g^{*kl}\underset{E}{n}{}^{h}v_{;h}\,,$$

where $g^{*kl}=g^{\alpha\beta}B^{k}_{\alpha}B^{l}_{\beta}$. Making use of the Ricci identity [24] and (2. 1) we can easily find

$$R_{ikjl}\underset{E}{n}{}^{i}v^{j}g^{*kl}=-mf'(v)\underset{E}{n}{}^{i}v_{;i}\,.$$

Consequently by means of (1. 10), (1. 12) and (2. 5) we have

$$\begin{aligned}w^{\alpha}{}_{;\alpha}=-\Big[&mH_{1;\delta}v^{\delta}+R_{ikjl}\underset{E}{n}{}^{i}\underset{E}{n}{}^{j}g^{*kl}\phi+mf'(v)\phi\\&+m\big\{mH_1^2-(m-1)\underset{E}{H_2}\big\}\phi+mH_1f(v)\Big]\end{aligned}$$

Then, by virtue of Green's theorem we obtain

$$\begin{aligned}\int_{V^m}\Big[&mH_{1;\delta}v^{\delta}+R_{ikjl}\underset{E}{n}{}^{i}\underset{E}{n}{}^{j}g^{*kl}\phi+mf'(v)\phi\\&+m\big\{mH_1^2-(m-1)\underset{E}{H_2}\big\}\phi+mH_1f(v)\Big]dA=0\,.\end{aligned}\tag{II}$$

If the first mean curvature H_1 of V^m is constant, we have from (I) and (II)

$$\int_{V^m}mH_1f(v)dA+\int_{V^m}mH_1^2\phi dA=0\,.$$

and

$$\int_{V^m}\Big[R_{ikjl}\underset{E}{n}{}^{i}\underset{E}{n}{}^{j}g^{*kl}\phi+mf'(v)\phi+m\big\{mH_1^2-(m-1)\underset{E}{H_2}\big\}\phi+mH_1f(v)\Big]dA=0\,.$$

Eliminating $\int_{V^n}mH_1f(v)dA$ from above two equations, we obtain

$$\int_{V^m}\big\{R_{ikjl}\underset{E}{n}{}^{i}\underset{E}{n}{}^{j}g^{*kl}+m(m-1)(H_1^2-\underset{E}{H_2})+mf'(v)\big\}\phi dA=0\,.\tag{2. 6}$$

Hence we have

Theorem 2. 1. *Let R^n be an n-dimensional Riemann space which admits a non-constant scalar field v such that*

$$v_{;i;j}=f(v)g_{ij}$$

and V^m a closed orientable submanifold such that

(i) $H_1 = const.$,

(ii) $v_{;i}\underset{E}{n}^i$ *has fixed sign on* V^m,

(iii) v^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha=1, 2, \cdots, m)$ *and* $\underset{E}{n}^i$ *at each point on* V^m,

(iv) $mf'(v)+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^j g^{*kl}\geqq 0$ *at each point on* V^m.

Then every point of V^m *is umbilic with respect to Euler-Schouten unit vector* $\underset{E}{n}^i$.

Proof. By means of (1. 10) and (1. 11) we have

$$H_1^2-\underset{E}{H}_2=\frac{1}{m^2(m-1)}\sum_{\alpha<\beta}(\underset{E}{k}_\alpha-\underset{E}{k}_\beta)^2\geqq 0\,. \tag{2. 7}$$

Hence from (2. 6) and our hypothesis it should be satisfied that $H_1^2-\underset{E}{H}_2=0$. Therefore by means of (2. 7) we obtain $\underset{E}{k}_1=\underset{E}{k}_2=\cdots=\underset{E}{k}_m$.

Theorem 2. 2. *Let R^n be an n-dimensional Riemann space which admits a non-constant scalar field v such that*

$$v_{;i;j} = kvg_{ij} \qquad (k = const.) \tag{2. 8}$$

and V^m a closed orientable submanifold such that

(i) $H_1 = const.$,

(ii) $v_{;i}\underset{E}{n}^i$ *has fixed sign on* V^m,

(iii) v^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha=1, 2, \cdots, m)$ *and* $\underset{E}{n}^i$ *at each point on* V^m,

(iv) $mk+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^j g^{*kl}\geqq 0$ *at each point on* V^m,

where $v\neq const.$ *on* V^m. *Then* V^m *is isometric to a sphere.*

Proof. By virtue of Theorem 2. 1, every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n}^i$. Since $H_1=const.$, we have

$$\underset{E}{b}_{\alpha\beta} = \lambda g_{\alpha\beta} \qquad (\lambda = const.) \tag{2. 9}$$

Then from (2. 2) and (2. 9) we get

$$v_{;\alpha;\beta} = (kv + \lambda v_{;i}\underset{E}{n}^i)g_{\alpha\beta}\,. \tag{2. 10}$$

Covariantly differentiating the scalar $v_{;i}\underset{E}{n}^i$ we have

$$(v_{;i}\underset{E}{n}^i)_{;\alpha} = v_{;i;j}\underset{E}{n}^i B^j_\alpha + v_{;i}\underset{E}{n}^i{}_{;\alpha} + v_{;i}\Gamma''{}^P_{E\alpha}\underset{P}{n}^i .$$

By means of our hypothesis (iii), (1. 15), (1. 17), (2. 8) and (2. 9) we have

$$(v_{;i}\underset{E}{n}^i)_{;\alpha} = -\lambda v_{;\alpha} .$$

Hence we obtain

$$(2.\ 11) \qquad v_{;i}\underset{E}{n}^i = -\lambda v + c . \qquad (c = const.)$$

Substituting (2. 11) into (2. 10) we have

$$(2.\ 12) \qquad v_{;\alpha;\beta} = \left\{(k-\lambda^2)v + c\lambda\right\} g_{\alpha\beta} .$$

If $k-\lambda^2=0$, then from (2. 12) we have $\Delta v = mc\lambda$, where Δ means the Laplacian operator. This is impossible unless $v = const.$ [16]. Then $k-\lambda^2 \neq 0$, and from (2. 12) we have

$$(2.\ 13) \qquad \left(v + \frac{c\lambda}{k-\lambda^2}\right)_{;\alpha;\beta} = (k-\lambda^2)\left(v + \frac{c\lambda}{k-\lambda^2}\right) g_{\alpha\beta} .$$

Therefore we obtain

$$\Delta\left(v + \frac{c\lambda}{k-\lambda^2}\right) = m(k-\lambda^2)\left(v + \frac{c\lambda}{k-\lambda^2}\right).$$

Consequently it follows that $k-\lambda^2 < 0$ [17]. Hence, by virtue of Obata's theorem V^m is isometric to a sphere.

Theorem 2. 3. *Let R^n be an n-dimensional Riemann space which admits a non-constant scalar field v such that*

$$(2.\ 14) \qquad v_{;i;j} = kg_{ij} \qquad (k = const.)$$

and V^m a closed orientable submanifod such that

(i) $H_1 = const.$,

(ii) $v_{;i}\underset{E}{n}^i$ *has fixed sign on* V^m,

(iii) v^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha = 1, 2, \cdots, m)$ *and* $\underset{E}{n}^i$ *at each point on* V^m,

(iv) $R_{ikjl}\underset{E}{n}^i\underset{E}{n}^j g^{*kl} \geqq 0$ *at each point on* V^m,

where $v \neq const.$ on V^m. Then V^m is isometric to a sphere.

Proof. By virtue of Theorem (2. 1), every point of V^m is umbilic with respect to Euler-Schuten unit vector $\underset{E}{n}^i$. Since $H_1 = const.$, we have

$$\underset{E}{b}_{\alpha\beta} = \lambda g_{\alpha\beta}. \qquad (\lambda = const.)$$

Hence from (2. 2) and (2. 14) we get

$$(2.15)\qquad v_{;\alpha;\beta}=(k+\lambda v_{;i}\underset{E}{n^i})g_{\alpha\beta}\,.$$

As we have (2. 11), from (2. 15) it follows that

$$(2.16)\qquad v_{;\alpha;\beta}=(-\lambda^2 v+c\lambda+k)g_{\alpha\beta}\,.$$

If $\lambda=0$, from (2. 16) we get $\Delta v=m(c\lambda+k)$ and this is impossible unless $v=$ *const.* [16]. Therefore $\lambda\neq 0$ and (2.16) is rewritten as follows:

$$\left(v-\frac{c\lambda+k}{\lambda^2}\right)_{;\alpha;\beta}=-\lambda^2\left(v-\frac{c\lambda+k}{\lambda^2}\right)g_{\alpha\beta}\,.$$

Consequently by virtue of Obata's theorem, V^m is isometric to a sphere.

Remark. When $m=n-1$, that is, V^m is a closed orientable hypersurface in R^n, Euler-Schouten unit vector $\underset{E}{n^i}$ is the unit normal vector n^i of V^{n-1} and $g^{*kl}=g^{kl}-n^k n^l$. In this case our hypothesis (iii) in the above three theorems are satisfied identically and (iv) becomes

$$mf'(v)+R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}=(mf'(v)g_{ij}+R_{ij})n^i n^j\geqq 0\,.$$

Therefore if $m=n-1$, our results coincide with those of Theorem 0. 6 (K. Yano).

§ 3. Properties of the vector field of M^n.

Let ξ^i be a vector field in R^n such that

$$\xi_{i;j}+\xi_{j;i}=2\Phi g_{ij}$$

where Φ is a scalar field in R^n. Then ξ^i is called a conformal Killing vector field and a continuous one-parameter group G generated by an infinitesimal transformation

$$\bar{x}^i=x^i+\xi^i\delta\tau$$

is called a conformal transformation group. If $\Phi=c$ ($c=const.$), ξ^i is called a homothetic Killing vector field and the group G is called a homothetic transformation group [18].

With respect to a scalar field σ in R^n, if a system of differential equations

$$\frac{\partial f}{\partial x^i}=e^{2\sigma}\xi_i$$

is integrable, then R^n admits a family of hypersurfaces defined by

$$f(x^1,x^2,\cdots,x^n)=\text{const.},$$

where $f(x^i)$ is a solution of the above system of differential equations. In this case every hypersurface belonging to the family is orthogonal to the trajectories of the group G.

Theorem 3.1. *Let* ξ^i *and* σ *be a vector field and a scalar field in* R^n *respectively. In order that* ξ^i *is a conformal Killing vector field and a system of differential eqnations*

$$\frac{\partial f}{\partial x^i} = e^{2\sigma}\xi_i \tag{3.1}$$

is integrable, it is necessary and sufficient that there exist a scalar field Φ *in* R^n *snch that*

$$\xi_{i;j} = \Phi g_{ij} + \sigma_{;i}\xi_j - \sigma_{;j}\xi_i . \tag{3.2}$$

Proof. If ξ^i is a conformal Killing vector field, there exists a scalar field Φ and

$$\xi_{i;j} + \xi_{j;i} = 2\Phi g_{ij} . \tag{3.3}$$

Covariantly differentiating (3.1) with respect to x^j, we obtain

$$f_{;i;j} = e^{2\sigma}(2\sigma_{;j}\xi_i + \xi_{i;j}) . \tag{3.4}$$

If the system of differential equations (3.1) is integrable, by means of its integrability conditions and (3.4) we have

$$2\sigma_{;j}\xi_i + \xi_{i;j} - 2\sigma_{;i}\xi_j - \xi_{j;i} = 0 . \tag{3.5}$$

Consequently, by means of (3.3) and (3.5) we obtain (3.2).

Conversely, if there exists a scalar field Φ and if (3.2) is satisfied we can easily obtain (3.3). Hence ξ^i is a conformal Killing vector field. On the other hand we have

$$(e^{2\sigma}\xi_i)_{;j} - (e^{2\sigma}\xi_j)_{;i} = e^{2\sigma}(2\sigma_{;j}\xi_i + \xi_{i;j} - 2\sigma_{;i}\xi_j - \xi_{j;i}) .$$

By means of (3.2) we find that the right hand side of the above relation vanishes identically. Therefore a system of differential equations (3.1) is integrable.

Let M^n be an n-dimensional Riemann space admitting a conformal Killing vector field ξ^i and a scalar field σ such that a system of differential equations (3.1) is integrable. Then by virtue of Theorem 3.1, M^n is characterized by admitting a vector field ξ^i and scalar field Φ and σ such that (3.2) is satisfied.

As a special case of Theorem 3.1 we have the following Corollary:

Corollary 3.1. *Let* ξ^i *and* σ *be a vector field and a scalar field such that* $\sigma_{;i} = \mu\xi_i$. *In order that* ξ^i *is a conformal Killing vector field and a system of differential equations* (3.1) *is integrable, it is necessary and suf-*

ficient that there exist a scalar field Φ such that $\xi_{i;j}=\Phi g_{ij}$.

Remark. Let a point O of an n-dimensional Euclidean space E^n be origin of cartesian coordinate system in E^n and y^i the coordinate of E^n with respect to the coordinate system. If we take the position vector y^i as the vector ξ^i, then we have

$$y_{i;j}=g_{ij},$$

where g_{ij} denotes covariant compoment of the metaic tensor in E^n with respect to the above cartesian coordinate system. Hence the vector field y^i is a homothetic Killing vector field in E^n, where we have $\Phi=1$. The trajectories of the homothetic transformation group G are straight lines which pass through the origin O. If we put $\sigma=\frac{1}{4}g_{jh}y^jy^h$, we have

$$\sigma_{;i}=\frac{1}{2}y_i.$$

This is the case when $\mu=\frac{1}{2}$ in Corollary 3. 1. Consequently a system of differential equations

$$\frac{\partial f}{\partial y^i}=e^{2\sigma}y_i$$

is integrable. If we put

$$f(y^1, y^2, \cdots, y^n)=e^{\frac{1}{2}(g_{jh}y^jy^h)},$$

we can easily verify that $f(y^i)$ satisfies the above system of differential equations. In this case a family of hypersurfaces defined by $f(y^1, y^2, \cdots, y^n)=const.$ is a family of hyperspheres for which the origin O is common centre. It is evident that each hypersphere belonging to this family be orthogonal to the trajectories of the homothetic transformation group G. In consequence of the above observations, E^n may be considered as a special case of M^n.

§ 4. Integral formulas for a closed orientable submanifold in M^n. Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and scalar fields Φ and σ such that

$$\xi_{i;j}=\Phi g_{ij}+\sigma_{;i}\xi_j-\sigma_{;j}\xi_i \tag{4. 1}$$

and V^m an m-dimensional closed orientable submanifold in M^n whose local expression is $x^i=x^i(u^\alpha)$. In this section we derive some integral formulas which are valid for V^m in M^n.

To the vector ξ^i, there is a covariant vector $\bar{\xi}_\alpha$ of V^m with the components

$$\bar{\xi}_\alpha = \xi_i B^i_\alpha .$$

Covariantly differentiating the vector $\bar{\xi}_\alpha$, we have

(4. 2) $$\bar{\xi}_{\alpha;\beta} = \xi_{i;j} B^i_\alpha B^j_\beta + \xi_i H_{\alpha\beta}{}^i .$$

Multiplying (4. 2) by $g^{\alpha\beta}$ and contracting, by means of (1. 9) and (4. 1) we obtain

$$\bar{\xi}^\alpha{}_{;\alpha} = (\Phi g_{ij} + T_{ij}) B^i_\alpha B^j_\beta g^{\alpha\beta} + m H_1 \xi_i \underset{E}{n}^i ,$$

where we put

$$T_{ij} = \sigma_{;i} \xi_j - \sigma_{;j} \xi_i .$$

Since T_{ij} is skew-symmetric with respect to its indices, we obtain

$$\bar{\xi}^\alpha{}_{;\alpha} = m\Phi + m H_1 \xi_i \underset{E}{n}^i .$$

As V^m is closed and orientable, by virtue of Green's theorem we get the following integral formula:

$$\int_{V^m} \Phi dA + \int_{V^m} H_1 \xi_i \underset{E}{n}^i dA = 0 , \qquad (\mathrm{I}')$$

where dA is the area element of V^m.

Now we put

$$\eta_\alpha = \left(\frac{\partial \underset{E}{n}^i}{\partial x^j} B^j_\alpha + \Gamma^i_{hj} \underset{E}{n}^h B^j_\alpha \right) \xi_i .$$

By means of (1. 16) we have

(4. 3) $$\eta_\alpha = (\underset{E}{n}^i{}_{;\alpha} + \Gamma''{}_E{}^P{}_\alpha \underset{P}{n}^i) \xi_i .$$

We assume that at each point of V^m the vector ξ^i is contained in the vector space spanned by $m+1$ independent vectors B^i_α $(\alpha = 1, 2, \cdots, m)$ and $\underset{E}{n}^i$. Then we put

(4. 4) $$\xi^i = \bar{\xi}^\alpha B^i_\alpha + \rho \underset{E}{n}^i .$$

By means of (1. 15), (1. 17) and (4. 3) we have

(4. 5) $$\eta_\alpha = \underset{E}{n}^i{}_{;\alpha} \xi_i = - \underset{E}{b}{}^\gamma_\alpha B^i_\gamma \xi_i .$$

From the definition of η_α the subscript E in (4. 5) does not mean vector component. Accordingly, covariantly differentiating the covariant vector η^α we get

$$\eta_{\alpha;\beta} = -(\underset{E}{b}{}^\gamma_{\alpha;\beta} B^i_\gamma \xi_i + \Gamma''{}_E{}^P{}_\beta \underset{P}{b}{}^\gamma_\alpha B^i_\gamma \xi_i + \underset{E}{b}{}^\gamma_\alpha H_{\gamma\beta}{}^i \xi_i + \underset{E}{b}{}^\gamma_\alpha B^i_\gamma B^j_\beta \xi_{i;j}) .$$

Putting

$$F_{\alpha\beta}=\Gamma''^{P}_{E\beta}\underset{P}{n}^{i}{}_{;\alpha}\xi_i=-\Gamma''^{P}_{E\beta}\underset{P}{b}{}^{\gamma}_{\alpha}B^i_\gamma\xi_i\,, \tag{4.6}$$

by means of (4. 1) we have

$$\begin{aligned}\eta^{\alpha}{}_{;\alpha}=&-g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha;\beta}B^i_\gamma\xi_i+F^{\alpha}_{\alpha}\\&-g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha}H_{\gamma\beta}{}^{i}\xi_i-g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha}B^i_\gamma B^j_\beta(\Phi g_{ij}+T_{ij})\,.\end{aligned} \tag{4.7}$$

By means of (1. 2), (1. 10) and the equations of Codazzi, the first term of the right hand side of (4. 7) becomes

$$g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha;\beta}B^i_\gamma\xi_i=mH_{1;\delta}\xi^{\delta}+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^jg^{*kl}\rho-R_{ikjl}\underset{E}{n}^i\xi^jg^{*kl}\,.$$

Making use of the Ricci identity we have

$$R_{jlik}\xi^j\underset{E}{n}^ig^{*kl}=(\xi_{l;i;k}-\xi_{l;k;i})\underset{E}{n}^ig^{*kl}\,.$$

Hence by means of (4. 1) we get

$$\begin{aligned}R_{ikjl}\underset{E}{n}^i\xi^jg^{*kl}&=(\Phi_{;k}g_{li}+T_{li;k}-\Phi_{;i}g_{lk}-T_{lk;i})\underset{E}{n}^ig^{*kl}\\&=T_{li;k}\underset{E}{n}^ig^{*kl}-m\Phi_{;i}\underset{E}{n}^i\,.\end{aligned}$$

Consequently we have

$$g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha;\beta}B^i_\gamma\xi_i=mH_{1;\delta}\xi^{\delta}+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^jg^{*kl}\rho-T_{li;k}\underset{E}{n}^ig^{*kl}+m\Phi_{;i}\underset{E}{n}^i\,. \tag{4.8}$$

By means of (1. 7), (1. 12) and (4. 4) the third term of the right hand side of (4. 7) becomes

$$g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha}H_{\gamma\beta}{}^{i}\xi_i=m\left\{mH_1^2-(m-1)\underset{E}{H}_2\right\}\rho\,. \tag{4.9}$$

From (1. 10) and skew-symmetric property of T_{ij} the last term of the right hand side of (4. 7) is reduced as follows:

$$g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha}B^i_\gamma B^j_\beta(\Phi g_{ij}+T_{ij})=m\Phi H_1\,. \tag{4.10}$$

By virtue of (4. 7), (4. 8), (4. 9) and (4. 10) we have

$$\begin{aligned}-\eta^{\alpha}{}_{;\alpha}=&\,mH_{1;\delta}\xi^{\delta}+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^jg^{*kl}\rho-T_{li;k}\underset{E}{n}^ig^{*kl}\\&+m\Phi_{;i}\underset{E}{n}^i-F^{\alpha}_{\alpha}+m\left\{mH_1^2-(m-1)\underset{E}{H}_2\right\}\rho+m\Phi H_1\,.\end{aligned}$$

Hence, by means of Green's theorem we get the following integral formula:

$$\begin{aligned}\int_{V^m}\Big[&mH_{1;\delta}\xi^{\delta}+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^jg^{*kl}\rho-T_{li;k}\underset{E}{n}^ig^{*kl}+m\Phi_{;i}\underset{E}{n}^i\\&-F^{\alpha}_{\alpha}+m\left\{mH_1^2-(m-1)\underset{E}{H}_2\right\}\rho+m\Phi H_1\Big]\,dA=0\,.\end{aligned} \tag{4.11}$$

From (4. 4) we have

$$\rho = \xi_i \underset{E}{n}^i .$$

Covariantly differentiating the scalar ρ, by means of (4. 1) and (4. 5) we get

$$\rho_{;\alpha} = T_{ij} \underset{E}{n}^i B^j_\alpha + \eta_\alpha .$$

Then we have

$$\rho_{;\alpha;\beta} = T_{ij;k} \underset{E}{n}^i B^j_\alpha B^k_\beta + T_{ij} \underset{E}{n}^i{}_{;\beta} B^j_\alpha + T_{ij} \Gamma''{}^P_{E\beta} \underset{P}{n}^i B^j_\alpha + T_{ij} \underset{E}{n}^i H_{\alpha\beta}{}^j + \eta_{\alpha;\beta} .$$

Multiplying the above relations by $g^{\alpha\beta}$ and contracting, by virtue of (1. 9) and (1. 15) we get

$$\begin{aligned} \rho^\alpha{}_{;\alpha} = {} & T_{ij;k} \underset{E}{n}^i g^{*jk} - T_{ij} \underset{E}{b}{}^\gamma_\beta B^i_\gamma B^j_\alpha g^{\alpha\beta} \\ & + T_{ij} \Gamma''{}^P_{E\beta} \underset{P}{n}^i B^j_\alpha g^{\alpha\beta} + m H_1 T_{ij} \underset{E}{n}^i \underset{E}{n}^j + \eta^\alpha{}_{;\alpha} . \end{aligned} \tag{4. 12}$$

By means of the skew-symmetric property of T_{ij}, the second and the fourth term of the right hand side of (4. 12) are vanish and the first term is rewritten as follows:

$$T_{ij;k} \underset{E}{n}^i g^{*jk} = - T_{ji;k} \underset{E}{n}^i g^{*kj} . \tag{4. 13}$$

Now, we calculate the third term of the right hand side of (4. 12). By virtue of (4. 1) it follows that

$$T_{ij} \Gamma''{}^P_{E\beta} \underset{P}{n}^i B^j_\alpha g^{\alpha\beta} = \xi_{i;j} \Gamma''{}^P_{E\beta} \underset{P}{n}^i B^j_\alpha g^{\alpha\beta} .$$

By means of (1. 17) and (4. 4) we get

$$\xi_{i;j} \Gamma''{}^P_{E\beta} \underset{P}{n}^i B^j_\alpha = - \Gamma''{}^P_{E\beta} \underset{P}{n}^i{}_{;\alpha} \xi_i .$$

Consequently from (4. 6) we have

$$T_{ij} \Gamma''{}^P_{E\beta} \underset{P}{n}^i B^j_\alpha g^{\alpha\beta} = - F^\alpha_\alpha . \tag{4. 14}$$

Hence by means of (4. 12), (4. 13) and (4. 14) we obtain the following integral formula:

$$\int_{V^m} (T_{li;k} \underset{E}{n}^i g^{*kl} + F^\alpha_\alpha) dA = 0 . \tag{4. 15}$$

By means of (4. 11) and (4. 15) we obtain the following integral formula for V^m in M^n:

$$\int_{V^m}\Big[mH_{1;\delta}\xi^{\delta}+m\Phi_{;i}\underset{E}{n}^i+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^j g^{*kl}\rho \\ +m\{mH_1^2-(m-1)\underset{E}{H_2}\}\rho+m\Phi H_1\Big]dA=0\,. \tag{II'}$$

§ 5. Some properties of closed orientable submanifolds in M^n. We suppose that V^m be an m-dimensional closed orientable submanifold in M^n and the first mean curvature H_1 of V^m is constant. Then from the integral formula (I') and (II') we gst

$$\int_{V^n} m\Phi H_1 dA+\int_{V^m} mH_1^2\rho dA=0$$

and

$$\int_{V^m}\Big[m\Phi_{;i}\underset{E}{n}^i+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^j g^{*kl}+m\,\{mH_1^2-(m-1)\underset{E}{H_2}\}\,\rho+m\Phi H_1\Big]dA=0\,.$$

Eliminating $\int_{V^m} m\Phi H_1 dA$ from above two equations, we obtain

$$\int_{V^n}\{m(m-1)(H_1^2-\underset{E}{H_2})\rho+m\Phi_{;i}\underset{E}{n}^i+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^j g^{*kl}\rho\}\,dA=0\,. \tag{5. 1}$$

Theorem 5. 1. *Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and scalar fields Φ and σ such that*

$$\xi_{i;j}=\Phi g_{ij}+\sigma_{;i}\xi_j-\sigma_{;j}\xi_i\,,$$

and V^m an m-dimensional closed orientable submanifold in M^n such that

(i) $H_1=const.$,

(ii) *ξ^i is contained in the vector space spanned by $m+1$ vectors B_α^i $(\alpha=1,2,\cdots,m)$ and $\underset{E}{n}^i$ at each point on V^m*[8],

(iii) *$m\Phi_{;i}\underset{E}{n}^i+R_{ikjl}\underset{E}{n}^i\underset{E}{n}^j g^{*kl}\rho\geqq 0$ (or $\leqq 0$) and $\rho>0$ (or <0) at each point on V^m.*

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n}^i$.

Proof. By virtue of (2. 7), (5. 1) and our hypothesis it should be satisfied that

$$H_1^2-\underset{E}{H_2}=0\,.$$

8) If $m=n-1$ we have Theorem 5.1 without hypothesis (ii) because it is satisfied identically for a hypersurface. This remark is available for every theorems in § 5 and § 6 of the present paper.

Consequently we get $\underset{E}{k_1}=\underset{E}{k_2}=\cdots=\underset{E}{k_m}$.

Theorem 5.2. *Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and scalar fields Φ (Φ is not const.) and σ such that*

$$\xi_{i;j}=\Phi g_{ij}+\sigma_{;i}\xi_j-\sigma_{;j}\xi_i$$
$$\Phi_{;i}=\kappa\xi_i$$

and V^m an m-dimensional closed orientable submanifold in M^n such that

(i) $H_1=const.$,

(ii) *ξ^i is contained in the vector space spanned by $m+1$ vectors B^i_α ($\alpha=1,2,\cdots,m$) and $\underset{E}{n^i}$ at each point on V^m,*

(iii) *$\xi_i\underset{E}{n^i}$ has fixed sign on V^m,*

(iv) $m\kappa+R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl}\geqq 0$ *at each point on V^m.*

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$.

Proof. By means of (5.1) and our hypothsis we should have $H_1^2-\underset{E}{H_2}=0$. Hence we obtain $\underset{E}{k_1}=\underset{E}{k_2}=\cdots=\underset{E}{k_m}$.

Let S^2 be a sphere in E^3 and P be a point on S^2. We denote by ξ^i a field of tangent vectors to great circles passing through the point P. This is an example of a vector fields which satisfy the hypothesis in Theorem 5.2.

If we suppose that $\kappa=0$ in Theorem 5.2, we have $\Phi=c$ (c=const.) and a continuous one-parameter group of transformations generated by an infinitesimal transformation $\bar{x}^i=x^i+\xi^i\delta\tau$ is homothetic transformation group. In this case we have the following theorem analogous to Theorem 0.5:

Theorem 5.3. *Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and a scalar field σ such that*

$$\xi_{i;j}=cg_{ij}+\sigma_{;i}\xi_j-\sigma_{;j}\xi_i\,,$$

where c=const., and V^m an m-dimensional closed orientable submanifold such that

(i) $H_1=const.$,

(ii) *ξ^i is contained in the vector space spanned by $m+1$ vectors B^i_α ($\alpha=1,2,\cdots,m$) and $\underset{E}{n^i}$ at each point on V^m,*

(iii) *$\xi_i\underset{E}{n^i}$ has fixed sign on V^m,*

(iv) $R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl} \geqq 0$ *at each point on* V^m.

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$.

§6. Certain characterizations of an m-dimensional sphere in M^n.

In this section we give certain conditions for an m-dimensional closed orientable submanifold V^m in M^n to be isometric to a sphere.

Theorem 6.1. *Let M^n be an n-dimensional Riemann space admitting a vector field ξ^i and scalar fields Φ (Φ is not constant) and σ such that*

$$\xi_{i;j} = \Phi g_{ij} + \sigma_{;i}\xi_j - \sigma_{;j}\xi_i, \tag{6.1}$$

$$\Phi_{;i} = c\xi_i \qquad (c=\text{const.}\neq 0) \tag{6.2}$$

and V^m an m-dimensional closed orientable submanifold such that

(i) $H_1 = const.$,

(ii) $\xi_i\underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* $m+1$ *vectors* B^i_α $(\alpha=1, 2, \cdots, m)$ *and* $\underset{E}{n^i}$ *at each point on* V^m,

(iv) $mc + R_{ikjl}\underset{E}{n^i}\underset{E}{n^j}g^{*kl} \geqq 0$ *at each point on* V^m,

where $\Phi\neq const.$ on V^m. Then V^m is isometric to a sphere.

Proof. By virtue of Theorem 5.2, we can see that every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$. By means of (6.2), we have

$$\Phi_{;i;j} = c\xi_{i;j}. \tag{6.3}$$

This shows that $\xi_{i;j}$ is symmetric with respect to its indices. Hence, from (6.1) it should be satisfied that

$$\sigma_{;i}\xi_j - \sigma_{;j}\xi_i = 0.$$

Consequently we have

$$\xi_{i;j} = \Phi g_{ij}. \tag{6.4}$$

By means of (6.3) and (6.4) we obtain

$$\Phi_{;i;j} = c\Phi g_{ij}.$$

Because of (6.2) and hypothesis (iii), the vector Φ^i $(=g^{ij}\Phi_{;j})$ is contained in the vector space spanned by $m+1$ vectors B^i_α $(\alpha=1, 2, \cdots, m)$ and $\underset{E}{n^i}$. Therefore, by virtue of Theorem 2.2, V^m is isometric to a sphere.

Theorem 6.2. *Let M^n be an n-dimensional Riemann space admitting a homothetic Killing vector field ξ^i and a scalar field σ such that $\sigma_{;i}=\mu\xi_i$. We suppose that a system of differential equations $\frac{\partial f}{\partial x^i}=e^{2\sigma}\xi_i$ is integrable and V^m is an m-dimensional closed orientable submanifold such that*

(i) $H_1=const.$,

(ii) $\xi_i \underset{E}{n}^i$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* $m+1$ *vectors* B_α^i $(\alpha=1,2,\cdots,m)$ *and* $\underset{E}{n}^i$ *at each point on* V^m,

(iv) $R_{ikjl}\underset{E}{n}^i\underset{E}{n}^j g^{*kl}\geqq 0$ *at each point on* V^m,

where $\xi_\alpha\not\equiv 0$ *on* V^m. *Then* V^m *is isometric to a sphere.*

Proof. From our hypothesis and Theorem 5.3, we can see that every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n}^i$. By virtue of Corollary 3.1, there exists a constant field c in M^n and we have

$$\xi_{i;j}=cg_{ij}$$

Then $\xi_{i;j}$ is symmetric with respect to its indices and we can see that there exists a scalar field φ satisfying $\varphi_{;i}=\xi_i$. Consequently we obtain

$$\varphi_{;i;j}=cg_{ij}\,.$$

Furthermore, from our hypothesis (iii) the vector φ^i $(=g^{ij}\varphi_{;j})$ is contained in the vector space spanned by $m+1$ vectors B_α^i $(\alpha=1,2,\cdots,m)$ and $\underset{E}{n}^i$. Then by virtue of Theorem 2.3, V^m is isometric to a sphere.

Theorem 6.3. *Let M^n be an n-dimensional constant Riemann curvature space admitting a conformal Killing vector field ξ^i and a scalar field σ such that $\sigma_{;i}=\mu\xi_i$. We suppose that a system of differential equations $\frac{\partial f}{\partial x^i}=e^{2\sigma}\xi_i$ is integrable and V^m is an m-dimensional closed orientable submanifold such that*

(i) $H_1=const.$,

(ii) $\xi_i \underset{E}{n}^i$ *has fixed sign on* V^m,

(iii) ξ^i *is contained in the vector space spanned by* $m+1$ *vectors* B_α^i $(\alpha=1,2,\cdots,m)$ *and* $\underset{E}{n}^i$ *at each point on* V^m,

where $\xi_\alpha\not\equiv 0$ *on* V^m. *Then* V^m *is isometric to a sphere.*

Proof. From our hypothesis and Theorem 0.3, we can see that every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$. By virtue of Corollary 3. 1, there exists a scalar field Φ in M^n and we have

(6. 5) $$\xi_{i;j} = \Phi g_{ij}.$$

Hence, by means of the Ricci identity it follows that

$$R_{lijk}\xi^l = \Phi_{;k} g_{ij} - \Phi_{;j} g_{ik}.$$

Since M^n is constant Riemann curvature space we have

$$R_{lijk} = c(g_{lj}g_{ik} - g_{lk}g_{ij}). \qquad (c = \text{const.})$$

Then from above two relations we get

(6. 6) $$\Phi_{;i} = c\xi_i.$$

By means of (6. 5) and (6. 6) we obtain

$$\Phi_{;i;j} = c\Phi g_{ij}.$$

Because of (6. 6) and hypothesis (iii), the vector Φ^i $(=g^{ij}\Phi_{;j})$ is contained in the vector space spanned by $m+1$ vectors B^i_α $(\alpha = 1, 2, \cdots, m)$ and $\underset{E}{n^i}$. Consequently, by virtue of Theorem 2. 2, V^m is isometric to a sphere.

Theorem 6. 4. *Let M^n be an n-dimensional constant Riemann curvature space admitting a conformal Killing vector field ξ^i and a scalar field σ such that $\sigma_{;i} = \mu\xi_i$. We suppose that a system of differential equations $\dfrac{\partial f}{\partial x^i} = e^{2\sigma}\xi_i$ is integrable and V^m is an m-dimensional closed submanifold such that*

(i) *$\underset{E}{k_1}, \underset{E}{k_2}, \cdots, \underset{E}{k_m} > 0$ on V^m and $\underset{E}{H_\nu} = $ const. for any ν $(1 < \nu \leqq m-1)$,*

(ii) *$\xi_i \underset{E}{n^i}$ has fixed sign on V^m,*

(iii) *ξ^i is contained in the vector space spanned by $m+1$ vectors B^i_α $(\alpha = 1, 2, \cdots, m)$ and $\underset{E}{n^i}$ at each point on V^m,*

where $\xi_\alpha \not\equiv 0$ on V^m. Then V^m is isometric to a sphere.

Proof. From our hypothesis and Theorem 0. 4, we can see that every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$. In this case, because of the relation

$$\underset{E}{H_1}\underset{E}{H_\nu} - \underset{E}{H_{\nu+1}} = \frac{\nu!(m-\nu-1)!}{mm!}\sum \underset{E}{k_{\alpha_1}}\underset{E}{k_{\alpha_2}}\cdots\underset{E}{k_{\alpha_{\nu-1}}}(\underset{E}{k_{\alpha_\nu}} - \underset{E}{k_{\alpha_{\nu+1}}})^2 \quad \text{(cf. [10])}$$

it follows that

$$\underset{E}{H_1}\underset{E}{H_\nu} - \underset{E}{H_{\nu+1}} = 0 \qquad (1 \leqq \nu \leqq m-1).$$

Consequently we get $\underset{E}{H_\nu} = (\underset{E}{H_1})^\nu$ for any ν. Then $\underset{E}{H_\nu}$ = const. means H_1 = const. Therefore, by virtue of Theorem 6.3, V^m is isometric to a sphere.

Remark. By virtue of the Remark stated at the end of §3, E^n admits a homothetic Killing vector field ξ^i and a scalar field σ such that $\sigma_{;i} = \mu\xi_i$, where $\mu = \frac{1}{2}$. If ξ^i is a proper homothetic Killing vector, then constant Riemann curvature space M^n becomes E^n. Let V^{n-1} be a closed convex hypersurface in E^n. We take the origin O of cartesian coordinate system of E^n in the interior of V^{n-1}. Then hypothesis (ii) and (iv) of Theorem 6.2 is satisfied. It is evident that the hypothesis (iii) of Theorem 6.2 is satisfied identically for a hypersurface. Because of the above observation, when M^n is E^n, Theorem 6.2 give us the fact that a closed convex hypersurface with constant mean curvature H_1 in E^n is isometric to a sphere. This means that when M^n is E^n, Theorem 6.2 coincides with Liebmann-Süss theorem.

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