ON THE TOPOLOGY OF SOME ALMOST CONTACT MANIFOLDS

Dedicated to Professor A. Komatsu for his 60th birthday

By

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1. Introduction and results.

We mean by an *almost contact structure* of a real orientable differentiable manifold M^{2n+1} a reduction of the structural group $SO(2n+1)$ of the tangent bundle $\tau(M)$ to $U(n)\times 1$. (Cf. J. W. Gray [2] and S. Sasaki [6].) If M has the almost contact structure, we say also that M is *almost contact.* Gray has proved that a 5-dimensional real orientable manifold is almost contact if and only if its third integral Stiefel-Whitney class is zero. The almost contact structure is a quite analogous to an almost complex structure. Passing to stable class of the tangent bundle both determine a stable complex structure. Recently, the study of stable complex structures of real vector bundles to find almost complex structures of even dimensional orientable differentiable manifolds is advanced by E. Thomas [10] and W. A. Sutherland [8] and others. In this note, we deduce some results on the existence of almost contact structures using their results. Our main theorem is,

Theorem 1. 1. *Let M be a real orientable differentiable manifold of odd dimension. The tangent bundle $\tau(M)$ has a stable complex structure if and only if M is almost contact.*

This theorem is proved by considering an induced map of Postnikov systems for fibre maps between classifying spaces. Our technique follows D. W. Kahn [4]. Let w_i denote the i-th Stiefel-Whitney class and δ the Bockstein coboundary operator associated with the exact sequence $Z\to Z\to Z_2\to 0$. Let a be a point of M. If $M-a$ has an almost contact structure then we say that M has an *almost contact structure except a point.* As applications of Theorem 1. 1, we obtain the following:

Theorem 1. 2. *Let M denote a closed real orientable differentiable manifold of dimension 9 such that $w_4(M)=0$. Then M has an almost contact structure except a point if and only if*

$$\delta w_2(M)=0\,.$$

Theorem 1.3. *Let M be a closed real orientable differentiable manifold of dimension 9 such that $H_1(M;\,Z_2)=0$. Then M has an almost contact structure except a point if and only if*

$$\delta w_2(M)=\delta w_6(M)=0\,.$$

The proofs of the theorems use Thomas's result on the existence of stable complex structures over an 8-skeleton of a base space for a vector bundle over a CW-complex and is completed by using Theorem 1.1.

Remark. If M is a real orientable manifold of dimension 7, by the Thomas's result and Theorem 1.1 we can easily verify that M is almost contact if and only if $\delta w_2(M)=0$. In the case dim $M=5$, we have the result of the same type which is already stated in the above. In the case dim $M=3$, M is an almost contact. (See also J. W. Gray [2].)

Theorem 1.1 is also applied to translate results of W. A. Sutherland [8] on the stable almost complex structure into those of the almost contact structure. Let $\xi=(B,\,S^n,\,\pi)$ be a differentiable r-sphere bundle over the standard n-sphere with the structural group $SO(r+1)$, where $n,\ r>0$ and $n+q$ is odd. It is obvious that B is an orientable differentiable manifold. Let $\theta\in\pi_{n-1}(S^r)$ be the transgression of the generator of $\pi_n(S^n)$ in the homotopy sequence of ξ. For any finite CW-complex A with a base point let $\widetilde{KO}^*(A)$ denote the Grothendieck reduced ring of real vector bundles over A. Let ξ' be the $(r+1)$-vector bundle associated to ξ and let $\{\xi'\}\in\widetilde{KO}(S^n)$ be the stable class of ξ'. We have,

Theorem 1.4. *The manifold B is almost contact if and only if one of the following three conditions is satisfied:*

(1) $n\not\equiv 0,\ 1$ *mod* 8;

(2) $\{\xi'\}$ *is divisible by* 2;

(3) $\theta^*:\ \widetilde{KO}^{-2}(S^r)\longrightarrow\widetilde{KO}^{-2}(S^{n-1})$ *is not zero.*

Remark. There is such an example that $n\equiv 1$ mod 8, $r\equiv 6$ mod 8, $\{\xi'\}$ is not divisible by 2 and $\theta^*\widetilde{KO}^{-2}(S^r)$ is not zero. (See, e.g., W. A. Sutherland [8] p. 708 and M. Kervaire [5]). Counter examples for conditions of the theorem are easily found, because $\widetilde{KO}(S^n)=Z$ for $n=0$ mod 8, $\widetilde{KO}(S^n)=Z_2$ for $n=1$ mod 8 and $\pi_{n-1}(S^r)=0$ for $r\geq n$.

Similar considerations are made for a differentiable vector bundle and a differentiable sphere bundle over a differentiable manifold with a stable complex tangent bundle. From Theorem 1.1 and the Thomas's result, we

obtain,

Theorem 1.5. *Suppose that M is a real orientable differentiable manifold of dimension 7, such that $\delta w_2(M)=0$. Let r be an even positive integer.* (**a**) *Let E denote a total space of a real orientable differentiable r-vector bundle ξ over M. Then the orientable differentiable manifold E is almost contact if and only if $\delta w_2(\xi)=0$.* (**b**) *Let B denote a total space of an r-sphere bundle over M with the structural group $SO(r+1)$, to which a real orientable differentiable $(r+1)$-vector bundle η over M is associated. Then the orientable differentiable manifold B is almost contact if $\delta w_2(\eta)=0$.*

Remark. Even if dim $M=4, 5$, or 6 and r is a positive integer such that dim $M+r$ is odd, the theorem is true. In the case dim $M=1, 2$ or 3, all conditions for the Stiefel-Whitney classes in the theorem are trivially satisfied and the consequences also hold.

All theorems stated in the above are proved in Section 3.

2. Stable complex structures of vector bundles.

Let A be a pathwise connected CW-complex and let ξ be a real orientable vector bundle over A. We say that ξ has a *stable* complex structure if it is stably isomorphic to the real vector bundle ω_R underlying some complex vector bundle ω over A. $BSO(n)$ denotes a classifying space of real orientable n-dimensional vector bundle, BSO a classifying space of stable real orientable vector bundles. $BU(n)$ denotes a classifying space of complex vector bundles of complex dimension n, BU a classifying space of stable complex vector bundles. We have a commutative diagram below, where all arrows are induced by the natural inclusions,

$$\begin{array}{ccc} BU(n) & \xrightarrow{\alpha} & BU \\ \downarrow{\scriptstyle \lambda_n} & & \downarrow{\scriptstyle \lambda} \\ BSO(2n+1) & \xrightarrow{\beta} & BSO\,. \end{array}$$

We can regard λ_n, λ as fibre maps and obtain the following commutative diagram,

$$\begin{array}{ccc} SO(2n+1)/U(n) & \xrightarrow{\varphi} & SO/U \\ \downarrow & & \downarrow \\ BU(n) & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle \lambda_n} & & \downarrow{\scriptstyle \lambda} \\ BSO(2n+1) & \xrightarrow{\beta} & BSO\,, \end{array}$$

where $SO(2n+1)/U(n)$, SO/U are fibres and φ is a natural inclusion. It is easy to verify that φ induces an isomorphism,

$$(1) \qquad \pi_i(SO(2n+1)/U(n)) \cong \pi_i(SO/U) \qquad 0 \leq i \leq 2n\,.$$

Now we use the notion of Postnikov systems of fibrations. The Postnikov system of the fibration is constructed by the induction on the dimensions of homotopy groups of fibre. (See, e.g., E. Thomas [9].) We assume all spaces have homotopy types of countable simply connected CW-complexes as far as Postnikov systems are concerned. Let $p^X: X \to \bar{X}$ and $p^Y: Y \to \bar{Y}$ be fibrations and $(f, \bar{f})$ a fibre map from the first fibration to the second, that is, $f: X \to Y$ and $\bar{f}: \bar{X} \to \bar{Y}$ are continuous maps such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p^X}\downarrow & & \downarrow{\scriptstyle p^Y} \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{Y}\,. \end{array}$$

By using the covering homotopy property[1] of the fibration p^Y, we change f by homotopy so that the above diagram is exactly commutative. If we denote by F_x^X and $F_{\bar{f}(x)}^Y$ fibres over $x \in \bar{X}$ and $\bar{f}(x) \in \bar{Y}$ respectively. We can easily see that $f(F_x^X) \subset F_{\bar{f}(x)}^Y$. Let $\{X_i, p_i^X, \pi_{i+1,i}^X\}$ and $\{Y_i, p_i^Y, \pi_{i+1,i}^Y\}$ denote Postnikov systems of the fibrations p^X and p^Y respectively. By an almost same but slightly general version of Theorems 2.1 and 2.2 of D. W. Kahn [4], there is a diagram,

$$(2) \qquad \begin{tikzcd} X \arrow[rrr, "f"] \arrow[d, dashed] \arrow[dd, bend right=60, "p^X"'] \arrow[ddd, bend right=70, "p^X"'] & & & Y \arrow[d, dashed] \arrow[dd, bend left=60, "p^Y"] \arrow[ddd, bend left=70, "p^Y"] \\ X_n \arrow[rr, "f_n"] \arrow[d, "\pi^X_{n,n-1}"] \arrow[dd, bend right=60, "p^X_{n-1}"'] & & Y_n \arrow[d, "\pi^Y_{n,n-1}"'] \arrow[dd, bend left=60, "p^Y_{n-1}"] & \\ X_{n-1} \arrow[rr, "f_{n-1}"'] \arrow[d, dashed] & & Y_{n-1} \arrow[d, dashed] & \\ \bar{X} \arrow[rr, "\bar{f}"] & & \bar{Y}, & \end{tikzcd}$$

in which the rectangular diagrams are commutative and all other diagrams are homotopy commutative. Furthermore, if we denote the k-invariants of the two Postnikov systems by k_n^X and k_n^Y for $n \geq 2$, then we have

$$(3) \qquad f_c^* k_n^X = f_{n-2}^* k_n^Y\,,$$

1) Since we have assumed that X is a countable CW-complex, the covering homotopy of any homotopy from X to $\bar{Y}$ exists for p^Y if it is a (locally trivial) fibre bundle. See, e.g., D. Husemoller [3] p. 50.

where f_c^* is the coefficient homomorphism induced by

$$(f|F_x^X)_{\#}: \pi_{n-1}(F_x^X) \longrightarrow \pi_{n-1}(F_{\bar{f}(x)}^Y).$$

We call the diagram (2) an *induced map of Postnikov systems* for the fibre map $(f, \bar{f})$.

We consider the induced map of Postnikov systems for the fibre map (α, β), that is, we put in the diagram (2) $f=\alpha$, $\bar{f}=\beta$, $X=BU(n)$, $\bar{X}=BSO(2n+1)$, $Y=BU$, $\bar{Y}=BSO$, $\lambda_n=p^X$ and $\lambda=p^Y$. The fibre F_x^X is $SO(2n+1)/U(n)$ and the fibre $F_{\bar{f}(x)}^Y$ is SO/U. Let A be a CW-complex of dimension $(2n+1)$ and ξ be a real $(2n+1)$-vector bundle over A. We consider a stable complex structure of ξ.

Lemma 2. 1. *ξ has a stable complex structure if and only if the structural group of ξ is reduced to $U(n)\times 1$.*

Remark. If ξ is a real $2n$-vector bundle over a CW-complex of dimension $2n$, a stable complex structure of ξ does not necessarily imply a structure of a complex n-vector bundle. For example, the standard $2n$-sphere S^{2n} for $n\geq 4$ does not admit an almost complex structure by Proposition 15. 1 of A. Borel and J.-P. Serre [1].

Proof. Let $\{\xi\}$ denote the stable class of ξ. We identify a vector bundle with its classifying map. We regard α as a fibre map with fibre $U/U(n)$,

$$U/U(n) \longrightarrow BU(n) \xrightarrow{\alpha} BU.$$

Since the fibre $U/U(n)$ is $2n$-connected, by the exact homotopy sequence of the fibre map and by a theorem of J. H. C. Whitehead [11] α induces a $2n$-homotopy equivalence,

$$BU(n) \simeq_{2n} BU. \tag{4}$$

In a similar manner, regarding β as a fibre map with fibre $SO/SO(2n+1)$,

$$SO/SO(2n+1) \longrightarrow BSO(2n+1) \xrightarrow{\beta} BSO,$$

it is verified that β induces a $2n$-homotopy equivalence,

$$BSO(2n+1) \simeq_{2n} BSO. \tag{5}$$

Suppose that ξ has a stable complex structure, that is, $\{\xi\}$ has a lift η for the map λ. By (4), the restriction of η on the $2n$-skeleton A^{2n} of A, denoted by $\eta|A^{2n}$, is lifted for the map α. We denote this lift by η'. By the commutativity except the triangle η', λ_n, $\xi|A^{2n}$ in the diagram,

(6)
$$\begin{tikzcd}
 & BU(n) \arrow[r, "\alpha"] \arrow[d, "\lambda_n"] & BU \arrow[d, "\lambda"] \\
 & BSO(2n+1) \arrow[r, "\beta"] & BSO \\
A^{2n} \arrow[uur, "\eta'"] \arrow[ur, "\eta|A^{2n}"] \arrow[r, "\xi|A^{2n}"] \arrow[urr, bend right, "\{\xi\}|A^{2n}"'] & &
\end{tikzcd}$$

and by (5), $\lambda_n \cdot \eta'$ is homotopic to the restriction $\xi|A^{2n}$ of ξ on A^{2n}. We change ξ by homotopy if necessary and we can assume that η' is an exact lift of $\xi|A^{2n}$, that is $\lambda_n \cdot \eta' = \xi|A^{2n}$. The diagram (6) is homotopy commutative. By (2), we can see easily that $(p^{BU}_{2n-1}) \cdot (\eta|A^{2n}) \simeq (f_{2n-1})(p^{BU(n)}_{2n-1}) \cdot \eta'$. The map $(p^{BU(n)}_{2n-1}) \cdot \eta'$ is extended to a lift of ξ, $\eta'' : A \to X_{2n-1}$ and we have

$$(p^{BU}_{2n-1}) \cdot \eta \simeq f_{2n-1} \cdot \eta'' .$$

From this relation and (3), it follows that

$$\begin{aligned}
(7) \qquad (\eta)^* (p^{BU}_{2n-1})^* k^{BU}_{2n+1} &= (\eta'')^* (f_{2n-1})^* k^{BU}_{2n+1} \\
&= (\eta'')^* \alpha_c^* k^{BU(n)}_{2n+1} \\
&= \alpha_c^* (\eta'')^* k^{BU(n)}_{2n+1} ,
\end{aligned}$$

since α_c^* is the coefficient homomorphism. But $\eta|A^{2n}$ is extended to the map η over the whole complex A and hence the left side of (7) is zero. The homomorphism α_c^* is induced by the map φ and is the isomorphism $\pi_{2n}(SO(2n+1)/U(n)) \cong \pi_{2n}(SO/U)$ of (1). Therefore by (7), we have

$$(\eta'')^* k^{BU(n)}_{2n+1} = 0 ,$$

which is the obstruction to extend the lift η' of $\xi|A^{2n}$ over the whole complex A. Thus the map ξ is lifted to $\bar{\eta} : A \to BU(n)$, which means the reduction of the structural group of ξ to $U(n) \times 1$.

The converse is obvious.

3. Almost contact structures of manifolds.

We apply Lemma 2.1 in the above to find almost contact structures on some odd dimensional manifolds.

Proof of Theorem 1.1. We regard the tangent bundle $\tau(M)$ of the given differentiable manifold as the real $(2n+1)$-vector bundle ξ in Lemma 2.1. The result follows immediately.

To prove Theorems 1.2 and 1.3, we state a theorem of E. Thomas about the existence of a stable complex structure of the real vector bundle

over a CW-complex. Let ξ be a real orientable vector bundle over a CW-complex A. Define $R_1(\xi)\subset H^4(A\,;\,Z)$ to be the set of all classes $c_2(\omega)$, where ω runs over all stable complex structures on ξ restricted to the 6-skeleton A^6 of A. Let Ω denote (non-stable) secondary operation associated with the relation,

$$Sq^2(\delta Sq^2)=0 \tag{8}$$

on integral classes of dimension 4, such that for the second Chern class $2x\in H^4(QP^2\,;\,Z)$ of the 2-dimensional quaternion projective space QP^2, x^2 mod $2\in\Omega(2x)$. Theorem 1.2 of E. Thomas [10] says that ξ restricted to the 8-skeleton A^8 of A has a stable complex structure if and only if

$$\delta w_2(\xi)=\delta w_6(\xi)=0 \tag{9}$$

and

$$w_8(\xi)+w_4(\xi)^2+w_2(\xi)^2\,w_4(\xi)\in\Omega(R_1(\xi))\,. \tag{10}$$

We refer this theorem to Theorem (T).

The proof of Theorem 1.2 closely follows that of Theorem 1.6 of E. Thomas [10].

Proof of Theorem 1.2. Let a be a point of M. Because of the isomorphism $H^3(M\,;\,Z)\cong H^3(M-a\,;\,Z)$ induced by the natural inclusion map, $\delta w_2(M)=0$ is certainly a necessary condition for M to have an almost contact structure except a point. In the following, we show that this condition is sufficient under the hypothesis of the theorem.

Let $v\in R_1(M)=R_1(\tau(M))$. Then we have

$$v \bmod 2=w_4(M)=0$$

and there is a class $x\in H^4(M\,;\,Z)$ such that $v=2x$. Thus by Lemmas 1.1 and 1.5 of E. Thomas [10], we have

$$\Omega(R_1(M))=\Omega(v)=\Omega(2x)=\{x^2\}\,.$$

At first, we show that x^2 mod $2\in Sq^2H^6(M\,;\,Z)$ for all $x\in H^4(M\,;\,Z)$, which implies that $\Omega(R_1(M))=0$. For this it suffices to show (by Poincaré duality) for a fixed x and for any $y\in H^1(M\,;\,Z)$, that

$$x^2\cdot y=(Sq^2(x\cdot u))\cdot y\,,$$

where $u\in H^2(M\,;\,Z)$ is a class such that u mod $2=w_2(M)$. Now we obtain

$$x^2\cdot y=Sq^4(x)\cdot y=Sq^4(x\cdot y)+Sq^3(x)\cdot Sq^1(y)\,. \tag{11}$$

Let $V_i\in H^i(M\,;\,Z_2)$ denote the Wu class, $i\geq 0$. That is, V_i is the unique class such that for all $u\in H^{9-i}(M\,;\,Z_2)$,

(12) $$Sq^i(u) = u \cdot V_i .$$

(Note that $V_0 = 1$, $V_i = 0$ for $i \geq 5$.) We have the Wu's formula (see W. T. Wu [12] or E. Spanier [7]),

$$w_k(M) = \sum_{i=0}^{k} Sq^i V_{k-i} , \qquad k \geq 0 .$$

In particular, we obtain

$$V_1 = w_1(M) = 0 ,^{2)}$$
$$V_2 = w_2(M) ,$$
$$V_3 = w_3(M) + Sq^1 V_2 = 0 .$$

(13) $$V_4 = w_4(M) + w_2(M)^2 = w_2(M)^2 ,^{3)}$$

(14) $$w_6(M) = 0 , \qquad \delta w_6(M) = 0 ,$$

(15) $$w_8(M) = (w_4(M) + w_2(M)^2)^2 = w_2(M)^4 .$$

From (12) and (13), it follows that

(16) $$Sq^4(x \cdot y) = x \cdot y \cdot V_4 = x \cdot y \cdot w_2(M)^2 .$$

By the Cartan formula, we have

$$Sq^2(Sq^2(x) \cdot y) = Sq^2 Sq^2(x) \cdot y + Sq^3(x) \cdot Sq^1(y) + Sq^2(x) \cdot Sq^2(y) .$$

But we get $Sq^2(y) = 0$, since $y \in H^1(M ; Z)$ and get $Sq^2 Sq^2(x) = Sq^3 Sq^1(x) = 0$, since x is an integral class. Therefore, we have

(17) $$Sq^2(Sq^2(x) \cdot y) = Sq^3(x) \cdot Sq^1(y) .$$

From (11), (17), (16) and (12), it follows that

$$x^2 \cdot y = x \cdot y \cdot w_2(M)^2 + (Sq^2(x) \cdot y) \cdot w_2(M) .$$

Finally, we obtain

$$x \cdot w_2(M)^2 + Sq^2(x) \cdot w_2(M) = Sq^2(x \cdot u)$$

and so

$$x^2 \cdot y = Sq^2(x \cdot u) \cdot y ,$$

namely,

(18) $$\Omega(R_1(M)) = 0 .$$

By (15), we get

$$w_8(M) = u^4 \bmod 2 = Sq^2(u^3) \in Sq^2 H^6(M ; Z) .$$

2) Since M is orientable by the assumption of the theorem, we have $W_1(M) = 0$.
3) We have assumed in theorem that $W_4(M) = 0$.

Therefore, from (18) and the assumption $w_4(M)=0$, it follows that

$$w_8(M)+(w_4(M))^2+w_2(M)^2w_4(M)\in\Omega(R_1(M)).$$

We apply to both sides of this formula the homomorphism, $H^8(M;Z_2)\to H^8(M-a;Z_2)$ induced by the natural inclusion map, and then by the naturality of the Stiefel-Whitney classes and of the operations used in the above formula, we have

$$w_8(M-a)+(w_4(M-a))^2+(w_2(M-a))^2\cdot w_4(M-a)\in\Omega(R_1(M-a)).$$

Moreover, using the isomorphisms, $H^3(M;Z)\cong H^3(M-a;Z)$ and $H^7(M;Z)\cong H^7(M-a;Z)$ induced by the natural inclusion map, we derive from the condition $\delta w_2(M)=0$ and (14), that

$$\delta w_2(M-a)=\delta w_6(M-a)=0.$$

By Theorem (T), $\tau(M)$ restricted to the 8-skeleton of $M-a$ has a stable complex structure. The next obstruction to extend the stable complex structure lies in $H^9(M-a;Z_2)$ which is zero. Hence, the restriction $\tau(M)|M-a$ of $\tau(M)$ on $M-a$ has a stable complex structure over all of $M-a$. Therefore, from Theorem 1.1 it follows that $\tau(M-a)=\tau(M)|M-a$ is almost contact, that is, M has an almost contact structure except a point.

Proof of Theorem 1.3. By the same argument in the proof of Theorem 1.2, we deduce that $\delta w_2(M)=0$ is a necessary condition for M to have an almost contact structure except a point. We consider to prove the sufficiency of this condition. By the assumption, $H^1(M;Z_2)=0$ and Poincaré duality, we have $H^8(M;Z_2)=0$. Therefore, the condition (10) on the secondary operation of Theorem (T) for $\tau(M)$ is trivially satisfied. Moreover, by the assumption we have $\delta w_2(M)=\delta w_6(M)=0$. If G denote any coefficient group, the homomorphism $\varphi^*: H^i(M;G)\to H^i(M-a;G)$ is isomorphic for $0\leq i\leq 7$ and injective for $i=8$, and $H^9(M-a;G)$ is zero. By making use of these facts, and by the same arguments as the last part of the proof of Theorem 1.2, the proof of Theorem 1.3 is completed.

Proof of Theorem 1.4. Theorem 1.2 of W. A. Sutherland [8] shows that the tangent bundle $\tau(B)$ of the manifold B (the total space of the r-sphere bundle over the n-sphere) has a stable complex structure if and only if one of the three conditions of the theorem. By Theorem 1.1, a stable complex structure over $\tau(B)$ is reduced to an almost contact structure of B and the proof of Theorem 1.4 is completed.

Proof of Theorem 1.5. Since M is orientable, we have $H^7(M;Z)=Z$. But $\delta w_6(M)$, $\delta w_6(\xi)$ and $\delta w_6(\eta)\in H^7(M;Z)$ is an element of order 2, and

therefore they must be zero. Moreover, we have assumed that $\delta w_2(M)=0$. Since $H^8(M;\, Z_2)=0$, the conditions of Theorem (T) of Thomas, for $\tau(M)$ are all satisfied. Therefore, $\tau(M)$ has a stable complex structure.

(a) Let $\pi:\ E\to M$ be the projection of the r-vector bundle ξ and let $\bar{\xi}$ be the tangent bundle along the fibre of ξ. We denote by $\pi^!$ the correspondence to a bundle induced by π from a bundle over M. It is easy to verify that

$$\tau(E)=\pi^!\tau(M)\oplus\bar{\xi}\,.$$

$\pi^!\tau(M)$ has a stable complex structure since $\tau(M)$ has it. If M_0 denote the 0-section of $\pi:\ E\to M$, ξ coincides with the restriction $\bar{\xi}|M_0$ of $\bar{\xi}$ on M_0 and hence by the homotopy classification theorem, we have $\bar{\xi}\cong\pi^!\xi$. Therefore, $\tau(E)$ has a stable complex structure if and only if ξ has a stable complex structure. Moreover, by Theorem (T), ξ has a stable complex structure if and only if

$$\delta w_2(\xi)=0\,.$$

Theorem 1. 1 says that $\tau(E)$ is almost contact if and only if it has a stable complex structure and thus the result is proved.

(b) Let π be the projection of the given r-sphere bundle over n-sphere and let $\bar{\eta}$ be the tangent bundle along the fibre of the r-sphere bundle. We have denoted by B the total space of the r-sphere bundle. It follows that

$$\tau(B)=\pi^!\tau(M)\oplus\bar{\eta}\,,\qquad \pi^!\eta\cong\bar{\eta}\oplus 1\,.$$

If η has a stable complex structure, $\tau(B)$ has a stable complex structure since $\pi^!\tau(M)$ has a stable complex structure. Again by Theorem (T), η has a stable complex structure if and only if

$$\delta w_2(\eta)=0\,.$$

From Theorem 1. 1, it follows that B is almost contact if $\delta w_2(\eta)=0$.

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