A QUASI-DECOMPOSABLE ABELIAN GROUP WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS AND PROPER ISOMRPHIC SUBGROUPS II

By

Takashi ITO and John M. IRWIN

1. Introduction

This paper is a continuation of our paper [2] with the same title. In [2] one of theorems was stated without detailed proof. The purpose of this paper is just to give a complete proof of it. So we shall omit an origin of our problem and general background for it, see the first section of [2] and references in [2].

Let $p>1$ be a fixed prime number, $C(p^n)$ be a cyclic group of order p^n, $\sum$ be the direct sum of cyclic groups $C(p^n)$, Π be the direct product of cyclic groups $C(p^n)$ and C be the torsion group of Π, that is, $\sum$ is the standard basic group and C is the torsion completion of $\sum$.

Our theorem is as following, that is Theorem 3 in [2].

Theorem. *There exists a pure subgroup G of C which contains $\sum$ and satisfies properties;*

1) *G has no proper isomorphic quotient groups,*
2) *G has no proper isomorphic subgroups,*
3) *G has a decomposition $G_1 \oplus G_2$ such that G_1 and G_2 are not bounded.*

2. Group Homomorphisms on $C[p]$.

The p-socle $C[p]$ of C is a vector space over the prime field of characteristic p and can be topologized as a totally disconnected compact topological group, because Π is clearly a totally disconnected compact topological group with respect to the product topology of compact discrete topologies and the p-socle $C[p]$ of C is the closed subgroup $\{x|x \in \Pi,\ px=0\}$ of Π. Actually $U_n=\{x|x \in C[p]$ and the hight of $x \geqq n\}=(p^nC)[p]$ $(n=1,2\cdots)$ are open compact subgroups of $C[p]$ and $\{U_n\}$ is a fundamental system of 0-neighborhoods in $C[p]$. These two structures on $C[p]$ which are a vector space and a totally disconnected compact group are used in an essential way.

Every continuous group homomorphism T on $C[p]$ defines compact subgroups $E_q(T)=\{x|x \in C[p]$ and $Tx=qx\}$ $(0 \leqq q<p)$ and the compact subgroup

$E(T)=E_0(T)\oplus E_1(T)\oplus\cdots\oplus E_{p-1}(T)$. We call T is a *singular* homomorphism if $E(T)$ is an open compact subgroup of $C[p]$. For instance a continuous projection on $C[p]$ is singular. If a continuous group homomorphism T on $C[p]$ has a dense subgroup which is invariant under T and on which T is one to one, T is called a *semi-isomorphism* on $C[p]$.

In the final step in our proof of the theorem we shall use following proposition which is shown in [2] (Theorem 1 of [2]).

Proposition. *Let G be a pure subgroup of C which contains $\sum$ and $G[p]$ be the p-socle of G.*

1) *If $G[p]$ is not invariant under any non-singular onto homomorphism on $C[p]$, then G has no proper isomorphic quotient groups.*

2) *If $G[p]$ is not invariant under any non-singular semi-isomorphism on $C[p]$, then G has no proper isomorphic subgroups.*

Therefore our main work in this paper is to show existence of a group Q which is between $\sum[p]$ and $C[p]$, and which is not invariant under any non-singular homomorphism but invariant under the canonical projection P_e onto even coordinates (see Lemma 2). In our proof of this fact we have some difficulity which comes from non-commutativity between non-singular homomorphisms and P_e. So we classify degree of non-commutativity in the following section.

3. Proof of Theorem

We introduce eight classes (essentially four) of non-singular homomorphisms depending on degree of non-commutativity. Hereafter we shall use notations $A^e=P_e(A)$ ($A^o=(I-P_e)(A)$) for a subset A of $C[p]$ and $x^e=P_ex$ ($x^o=x-P_ex$) for an element x in $C[p]$. Every continuous homomorphism T on $C[p]$ defines compact subgroups

$$F^o(T)=\left\{x|x\in C[p]^o \text{ and } (Tx)^e=0\right\},$$

$$E^o_{q,n}(T)=\left\{x|x\in U^o_n \text{ and } (Tx)^o=qx\right\} \quad \text{for} \quad q=0,1,\cdots,p-1;$$

$n=1,2,\cdots$. $F^e(T)$ and $E^e_{q,n}(T)$ are defined similarly. Let $\tilde{E}^o_n(T)=E^o_{0,n}(T)\oplus E^o_{1,n}(T)\oplus\cdots\oplus E^o_{p-1,n}(T)$ and $\tilde{E}^e_n(T)=E^e_{0,n}(T)\oplus E^e_{1,n}(T)\oplus\cdots\oplus E^e_{p-1,n}(T)$.

Following eight types about T are defined;

- I^o *there exists N such that $F^o(T)\supset U^o_N$ and $\tilde{E}^o_n(T)\not\supset U^o_m$ for all n, $m\geqq N$,*
- II^o *$F^o(T)\not\supset U^o_n$ for all $n=1,2,\cdots$ but there are N and q such that $E^o_{q,N}(T)\supset U^o_N$,*
- III^o *$F^o(T)\not\supset U^o_n$ for all $n=1,2,\cdots$ and $E^o_{q,n}(T)\not\supset U^o_m$ for all $n,m=$*

$1, 2, \cdots$ and $q=0, 1, \cdots, p-1$, but there are $N' \geqq N$ such that $\tilde{E}_N^{o}(T) \supset U_{N'}^{o}$,

IV^{o} $F^{o}(T) \not\supset U_n^{o}$ for all $n=1, 2, \cdots$ and $\tilde{E}_n^{o}(T) \not\supset U_m^{o}$ for all $n, m = 1, 2, \cdots$.

I^{e}, II^{e}, III^{e} and IV^{e} are defined by exchanging odd and even.

We note that every non-singular homomorphism is at least of one of the above eight types. Because, suppose T is of none of the above eight types, then there are N, n and m such that $n, m \geqq N$ and

1) $F^{o}(T) \supset U_N^{o}$ and $F^{e}(T) \supset U_N^{e}$,

2) $\tilde{E}_n^{o}(T) \supset U_m^{o}$ and $\tilde{E}_n^{e}(T) \supset U_m^{e}$.

Then 1) implies $E_{q,n}^{o}(T) \subset E_q(T)$ and $E_{q,n}^{e}(T) \subset E_q(T)$ for all $q=0, 1, \cdots, p-1$. So 2) implies $E(T)=E_0(T) \oplus \cdots \oplus E_{p-1}(T) \supset U_m^{o} \oplus U_m^{e} = U_m$. This means that T is a singular homomorphism.

In order to construct Q explained at the end of section 2 we need the following;

Lemma 1. *For an arbitrary non-singular homomorphism T we can find a one-parameter family $\Delta(T)=\{x_t | 0 \leqq t \leqq 1\}$ of elements in $C[p]$ which has one of the following six properties*; 1^{o}, 2^{o}, 3^{o}, 1^{e}, 2^{e} and 3^{e},

1^{o} *x_t, $Tx_t \in C[p]^{o}$ for all $0 \leqq t \leqq 1$ and four elements x_s, x_t, Tx_s and Tx_t are linearly independent for arbitrary $s \neq t$,*

2^{o} *there exists q, $0 \leqq q \leqq p-1$ such that $x_t \in C[p]^{o}$ and $Tx_t - qx_t \in C[p]^{e}$ for all $0 \leqq t \leqq 1$ and four elements x_s, x_t, $Tx_s - qx_s$ and $Tx_t - qx_t$ are linearly independent for arbitrary $s \neq t$,*

3^{o} *$x_t \in C[p]^{o}$ for all $0 \leqq t \leqq 1$ and six elements x_s, x_t, $(Tx_s)^{o}$, $(Tx_s)^{e}$, $(Tx_t)^{o}$ and $(Tx_t)^{e}$ are linearly independent for arbitrary $s \neq t$.*

1^{e}, 2^{e} *and* 3^{e} *are dual properties of* 1^{o}, 2^{o} *and* 3^{o} *by exchanging odd for even.*

Proof. Of course the choice of these six properties about $\Delta(T)$ depends on types of T defined above. Therefore the proof is divided into four cases, a)$\sim$d).

The following fact (*) which was shown already in [2] (see b) of the proof of Lemma 3 in [2]) will be used in cases a), b) and d).

(*) Suppose x_0 and Tx_0 are linearly independent and $\{z_0, z_1, \cdots, z_{p^2-1}\}$ is a group $[x_0] \oplus [Tx_0]$. Then we can take U_M such that $z_i + U_M + T(U_M)$ $(0 \leqq i \leqq p^2-1)$ are disjoint, and if $\{y_t | 0 \leqq t \leqq 1\} \subset U_M$ satisfies the condition that $y_t - y_s$ and $T(y_t - y_s)$ are linearly independent for arbitrary $s \neq t$, then four elements $x_0 + y_s$, $x_0 + y_t$, $T(x_0 + y_s)$ and $T(x_0 + y_t)$ are linearly independent for srbitrary $s \neq t$.

a) *If T is of type I°, then $\Delta(T)$ having property* 1° *exists.*

If a compact group $U_n^\circ/\tilde{E}_n^\circ(T)$ is finite, then $\tilde{E}_n^\circ(T) \supset U_m^\circ$ for some m, which contradicts our assumption. Therefore, since a compact infinite group has at least cardinality $\aleph$, $\dim U_n^\circ/\tilde{E}_n^\circ(T) = \aleph$ for all $n=1,2,\cdots$, then $U_n^\circ = \tilde{E}_n^\circ(T) \oplus D_n$ for all $n=1,2,\cdots$ and $\dim D_n = \aleph$ for all $n=1,2,\cdots$. Choose $0 \neq x_0 \in D_N$, then $Tx_0 \in C[p]^\circ$ since $D_N \subset U_N^\circ \subset F^\circ(T)$ and x_0 and Tx_0 are linearly independent. By the above (*) we can determine $M \geqq N$ and choose a basis $\{y_t | 0 \leqq t \leqq 1\}$ of D_M, then $Ty_t \in C[p]^\circ$ for all $0 \leqq t \leqq 1$ and for $s \neq t$ $y_s - y_t$ and $T(y_s - y_t)$ are linearly independent, because $D_M \cap \tilde{E}_M^\circ(T) = \{0\}$. Therefore by (*) $\Delta(T) = \{x_0 + y_t | 0 \leqq t \leqq 1\}$ has the property 1°.

b) *If T is of type II°, then $\Delta(T)$ having property* 2° *exists.*

Our assumption $F^\circ(T) \not\supset U_n^\circ$ for all $n=1,2,\cdots$ implies that compact groups $U_n^\circ/F^\circ(T) \cap U_n^\circ$ for all $n=1,2,\cdots$ are infinite, so $U_n^\circ = (F^\circ(T) \cap U_n^\circ) \oplus H_n$ for all $n=1,2,\cdots$ and $\dim H_n = \aleph$. And we have $(Ty)^e \neq 0$ for all $0 \neq y \in H_n$. Let $\tilde{T} = T - q(I-P_e)$, then our assumption $E_{q,N}^\circ(T) \supset U_N^\circ$ implies $E_{0,N}^\circ(\tilde{T}) = U_N^\circ$, and $\tilde{T}x = (Tx)^e$ for all $x \in U_N^\circ$. Choose $0 \neq x_0 \in H_N$, then $0 \neq \tilde{T}x_0 \in C[p]^e$ and x_0 and $\tilde{T}x_0$ are linearly independent. We can apply (*) to $\tilde{T}$, so M can be chosen as in (*) and $\{y_t | 0 \leqq t \leqq 1\}$ can be a basis of H_N. Then $\tilde{T}y_t \in C[p]^e$ for all $0 \leqq t \leqq 1$ and for $s \neq t$, $y_s - y_t$ and $\tilde{T}(y_s - y_t)$ are linearly independent, because $y_s - y_t \in H_N \subset U_N^\circ$ and $\tilde{T}(y_s - y_t) \in C[p]^e$. Therefore by (*) we can see that $\Delta(T) = \{x_0 + y_t | 0 \leqq t \leqq 1\}$ is a one-parameter family having the property 2°.

The following fact (**) will be used in cases c) and d).

(**) Suppose $0 \neq x_0 \in C[p]^\circ$ such that x_0, $(Tx_0)^\circ$ *and* $(Tx_0)^e$ are linearly independent and suppose $\{z_0, z_1, \cdots, z_{p^3-1}\}$ is the group $[x_0] \oplus [(Tx_0)^\circ] \oplus [(Tx_0)^e]$. Then we can take U_M such that $z_i + U_M + (T(U_M))^\circ + (T(U_M))^e$ $(0 \leqq i \leqq p^3-1)$ are mutually disjoint, and if $\{y_t | 0 \leqq t \leqq 1\} \subset U_M^\circ$ satisfies the condition that three elements $y_t - y_s$, $(T(y_t - y_s))^\circ$ and $(T(y_t - y_s))^e$ are linearly independent for arbitrary $s \neq t$. Thus $\Delta(T) = \{x_0 + y_t | 0 \leqq t \leqq 1\}$ is a one-parameter family having the property 3°.

Now (**) can be shown as follows; it is clear that there exists U_N such that $z_i + U_N$ $(0 \leqq i \leqq p^3-1)$ are mutually disjoint. By continuity of P_eT and $(I-P_e)T$ there is $M \geqq N$ such that $(T(U_M))^e$, $(T(U_M))^\circ \subset U_N$, then U_M is as desired, since $z_i + U_M + (T(U_M))^\circ + (T(U_M))^e \subset z_i + U_N + U_N + U_N = z_i + U_N$ $(0 \leqq i \leqq p^3-1)$. Next, suppose $n_1(x_0+y_t) + n_2(T(x_0+y_t))^\circ + n_3(T(x_0+y_t))^e = n_1'(x_0 + y_s) + n_2'(T(x_0+y_s))^\circ + n_3'(T(x_0+y_s))^e$ for some $s \neq t$, where n_1, n_2, n_3, n_1', n_2' and n_3' are integers, then $n_1x_0 + n_2(Tx_0)^\circ + n_3(Tx_0)^e = z_i$ for some i and $n_1'x_0 + n_2'(Tx_0)^\circ + n_3'(Tx_0)^e = z_j$ for some j. So $z_i + n_1y_t + n_2(Ty_t)^\circ + n_3(Ty_t)^e = z_j + n_1'y_s + n_2'(Ty_s)^\circ + n_3'(Ty_s)^e$ and $n_1y_t + n_2(Ty_t)^\circ + n_3(Ty_t)^e$, $n_1'y_s + n_2'(Ty_s)^\circ + n_3'(Ty_s)^e \in U_M + (T(U_M))^\circ$

$+(T(U_M))^e$. By our choice of U_M, z_i must be equal to z_j, this implies $n_1=n_1'$ mod p, $n_2=n_2'$ mod p and $n_3=n_3'$ mod p. Therefore we have $n_1(y_t-y_s)+n_2(T(y_t-y_s))^o+n_3(T(y_t-y_s))^e=0$, but our assumption about $\{y_t\}$ implies $n_1=n_2=n_3=0$ mod p. Hence $\Delta(T)=\{x_0+y_t|0\leqq t\leqq 1\}$ satisfies property 3°.

c) *If T is of type III°, then $\Delta(T)$ having property 3° exists.*

$F^o(T)\not\supset U_n^o$ for all $n=1,2,\cdots$ yields a decomposition of U_n^o such that

(1) $U_n^o=(F^o(T)\cap U_n^o)\oplus H_n$ and $\dim H_n=\aleph$ for all $n=1,2,\cdots$. $\tilde{E}_N^o(T)\supset U_{N'}^o$ yields a decomposition of U_n^o for $n\geqq N'$ such that

(2) $U_n^o=\tilde{E}_n^o(\tilde{T})\oplus D_n$ and $\dim D_n<\aleph_0$ for all $n\geqq N'$.

Because, $U_n^o/\tilde{E}_n^o(T)\subset U_{N'}^o/\tilde{E}_n^o(T)\subset\tilde{E}_N^o(T)/\tilde{E}_n^o(T)$

$$\cong\sum_{q=0}^{p-1}(E_{q,N}^o(T)/E_{q,n}^o(T)),\text{ but for each } q,\ E_{q,N}^o(T)/E_{q,n}^o(T)$$

$$=E_{q,N}^o(T)/E_{q,N}^o(T)\cap U_n^o\cong(E_{q,N}^o(T)+U_n^o)/U_n^o\subset U_N^o/U_n^o \text{ and } U_N^o/U_n^o$$

is finite.

And assumptions $\tilde{E}_N^o(T)\supset U_{N'}^o$ and $E_{q,n}^o(T)\not\supset U_m^o$ for all $n,m=1,2,\cdots$ and $q=0,1,\cdots,p-1$ imply that there are $q'\neq q''$ not depending on n such that

(3) $\dim E_{q',n}^o(T)=\dim E_{q'',n}^o(T)=\aleph$ for all $n\geqq N'$.

This (3) can be shown easily, see a) in the proof of Lemma 3 in [2].

(1) and (2) imply that there is q''' not depending on n such that

(4) $\dim(E_{q''',n}^o(T)/E_{q''',n}^o(T)\cap F^o(T))=\aleph$ for all $n\geqq N'$.

Because, suppose

$$E_{q,N'}^o(T)/E_{q,N'}^o(T)\cap F^o(T)\ \text{are finite for all } 0\leqq q\leqq p-1.$$

Then, since

$$E_{q,N'}^o(T)/E_{q,N'}^o(T)\cap F^o(T)\cong(E_{q,N'}^o(T)+F^o(T)\cap U_{N'}^o)/F^o(T)\cap U_{N'}^o,$$

$$\left(\sum_{q=0}^{p-1}E_{q,N'}^o(T)+F^o(T)\cap U_{N'}^o\right)\Big/F^o(T)\cap U_{N'}^o=(\tilde{E}_{N'}^o(T)+F^o(T)\cap U_{N'}^o)/F^o(T)\cap U_{N'}^o$$

must be finite. However $D_{N'}$ is finite by (2), so we see that $(\tilde{E}_{N'}^o(T)+D_{N'}+F^o(T)\cap U_{N'}^o)/F^o(T)\cap U_{N'}^o=U_{N'}^o/F^o(T)\cap U_{N'}^o$ is finite. This contradicts the facts that $U_{N'}^o/F^o(T)\cap U_{N'}^o\cong H_{N'}$ and $H_{N'}$ is infinite by (1). Therefore we have that there exists q''' such that $E_{q''',N'}^o(T)/E_{q''',N'}^o(T)\cap F^o(T)$ is finite. If $n\geqq N'$, then $E_{q''',N'}^o(T)/E_{q''',n}^o(T)\cap F^o(T)$ is infinite, because $E_{q''',n}^o(T)\subset E_{q''',N'}^o(T)$. On the other hand $E_{q''',N'}^o(T)/E_{q''',n}^o(T)$ is finite, because $E_{q''',N'}^o(T)/E_{q''',n}^o(T)=E_{q''',N'}^o(T)/E_{q''',N'}^o(T)\cap U_n^o\cong(E_{q''',N'}^o(T)+U_n^o)/U_n^o\subset U_{N'}^o/U_n^o$ and $U_{N'}^o/U_n^o$ is finite. These two facts imply clearly that $E_{q''',n}^o(T)/E_{q''',n}^o(T)\cap F^o(T)$ is infinite for all $n\geqq N'$. Therefore (4) is shown (note that every compact infinite group has at least cardinality $\aleph$).

Without loss of generality we can assume $q'''=q'$ in (3) and combining

(3) and (4) we can see that there exist $q'\neq q''$ not depending on n such that for all $n\geqq N'$ we have

(5) $\dim E^{\circ}_{q',n}(T)=\dim E^{\circ}_{q'',n}(T)=\dim(E^{\circ}_{q',n}(T)/E^{\circ}_{q',n}(T)\cap F^{\circ}(T))=\aleph$.

Since $E^{\circ}_{q',n}(T)$ and $E^{\circ}_{q'',n}(T)$ $(n=1,2,\cdots)$ are decreasing and $\bigcap\limits_{n=1}^{\infty}U^{\circ}_{n}=\{0\}$, the following three cases may occur.

(6)
(α) $E^{\circ}_{q'',n}(T)\cap F^{\circ}(T)$ is infinite for all $n\geqq N'$,
(β) $E^{\circ}_{q'',n}(T)\cap F^{\circ}(T)$ is finite but $E^{\circ}_{q',n}(T)\cap F^{\circ}(T)$ is infinite for all sufficiently large $n\geqq N'$,
(γ) $E^{\circ}_{q'',n}(T)\cap F^{\circ}(T)=E^{\circ}_{q',n}(T)\cap F^{\circ}(T)=\{0\}$ for all sufficiently large $n\geqq N'$.

We want to show that

(7) If n satisfies (6), then there exists $\{y_t|0\leqq t\leqq 1\}\subset U^{\circ}_{n}$ such that y_s-y_t, $(T(y_s-y_t))^{\circ}$ and $(T(y_s-y_t))^{e}$ are linearly independent for arbitrary $s\neq t$.

Suppose n satisfies (α). Then $\dim(E^{\circ}_{q'',n}(T)\cap F^{\circ}(T))=\aleph$. By (5) $E^{\circ}_{q',n}(T)$ has a decomposition $E^{\circ}_{q',n}(T)=(E^{\circ}_{q',n}(T)\cap F^{\circ}(T))\oplus A_n$ and $\dim(A_n)=\aleph$. Let $\{z'_t|0\leqq t\leqq 1\}$ be a basis of A_n and $\{z''_t|0\leqq t\leqq 1\}$ be a basis of $E^{\circ}_{q'',n}(T)\cap F^{\circ}(T)$, then $\{y_t=z'_t+z''_t|0\leqq t\leqq 1\}$ has the desired property. Because, if $n_1(y_s-y_t)+n_2(T(y_s-y_t))^{\circ}+n_3(T(y_s-y_t))^{e}=0$, then $n_1(y_s-y_t)+n_2(T(y_s-y_t))^{\circ}=0$ and $n_3(T(y_s-y_t))^{e}=0$. $n_1(y_s-y_t)+n_2(T(y_s-y_t))^{\circ}=0$ implies that $(n_1+n_2q')(z'_s-z'_t)+(n_1+n_2q'')(z''_s-z''_t)=0$, but $z'_s-z'_t\neq 0$, $z''_s-z''_t\neq 0$ and $E^{\circ}_{q',n}(T)\cap E^{\circ}_{q'',n}(T)=\{0\}$, so $n_1+n_2q'=n_1+n_2q''=0 \mod p$. This implies $n_1=n_2=0 \mod p$ since $q'\neq q'' \mod p$. And $n_3(T(y_s-y_t))^{e}=0$ implies $0=n_3(T(z'_s-z'_t))^{e}+n_3(T(z''_s-z''_t))^{e}=n_3(T(z'_s-z'_t))^{e}$, but $(T(z'_s-z'_t))^{e}\neq 0$ since $0\neq z'_s-z'_t\in A_n$, therefore we have $n_3=0 \mod p$. Suppose n satisfies (β). Then automatically $\dim(E^{\circ}_{q'',n}(T)/E^{\circ}_{q'',n}(T)\cap F^{\circ}(T))=\aleph$. Therefore the same argument as in (α) applies by replacing q'' with q'. Suppose n satisfies (γ). Let φ_n be a canonical projection onto H_n from U°_{n}, which is well defined by a decomposition $U^{\circ}_{n}=(F^{\circ}(T)\cap U^{\circ}_{n})\oplus H_n$ in (1). The condition in (γ) means that φ_n is one to one on $E^{\circ}_{q',n}(T)$ and $E^{\circ}_{q'',n}(T)$. This implies $\dim(\varphi_n(E^{\circ}_{q',n}(T)))=\dim(\varphi_n(E^{\circ}_{q'',n}(T)))=\aleph$. Let $X'_n=\varphi_n(E^{\circ}_{q',n}(T))$, $X''_n=\varphi_n(E^{\circ}_{q'',n}(T))$, $X'_n=Y'_n\oplus(X'_n\cap X''_n)$ and $X''_n=Y''_n\oplus(X'_n\cap X''_n)$. If $\dim(X'_n\cap X''_n)=\aleph$, then choose $\{y_t|0\leqq t\leqq 1\}\subset U^{\circ}_{n}$ as follows. Let $\{u_t|0\leqq t\leqq 2\}$ be a basis of $X'_n\cap X''_n$. Then there exists uniquely z'_t in $E^{\circ}_{q',n}(T)$ such that $u_t=\varphi_n(z'_t)$ for all $0\leqq t\leqq 1$ and uniquely z''_t in $E^{\circ}_{q'',n}(T)$ such that $u_{t+1}=\varphi_n(z''_t)$ for all $0\leqq t\leqq 1$. Let $y_t=z'_t+z''_t$ for $0\leqq t\leqq 1$, then it is easy to see the linear independence of (y_s-y_t) and $(T(y_s-y_t))^{\circ}$ for $s\neq t$

(see case (α)). And $(T(y_s-y_t))^e=(T(z_s'-z_t'+z_s''-z_t''))^e=(T(u_s-u_t+u_{s+1}-u_{t+1}))^e \neq 0$, because $0\neq u_s-u_t+u_{s+1}-u_{t+1}\in H_n$. Therefore y_s-y_t, $(T(y_s-y_t))^o$ and $(T(y_s-y_t))^e$ are linearly independent for arbitrary $s\neq t$. If $\dim(X_n'\cap X_n'')<\aleph$, then $\dim(Y_n')=\dim(Y_n'')=\aleph$. Let $\{u_t|0\leqq t\leqq 1\}$ be a basis of Y_n' and $\{v_t|0\leqq t\leqq 1\}$ be a basis of Y_n'', then $\{u_t\}\cup\{v_t\}$ is a linearly independent system in H_n. We can take unique z_t' in $E^o_{q',n}(T)$ and z_t'' in $E^o_{q'',n}(T)$ such that $u_t=\varphi_n(z_t')$ and $v_t=\varphi_n(z_t'')$ for all $0\leqq t\leqq 1$. Then the system $\{y_t=z_t'+z_t''|0\leqq t\leqq 1\}$ is as desired, which is proved in the same way as above.

Now we can present the proof of c). By (7) there exists $0\neq x_0\in C[p]^o$ such that x_0, $(Tx_0)^o$ and $(Tx_0)^e$ are linearly independent. By (**) we can determine a large M which satisfies the condition stated in (**). Without loss of generality we can assume this M satisfies one of (α), (β) and (γ), because M can be replaced by a larger one if it is necessary. (7) guarantees the existence of $\{y_t|0\leqq t\leqq 1\}\subset U_M^o$ for which y_s-y_t, $(T(y_s-y_t))^o$ and $(T(y_s-y_t))^e$ are linearly independent. Therefore (**) shows us that $\Delta(T)=\{x_0+y_t|0\leqq t\leqq 1\}$ is a one-parameter family having property 3°.

d) *If T is of type IV° then $\Delta(T)$ having property* 1° *or* 3° *exists.*

Our assumptions yield decompositions of U_n^o;

$$U_n^o=(F^o(T)\cap U_n^o)\oplus H_n \text{ and } \dim H_n=\aleph \quad \text{for all} \quad n=1,2,\cdots,$$
$$U_n^o=\tilde{E}_n^o(T)\oplus D_n \text{ and } \dim D_n=\aleph \quad \text{for all} \quad n=1,2,\cdots.$$

We can see that for all $n=1,2,\cdots$

$$\dim(F^o(T)\cap U_n^o/F^o(T)\cap\tilde{E}_n^o(T))=\aleph \text{ or } \dim(H_n/H_n\cap\tilde{E}_n^o(T))=\aleph.$$

Because, suppose both dimensions are less than $\aleph$, then $\dim((F^o(T)\cap U_n^o+\tilde{E}_n^o(T))/\tilde{E}_n^o(T))<\aleph$ and $\dim((H_n+\tilde{E}_n^o(T))/\tilde{E}_n^o(T))<\aleph$ since $F^o(T)\cap U_n^o/F^o(T)\cap\tilde{E}_n^o(T)\cong(F^o(T)\cap U_n^o+\tilde{E}_n^o(T))/\tilde{E}_n^o(T)$ and $H_n/H_n\cap\tilde{E}_n^o(T)\cong(H_n+\tilde{E}_n^o(T))/\tilde{E}_n^o(T)$. Therefore $\aleph>\dim(((F^o(T)\cap U_n^o)+H_n+\tilde{E}_n^o(T))/\tilde{E}_n^o(T))=\dim(U_n^o/\tilde{E}_n^o(T))=\dim D_n=\aleph$. This is a contradiction. Next, if $\dim(F^o(T)\cap U_n^o/F^o(T)\cap\tilde{E}_n^o(T))<\aleph$ for some n, then $\dim(F^o(T)\cap U_m^o/F^o(T)\cap\tilde{E}_m^o(T))<\aleph$ for all $m\geqq n$. Because $F^o(T)\cap U_m^o/F^o(T)\cap\tilde{E}_m^o(T)=F^o(T)\cap U_m^o/F^o(T)\cap U_m^o\cap\tilde{E}_n^o(T)\cong(F^o(T)\cap U_m^o+\tilde{E}_n^o(T))/\tilde{E}_n^o(T)\subset(F^o(T)\cap U_n^o+\tilde{E}_n^o(T))/\tilde{E}_n^o(T)\cong F^o(T)\cap U_n^o/F^o(T)\cap\tilde{E}_n^o(T)$. Therefore the following two cases may occur;

(δ) $\dim(F^o(T)\cap U_n^o/F^o(T)\subset\tilde{E}_n^o(T))=\aleph$ for all $n=1,2,\cdots$,

(ε) There exists N such that $\dim(H_n/H_n\cap\tilde{E}_n^o(T))=\aleph$ for all $n\geqq N$.

If (δ) happens, then we can show the existence of $\Delta(T)$ having property 1°. (δ) implies that

$$F^o(T)\cap U_n^o=(F^o(T)\cap\tilde{E}_n^o(T))\oplus X_n \text{ and } \dim(X_n)=\aleph \text{ for all } n=1,2,\cdots.$$

We take $0\neq x_0\in X_1$, then $(Tx_0)^e=0$ and x_0 and Tx_0 are linearly independent.

According to (*) we can determine M which satisfies the condition stated in (*). For this M we take a basis $\{y_t|0\leqq t\leqq 1\}$ of X_M, then $\{y_t\}\subset U_M^o$ and $\{Ty_t\}\subset C[p]^o$. Suppose $n_1(y_s-y_t)+n_2T(y_s-y_t)=0$ for $s\neq t$, then we have $n_1=n_2=0 \bmod p$, because, if $n_2\neq 0 \bmod p$, then $T(y_s-y_t)=q(y_s-y_t)$ for some q, $0\leqq q\leqq p-1$. This leads to a contradiction $0\neq y_s-y_t\in \tilde{E}_M^o(T)\cap X_M^o=\{0\}$. By (*) $\Delta(T)=\{x_o+y_t|0\leqq t\leqq 1\}$ is a one-parameter family having property 1^o. If (ε) happens, then we can show the existence of $\Delta(T)$ having property 3^o. (ε) implies that

$$H_n=(H_n\cap \tilde{E}_n^o(T))\oplus Y_n \quad \text{and} \quad \dim Y_n=\aleph \quad \text{for all} \quad n\geqq N.$$

We can take $0\neq x_0\in X_N$, then we can see that x_0, $(Tx_0)^o$ and $(Tx_0)^e$ are linearly independent. According to (**) we can determine M. Take a basis $\{y_t|0\leqq \mathrm{t}\leqq 1\}$ from Y_M, then we see also that y_s-y_t, $(T(y_s-y_t))^o$ and $(T(y_s-y_t))^e$ are linearly indepenpent for $s\neq t$. By (**) $\Delta(T)=\{x_0+y_t|0\leqq \mathrm{t}\leqq 1\}$ is a one-parameter family having property 3^o.

Exchanging odd and even in a)$\sim$d) we have

e) *If T is of one of types $I^e\sim IV^e$, then $\Delta(T)$ having one of properties $1^e\sim 3^e$ exists.*

Q. E. D.

Now we are in a position to show existence of a group Q which plays fundamental role to our ideas.

Lemma 2. *For any family $\{T_\lambda|\lambda\in\Lambda\}$ of non-singular homomorphisms on $C[p]$ there exists a subgroup Q between $\sum[p]$ and $C[p]$ such that Q is not invariant under any $T_\lambda(\lambda\in\Lambda)$ but invariant under the canonical projection P_e onto even coordinates.*

Proof. $\{T_\lambda|\lambda\in\Lambda\}$ is a given family of non-singular homomorphisms on $C[p]$. We can assume that Λ is a well ored set of ordinal numbers which are less than the first ordinal number whose cardinality is $\aleph$. Choose $c\in C[p]$ but c^o, $c^e\notin\sum[p]$. By transfinite induction we can construct the following family of subgroups $R_\lambda(\lambda\in\Lambda)$;

a) $\sum[p]=R_0\subset R_\lambda\subset R_\mu\subset C[p]$ if $0\leqq\lambda<\mu$ $(\lambda,\mu\in\Lambda)$,

b) card $R_\lambda\leqq$ (card $\lambda)\cdot\aleph_0<\aleph$ for all $\lambda\in\Lambda$,

c) R_λ is invariant under P_e for all $\lambda\in\Lambda$,

d) c^o and $c^e\notin R_\lambda$ but there exists $x_\lambda\in R_\lambda\cap\Delta(T_\lambda)$ such that $c^o-T_\lambda x_\lambda$ or $c^e-T_\lambda x_\lambda$ or $c-T_\lambda x_\lambda\in R_\lambda$ for all $\lambda\in\Lambda$.

Suppose R_λ has been constructed for all $\lambda<\mu\in\Lambda$. Let $R'_\mu=\bigcup_{\lambda<\mu} R_\lambda$. Then card $R'_\lambda\leqq$ (card $\lambda)\cdot\aleph_0<c$ and R'_λ is invariant under P_e and c^o and $c^e\notin R'_\lambda$. By Lemma 1 $\Delta(T_\mu)$ having one of properties $1^o\sim 3^o$ and $1^e\sim 3^e$ exists. Suppose $\Delta(T_\mu)$

has property 1°, then we can find $x_\mu \in \Delta(T_\mu)$ such that $(R'_\mu + [c^o] + [c^e]) \cap ([x_\mu] \oplus [T_\mu x_\mu]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^o - T_\mu x_\mu]$, then clearly R_μ satisfies above a), b) and c). And c^o and $c^e \notin R_\mu$ also holds. Suppose $c^o \in R_\mu$, then $c^o = x + nx_\mu + m(c^o - T_\mu x_\mu)$ for some $x \in R'_\mu$ and some integers n and m, so $-x + (1-m)c^o = nx_\mu - mT_\mu x_\mu$, but by our choice of x_μ, $nx_\mu - mT_\mu x_\mu = 0$ and $x + (m-1)c^o = 0$. This implies $n = m = 0 \bmod p$ and $c^o = x \in R'_\mu$ which is a contradiction. Suppose $c^e \in R_\mu$, then $c^e = x + nx_\mu + m(c^o - T_\mu x_\mu)$ for some $x \in R'_\mu$ and some integers n and m, but x_μ and $T_\mu x_\mu \in C[p]^o$, so $c^e = x \in R'_\mu$ which is also a contradiction. Suppose $\Delta(T_\mu)$ has property 2°, then we can find $x_\mu \in \Delta(T_\mu)$ such that $(R'_\mu + [c^o] + [c^e]) \cap ([x_\mu] \oplus [T_\mu x_\mu - qx_\mu]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^e - T_\mu x_\mu + qx_\mu]$, then clearly R_μ satisfies above a), b) and c). And c^o and $c^e \notin R_\mu$ also holds. Suppose $c^o \in R_\mu$, then $c^o = x + nx_\mu + m(c^e - T_\mu x_\mu + qx_\mu)$ for some $x \in R'_\mu$ and some integers n and m, but $x_\mu \in C[p]^o$ and $T_\mu x_\mu - qx_\mu \in C[p]^e$, hence we have $c^o = x^o + nx_\mu$, that is, $-x^o + c^o = nx_\mu$. Our choice of x_μ implies $nx_\mu = 0 = -x^o + c^o$, so we have $c^o = x^o \in R'_\mu$ which is a contradiction. Suppose $c^e \in R_\mu$, then $c^e = x + nx_\mu + m(c^e - T_\mu x_\mu + qx_\mu)$ for some $x \in R'_\mu$ and some integers n and m. Hence $-x + (1-m)c^e = nx_\mu - m(T_\mu x_\mu - qx_\mu)$, but by our choice of x_μ we see $-x + (1-m)c^e = 0 = nx_\mu - m(T_\mu x_\mu - qx_\mu)$. This implies $n = m = 0 \bmod p$, so $c^e = x \in R'_\mu$ which is also a contradiction. Suppose $\Delta(T_\mu)$ has property 3°, then we can find $x_\mu \in \Delta(T_\mu)$ such that $(R'_\mu + [c^o] + [c^e]) \cap ([x_\mu] \oplus [(T_\mu x_\mu)^o] \oplus [(T_\mu x_\mu)^e]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^o - (T_\mu x_\mu)^o] + [c^e - (T_\mu x_\mu)^e]$. Then R_μ clearly satisfies a), b) and c). And c^o and $c^e \notin R_\mu$ can be seen as follows. Suppose $c^o = x + nx_\mu + m(c^o - (T_\mu x_\mu)^o) + m'(c^e - (T_\mu x_\mu)^e)$ for some $x \in R'_\mu$ and integers n, m and m', then $c^o = x^o + nx_\mu + m(c^o - (T_\mu x_\mu)^o)$, so $-x^o + (1-m)c^o = nx_\mu - m(T_\mu x_\mu)^o$. This implies $nx_\mu - m(T_\mu x_\mu)^o = 0 = -x^o + (1-m)c^o$ by our choice of x_μ. Hence $m = 0$ and $c^o = x^o \in R'_\mu$ which is a contradiction. We can see also $c^e \notin R_\mu$ for same reason. And x_μ and $c - T_\mu x_\mu \in R_\mu$ is clear. The construction of R_μ for $\Delta(T_\mu)$ having one of properties $1^e \sim 3^e$ is exactly similar by exchanging odd for even.

Let $Q = \bigcup_{\lambda \in \Lambda} R_\lambda$. Then the above properties a)$\sim$d) for all R_λ guarantee that Q is a subgroup between $\sum [p]$ and $C[p]$ not invariant under any $T_\lambda (\lambda \in \Lambda)$ but invariant under P_e.

Q. E. D.

Proof of Theorem. We have clearly $C = C^e \oplus C^o$ and $\sum = \sum^e \oplus \sum^o$. By Lemma 2 there exists a subgroup Q between $\sum [p]$ and $C[p]$ such that Q is not invariant under any non-singular homomorphisms on $C[p]$ but is invariant under P_e, therefore $\sum^e [p] = \sum [p]^e \subset Q^e \subset C[p]^e = C^e[p]$, $\sum^o [p] = \sum [p]^o \subset Q^o \subset C[p]^o = C^o[p]$ and $Q = Q^e \oplus Q^o$. We can show that there exists a pure snbgroup G_1 (G_2) of C^e (C^o) which contains $\sum^e$ ($\sum^o$) and $G_1[p] = Q^e$

$(G_2[p]=Q^o)$. This fact is known as the purification property, see Lemma 1 in [2] and more general form in [1]. Clearly G_1 and G_2 are not bounded. Let $G=G_1\oplus G_2$, then G is a pure subgroup of C which contains $\sum$ and $G[p]=G_1[p]\oplus G_2[p]=Q^e\oplus Q^o=Q$. By Proposition G has the properties 1) and 2) in Theorem.

Q. E. D.

References

[1] J. M. IRWIN, F. RICHMAN and E. A. WALKER: Countable direct sums of torsion complete groups, Proc. Amer. Math. Soc., 17 (1966) pp. 763–766.

[2] T. ITO and J. M. IRWIN: A quasi-decomposable abelian group without proper isomorphic quotient groups and proper isomorphic subgroups, to appear in Pacific Journal of Mathematics.

Department of Mathematics,
Hokkaido University
Wayne State University

(Received Dec. 24, 1968)