A QUASI-DECOMPOSABLE ABELIAN GROUP
WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS
AND PROPER ISOMORPHIC SUBGROUPS II

By

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1. Introduction
This paper is a continuation of our paper [2] with the same title. In [2] one of theorems was stated without detailed proof. The purpose of this paper is just to give a complete proof of it. So we shall omit an origin of our problem and general background for it, see the first section of [2] and references in [2].

Let $p>1$ be a fixed prime number, $C(p^n)$ be a cyclic group of order $p^n$, $\Sigma$ be the direct sum of cyclic groups $C(p^n)$, $\Pi$ be the direct product of cyclic groups $C(p^n)$ and $C$ be the torsion group of $\Pi$, that is, $\Sigma$ is the standard basic group and $C$ is the torsion completion of $\Sigma$.

Our theorem is as following, that is Theorem 3 in [2].

Theorem. There exists a pure subgroup $G$ of $C$ which contains $\Sigma$ and satisfies properties;
1) $G$ has no proper isomorphic quotient groups,
2) $G$ has no proper isomorphic subgroups,
3) $G$ has a decomposition $G_1 \oplus G_2$ such that $G_1$ and $G_2$ are not bounded.

The $p$-socle $C[p]$ of $C$ is a vector space over the prime field of characteristic $p$ and can be topologized as a totally disconnected compact topological group, because $\Pi$ is clearly a totally disconnected compact topological group with respect to the product topology of compact discrete topologies and the $p$-socle $C[p]$ of $C$ is the closed subgroup $\{x|x \in \Pi, px=0\}$ of $\Pi$. Actually $U_n=\{x|x \in C[p]$ and the hight of $x \geq n\}=(p^nC)[p]$ $(n=1,2\cdots)$ are open compact subgroups of $C[p]$ and $\{U_n\}$ is a fundamental system of 0-neighborhoods in $C[p]$. These two structures on $C[p]$ which are a vector space and a totally disconnected compact group are used in an essential way.

Every continuous group homomorphism $T$ on $C[p]$ defines compact subgroups $E_q(T)=\{x|x \in C[p]$ and $Tx=qx\}$ $(0 \leq q < p)$ and the compact subgroup
$E(T) = E_0(T) \oplus E_1(T) \oplus \cdots \oplus E_{p-1}(T)$. We call $T$ a *singular* homomorphism if $E(T)$ is an open compact subgroup of $C[p]$. For instance a continuous projection on $C[p]$ is singular. If a continuous group homomorphism $T$ on $C[p]$ has a dense subgroup which is invariant under $T$ and on which $T$ is one to one, $T$ is called a *semi-isomorphism* on $C[p]$.

In the final step in our proof of the theorem we shall use following proposition which is shown in [2] (Theorem 1 of [2]).

**Proposition.** Let $G$ be a pure subgroup of $C$ which contains $\Sigma$ and $G[p]$ be the $p$-socle of $G$.

1) If $G[p]$ is not invariant under any non-singular onto homomorphism on $C[p]$, then $G$ has no proper isomorphic quotient groups.

2) If $G[p]$ is not invariant under any non-singular semi-isomorphism on $C[p]$, then $G$ has no proper isomorphic subgroups.

Therefore our main work in this paper is to show existence of a group $Q$ which is between $\Sigma[p]$ and $C[p]$, and which is not invariant under any non-singular homomorphism but invariant under the canonical projection $P_e$ onto even coordinates (see Lemma 2). In our proof of this fact we have some difficulty which comes from non-commutativity between non-singular homomorphisms and $P_e$. So we classify degree of non-commutativity in the following section.

### 3. Proof of Theorem

We introduce eight classes (essentially four) of non-singular homomorphisms depending on degree of non-commutativity. Hereafter we shall use notations $A^e = P_e(A)$ ($A^o = (I - P_e)(A)$) for a subset $A$ of $C[p]$ and $x^e = P_e x$ ($x^o = x - P_e x$) for an element $x$ in $C[p]$. Every continuous homomorphism $T$ on $C[p]$ defines compact subgroups

$$F^o(T) = \{ x | x \in C[p]^o \text{ and } (Tx)^o = 0 \},$$

$$E_{q,n}^o(T) = \{ x | x \in U_n^o \text{ and } (Tx)^o = qx \} \quad \text{for } q = 0, 1, \ldots, p-1; n = 1, 2, \ldots. \quad F^e(T) \text{ and } E_{q,n}^e(T) \text{ are defined similarly.}$$

Let $\tilde{E}_n^o(T) = E_{0,n}^o(T) \oplus E_{1,n}^o(T) \oplus \cdots \oplus E_{p-1,n}^o(T)$ and $\tilde{E}_n^e(T) = E_{0,n}^e(T) \oplus E_{1,n}^o(T) \oplus \cdots \oplus E_{p-1,n}^e(T)$.

Following eight types about $T$ are defined;

$I^o$ there exists $N$ such that $F^o(T) \supset U_N^o$ and $\tilde{E}_n^o(T) \not\supset U_m^o$ for all $n, m \in N$,

$II^o$ $F^o(T) \not\supset U_n^o$ for all $n = 1, 2, \ldots$ but there are $N$ and $q$ such that $E_{q,N}^o(T) \supset U_N^o$,

$III^o$ $F^o(T) \not\supset U_n^o$ for all $n = 1, 2, \ldots$ and $E_{q,n}^o(T) \not\supset U_m^o$ for all $n, m =$
1, 2, \ldots \text{ and } q=0, 1, \ldots, p-1, \text{ but there are } N \geq N \text{ such that } \tilde{E}^o_n(T) \\ \supset U^o_n, \\
 IV^o F^o(T) \not\supset U^o_n \text{ for all } n = 1, 2, \ldots \text{ and } \tilde{E}^o_n(T) \not\supset U^o_m \text{ for all } n, m = 1, 2, \ldots.

I^o, II^o, III^o \text{ and } IV^o \text{ are defined by exchanging odd and even.}

We note that every non-singular homomorphism is at least of one of the above eight types. Because, suppose } T \text{ is of none of the above eight types, then there are } N, n \text{ and } m \text{ such that } n, m \geq N \text{ and}

1) F^o(T) \supset U^o_n \text{ and } F^e(T) \supset U^e_n, \\
2) \tilde{E}^o_n(T) \supset U^o_n \text{ and } \tilde{E}^e_n(T) \supset U^e_n.

Then 1) implies } E^o_q(T) \subset E^o_q(T) \text{ and } E^e_q(T) \subset E^e_q(T) \text{ for all } q = 0, 1, \ldots, p-1. \text{ So 2) implies } E(T) = E^o(T) \oplus \cdots \oplus E^e_{p-1}(T) \supset U^o_n \oplus U^e_m = U_m. \text{ This means that } T \text{ is a singular homomorphism.}

In order to construct } Q \text{ explained at the end of section 2 we need the following:}

**Lemma 1.** For an arbitrary non-singular homomorphism } T \text{ we can find a one-parameter family } A(T) = \{x_t|0 \leq t \leq 1\} \text{ of elements in } C[p] \text{ which has one of the following six properties; } 1^o, 2^o, 3^o, 1^e, 2^e \text{ and } 3^e,

1^o x_t, Tx_t \in C[p]^o \text{ for all } 0 \leq t \leq 1 \text{ and four elements } x_s, x_t, Tx_s \text{ and } Tx_t \text{ are linearly independent for arbitrary } s \neq t, \\
2^o \text{ there exists } q, 0 \leq q \leq p-1 \text{ such that } x_t \in C[p]^o \text{ and } Tx_t - qx_t \in C[p]^o \text{ for all } 0 \leq t \leq 1 \text{ and four elements } x_s, x_t, Tx_s - qx_s \text{ and } Tx_t - qx_t \text{ are linearly independent for arbitrary } s \neq t, \\
3^o x_t \in C[p]^o \text{ for all } 0 \leq t \leq 1 \text{ and six elements } x_s, x_t, (Tx_s)^o, (Tx_t)^o, (Tx_s)^e \text{ and } (Tx_t)^e \text{ are linearly independent for arbitrary } s \neq t.

1^e, 2^e \text{ and } 3^e \text{ are dual properties of } 1^o, 2^o \text{ and } 3^o \text{ by exchanging odd for even.}

**Proof.** Of course the choice of these six properties about } A(T) \text{ depends on types of } T \text{ defined above. Therefore the proof is divided into four cases, a) \sim d).}

The following fact (*) which was shown already in [2] (see b) of the proof of Lemma 3 in [2]) will be used in cases a), b) and d).

(*) Suppose } x_0 \text{ and } Tx_0 \text{ are linearly independent and } \{x_0, x_1, \ldots, x_{p^2-1}\} \text{ is a group } [x_0] \oplus [Tx_0]. \text{ Then we can take } U_M \text{ such that } z_i + U_M + T(U_M) \text{ for } 0 \leq i \leq p^2-1 \text{ are disjoint, and if } \{y_t|0 \leq t \leq 1\} \subset U_M \text{ satisfies the condition that } y_t - y_s, \text{ and } T(y_t - y_s) \text{ are linearly independent for arbitrary } s \neq t, \text{ then four elements } x_0 + y_s, x_0 + y_t, T(x_0 + y_s) \text{ and } T(x_0 + y_t) \text{ are linearly independent for
arbitrary $s \neq t$.

a) If $T$ is of type $I^0$, then $\Delta(T)$ having property $1^0$ exists.

If a compact group $U^\circ_n/\tilde{E}^\circ_n(T)$ is finite, then $\tilde{E}^\circ_n(T)\supset U^\circ_n$ for some $m$, which contradicts our assumption. Therefore, since a compact infinite group has at least cardinality $\aleph$, $\dim U^\circ_n/\tilde{E}^\circ_n(T)=\aleph$ for all $n=1, 2, \cdots$, then $U^\circ_n=\tilde{E}^\circ_n(T)+D_n$ for all $n=1, 2, \cdots$ and $\dim D_n=\aleph$ for all $n=1, 2, \cdots$. Choose $0 \neq x_0 \in D_n$, then $Tx_0 \in C[p]^o$ since $D_N \subset U^\circ_n \subset F^o(T)$ and $x_0$ and $Tx_0$ are linearly independent. By the above (*) we can determine $M \geq N$ and choose a basis $\{y_i|0 \leq t \leq 1\}$ of $D_M$, then $T y_i \in C[p]^o$ for all $0 \leq t \leq 1$ and for $s \neq t$ $y_s-y_t$ and $T(y_s-y_t)$ are linearly independent, because $D_M \cap \tilde{E}^\circ_n(T)=\{0\}$. Therefore by (*) $\Delta(T)=\{x_0+y_t|0 \leq t \leq 1\}$ has the property $1^0$.

b) If $T$ is of type $II^0$, then $\Delta(T)$ having property $2^0$ exists.

Our assumption $F^o(T)\not\supset U^\circ_n$ for all $n=1, 2, \cdots$ implies that compact groups $U^\circ_n/F^o(T) \cap U^\circ_n$ for all $n=1, 2, \cdots$ are infinite, so $U^\circ_n=(F^o(T) \cap U^\circ_n)\oplus H_n$ for all $n=1, 2, \cdots$ and $\dim H_n=\aleph$. And we have $(Ty)^o \neq 0$ for all $0 \neq y \in H_n$. Let $\tilde{T}=T-q(I-P_e)$, then our assumption $E^o_{0,N}(T) \supset U^\circ_n$ implies $E^o_{0,N}(\tilde{T})=U^\circ_n$, and $\tilde{T} x=(Tx)^o$ for all $x \in U^\circ_n$. Choose $0 \neq x_0 \in H_n$, then $0 \neq \tilde{T} x_0 \in C[p]^o$ and $x_0$ and $\tilde{T} x_0$ are linearly independent. We can apply (*) to $\tilde{T}$, so $M$ can be chosen as in (*) and $\{y_i|0 \leq t \leq 1\}$ can be a basis of $H_N$. Then $\tilde{T} y_i \in C[p]^o$ for all $0 \leq t \leq 1$ and for $s \neq t$, $y_s-y_t$ and $\tilde{T}(y_s-y_t)$ are linearly independent, because $y_s-y_t \in H_N \subset U^\circ_n$ and $\tilde{T}(y_s-y_t) \in C[p]^o$. Therefore by (*) we can see that $\Delta(T)=\{x_0+y_t|0 \leq t \leq 1\}$ is a one-parameter family having the property $2^0$.

The following fact (***) will be used in cases c) and d).

(***) Suppose $0 \neq x_0 \in C[p]^o$ such that $x_0$, $(Tx_0)^o$ and $(Tx_0)^o$ are linearly independent and suppose $\{z_0, z_1, \cdots, z_{p^3-1}\}$ is the group $[x_0]+[(Tx_0)^o]+[(Tx_0)^o]$. Then we can take $U_M$ such that $z_t+U_M+(T(U_M))^o+(T(U_M))^o \not\subset U^\circ_n \subset (0 \leq i \leq p^3-1)$ are mutually disjoint, and if $\{y_i|0 \leq t \leq 1\} \subset U^\circ_n$ satisfies the condition that three elements $y_t-y_s$, $(Ty_t-y_s)^o$ and $(Ty_t-y_s)^o$ are linearly independent for arbitrary $s \neq t$. Thus $\Delta(T)=\{x_0+y_t|0 \leq t \leq 1\}$ is a one-parameter family having the property $3^0$.

Now (***) can be shown as follows; it is clear that there exists $U_N$ such that $z_t+U_N \not\subset (0 \leq i \leq p^3-1)$ are mutually disjoint. By continuity of $P_e T$ and $(I-P_e) T$ there is $M \geq N$ such that $(T(U_M))^o$, $(T(U_M))^o \subset U_N$, then $U_M$ is as desired, since $z_t+U_M+(T(U_M))^o+(T(U_M))^o \subset z_t+U_N+U_N=U_N \not\subset (0 \leq i \leq p^3-1)$. Next, suppose $n_i(x_0+y_t)+n_2(T(x_0+y_t))^o+n_3(T(x_0+y_t))^o=n_i(x_0+y_t)+n_2(T(x_0+y_t))^o+n_3(T(x_0+y_t))^o$ for some $s \neq t$, where $n_1$, $n_2$, $n_3$, $n_1'$, $n_2'$ and $n_3'$ are integers, then $n_1 x_0+n_2(Tx_0)^o+n_3(Tx_0)^o=z_t$ for some $i$ and $n_1 x_0+n_2(Tx_0)^o+n_3(Tx_0)^o=z_t$ for some $j$. So $z_t+n_1 y_t+n_2(Ty_t)^o+n_3(Ty_t)^o=z_t+n_1 y_t+n_2(Ty_t)^o+n_3(Ty_t)^o$ and $n_1 y_t+n_2(Ty_t)^o+n_3(Ty_t)^o+n_1' y_t+n_2'(Ty_t)^o+n_3'(Ty_t)^o \in U_M+(T(U_M))^o$.
+(T(U_M))^n. By our choice of U_M, z_t must be equal to z_j, this implies n_1 = n'_1 mod p, n_2 = n'_2 mod p and n_3 = n'_3 mod p. Therefore we have n_1(y_t - y_s) + n_2(T(y_t - y_s))^o + n_3(T(y_t - y_s))^o = 0, but our assumption about \{y_i\} implies n_1 = n_2 = n_3 = 0 mod p. Hence \(\Delta(T) = \{x_0 + y_t|0 \leq t \leq 1\}\) satisfies property 3°.

c) If \(T\) is of type III^o, then \(\Delta(T)\) having property 3° exists.

\(F^o(T) \not\supset U_n^o\) for all \(n = 1, 2, \cdots\) yields a decomposition of \(U_n^o\) such that

(1) \(U_n^o = (F^o(T) \cap U_n^o) + H_n\) and \(\dim H_n = \aleph\) for all \(n = 1, 2, \cdots\). \(\tilde{E}_n^o(T)\) yields a decomposition of \(U_n^o\) for \(n \geq N\) such that

(2) \(U_n^o = \tilde{E}_n^o(T) + D_n\) and \(\dim D_n < \aleph\) for all \(n \geq N\).

Because, \(U_n^o/\tilde{E}_n^o(T) \subset U_n^o/\tilde{E}_n^o(T) \subset \tilde{E}_n^o(T)/\tilde{E}_n^o(T)\)

\(= \sum_{q=0}^{p-1} (E_{q,N}^o(T)/E_{q,n}^o(T)),\) but for each \(q, E_{q,N}^o(T)/E_{q,n}^o(T)\)

\(= E_{q,N}^o(T)/E_{q,n}^o(T) \cap U_n^o = (E_{q,N}^o(T) + U_n^o)/U_n^o \subset U_n^o/\tilde{E}_n^o(T)\)

is finite.

And assumptions \(\tilde{E}_n^o(T) \supset U_n^o\) and \(E_{q,n}^o(T) \not\supset U_m^o\) for all \(n, m = 1, 2, \cdots\) and \(q = 0, 1, \cdots, p - 1\) imply that there are \(q' \neq q''\) not depending on \(n\) such that

(3) \(\dim E_{q',n}^o(T) = \dim E_{q'',n}^o(T) = \aleph\) for all \(n \geq N\).

This (3) can be shown easily, see a) in the proof of Lemma 3 in [2]. (1) and (2) imply that there is \(q'''\) not depending on \(n\) such that

(4) \(\dim (E_{q'''}^o,T)/E_{q'''}^o,n(T) \cap F^o(T) = \aleph\) for all \(n \geq N\).

Because, suppose

\(E_{q,N}^o(T)/E_{q,N}^o(T) \cap F^o(T)\) are finite for all \(0 \leq q \leq p - 1\).

Then, since

\(E_{q,N}^o(T)/E_{q,N}^o(T) \cap F^o(T) \equiv (E_{q,N}^o(T) + F^o(T) \cap U_n^o)/F^o(T) \cap U_n^o,\)

\((\sum_{q=0}^{p-1} E_{q,N}^o(T) + F^o(T) \cap U_n^o)/F^o(T) \cap U_n^o = (\tilde{E}_n^o(T) + F^o(T) \cap U_n^o)/F^o(T) \cap U_n^o,\)

must be finite. However \(D_N\) is finite by (2), so we see that

\((\tilde{E}_n^o(T) + D_N + F^o(T) \cap U_n^o)/F^o(T) \cap U_n^o = U_n^o/F^o(T) \cap U_n^o\) is finite. This contradicts the facts that \(U_n^o/F^o(T) \cap U_n^o \equiv H_n\), and \(H_n\) is infinite by (1). Therefore we have that there exists \(q''''\) such that \(E_{q'',N}^o(T)/E_{q'',N}^o(T) \cap F^o(T)\) is finite. If \(n \geq N\), then \(E_{q'',N}^o(T)/E_{q'',N}^o(T) \cap F^o(T)\) is infinite, because \(E_{q'',N}^o(T) \supset E_{q'',N}^o(T)\). On the other hand \(E_{q'',N}^o(T)/E_{q'',N}^o(T)\) is finite, because \(E_{q'',N}^o(T)/E_{q'',N}^o(T) = E_{q'',N}^o(T)/E_{q'',N}^o(T) \cap U_n^o \equiv (E_{q'',N}^o(T) + U_n^o)/U_n^o \subset U_n^o/\tilde{E}_n^o(T)\) and \(U_n^o/\tilde{E}_n^o(T)\) is finite. These two facts imply clearly that \(E_{q'',N}^o(T)/E_{q'',N}^o(T) \cap F^o(T)\) is infinite for all \(n \geq N\). Therefore (4) is shown (note that every compact infinite group has at least cardinality \(\aleph\)).

Without loss of generality we can assume \(q'''' = q\)' in (3) and combining
(3) and (4) we can see that there exist \( q' \neq q'' \) not depending on \( n \) such that for all \( n \geq N' \) we have

\[
\dim E_{q',n}^o(T) = \dim E_{q'',n}^o(T) = \dim (E_{q',n}^o(T)/E_{q'',n}^o(T) \cap F^o(T)) = \aleph.
\]

Since \( E_{q',n}^o(T) \) and \( E_{q'',n}^o(T) \) \((n = 1, 2, \ldots)\) are decreasing and \( \bigcap_{n=1}^\infty U_n = \{0\} \), the following three cases may occur.

\( \alpha \)  \( E_{q',n}^o(T) \cap F^o(T) \) is infinite for all \( n \geq N' \),

\( \beta \)  \( E_{q'',n}^o(T) \cap F^o(T) \) is finite but \( E_{q',n}^o(T) \cap F^o(T) \) is infinite for all sufficiently large \( n \geq N' \),

\( \gamma \)  \( E_{q',n}^o(T) \cap F^o(T) = E_{q'',n}^o(T) \cap F^o(T) = \{0\} \)

for all sufficiently large \( n \geq N' \).

We want to show that

\( \text{(7)} \)  If \( n \) satisfies (6), then there exists \( \{y_t|0 \leq t \leq 1\} \subset U_n \) such that \( y_s - y_t, (T(y_s - y_t))^o \) and \( (T(y_s - y_t))^o \) are linearly independent for arbitrary \( s \neq t \).

Suppose \( n \) satisfies (\( \alpha \)). Then \( \dim (E_{q',n}^o(T)/E_{q'',n}^o(T) \cap F^o(T)) = \aleph \). By (5) \( E_{q',n}^o(T) \) has a decomposition \( E_{q',n}^o(T) = (E_{q',n}^o(T) \cap F^o(T)) \oplus A_n \) and \( \dim (A_n) = \aleph \). Let \( \{z_t|0 \leq t \leq 1\} \) be a basis of \( A_n \) and \( \{z'_t|0 \leq t \leq 1\} \) be a basis of \( E_{q',n}^o(T) \cap F^o(T) \), then \( \{y_t = z'_t + z_t|0 \leq t \leq 1\} \) has the desired property. Because, if \( n_1(y_s - y_t) + n_2(T(y_s - y_t))^o + n_3(T(y_s - y_t))^o = 0 \), then \( n_1(y_s - y_t) + n_2(T(y_s - y_t))^o = 0 \) and \( n_3(T(y_s - y_t))^o = 0 \) implies that \( (n_1 + n_2 q') \) \( (z'_s - z'_t) \) \( + (n_1 + n_2 q') \) \( (z'_s - z'_t) \) \( = 0 \), but \( z'_s - z'_t \neq 0 \), \( z'_s - z'_t \neq 0 \) and \( E_{q',n}^o(T) \cap E_{q'',n}^o(T) = \{0\} \), so \( n_1 + n_2 q' = n_1 + n_2 q' = 0 \) mod \( p \). This implies \( n_1 = n_2 = 0 \) mod \( p \) since \( q' \neq q'' \) mod \( p \). And \( n_3(T(y_s - y_t))^o = 0 \) implies \( 0 = n_3(T(z'_s - z'_t))^o + n_3(T(z'_s - z'_t))^o = n_3(T(z'_s - z'_t))^o \), \( (T(z'_s - z'_t))^o \neq 0 \) since \( 0 \neq z'_s - z'_t \in A_n \), therefore we have \( n_1 = 0 \) mod \( p \). Suppose \( n \) satisfies (\( \beta \)). Then automatically \( \dim (E_{q',n}^o(T)/E_{q'',n}^o(T) \cap F^o(T)) = \aleph \). Therefore the same argument as in (\( \alpha \)) applies by replacing \( q'' \) with \( q' \). Suppose \( n \) satisfies (\( \gamma \)). Let \( \varphi_n \) be a canonical projection onto \( H_n \) from \( U_n \), which is well defined by a decomposition \( U_n = (F^o(T) \cap U_n) \oplus H_n \) in (1). The condition in (\( \gamma \)) means that \( \varphi_n \) is one to one on \( E_{q',n}^o(T) \) and \( E_{q'',n}^o(T) \). This implies \( \dim (\varphi_n(E_{q',n}^o(T))) = \dim (\varphi_n(E_{q'',n}^o(T))) = \aleph \). Let \( X_s = \varphi_n(E_{q',n}^o(T)), X''_s = \varphi_n(E_{q'',n}^o(T)), X'_n = Y_n \oplus (X'_n \cap X''_n) \) and \( X''_n = Y''_n \oplus (X'_n \cap X''_n) \). If \( \dim (X'_n \cap X''_n) = \aleph \), then choose \( \{y_t|0 \leq t \leq 1\} \subset U_n \) as follows. Let \( \{u_t|0 \leq t \leq 2\} \) be a basis of \( X'_n \cap X''_n \). Then there exists uniquely \( z_t \) in \( E_{q',n}^o(T) \) such that \( u_t = \varphi_n(z_t) \) for all \( 0 \leq t \leq 1 \) and uniquely \( z''_t \) in \( E_{q'',n}^o(T) \) such that \( u_{t+1} = \varphi_n(z''_t) \) for all \( 0 \leq t \leq 1 \). Let \( y_t = z_t + z''_t \) for \( 0 \leq t \leq 1 \), then it is easy to see the linear independence of \( (y_s - y_t) \) and \( (T(y_s - y_t))^o \) for \( s \neq t \).
(see case (α)). And \((T(y_{s}-y_{t}))^{o}=(T(z_{t}^{o}-z_{t}^{r}+z_{t}^{r}-z_{t}^{r}))^{o}=(T(u_{s}-u_{t}+u_{s+1}-u_{t+1}))^{o}\) \(\neq 0\), because \(0\neq u_{s}-u_{t}+u_{s+1}-u_{t+1}\in H_{n}\). Therefore \(y_{s}-y_{t}\), \((T(y_{s}-y_{t}))^{o}\) and \((T(y_{s}-y_{t}))^{c}\) are linearly independent for arbitrary \(s\neq t\). If \(\dim (X_{n}\cap X_{n}'')<\aleph\), then \(\dim (Y_{n})=\dim (Y_{n}'')=\aleph\). Let \(\{u_{t}|0\leq t\leq 1\}\) be a basis of \(Y_{n}\) and \(\{v_{t}|0\leq t\leq 1\}\) be a basis of \(Y_{n}''\), then \(\{u_{t}\} \cup \{v_{t}\}\) is a linearly independent system in \(H_{n}\). We can take unique \(z_{i}'\) in \(E_{n}^{o}(T)\) and \(z_{i}''\) in \(E_{n}^{o}(T)\) such that \(u_{t}=\varphi_{n}(z_{i}')\) and \(v_{t}=\varphi_{n}(z_{i}'')\) for all \(0\leq t\leq 1\). Then the system \(\{y_{s}=z_{s}'+z_{s}'|0\leq t\leq 1\}\) is as desired, which is proved in the same way as above.

Now we can present the proof of c). By (7) there exists \(0\neq x_{0}\in C[p]^{o}\) such that \(x_{0}, (Tx_{0})^{o}\) and \((Tx_{0})^{c}\) are linearly independent. By (**) we can determine a large \(M\) which satisfies the condition stated in (**). Without loss of generality we can assume this \(M\) satisfies one of (α), (β) and (7), because \(M\) can be replaced by a larger one if it is necessary. (7) guarantees the existence of \(\{y_{t}|0\leq t\leq 1\}\subset U_{\aleph}^{o}\) for which \(y_{s}-y_{t}\), \((T(y_{s}-y_{t}))^{o}\) and \((T(y_{s}-y_{t}))^{c}\) are linearly independent. Therefore (**') shows us that \(\Delta(T)=\{x_{0}+y_{t}|0\leq t\leq 1\}\) is a one-parameter family having property \(3^{o}\).

d) **If \(T\) is of type \(IV^{o}\) then \(\Delta(T)\) having property \(1^{o}\) or \(3^{o}\) exists.**

Our assumptions yield decompositions of \(U_{n}^{o}\);

\[
U_{n}^{o} = (F^{o}(T)\cap U_{n}^{o}) \oplus H_{n} \quad \text{and} \quad \dim H_{n} = \aleph \quad \text{for all} \quad n=1,2,\cdots,
\]

\[
U_{n}^{o} = \tilde{E}_{n}^{o}(T) \oplus D_{n} \quad \text{and} \quad \dim D_{n} = \aleph \quad \text{for all} \quad n=1,2,\cdots.
\]

We can see that for all \(n=1,2,\cdots\)

\[
\dim (F^{o}(T)\cap U_{n}^{o}/F^{o}(T)\cap \tilde{E}_{n}^{o}(T)) = \aleph \quad \text{or} \quad \dim (H_{n}/H_{n}\cap \tilde{E}_{n}^{o}(T)) = \aleph.
\]

Because, suppose both dimensions are less than \(\aleph\), then

\[
\dim ((F^{o}(T)\cap U_{n}^{o}+\tilde{E}_{n}^{o}(T))/\tilde{E}_{n}^{o}(T))<\aleph \quad \text{and} \quad \dim ((H_{n}+\tilde{E}_{n}^{o}(T))/\tilde{E}_{n}^{o}(T))<\aleph
\]

since

\[
F^{o}(T)\cap U_{n}^{o}/F^{o}(T)\cap \tilde{E}_{n}^{o}(T) \cong (F^{o}(T)\cap U_{n}^{o}+\tilde{E}_{n}^{o}(T))/\tilde{E}_{n}^{o}(T) \quad \text{and} \quad H_{n}/H_{n}\cap \tilde{E}_{n}^{o}(T)
\]

\[
\cong (H_{n}+\tilde{E}_{n}^{o}(T))/\tilde{E}_{n}^{o}(T). \quad \text{Therefore} \quad \aleph > \dim ((F^{o}(T)\cap U_{n}^{o}+H_{n}+\tilde{E}_{n}^{o}(T))/\tilde{E}_{n}^{o}(T))
\]

\[
= \dim (U_{n}^{o}/\tilde{E}_{n}^{o}(T)) = \dim D_{n} = \aleph. \quad \text{This is a contradiction.} \quad \text{Next, if} \quad \dim (F^{o}(T)\cap U_{n}^{o}/F^{o}(T)\cap \tilde{E}_{n}^{o}(T))<\aleph \quad \text{for some} \quad n, \quad \text{then} \quad \dim (F^{o}(T)\cap U_{n}^{o}/F^{o}(T)\cap \tilde{E}_{n}^{o}(T))<\aleph
\]

for all \(m\geq n\). Because

\[
(F^{o}(T)\cap U_{m}^{o}/F^{o}(T)\cap \tilde{E}_{n}^{o}(T)) = F^{o}(T)\cap U_{m}^{o}/F^{o}(T)\cap \tilde{E}_{n}^{o}(T)\subset (F^{o}(T)\cap U_{m}^{o}+\tilde{E}_{n}^{o}(T))/\tilde{E}_{n}^{o}(T) \cong F^{o}(T)\cap U_{m}^{o}/F^{o}(T)\cap \tilde{E}_{n}^{o}(T).
\]

Therefore the following two cases may occur;

\[
(\delta) \quad \dim (F^{o}(T)\cap U_{n}^{o}/F^{o}(T)\subset \tilde{E}_{n}^{o}(T)) = \aleph \quad \text{for all} \quad n=1,2,\cdots,
\]

\[
(\epsilon) \quad \text{There exists} \quad N \quad \text{such that} \quad \dim (H_{n}/H_{n}\cap \tilde{E}_{n}^{o}(T)) = \aleph \quad \text{for all} \quad n\geq N.
\]

If (δ) happens, then we can show the existence of \(\Delta(T)\) having property \(1^{o}\). (δ) implies that

\[
F^{o}(T)\cap U_{n}^{o} = (F^{o}(T)\cap \tilde{E}_{n}^{o}(T)) \oplus X_{n} \quad \text{and} \quad \dim (X_{n}) = \aleph \quad \text{for all} \quad n=1,2,\cdots.
\]

We take \(0\neq x_{0}\in X_{1}\), then \((Tx_{0})^{o}=0\) and \(x_{0}\) and \(Tx_{0}\) are linearly independent.
According to (*) we can determine $M$ which satisfies the condition stated in (*). For this $M$ we take a basis $\{y_{i}|0 \leq t \leq 1\}$ of $X_{M}$, then $\{y_{i}\} \subset U_{y}$ and $\{T y_{i}\} \subset C[p]$. Suppose $n_{1}(y_{s}-y_{i})+n_{2}(T y_{s}-y_{i})=0$ for $s \neq t$, then we have $n_{1}=n_{2}=0$ mod $p$, because, if $n_{2} \neq 0$ mod $p$, then $T(y_{s}-y_{i})=q(y_{s}-y_{i})$ for some $q$, $0 \leq q \leq p-1$. This leads to a contradiction $0 \neq y_{s}-y_{i} \in E_{y}(T) \cap X_{y} = \{0\}$. By (*) $\Delta(T) = \{x_{0} + y_{i}|0 \leq t \leq 1\}$ is a one-parameter family having property $1^{o}$. If (e) happens, then we can show the existence of $\Delta(T)$ having property $3^{o}$. (e) implies that

$$H_{n} = (H_{n} \cap E_{n}^{o}(T)) \oplus Y_{n} \text{ and } \dim Y_{n} = \aleph \text{ for all } n \geq N.$$  

We can take $0 \neq x_{0} \in X_{N}$, then we can see that $x_{0}$, $(Tx_{0})^{o}$ and $(Tx_{0})^{e}$ are linearly independent. According to (***) we can determine $M$. Take a basis $\{y_{i}|0 \leq t \leq 1\}$ from $Y_{M}$, then we see also that $y_{s}-y_{i}$, $(T y_{s}-y_{i})^{o}$ and $(T y_{s}-y_{i})^{e}$ are linearly independent for $s \neq t$. By (***) $\Delta(T) = \{x_{0} + y_{i}|0 \leq t \leq 1\}$ is a one-parameter family having property $3^{o}$.

Exchanging odd and even in a)~d) we have

c) If $T$ is of one of types $I^{o} \sim IV^{o}$, then $\Delta(T)$ having one of properties $1^{o} \sim 3^{o}$ exists.

Q. E. D.

Now we are in a position to show existence of a group $Q$ which plays fundamental role to our ideas.

**Lemma 2.** For any family $\{T_{i}|\lambda \in \Lambda\}$ of non-singular homomorphisms on $C[p]$ there exists a subgroup $Q$ between $\sum[p]$ and $C[p]$ such that $Q$ is not invariant under any $T_{i}(\lambda \in \Lambda)$ but invariant under the canonical projection $P_{e}$ onto even coordinates.

**Proof.** $\{T_{i}|\lambda \in \Lambda\}$ is a given family of non-singular homomorphisms on $C[p]$. We can assume that $\Lambda$ is a well ordered set of ordinal numbers which are less than the first ordinal number whose cardinality is $\aleph$. Choose $c \in C[p]$ but $c^{e} \notin \sum[p]$. By transfinite induction we can construct the following family of subgroups $R_{i}(\lambda \in \Lambda)$;

a) $\sum[p] = R_{0} \subset R_{1} \subset R_{\mu} \subset C[p]$ if $0 \leq \lambda < \mu$ ($\lambda, \mu \in \Lambda$),
b) card $R_{i} \leq (\text{card } \lambda) \cdot \aleph_{0} < \aleph$ for all $\lambda \in \Lambda$,
c) $R_{i}$ is invariant under $P_{e}$ for all $\lambda \in \Lambda$,
d) $c^{o}$ and $c^{e} \notin R_{i}$ but there exists $x_{i} \in R_{i} \cap \Delta(T_{i})$ such that $c^{o}-T_{i} x_{i}$ or $c^{e}-T_{i} x_{i}$ or $c-T_{i} x_{i} \in R_{i}$ for all $\lambda \in \Lambda$.

Suppose $R_{i}$ has been constructed for all $\lambda < \mu \in \Lambda$. Let $R_{i}^{\prime} = \bigcup _{\lambda < \mu} R_{i}$. Then card $R_{i}^{\prime} \leq (\text{card } \lambda) \cdot \aleph_{0} < c$ and $R_{i}^{\prime}$ is invariant under $P_{e}$ and $c^{o}$ and $c^{e} \notin R_{i}^{\prime}$. By Lemma 1 $\Delta(T_{\mu})$ having one of properties $1^{o} \sim 3^{o}$ and $1^{e} \sim 3^{e}$ exists. Suppose $\Delta(T_{\mu})$
has property $1^o$, then we can find $x_{\mu}\in A(T_{\mu})$ such that $(R'_{\mu}+[c^o]+[c^e])\cap ([x_{\mu}]\oplus [T_{\mu}x_{\mu}])=\{0\}$. Let $R_{\mu}=R'_{\mu}+[x_{\mu}]+[c^o-T_{\mu}x_{\mu}]$, then clearly $R_{\mu}$ satisfies above a), b) and c). And $c^o$ and $c^e \notin R_{\mu}$ also holds. Suppose $c^e \in R_{\mu}$, then $c^e=x+nx_{\mu}+m(c^o-T_{\mu}x_{\mu})$ for some $x \in R'_{\mu}$ and some integers $n$ and $m$, so $-x+(1-m)c^o=mx_{\mu}-mT_{\mu}x_{\mu}$, but by our choice of $x_{\mu}$, $nx_{\mu}-mT_{\mu}x_{\mu}=0$ and $x+(m-1)c^o=0$. This implies $n=m=0$ mod $p$ and $c^o=x \in R'_{\mu}$ which is a contradiction. Suppose $c^e \in R_{\mu}$, then $c^e=x+nx_{\mu}+m(c^o-T_{\mu}x_{\mu})$ for some $x \in R'_{\mu}$ and some integers $n$ and $m$, but $x_{\mu}$ and $T_{\mu}x_{\mu} \in C[p]^o$, so $c^e=x \in R'_{\mu}$ which is also a contradiction. Suppose $A(T_{\mu})$ has property $2^o$, then we can find $x_{\mu} \in A(T_{\mu})$ such that $(R'_{\mu}+[c^o]+[c^e])\cap ([x_{\mu}]\oplus [T_{\mu}x_{\mu}-qx_{\mu}])=\{0\}$. Let $R_{\mu}=R'_{\mu}+[x_{\mu}]+[c^o-T_{\mu}x_{\mu}+qx_{\mu}]$, then clearly $R_{\mu}$ satisfies above a), b) and c). And $c^o$ and $c^e \notin R_{\mu}$ also holds. Suppose $c^e \in R_{\mu}$, then $c^e=x+nx_{\mu}+m(c^o-T_{\mu}x_{\mu}+qx_{\mu})$ for some $x \in R'_{\mu}$ and some integers $n$ and $m$, but $x_{\mu} \in C[p]^o$ and $T_{\mu}x_{\mu}-qx_{\mu} \in C[p]^e$, hence we have $c^o=x^o+nx_{\mu}$, that is, $-x^o+c^o=mx_{\mu}$. Our choice of $x_{\mu}$ implies $nx_{\mu}=0=-x^o+c^o$, so we have $c^o=x^o \in R'_{\mu}$ which is a contradiction. Suppose $c^e \in R_{\mu}$, then $c^e=x+nx_{\mu}+m(c^o-T_{\mu}x_{\mu}+qx_{\mu})$ for some $x \in R'_{\mu}$ and some integers $n$ and $m$. Hence $-x+(1-m)c^o=mx_{\mu}-m(T_{\mu}x_{\mu}-qx_{\mu})$. This implies $n=m=0$ mod $p$, so $c^e=x \in R'_{\mu}$ which is also a contradiction. Suppose $A(T_{\mu})$ has property $3^o$, then we can find $x_{\mu} \in A(T_{\mu})$ such that $(R'_{\mu}+[c^o]+[c^e])\cap ([x_{\mu}]\oplus [(T_{\mu}x_{\mu})^o]\oplus [(T_{\mu}x_{\mu})^e])=\{0\}$. Let $R_{\mu}=R'_{\mu}+[x_{\mu}]+[c^o-(T_{\mu}x_{\mu})^o]+[c^o-(T_{\mu}x_{\mu})^e]$. Then $R_{\mu}$ clearly satisfies a), b) and c). And $c^o$ and $c^e \notin R_{\mu}$ can be seen as follows. Suppose $c^e=x+nx_{\mu}+m(c^o-(T_{\mu}x_{\mu})^o)+m'(c^o-(T_{\mu}x_{\mu})^e)$ for some $x \in R'_{\mu}$ and integers $n$, $m$ and $m'$, then $c^o=x^o+nx_{\mu}+m(c^o-(T_{\mu}x_{\mu})^o)$, so $-x^o+(1-m)c^o=mx_{\mu}-m(T_{\mu}x_{\mu})^o$. This implies $nx_{\mu}-m(T_{\mu}x_{\mu})^o=0=-x^o+(1-m)c^o$ by our choice of $x_{\mu}$. Hence $m=0$ and $c^e=x^o \in R'_{\mu}$ which is a contradiction. We can see also $c^e \notin R_{\mu}$ for same reason. And $x_{\mu}$ and $c-T_{\mu}x_{\mu} \in R_{\mu}$ is clear. The construction of $R_{\mu}$ for $A(T_{\mu})$ having one of properties $1^o \sim 3^o$ is exactly similar by exchanging odd for even.

Let $Q=\bigcup_{\mu \in \Lambda} R_{\mu}$. Then the above properties a)~d) for all $R_{\mu}$ guarantee that $Q$ is a subgroup between $\Sigma[p]$ and $C[p]$ not invariant under any $T_{\mu}(\lambda \in \Lambda)$ but invariant under $P$. Q. E. D.

**Proof of Theorem.** We have clearly $C=C^o \oplus C^e$ and $\Sigma=\Sigma^o \oplus \Sigma^e$. By Lemma 2 there exists a subgroup $Q$ between $\Sigma[p]$ and $C[p]$ such that $Q$ is not invariant under any non-singular homomorphisms on $C[p]$ but is invariant under $P$, therefore $\Sigma^o[p]=\Sigma[p]^o \subset Q^o \subset C[p]^o=C^o[p]$, $\Sigma^e[p]=\Sigma[p]^e \subset Q^e \subset C[p]^e=C^e[p]$ and $Q=Q^o \oplus Q^e$. We can show that there exists a pure subgroup $G_1(G_2)$ of $C^o(C^e)$ which contains $\Sigma^o(\Sigma^e)$ and $G_1[p]=Q^o$.
(\(G_2[p]=Q^o\)). This fact is known as the purification property, see Lemma 1 in [2] and more general form in [1]. Clearly \(G_1\) and \(G_2\) are not bounded. Let \(G=G_1 \oplus G_2\), then \(G\) is a pure subgroup of \(C\) which contains \(\Sigma\) and \(G[p]=G_1[p] \oplus G_2[p]=Q^e \oplus Q^o=Q\). By Proposition \(G\) has the properties 1) and 2) in Theorem.

Q. E. D.

References


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