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# On equilibrium existence theorem based on an infinite dimensional Gale-Nikaido Lemma.[1]

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## 概 要

This paper considers several versions of the infinite dimensional Gale-Nikaido lemma and establishes the existence of competitive equilibrium in a simplified Bewley(1972)' economy with $l_\infty$ as the commodity space with applying an infiinite dimensional Gale-Nikaido Lemma obtained in this paper.

# 1 Introduction

In the model of a market economy with finite number of commodities he constructed, Walrus(1874,77) established the existence of a competitive equilibrium in the model by showing that the number of the equations and that of the unknowns in the model are same owing to the homogeneity and Walras law of the mode. It is, however, not enough to establish the existence of a competitive equilibrium in Walras' market economy model based on counting the numbers of the equations and the unknowns in the model since the equality of the numbers of the equations and that of the unknown of the system does not yield the solutions, and, even solutions exist, they may be meaningless when they included negative values. Wald in 1930's showed the existence of consistent a non-negative solution for the Cassel system, which is a Walras model treated in Cassel(1924).[1] Later in 1950's Arrow-Debreu(1954), McKenzie(1954), Gale(1955), and Nikaido(1956a) established the existence of a competitive equilibrium in generalized Laureation Model. Arrow-Debreu(1954) translated this problem into the existence of an equilibrium in abstract game treated in Debreu(1952) and proved the existence of competitive equilibrium in Walrasian model by applying the result on the existence of an equilibrium in an abstract economy in Debreu(1952).[2] McKenzie(1954) proved the existence of world free trade equilibrium in the Graham model of international trade with many goods and countries based on Kakutani's fixed point theorem and the price adjustment mechanism that changes the prices of goods according to sign of their excess demands. It should be mentioned here that McKenzie(1954) used first time a fixed point theorem directly to prove the existence of competitive equilibrium for a competitive economy.[3]

Arrow-Debreu(1954) shows that a competitive equilibrium in an economy restricted in a sufficient large subset are same as that of original unrestricted economy, and hence Arrow-

[1]Walras concluded that the equality of number of the equations and that of unknowns was enough for the existence of competitive equilibrium in his model, and he thought that such equilibrium was attained based on the tatonnament process of market price adjusment mechanism. The issue that the Cassel system has a meaningful non-negative solution was treated by Zeuthen, Neisser, and Schlesinger, and solved finally by Wald in 1930's.

[2]Nash(1950) proves Nash equilibrium existence theorem with using Kakutani's fixed point theorem. Nash equilibrium existence theorem in abstract economies by Debreu(1952) is proved based on the fixed point theorem of Eilenberg-Montgomery-Begle type which is more general than Kakutani's fixed point theorem. The upper-semicontinuity of optimal policy correspondences in the models obtained from applying Berge's maxmimum theorem, however, implies those in abstract economies, and hence the Nash equilibrium existence theorem in abstract economies considered in Arrow-Debreu(1954) is indeed proved basically with Kakutani's fixed point theorem instead of Eirenberg-Montgomery-Begle type general fixed point theorem. In 1970's, Shafer-Sonnenshein(1975) and Gale-MasColell(1975) proved the existence of Nash equilibrium in non-transitive sbstract economies, and from this viewpoint, it is Kakutani's fixed point theoremis that is used to prove the equilibrium existence theorem in non-transitive abstract economies.

[3]McKenzie mentioned his paper(1998) that when he was staying at Cowles Foundation in Chicago in early 1950's, he was informed from M. Slater, who is very famous with his non-linear programming result, that Kakutani's fixed point theorem must be relevant to equilibrium existence theorem in competitive economies. Although Nash(1950) uses Kakutani's fixed point theorem to prove the existence of a Nash equilibrium in non-cooperative n-person games, at that time, however, it is not appearent how Nash equilibrium is relating to competitive equilibrium, which is indeed shown later in Arrow-Debreu(1954). From this viewpoint, it is still McKenzie(1954) which uses Kakutani's fixed point theorem directly to prove the exsitence of competitive equilibrium in walrasian competitive economy although McKenzie(1954) uses Graham's model of international trade with many goods and countries. It is quite intersting that both of Nash(1951) and McKenzie(1959) switch to use Brouwer's fixed point theorem of continuous functions from Kakutani's fixed point theorem of muluti-valued correspondences.

Debreu(1954) translates the problem of establishing the existence of competitive equilibrium into that of establishing the existence of competitive equilibrium in an economy restricted in a sufficient large subset. Gale(1955) and Nikaido(1956a), based on this fact,then establish Gale-Nikaido's lemma as a tool to show the existence of competitive equilibrium for an excess demand correspondence in an economy restricted over a sufficiently large subset. Once the conditions of correspondences in this lemma are satisfied, than an economy restricted over a sufficiently large subset has a competitive equilibrium. Gale(1955) uses KKM lemma for the proof of the lemma, on the other hand, Nikaido(1956a) uses Kakutani's fixed point theorem for the same purpose. [4] In particularly, owing to the clearness and the relevance with price adjustment mechanism of the proof by Nikaido(1956a), Gale-Nikaido's lemma and its proof by Nikaido(1956a) are reproduced in Debreu(1959), the fundamental textbook on classical general equilibrium theory, which also includes the proofs by Gale(1955) and Nikaido(1956a) as well. The similar method of the proof is also preserved in Nikaido(1960), the earliest Japanese book on general equilibrium theory. [5] After Debreu(1959), many text books on mathematical economics follows, up to the present, this method of the proof for the existence of competitive equilibrium in economies with finite number of goods and agents. [6]

So far, the economies under consideration are classical in a sense that the number of goods in these economies is finite. As in explained in Debreu(1959), the number of character of goods, that of time when trades take place, that of place where trades take place, and that of states of nature are all finite in these economies. When, however, one of these number is infinite, these classical economies can not handle the existence of competitive equilibrium in a situation. Debreu(1954) then analyses the fundamental theorems in welfare economics, which is on the relation between Pareto optimum and competitive equilibrium in economies with general linear space as the commodity space.[7] In classical economies with finite number of goods, the commodity spaces and the price spaces are same finite dimensional Euclidean spaces, the important aspect in Debreu(1954) is that it distinguishes explicitly the commodity spaces and price spaces as the dual spaces of the commodity spaces in economies with general topological vector space as the commodity space, so that it interprets the dual mapping defined on the commodity space and price spaces as the value of the commodity

[4]Gale(1855)'s proof indeed use KKM lemma and the apporoximation method based on Brouwer's fixed poin theorem employed in the proof of the fixed point theorem in Kakutani(1941), while Kuhn(1956) proves Gale-Nikaido's lemma directly with using Eilenberg-Montgomery's fixed point theorem. As shown in Nikaido(1956a), however, Kakutani's fixed point theorem is sufficient for the proof of Gale-Nikaido's Lemma. Note that Uzawa-Suzumura's equivalent theorem shows Gale-Nikaido's lemma is equivalent to Kakutani's fixed point theorem. Border(1985) explains the relation between Sperner's lemma, KKM lemma, Brouwer's fixed point theorem, Kakutani's fixed point theorem, and Gale-Nikaido's lemma.

[5]This book is traslated into English as Nikaido(1970).

[6]Although non-negative orthant in finite dimensional Euclidean space is employed in Gale-Nikaido's original result, Debreu(1956) extends the result in general closed convex cone in finite dimensional Euclidean spaces, so that this result is sometimes called Gale-Nikaido-Debreu's lemma(GND-lemma).

[7]The 2nd fundamental theorem of welfare economics in economies with general linear topological vector spaces as the commodity spacesis proved in Debreu(1954) with using a form of separation theorem of convex sets and the non-empty interiority of the production set. The latter condition is indeed one of the issues discussed in 1980's as the interiority condition. From the part mentioned in Nikaido(1957b) on Nikaido(1956b), the extension from Nikaido(1956b) to Nikaido(1957b) is mainly on this interiority condition. Debreu(1954) emplyed as the continuity of preferences that used in Herstein-Milnor(1952) for expected utility representation, which is weaker than those, upper-semicontinuity, usually employed. Although the reason is unclear, any of Nikaido(1956b,57b,59) did not refer to Debreu(1954).

bundle evaluated with the price system in the price space.

Debreu(1954), however, did not consider the existence of competitive equilibrium in the economies whose commodity space is a genera topological vector space. The existence of competitive equilibrium in economies with infinitely many commodities is later analyzed in an economy with $s$, the space of all sequences, as the commodity space by Peleg-Yarri(1970) and in an economy with $L_\infty$, the space of essentially bounded measurable functions, as the commodity space by Bewley(1972). The proof by Peleg-Yarri(1970) employes Scarf's theorem on the non-emptyness of core allocations and Debreu-Scarf's Limit theorem on the convergence of core, on the other hand, the proof by Bewley(1972) uses the fact that the competitive equilibrium in the original economy with infinite number of goods is the limit of the sequence(net) of those in the finite dimensional subeconomies based on the weak $*$ compactness of the set of feasible allocations and the set of price set.[8] The original paper(1969,92) of Bewley(1972) employs Negishi(1960)'method based on the 2nd fundamental theorem of welfare economics to prove the existence of competitive equilibrium in the economies with an infinite number of goods. [9] The existence of competitive equilibrium in economies with infinitely many goods then became one of the major problems in general equilibrium theory in 1980's.[10]

As in Gale(1955) and Nikaido(1956a), the proof of the existence of competitive equilibrium in economies with infinitely many goods may be proved based on the Gale-Nikaido's lemma in an infinite dimensional space. For this purpose, infinite dimensional Gale-Nikaido's lemma is necessary and indeed this is the main purpose of Nikaido(1956b,57b,59). Nikaido(1956b,57b) considers this problem in an infinite dimensional normed space and then Nikaido(1959) extend this to a locally convex topological vector space.[11] The proof in Nikaido(1956b,57b) use the finite intersection property of the set of equilibrium prices in the finite dimensional subeconomies with finite number of goods, which is derived from the weak$*$ compactness of the excess supply correspondence. Since this proof uses the approximation of the original economy by the sequence of the finite dimensional subeconomies, it has the same spirit with Bewley(1972)'s. Bewley(1972) applies a competitive equilibrium existence theorem to the sequence of finite dimensional subeconomies, on the other hand, Nikaido(1956b,57b) employs Gale-Nikaido's lemma to the sequence of finite dimensional subecnomies. Nikaido(1959) establishes first the general theorem which covers the Nash equilibrium existence theorem

[8]Bewley(1972) refers to a result in Krein - Rutman(1948) to extend the equilibrium prices in finite dimensional subecnomies over the entire original economy, Nikaido(1957b) also refers Krein - Rutman(1948) in its reference. It is, however, unclear how and where Krein - Rutman(1948)'s result is used in Nikaido(1957b). Note also that since $L_\infty$ contains $||\cdot||_\infty-$ interior point in its positive orthant $L_\infty^+$, the use of standard separation theorem on convex sets does not have any troubles, but the set of sequeces $s$ does not have any interir point (with respect to product topology) in its potive orthant $s^+$, the standard separation theorem on convex sets does not apply to such a situation so that the appropriate device such as used in Peleg-Yarri(1970) is required.

[9]Bewley(1969,92) proves the existence of competitive equiibiurm in a pure exchange economy with inifinitely many goods, and Magill(1981) then extend this result to a production economy with using Negishi(1960)'s method.

[10]The problem of existence of competitive equilibrium in economies with infinitely many goods is explained in the survey chapter by MasColell-Zame(1991) and the book by Aliprantis-Brown-Burkinshaw(1989).

[11]Unofrtunately, Nikaido(1956b) is no longer available nowadays. Butsince Nikaido(1957b) is referred as the revision of Nikaido(1956b), the content of Nikaido(1956b) is obtained from the parts in Nikaido(1957b) where Nikaido(1956b) is mentioned.

and Minimax theorem in a general topological vector space, then applies this general result to the sequence of finite dimensional subeconomies as in Nikaido(1956b,57b). The proof of the general theorem, however, is complicated. [12]

It is quite strange that none refers to any of these Nikaido's works in the literature on economies of infinite dimensional commodity spaces developed in 1980's.[13] There are several reasons why this occurs. The first reason is that Nikaido(1956b,57b) were only discussion papers and did not published in any journals. Since the title of Nikaido(1956b) contains explicitly "existence of competitive equilibrium in infinitely many commodities", it would contribute a lot the later literature on infinite dimensional commodity spaces when it was published in economics journals. Although Nikaido(1959), on the other hand, was published in Japanese mathematics journal, it first derives a general result covering Nash equilibrium existence theorem and minimax theorem in general topological vector spaces, and then applies it to prove Gale-Nikaido's lemma in infinite dimensional commodity spaces. Since this general infinite dimensional Gale-Nikaido's lemma is not the main purpose the paper, the title does not include words relating the existence of competitive equilibrium for infinite commodity spaces, so that later literature on infinite commodity spaces missed it completely.[14] Since Nikaido(1968), one of the text book on general equilibrium after Debreu(1959), contains Nikaido(1959) in its reference and was referred by many GE theorist, they may remind Nikaido(1959) as one of the early attempt done for infinite commodity case when its title contains word relating to infinite dimensional Gale-Nikaido's lemma.[15]

Moreover, it is another reason why Nikaido(1956b,57b,59) were neglected that Nikaido(1956b,57b,59) did not give any examples of economic models which satisfies the infinite dimensional Gale-Nikaido's lemma established there. The main purpose of Gale(1955) and Nikaido(1956a) is not to establish Gale-Nikaido's lemma but to show the existence of competitive equilibrium in economies with finite number of goods is derived from Gale-Nikaido's lemma. It seems more appropriate to give an example of economic models which is shown to have a competitive equilibrium from the infinite dimensional Gale-Nikaido' lemma in Nikaido(1956b,57b,59).

From this viewpoint, this paper picks up a simple model of Bewley(1972) with $l_\infty$ as the commodity space, and shows as in Gale(1955) and Nikaido(1956a) that the existence of competitive equilibrium in such an economy with applying infinite dimensional Gale-Nikaido's lemma established as in Nikaido(1956b,57b,59). Since, however, the linear topologies used in Nikaido(1956b,57b,59) and that in Bewley(1972) are different in some sense, the original infinite dimensional Gale-Nikaido's lemma as of Nikaido(1956b,57b,59) can not apply directly to the simple Bewley's model in this paper, and a small change in the original form

[12] Although Nikaido(1957b) remarked that the proof done in the case of normed space there can be easily extends to the case of general topological vector spaces, the similar method is indeed does not employed in Nikaido(1959). Kubota(2007) then follows Nikaido(1957b)'s remark to prove Gale-Nikado's lemma in a general local concex topological vector space.

[13] Urai(2000) is the only exception. Urai(2010) later collects and reviews the relations between fixed point theorems, Gale-Nikaido's lemma, existence of competitive equilibiurm in inifinite dimensional commodity spaces.

[14] Indeed, Bojan(1974) and Elbarkoky(1977) are referred many times in the literature on infinitely many commodities case, particularly, MasColell-Zame(1991).

[15] Aliprantis-Brown(1983), one of the early literature, refers the argument on gross substitutability of excess demand functions in Nikado(1968) in relation to the property of revealed preferences of excess demand functions.

of infinite dimensional Gale-Nikaido's lemma is required. This paper, then, establish a revise version of infinite dimensional Gale-Nikaido's lemma based on Florenzano(1983) which also has the same reasoning. Then, this paper apply the lemma to the simple Bewley's model and establish the existence of competitive equilibrium in such an economy just as in the classical finite dimensional case of Gale(1955) and Nikaido(1956a).[16]

In order to establish the existence of competitive equilibrium in the simple Bewley's model with using the modified version of infinite dimensional Gale-Nikaido's lemma, it is necessary to consider the excess demand correspondence in an economy restricted over an bounded region as in the case with a finite number of commodities. The difference in the linear topology employed in Nikaido(1956b,57b,59) and that in Bewley(1972), however, implies that it is not possible to define the excess demand correspondence over the original price so that the extension of the price set is necessary. This point means that the original infinite dimensional Gale-Nikaido's lemma in Nikaido(1956b,57b,59) can not apply directly to the Bewley's economy but a slight change is necessary although the similar proof is employed.[17] Then the application of this modified infinite dimensional Gale-Nikaido's lemma to the Bewley's economy with $l_\infty$ as the commodity space establishes the existence of competitive equilibrium in this economy. Although Nikaido(1956b,57b,59) did not give any example of the economy where the existence of competitive equilibrium was shown by applying the original infinite dimensional Gale-Nikaido's lemma in Nikaido(1956b,57b,59), this simplified Bewley's economy is considered as an example of the economy where the existence of competitive equilibrium is established by applying an infinite dimensional Gale-Nikaido's lemma. This aspect is the main point of this paper so that it achieves an important role in the sense that it reinforces the missing aspects of Nikaido(1956b,57b,59).

# 2 Infinite Dimensional Gale-Nikaido's Lemma : Model

In this paper, a version of infinite dimensional Gale-Nikaido's lemma which is applied to a simplified Bewley(1972)'s infinite dimensional economy with $l_\infty$ as the commodity space is established in order to show that this economy has a competitive equilibrium. For this purpose, this section presents the framework where a version of infinite dimensional Gale-Nikaido's lemma is established, and the following two sections include the proofs of this result and the results necessary for this result. Then section 6 constructs a simplified Bewley(1972)'s infinite dimensional economy with $l_\infty$ as the commodity space to which the infinite dimensional Gale-Nikaido's lemma is applied and shows the existence of competitive

[16]Florenzano(1983) applies its infinite dimensional Gale-Nikaido's lemma to Bewley(1972)'s model to show the existence of competitive equilibiurm in such an economy. Frolenzano(1983) uses a pure exchange economy of Bewley(1972)'s model, on the otherhand, this paper uses a production of Bewley(1972)'s model.

[17]Florenzano(1983) also takes into account of this point. It is quite interesting how Prof. Nikaido might take this point into account when Prof. Nikaido would try to consider an economic example with an infinite number of commodities where the existence of competitive equilibrium is derived from appling infinite dimensional Gale-Nikaido's lemma of Nikaido(1956b,57b,59). Since the economic model with $l_\infty$ as its commodity space is easily obtained from the framework of Nikaido(1956b), it seems that Prof. Nikaido might find an interesting result on the existence of competitive equilibrium in an economy with an infinite number of goods such as one with $l_\infty$ when he would drive himself to this direction as well as the direction to Nikaido(1967b,59) of extending the mathematical result.

equilibrium in this economy. Note that $l_\infty$ is the infinite dimensional space of the set of all bounded sequences with $||\cdot||_\infty-$norm and that $l_\infty$ is the norm dual space of the infinite dimensional space of the set of summable sequences with $||\cdot||_1-$norm, that is $l_\infty = (l_1, ||\cdot||_1)^*$.[18] On the other hand, the norm dual space of $l_\infty$ is the set of bounded and finitely additive measures on $N$, that is $ba = (l_\infty, ||\cdot||_\infty)^*$.[19] $l_\infty$ is used as the commodity space of the section of economic model and $ba$ is used as the price space there, respectively. In the section of the economic model with $l_\infty$ as the commodity space, the excess demand correspondence is shown to be non-empty on $l_1$, but it may not be non-empty on entire $ba$.[20] It is natural from the viewpoint of economics to use $l_1$ as the price space of the economic model with $l_\infty$ as the commodity space. As in Bewley(1972) and Lucas-Prescott(1972), this paper follows the two step argument where the existence of equilibrium price is shown first in $ba$ and then this equilibrium price is shown to be in indeed in $l_1$.[21]

In the following, the general model of an infinite dimensional Gale-Nikaido lemma is explained based on Nikaido(1959, section 5).[22] Let $E$ be a (real Hausdorff) locally convex topological vector space.[23] Also let $\tau$ be its linear topology. $\tau$ has a fundamental neighborhood system consisting of convex and balanced subsets of the origin.[24] Let $E^*$ be the dual space of $E$ that is the set of all $\tau-$continuous linear functions on $E$. Let $(\cdot,\cdot) : E\times E^* \to R, (x,p) \in E\times E^* \to x\cdot p$ be the bilinear form on $E\times E^*$. The weak topology $\sigma(E, E^*)$ of $E$ is the weakest local convex linear topology on $E$ which makes $(\cdot, p) : E \to R$ be continuous with respect to this topology for $p \in E^*$. Mackey topology $\tau(E, E^*)$ of $E$ is the strongest local convex linear topology on $E$ which makes $(\cdot, p) : E \to R$ be continuous with respect to this topology for $p \in E^*$.[25] Similarly, the weak $(*)$ topology $\sigma(E^*, E)$ of $E^*$

[18] $l_\infty$ is the normed space of sequences whose $||x||_\infty = \sup_{t\in N} |x_t|$ is finite and $l_1$ is the normed space of sequences whose $||x||_1 = \Sigma_{t=1}^\infty |x_t|$ is finte. Notice that $int_{||\cdot||_\infty}(l_\infty^+) \neq \emptyset$ but $int_{||\cdot||_1}(l_1^+) = \emptyset$. When the $*$ weak $\sigma(l_\infty, l_1)-$topology on $l_\infty$ and the $||\cdot||_1-$norm topology on $l_1$, the evaluation mapping $(\cdot) : l_\infty \times l_1 \to R$ becomes continuous with respect to the product topology on $B\times l_1$ where $B$ is a $||\cdot||_\infty-$bounded set. Thus, the evaluation mapping $(\cdot)$ is continuous with respect to $(x,p)$ simaltineously on $B\times l_1$. See Aliprantis-Border(1999, p.260, corollay 6.47). Then, the evaluation mapping $(\cdot)$ is continuous with respect to $(x,p)$ simaltineously on $B\times R^n$ when $R^n$ is considered as a subset of $l_1$. Also since $l_1$ is a separable normed space, the $*$ weak $\sigma(l_\infty, l_1)-$topology on norm bounded ball $B$ is metrizable and the $*$ weak $\sigma(l_\infty, l_1)-$closeure on $B$ is characterized with the limit of $*$ weak $\sigma(l_\infty, l_1)-$converging sequences.

[19] Since the Banach limit is in $ba$ but not in $l_1$, $ba\backslash l_1 \neq \emptyset$ holds. Indeed Yoshida-Hewitt decomposition theorem says that any element in 吉 $ba$ is uniquely decomposed to an element in $l_1$ and a purely finitely addive measure. This purely finitely addive measure is continuous with respect to $||\cdot||_\infty-$norm and its value does not change even when the elements of any finite coordinates of a point in $l_\infty$ switch to 0. The Banach limit is a typical example of furely finitely additive measures on $l_\infty$.

[20] Nikaido(1956,57b,59) chooses the linear topology when $ba$ is the price space. Here, however, $l_1$ is used as the basic price space so that the choice of the underlying linear topology needs a slight modification as in Florezano(1983). This point is discussed later in this paper.

[21] Florenzano(1983) also mentions that it is sufficient to follow the argument used in Bewley(1972) where an equilibrium price in $ba$ is converted to one in $l_1$ once an equilibrium price is found in $ba$. For this conversion, see Bewley(1972) and Lucas-Prescott(1972).

[22] If necesary, some results of Nikaido(1957b) are mentioned as well.

[23] Nikaido(1956b,57b) uses an infinite dimensional normed space as the commodity space $E$. Nikaido(1959), then, uses an infinite dimensional general topological vector space as the commodity space $E$.

[24] Since $\tau$ is a linear topology, the vecor operations are continuous with respect to $\tau$. Moreover, since $\tau$ is local convex, it is generated by a family of semi-norms on $E$.

[25] Then it is well-known that from Mackey-Allen theorem $\sigma(\mathrm{E}, \mathrm{E}^*) \leq \tau \leq \tau(\mathrm{E}, \mathrm{E}^*)$ holds. Also separation theorem of convex sets implies that these linear topologies have the same famility of closed convex sets. Moreover, Mackey theorem implies that these linear topologies have the same famility of bounded sets. The

is the weakest local convex linear topology on $E^*$ which makes $(x,\cdot) : E^* \to R$ be continuous with respect to this topology for $x \in E$. Mackey topology $\tau(E^*, E)$ of $E^*$ is the strongest local convex linear topology on $E^*$ which makes $(x,\cdot) : E^* \to R$ be continuous with respect to this topology for $x \in E$.[26] There is a locally convex linear topological space $(G,\kappa)$ whose dual space $(G,\kappa)^*$ is $E$. Then $G \subset E^*$ holds.[27] When a partial order $\geq$ on $E$ is consistent with the linear structure of $E$, there is a convex cone $P$ whose vertex is the origin such that $x \geq y \Leftrightarrow x - y \in P$ holds. Then this convex cone $P$ is express as $P = E^+ = \{x \in E : x \geq 0\}$. This $E^+$ is called the positive cone of $E$ with respect to equal to $\geq$. [28] An partial order $\geq$ on $E^*$ is derived from the order $\geq$ on $E$ by defining $p \geq q \Leftrightarrow p \cdot x \geq q \cdot x \ \forall x \in E,\ p, q \in E^*$. The positive cone $(E^*)^+$ is defined as $\{x \in E^* : p \geq 0\}$.[29]

For a $*$ weak $\sigma(E,G)-$closed convex cone$P(\subset E)$, the dual cone $(P)^*$ $(\subset G)$is defined as $(P)^* = \{p \in G : x \cdot p \leqq 0 \ \forall x \in P\}$. Since the definition of weak topology $\sigma(G,E)$ implies that $x \in P$ is continuous with respect to weak $\sigma(G,E)$, $P^*$ is weak $\sigma(G,E)-$closed. The second dual cone $(P)^{**}$ with respect to $G$ is defined as $\{x \in E : x \cdot p \leqq 0 \ \forall p \in (P)^*(\subset G)\}$. Since the definition of $*$ weak $\sigma(E,G)$ topology implies that $p \in P^*$ is continuous with respect to $*$ weak $\sigma(E,G)$, $P^{**}$ is $*$ weak $\sigma(E,G)-$closed.[30]

For a given convex cone $P$ with the origin as the vertex, $< (P)^* >$ is defined as $(P)^* \backslash \{0\} = \{p \in (P)^* : p \neq 0\}$, the set of non-zero elements in $(P)^*$. Suppose that the local convex space $E$ discussed so far is the commodity space of an economy and there is an excess de-

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weak topology $\sigma(\mathrm{E}, \mathrm{E}^*)$ is the linear topology generated by a family of semi-norms $\{|x \cdot p| : p \in \mathrm{E}^*\}$. The Mackey topology $\tau(\mathrm{E}, \mathrm{E}^*)$ is the linear topology generalted by a family of semi-norms $\{\sup_{p \in V^*} |x \cdot p| : V^* \subset \mathrm{E}^*, V^*$ is convex balanced and $*$ weak $\sigma(\mathrm{E}^*, \mathrm{E})-$compact$\}$, where $*$ weak $\sigma(\mathrm{E}^*, \mathrm{E})-$topology is defined just next in the text.

[26]Debreu(1954) is the first paper which uses a general topological vector space as the underlying commodity space in the mathmatical economics literature, but it does not use weak topology and $*$ weak topology. Nikaido(1956b,57b), on the other hand, use a particular infinite dimensional normed space as the commodity space, but they use the weak topology and $*$ weak topology, and even Banach-Alaoglu theorem on the $*$ weak compactness of the unit ball of the dual space. A Cowles commision discussion paper (mathematics) by Herstein dated 1953 explains the weak topology and $*$ weak topology of a Banach space, and even Banach-Alaoglu theorem. MacKey topology and Yoshida-Hewitt decomposition theorem on $l_\infty^* = ba$ are used explicitly in Hildenbrand(1974) and becomes popular from then. Nikaido((1956b,57b,59), however, do not use Mackey topology nor Yoshida-Hewitt decomposition theorem.

[27]The relation $(l_1, ||\cdot||_1)^* = l_\infty$ and $l_1 \subset (l_\infty, ||\cdot||_\infty)^* = ba$ is one of typical examples.

[28]A partial order $\geq$ is a vinary relation satisfying reflexivity and transitivity,where the former is $x \geq x$ $\forall x \in E$ and the latter is $x \geq y, y \geq z, \to x \geq z \ \forall x, y, z \in E$. An order $\geq$ is called consistent with the linear structure of $E$ when $x \geq y \to x + z \geq y + z \ \forall z \in E$ and $x \geq y \to \lambda x \geq \lambda y \ \forall \lambda > 0$ hold. A linear space with a consistent order is called an *ordered linear space*. When the positive cone $E^+$, defined by $\{x \in E : x \geq 0\}$, of an ordered linear space $E$ with $\tau-$topology is $\tau-$closed, $E$ is called an *ordered topological vector space*.

[29]As to these relations, $(E,\tau)$ corresponds to $(l_\infty, ||\cdot||_\infty)$, $E^*$ to $ba$, and $(G,\kappa)$ to $(l_1, ||\cdot||_1)$. An order $\geq$ on $l_1$ is given by $p = (p_t) \geq p' = (p'_t) \Leftrightarrow p_t \geq p'_t \ \forall t = 1, 2, \cdots$ and an order $\geq$ on $l_\infty$ is bu $x = (x_t) \geq x' = (x'_t) \Leftrightarrow x_t \geq x'_t \ \forall t = 1, 2, \cdots$. Then $l_1^+ = \{p \in l_1 : p \geq 0\}$ and $l_\infty^+ = \{x \in l_\infty : x \geq 0\} = \{x \in l_\infty : (x,p) \geqq 0$ $\forall p \in l_1^+\}$ follow. $e = (1, 1, \cdots) \in int_{||\cdot||_\infty}(l_\infty^+)$ implies $int_{||\cdot||_\infty}(l_\infty^+) \neq \emptyset$. Also $l_1 = l_1^+ - l_1^+$ and $l_\infty = l_\infty^+ - l_\infty^+$ follow. Moreover, $ba^+ = \{\pi \in ba : (x,\pi) \geq 0 \ \forall x \in l_\infty\}$ holds.

[30]The following is a usually procedure including (1957b,59). For a $\tau-$closed convex cone $P(\subset E)$, its duak cone $P^*$ is defined by $\{p \in E^* : x \cdot p \leqq 0 \ \forall x \in P\}$, and its second dual cone $P^{**}$ is defined by $\{x \in E : x \cdot p \leqq 0$ $\forall p \in P^*\}$. Since the definition of $*$ weak $\sigma(E^*, E)-$topology implies that $x \in P$ is continuous with respect to $*$ weak $\sigma(E^*, E)$, $P^*$ is $*$ weak $\sigma(E^*, E)-$ closed. Similarly, the definition of weak $\sigma(E, E^*)-$topology implies that $p \in P^*$ is continuous with respect to weak $\sigma(E, E^*)$, $P^{**}$ is weak $\sigma(E, E^*)-$ closed. Since an equilibrium price is shown to be in $l_1$, the linear topology is chosen so as to be consistent with the choice of $l_1$ as the price space and hence here uses $G$ instead of $E^*$ as in the text. Nikaido(1957b,59) show the result with $\tau$ and $P^*$ corresponding to those up to lemma 6 obtained with $\sigma(E,G)$ and $(P)^*$ in the text here.

mand correspondence $\phi :< (P)^* >\rightarrow E\backslash\{\emptyset\}$ for this economy, where $\phi(p) \neq \emptyset$ is assumed for $p \in< (P)^* >$.[31] Then the problem is reduced to under what conditions the existence of $\overline{p} \in< (P)^* >$ with $(\phi(\overline{p}) \cap P) \neq \emptyset$ holds. This is given by an version of infinite dimensional Gale-Nikaido Lemmas in Nikaido(1956b,57b,59).[32] Several modifications are, however, necessary when an infinite dimensional Gale-Nikaido lemma is applied to a simplified version of Bewley(1972)'economy with $l_\infty$ as the commodity space to show the existence of competitive equilibrium in this economy. This modified version of an Infinite dimensional Gale-Nikaido lemma is treated later and corresponds to that in Florenzano(1983).

# 3 Several Lemmas

This section establishes several lemmas which are necessary for a modified version of an infinite dimensional Gale-Nikaido lemma. The following results hold for a locally convex topological vector space $E$.

**Lemma 1** For a $*$ weak $\sigma(E,G)-$closed convex cone $P$, not equal to $E$, $< (P)^* >\neq \emptyset$.[33]

**Lemma 2** For a convex cone $P$ in $E$, $x \in cl_{\sigma(E,G)}(P)$ is equivalent to $x \cdot p \leqq 0$ $\forall p \in P^*(\subset G)$.[34]

**Lemma 3** For an interior point $u$ of a convex cone $P$ in $E$ with respect to $*$ weak $\sigma(E,G)$ topology, $u \cdot p < 0$ follows $\forall p \in< (P)^* >$.[35]

For a convex cone $P$ in $E$ or $G$, it is called *pointed* when it does not include $x$ and $-x$ at the same time except $x = 0$. In this situation, convex cone $P$ does not contain any straight line passing though the origin 0 and $P \cap (-P) = \{0\}$ holds. Then the following holds.

**Lemma 4** When the interior of a convex cone $P(\subset E)$ with respect to $*$ weak $\sigma(E,G)$ is non-empty, $(P)^*$ is pointed.[36]

[31]Nikaido(1956,57b,59) uses $\phi :< P^* >\rightarrow E\backslash\{\emptyset\}$ as an excess supply correspondence. Here, on the other hand, uses $\phi :< (P)^* >\rightarrow E\backslash\{\emptyset\}$ as an excess demand correspondence, instead.

[32]When the $*$ weak $\sigma(l_\infty, l_1)-$topology is used on $l_\infty$, the non-emptyness of ecxess demand points on $< (P)^* >$ holds. When the weak $\sigma(l_\infty, l_1)-$topology is used on $l_\infty$ instead, the non-emptyness of ecxess demand points on $< (P)^* >$ does not follow. In the literatures on economies with infinite dimensional commodity spaces, there are many examples pointed out where excess demand correspondences are empty for some prices.

[33]Nikaido(1959, lemma 4).This follows from a form of separation theorems for convex sets. See Nikaido(1957b, lemma 2) and Here, however, uses $*$ weak $\sigma(E,G)$ as the topology used in Nikaido(1957b,59).

[34]Nikaido(1957b lemma 3) and Nikaido(1959, lemma 5). The duality theorem on convex cone and that on second dual cone imply $cl_\kappa(P) = (P)^{**}$. As in the previous lemma, here uses $*$ weak $\sigma(E,G)$ as the topology used in Nikaido(1957b,59).

[35]Nikaido(1957b, lemma 1) and Nikaido(1959, lemma 6). If $u \cdot p = 0$ holds for a $p \in< (P)^* >$, $u \cdot p > 0$ holds for some $u' \in P$ from the $*$ weak $\sigma(E,G)(\kappa)-$continuity and $u \in int_{\sigma(E,G)(\kappa)}(P)$. This is, however, a contradiction to $p \in (P)^*$. As in the previous lemmas, here uses $*$ weak $\sigma(E,G)$ as the topology used in Nikaido(1957b,59).

[36]Nikaido(1957b, lemma 4) and Nikaido(1959, lemma 7). For $p \in (P)^*\backslash\{0\} =< (P)^* >$ and $u \in int_{\sigma(E,G)}P$, lemma 3 implies $-u \cdot p > 0$ and hence $-u \notin (P)^*$, which implies the pointedness of $(P)^*$. As in the previous lemmas, here uses $*$ weak $\sigma(E,G)$ as the topology used in Nikaido(1957b,59).

**Lemma 5** When the interior of a convex cone $P(\subset E)$ with respect to $*$ weak $\sigma(E,G)$ is non-empty and $cl_{\sigma(E,G)}(P)$ is not equal to $E$, $<(P)^*>$ is non-empty and convex. [37]

Suppose that $P(\subset E)$ is $*$ weak $\sigma(E,G)-$closed convex cone satisfying is $P\cap(-P)\neq P$. Since $P\cap(-P)$ is the maximal linear subspace in $P$, $P\cap(-P)\neq P$ implies that $P$ is not an linear subspace. Moreover, when $P$ has a $*$ weak $\sigma(E,G)-$interior point, $P\cap(-P)\neq P$ implies $P\neq E$. In the following, however, own that any convex cone $P$ satisfying $P\cap(-P)\neq P$ can be approximated by a family of $*$ weak $\sigma(E,G)-$closed convex cones, not equal to $E$, with non-empty $*$ weak $\sigma(E,G)-$interior.

Let $u$ be any elements of $P\backslash(P\cap(-P))$. Then $u\neq 0$ holds. Since $*$ weak $\sigma(E,G)$ topology is locally convex linear topology, $-u\notin P$ and $*$ weak $\sigma(E,G)-$closedness of $P$ implies that there is a convex balanced neighborhood $U$ satisfying $(-u+U)\cap P=\varnothing$. For any convex balanced $*$ weak $\sigma(E,G)-$neighborhood $V(\subset U)$, let $Q(V)$ be $\cup_{\lambda>0}\lambda(u+V)$, the convex cone generated by $u+V$ and $P(V)$ be $cl_{\sigma(E,G)}(P+Q(V))(\supset P)$. Then, $P(V)$ is $*$ weak $\sigma(E,G)-$closed and has non-empty $*$ weak $\sigma(E,G)-$interior from $(u+V)\subset Q(V)\subset P(V)$. The following holds.

**Lemma 6** $(i)$ For any convex balanced $*$ weak $\sigma(E,G)-$neighborhood $V(\subset U)$, $-u\notin P(V)$ holds. $(ii)$ $P=\cap\{P(V):$ all convex balanced $*$ weak $\sigma(E,G)-$neighborhood $V(\subset U)\}$.[38]

In lemma 6, $P(V)$ is $*$ weak $\sigma(E,G)-$closed and $-u\notin P(V)$ implies $P(V)\neq E$. Thus, lemma 5 can apply to $P(V)$ in lemma 6.

[37] Nikaido(1957b, lemma 7) and Nikaido(1959, lemma 8). Since $*$ weak closed convex cone $cl_{\sigma(E,G)}(P)$ is not equal to $E$, lemma 1 implies $<((cl_{\sigma(E,G)}(P)))^*>\neq\varnothing$. Since $P\subset(cl_{\sigma(E,G)}(P))$ implies $((cl_{\sigma(E,G)}(P)))^*\subset(P)^*$, $p\in<((cl_{\sigma(E,G)}(P)))^*>$ implies $p\neq 0$ and $p\in(P)^*$ and hence $p\in<(P)^*>\neq\varnothing$ holds. For $p,q\in<(P)^*>,\lambda\in(0,1)$, the convexity of $(P)^*$ implies $\lambda p+(1-\lambda)q\in(P)^*$. For $u\in int_{\sigma(E,G)}(P)$, lemma 3 implies $u\cdot p<0, u\cdot q<0$ so that $u\cdot(\lambda p+(1-\lambda)q)=\lambda(u\cdot p)+(1-\lambda)(u\cdot q)<0$ holds. Then $\lambda p+(1-\lambda)q)\neq 0$ and $\lambda p+(1-\lambda)q)\in<(P)^*>$ hold and hence $<(P)^*>$ is convex. Although Nikaido(1959, lemma 8) assumes that $int_\tau(P)$ and $E$ are not equal, here uses $\sigma(E,G)$ as $\tau$ in Nikaido(1959) and assumes that $cl_{\sigma(E,G)}(P)$ and $E$ are not equal.

[38] Nikaido(1957b, lemma 7) and Nikaido(1959, lemma 8). Here uses $*$ weak $\sigma(E,G)$ as the topology used in Nikaido(1957b,59). The proof is following. All neighborhoods below are chosen with respect to $*$ weak $\sigma(E,G)$. The proof of $(i)$ : Suppose $-u\in P(V)$ holds. Then $-u=x+\lambda(u+a)+b$ holds for some $x\in P,\lambda\geqq 0,a,b\in V$ and hence $-u=x/(1+\lambda)+\lambda a/(1+\lambda))+b/(1+\lambda)$. Since $P$ is cone, $x/(1+\lambda)\in P$ holds. Moreover, since $V$ is convex, $[\lambda a/(1+\lambda))+b/(1+\lambda)]\in V$ holds. Then the balancedness of $V$ implies $(-u+V)\cap P\neq\varnothing$ and hence $(-u+U)\cap P\neq\varnothing$ holds from $V\subset U$, which contradicts to the choice of $U$.

The proof of $(ii)$ : For a convex balanced neighborhood $V(\subset U)$ of 0, $P(V)\supset P$ holds from the definition of $P(V)$ and hence $P\subset\cap\{P(V):$ all convex and balanced neighborhood $V(\subset U)\}$ holds. Suppose that for any convex and balanced neighborhood $V(\subset U)$ of 0, $y\in P(V)$ and $y\notin P$ hold. $y\in P(V)$ implies $y=x_V+\lambda_V(u+a_V)+b_V$ for some $x_V\in P,\lambda_V\geqq 0,a_V,b_V\in V$. Let $c_V$ be equal to $y-(x_V+\lambda_V u)=\lambda_V a_V+b_V$ . Since $P$ is aconvex cone and contains $x_V$ and $u$, $x_V+\lambda_V u\in P$ holds. $y\notin P$ implies from $*$ weak $\sigma(E,G)-$closedness of $P$、$(y+W)\cap P=\varnothing$ for a convex and balanced neighborhood $W(\subset U)$ of 0. Then $y+c_V=(x_V+\lambda_V u)\in P$ implies $c_V\notin W$. $\lambda_V\to\infty(V\to 0)$ is shown to hold first. For this purpose, it is enough to show that for a given $n\in N$, $\lambda_V>n-1$ holds for any convex and balanced neighborhood( of 0) $V\subset W/n$. From the convexity of $V$, $c_V/(1+\lambda_V)=[\lambda_V a_V/(1+\lambda_V))+b_V/(1+\lambda_V)]\in V\subset W/n$ holds and hence $c_V\in W(1+\lambda_V)/n$ holds. Thus $c_V\notin W$ implies $(1+\lambda_V)/n>1$ and hence $\lambda_V>n-1$. Since $c_V/(1+\lambda_V)\in V$ implies $c_V/\lambda_V=y/\lambda_V-(x_V/\lambda_V+u)\in(1+1/\lambda_V)V$ and hence $x_V/\lambda_V+u\in y/\lambda_V+(1+1/\lambda_V)V$. Since $V\to 0\Longrightarrow\lambda_V\to\infty$ holds, $V\to 0\Longrightarrow y/\lambda_V+(1+1/\lambda_V)V\to 0$ and hence $x_V/\lambda_V+u\to 0$. Thus $\lim_{V\to 0}x_V/\lambda_V=-u$ follows. Since $P$ is cone, $x_V\in P$ implies $x_V/\lambda_V\in P$ and hence $\lim_{V\to 0}x_V/\lambda_V=-u\in cl_\tau(P)=P$, which contradicts to $-u\notin P$. Therefore $P\supset\cap\{P(V):$ any convex balanced $*$ weak $\sigma(E,G)-$neighborhood$V(\subset U)$ of 0$\}$ holds and $P=\cap\{P(V):$any convex, balanced $*$ weak $\sigma(E,G)-$neighborhood $V(\subset U)$ of 0$\}$.

The relation between upper hemi-continuity and closedness of correspondences is discussed briefly here. Let $S$ and $X$ be two (real Hausdorff) topological spaces, and $\varphi : S \to X\backslash\{\emptyset\}$ be a correspondence from $S$ to $X$. Then, (1)∶When for any $p \in S$ and open neighborhood $U(\varphi(p))$ there is an open neighborhood $V(p)$ of $p$ such that $\varphi(q) \subset U(\varphi(p))$ holds for any $q \in V(p)$, $\varphi$ is called *upper hemi − continuous*$(u.h.c)$.[39] (2) ∶ When the graph of $\varphi$, $G_\varphi = \{(p, x) \in S \times X : x \in \varphi(p)\}$, is closed in $S \times X$ with respect to the product topology, $\varphi$ is called *closed* or to have closed graph. The composition of two upper hemi-continuous correspondences is also upper hemi-continuous as well. Also continuous functions are upper hemi-continuous when they are considered as correspondences. Although these two continuity concepts of correspondences are not equivalent generally, under the compactness of $S$, its equivalence is characterized as following.

**Lemma 7** Suppose that $S$ is compact. Then the following two conditions are equivalent:$(a)$ : $\varphi$ is $u.h.c$ and $\varphi(p)$ is compact $\forall p \in S$, i.e, $\varphi$ is compact-valued.$(b)$ : $\varphi$ is closed and there is a compact set $T$ such that $\forall p \in S$, $\varphi(p)$ is contained in $T$. [40]

Since this paper follows the method of proof done in Nikaido(1957b), which uses the famous Gale-Nikaido lemma in finite dimensional Euclidean spaces, this result is stated here. Let $S^n$ be the unit simplex $\{p \in R^n : p \geqq 0, \sum_{i=1}^n p_i = 1\}$ of the $n-$dimensional Euclidean space $R^n$, $R^n_+$ be the non-negative orthant of $R^n$, $\{x \in R^n : x \geqq 0\}$, and $\phi : S^n \to R^n\backslash\{\emptyset\}$ be an excess supply correspondence in an economy with $R^n$ as its commodity space.

**Lemma 8(Gale-Nikaido)** Suppose that an excess demand correspondence $\phi : S^n \to R^n\backslash\{\emptyset\}$ is non-empty, compact, and convex-valued $u.h.c.$ and $\forall p \in S^n$, $x \in \phi(p) \to p \cdot x \leqq 0$, i.e., the weak Walras' law holds. Then there is a $\overline{p} \in S^n$ satisfying $\phi(\overline{p}) \cap (-R^n_+) \neq \emptyset$, and hence, $\overline{x} \leqq 0$ for some $\overline{x} \in \phi(\overline{p})$.[41]

The following lemma is also necessary.[42]

**Lemma 9** Suppose that a convex cone $P$ in $E$ has non-empty interior with respect to $\tau$. Then for $u \in int_\tau(P)$, $\Delta^* = \{p \in P^* : p \cdot u = -1\}$ is $*$ weak $\sigma(E^*, E)-$compact and $P^* = \cup_{\lambda \geqq 0} \lambda\Delta^*$ holds, where $P^*$ $(\subset E^*)$ is the dual cone of $P$ in $E^*$, $\{p \in E^* : p \cdot x \leqq 0 \forall x \in P\}$.[43]

With the aid of above results, the following section proves an modified infinite dimensional Gale-Nikaido Lemma which is necessary for the purpose of this paper.

[39] $\varphi$ is called *upper demi − continuous*$(u.d.c)$ when the similar condition holds only for any open half space $H(\varphi(p))$ conaining $\varphi(p)$ instead of any open neighborhood $U(\varphi(p))$ conaining $\varphi(p)$ and it is employed in Yannelis(1985). Yannelis(1985) extends the result on infinite dimensional Gale-Nikaido lemma with upper hemi-continuous correspondences in this paper to the case with upper demi-continuous correspondences.

[40] See Nikaido(1957b pp.8-9), Nikaido(1959 p.361), or Klein-Thompson(1984 p.78) for the proof. Note that $\varphi(S) = \cup_{p \in S}\varphi(p)$ is also compact in $(a)$ and that $(b) \to (a)$ holds generally without the compactness of $S$.

[41] See Nikaido(1956a pp.139-41), Nikaido(1957b lemma)、Debreu(1959 (1) pp.82-3), or Nikido(1968 Theorem 16.9 pp.265-6) for the proof.

[42] The proof this lemma uses Banach-Alaoglu theorem, which says that when $U$ is an neighborhood of the origin, $U^o = \{p \in E^* : |p \cdot x| \leqq 1\}$, the polar of $U$, is $*$weeak $\sigma(E^*, E)-$comact. when $E$ is a normed space, unit closed ball $B^* = \{p \in E^* : ||p|| \leqq 1\}$ is $*$weeak $\sigma(E^*, E)-$comact in $E^*$.

[43] Florenzano(1983 proposition 2). The proof is as following. Since $P^*$ is $*$ weak $\sigma(E^*, E)-$closed and $u \in P$ is $*$ weak $\sigma(E^*, E)-$continuous linear functional, this continuity implies the $*$ weak $\sigma(E^*, E)-$closedness of $\Delta^*$. Also since $u$ is in the $\tau-$ of $P$, $(u+W) \subset P$ holds for a convex balanced neighborhood $W$ of the origin 0. Then the balancedness of $W$ implies that $p \cdot (u + w) \leqq 0$ and $p \cdot (u - w) \leqq 0$ hold $\forall w \in W, p \in P^*$. Since

# 4 Gale-Nikaido's Lemma in Infinite Dimensional Commodity Space Model： Proof

This section proves a version of the infinite dimensional Gale-Nikaido lemma obtained in Nikaido(1957b,59). Let $(E, \tau)$ be a locally convex topological vector space such that $(G, \kappa)^* = E$ holds for a locally convex linear topological space $(G, \kappa)$ as before. Also let $P$ be a $*$ weak $\sigma(E, G)-$closed convex cone with the vertex as the origin satisfying $(-P \cap P) \neq P$. [44] The following is a version of the infinite dimensional Gale-Nikaido lemma proved in this section.

**Theorem 1** Suppose that a correspondence $\varphi :< (P)^* > \rightarrow E \backslash \{\emptyset\}$ satisfied the following conditions: $(i)$ : $\varphi(p)$ is non-empty, convex, and $*$ weak $\sigma(E, G)-$closed $\forall p \in < (P)^* >$. $(ii)$ : Let $L$ be any finite dimensional convex cone of $(P)^*$, $< (P)^* > \cap L$ and $E$ be endowed with the weak $\sigma(G, E)$ topology and $*$ weak $\sigma(E, G)-$topology, respectively. Then, $\varphi :< (P)^* > \cap L \rightarrow E \backslash \{\emptyset\}$, the correspondence $\varphi$ restricted on $< (P)^* > \cap L$ is *u.h.c.*. $(iii)$ : $\cup_{p \in <(-P)^*>} \varphi(p)$, the image of $< (P)^* >$ through $\varphi$, is $*$ weak $\sigma(E, G)-$compact. $(iv)$ : $x \in \phi(p) \rightarrow p \cdot x \leqq 0$ holds $\forall p \in < (P)^* >$(weak Walras' law). Then, $\varphi(\overline{p}) \cap P \neq \emptyset$ holds for some $\overline{p} \in < (P)^* >$.

**Proof of the theorem.** Let $V$ be a convex balanced $*$ weak $\sigma(E, G)-$ neighborhood of the origin 0. Then, $P \subset P(V)$ implies $(P(V))^* \subset (P)^*$. Any $P(V)$ contains $u$ as an interior point with respect to $*$ weak $\sigma(E, G)-$topology but does not $-u$. Thus, $P(V)$ satisfies the condition of lemma 5 and hence $< (P(V))^* >$ is non-empty and convex. Choose a finite set $F = \{p_1, \cdots, p_m\}$ from $< (P(V))^* >$. Then $co(F) \subset < (P(V))^* >$ holds from the convexity of $< (P(V))^* >$. Define here two continuous functions: Let $\alpha : E \rightarrow R^m$ be $\alpha_i(x) = p_i(x), i = 1, \cdots, m$. Since $p_i \in G$ implies that $p_i$ is $*$ weak $\sigma(E, G)-$continuous linear functional on $E$, $\alpha_i : E \rightarrow R$ is also $*$ weak $\sigma(E, G)-$continuous and hence $\alpha : E \rightarrow R^m$ is $*$ weak $\sigma(E, G)-$continuous as well, where $\alpha(x) = (\alpha_i(x))_{i=1}^m$. Define next $\beta : S^m \rightarrow < (P(V))^* >$ by $\beta(w) = \sum_{i=1}^m w_i p_i$. Since $\beta$ is a convex combination of $p_i, i = 1, \cdots, m$, its is continuous from a property of linear topology when $G$ is endowed with the weak $\sigma(G, E)-$topology with respect to $\beta$. $co(F) \subset < (P(V))^* >$ implies $\beta(w) \in < (P(V))^* >$. Let $\phi : S^m \rightarrow R^m$ be the composite function of $\alpha, \beta, \varphi$ defined by $\phi(w) = \alpha(\varphi(\beta(w)))$. Then $\phi(w)$ is the value of the excess demand $\varphi(\beta(w))$ at the price $\beta(w)$of a convex combination of $p_1, \cdots, p_m$, $(p_1(\varphi(\beta(w)), \cdots, p_m(\varphi(\beta(w))))$, evaluated at each $p_1, \cdots, p_m$. $\beta(S^m) = co(F) \subset < (P(V))^* >$ implies the *u.h.c.*of $\varphi$ on $\beta(S^m)$ from $(ii)$.

$p \cdot u \leqq 0$ holds, letting $w \neq 0, p \neq 0$ gives $p \cdot w \neq 0$ and hence $p \cdot u = 0$ does not hold. Thus, $p \cdot u < 0$ holds $\forall p \in < P^* >$ and hence $p/(p \cdot u) \in \Delta^*$ holds. For a given $w \in W$, $q(u + w) = q(u) + q(w) = -1 + q(w) \leqq 0$ holds $\forall q \in \Delta^*$ and hence $q(w) \leqq 1$ holds. Since the balancedness of $W$ implies $-w \in W$, similar argument implies $q(-w) \leqq 1$ and hence $q(w) \geqq -1$ holds. Thus, $|q(w)| \leqq 1$ holds. Since this implies that $\Delta^*$ is in the polar set$W^\circ$ of $W$ and Banach-Alaoglu theorem implies the $*$ weak $\sigma(E^*, E)-$comapctness of $W^\circ$, the $*$ weak $\sigma(E^*, E)-$closedness of $\Delta^*$ implies that its $*$ weak $\sigma(E^*, E)-$compactness. Since $P^*$ is a convex cone, $\Delta^* \subset P^*$ implies $\lambda \Delta^* P^* \ \forall \lambda \geqq 0$ and hence $\cup_{\lambda \geqq 0} \lambda \Delta^* \subset P^*$. On the other hand, since $0 \in (P^* \cap (\cup_{\lambda \geqq 0} \lambda \Delta^*))$ holds, lemma 3 implies $p \cdot u < 0 \ \forall p \in < P^* >$. Then $p \cdot u/(-p \cdot u) = -1$ holds and hence $p \in \lambda' \Delta^*$ for $\lambda' = 1/(-p \cdot u) > 0$. Thus, $P^* \subset (\cup_{\lambda \geqq 0} \lambda \Delta^*)$ and hence $P^* = (\cup_{\lambda \geqq 0} \lambda \Delta^*)$ holds.

[44] $G \subset E^*$ implies that $*$ weak $\sigma(E, G)-$topology on $E$ is coaser than the weak $\sigma(E, E^*)-$topology on $E$. Also $P$ is a weak $\sigma(E, E^*)-$closed conevx cone and the weak $\sigma(E, E^*)-$topology on $E$ is coarser than the original $\tau-$topology. Thus, $P$ is also $\tau-$closed as well.

It is first shown that $\phi : S^m \to R^m$ satisfied the conditions of lemma 8. Let $\beta(w) = q$. Since $\varphi(q) \neq \emptyset$ implies that the evaluation of $\varphi(q)$ at each $p_i, i = 1, \cdots, m$, exists, $\phi(w) \neq \emptyset$ holds. Let $r, r' \in \phi(w), \gamma \in (0,1)$ and $r'' = \gamma r + (1-\gamma)r''$. $r, r' \in \phi(w)$ implies $r = \alpha(y), r' = \alpha(y')$ for some $y, y' \in \varphi(\beta(w))$. The convexity of $\varphi(\beta(w))$ implies $\gamma y + (1-\gamma)y' \in \varphi(\beta(w))$ and hence $\alpha(\gamma y + (1-\gamma)y') \in \phi(w)$. Since $\alpha$ is linear, $\alpha(\gamma y + (1-\gamma)y') = \gamma\alpha(y) + (1-\gamma)\alpha(y') = \gamma r + (1-\gamma)r' = r'' \in \phi(w)$ holds and hence $\phi(w)$ is convex. Since $\varphi(q)$ is $*$ weak $\sigma(E,G)-$compact and each $p_i$ is $*$ weak $\sigma(E,G)-$continuous, Weierstrauss theorem implies the compactness of each $p_i(\varphi(q))(\subset R)$ and hence that of $\phi(w)(\subset R^m)$. The *u.h.c.* of $\phi$ holds since the continuity as functions of $\alpha$ and $\beta$ implies their *u.h.c.* when they are considered as two correspondences, and the composite of two *u.h.c.* correspondences is *u.h.c.*. For $r \in \phi(w)$, there is $z \in \varphi(\beta(w))$ satisfying $r = \alpha(z) = (p_1(z), \cdots, p_m(z))$. $[\beta(w)](z) \leqq 0$ holds from $(iv)$. Then $w \cdot r = \sum_{i=1}^m p_i(z)w_i = (\sum_{i=1}^m w_i p_i)(z) = [\beta(w)](z) \leqq 0$ holds and hence weak Walras' law holds for $\phi$ as well. Thus, $\phi : S^m \to R^m$ satisfies the conditions of lemma 8.

Then from lemma 8 there is $\overline{w} \in S^m$ with $\phi(\overline{w}) \cap (-R^m_+) \neq \emptyset$ and hence $(p_1(\overline{x}), \cdots, p_m(\overline{x})) \leqq 0$ holds for $\overline{x} \in \varphi(\beta(\overline{w}))$. $\beta(\overline{w}) \in < (P(V))^* > \subset < (P)^* >$ implies $\overline{x} \in \varphi(\beta(\overline{w})) \subset \Gamma = \cup_{p \in <(P)^*>} \varphi(p)$. Define $P(V,F) = \{x \in E : p_i(x) \leqq 0, p_i \in F\} = \{x \in E : p_i(x) \leqq 0, i = 1 \cdots, m\}$. Then $\overline{x} \in P(V,F)$ holds and hence $(P(V,F) \cap \Gamma) \neq \emptyset$ holds. Since $p_i$ is $*$ weak $\sigma(E,G)-$continuous, $P(V,F)$ is $*$ weak $\sigma(E,G)-$closed. Note that this result holds for any finite set $F$ in $< (P(V))^* >$. Consider a family of $\{P(V,F) : F \subset \Delta^-, \#F < \infty\}$. Then this family satisfies the finite intersection property on $\Gamma$, which is $*$ weak $\sigma(E,G)-$compact from $(iii)$, and hence $[(\cap_{F \subset <(-P(V))^*>} P(V,F)) \cap \Gamma] \neq \emptyset$ holds. Since $P(V)$ is $*$ weak $\sigma(E,G)-$closed convex cone, lemma 2 implies $P(V) = \{x \in E : p(x) \leqq 0, \forall p \in (P(V))^*\} \subset \{x \in E : p(x) \leqq 0, \forall p \in < (P(V))^* >\} \subset P(V,F)(= \{x \in E : p(x) \leqq 0, p \in F, \#F < \infty, F \subset < (P(V))^* >\})$, and hence $P(V) \subset (\cap_{F \subset <(P(V))^*>} P(V,F))$. On the other hand, letting $F = \{p\}$ for $p \in (P(V))^*$ implies $(\cap_{F \subset <(P(V))^*>} P(V,F)) \subset (\cap_{\{p\} \subset <(P(V))^*>} P(V,\{p\})) = \{x \in E : p(x) \geqq 0, \forall p \in < (P(V))^* >\} = P(V)$ and hence $P(V) = (\cap_{F \subset <(P(V))^*>} P(V,F))$ holds. Then $[(\cap_{F \subset <(P(V))^*>} P(V,F)) \cap \Gamma] \neq \emptyset$ implies $(P(V) \cap \Gamma) \neq \emptyset$. This result holds for any convex balanced $*$ weak $\sigma(E,G)-$neighborhood of 0. Let $N^*(0)$ be a family of convex balanced $*$ weak $\sigma(E,G)-$neighborhood of 0 and $P(V)$ be $\{P(V) : V \in N^*(0)\}$. Then the property of neighborhood implies that the finite intersection property holds for this family $P(V)$ on $\Gamma$and hence $[(\cap_{V \in N^*(0)} P(V)) \cap \Gamma] \neq \emptyset$ holds. Since $(ii)$ of lemma 6 implies $(\cap_{V \in N^*(0)} P(V)) = P$ and hence $(\varphi(\overline{p}) \cap P) \neq \emptyset$ holds for some $\overline{p} \in < (P)^* >$. This is the result to be shown. $\square$

When this result in the infinite dimensional case is compared with lemma 8 in the finite dimensional case, there is a slight difference between these two cases. In lemma 8, $\varphi$ is assumed to be *u.h.c.*on the entire domain, on the other hand, in theorem 1, $\varphi$ is assumed to be *u.h.c.* only on any finite dimensional convex subcone $L(\subset < (P)^* >)$ instead of being *u.h.c.* on entire domain $< (P)^* >$. Note that a finite dimensional convex subcone $L(\subset < (P)^* >)$ is intersection of $(P)^*$ and the linear subspace generated by a finite set of linear independent vectors $\{p_1, \cdots, p_m\}$ in $(P)^*$. Since any locally convex topology on a finite dimensional space is equivalent to that of Euclidean topology of the same finite dimensional space and the dual space of finite dimensional Euclidean space is indeed itself, i.e., $(R^m)^* = R^m$, $(ii)$ in the

theorem 1 means that the similar condition hold in lemma 8 of a finite dimensional case holds on a finite dimensional convex subcone $L(\subset< (P)^* >)$. When the $u.h.c.$of $\varphi$ holds on entire $< (P)^* >$, it also holds on a finite dimensional convex subcone $L(\subset< (P)^* >)$, but the converse does not necessary holds. It is shown later that this condition $(ii)$ holds indeed in a simplified Bewley(1972)'economy with $l_\infty$ as the commodity space, treated in section 6.[45] Note, however, that the condition $(iii)$ does not necessarily hold in the simplified Bewley(1972)'economy, and hence a further modification is required. This is indeed treated in the next section.[46]

The above proof basically follows the originals of Nikaido(1957b,1959(convex-valued case)). The difference between the above result and the original is as following: The original uses the polar cone $P^*$ in $E^*$ of $\tau-$closed convex cone $P$ of $E$ instead of $(P)^*$ in $G$, weak $\sigma(E, E^*)-$topology on $E$ instead of $*$ weak $\sigma(E, G)-$topology on $E$, and $*$ weak $\sigma(E^*, E)-$topology on $E^*$ instead of weak $\sigma(G, E)-$topology on $G$, in order to establish the existence of $\overline{p} \in< (P)^* >$ with $(\varphi(\overline{p}) \cap P) \neq \emptyset$ in a similar proof.[47] The original theorem also uses the condition corresponding to $(iii)$ since the original does not appeal to any kind of compactness of the price set.

Here considers the condition $(ii)$ and $(iii)$. Let $P$ be a $*$ weak $\sigma(E, G)-$closed convex cone $P$ with $int_{\sigma(E,G)}(P) \neq \emptyset$. Let $u \in int_{\sigma(E,G)}(P)$ and define $C_u = \{p \in (P)^* : p \cdot u = -1\}$. Then $(E, \sigma(E, G))^* = G$ and lemma 9 imply that $C_u$ is $*$ weak $\sigma(G, E)-$compact. Consider the following condition: $(ii)' : \varphi :< (P)^* >\rightarrow E$ is $u.h.c.$ on $< (P)^* > (\subset G)$ where $G$ is endowed with $*$ weak $\sigma(G, E)-$topology and $E$ is endowed with $*$ weak $\sigma(E, G)-$topology besides $(i)$ and $(iv)$. Trivially $(ii)' \rightarrow (ii)$ follows as mentioned already. Define $\psi :< (P)^* >\rightarrow E$ by $\psi(p) = \varphi(p/(-p \cdot u))$. $\psi$ satisfies $(i)$ and $(iv)$, and $\psi(< (P)^* >) = \varphi(C_u)$ holds. Since $C_u$ is $*$ weak $\sigma(G, E)-$compact and $\varphi$ is $u.h.c.$ on $C_u$ with respect to $*$ weak $\sigma(G, E)-$topology, from lemma 7 $\varphi(C_u)$ is $*$ weak $\sigma(E, G)-$compact and hence so is $\psi(< (P)^* >)$. Thus, $\psi$ satisfies $(iii)$. Since $p/(-p \cdot u)$ is $*$ weak $\sigma(G, E)-$continuous on $< (P)^* >$, $\psi$ is also $*$ weak $\sigma(G, E) - u.h.c.$ on $< (P)^* >$. Thus, $\psi :< (P)^* >\rightarrow E$ satisfied $(i) - (iv)$ and hence $(\psi(\overline{p}) \cap P) = (\varphi(\overline{p}/(-\overline{p} \cdot u)) \cap P) \neq \emptyset$ holds for some $\overline{p} \in< (P)^* >$. This means that when $\varphi :< (P)^* >\rightarrow E$ satisfies only $(i), (ii)', (iv)$, $(\varphi(\overline{p}/(-\overline{p} \cdot u)) \cap P) \neq \emptyset$ holds for a $\overline{p}/(-\overline{p} \cdot u) \in C_u \subset< (P)^* >$. In this situation, $(iii)$ is unnecessary to $\varphi$ to show the existence of $\overline{p} \in< (P)^* >$ with $(\varphi(\overline{p}) \cap P) \neq \emptyset$. On the other hand, the above proof of theorem 1 does not use the $*$ weak $\sigma(G, E)-$compactness of $C_u$, it uses, $(iii)$, the $*$ weak $\sigma(E, G)-$compactness of $\Gamma = \cup_{p \in <(P)^*>} \varphi(p)$, instead. This $(iii)$ is crucial to the above proof of theorem 1 in showing the existence of $\overline{p} \in< (P)^* >$ with $(\varphi(\overline{p}) \cap P) \neq \emptyset$. [48]

Note that the theorem 1 assumes $(iii)$ directly since generally $(ii)$ does not necessarily yield $(iii)$. It is, however, necessary to assume $(iii)'$ :the $*$ weak $\sigma(E, G)-$relative compactness of $\Gamma = \cup_{p \in <(P)^*>} \varphi(p)$ instead of $(iii)$ since $(iii)$ does not hold when the existence of competitive

[45]Both of Florenzano(1983) and Aliplantis-Brown(1983) mention that $(ii)$ holds in a Bewley(1972) type infinite dimensional economy.

[46]Florenzano(1983) considers the similar aregument in a Bewley(1972) type infinite dimensional economy. This point is discussed in the next section.

[47]See Kubota(2007) for the detail.

[48]Of course, in the case where an economy with $l_\infty$ as the commodity space and with $l_1$ as the price space, when $(iii)$ holds, the existence of equilibrium price $\overline{p} \in l_1$ is shown directly even without appealing the indirect argument using Yoshida-Hewitt deconposition theorem as in Bewley(1972) and Lucas-Prescott(1972).

equilibrium is established in an economy with $l_\infty$ as the commodity space in the section 6. Then this modification requires several changes on the consequence of the existence of $\overline{p} \in < (P)^* >$ with $(\varphi(\overline{p}) \cap P) \neq \emptyset$ in the sense as explained in the next section. Thus, when $(iii)'$ is assumed instead of $(iii)$ to establish he existence of competitive equilibrium in an economy with $l_\infty$ as the commodity space with using a version of the infinite dimensional Gale-Nikaido lemma, it is necessary to modify further the infinite dimensional Gale-Nikaido lemma obtained above as the theorem 1 with taking account of this point. This is indeed done in Florenzano(1983) and is treated in the next section.

Aliprantis-Brown(1983) uses an excess demand function in an infinite dimensional economy to show the existence of competitive equilibrium in such an economy. Aliprantis-Brown(1983) uses explicitly an order structure $\geq$ on an infinite dimensional commodity space $E$ which express an inequality relation of quantity of goods between two commodity bundles. They also use the linear topology $\tau$ on $E$ which is consistent with the order structure $\geq$. When an partial(reflexive and transitive) order $\geq$ on $E$ is consistent with the linear structure of $E$, $E$ is called an ordered linear space and the convex cone $P = E_+ = \{x \in E : x \geq 0\}$ is called the positive cone of $E$.[49] When the positive cone $E_+$ of $E$ is $\tau-$closed, the ordered linear space $E$ is called an ordered topological vector space.[50]

The main result of Aliprantis-Brown(1983) is stated as follows. Suppose that $(E, E')$ is a Riesz dual system and the price set $\Pi$ in $\tilde{E_+}$ is non-empty convex and $*$ weak $(E', E)-$compact. Suppose further that $S = \{p \in \Pi : p \gg 0\}$, the set of strongly positive elements in $\Pi$ is $*$ weak $(E', E)-$dense in $\Pi$ and the cone $\cup_{\lambda \geqq 0} \lambda S$ generated by $S$ is $*$ weak $(E', E)-$dense in $E_+^-$. Let $\varphi : D \to E$ be an excess demand function defined from an convex set $D(\subset \Pi)$ to $E$. Suppose that $\varphi : D \to E$ satisfies the following four conditions: $(a)$ : $D$ is $*$ weak $(E', E)-$dense in $\Pi$(dense condition), $(b)$ : there is a local convex topology $\tau$ on $E$ such that $\varphi : (D, *$ weak $\sigma(E', E)) \to (E, \tau)$ is continuous on $D$(continuity condition), $(c)$ : there is some $q \in D$ with $\limsup_{\alpha\uparrow} q \cdot \varphi(p^\alpha) > 0$ when $p^\alpha \to p$(with respect to $*$ weak $\sigma(E', E)) \in \Pi \backslash D$ holds for a net $\{p^\alpha\} \subset D$(boundary condition), $(d)$ : $p \cdot \varphi(p) = 0$ holds for $p \in D$(Walras' law). Then, $\{p \in \Pi : \varphi(p) \leqq 0\}$ is non-empty and $*$ weak $(E', E)-$compact. [51] The proof of this result

[49]When the $\sup(x, y)$ and $\inf(x, y)$ exist for any pair of $x, y \in$ E for an partial order $\geq$ on an ordered linear space E so that E is a *lattice*, E is called a *linear lattice* or *Riesz space*. $\sup(x, 0) = |x|$ is called the abusolute value of $x$, $[x, y] = \{z \in$ E $: x \leq z \leq y\}$ is called an *ordered interval* of $x$ and $y$. When $x, y \in A \to [x, y] \subset A$ holds for $A \subset$ E, $A$ is called *Solid*. When $|x| \leqq |y|, y \in A \to x \in A$ holds for $A$ in a Riesz space E, $A$ is called an *idearl*. When the images of ordered intervals by $\phi$ are bouned for an linear functional $\phi$ on E, $\phi :$ E $\to R$ is called *order bounded*. The set of all order bounded linear functionals on E is called *order dual* is denoted by $\mathrm{E}^-$. When $x \in \mathrm{E}_+$ implies $\phi(x) \geqq \varphi(x)$ for $\phi, \varphi \in \mathrm{E}^-$, an order is defined on $\mathrm{E}^-$ and denoted by $\phi \geqq \varphi$. In particular, $\phi$ is called positive when $\phi \geqq 0$ holds. Denote $\mathrm{E}_+^- = \{\phi \in \mathrm{E}^- : \phi \geqq 0\}$, the set of all positive linear functionals on E. $\phi$ is called *strictly positive* and denoted by $\phi \gg 0$ when $x > 0$ implies $\phi(x) > 0$.

[50]When an ordered topological vector space E is also a lattice and algebraic operations on $\sup(x, y)$ and $\inf(x, y)$ are continuous with respect to the linear topology $\tau$ on E, E is called an *topological linear lattice* or an *topological Riesz space*. This linear topology $\tau$ is called *locally solid* since the fundamental neighborhood system of the origin is composed of convex balanced and solid subsets of the origin. When $(L, L')$ is a pair of Riesz spaces and $L'$ is the dual space of $L$ with respect to an locally solid linear toplogy on $L$, $(L, L')$ is called a *Riesz dual system*. そして、When $(L, L')$ is a Riesz dual system, it is know that $L'_+$ is $*$ weak $\sigma(L', L)-$dense in $L_+^-$. Since $\tau(l_\infty, l_1)$ is locally solid, $(l_\infty, l_1)$ is a Riesz dual system. Moreover, since Banach-Alaoglu Theorem implies any order interval is $\sigma(l_\infty, l_1)-$compact, $(l_\infty, l_1)$ is called *symmetric* in particular. Note that $(l_\infty, ba)$ is a Riesz dual system but not symmetric.

[51]Aliprantis-Brown(1983, p.198, theorem 3.6).

in Aliprantis-Brown(1983) uses a similar finite dimensional approximation method as that used in the above proof of the theorem 1 based on Nikaido(1956b,57b,59). Note that since a simplified Bewley(1972)'s model with $l_\infty$ as the commodity space considered in section 6 does not necessarily satisfy the continuity condition $(b)$, the result of Aliprantis-Brown(1983) stated here can not apply to it directly. But since the simplified Bewley(1972)'s model, however, satisfies the finite dimensional continuity condition as $(ii)$ of the Theorem 1, to show the existence of competitive equilibrium in the simplified Bewley(1972)'s model, Aliprantis-Brown(1983) uses a similar finite dimensional approximation method as that used in the above proof of the theorem 1, where the existence of competitive equilibrium in a finite dimensional subecnomy is established based on the finite dimensional continuity condition of the excess demand function obtained in the simplified Bewley(1972)'s economy. This is also done in Florenzano(1983). [52] This paper also considers this result in section 6 later.

# 5 Gale-Nikaido's Lemma in Florenzano(1983):Proof

This section considers an infinite dimensional Gale-Nikaido lemma with the condition $(iii)'$ : the $*$ weak $\sigma(E,G)-$relative compactness of $\Gamma = \cup_{p\in<(P)^*>}\varphi(p)$ instead of that with the condition $(iii)$ :the $*$ weak $\sigma(E,G)-$compactness of $\Gamma = \cup_{p\in<(P)^*>}\varphi(p)$ in the last section. This result is also considered in Florenzano(1983). Florenzano(1983) shows first the result similar to the theorem 1, which is stated as the following theorem 2 below, with using the finite dimensional approximation based on $(ii)$. Then Florenzano(1983) applies it to establish the existence of a competitive equilibrium in an economy with the normed space $E$, which is the norm dual of a normed space $G$, as the commodity space. This paper also follows this procedure used in Florenzano(1983) to establish the existence of a competitive equilibrium in an simplified Bewley(1972)'s economy with $l_\infty$, which is the norm dual of a normed space $l_i$, as the commodity space.

Note, however, that although the theorem 1 above gives rise to an equilibrium price $\overline{p} \in <(P)^*>$ satisfying $(\varphi(\overline{p}) \cap P) \neq \emptyset$ directly by virtue of $(iii)$, the result in this section with $(iii)'$ does not give rise to such a price Instead, it is shown that an equilibrium price $\overline{p}$ is the limit of a net of prices such that $(\widetilde{\varphi}(\overline{p}) \cap P) \neq \emptyset$ holds for an extension of $\varphi(\overline{p})$ in a sense defined below. For this purpose, it is necessary to make the price space a $^*$ weak $\sigma(E,G)-$compact for this purpose so that the underlying space $E$ is to be a locally convex space with $int_\tau(P) \neq \emptyset$. [53] Since the finite dimensional approximation argument based on $(iii)$ and lemma 6 do not work well in this situation, the interiority condition $int_\tau(P) \neq \emptyset$ is necessary.[54]

[52]Since Florenzano(1983) refers to Aliprantis-Brwon(1983) on this point, the main purpose of Florenzano(1983) is considered to be on this point.

[53]It seems that Nikaido(1956b) considers an economy with a infinite dimensional normed space $E$ with this interiority condition on $P$ from the note in Nikaido(1957b) on this point and it shows a first version of an inifinite dimensional Gale-Nikaido lemma with using $*$ weak $\sigma(E,E^*)-$topology on $E$ based on this interiority condition on $P$. If Prof.Nikaido tried to apply this version of an infinite dimansional Gale-Nikaido lemma to an economy with $l_\infty$ as the commodity sapce to show the existence of a competitive equilibrium in this economy, he may establish a different version of an infinite dimensional Gale-Nikaido lemma from those of Nikaido(1956b, 1957b, 59), which may look close to the theorem 2 of this section.

[54]Of course, when the excess demand correspondence is defined on a a weak $\sigma(G,E)-$compactset, the

Define $\Delta^* = \{p \in P^* : p \cdot u = -1\}$ as the general price set with using $int_\tau(P) \neq \emptyset$. Lemma 9 implies the $^*$ weak $\sigma(E^*, G)-$compactness of $\Delta^*$. Although the excess demand correspondence is assumed to be defined only on $\Delta = (\Delta^* \cap G)$, not necessarily on $\Delta^*$, an equilibrium price is shown to be in $\Delta^*$, not necessarily in $\Delta$ since the equilibrium price is obtained indeed as the limit of a net of prices in $\Delta$. Then it is necessary to extend the excess demand to be defined on $\Delta^*$ to make this limiting argument work. [55] In an economy with $l_\infty$ as the commodity space, the non-emptyness of a demand correspondence is established on $l_1$ as the price space with using an appropriate topology on $l_\infty$, *i.e.*, $^*$ weak $\sigma(l_\infty, l_1)-$topology as in the proof in the case of finite dimensional commodity spaces. The non-emptyness of a demand correspondence, however, is not established on $ba$ as the price space even with using the weak $\sigma(l_\infty, ba)-$topology on $l_\infty$. Thus it is necessary to extend the excess demand correspondences defined on $l_1$ to the one on $ba$.

Let $(S, \tau)$ and $(X, \tau_X)$ be two (Hausdorff) topological spaces, and $\varphi : S \to X\backslash\{\emptyset\}$ be a correspondence between these two spaces. Let $(S', \tau')$ be a (Hausdorff) topological space with $S' \supset S$. Consider the closure $cl_{\tau' \times \tau_X}(G_\varphi)(\subset S' \times X)$ of the graph, $G_\varphi = \{(p, x) \in S \times X : x \in \varphi(p)\}(\subset S' \times X)$, with respect to the product topology on $S' \times X$ . Define a correspondence $\widetilde{\varphi} : T \to X\backslash\{\emptyset\}$ by $x \in T \to \widetilde{\varphi}(p) = \{x \in X : (p, x) \in cl_{\tau' \times \tau_X}(G_\varphi)\}$ for $T = \{p \in S' : \exists x \in X, (p, x) \in cl_{\tau' \times \tau_X}(G_\varphi)\}(\subset cl_{\tau'}(S))$. [56] Although this definition implies that $\varphi : S \to X\backslash\{\emptyset\}$ is extended to $\widetilde{\varphi} : T \to X\backslash\{\emptyset\}$, $\widetilde{\varphi}(p) = \varphi(p)$ may not hold for $p \in S$ and only $\widetilde{\varphi}(p) \supset \varphi(p)$ holds for $p \in S$ generally. Here $(\Delta, \sigma(G, E))$ and $(\Delta^*, \sigma(E^*, E))$ correspond to $(S, \tau)$ and $(\Delta^*, \sigma(E^*, E))$, respectively.[57]

Let $E$ be a locally convex topological space and $P(\neq E)$ be a $\sigma(E, G)-$closed convex cone $(\subset E)$ with $int_\tau(P) \neq \emptyset$. Since $G \subset E^*$ implies that the weak $\sigma(E, G)-$topology is coarser than the weak $\sigma(E, E^*)-$topology on $E$, $P$ is a $\sigma(E, E^*)-$closed convex cone. Moreover, since $*$ weak $\sigma(E, E^*)-$topology is coarser than the original $\tau-$topology on $E$, $P$ is also $\tau-$closed as well. Note that $\Delta^* = \{p \in P^* : p \cdot u = -1\}$ and $\Delta = \Delta^* \cap G$. Since $u \in int_\tau(P)$, the $\tau-$closedness of a convex cone $P$, and $P \neq E$ implies $P$ satisfies the conditions of lemma 5, its conclusion implies $< P^* >$ is non-empty and convex. Then

finite dimensional approximation method based on the lemma 6 works well as in the finite dimensional case.

[55] Florenzano(1983 definition 1 p.211). Florenzano(1983 lemma 1 pp.212-3) assumes the excess demand correspondence is *u.h.c.* on the finite dimensional subspaces of $\Delta^*$, not on entire $\Delta^*$ although it is defined on $\Delta^*$ as in Nikaido(1957b,59), and uses this kind of an extension of the excess demand correspondence to compensate its *u.h.c.* on the entire $\Delta^*$. Note that Florenzano(1983 corollary 1 pp.213-4) shows that when the excess demand correspondence is *u.h.c.*on $\Delta^*$, this extended excess demand correspondence coincides with the original extended excess demand correspondence. Note, however, that although Florenzano(1983, lemma 1 pp.212-3) uses the excess demand correspondence defined on $\Delta^*$, its proof uses only that the excess demand correspondence defined on $\Delta$. From this reason, the excess demand correspondence is defined only on $\Delta$ and then is extended over $\Delta^*$ in the text here.

[56] Note that Nikaido(1957a) uses an extension of a correspondence following the one used in Kuhn(1956).

[57] Florenzano(1983) applies the lemma 1 of Florenzano(1983, pp.212-3) to an economy with $l_\infty$ as the commodity space to show the existence of a quasi- equilibrium in such an economy. It uses the $*$ weak $\sigma(l_\infty, l_1)$ topology on $l_\infty$ and show that as in the finite dimensional commodity space case the excess demand is well-defined on $l_1$, corresponding to $G$ here, as the price space. It, however, does not show that the excess demand is well-defined on $ba$, corresponding to $E$ here, as the price space. Since Florenzano(1983 lemma pp.212-3) assumes that the excess demand is well-defined on $\Delta^*$, when the $*$ weak $\sigma(l_\infty, l_1)$ topology is used on on $l_\infty$ so that the excess demand is well-defined only on $\Delta$, in fact, Florenzano(1983 lemma pp.212-3) does not apply to this economy. In the proof of Florenzano(1983 lemma pp.212-3), however, the non-emptyness of the excess demand only on $\Delta$ is sufficeint, the extension of a correspondence in the text is employed here.

lemma 7 implies that $\Delta^*$ is non-empty convex and $*$ weak $\sigma(E^*, E)-$compact. Since $P$ is a $*$ weak $\sigma(E, G)-$closed convex cone with $P \neq E$, lemma 1 implies that $0 \geqq p \cdot x \ \forall x \in P$ holds for a $p \in G \backslash \{0\}$. Then $p \in < P^* >$ holds from $G \subset E^*$. Since $p$ is considered to be in $E^*$, $u \in int_\tau(P)$ and lemma 3 implies $p \cdot u < 0$ and hence $p \cdot u/(-p \cdot u) = -1$ holds. Then $(-p \cdot u) > 0$ implies $p/(-p \cdot u) \in \Delta = (\Delta^* \cap G)$,and hence $\Delta$ is non-empty and convex. The result in Florenzano(1983) is stated as follows.

**Theorem 2(Florenzano(1983))** Suppose that a correspondence $\varphi : \Delta \to E \backslash \{\emptyset\}$ satisfies the following conditions: $(i)' : \varphi(p)$ is non-empty convex and $*$ weak $(\sigma(E, G)-$compact $\forall p \in \Delta$. $(ii)'' :$ Let $L$ be a finite dimensional subcone of $(P^* \cap G)$ and use the weak $\sigma(G, E)-$topology on $\Delta \cap L$ and the $*$ weak $\sigma(E, G)-$topology on $E$, respectively. The correspondence of $\varphi$ restricted on $\Delta \cap L$, $\varphi : \Delta \cap L \to E \backslash \{\emptyset\}$, is then $u.h.c.$ on $(\Delta \cap L)$. $(iii)' :$ The image of $\Delta$ by $\varphi$, $\Gamma = \cup_{p \in \Delta^-} \varphi(p)$ is in a $*$ weak $\sigma(E, G)-$compact subset $\Psi$. $(iv) : x \in \phi(p) \to p \cdot x \leqq 0$ holds $\forall\ p \in \Delta$(weak Walras' law). Let $\widehat{\Delta} = \{p \in \Delta^* : \exists x \in E, (p, x) \in cl_{\sigma(E^*,E) \times \sigma(E,G)}(G_\varphi)\}(\subset cl_{\sigma(E^*,E)}(\Delta))$ and define an extended correspondence of $\varphi$ from $\Delta$ to $\widehat{\Delta}$, $\widetilde{\varphi} : \widehat{\Delta} \to \Psi$ by $p \in \widehat{\Delta} \to \widetilde{\varphi}(p) = \{x \in X : (p, x) \in cl_{\sigma(E^*,E) \times \sigma(E,G)}(G_\varphi)\}$. Then there is $\overline{p} \in \widehat{\Delta}(\subset \Delta^*)$ satisfying $\widetilde{\varphi}(\overline{p}) \cap P \neq \emptyset$.[58]

**Proof of the theorem.** Let $F = \{p_1, \cdots, p_m\}$ be a finite set from $\Delta$. From the convexity of $\Delta$, $co(F) \subset \Delta$ holds. Define two continuous functions as following. First let $\alpha : E \to R^m$ be $\alpha_i(x) = p_i(x), i = 1, \cdots, m$. Since $p_i \in G$ implies that it is a $*$ weak $\sigma(E, G)-$continuous linear functional on $E$, $\alpha_i : E \to R$ is also $*$ weak $\sigma(E, G)-$continuous, $i = 1, \cdots, m$ and hence $\alpha : E \to R^m$ is $*$ weak $\sigma(E, G)-$continuous as well. Next let $\beta : S^m \to \Delta$ be defined by $\beta(w) = \sum_{i=1}^m w_i p_i \ \forall = (w_1, \cdots, w_m) w \in S^m$. Since $\beta$ is a convex combination of $p_i, i = 1, \cdots, m$, a property of linear topology implies that $\beta$ is continuous on $S^m$ when $\sigma(G, E)-$topology is used on the range $G$. Also $\beta(w) \in \Delta$ holds from $co(F) \subset \Delta$. Let $\phi : S^m \to R^m$ be the composite mapping $\alpha \circ \varphi \circ \beta$ of $\alpha, \beta, \varphi$. From the expression $\phi(w) = \alpha(\varphi(\beta(w)))$, $\phi(w)$ is expressed as a vector $(\alpha_1(\varphi(\beta(w)), \cdots, \alpha_m(\varphi(\beta(w)))) = (p_1(\varphi(\beta(w)), \cdots, p_m(\varphi(\beta(w))))$. Thus, $\phi(w)$ is a vector of the values of the excess demand at the price $\beta(w)$ of a convex combination of the prices $p_1, \cdots, p_m$ with the weight equal to $w$, evaluated at each of $p_1, \cdots, p_m$. Since $\beta(S^m) = co(F) \subset \Delta$ holds, $(iii)$ implies that $\varphi$ is $u.h.c.$ on $\beta(S^m)$. Then this $\phi : S^m \to R^m$ also satisfies the condition of lemma 8 as in the proof of theorem 1, and hence $\phi(\overline{w_F}) \cap (-R^m_+) \neq \emptyset$ holds for a $\overline{w_F} \in S^m$. Thus, $(p_1(\overline{x_F}), \cdots, p_m(\overline{x_F})) \leqq 0$ holds for $\overline{x_F} \in \varphi(\beta(\overline{w_F}))$. From $\beta(\overline{w_F}) \in \Delta(\subset \Delta^*)$, $\overline{x_F} \in \varphi(\beta(\overline{w_F})) \subset \Gamma = \cup_{p \in \Delta} \varphi(p) \subset \Psi$ holds.

Let $\overline{\Gamma} = cl_{\sigma(E,G)}(\Gamma)$ be a the $*$ weak $\sigma(E, G)-$closure of $\Gamma$ and $P(F) = \{x \in E : p_i(x) \leqq 0, p_i \in F\} = \{x \in E : p_i(x) \leqq 0, i = 1 \cdots, m\}$. Then, $\overline{x_F} \in P(F)$ and hence $(P(F) \cap \overline{\Gamma}) \supset (P(F) \cap \Gamma) \neq \emptyset$ holds.[59] Since$p_i$ is $*$ weak $\sigma(E, G)-$continuous, $P(F)$ is $*$

[58]Florenzano(1983 lemma 1 pp.212-3). It uses a linear toplogy which is coarser than the $*$ weak $\sigma(E, E^*)$-topology. Here uses the $*$ weak $\sigma(E, G)$- topology for this topology. Also, although Florenzano(1983 lemma 1 pp.212-3) assumes that the excess demand is well-defined on $\Delta^*$, here assumes that the excess demand is well-defined only on $\Delta$.

[59]Up to this point, there is no difference btween the proof of Nikaido(1957b,59) and that of Florenzano(1983). There is, however, a difference between two in the following. Here folows that of Nikaido(1957b,59), and that of Florenzano(1983) is discussed later after this proof.

weak $\sigma(E,G)-$closed. This result holds for any finite subset $F$ of $\Delta$. Consider the family $\{F : F \subset \Delta, \#F < \infty\}$ of such finite subsets. Then $(P(F) \cap \overline{\Gamma}) \supset (P(F) \cap \Gamma) \neq \emptyset$ implies that this family has the finite intersection property on $\overline{\Gamma}$. Since $\Gamma$ is $*$ weak $\sigma(E,G)-$relatively compact, $\overline{\Gamma}$ is $*$ weak $\sigma(E,G)-$compact, $[(\cap_{F \subset \Delta} P(F)) \cap \overline{\Gamma}] \neq \emptyset$ holds. Since $P$ is a $*$ weak $\sigma(E,G)-$closed convex cone, when $*$ weak $\sigma(E,G)-$topology is endowed on $E$, $(P)^* = \{p \in G : p \cdot x \leqq 0, \forall p \in P\}$ and lemma 2 imply $P = \{x \in E : p(x) \leqq 0, \forall p \in (P)^*\} = \{x \in E : p(x) \leqq 0, \forall p \in < (P)^* >\} \subset \{x \in E : p(x) \leqq 0, p \in F, \#F < \infty, F \subset \Delta\}$ and hence $P \subset (\cap_{F \subset \Delta} P(F))$. On the other hand, letting $F = \{p\}$ for $p \in (P)^*$ gives rise to $(\cap_{F \subset \Delta} P(F)) \subset (\cap_{\{p\} \subset \Delta} P(\{p\})) = \{x \in E : p(x) \geqq 0, \forall p \in < (P)^* >\} = P$, and hence $P = (\cap_{F \subset \Delta} P(F))$. Thus, $(P \cap \overline{\Gamma}) \neq \emptyset$ follows from $[(\cap_{F \subset \Delta} P(F)) \cap \overline{\Gamma}] \neq \emptyset$.

Take $\overline{x} \in (P \cap \overline{\Gamma})$. Also let $N^*(0)$ be fundamental system of convex balanced weak $\sigma(E,G)-$neighborhoods of the origin 0. Then $\overline{x} \in \overline{\Gamma}$ implies that $[(\overline{x} + V) \cap \Gamma] \neq \emptyset$ holds $\forall V \in N^*(0)$ and hence $\forall V \in N^*(0)\ \exists x_V \in \Gamma \quad x_V \in (\overline{x} + V)$ follows. Then $x_V \in \Gamma = \cup_{p \in \Delta} \varphi(p)$ implies that $\exists p_V \in \Delta (\subset \Delta^*), x_V \in \varphi(p_V)$. Let be a direct set $(N^*(0), \prec)$ by defining $V \prec V' \Longleftrightarrow V \subset V'$. Then $(x_V, p_V)_\prec$ is a net. Note first that $\forall V \in N^*(0)\ \exists x_V \in \Gamma$ $x_V \in (\overline{x} + V)$ implies $x_V \to \overline{x}(\prec\uparrow)$. Since $\Delta^*$ and $\overline{\Gamma}$ are $*$ weak $\sigma(E^*, E)-$compact and $*$ weak $\sigma(E,G)-$compact, respectively, there is a converging subnet from the net $(x_V, p_V)_\prec$ so that $(x_V, p_V) \to (\overline{x}, \overline{\pi}) \in \overline{\Gamma} \times \Delta^*$ holds.[60] Since $x_V \in \varphi(p_V), p_V \in \Delta$ imply $(p_V, x_V) \in G_r(\varphi)$, $(\overline{\pi}, \overline{x}) \in (cl_{\sigma(E^*,E) \times \sigma(E,G)}(G_r(\varphi)))(\subset \Delta^* \times \overline{\Gamma})$ holds and hence $\overline{x} \in \widetilde{\varphi}(\overline{\pi}), \overline{\pi} \in \widehat{\Delta}$ from the definition of the extension $\widetilde{\varphi}$ of the correspondence $\varphi$. Since $\overline{x} \in (P \cap \overline{\Gamma})$ implies $\overline{x} \in (\widetilde{\varphi}(\overline{\pi}) \cap P)$, $(\overline{x} \in)(\widetilde{\varphi}(\overline{\pi}) \cap P) \neq \emptyset$ holds for $\exists \overline{\pi} \in \widehat{\Delta} (\subset \Delta^*)$. This is the result to be shown. □

Note first that, as theorem 1 of the previous section, $\varphi$ is assumed to be *u.h.c.* on $(L \cap \Delta)$ for a finite dimensional convex subcone $L$ instead of on $\Delta$. This is same as of theorem 1 in the previous section and corresponds to the condition of lemma 9. Since a finite dimensional convex subcone $L$ of $(P^* \cap G)$ is the intersection of the linear subspace $[p_1, \cdots, p_m]$ given by a linearly independent finite points $\{p_1, \cdots, p_m\}$ and $(P^* \cap G)$, $(L \cap \Delta)$ is indeed the convex hull $co(\{p_1, \cdots, p_m\})$ given by $\{p'_1, \cdots, p'_m\}$, where $\{p'_1, \cdots, p'_m\}$ is the one adjusted of $\{p_1, \cdots, p_m\}$ such that $\{p'_1, \cdots, p'_m\}$ belongs to $\Delta$. Since the excess demand correspondence $\varphi$ is assumed to be defined on $\Delta$, which is a subset of $\Delta^*$, instead of on $\Delta^*$. From this reason, $\varphi$ defined on $\Delta$ need to be extended to $\widetilde{\varphi}$ on $\Delta^*$. Note, however, that the conclusion of the existence of $\overline{p} \in \Delta^*$ with $\widetilde{\varphi}(\overline{p}) \cap P \neq \emptyset$ does not say that $\overline{p} \in \Delta$ holds. It says only that $\widetilde{\varphi}(\overline{p}) \cap P \neq \emptyset$ holds for $\overline{p} \in \Delta^*$ when $\widetilde{\varphi}$ on $\Delta^*$ is used. This is also a difference between theorem 2 and theorem 1 which is corresponding to Nikaido(1956b,57b,59). Note also that $\Gamma = \cup_{p \in \Delta} \varphi(p)$ is assumed to be included in a $*$ weak $\sigma(E,G)-$compact subset $\Psi$, instead of $\Gamma$ being $*$ weak $\sigma(E,G)-$compact. This is also a difference between theorem 2 and theorem 1.

Although the above proof uses the similar proof of theorem 1 which is based on the one of Nikaido(1957b,59), it uses the $*$ weak $\sigma(E^*, E)-$compactness of $\Delta^*$ instead of $*$ weak $\sigma(E,G)-$compactness of $\Gamma = \cup_{p \in \Delta} \varphi(p)$, which is used in the proof the theorem 1 and those in Nikaido(1957b,59). As the result, $\overline{\pi} \in \Delta^*$ follows but $\overline{\pi} \in \Delta$ does not necessarily follow.[61]

[60] Followling the convension which makes the notion simple, $(x_V, p_V)_\prec$ is treated itself as this converging subnet.

[61] When $\Psi = \Gamma$ and hence $\Delta = \Delta^*$ holds, $\overline{\pi} \in \Delta$ holds. From the argument used in Bewley(1972), in an economy with $l_\infty$ as the commodity space, the proof of the existence of a competitive equilibrium uses the

When the method of the above proof is used in a more general case with $(P \cap -P) \neq P$ instead of $int_\tau(P) \neq \emptyset$, $(P \cap \overline{\Gamma}) \neq \emptyset$ is shown to hold only. Thus $\exists x \in P\ \forall V \in N^*(0)$ $\exists p_V \in < P^* >, x_V \in (\varphi(p_V) \cap (x+V))$ follows only. Then although $x_V \to x$ holds for the net $(x_V)$, the net $(p_V)$ may not have the limit. [62]

The part after showing $(P(F) \cap \overline{\Gamma}) \neq \emptyset$ in the above proof is the one different from the proof of Florenzano(1983). The corresponding part of Florenzano(1983、p.213) goes as follows. Consider a family $F = \{F : F \subset \Delta, \#F < \infty\}$ where $F$ satisfies $(P(F) \cap \overline{\Gamma}) \neq \emptyset$. Then $(F, \prec)$ becomes a directed set with an order $F \prec F' \Longleftrightarrow F \subset F'$, and hence $(\overline{x_F}, \beta(\overline{w_F}))_\prec$ is a net for $\overline{x_F} \in (P(F) \cap \Psi), \beta(\overline{w_F}) \in \Delta (\subset \Delta^*)$. Since $\Delta^*$ and $\Psi$ are $*$ weak $\sigma(E^*, E)-$compact and $*$ weak $\sigma(E, G)-$compact, respectively, $(\overline{x_F}, \beta(\overline{w_F}))$ has a converging subnet with respect to the product $*$ weak $\sigma(E^*, E) \times *$ weak $\sigma(E, G)-$topology, and hence there is $(\overline{x}, \overline{\pi}) \in \Psi \times \Delta^*$ with $(\overline{x_F}, \beta(\overline{w_F})) \to (\overline{x}, \overline{\pi})$. $\overline{x_F} \in \varphi(\beta(\overline{w_F}))$ and $\beta(\overline{w_F}) \in \Delta$ imply $(\beta(\overline{w_F}), \overline{x_F}) \in G_r(\varphi)$ and hence $(\overline{\pi}, \overline{x}) \in cl_{\sigma(E^*,E) \times \sigma(E,G)}(G_r(\varphi))$, or $\overline{x} \in \widetilde{\varphi}(\overline{\pi})$. Suppose that $\overline{x} \notin P$ occurs. Since $P$ is $*$ weak $\sigma(E, G)-$closed convex cone and hence $\tau-$closed, lemma 1 implies that there is $q \in (P)^* \backslash \{0\} (\subset P^* \backslash \{0\})$ satisfying $q \cdot \overline{x} > 0 \geqq q \cdot y\ \forall y \in P$. Then $q \in P^* \backslash \{0\}$ and $u \in int_\tau(P)$ imply $q \cdot u < 0$ and $q \in \Delta$. Since $\overline{x_F} \to \overline{x}$ holds, the definition of $*$ weak $\sigma(E, G)-$topology implies $q \cdot \overline{x_F} \to q \cdot \overline{x} > 0 (\prec \uparrow)$. Then $q \cdot \overline{x_{F'}} > 0$ holds $\forall F' (\in F) \succ F$ with $q \in F$, and hence $x_{F'} \notin P(F')$. Since, however, $\overline{x_F} \in P(F)$ holds for any finite subset $F$ of $\Delta$, this is a contradiction and hence $\overline{x} \in P$ holds. Thus, $(\overline{x} \in)(\widetilde{\varphi}(\overline{\pi}) \cap P) \neq \emptyset$ holds for $\overline{\pi} \in \Delta^*$. Note that this proof of Florenzano(1983) uses a family $F = \{F : F \subset \Delta, \#F < \infty\}$ with each $F$ satisfying $(P(F) \cap \overline{\Gamma}) \neq \emptyset$, on the other hand, the above proof of theorem 2 uses $N^*(0)$ of convex balanced $*$ weak $\sigma(E, G)-$neighborhoods of the origin 0 as a fundamental system of neighborhoods of 0.

In the next section, this result applies to a simplified Bewley(1972) economy with $l_\infty$ as the commodity space to establish the existence of a competitive equilibrium in this economy. This is an example of an economy to which an modified version of infinite dimensional Gale-Nikaido lemmas of Nikaido(1956b,57b,59) according to Florenzano(1983) applies to establish the existence of a competitive equilibrium in this economy.

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following two step argument. An equilibriu price is found first in the norm dual space $(l_\infty)^* = ba$ and then its $l_1-$part, owing to Yoshida-Hewitt theorem, is also an equilibrium price. When $\Psi = \Gamma$ holds, theorem 1 and theorem 2 imply that an equilibrium price exsits in $l_1$ from the first even without using this two stage argument. The next section considers the existence of a competitive equilibrium an economy with $l_\infty$ as the commodity space.

[62] When $\Delta_V = \{p \in (P(V)^* \cap G) : p \cdot u = -1\} (\subset (\cup_{V \in N^*(0)} \Delta_{V'}) \subset \Delta \subset \Delta^*)$ と a $*$ weak $\sigma(E, G)-$compact subset, is used for $V \in N^*(0)$, $V \supset V' \Longleftrightarrow \Delta_V \subset \Delta_{V'}$ holds. Since $u \in int_\tau(P)$ is not assumed, $\cup_{V \in N^*(0)} \Delta_{V'}$ and $\Delta^*$ are not $*$ weak $\sigma(E, G)-$compact. Thus, even when $p_V \in \Delta_V$ holds, $(p_V)$ may not have any $*$ weak $\sigma(E, G)-$converging subnet. From this point, the above theorem 2 assumes $u \in int_\tau(P)$ to make $\Delta^*$ $*$ weak $\sigma(E^*, E)-$compact, as the result, $(p_V) (\subset \Delta^*)$ has a converging subnet and hence $p_V \to \overline{p} \in \Delta^*$ holds. This fact is crucial in the theorem 2. In the finite dimensional commodity space case, from the compactness of the unit ball $B = \{p : ||p|| = 1\}$, since $||p_V|| = 1$ holds always even when $P$ is approximated from outside by $\{P_V\}$, the existence of $\overline{p}$ with $p_V \to \overline{p}, ||\overline{p}|| = 1$ follows and hence there is no problem as to this point. When the domain $\Delta$, on which the excess demand correspondence is defined, is weak $\sigma(G, E)-$compact from the first, $p_V \to \overline{p} \in \Delta$ follows even without assuming $u \in int_\tau(P)$.

# 6 Existence of Competitive Equilibrium in Infinite Dimensional Commodity Space Model : Application to Bewley(1972)'s Model

This section establishes the existence of a competitive equilibrium in an simplified Bewley(1972)'s economy with $l_\infty$ as the commodity space with applying the theorem 2 on an infinite dimensional Gale Nikaido lemma modified according to Florenzano(1983) in the previous section. [63] Florenzano(1983) uses a pure exchange economy, here, on the other hand, uses a production economy. The present proof goes as in the finite dimensional commodity space case as of Gale(1955), Nikaido(1956), and Debreu(1959) in the sense that the existence of a competitive equilibrium is established first in an economy restricted over an bounded region based on an application of the theorem 2 on an modified infinite dimensional Gale-Nikaido lemma and, then, this competitive equilibrium is also the one in the unrestricted original economy. In this process, however, the compactness of the price space which is consistent with the linear topology on the commodity space is not obtained and the *u.h.c.* of the excess demand holds only on finite dimensional subspace of the price set. These two facts turn clear why the modification is done as the theorem 2 to the original versions of the infinite dimensional Gale-Nikaido lemma according to Florenzano in the previous section.

Consider an following economy. There are finite $H$ consumers whose consumption set is $l_\infty^+$, whose preference relation $R^h$, satisfying reflexivity, completeness, and transitivity, is defined on $l_\infty^+$, and whose initial endowment $\omega^h \in l_\infty^+$. There is also an aggregate production set $Y$ which is a $*$ weak $\sigma(l_\infty, l_1)-$closed convex cone with the vertex at the origin 0. $R^h$ satisfies the condition that $R^h(x)$ is $*$ weak $\sigma(l_\infty, l_1)-$closed for $x \in l_\infty^+$ and , the lower inverse $(R^h)^-(x)$ of $R^h$ at $x(= \{y \in l_\infty^+ : x \in R^h(y)\})$ is $||\cdot||_\infty-$closed.[64] The preference relation $R^h$ is assumed to be strongly monotonic in the sense that $x > y \to x \in P^h(y)$ and convex in the sense that $x' \in P^h(x), \alpha \in (0,1) \to (\alpha x' + (1-\alpha)x) \in P^h(x)$. Then $R^h(x)$ becomes a convex set for $x \in l_\infty^+$. As to $h$'s initial endowments $\omega^h$, $\omega^h \geqq \alpha^h e$ holds for $\alpha^h > 0, h = 1, \cdots, H$. Denote $\omega = \sum_{h=1}^{H} \omega^h$, the aggregate initial endowment of the economy. The aggregate production set $Y$ also satisfies the free disposal, $-(l_\infty^+) \subset Y$. $(\omega + Y) \cap l_\infty^+$ is assumed to be $||\cdot||_\infty-$bounded. Then since $e = (1, 1, \cdots) \in int_{||\cdot||_\infty-}(l_\infty^+)$ implies $int_{||\cdot||_\infty}(l_\infty^+) \neq \emptyset$ and hence $int_{||\cdot||_\infty}(Y) \neq \emptyset$. Moreover, there is some $m > 0$ such that $z \leqq me$ holds for $z \in ((\omega + Y) \cap l_\infty^+)$. Since this latter condition implies that the set of feasible allocations is in a $||\cdot||_\infty-$bounded subset, a competitive allocation is in this $||\cdot||_\infty-$bounded subset. Thus, it is enough to show the existence of a competitive equilibrium in this $||\cdot||_\infty-$bounded region of the economy in order to prove the existence of a competitive equilibrium in the original unrestricted economy. Note

[63] As already noted, $l_\infty$ is the set of all bounded sequences and is a normed space with $||\cdot||_\infty-$norm. The norm dual space of $l_\infty$ is the set $ba$ of bounded finitely additive measures on $N$. Also $l_1$ is the set of all summable sequences and is a normed space with $||\cdot||_1-$norm. The norm dual space of $l_1$ is $l_\infty$. Note that $ba$ contains $l_1$ as its subset. Yoshida-Hewitt theorem states that an element of $ba$ is decomposed into $l_1$ and $pfa$, the set of purely finitely additive measures on $N$. An element of $pfa$ has a property that its value does not change even the elements of any finite coordinates on $N$ are switched all to zero.

[64] As mentioned later, when an equilibrium price is found in $l_1$, the lower inverse $(R^h)^-(x)$ of $R^h$ at $x$ needs to be $*$ weak $\sigma(l_\infty, l_1)-$closed.

that $(0 \leqq)\omega^h \leqq \omega \in ((\omega + Y) \cap l_\infty^+)$ implies $\omega^h \leqq me$, $h = 1, \cdots, H$.

Define the general price set $\Pi = \{p \in Y^* : p \cdot (-e) = -1\}(\neq \emptyset)$ for $e = (1, 1, \cdots) \in int_{||\cdot||_\infty}(l_\infty^+)$, lemma 1 and $-e \in int_{||\cdot||_\infty}(Y)$ imply the $*$ weak $\sigma(ba, l_\infty)-$compactness. Also define $\Delta = \{p \in (Y^* \cap l_1) : p \cdot (-e) = -1\}(\subset \Pi)$, the price set restricted to $l_1$. Then $Y \neq l_\infty$ and lemma 1 imply $\Delta \neq \emptyset$. Note that lemma 9 can not apply to $\Delta$, $\Delta$ is not necessarily $*$ weak $\sigma(l_1, l_\infty)-$compact. As mentioned in the previous section, this is the main reason why the extension of the graph of the excess demand correspondence is required as an modification to theorem 2 of the original infinite dimensional Gale-Nikaido lemmas in Nikaido(1957b,1959) according to Florenzano(1983).

Define consumer $h$'s budget correspondence $B^h : \Pi \to l_\infty^+$ by $B^h(p) = \{x \in l_\infty^+ : p \cdot x \leqq p \cdot \omega^h\}$ for $p \in \Pi$. $0 \in B^h(p)$ implies $B^h(p) \neq \emptyset$ $\forall p \in \Pi$. Since aggregate profit of the firms is 0 from $p \in Y^*$ $\forall p \in \Pi$, it does not matter whether the disposal income of a consumer contains the profits of the firms or not. $B^h(p)$ is convex from the linearity of $p \cdot x$ and is weak $\sigma(l_\infty, ba)-$closed from the weak $\sigma(l_\infty, ba)-$continuity of $p \cdot x$ with respect to $x$.[65] Define consumer $h$'s demand correspondence $\varphi^h : \Pi \to l_\infty^+$ by $\varphi^h(p) = \{x \in B^h(p) : y \in B^h(p) \to x \in R^h(y)\}$ $\forall p \in \Pi$.[66] Consider $\alpha x + (1 - \alpha)x'$ for $x, x' \in \varphi^h(p)$ and $\alpha \in (0, 1)$. From the linearity of $p \cdot x$, $B^h(p)$ is convex and hence $\alpha x + (1 - \alpha)x') \in B^h(p)$. Since $x$ and $x'$ are indifferent from the definition of $\varphi^h(p)$, $x, x' \in R^h(x) = R^h(x')$ holds. Since also the convexity of preferences implies the convexity of $R^h(x)$, $(\alpha x + (1 - \alpha)x') \in R^h(x)$ holds. Since then from $x(x') \in \varphi^h(p)$ $y \in B^h(p) \to x(x') \in R^h(y)$ holds, $y \in B^h(p) \to (\alpha x + (1 - \alpha)x') \in R^h(x)$ holds. Thus, $(\alpha x + (1 - \alpha)x') \in \varphi^h(p)$ follows and $\varphi^h(p)$ is convex. When $\Delta$ is used as the price set, $B^h(p)$ is $*$ weak $\sigma(l_\infty, l_1)-$closed from the $*$ weak $\sigma(l_\infty, l_1)-$continuity of $p \cdot x$ on $l_\infty$. Since the strict monotonicity of the preferences implies $x + \alpha e > x$ for $\alpha > 0$ and $x + \alpha e \in P^h(x)$ holds. Since$\alpha \to 0 \Longrightarrow \alpha(p \cdot e) \to 0 \Longrightarrow p \cdot (x + \alpha e) = p \cdot x + \alpha(p \cdot e) \to p \cdot x$, $p \cdot x = p \cdot \omega^h$ holds $p \cdot (x + \alpha e) < p \cdot \omega^h$ holds for $\alpha(> 0)$ sufficiently close to 0 when $p \cdot x < p \cdot \omega^h$. Then there is a preferred point $x + \alpha e \in P^h(x)$ to $x(\in \varphi^h(p))$ in the budget set $B^h(p)$, which is a contradiction to $x(\in \varphi^h(p))$. Thus, $p \cdot x = p \cdot \omega^h$ holds at $x \in \varphi^h(p)$.

Define the (aggregate) excess demand correspondence $\psi : \Pi \to l_\infty$ by $\to \psi(p) = \sum_{h=1}^{H} \phi^h(p) - \sum_{h=1}^{H} \omega^h = \phi(p) - \omega$ for $p \in \Pi$. Since $x \in \varphi^h(p) \Longrightarrow p \cdot x = p \cdot \omega^h$ holds $\forall h = 1, \cdots, H$, $p \cdot \psi(p) = p \cdot (\sum_{h=1}^{H} \phi^h(p) - \sum_{h=1}^{H} \omega^h) = \sum_{h=1}^{H} (p \cdot \phi^h(p) - p \cdot \omega^h) = 0$ holds $\forall p \in \Pi$. Thus, Walras' law holds to $\psi$.

Let $\widetilde{x} = (x^1, \cdots, x^H) \in (l_\infty^+)^H$ be an allocation of the economy and such a $\widetilde{x} = (x^1, \cdots, x^H)$ be called feasible when $(\sum_{h=1}^{H} x^h - \sum_{h=1}^{H} \omega^h) \in Y$ holds. Define $(\widetilde{\overline{x}}, \overline{p}) = (\overline{x^1}, \cdots, \overline{x^H}, \overline{p})$ as

[65] From the definition of $*$ weak $\sigma(l_\infty, l_1)$ -topology $p \cdot x$ is continuous on $l_\infty$ with respect to $*$ weak $\sigma(l_\infty, l_1)-$topology for $p \in l_1$, $\{x \in l_\infty : (x, p) \geqq 0\}$ is $*$ weak $\sigma(l_\infty, l_1)-$closed and hence so is $l_\infty^+ = \{x \in l_\infty : x \geq 0\} = \{x \in l_\infty : (x, p) \geqq 0 \ \forall p \in l_1^+\} = \cap_{p \in l_1^+}\{x \in l_\infty : (x, p) \geqq 0\}$. Then, again from the $*$ weak $\sigma(l_\infty, l_1)$ -continuoity of $p \cdot x$ on $l_\infty$, $B^h(p)$ is $*$ weak $\sigma(l_\infty, l_1)-$closed.

[66] When the strict preference relation $P^h(x)$ derived from $R^h(x)$ is used, $\varphi^h(p) = \{x \in B^h(p) : B^h(p) \cap P^h(x)) = \emptyset\}$ follows.

a competitive equilibrium of the economy when $\widetilde{\overline{x}} = (\overline{x^1}, \cdots, \overline{x^H}) \in \widetilde{X}, \overline{x^h} \in \varphi^h(\overline{p}), h = 1, \cdots, H, \overline{p} \in \Pi$ hold. [67] Then the aggregate supply $\overline{y}$ is given by $\overline{y} = (\sum_{h=1}^{H} \overline{x^h} - \sum_{h=1}^{H} \omega^h)(\in Y)$ at this situation.[68] Let $\widetilde{X} = \{\widetilde{x} \in (l_\infty^+)^H : (\sum_{h=1}^{H} x^h - \sum_{h=1}^{H} \omega^h) \in Y\}$ be the set of feasible allocations of the economy. Since $0 \in Y$ and $\omega^h \in (l_\infty^+)$ imply $\widetilde{\omega} = (\omega^1, \cdots, \omega^H) \in \widetilde{X}$, $\widetilde{X} \neq \emptyset$ holds. The $*$ weak $\sigma(l_\infty, l_1)-$continuity of the vector operation and the $*$ weak $\sigma(l_\infty, l_1)-$closedness of $(l_\infty^+)$ and $Y$ imply that $\widetilde{X}$ is product $*$ weak $\times_1^H \sigma(l_\infty, l_1)-$closed. Let $\widetilde{X^h} = \{x^h \in l_\infty^+ : (\sum_{h=1}^{H} x^h - \sum_{h=1}^{H} \omega^h) \in Y, \exists x^k \geqq 0, k = 1, \cdots, H\}$ be the set of feasible consumptions to consumer $h$.[69] Then $\widetilde{X^h}$ is convex from the linearity of the vector operations and the convexity of $l_\infty^+$ and $Y$. Since from its definition for $x^h \in \widetilde{X^h}$ there is $x^k (\geqq 0)\ \forall k \neq h$ satisfying $\sum_{k=1}^{H} x^k \leqq me - \sum_{h=1}^{H} \omega^h \leqq me$, $(0 \leqq) x^h \leqq me - \sum_{k \neq h} x^k \leqq me$ holds and hence $\widetilde{X^h}$ is $||\cdot||_\infty-$bounded.[70] Since $\widetilde{X^h}$ is $||\cdot||_\infty-$bounded $\forall h = 1, \cdots, H$, $\widetilde{X^h} \subset int_{||\cdot||_\infty}(B)$ holds $\forall h = 1, \cdots, H$ for some $||\cdot||_\infty-$closed ball $B$ in $l_\infty$. Note that since $(\sum_{h=1}^{H} \omega^h - \sum_{h=1}^{H} \omega^h) = 0 \in Y$ and $0 \leqq \omega^h$ hold, $\omega^h \in \widetilde{X^h}$ and hence $\omega^h \in int_{||\cdot||_\infty}(B)$ hold $\forall h = 1, \cdots, H$.

Define $\widetilde{B^h} : \Pi \to l_\infty^+$, the consumer $h$'s budget correspondence restricted to $B$, by $\widetilde{B^h}(p) = \{x \in (l_\infty^+ \cap B) : p \cdot x \leqq p \cdot \omega^h\} = (B^h(p) \cap B)(\neq \emptyset)$ for $p \in \Pi$. Note first that $\omega^h \in B$ implies $\omega^h \in \widetilde{B^h}(p)$ $h = 1, \cdots, H$. Since both of $B$ and $B^h(p)$ are convex, $\widetilde{B^h}(p)$ is convex as well for $p \in \Pi$. Since $B$ is $*$ weak $\sigma(l_\infty, l_1)-$compact by Banach-Alaoglu theorem and $B^h(p)$ is $*$ weak $\sigma(l_\infty, l_1)-$closed for $p \in \Delta$, $\widetilde{B^h}(p)$ is $*$ weak $\sigma(l_\infty, l_1)-$compact for $p \in \Delta$.[71] Similarly define $\widetilde{\varphi^h} : \Pi \to l_\infty^+$, the consumer $h$'s demand correspondence restricted to $B$, by $\widetilde{\varphi^h}(p) = \{x \in \widetilde{B^h}(p) : y \in \widetilde{B^h}(p) \to x \in R^h(y)\}$ for $p \in \Pi$. As in the proof of the convexity of $\varphi^h(p)$, the convexity of $\widetilde{B^h}(p)$ and the convexity of the preferences imply the convexity of $\widetilde{\varphi^h}(p)$ as well. Note, however, that $\widetilde{\varphi^h}$ is shown to be non-empty only on $\Delta$

[67] Here uses $\overline{p} \in \Pi$ since $\overline{p} \in \Delta$ does not necessarily hold.

[68] Since, as aready mentioned, $\overline{p} \cdot \overline{x^h} = \overline{p} \cdot \omega^h$ holds $\forall h = 1, \ldots, H$ from the strict monotonicity of the preferences, $\overline{p} \cdot \overline{y} = \overline{p} \cdot \sum_{h=1}^{H} \overline{x^h} - \overline{p} \cdot \sum_{h=1}^{H} \omega^h = 0$ holds for the aggregate supply $\overline{y} = (\sum_{h=1}^{H} \overline{x^h} - \sum_{h=1}^{H} \omega^h)(\in Y)$. Thus, $\overline{y}$ attains the profit maximization over $Y$ at $\overline{p}$.

[69] Obviously, $\widetilde{X} \subset \times_{h=1}^{H} \widetilde{X^h}$ holds.

[70] Indeed, $\widetilde{X^h}$ is $*$ weak $\sigma(l_\infty, l_1)-$compact. Let $x^{h\alpha} \in \widetilde{X^h}, x^{h\alpha} \to_{\sigma(l_\infty, l_1)} x^h (\alpha \uparrow)$. By the definition of $\widetilde{X^h}$, $(\Sigma_{k=1}^{H} x^{k\alpha} - \Sigma_{k=1}^{H} \omega^k) \in Y$ holds for some $(x^{k\alpha}) \in \widetilde{X^k}\ \forall k \neq h$. Since $\widetilde{X^k}$ is $||\cdot||_\infty-$bounded, from Banach-Alaoglu theorem, $\widetilde{X^k}$ is relatively $*$ weak $\sigma(l_\infty, l_1)-$compact and hence $x^{k\alpha} \to_{\sigma(l_\infty, l_1)} x^k (\alpha \uparrow)$ holds. Since $l_\infty^+$ is $*$ weak $\sigma(l_\infty, l_1)-$closed, $x^k \in l_\infty^+$ holds from $x^{k\alpha} \in l_\infty^+$. Then since $(\Sigma_{k=1}^{H} x^{k\alpha} - \Sigma_{k=1}^{H} \omega^k) \to_{\sigma(l_\infty, l_1)} (\Sigma_{k=1}^{H} x^k - \Sigma_{k=1}^{H} \omega^k)$ holds and $Y$ is $*$ weak $\sigma(l_\infty, l_1)-$closed, $(\Sigma_{k=1}^{H} x^k - \Sigma_{k=1}^{H} \omega^k) \in Y$ follows and hence $x^h \in \widetilde{X^h}$ holds. Thus, $\widetilde{X^h}$ is $*$weak $\sigma(l_\infty, l_1)-$compact. Then, Since $\widetilde{X} \subset \times_{k=1}^{H} \widetilde{X^k}$ implies that $\widetilde{X}$ is relative $*$ weak product $\times_{k=1}^{H} \sigma(l_\infty, l_1)-$ compact and $\widetilde{X}$ is $*$ weak product $\times_{k=1}^{H} \sigma(l_\infty, l_1)-$ closed, $\widetilde{X}$ is $*$ weak product $\times_{k=1}^{H} \sigma(l_\infty, l_1)-$ compact.

[71] Note that $\widetilde{B^h}(p)$ is not weak $\sigma(l_\infty, ba)-$compact for $p \in \Delta^*$. $\widetilde{B^h}(p)$ is $*$ weak $\sigma(l_\infty, l_1)-$compact only for $p \in \Delta$.

below.[72] Define $\widetilde{\psi} : \Pi \to l_\infty$ as the aggregate excess demand correspondence restricted to $B$, by $\widetilde{\psi}(p) = \sum_{h=1}^{H} \widetilde{\phi^h}(p) - \sum_{h=1}^{H} \omega^h$ for $p \in \Pi$. Since $\widetilde{\phi^h}(p)$ is convex for $h = 1, \cdots, H$, so is $\widetilde{\psi}(p)$ as well. Since from the budget constraint of the consumers $p \cdot \widetilde{\phi^h}(p) \leqq p \cdot \omega^h$ holds for $h = 1, \cdots, H$, $p \cdot \widetilde{\psi}(p) \leqq 0$, weak Walras' law holds for $p \in \Pi$. [73] Since $\widetilde{\phi^h}(p) \subset B$, $h = 1, \cdots, H$ implies that $\widetilde{\psi}(p)$ is $||\cdot||_\infty-$bounded,$\widetilde{\psi}(p)$ is contained in a $||\cdot||_\infty-$closed ball $\Psi$ for $p \in \Pi$. $\Psi$ is $*$ weak $\sigma(l_\infty, l_1)-$compact by Banach-Alaoglu theorem. $(\cup_{p\in\Delta}\widetilde{\psi}(p)) \subset \Psi$ holds.[74]

The non-emptyness of the modified demand correspondence $\widetilde{\varphi^h}$ on $\Delta$ is shown first.

**Lemma 10** The demand correspondence $\widetilde{\varphi^h} : \Delta \to (l_\infty^+ \cap B)$ is non-empty convex and $*$ weak $\sigma(l_\infty, l_1)-$compact-valued.

**Proof.** The convexity of $\widetilde{\varphi^h}(p)$ is already given above. Let $\widetilde{R^h}(x) = (\widetilde{B^h}(p) \cap R^h(x))$ for $x \in \widetilde{B^h}(p)$ and $p \in \Delta$. Since $R^h(x)$ is $*$ weak $\sigma(l_\infty, l_1)-$closed and $\widetilde{B^h}(p)$ is $*$ weak $\sigma(l_\infty, l_1)-$compact, $\widetilde{R^h}(x)$ is $*$ weak $\sigma(l_\infty, l_1)-$compact as well. Then, $\{\widetilde{R^h}(x) : x \in \widetilde{B^h}(p)\}$ is a family of $*$ weak $\sigma(l_\infty, l_1)-$closed sets on the $*$ weak $\sigma(l_\infty, l_1)-$compact set $\widetilde{B^h}(p)$. Since the completeness and transitivity of the preference $R^h$ imply that there is a maximal point over a set of finite consumption points, $\{\widetilde{R^h}(x) : x \in \widetilde{B^h}(p)\}$ has the finite intersection property on $\widetilde{B^h}(p)$. Thus, $\cap_{x\in\widetilde{B^h}(p)}\widetilde{R^h}(x) \neq \emptyset$ follows from the the $*$ weak $\sigma(l_\infty, l_1)-$compactness of $\widetilde{B^h}(p)$. Since $\cap_{x\in\widetilde{B^h}(p)}\widetilde{R^h}(x) = \widetilde{\varphi^h}(p)$ holds, it is non-empty and $*$ weak $\sigma(l_\infty, l_1)-$compact. □

Note that $\widetilde{\varphi^h}(p) \neq \emptyset$ is established only for $p \in \Delta$ in the above lemma. Since this lemma implies that $\widetilde{\psi}$, the modified excess demand correspondence restricted over $B$, is well-defined on $\Delta$, here consider $\widetilde{\psi} : \Delta \to l_\infty$ instead of $\widetilde{\psi} : \Pi \to l_\infty$. $\widetilde{\psi} : \Delta \to l_\infty$ is shown below to satisfy $(i)', (ii)', (iii)', (iv)'$ of theorem 2.

**Lemma 11** $\widetilde{\psi} : \Delta \to l_\infty$ satisfies $(i)', (ii)'', (iii)', (iv)'$ of theorem 2.[75]

**Proof.** Note first that $(\cup_{p\in\Delta}\widetilde{\psi}(p)) \subset \Psi$ follows from the definition of $\Psi$ as already mentioned. Since $\widetilde{\varphi^h}(p) \neq \emptyset$ holds for $p \in \Delta$ from lemma 10 and $\widetilde{\psi}(p) = \sum_{h=1}^{H} \widetilde{\phi^h}(p) - \sum_{h=1}^{H} \omega^h$

[72]Note that $\widetilde{\varphi^h}$ is not shown to be non-empty on $\Pi$ due to the lack of the compactness condition on the relevent sets.

[73]When $\widetilde{\phi^h}(p) \in int_{||\cdot||_\infty}(B)$ occurs $h = 1, \ldots, H$, $p \cdot \phi^h(p) = p \cdot \omega^h$ follows from the strict monotonicity of the preferences as in the case of $\phi^h(p)$. Thus, $p \cdot \widetilde{\psi}(p) = 0$ holds at this situation. When, in particular, $\widetilde{\phi^h}(p) \in \widetilde{X^h}(\subset int_{||\cdot||_\infty}(B))$ holds $h = 1, \ldots, H$, $p \cdot \widetilde{\psi}(p) = 0$ holds. Note that it is enough to have $p \cdot \widetilde{\psi}(p) \leqq 0$ $\forall p \in \Pi$ here and $p \cdot \widetilde{\psi}(p) = 0$ $\forall p \in \Pi$ is not required here.

[74]Since $(\cup_{p\in\Delta}\widetilde{\psi}(p)) \subset \Psi$ holds and $\Psi$ is $*$weak $\sigma(l_\infty, l_1)-$compact, $(\cup_{p\in\Delta}\widetilde{\psi}(p))$ is relatively $*$ weak $\sigma(l_\infty, l_1)-$compact although $(\cup_{p\in\Delta}\widetilde{\psi}(p))$ itself is not necessarily $*$ weak $\sigma(l_\infty, l_1)-$compact. Because of this reason, theorem 1 can not apply directly to the economy here. This is also a reason why theorem 1 is modified to theorem 2 to take into account of this aspect. Of course, although when $(\cup_{p\in\Delta}\widetilde{\psi}(p))$ is $*$ weak $\sigma(l_\infty, l_1)-$compact, theorem 1 derives directlt the existence of equilibrium price in $l_1$, the $*$ weak $\sigma(l_\infty, l_1)-$compactness of $(\cup_{p\in\Delta}\widetilde{\psi}(p))$ in this economy is not shown yet.

[75]Since this simple result is not explicitly proved in Florenzano(1983) and Aliplantis-Brown(1983), it is included here.

holds from the definition, $\widetilde{\psi}(p) \neq \emptyset$ holds for $p \in \Delta$. Since $\widetilde{\phi^h}(p)$ is $*$ weak $\sigma(l_\infty, l_1)-$compact from lemma 10, so is $\widetilde{\psi}(p)$ as well from the continuity of vector operations. Since $\Delta \subset \Pi$ holds from the definition and $\widetilde{\psi}$ satisfies Walras' law on $\Pi$, $\widetilde{\psi}$ satisfies Walras's law on $\Delta$ as well. Thus, $\widetilde{\psi} : \Delta \to l_\infty$ satisfies $(i)', (iii)', (iv)'$ of theorem 2, and hence it is enough to show that $\widetilde{\psi}$ satisfies $(ii)''$.

Take a finite set of $\{p_1, \cdots, p_l\}$ from $\Delta$ and let $\Delta^l$ be $co\{p_1, \cdots, p_l\} = \Delta^l (\subset \Delta)$, the convex full of $\{p_1, \cdots, p_l\}$. It is enough to show that $\widetilde{\psi}$ is *u.h.c.* on $\Delta^l$ since $\widetilde{\psi}$ is homogeneous degree 0 in $p$. Note that the finite sum of compact-valued *u.h.c.* correspondences is also compact-valued *u.h.c.*. Then since $\widetilde{\psi}(p) = \sum_{h=1}^{H} \widetilde{\phi^h}(p) - \sum_{h=1}^{H} \omega^h$ holds and $\widetilde{\phi^h}(p)$ is $*$ weak $\sigma(l_\infty, l_1)-$compact for $p \in \Delta$, $\widetilde{\psi}$ is also *u.h.c.* for $p \in \Delta$ when so is $\widetilde{\phi^h}$. Thus it is enough to show that $\widetilde{\phi^h}$ is *u.h.c.* on $\Delta^l$. Then since $\widetilde{\phi^h}(\cdot) \subset B$ holds, from $(b) \to (a)$ in lemma 7, it is enough to show that the graph of $\widetilde{\phi^h}$ is product $\sigma(l_1, l_\infty) \times \sigma(l_\infty, l_1)-$closed in $\Delta^l \times \Psi$. Since $\Delta^l$ is a finite dimensional (compact) subset of $l_1$, both of the weak $\sigma(l_1, l_\infty)-$topology and the $||\cdot||_1-$norm topology on $\Delta^l$ is considered as equivalent to the Euclidean topology on $\Delta^l$, and hence it is unnecessary to distinguish the weak $\sigma(l_1, l_\infty)-$topology and the $||\cdot||_1-$norm topology on $\Delta^l$. Thus, it is enough to show that the graph of $\widetilde{\phi^h}$ is product $||\cdot||_1 \times \sigma(l_\infty, l_1)-$closed in $\Delta^l \times \Psi$. Since $B$ is $*$ weak $\sigma(l_\infty, l_1)-$compact, $p \cdot x$ is jointly continuous on $l_1 \times B$ with respect to product $||\cdot||_1 - \times\sigma(l_\infty, l_1)-$topology. [76] Moreover, since $B$ is $||\cdot||_\infty-$bounded and the $*$ weak $\sigma(l_\infty, l_1)-$topology on a $||\cdot||_\infty-$bounded subset is metrizable, the $*$ weak $\sigma(l_\infty, l_1)-$topology on $B$ is metrizable. Thus, the product $||\cdot||_1 - \times\sigma(l_\infty, l_1)-$topology on $(B \times \Delta^l)$ is characterized by sequences.

Let $(p^n, x^{hn}) \to_{||\cdot||_1 \times \sigma(l_\infty, l_1)} (\widehat{p}, \widehat{x^h})$ for $(p^n, x^{hn})_{n=1}^{\infty} \in Gr(\widetilde{\phi^h})(\subset (\Delta^l \times B))$. Since $\Delta^l$ is compact in the sense equivalent to the finite dimensional Euclidean topology, $\widehat{p} \in \Delta^l$ holds. $x^{hn} \in (l_\infty^+ \cap B)$ ℃ Since $(l_\infty^+ \cap B)$ is $*$ weak $\sigma(l_\infty, l_1)-$compact, it is also $*$ $\sigma(l_\infty, l_1)-$closed. Then, since $x^{hn} \to_{\sigma(l_\infty, l_1)} \widehat{x^h}$ holds for $x^{hn} \in (l_\infty^+ \cap B)$, $\widehat{x^h} \in (l_\infty^+ \cap B)$ follows. Also when $(p^n, x^{hn}) \to (\widehat{p^n}, \widehat{x^h})$ occurs, since $p^n \cdot x^{hn} \leqq p^n \cdot \omega^h$ holds and $p \cdot x$ is jointly continuous on $\Delta^l \times B(\supset (p^n, x^{hn})_{n=1}^{\infty})$ with respect to the product $||\cdot||_1 \times *$ weak $\sigma(l_\infty, l_1)-$topology, $\widehat{p} \cdot \widehat{x^h} \leqq \widehat{p} \cdot \omega^h$ holds and hence $\widehat{x^h} \in \widetilde{B^h}(\widehat{p})$ holds. Thus it is enough to show $z \in \widetilde{B^h}(\widehat{p}) \to \widehat{x^h} \in R^h(z)$ in order to establish $\widehat{x^h} \in \widetilde{\phi^h}(p^n)$. Suppose contrary that $z \in P^h(\widehat{x^h})$ occurs for some $z \in \widetilde{B^h}(\widehat{p})$. $\widehat{p} \cdot z \leqq \widehat{p} \cdot \omega^h$ holds from $z \in \widetilde{B^h}(\widehat{p})$. Since $\omega^h \in int_{||\cdot||_\infty}(l_\infty^+)$ is assumed, the free disposal assumption implies $-\omega^h \in int_{||\cdot||_\infty}(Y)$. Thus, $\widehat{p} \cdot (-\omega^h) < 0$ holds from $\widehat{p} \in Y^*$ and hence $\widehat{p} \cdot \omega^h > 0$ holds.[77] Then since $\widehat{p} \cdot \alpha z \leqq \widehat{p} \cdot \alpha\omega^h < \widehat{p} \cdot \omega^h$ holds from $\alpha z \in \widetilde{B^h}(\widehat{p})$ for $\alpha \in (0, 1)$, $p^n \to \widehat{p}$ implies that $p^n \cdot \alpha z < p^n \cdot \omega^h$ holds for $n$, sufficiently large. From the property of a demand point, $x^{hn} \in \widetilde{\phi^h}(p^n)$ implies $x^{hn} \in R^h(\alpha z)$. Moreover, $R^h(\alpha z)$ is $*$ weak $\sigma(l_\infty, l_1)-$closed from the continuity of the preferences. Thus $\widehat{x^h} \in R^h(\alpha z)$ holds from $x^{hn} \to \widehat{x^h}(n \to \infty)$and hence $\alpha z \in (R^h)^{-1}(\widehat{x^h})$ holds. Since $(R^h)^{-1}(\widehat{x^h})$ is $||\cdot||_\infty-$closed from the continuity of the preferences and $\alpha z \to_{||\cdot||_\infty} z(\alpha \to 1)$ holds, $z \in (R^h)^{-1}(\widehat{x^h})$ holds and

[76] See footnote 18.

[77] Lemma 3 uses $(E, G)$, on the other hand, Nikaido(1957b,59) use $(E, E^*)$ to show the result corresponding to Lemma 3. Here uses $E = l_\infty$ with $||\cdot||_\infty-$topology to get the result corresponding to the original ones in Nikaido(1957b,59).

hence $\widehat{x^h} \in R^h(z)$ holds. This is, however, a contradiction to $z \in P^h(\widehat{x^h})$. Thus, $\widehat{x^h} \in \widetilde{\phi^h}(p^n)$ holds. This implies that $Gr(\widetilde{\phi^h})$ is product $||\cdot||_1 \times \sigma(l_\infty, l_1)-$closed. Thus, $\widetilde{\phi^h}$ is *u.h.c.* on $\Delta^l$ and hence $\widetilde{\psi}$ is also *u.h.c.* on $\Delta^l$, which implies $(ii)''$. $\square$

Since $\widetilde{\psi} : \Delta \to l_\infty$ is shown to satisfy $(i)', (ii)'', (iii)', (iv)'$ of the theorem 2 from these arguments, it applies to $\widetilde{\psi}$ and its conclusion implies that the economy with $l_\infty$ as the commodity space has a competitive equaling. The following is the equilibrium existence theorem in this economy.

**Theorem 3(Bewley(1972))** Suppose that a production economy with $l_\infty$ as the commodity space and with a finite number of consumers satisfies the following conditions: (1) : the consumption set of consumer $h$ is non-negative orthant $l_\infty^+$ and his preference $R^h$ defined on $l_\infty^+$ satisfies the reflexivity, completeness, and transitivity. The initial endowment $\omega^h (\in l_\infty^+)$ of consumer $h$ satisfies $\omega^h \geqq \alpha^h e$ for some $\alpha^h > 0$. (2) : $R^h(x)$ is convex and $*$ weak $\sigma(l_\infty, l_1)-$closed for $x \in l_\infty^+$ and the lower inverse $(R^h)^{-1}(x)$ of $R^h$ at $x = \{y \in l_\infty^+ : x \in R^h(y)\}$ is $||\cdot||_\infty-$closed for $x \in l_\infty^+$, respectively. The preference $R^h$ satisfy the strict monotonicity in the sense that $x > y \to x \in P^h(y)$ and the convexity in the sense that $x' \in P^h(x)$ and $\alpha \in (0,1) \to \alpha x' + (1-\alpha)x) \in P^h(x)$. (3) : the aggregate production set $Y$ is a $*$ weak $\sigma(l_\infty, l_1)-$closed convex cone with the vertex at the origin 0 and satisfies the free disposal condition in the sense that $-(l_\infty^+) \subset Y$ holds. (4) : $(\omega + Y) \cap l_\infty^+$ is $||\cdot||_\infty-$bounded. Then this economy has a competitive equilibrium whose price belong to $ba^+ \backslash \{0\}$.[78]

**Proof.** The application of theorem 2 to $\widetilde{\psi}$ gives rise to $\exists(\overline{p}, \overline{x}) \in (cl_{(\sigma(ba,l_\infty)\times\sigma(l_\infty,l_1)}(Gr(\widetilde{\psi}))(\subset (\Pi \times \Psi))$ with $\overline{x} \in Y$. Note first that $\overline{p}$ does not necessarily belong to $\Delta$ and it is shown to belong to $\Pi$ only here. $\overline{x} \in Y$ and $\overline{p} \in \Pi \subset Y^*$ imply $\overline{p} \cdot \overline{x} \leqq 0$. Define a directed set $(N(0), \prec)$ by $(U,V) \prec (U',V') \iff (U,V) \subset (U',V')$ for a family of balances convex $*$ weak $\sigma(l_\infty, l_1)-$neighborhoods of 0, $(p^\alpha, x^\alpha) \to_{(\sigma(ba,l_\infty)\times\sigma(l_\infty,l_1)} (\overline{p}, \overline{x})(\alpha \uparrow)$ holds for a net $(p^\alpha, x^\alpha) \in Gr(\widetilde{\psi})(\subset (\Pi \times \Psi))$ from $(\overline{p}, \overline{x}) \in (cl_{(\sigma(ba,l_\infty)\times\sigma(l_\infty,l_1)}(Gr(\widetilde{\psi}))(\subset ((\Pi \times \Psi))$. $(p^\alpha, x^\alpha) \in Gr(\widetilde{\psi})$ implies $x^\alpha = \sum_{h=1}^H x^{h\alpha} \in \widetilde{\psi}(p^\alpha) = \displaystyle\sum_{h=1}^H \widetilde{\phi^h}(p) - \sum_{h=1}^H \omega^h$ and hence $z^{h\alpha} = x^{h\alpha} + \omega^h \in \widetilde{\phi^h}(p^\alpha)(\subset (B \cap l_\infty^+))$ holds $\forall h = 1, \cdots, H$. Since $(x^{h\alpha})_\alpha$ is also $||\cdot||_\infty-$bounded, there is a $*$ weak $\sigma(l_\infty, l_1)-$converging subnet such that $x^{h\alpha} \to_{\sigma(l_\infty,l_1)} \overline{x^h}(\alpha \uparrow)$ holds for $\overline{x^h} \in (B \cap l_\infty^+)$ $\forall h = 1, \cdots, H$. Then $x^\alpha = \sum_{h=1}^H x^{h\alpha} \to_{\sigma(l_\infty,l_1)} x = \sum_{h=1}^H \overline{x^h}(\alpha \uparrow)$. Since $x^\alpha = \sum_{h=1}^H x^{h\alpha} \to_{\sigma(l_\infty,l_1)} \overline{x}(\alpha \uparrow)$ holds, $x = \overline{x}$ holds from the uniqueness of the limit of the net. Then $\overline{x} = \sum_{h=1}^H \overline{x^h} \in Y$ follows. Let $\overline{z^h} = \overline{x^h} + \omega^h (\in l_\infty^+)$ $\forall h = 1, \cdots, H$. Since $\sum_{h=1}^H \overline{z^h} - \displaystyle\sum_{h=1}^H \omega^h = \sum_{h=1}^H \overline{x^h} = \overline{x} \in Y$ holds $\forall h = 1, \cdots, H$, $\widetilde{\overline{z}} = (\overline{z^h}) \in \widetilde{X}$ holds. At the same

[78] Bewley(1972, theorem 1) uses a general convex production economy with $L_\infty$ as the commodity space and with a finite number of consumers and firms to have an equilibrium price belongs to $ba \backslash \{0\}$. Then Bewley(1972, theorem 3) uses a convex cone production economy with $L_\infty$ as the commodity space to have an equilibrium price belongs to $L_1 \backslash \{0\}$. This theorem 3 here uses a production economy with $l_\infty$ as the commodity space and with a finite number of sonsumers and a convex aggregate production set as in Bewley(1972, theorem 3) although here uses $l_\infty$ as the commodity space instead of general $L_\infty$. Note, however, that it is quite straightforward to extend the theorem 3 here with $l_\infty$ to the one with general $L_\infty$ since the argmenents of $l_\infty$ case and of $L_\infty$ case are quite parallel.

time, $\overline{z^h} \in \widetilde{X^h}(\subset int_{||\cdot||_\infty}(B))$ holds $\forall h = 1, \cdots, H$ as well. Note that $\overline{z^h} \in int_{||\cdot||_\infty}(B)$ holds $\forall h = 1, \cdots, H$.

$(\overline{p}, \widetilde{\overline{z}})$ is shown first to be a competitive equilibrium in an economy with $(B \cap l_\infty^+)$ as the consumption set of the consumers. For this purpose, it is enough to show that $\overline{z^h} \in \widetilde{\phi^h}(\overline{p})$ holds and hence $\overline{z^h}$ is a demand point over $\widetilde{B^h}(\overline{p})$ $\forall h = 1, \cdots, H$ since the other two conditions follow trivially. Let $z' \in (P^h(\overline{z^h}) \cap B)$. Since $x^{h\alpha} \to_{\sigma(l_\infty, l_1)} \overline{x^h}(\alpha \uparrow)$ holds $z^{h\alpha} = x^{h\alpha} + \omega^h \to_{\sigma(l_\infty, l_1)} \overline{z^h} = \overline{x^h} + \omega^h(\alpha \uparrow)$ $\forall h = 1, \cdots, H$. Since $(P^h)^{-1}(z')(\ni \overline{z^h})$ is $^*$ weak$\sigma(l_\infty, l_1)-$open in $l_\infty^+$ from the continuity of the preferences, $(P^h)^{-1}(z')(\ni z^{h\alpha})$ and hence $z' \in P^h(z^{h\alpha})(\cap B)$ hold $\forall$ sufficiently large $\alpha$. Then $z^{h\alpha} \in \widetilde{\phi^h}(p^\alpha)$ and the definition of demand points imply $p^\alpha \cdot z' > p^\alpha \cdot \omega^h$ $\forall$ sufficiently large $\alpha$. This implies $\overline{p} \cdot z' \geqq \overline{p} \cdot \omega^h$ since $p^\alpha \to_{\sigma(ba, l_\infty)} \overline{p}(\alpha \uparrow)$ holds. Since $\overline{z^h} \in int_{||\cdot||_\infty}(B)$ holds, the strict monotonicity of the preferences implies $\overline{z^h} + \beta e \in (P^h(\overline{z^h}) \cap B)$ for $\beta(> 0)$ sufficiently close to 0. Then $\overline{p} \cdot (\overline{z^h} + \beta e) \geqq \overline{p} \cdot \omega^h$ holds for $\beta(> 0)$ sufficiently close to 0. Letting $\beta \to 0$ gives rise to $\overline{p} \cdot \overline{z^h} \geqq \overline{p} \cdot \omega^h$. Since this holds for $h = 1, \cdots, H$, $\overline{p} \cdot \sum_{h=1}^H (\overline{z^h} - \omega^h) \geqq 0$ holds. But $\overline{p} \cdot \sum_{h=1}^H (\overline{z^h} - \omega^h) = \overline{p} \cdot \sum_{h=1}^H \overline{x^h} = \overline{p} \cdot \overline{x} \leqq 0$ holds from $\overline{p} \in \Pi \subset Y^*$. Thus, $\overline{p} \cdot \sum_{h=1}^H \overline{x^h} = \overline{p} \cdot \sum_{h=1}^H (\overline{z^h} - \omega^h) = 0$ holds. Then $\overline{p} \cdot (\overline{z^h} - \omega^h) = 0$ and hence $\overline{p} \cdot \overline{z^h} = \overline{p} \cdot \omega^h$ holds $\forall h = 1, \cdots, H$. This implies $\overline{z^h} \in \widetilde{B^h}(\overline{p})$ $\forall h = 1, \cdots, H$ and hence $\overline{z^h}$ satisfies the budget constraint in the economy restricted over $B$. Suppose that $\overline{p} \cdot z' = \overline{p} \cdot \omega^h (> 0)$ holds for some $z' \in (P^h(\overline{z^h}) \cap B)$. Since the continuity of the preferences implies $P^h(\overline{z^h})$ is $||\cdot||_\infty-$open in$l_\infty^+$, $\beta z' \in (P^h(\overline{z^h}) \cap B)$ holds for $\beta \in (0, 1)$ sufficiently close to 1. Then $\overline{p} \cdot \beta z' = \beta \overline{p} \cdot z' = \beta \overline{p} \cdot \omega^h < \overline{p} \cdot \omega^h$ holds. But this is a contradiction to $z' \in (P^h(\overline{z^h}) \cap B) \Longrightarrow \overline{p} \cdot z' \geqq \overline{p} \cdot \omega^h$, which is already shown above. Thus, $z' \in (P^h(z^{h\alpha}) \cap B) \Longrightarrow \overline{p} \cdot z' > \overline{p} \cdot \omega^h = \overline{p} \cdot \overline{z^h}$ follows and hence $\overline{z^h} \in \widetilde{\phi^h}(\overline{p})$ holds for $h = 1, \cdots, H$. Note that $y \in Y$ and $\overline{p} \in \Delta \subset Y^*$ imply $\overline{p} \cdot y \leqq 0$. Thus $\overline{x}$ satisfies the aggregate profit maximization condition over $Y$ at $\overline{p}$ since $\overline{x} \in Y$ and $\overline{p} \cdot \overline{x} = \overline{p} \cdot \sum_{h=1}^H \overline{x^h} = 0$ hold. Therefore, $(\overline{p}, \widetilde{\overline{z}})$ is a competitive equilibrium in the economy with $(B \cap l_\infty^+)$ as the budget set of the consumers.

$(\overline{p}, \widetilde{\overline{z}})$ is shown indeed to be a competitive equilibrium in the original unrestricted economy with $l_\infty^+$ as the consumption set of the consumers. It is enough to show $\overline{z^h} \in \widetilde{\phi^h}(\overline{p}), h = 1, \cdots, H$ for this purpose. Since $\overline{z^h}$ is already shown to be a demand point on $\widetilde{B^h}(\overline{p})$, it is enough to show that $z' \in (P^h(\overline{z^h}) \cap B^c) \Longrightarrow \overline{p} \cdot z' > \overline{p} \cdot \omega^h = \overline{p} \cdot \overline{z^h}$ holds $\forall h = 1, \cdots, H$ for the purpose. Suppose that $\overline{p} \cdot z' \leqq \overline{p} \cdot \omega^h = \overline{p} \cdot \overline{z^h}$ occurs for some $z' \in (P^h(\overline{z^h}) \cap B^c)$. Consider $\beta z' + (1 - \beta)\overline{z^h}$ for $\beta \in (0, 1)$. Then $\overline{p} \cdot (\beta z' + (1 - \beta)\overline{z^h}) = \beta \overline{p} \cdot z' + (1 - \beta)\overline{p} \cdot \overline{z^h} \leqq \overline{p} \cdot \omega^h$ holds and hence $(\beta z' + (1 - \beta)\overline{z^h}) \in P^h(\overline{z^h})$ holds from the convexity of the preferences. Since, however, $\overline{z^h} \in int_{||\cdot||_\infty}(B)$ implies $(\beta' z' + (1 - \beta')\overline{z^h}) \in B$ for $\beta' \in (0, 1)$, sufficiently close to 0, $(\beta' z' + (1 - \beta')\overline{z^h}) \in (P^h(\overline{z^h}) \cap B)$ holds. Then $\overline{z^h} \in \widetilde{\phi^h}(\overline{p})$ implies $\overline{p} \cdot (\beta z' + (1 - \beta)\overline{z^h}) > \overline{p} \cdot \omega^h$, which is a contradiction. thus, $z' \in P^h(\overline{z^h}) \Longrightarrow \overline{p} \cdot z' > \overline{p} \cdot \omega^h = \overline{p} \cdot \overline{z^h}$ follows and $\overline{z^h} \in \phi^h(\overline{p})$ holds $\forall h = 1, \cdots, H$. Therefore $(\overline{p}, \widetilde{\overline{z}})$ is a competitive equilibrium in the original economy with $l_\infty^+$ as the consumption set of the consumers. $\square$

Thus the existence of a competitive equilibrium in an economy with $l_\infty$ as the commodity space is shown with the method applying the modified infinite dimensional Gale-Nikaido lemma as in the finite dimensional commodity case of Gale(1955) and Nikaido(1956). Although Nikaido(1956b,57b,59) establish several version of infinite dimensional Gale-Nikaido

lemmas in infinite dimensional commodity space economies, they do not give any example of an economy whose competitive equilibrium is shown with applying an infinite dimensional Gale-Nikaido lemma established there. Here gives such an example of an economy with $l_\infty$ as the commodity space. Florenzano(1983) uses a production economy with $l_\infty$ as the commodity space but its production set is in $-(l_\infty^+)$, the non-positive orthant of $l_\infty$. Here uses a production economy with $l_\infty$ as the commodity space and its production set contains $-(l_\infty^+)$ as the free disposal assumption.[79]

The above proof of theorem 3 follows the argument used in the classical finite dimensional case such of Gale(1955) and Nikaido(1956a). There is, however, a difference between the finite dimensional case and the infinite dimensional case treated above in a sense that in the latter case the excess demand correspondence is well-defined only on $\Delta$ so that it needs to be extended so as to be well-defined on the entire price set $\Pi(\supset \Delta)$.[80] Because of this reason, the original infinite dimensional Gale-Nikaido lemma in Nikaido(1957b, 59) can not apply directly to the above economy with $l_\infty$ as the commodity space to find a competitive equilibrium in such an economy and several modifications are required on the choice of linear topologies and the price set as in Florenzano(1983). The spirit of the proof of this modified version of infinite dimensional Gale-Nikaido lemma is, however, basically same as the one in the original version of infinite dimensional Gale-Nikaido lemma in Nikaido(1957b, 59).

Although an equilibrium price $\overline{p}$ in theorem 3 is shown to belongs only to $(\Pi \subset)ba^+\backslash\{0\}$, the argument from Bewley(1972) and Prescott-Lucas(1972) transform it as the one belonging only to $\Pi \in l_1^+\backslash\{0\}$. For this purpose, it is necessary to modify the $||\cdot||_\infty-$ closedness of the lower inverse $(R^h)^{-1}(x) = \{y \in l_\infty^+ : x \in R^h(y)\}$ at $x$, (2) in theorem 3, to its $*$ weak$\sigma(l_\infty, l_1)-$closedness and to add, to (3), $y \in Y \rightarrow \exists t' \in N, \forall y > t', y(t) \in Y$(the possibility of stop production at any time), where $y(t) = (y_1, y_2, \cdots, y_t, 0, 0, \cdots)$ for $y = (y_t) = (y_1, y_2, \cdots)$. Note that $\overline{p} = \overline{p}_c + \overline{p}_{pfa}$ holds when Yoshida-Hewitt decomposition theorem applies to $\overline{p}$, where $\overline{p}_c$ is its $l_1-$part and $\overline{p}_{pfa}$ is its $pfa-$part, respectively. Then, with these modified conditions, this $\overline{p}_c-$part of $\overline{p}$ is shown to be also an equilibrium price. Thus, an equilibrium price is found in $\Delta$. [81]

In the above example of the economy with $l_\infty$ as the commodity space, the positive $l_\infty^+$ contains $||\cdot||_\infty-$interior points and hence the aggregate production set $Y$ also contains such $||\cdot||_\infty-$interior points from the free disposal condition. This case, then, is considered as the case of Nikaido(1956b) where $||\cdot||_\infty-$topology is used as $\tau-$topology and the closed cone $P$ contains $||\cdot||_\infty-$interior points so that the infinite dimensional Gale-Nikaido lemma of Nikaido(1956b) is more appropriate to be used.[82] Thus, when Prof. Nikaido would consider

[79]Although here use several strong conditions such as $l_\infty^+$ as the consumption set and $\omega^h$ satisfying $\omega^h \in int_{||\cdot||_\infty}(l_\infty^+)$, these conditions are weaken easily to a general convex consumption set and to the irreducibility condition of McKenzie(1959, 81) and esource-relatedness condition of Arrow-Hahn(1971). Also for finding an equilibrium price in $l_1^+$ with the general convex consumption sets case, it is enough to put the exclusion condition to the consumption sets of Back(1984) besides the exclusion condition on production sets of Bewley(1972).

[80]Note that Nikaido(1957a) uses a concept of an extension of a correspondence following the one used in Kuhn(1956).

[81]See Bewley(1972 theorem 3) and Prescott-Lucas(1972) for the detail of this argument. See also Lucas-Stokey(1989 ch.16 and 17).

[82]Unfortunately, Nikaido(1956b) is unavailable nowadays and the content of this paper is only guessed from that of Nikaido(1957b).

an economy example such as the $l_\infty$ case to which Nikaido(1956b)'s infinite dimensional Gale-Nikaido lemma to show the existence of competitive equilibrium in such an economy, besides going to the direction to Nikaido(1957b,59) dispensing with this interiority condition of $P$, he might realize to modify his (1956b)'result such as in Florenzano(1983) as did here. Then for this purpose he might establish an infinite dimensional Gale-Nikaido lemma as of Florezano(1983) and this paper. If this would occur, Nikaido(1956b,57b,59) might be treated as important early attempts as of Debreu(1954) in the infinite dimensional commodity space literature. Of course, it is also quite interesting to find an economic example where the closed cone $P$ does not satisfy this interiority condition since this is a main reason why Nikaido(1956b) is generalized to Nikaido(1956b,59).

The proof the theorem done in Bewley(1972 theorem 1) uses the method of approximating a candidate for a competitive equilibrium of the original infinite dimensional economy by a sequence(net) of competitive equilibria in finite dimensional subeconomies, where the Arrow-Debreu-McKenzie existence theorem applies to each to these finite dimensional subeconomies to have one in these subeconomies. On the other hand, this paper, first establishes an infinite dimensional Gale-Nikaido lemma by applying the finite dimensional Gale-Nikaido lemma to finite dimensional subeconomies, and then this infinite dimensional Gale-Nikaido applies to the original infinite dimensional economy to establish the existence of a competitive equilibrium in the economy. In this sense, the approach here is indirect. Note that there is Fan-Glicksberg fixed point theorem(1952), which is a generalized version of Kakutani's fixed point theorem in locally convex topological vector spaces. The proof of this theorem uses the method of approximating a candidate for a fixed point of the original infinite dimensional space by a sequence(net) of fixed points in finite dimensional subeconomies, where Kakutani's fixed point theorem applies to each to these finite dimensional spaces to have ones in these subspace. It is quite interesting that all of the proofs in Fan-Glicksberg fixed point theorem, Bewley(1972)'s equilibrium existence theorem, Nikaido(1957b,59)'s infinite dimensional Gale-Nikaido lemmas use some kind of finite dimensional approximation of the original infinite dimensional space. That is, Kakutani's fixed point theorem(1941) in Fan-Glicksberg fixed point theorem(1952), Arrow-Debreu-McKenzie existence theorem(1954) in Bewley(1972)'s equilibrium existence theorem, and the dimensional Gale-Nikaido lemma(1955,56) in Nikaido's infinite dimensional Gale-Nikaido lemmas(1957b,59).[83] These common features are quite interesting from the viewpoint of mathematical structures.

[83]Note that in an infinite dimensional economy without ordered preferences, although the excess demand correspondence is not necessalily well-defined on the price set even in finite dimensional subecbomies as done in theorem 3 in the previous section so that Arrow-Debreu-McKenzie(1954) existence theorem can not apply to these subecnomies, instead, Gale-MacColl(1974)'s exsitence theorem or Schafer-Sonnnenschein(1975)'s existence theorem apply to these economy to have a competitive equilibrium in these subeconmies. Thus, as done in Zame(1987), the Bewley(1972)'s finite dimensional approximation method still applies to this case withou ordered preferences.

# 7 Conclusion

Although Nikaido(1956b,57b,59) do not gave any examples of an economy with an infinite dimensional commodity space, where the existence of competitive equilibrium in the economy is shown using an infinite dimensional Gale-Nikaido lemma, this paper gives as such an example an economy with $l_\infty$ as the commodity space. This economy with $l_\infty$ as the commodity space is picked up first in Debreu(1954) and is shown to have a competitive equilibrium in Bewley(1972). Since there is a slight difference in the choice of linear topologies used between Nikaido(1956b,57b,59) and Bewley(1972) in the economy with $l_\infty$ as the commodity space, the excess demand correspondence is not necessarily defined on the entire price set and hence it is extended over the entire price set, as explained before. From this viewpoint, when prof. Nikaido would try to construct an example of economy with an infinite dimensional commodity space to which an infinite dimensional Gale-Nikaido lemma apply to establish the existence of a competitive equilibrium in the economy, he might recognize that the original version of his infinite dimensional Gale-Nikaido lemma need to be modified such as in Florenzano(1983). Since Nikaido(1957b) already used the extension of correspondences, he would soon try to modify the original version of his infinite dimensional Gale-Nikaido lemma appropriately in applying it to the infinite dimensional commodity space economy he might use in late 1950's or early 1960's. Then since he might contribute as to this aspect to the issue on the existence of a competitive equilibrium in an infinite dimensional commodity space, he might be considered as the first person who established the existence of a competitive equilibrium in an economy with an infinite dimensional commodity space as a precursor to Peleg-Yarri(1970) and Bewley(1972) in the literature.

Of course, although Nikaido(1956b,57b,59) did not construct any example of infinite dimensional economies to which an infinite dimensional Gale-Nikaido lemma apply to establish the existence of a competitive equilibrium in these economies, the contributions done by Nikaido (1956b,57b,59) on the infinite dimensional Gale-Nikaido lemma are still quite impressive in a sense that these consider several economies with infinite dimensional commodity spaces explicitly and establish several versions of infinite dimensional Gale-Nikaido lemma just after original Gale(1955) and Nikaido(1956a) establish the one in an finite dimensional economy. [84]

This paper picks up $l_\infty$ as an economic example to which an infinite dimensional Gale-Nikaido lemma applies to show the existence of a competitive equilibrium in this economy. Although $l_\infty$ satisfies $int_{||\cdot||_\infty}(l_\infty^+) \neq \emptyset$, which is a case that Nikaido(1956b) supposes, the main purpose of Nikaido(1957b,59) is to establish an infinite dimensional Gale-Nikaido lemma in the case where this kind of the interiority condition does not hold as in that with the (uniform) properness condition of Mas-Colell(1986). Thus the next task is to find an economic example without the interiority condition where an infinite dimensional Gale-Nikaido lemma holds and is used to show the existence of a competitive equilibrium in this economy.

[84] After 1980's, the infinite dimensional Gale-Nikaido lemma is further generalized by Yannelis(1985), Mehta-Tarafdar(1987), and Urai(2000, 2010).

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