博士学位論文

Differential systems associated with partial differential equations of one and more unknown functions

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1 Introduction

In this thesis we will treat with differential systems and exterior differential systems associated with second order partial differential equations. A differential system D on a manifold M is a subbundle of the tangent bundle TM of M and an exterior differential system $\mathcal{I}$ on a manifold Σ is a differential ideal of the algebra of all differential forms on Σ. These topics are roughly divided into two parts: partial differential equations of one and more unknown functions.

In the former part we will study Monge-Ampère equations

(1.0.1) $$Az_{xx} + 2Bz_{xy} + Cz_{yy} + D + E(z_{xx}z_{yy} - z_{xy}^2) = 0,$$

where the capital letters denote functions of variables x, y, z, z_x, z_y. Monge-Ampère equations are described from viewpoints of differential system and exterior differential system. These equations (generally, single second order partial differential equations $F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0$ of one unknown function with two independent variables) are expressed as differential systems (R, D); (R, D) is defined as the restriction of the canonical system E of the Lagrange-Grassmann bundle $L(J)$ over a 5-dimensional contact manifold J, which is a geometric 2-jet bundle introduced by K. Yamaguchi ([Yam82]), to the hypersurface R of $L(J)$ defined by the given equation. On the other hand, these equations are expressed as certain exterior differential systems $\mathcal{I}$ on a 5-dimensional contact manifold J, which is called Monge-Ampère systems. One of the specialty of Monge-Ampère equations is that they can be considered on manifolds of less dimension than the original. Indeed we can integrate solutions of Monge-Ampère equations in 5 variable space (for example, see [For06], [Mon07]). Both Monge-Ampère systems and differential systems derived from Monge-Ampère equations have Monge characteristic systems $\mathcal{M}$, which are differential systems of rank 2. These characteristic systems are utilized for the above integral method, called Darboux's method (see also [IL03]). We will study the properties and relations of Monge characteristic systems of Monge-Ampère systems and differential systems, and determine the condition for that a partial differential equation is a Monge-Ampère equation by using these characteristic systems in Section 3.2. Note that the characterization of Monge-Ampère equation was studied by R. Gardner and N. Kamran in terms of differential invariants ([GK93]). Moreover, a generalization of hyperbolic Monge-Ampère systems is considered, which is called a hyperbolic exterior differential system ([BGH95a]) and a hyperbolic differential system is also defined. Their Monge characteristic systems is defined as a generalization of those of Monge-Ampère equations. We will generalize the results on Monge-Ampère equations and obtain a reduction theorem in Section 4. Namely, it is known that the prolongation of hyperbolic exterior differential systems and differential systems are hyperbolic differential systems ([BGH95a]). Conversely, given a hyperbolic differential system, we will construct a hyperbolic differential system or exterior differential system on a manifold of smaller dimension whose prolongation coincides with the given system under some conditions.

In the latter part we will study partial differential equations of m (≥ 2) unknown functions. According to Realization Lemma, which is established by N. Tanaka ([Yam82]), any differential system corresponds to a system of differential equations of first order (Section 2.2). Therefore partial differential equations of second order would be characterized as a structure of differential systems with some conditions.
K. Yamaguchi characterized second and higher order partial differential equations of one unknown functions in terms of differential systems, where this geometric structure is called a PD-manifold ([Yam82]). In contrast, we will characterize second order partial differential equations of m (≥ 2) unknown functions in Section 5. We will give an example of PD-manifolds of finite type and show that it is of irreducible type (I, S), which is introduced by Y. Se-ashi ([Sa88]). Finally, from the viewpoint of parabolic geometry ([YY07]), we will seek PD-manifolds associated with a simple graded Lie algebra of type (X_l, Δ_1), but see that there do not exist such PD-manifolds.

Now let us describe the contents of each sections. In Section 2 we recall definitions of differential systems and various systems, and the jet space $(J(M,n),C)$ of first order.

Differential systems play an important role with Realization Lemma in whole of this thesis, which is stated in Section 2.2. We describe a Monge-Ampère equation in terms of differential systems in Section 3.1.1 and exterior differential systems in Section 3.1.2, and define their Monge characteristic systems. We divide into the hyperbolic (Section 3.2.1) and parabolic cases (Section 3.2.2), and calculate derived systems of Monge characteristic systems and illustrate relations of their Monge characteristic systems (Theorem 3.3 and Corollary 3.4 in hyperbolic case and Theorem 3.11 in parabolic case). In Section 3.3, utilizing results in the previous section, we state the condition for that a given single second order partial differential equation is Monge-Ampère equation in terms of Monge characteristic systems (Theorem 3.12 and Theorem 3.17). Furthermore we consider hyperbolic differential systems and exterior differential systems in Section 4. Section 4.2 describes relations of Monge characteristic systems (Theorem 4.3 and 4.4) as in Section 3.2. In Section 4.3 we state reduction theorems for hyperbolic differential systems, which is converse to the prolongation theorem in [BGH95a]. Precisely, given a hyperbolic differential system, we will determine the condition for that there exists a hyperbolic exterior differential system or differential system whose prolongation coincides with the given system (Theorem 4.5 and 4.6). In Section 5.1 we recall definitions and notions of jet space $J^2(M,n)$ of second order, symbol algebras of differential systems and graded simple Lie algebras. In Section 5.2 we characterize second order partial differential equations of m (≥ 2) unknown functions in terms of differential systems (Theorem 5.3), called PD-manifolds of second order. We show an example of a PD-manifold of finite type in Section 5.3. In Section 5.4 we seek PD-manifolds of type (X_l, Δ_1) and determine a model equation for classical type.

Throughout this thesis we always assume the differentiability of class C^∞.

2 Preliminaries

In this section we recall the definitions of various differential systems and Realization Lemma, which are used in the whole of this thesis.

2.1 Differential systems and various systems

A *differential system* D or (M, D) is a subbundle of the tangent bundle TM of a manifold M. A differential system D is locally defined by linearly independent 1-forms $\varpi^1, \ldots, \varpi^r$ as follows:

$$D = \left\{ \varpi^1 = \cdots = \varpi^r = 0 \right\},$$

where r is the codimension of D. v is an *integral element* of the differential system D at a point $x \in M$ if v is a subspace of T_xM such that $\varpi^a|_v = 0$ and $d\varpi^a|_v = 0$ for all $1 \le a \le r$. An *integral manifold* of the differential system D is a submanifold $\iota : N \longrightarrow M$ such that $\iota^*\varpi^a = 0$ for all $1 \le a \le r$. A function f on M is a *first integral* of D if $df \equiv 0 \pmod{D^\perp}$, where $D^\perp$ is the annihilater subbundle of T^*M defined by

$$D^\perp(x) = \{\,\omega \in T_x^*M \mid \omega(X) = 0 \text{ for } X \in T_xM\,\} \quad \text{for } x \in M.$$

The *k-th derived system* $\partial^k D$ is defined inductively as follows: If $\partial^{k-1}D$ is a differential system, then

$$\partial^k \mathcal{D} = \partial^{k-1}\mathcal{D} + [\partial^{k-1}\mathcal{D}, \partial^{k-1}\mathcal{D}]$$

where $\partial^k\mathcal{D}$ is the space of sections of $\partial^k D$ and $[\ ,\ ]$ is Lie bracket for vector fields, and we put $\partial^0 D = D$ for convention. Precisely, $\partial^k D$ is defined in terms of sheaves (see [Yam82]). When ∂D coincides with D, D is said to be *completely integrable*.

The *k-th weak derived system* $\partial^{(k)}(D)$ of D is defined inductively by

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

where $\partial^{(0)}D = D$ and $\partial^{(k)}\mathcal{D}$ is the space of sections of $\partial^{(k)}D$. Let $D^{-(k+1)} = \partial^{(k)}D$ for $k \ge 0$. Note that $D^{-2} = \partial^{(1)}D = \partial^1 D$. A differential system (M, D) is *regular* if D^{-k} is a differential system on M for all $k \ge 2$. For a regular differential system (M, D), it is known that ([Tan70, Proposition 1.1], [Yam09, Section 2.4])

1. There exists a unique integer $\mu > 0$ such that

$$D = D^{-1} \subsetneq D^{-2} \subsetneq \cdots \subsetneq D^{-\mu+1} \subsetneq D^{-\mu} = \cdots = D^k$$

for all $k \ge \mu$,

2. $[\mathcal{D}^{-p}, \mathcal{D}^{-q}] \subset \mathcal{D}^{-(p+q)}$ for all $p, q > 0$,

where $\mathcal{D}^{-p}$ is the space of sections of D^{-p}. Note that $D^{-\mu}$ is the smallest completely integrable differential system that contains D.

The *Cauchy characteristic system* $\mathrm{Ch}(D)$ of D is defined by

$$\mathrm{Ch}(D)(x) = \left\{ X \in D(x) \mid X \lrcorner\, d\varpi^a \equiv 0 \pmod{\varpi_x^1, \ldots, \varpi_x^r} \text{ for } 1 \le a \le r \right\}$$

at each point $x \in R$. If $\mathrm{Ch}(D)$ is a differential system, it is the largest completely integrable system contained by D. Let $p : R \longrightarrow M$ be a differentiable map between smooth manifolds R and M and assume p is of constant rank. Let C be a differential system on M. Differential systems $p_*^{-1}(C)$ and $\mathrm{Ker}\,(p_*)$ are defined as follows:

$$p_*^{-1}(C)(x) = \{\, X \in T_xR \mid p_*(X) \in C(p(x)) \,\},$$
$$\mathrm{Ker}\,(p_*)(x) = \{\, X \in T_xR \mid p_*(X) = 0 \,\},$$

for $x \in R$. Note that $\mathrm{Ker}\,(p_*)$ is completely integrable. Let $\rho : P \longrightarrow Q$ be a submersion between smooth manifolds P and Q. We say that a differential system D on P *drops down* to Q if there exists a differential system C' on Q such that $\rho_*^{-1}(C') = D$.

2.2 Jet space $(J(M,n),C)$ of first order and Realization Lemma

Let M be a manifold of dimension $m+n$. Let denote $J(M,n)$ be the Grassmann bundle over M. Namely each fiber $J(M,n)_x$ over $x \in M$ is the Grassmannian $Gr(T_xM,n)$ of n-dimensional subspace of T_xM:

$$J(M,n) = \bigcup_{x\in M} J(M,n)_x \xrightarrow{\ \Pi\ } M$$

where Π is the canonical projection of $J(M,n)$ onto M. *The canonical system* C on $J(M,n)$, which is a differential system of codimension n, is defined by

$$C(u) = \Pi_*^{-1}(u) \quad \text{for } u \in J(M,n)$$

where the right hand side means the inverse image of the n-dimensional subspace u of $T_{\Pi(u)}M$ under the differential of Π at u.

Next we will give a canonical coordinate system (or inhomogeneous Grassmann coordinate) $(x^1,\dots,x^n,z^1,\dots,z^m,p_1^1,\dots,p_n^m)$ of $J(M,n)$. Let us fix a point u_o of $J(M,n)$. Let $(x^1,\dots,x^n,z^1,\dots,z^m)$ be a coordinate system on a neighborhood U of $\Pi(u_o)$ such that $dx^1,\dots,dx^n$ are linearly independent on u_o. Let $\hat{U}$ be the set of all elements $u \in \Pi^{-1}(U)$ such that $dx^1 \wedge \cdots \wedge dx^n\,|_u \neq 0$, which is a neighborhood of u_o. We take functions p_i^a for $1 \le a \le m$ and $1 \le i \le n$ on $\hat{U}$ so that $dz^b|_u - \sum_i p_i^b(u)\, dx^i|_u = 0$ for $u \in \hat{U}$ and $1 \le b \le m$. Thus we have achieved the coordinate system $(x^1,\dots,x^n,z^1,\dots,z^m,p_1^1,\dots,p_n^m)$ on $\hat{U}$. It follows that the canonical system C restricted on $\hat{U}$ is defined by the 1-forms

$$\varpi^a = dz^a - \sum_{i=1}^{n} p_i^a dx^i \quad \text{for } 1 \le a \le m \tag{2.2.1}$$

and the Cauchy characteristic system of C is trivial, i.e. $\mathrm{Ch}(C) = \{0\}$.

Through this thesis we utilize *Realization Lemma* ([Yam82]):

Realization Lemma. *Let R and M be manifolds and a map $p : R \longrightarrow M$. Let D be a differential system on R. Assume that p is of constant rank and $F = \mathrm{Ker}\, p_*$ is a subbundle of D of codimension n. Then there exists a unique map $\psi : R \longrightarrow J(M,n)$ satisfying $p = \Pi \circ \psi$ and $D = \psi_*^{-1}(C)$. Indeed ψ is defined by*

$$\psi(x) = p_*((D(x))) \quad \text{for } x \in R \tag{2.2.2}$$

and satisfies

$$\mathrm{Ker}\,(\psi_*)_x = F(x) \cap \mathrm{Ch}(D)(x).$$

Here, the right hand side of (2.2.2) *means the image of the differential p_* of the subspace $D(x)$ of T_xR, which is considered as a point of $J(M,n)$.*

This Lemma also says "any differential system is considered as a system of differential equations of first order." In fact, for a given differential system (M,D), let us choose p as the identity map $\mathrm{id} : M \longrightarrow M$. Then $\psi : M \longrightarrow J(M,n)$ is defined as (2.2.2), where $n = \mathrm{rank}\, D$, and we have $\mathrm{Ker}\,(\psi_*) = \{0\}$. Therefore M is immersed into $J(M,n)$ and $\psi_*^{-1}(C) = D$.

3 Monge-Ampère equations

3.1 Preliminaries

In this section we will recall definitions and notations of Lagrange-Grassmann bundle $L(J)$, differential systems associated with single second order partial differential equations, exterior differential systems and Monge-Ampère systems.

3.1.1 Lagrange-Grassmann bundle over contact manifolds and single second order partial differential equations

We will recall the definition of Lagrange-Grassmann bundle $(L(J),E)$ over a contact manifold (J,C) in order to treat with second order partial differential equations of one unknown function geometrically ([Yam82]).

Let (J,C) be a differential system of codimension 1, which implies that, for a point $u \in J$, there exists a 1-form θ around u such that $C = \{\theta = 0\}$ locally. Then (J,C) is called a *contact manifold* if $\theta \wedge (d\theta)^n$ is a volume form on J.

Let (J,C) be a $(2n+1)$-dimensional contact manifold. We now construct *Lagrange-Grassmann bundle* $L(J)$ over J and the *canonical system* E on $L(J)$ as follows: let $L(J)$ be the space consisting of all n-dimensional integral elements of C, namely

$$L(J) = \bigcup_{x \in J} L(J)_x \xrightarrow{\ \Pi\ } J$$

where $L(J)_x$ is the Grassmannian of all Lagrangian (or Legendrian) subspaces of the symplectic vector space $(C(x), d\theta_x)$ and Π is the canonical projection. The canonical system E on $L(J)$ is defined by

$$E(u) = \Pi_*^{-1}(u) \subset T_u(L(J)) \quad \text{for } u \in L(J).$$

We now take a coordinate system of $L(J)$ as follows: let us fix a point $u_o \in L(J)$. By Darboux's Theorem, there exists a canonical coordinate system $(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ on a neighborhood U of $\Pi(u_o) \in J$ such that $\theta = dz - \sum_{i=1}^n p_i dx^i$ and $dx^1 \wedge \cdots \wedge dx^n|_{u_o} \neq 0$. Let $\hat{U}$ be a neighborhood of u_o that consists of all points $u \in \Pi^{-1}(U)$ such that $dx^1 \wedge \cdots \wedge dx^n|_u \neq 0$. Let p_{ij} for $1 \leq i, j \leq n$ be functions on $\hat{U}$ such that $dp_i|_u - \sum_k p_{ik}(u)\, dx^k|_u = 0$ for all $u \in \hat{U}$ and $1 \leq i \leq n$. Since $d\theta|_u = 0$, we have $p_{ij} = p_{ji}$. Thus we have obtained the coordinate system $(x^i, z, p_i, p_{ij}\ (1 \leq i \leq j \leq n))$ on $\hat{U}$, is called the canonical coordinate system of $L(J)$. Then E is locally defined by

$$E = \{\, \varpi_0 = \varpi_1 = \cdots = \varpi_n = 0 \,\}$$

where $\varpi_0 = dz - \sum_{i=k}^n p_k dx^k$ and $\varpi_i = dp_i - \sum_{k=1}^n p_{ik} dx^k$ for $1 \leq i \leq n$.

Let us consider a single second order partial differential equation of one unknown function with two independent variables

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0. \tag{3.1.1}$$

Assume that the partial derivatives $\frac{\partial F}{\partial z_{xx}}, \frac{\partial F}{\partial z_{xy}}, \frac{\partial F}{\partial z_{yy}}$ of the function F with respect to z_{xx}, z_{xy}, z_{yy} are never simultaneously zero at each point. If we regard Equation (3.1.1) as a submanifold

$$R = \{\, F(x, y, z, p, q, r, s, t) = 0 \,\}$$

of $L(J)$ over 5-dimensional standard contact manifold $J = \mathbb{R}^5$ with coordinates (x, y, z, p, q, r, s, t), the equation provides the differential system (R, D) that is the restriction D of the canonical system E of $L(J)$ to R. The assumption on F then implies that the restriction of the projection $\Pi : L(J) \longrightarrow J$ to R is submersion. Generally, let J be a contact manifold of dimension 5 and (R, D) a differential system on a hypersurface R of $L(J)$ defined by the restriction of the canonical system E of $L(J)$ to R. Let $\rho : R \longrightarrow J$ denotes the restriction of $\Pi : L(J) \longrightarrow J$ and assume that ρ is submersion. If we write R as

$$R = \{ F(x, y, z, p, q, r, s, t) = 0 \}$$

with the canonical coordinate system (x, y, z, p, q, r, s, t) of $L(J)$, a 2-dimensional integral manifold of D transverse to fibers of ρ is the graph of a solution of a single second order partial differential equation $F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial x}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial y}) = 0$. We will also call such (R, D) a single second order partial differential equation in what follows.

Let (R, D) be a single second order partial differential equation and assume $\rho : R \longrightarrow J$ is submersion.

It is well-known that the structure equation of D is expressed as follows: let us fix a point $v_o \in R$. If the equation R is hyperbolic around v_o, the structure equation is

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\text{mod } \varpi_0), \\ d\varpi_1 \equiv \omega^1 \wedge \pi_{11} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \qquad\qquad \omega^2 \wedge \pi_{22} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

where $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{11}, \pi_{22}\}$ is a coframe around $v_o \in R$ ([BCG+91, p.277]). If the equation R is parabolic around v_o,

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\text{mod } \varpi_0), \\ d\varpi_1 \equiv \qquad\qquad \omega^2 \wedge \pi_{12} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \omega^1 \wedge \pi_{12} + \omega^2 \wedge \pi_{22} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

where $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{12}, \pi_{22}\}$ is a coframe around $v_o \in R$ ([BCG+91, p.275]).

Then, if R is hyperbolic or parabolic, the Monge characteristic system $\mathcal{M}_i$ of (R, D) are defined as

$$\mathcal{M}_i = \left\{ \varpi_0 = \varpi_1 = \varpi_2 = \omega^i = \pi_{ii} = 0 \right\} \quad \text{for } i = 1, 2$$

or

$$\mathcal{M} = \left\{ \varpi_0 = \varpi_1 = \varpi_2 = \omega^2 = \pi_{12} = 0 \right\},$$

respectively ([IL03, p.213]). Note that they are invariant under diffeomorphisms of R preserving D.

3.1.2 Exterior differential systems and Monge-Ampère systems

An *exterior differential system* on a manifold Σ is a differential ideal $\mathcal{I}$ on Σ, namely an algebraic ideal of the differential algebra of differential forms on Σ closed under exterior differentiation. Let $\mathcal{I} = \{\psi^1, \ldots, \psi^n\}_{\rm diff}$ denote an exterior differential system algebraically generated by differential forms $\psi^1, \ldots, \psi^n$ and their derivatives $d\psi^1, \ldots, d\psi^n$. Especially, we say that an exterior differential system is *Pfaffian* if the system is generated algebraically by 1-forms and those exterior derivatives. A differential system corresponds to a Pfaffian system.

Let $\mathcal{I}$ be an exterior differential system on Σ. For a point $p \in \Sigma$, an *integral element* v of $\mathcal{I}$ at p is a subspace v of $T_p\Sigma$ such that $\psi|_v = 0$ for all $\psi \in \mathcal{I}$. An *integral manifold* of an exterior differential system $\mathcal{I}$ on Σ is an immersed submanifold $\iota : M \hookrightarrow \Sigma$ such that $\iota^*\psi = 0$ for all $\psi \in \mathcal{I}$.

For a (classical) Monge-Ampère equation in coordinates description

$$Az_{xx} + 2Bz_{xy} + Cz_{yy} + D + E(z_{xx}z_{yy} - z_{xy}^2) = 0, \tag{3.1.2}$$

where the capital letters denote functions of variables x, y, z, z_x, z_y, we consider the following exterior differential system

$$\mathcal{I} = \{\,\theta, \Psi\,\}_{\rm diff},$$

where $\theta = dz - pdx - qdy$ and

$$\Psi = Adp \wedge dy + B(dq \wedge dy - dp \wedge dx) - Cdq \wedge dx + Ddx \wedge dy + Edp \wedge dq, \tag{3.1.3}$$

on the standard contact manifold $J = \mathbb{R}^5$ with the standard coordinate system (x, y, z, p, q). Then a 2-dimensional integral manifold of $\mathcal{I}$ on which $dx \wedge dy$ never vanishes is locally the graph of a solution of the Monge-Ampère equation (3.1.2).

Let J be a 5-dimensional contact manifold with contact form θ and Ψ a 2-form on J and suppose $\Psi \not\equiv 0 \pmod{\theta, d\theta}$. Then the exterior differential system

$$\mathcal{I} = \{\,\theta, \Psi\,\}_{\rm diff}$$

is called *a Monge-Ampère system* on J. By Darboux's Theorem, there exists a coordinate system (x, y, z, p, q) of J such that $\theta = dz - pdx - qdy$ and (3.1.3) holds. A 2-dimensional integral manifold of a Monge-Ampère system on which $dx \wedge dy$ never vanishes is the graph of a solution of a Monge-Ampère equation (3.1.2). For a point $u \in J$, $\mathcal{I}$ is called *hyperbolic*, *parabolic* or *elliptic* at u if $\mathcal{I}_u$ has two, one or no decomposable 2-covector, modulo θ_u, respectively. Since J is of dimension 5, a 2-covector $(\Psi + \lambda d\theta)_u$ is decomposable, modulo θ_u, if and only if $(\Psi + \lambda d\theta)_u^2 \equiv 0 \pmod{\theta_u}$. Then the relation

$$(3.1.4) \qquad (\Psi + \lambda d\theta)_u^2 = \Psi_u \wedge \Psi_u + 2\lambda \Psi_u \wedge d\theta_u + \lambda^2 d\theta_u \wedge d\theta_u \equiv 0 \qquad \pmod{\theta_u}$$

yields a quadratic equation in the variable λ. Because a root of the quadric equation satisfies (3.1.4), $\mathcal{I}$ is hyperbolic, parabolic or elliptic at $u \in J$ if the quadratic equation has two, one or no real roots, respectively. If $\mathcal{I}$ has a decomposable 2-form $\omega \wedge \pi$, modulo θ, then a *Monge characteristic system* $\mathcal{M}$ of $\mathcal{I}$ is defined as

$$\mathcal{H} = \{\, \theta = \omega = \pi = 0 \,\},$$

which is a differential system of rank 2 on J. Note that they are invariant under diffeomorphisms of J preserving $\mathcal{I}$.

Finally, we will see that, given a Monge-Ampère system $\mathcal{I}$, Monge-Ampère equation (R, D) is obtained as the *prolongation* of $\mathcal{I}$, and describe relations between the Monge characteristic systems of $\mathcal{I}$ and (R, D).

Let $\mathcal{I} = \{\theta, \Psi\}_{\text{diff}}$ be a Monge-Ampère system on J and let $L(J)$ be the Lagrange-Grassmann bundle over J. We obtain the prolongation (R, D) of $\mathcal{I}$ as follows: Let R be the set of all 2-dimensional integral elements of $\mathcal{I}$, which is a subsheaf of $J(J, 2)$ generally, where $J(J, 2)$ means the jet space of first order over J. Assuming R is a smooth manifold, we can define the differential system on R as the restriction of the canonical system $(J(J, 2), C)$ to R. Then (R, D) is called the *prolongation* of $\mathcal{I}$ (cf. [BCG+91], [IL03]).

Let us fix a point $v_o \in L(J)$ and take a coframe $\{\theta, \omega^1, \omega^2, \pi_1, \pi_2\}$ around $u_o = \pi(v_o)$ such that

$$d\theta \equiv \omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2 \qquad \pmod{\theta}.$$

We may assume $\omega^1 \wedge \omega^2 |_{v_o} \neq 0$. Let V be a neighborhood of v_o such that $\omega^1 \wedge \omega^2 |_v \neq 0$ for all $v \in V$. Then we can take fiber coordinate functions a, b, c on V such that

$$\begin{aligned} \pi_1|_v &= a(v)\, \omega^1|_v + b(v)\, \omega^2|_v, \\ \pi_2|_v &= b(v)\, \omega^1|_v + c(v)\, \omega^2|_v, \end{aligned}$$

for $v \in V$. Writing

(3.1.5)
$$\Psi = A\,\pi_1 \wedge \omega^2 + B\,(\pi_2 \wedge \omega^2 - \pi_1 \wedge \omega^1) - C\,\pi_2 \wedge \omega^1 + D\,\omega^1 \wedge \omega^2 + E\,\pi_1 \wedge \pi_2,$$

where the capital letters denote functions around u_o, we have

$$\Psi|_v = (Aa + 2Bb + Cc + D + E(ac - b^2))(v)\,\omega^1 \wedge \omega^2\,|_v.$$

Thus we have

$$\begin{aligned} R &= \{\, v \in V \mid \Psi|_v = 0 \,\} \\ &= \left\{ Aa + 2Bb + Cc + D + E(ac - b^2) = 0 \right\}, \end{aligned} \tag{3.1.6}$$

which is a subvariety of $L(J)$. Around each regular points of R, we may define D as the restriction of E to R. In this thesis we call the prolongation (R, D) of $\mathcal{I}$ the *corresponding* Monge-Ampère equation. In fact, as mentioned above, for a given Monge-Ampère system $\mathcal{I} = \{\theta, \Psi\}_{\text{diff}}$, we can take a coordinate system (x, y, z, p, q) such that $\theta = dz - pdx - qdy$ and set $\omega^1 = dx$, $\omega^2 = dy$, $\pi_1 = dp$, $\pi_2 = dq$, and then we set given Ψ as in Equation (3.1.5). Therefore we obtain the coordinate description (3.1.6) of the Monge-Ampère equation R.

Let $\mathcal{I}$ be a Monge-Ampère system and (R, D) the corresponding Monge-Ampère equation. Let $\mathcal{H}$ be a Monge characteristic system of $\mathcal{I}$. In the next section we will show that there exists a Monge characteristic system $\mathcal{M}$ of (R, D) such that

$$\mathcal{M} \subset \rho_*^{-1}(\mathcal{H}).$$

We call $\mathcal{M}$ the *corresponding* Monge characteristic system of (R, D).

3.2 Properties and relations of Monge characteristic systems of Monge-Ampère systems and equations

We will investigate relations between the Monge characteristic systems of Monge-Ampère systems and those of the corresponding Monge-Ampère equations by describing these structure equations in hyperbolic and parabolic cases individually. This observation will be utilized for the characterization of Monge-Ampère equations in Section 3.3.

3.2.1 Hyperbolic case

First, we will choose a coframe adapted for a Monge-Ampère system. Let $\mathcal{I} = \{\theta, \Psi\}_{\text{diff}}$ be a Monge-Ampère system and let (R, D) denote the corresponding Monge-Ampère equation. Let us fix a point $v_o \in R$. Assume $\mathcal{I}$ is hyperbolic around $u_o = \pi(v_o)$. Then we can take different functions λ_1 and λ_2 around u_o so that $\Psi + \lambda_1 d\theta$ and $\Psi + \lambda_2 d\theta$ are decomposable 2-forms, and hence take 1-forms $\omega^1, \omega^2, \pi_1', \pi_2'$ around u_o such that

$\omega^i \wedge \pi'_i \equiv \Psi + \lambda_i d\theta \pmod{\theta}$ for $i = 1, 2$. Since v_o is an integral element of $\mathcal{I}$, we have $\omega^1 \wedge \pi'_1 |_{v_o} = \omega^2 \wedge \pi'_2 |_{v_o} = 0$. Hence we may assume $\omega^1|_{v_o} \neq 0$ and $\omega^2|_{v_o} \neq 0$. Then $\pi_1|_{v_o}$ is a multiple of $\omega^1|_{v_o}$ and $\pi_2|_{v_o}$ is of $\omega^2|_{v_o}$. Since $\omega^1 \wedge \pi'_1 - \omega^2 \wedge \pi'_2 \equiv (\lambda_1 - \lambda_2)\, d\theta$ $\pmod{\theta}$ and $\lambda_1 - \lambda_2 \neq 0$, we have

$$d\theta \equiv \omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2 \qquad \pmod{\theta}$$

where $\pi_1 = \frac{1}{\lambda_1 - \lambda_2}\, \pi'_1$, $\pi_2 = -\frac{1}{\lambda_1 - \lambda_2}\, \pi'_2$. Since θ is a contact form, $\theta \wedge \omega^1 \wedge \omega^2 \wedge \pi_1 \wedge \pi_2 \neq 0$ around u_o. Hence $\{\theta, \omega^1, \omega^2, \pi_1, \pi_2\}$ is a coframe around u_o.

Next, we will choose a coframe adapted for (R, D). Let us take a neighborhood V of v_o such that $\omega^1 \wedge \omega^2 |_v \neq 0$ at each $v \in V$ and functions a, b, c on V such that

$$\begin{aligned} \pi_1|_v &= a(v)\, \omega^1|_v + b(v)\, \omega^2|_v, \\ \pi_2|_v &= b(v)\, \omega^1|_v + c(v)\, \omega^2|_v, \end{aligned}$$

for $v \in V$. Since $\omega^1 \wedge \pi_1 |_v = 0$, we have $b(v) = 0$. Thus

$$D = \{\, \varpi_0 = \varpi_1 = \varpi_2 = 0 \,\},$$

where $\varpi_0 = \rho^*\theta$, $\varpi_1 = \rho^*\pi_1 - a\rho^*\omega^1$, $\varpi_2 = \rho^*\pi_2 - c\rho^*\omega^2$.

For $i = 1, 2$, we can write

$$\begin{aligned} d\pi_i &\equiv \pi_1 \wedge (A_i\, \pi_2 + B_i\, \omega^1 + C_i\, \omega^2) + \pi_2 \wedge (E_i\, \omega^1 + F_i\, \omega^2) + G_i\, \omega^1 \wedge \omega^2, \\ d\omega^i &\equiv \pi_1 \wedge (H_i\, \pi_2 + I_i\, \omega^1 + J_i\, \omega^2) + \pi_2 \wedge (K_i\, \omega^1 + L_i\, \omega^2) + N_i\, \omega^1 \wedge \omega^2, \end{aligned} \tag{3.2.7}$$

modulo θ, where each capital letter with an additional character indicates smooth functions around u_o on J. Let us omit the pullback ρ^* in what follows. Then we have

$$\left.\begin{aligned} d\pi_1 - a d\omega^1 &\equiv \Gamma_1\, \omega^1 \wedge \omega^2 \\ d\pi_2 - c d\omega^2 &\equiv \Gamma_2\, \omega^1 \wedge \omega^2 \end{aligned}\right. \qquad \pmod{\varpi_0, \varpi_1, \varpi_2},$$

where $\Gamma_1 = A_1ac + C_1a - E_1c + G_1 - H_1a^2c - J_1a^2 + K_1ac - N_1a$, $\Gamma_2 = A_2ac + C_2a - E_2c + G_2 - H_2ac^2 - J_2ac + K_2c^2 - N_2c$. Therefore we obtain the following structure equation:

$$\left\{\begin{aligned} d\varpi_0 &\equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 && \pmod{\varpi_0}, \\ d\varpi_1 &\equiv \omega^1 \wedge \pi_{11} && \pmod{\varpi_0, \varpi_1, \varpi_2}, \\ d\varpi_2 &\equiv \qquad\qquad\quad \omega^2 \wedge \pi_{22} && \pmod{\varpi_0, \varpi_1, \varpi_2}, \end{aligned}\right. \tag{3.2.8}$$

where $\pi_{11} = da + \Gamma_1\, \omega^2$, $\pi_{22} = dc - \Gamma_2\, \omega^1$.

Lemma 3.1.

(3.2.9) $$\mathcal{M}_i \subset \rho_*^{-1}(\mathcal{H}_i) \quad \text{and} \quad \partial\mathcal{M}_i \subset \rho_*^{-1}(\mathcal{H}_i) \quad \text{for } i = 1, 2.$$

Proof. As we use the coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{11}, \pi_{22}\}$ taken above,

$$\begin{aligned} \mathcal{M}_i &= \left\{ \varpi_0 = \varpi_1 = \varpi_2 = \omega^i = \pi_{ii} = 0 \right\}, \\ \rho_*^{-1}(\mathcal{H}_i) &= \left\{ \rho^*\theta = \rho^*\omega^i = \rho^*\pi_i = 0 \right\} \\ &= \left\{ \varpi_0 = \varpi_i = \omega^i = 0 \right\}. \end{aligned}$$

By (3.2.7) and (3.2.8), we have $d\varpi_0 \equiv d\varpi_i \equiv d\omega^i \equiv 0 \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^i, \pi_{ii}}$. Thus

$$\partial\mathcal{M}_i \subset \left\{ \varpi_0 = \varpi_i = \omega^i = 0 \right\} = \rho_*^{-1}(\mathcal{H}_i).$$

□

For a Monge characteristic system $\mathcal{H}_i$, the system $\mathcal{M}_i$ satisfying (3.2.9) is called the *corresponding* Monge characteristic system.

Corollary 3.2. *If $\mathcal{H}_i$ has two independent first integrals, then $\mathcal{M}_i$ also has at least two.*

Here, "independent" means independence as function, namely there exists two first integrals f_1, f_2 of $\mathcal{H}_i$ such that $df_1 \wedge df_2 \neq 0$.

Though we obtain this corollary from the structure equation (3.2.8), to obtain more information, we need to analyze the structure equation in more detail:

Theorem 3.3. *Let $\mathcal{I}$ be a hyperbolic Monge-Ampère system on a 5-dimensional contact manifold J and let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the Monge characteristic systems of $\mathcal{I}$, and let (R, D) denote the corresponding Monge-Ampère equation and $\mathcal{M}_1$ and $\mathcal{M}_2$ the corresponding Monge characteristic systems respectively. Then, for $i = 1, 2$, $\partial\mathcal{M}_i$, $\partial^2\mathcal{M}_i$ and $\partial\mathcal{H}_i$ are differential systems, and satisfy that* $\operatorname{codim} \partial^2\mathcal{M}_i = 3$ *and*

$$\partial^2\mathcal{M}_i \subset \rho_*^{-1}(\partial\mathcal{H}_i).$$

Proof. Let us choose the coframe $\{\theta, \omega^1, \omega^2, \pi_1, \pi_2\}$ and $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{11}, \pi_{22}\}$ taken above. It follows from (3.2.7) that

$$\left.\begin{aligned} d\pi_i \equiv A_i\,\varpi_1 \wedge \varpi_2 &+ \varpi_1 \wedge (B_i\,\omega^1 + (A_i c + C_i)\,\omega^2) \\ &+ \varpi_2 \wedge ((-A_i a + E_i)\,\omega^1 + F_i\,\omega^2) \\ &+ (A_i ac + C_i a - E_i c + G_i)\,\omega^1 \wedge \omega^2 \\ d\omega^i \equiv H_i\,\varpi_1 \wedge \varpi_2 &+ \varpi_1 \wedge (I_i\,\omega^1 + (H_i c + J_i)\,\omega^2) \\ &+ \varpi_2 \wedge ((-H_i a + K_i)\,\omega^1 + L_i\,\omega^2) \\ &+ (H_i ac + J_i a - K_i c + N_i)\,\omega^1 \wedge \omega^2 \end{aligned}\right\} \pmod{\varpi_0}$$

and hence

$$\begin{aligned} d\varpi_1 \equiv \omega^1 \wedge \pi_{11} &+ \varpi_1 \wedge ((B_1 - I_1 a)\,\omega^1 + (A_1 c + C_1 - H_1 ac - J_1 a)\,\omega^2) \\ &+ \varpi_2 \wedge ((-A_1 a + E_1 + H_1 a^2 - K_1 a)\,\omega^1 + (F_1 - L_1 a)\,\omega^2) \\ &+ (A_1 - H_1 a)\,\varpi_1 \wedge \varpi_2 \qquad (\mathrm{mod}\ \varpi_0), \\ d\varpi_2 \equiv \omega^2 \wedge \pi_{22} &+ \varpi_1 \wedge ((B_2 - I_2 c)\,\omega^1 + (A_2 c + C_2 - H_2 c^2 - J_2 c)\,\omega^2) \\ &+ \varpi_2 \wedge ((-A_2 a + E_2 + H_2 ac - K_2 c)\,\omega^1 + (F_2 - L_2 c)\,\omega^2) \\ &+ (A_2 - H_2 c)\,\varpi_1 \wedge \varpi_2 \qquad (\mathrm{mod}\ \varpi_0), \end{aligned}$$

where

$$\begin{aligned} \pi_{11} &= da + (A_1 ac + C_1 a - E_1 c + G_1 - H_1 a^2 c - J_1 a^2 + K_1 ac - N_1 a)\,\omega^2, \\ \pi_{22} &= dc - (A_2 ac + C_2 a - E_2 c + G_2 - H_2 ac^2 - J_2 ac + K_2 c^2 - N_2 c)\,\omega^1. \end{aligned}$$

By definition, one Monge characteristic system is

$$\mathcal{H}_1 = \left\{ \theta = \omega^1 = \pi_1 = 0 \right\}.$$

Since the structure equation of $\mathcal{H}_1$ is

$$\begin{cases} d\theta \equiv \omega^2 \wedge \pi_2 \\ d\omega^1 \equiv -L_1\,\omega^2 \wedge \pi_2 \\ d\pi_1 \equiv -F_1\,\omega^2 \wedge \pi_2 \end{cases} \qquad (\mathrm{mod}\ \theta, \omega^1, \pi_1),$$

the first derived system of $\mathcal{H}_1$ is

$$\partial\mathcal{H}_1 = \left\{ \widetilde{\omega}^1 = \widetilde{\pi}_1 = 0 \right\},$$

where $\widetilde{\omega}^1 = \omega^1 + L_1\,\theta$, $\widetilde{\pi}_1 = \pi_1 + F_1\,\theta$, and hence $\partial\mathcal{H}_1$ is a differential system on J.

On the other hand, let us recall the corresponding Monge characteristic system

$$\mathcal{M}_1 = \left\{ \varpi_0 = \varpi_1 = \varpi_2 = \omega^1 = \pi_{11} = 0 \right\}.$$

Since the structure equation of $\mathcal{M}_1$ is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv 0 \\ d\varpi_2 \equiv \omega^2 \wedge \pi_{22} \\ d\omega^1 \equiv 0 \\ d\pi_{11} \equiv -(A_1 a - E_1 - H_1 a^2 + K_1 a)\,\omega^2 \wedge \pi_{22} \end{cases} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1, \pi_{11}),$$

the first derived system of $\mathcal{M}_1$ is

$$\partial\mathcal{M}_1 = \left\{ \varpi_0 = \varpi_1 = \omega^1 = \overline{\pi}_{11} = 0 \right\},$$

where $\overline{\pi}_{11} = \pi_{11} + (A_1 a - E_1 - H_1 a^2 + K_1 a)\,\varpi_2$, and hence $\partial\mathcal{M}_1$ is a differential system on R. Since

$$\begin{cases} d\varpi_0 \equiv \omega^2 \wedge \varpi_2 \\ d\varpi_1 \equiv -(F_1 - L_1 a)\,\omega^2 \wedge \varpi_2 \\ d\omega^1 \equiv -L_1\,\omega^2 \wedge \varpi_2 \end{cases} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1), \tag{3.2.10}$$

the second derived system of $\mathcal{M}_1$ is

$$\partial^2\mathcal{M}_1 \subset \left\{ \widehat{\varpi}_1 = \widehat{\omega}^1 = 0 \right\}, \tag{3.2.11}$$

where $\widehat{\varpi}_1 = \varpi_1 + (F_1 - L_1 a)\,\varpi_0$, $\widehat{\omega}^1 = \omega^1 + L_1\varpi_0$. We have

$$\begin{aligned} \rho^*(\widetilde{\omega}^1) &= \omega^1 + L_1\,\varpi_0 = \widehat{\omega}^1, \\ \rho^*(\widetilde{\pi}_1) &= \varpi_1 + a\,\omega^1 + F_1\,\varpi_0 = \widehat{\varpi}_1 + a\,\widehat{\omega}^1, \end{aligned} \tag{3.2.12}$$

and hence $\partial^2\mathcal{M}_1$ satisfies the inclusion

$$\partial^2\mathcal{M}_1 \subset \rho_*^{-1}(\partial\mathcal{H}_1) = \left\{ \rho^*\widetilde{\omega}^1 = \rho^*\widetilde{\pi}_1 = 0 \right\}. \tag{3.2.13}$$

Furthermore, since

$$d\overline{\pi}_{11} \equiv (acdA_1 + adC_1 - cdE_1 + dG_1 - a^2cdH_1 - a^2dJ_1 + acdK_1 - adM_1) \wedge \omega^2$$

modulo $\varpi_0, \varpi_1, \omega^1, \overline{\pi}_{11}, \omega^2 \wedge \varpi_2$, and

$$\begin{aligned} dA_1 \wedge \omega^2 &\equiv dC_1 \wedge \omega^2 \equiv dE_1 \wedge \omega^2 \equiv dG_1 \wedge \omega^2 \\ &\equiv dH_1 \wedge \omega^2 \equiv dJ_1 \wedge \omega^2 \equiv dK_1 \wedge \omega^2 \equiv dM_1 \wedge \omega^2 \equiv 0 \end{aligned}$$

modulo $\varpi_0, \varpi_1, \varpi_2, \omega^1, \overline{\pi}_{11}$, we have

$$d\overline{\pi}_{11} \equiv 0 \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1, \overline{\pi}_{11}, \omega^2 \wedge \varpi_2).$$

Thus $\partial^2\mathcal{M}_1$ is a differential system and $\operatorname{codim}\partial^2\mathcal{M}_1 = 3$.

Similarly, we can prove the claims in the case of $\mathcal{H}_2$ and $\mathcal{M}_2$. □

The following corollary is a key of characterization of Monge-Ampère equation (see Theorem 3.12)

Corollary 3.4.

$$\rho_*^{-1}(\mathcal{H}_i) = \partial\mathcal{M}_i + \mathrm{Ch}(\partial D) \quad \textit{for } i = 1, 2.$$

From (3.2.10), (3.2.11), (3.2.12) and (3.2.13), we obtain the following corollary:

Corollary 3.5. *If $\mathcal{M}_i$ has three independent first integrals, then $\mathcal{H}_i$ also has two.*

Remark 3.6. As it is seen in Corollary 3.2, if $\mathcal{H}_i$ has two independent first integrals, then $\mathcal{M}_i$ also has at least two. However, it is not always true that $\mathcal{H}_i$ also has two independent first integrals if $\mathcal{M}_i$ has two independent first integrals. For example, let us consider the hyperbolic Monge-Ampère equation ([Boo59], [Gou90], [For06])

$$r - t - \frac{np}{x} = 0,$$

where n is an integer. The Monge-Ampère system is

$$\left\{ \theta = dz - pdx - qdy, \Psi = dp \wedge dy + dq \wedge dx - \frac{np}{x} dx \wedge dy \right\}_{\mathrm{diff}}$$

and decomposable 2-forms are

$$\Psi \pm d\theta = \left(dp \mp dq - \frac{np}{x} dx \right) \wedge (dy \mp dx).$$

Then we have

$$d\theta = \omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2,$$

where $\omega_1 = \frac{1}{2}(dx - dy)$, $\omega_2 = \frac{1}{2}(dy + dx)$, $\pi_1 = dp - dq - \frac{np}{x} dx$, $\pi_2 = dq + dp - \frac{np}{x} dx$.

We obtain the derived systems $\partial^k\mathcal{H}_i$ for each $i = 1, 2$ as follows: Since the structure equation of $\mathcal{H}_1 = \{\theta = \omega^1 = \pi_1 = 0\}$ is

$$\begin{cases} d\theta \equiv \omega^2 \wedge \pi_2 \\ d\omega^1 = 0 \\ d\pi_1 \equiv \frac{n}{2x} \omega^2 \wedge \pi_2 \end{cases} \quad (\mathrm{mod}\ \theta, \omega^1, \pi_1),$$

the first derived system is

$$\partial\mathcal{H}_1 = \left\{ \omega^1 = \pi_1' = 0 \right\},$$

where $\pi_1' = \pi_1 - \frac{n}{2x}\theta$. Since the structure equation of $\partial\mathcal{H}_1$ is

$$\begin{cases} d\omega^1 = 0 \\ d\pi_1' \equiv \frac{n(n+2)}{4x^2} \omega^2 \wedge \theta \end{cases} \quad (\mathrm{mod}\ \omega^1, \pi_1'),$$

$\partial\mathcal{H}_1$ is completely integrable if and only if $n = 0$ or -2.

On the other hand, let us recall the corresponding Monge characteristic system

$$\mathcal{M}_1 = \left\{ \varpi_0 = \varpi_1 = \varpi_2 = \omega^1 = \pi_{11} = 0 \right\},$$

where $\varpi_0 = \rho^*\theta$, $\varpi_1 = \rho^*\pi_1 - a\rho^*\omega^1$, $\varpi_2 = \rho^*\pi_2 - c\rho^*\omega^2$ and let us omit the pullback ρ^* in what follows. Then we have

$$\begin{aligned} d\varpi_0 &= \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2, \\ d\varpi_1 &= \omega^1 \wedge \pi_{11} - \frac{n}{2x}\,\varpi_1 \wedge (\omega^1 + \omega^2) - \frac{n}{2x}\,\varpi_2 \wedge (\omega^1 + \omega^2), \\ d\varpi_2 &= \omega^2 \wedge \pi_{22} - \frac{n}{2x}\,\varpi_1 \wedge (\omega^1 + \omega^2) - \frac{n}{2x}\,\varpi_2 \wedge (\omega^1 + \omega^2), \end{aligned}$$

where $\pi_{11} = da - \frac{n(a-c)}{2x}\,\omega^2$, $\pi_{22} = dc + \frac{n(a-c)}{2x}\,\omega^1$. Since the structure equation of $\mathcal{M}_1$ is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv 0 \\ d\varpi_2 \equiv \omega^2 \wedge \pi_{22} \\ d\omega^1 = 0 \\ d\pi_{11} \equiv -\frac{n}{2x}\,\omega^2 \wedge \pi_{22} \end{cases} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1, \pi_{11}),$$

the first derived system is

$$\partial\mathcal{M}_1 = \left\{ \varpi_0 = \varpi_1 = \omega^1 = \widetilde{\pi}_{11} = 0 \right\},$$

where $\widetilde{\pi}_{11} = \pi_{11} + \frac{n}{2x}\,\varpi_2$. Since the structure equation of $\partial\mathcal{M}_1$ is

$$\begin{cases} d\varpi_0 \equiv \omega^2 \wedge \varpi_{22} \\ d\varpi_1 \equiv \frac{n}{2x}\,\omega^2 \wedge \varpi_2 \\ d\omega^1 = 0 \\ d\widetilde{\pi}_{11} \equiv -\frac{n}{2x^2}\,\omega^2 \wedge \varpi_2 \end{cases} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1, \widetilde{\pi}_{11}),$$

the second derived system is

$$\partial^2\mathcal{M}_1 = \left\{ \varpi_1' = \omega^1 = \pi_{11}' = 0 \right\},$$

where $\varpi_1' = \varpi_1 - \frac{n}{2x}\,\varpi_0$, $\pi_{11}' = \widetilde{\pi}_{11} + \frac{n}{2x^2}\,\varpi_0 = \pi_{11} + \frac{n}{2x^2}\,\varpi_0 + \frac{n}{2x}\,\varpi_2$. Since the structure equation of $\partial^2\mathcal{M}_1$ is

$$\text{(3.2.14)} \qquad \begin{cases} d\varpi_1' \equiv \frac{n(n+2)}{4x^2}\,\omega^2 \wedge \varpi_0 \\ d\omega^1 = 0 \\ d\pi_{11}' \equiv \frac{n(n+2)(n-4)}{8x^3}\,\omega^2 \wedge \varpi_0 \end{cases} \qquad (\mathrm{mod}\ \varpi_1', \omega^1, \pi_{11}'),$$

$\partial^2\mathcal{M}_1$ is completely integrable if and only if $n = -2$ or 0.

Let us continue the calculation except for the case of $n = -2$ or 0. Equation (3.2.14) implies

$$\partial^3\mathcal{M}_1 = \left\{ \omega^1 = \widehat{\pi}_{11} = 0 \right\},$$

where $\widehat{\pi}_{11} = \pi'_{11} - \frac{n-4}{2x}\,\varpi'_1 = \pi_{11} + \frac{n(n-2)}{4x^2}\,\varpi_0 - \frac{n-4}{2x}\,\varpi_1 + \frac{n}{2x}\,\varpi_2$. Since we have

$$d\widehat{\pi}_{11} \equiv -\frac{n(n+4)(n-2)}{8x^3}\,\omega^2 \wedge \varpi_0 + \frac{(n+4)(n-2)}{4x^2}\,\omega^2 \wedge \varpi_1 \qquad (\mathrm{mod}\ \omega^1),$$

$\partial^3\mathcal{M}_1$ is completely integrable if and only if $n = -4$ or 2. In the other cases, $\partial^4\mathcal{M}_1 = \{\omega^1 = 0\}$.

The case of $\mathcal{H}_2$ and $\mathcal{M}_2$ are as follows: $\partial\mathcal{H}_2 = \{\omega^2 = \pi'_2 = 0\}$, where $\pi'_2 = \pi_2 - \frac{n}{2x}\,\theta$, and

$$d\pi'_2 = \frac{n(n+2)}{4x^2}\,\omega^1 \wedge \theta.$$

On the other hand, we can obtain

$$\partial^2\mathcal{M}_2 = \left\{ \varpi'_2 = \omega^2 = \pi'_{22} = 0 \right\},$$

where $\varpi'_2 = \varpi_2 - \frac{n}{2x}\,\varpi_0$, $\pi'_{22} = \pi_{22} + \frac{n}{2x}\,\varpi_1 + \frac{n}{2x^2}\,\varpi_0$, and

$$\begin{aligned} d\varpi'_2 &\equiv \tfrac{n(n+2)}{4x^2}\,\omega^1 \wedge \varpi_0 \\ d\pi'_{22} &\equiv \tfrac{n(n+2)(n-4)}{8x^3}\,\omega^1 \wedge \varpi_0 \end{aligned} \qquad (\mathrm{mod}\ \varpi'_2, \omega^2, \pi'_{22}).$$

If $n \neq -2$ and 0, we have

$$\partial^3\mathcal{M}_2 = \left\{ \omega^2 = \overline{\pi}_{22} = 0 \right\},$$

where $\overline{\pi}_{22} = \pi'_{22} - \frac{n-4}{2x}\,\varpi'_2 = \pi_{22} + \frac{n(n-2)}{4x^2}\,\varpi_0 + \frac{n}{2x}\,\varpi_1 - \frac{n-4}{2x}\,\varpi_2$. Then

$$d\overline{\pi}_{22} \equiv -\frac{n(n+4)(n-2)}{8x^3}\,\omega^1 \wedge \varpi_0 + \frac{(n+4)(n-2)}{4x^2}\,\omega^1 \wedge \varpi_2 \qquad (\mathrm{mod}\ \omega^2).$$

For $i = 1, 2$, we have obtained

Table 1: The Number of Independent First Integrals of Each Monge Characteristic System

n	the number of independent first integrals of $\mathcal{M}_i$	the number of independent first integrals of $\mathcal{H}_i$
$-2, 0$	3	2
$-4, 2$	2	1
the others	1	1

3.2.2 Parabolic case

First, we choose a coframe adapted for a Monge-Ampère system: Let $\mathcal{I} = \{\theta, \Psi\}_{\text{diff}}$ be a Monge-Ampère system and let (R, D) denote the corresponding Monge-Ampère equation. Let us fix a point $v_o \in R$. Assuming that $\mathcal{I}$ is a parabolic system around $u_o = \pi(v_o)$, we can take a function λ around u_o such that $\Psi + \lambda d\theta$ is a decomposable 2-form. Hence we may suppose that $\Psi = \omega \wedge \pi$ is a decomposable 2-form. By definition, since the quadratic equation in a variable λ given by

$$(\Psi + \lambda d\theta)^2 = 2\lambda\Psi \wedge d\theta + \lambda^2 d\theta \wedge d\theta = 0$$

has the multiple root $\lambda = 0$, we have

$$\Psi \wedge d\theta = \omega \wedge \pi \wedge d\theta = 0.$$

This implies

$$d\theta \equiv \omega^1 \wedge \pi + \omega \wedge \pi_2 \qquad (\text{mod } \theta),$$

where ω^1 and π_2 are 1-forms around u_o. Because θ is a contact form, $\theta \wedge \omega^1 \wedge \pi \wedge \omega \wedge \pi_2 \neq 0$. Hence $\{\theta, \omega^1, \omega, \pi, \pi_2\}$ is a coframe around u_o. If $\omega|_{v_o}$ and $\pi|_{v_o}$ are simultaneously never zero, we may assume $\omega|_{v_o} \neq 0$. Since $d\theta|_{v_o} = 0$, it follows that $\omega^1 \wedge \omega\,|_{v_o}$ must be non-zero.

Namely, we may suppose $\omega^1 \wedge \omega\,|_{v_o} \neq 0$ except for the case that both $\omega|_{v_o}$ and $\pi|_{v_o}$ vanish (see Remark 3.8 below).

Secondly, let us take a neighborhood V of v_o such that $\omega^1 \wedge \omega\,|_v \neq 0$ at each $v \in V$. Since $\Psi|_v = 0$ for any $v \in V$, we can take fiber coordinates a, b, c on V such that

$$\pi|_v = a(v)\,\omega^1|_v + b(v)\,\omega|_v,$$
$$\pi_2|_v = b(v)\,\omega^1|_v + c(v)\,\omega|_v,$$

for $v \in V$. Since $\omega \wedge \pi\,|_v = 0$, we have $a(v) = 0$. Thus

$$D = \{\, \varpi_0 = \varpi_1 = \varpi_2 = 0 \,\},$$

where $\varpi_0 = \rho^*\theta$, $\varpi_1 = \rho^*\pi - b\rho^*\omega$, $\varpi_2 = \rho^*\pi_2 - b\rho^*\omega^1 - c\rho^*\omega$ and let us omit the pullback ρ^* in what follows.

Putting

$$\begin{aligned} d\pi &\equiv \pi \wedge (A\,\pi_2 + B\,\omega^1 + C\,\omega) + \pi_2 \wedge (E\,\omega^1 + F\,\omega) + G\,\omega^1 \wedge \omega \\ d\omega &\equiv \pi \wedge (H\,\pi_2 + I\,\omega^1 + J\,\omega) + \pi_2 \wedge (K\,\omega^1 + L\,\omega) + N\,\omega^1 \wedge \omega \end{aligned} \qquad (\mathrm{mod}\ \theta),$$

where each capital letter indicates smooth functions on J, we have

$$d\pi - b d\omega \equiv -(Ab^2 + Bb + Ec - Fb - G - Hb^3 - Ib^2 - Kbc + Lb^2 + Nb)\,\omega^1 \wedge \omega,$$

modulo ϖ_0, ϖ_1, ϖ_2. Hence we obtain the structure equation:

Lemma 3.7.

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \pi + \omega \wedge \varpi_2 & (\mathrm{mod}\ \varpi_0) \\ d\varpi_1 \equiv \omega \wedge \pi_{12} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2) \\ d\varpi_2 \equiv \omega^1 \wedge \pi_{12} + \omega \wedge \pi_{22} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2) \end{cases}$$

where $\pi_{12} = db + (Ab^2 + Bb + Ec - Fb - G - Hb^3 - Ib^2 - Kbc + Lb^2 + Nb)\,\omega^1$.

Remark 3.8. If both $\omega|_{v_o}$ and $\pi|_{v_o}$ vanish, it must satisfy $\omega^1 \wedge \pi_2\,|_{v_o} \neq 0$. We consider a neighborhood V of v_o such that $\omega^1 \wedge \pi_2\,|_v \neq 0$ at each $v \in V$.

$$D = \{\, \varpi_0 = \varpi_1 = \varpi_2 = 0 \,\},$$

where $\varpi_0 = \theta$, $\varpi_1 = \pi - a\,\omega^1 - b\,\pi_2$, $\varpi_2 = \omega - b\,\omega^1 - c\,\pi_2$. Since $\omega \wedge \pi\,|_v = 0$ for all $v \in V$, $R \cap V = \{ac - b^2 = 0\}$ and hence v_o is a singular point of $R \cap V$. Thus we omit a point v_o such that both $\omega|_{v_o}$ and $\pi|_{v_o}$ vanish.

Lemma 3.9.

$$\mathcal{M} \subset \rho_*^{-1}(\mathcal{H}).$$

Proof. As we use the coframe taken above,

$$\begin{aligned} \mathcal{M} &= \{\, \varpi_0 = \varpi_1 = \varpi_2 = \omega = \pi_{12} \,\}, \\ \rho_*^{-1}(\mathcal{H}) &= \{\, \theta = \omega = \pi = 0 \,\} \\ &= \{\, \varpi_0 = \varpi_1 = \omega = 0 \,\}, \end{aligned}$$

and hence our assertion follows. □

Corollary 3.10. *If $\mathcal{H}$ has two independent first integrals, then $\mathcal{M}$ also has at least two.*

In the same way as in the case of hyperbolic system, let us analyze the structure equation in more detail:

Theorem 3.11. *Let $\mathcal{I}$ be a parabolic Monge-Ampère system on a 5-dimensional contact manifold J and let (R, D) denote the corresponding Monge-Ampère equation. Then it follows that*

$$\rho_*^{-1}(\mathcal{H}) = \partial(\mathcal{M} + \mathrm{Ch}(\partial D)) \tag{3.2.15}$$

and the Monge characteristic system $\mathcal{H}$ of $\mathcal{I}$ is completely integrable if and only if the Monge characteristic $\mathcal{M}$ of (R, D) is completely integrable.

Moreover, if $\mathcal{M}$ does not coincide with $\partial\mathcal{M}$, and $\partial\mathcal{M}$ is a differential system on R, then it follows that

$$\partial^2\mathcal{M} = \rho_*^{-1}(\mathcal{H}).$$

Proof. Let us choose a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega, \pi_{12}, \pi_{22}\}$ taken above. By definition,

$$\mathcal{H} = \{\, \theta = \omega = \pi = 0 \,\}.$$

Since

$$\begin{cases} d\theta \equiv 0 \\ d\omega \equiv -E\,\omega^1 \wedge \pi_2 \\ d\pi \equiv -K\,\omega^1 \wedge \pi_2 \end{cases} \qquad (\mathrm{mod}\ \theta, \omega, \pi),$$

$\mathcal{H}$ is completely integrable if and only if E and K vanish locally.

On the other hand, let us start with the Monge characteristic system

$$\mathcal{M} = \{\, \varpi_0 = \varpi_1 = \varpi_2 = \omega = \pi_{12} = 0 \,\}.$$

of (R, D). Since

$$\begin{aligned} dA \wedge \omega^1 &\equiv dB \wedge \omega^1 \equiv dF \wedge \omega^1 \equiv dE \wedge \omega^1 \equiv dG \wedge \omega^1 \equiv dH \wedge \omega^1 \\ &\equiv dI \wedge \omega^1 \equiv dL \wedge \omega^1 \equiv dK \wedge \omega^1 \equiv dM \wedge \omega^1 \equiv db \wedge \omega^1 \equiv 0, \end{aligned}$$

modulo $\varpi_0, \varpi_1, \varpi_2, \omega^1, \overline{\pi}_{11}$, we have

$$d\pi_{12} \equiv (Kb - E)\,\omega^1 \wedge \pi_{22} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega, \pi_{12}).$$

Since $d\varpi_0 \equiv d\varpi_1 \equiv d\varpi_2 \equiv d\omega \equiv 0 \pmod{\varpi_0, \varpi_1, \varpi_2, \omega, \pi_{12}}$ and b is one of the fiber coordinates, $\mathcal{M}$ is completely integrable if and only if E and K vanish locally. Hence second assertion follows.

Since $d\varpi_0 \equiv d\varpi_1 \equiv d\omega \equiv 0$ and $d\varpi_2 \equiv \omega^1 \wedge \pi_{12} \pmod{\varpi_0, \varpi_1, \varpi_2, \omega}$, first assertion follows.

Moreover, let us suppose that $\mathcal{M}$ does not coincide with $\partial\mathcal{M}$ and $\partial\mathcal{M}$ is a differential system on R. Then

$$\partial\mathcal{M} = \{\, \varpi_0 = \varpi_1 = \varpi_2 = \omega = 0 \,\}.$$

Since the structure equation of $\partial\mathcal{M}$ is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv 0 \\ d\varpi_2 \equiv \omega^1 \wedge \pi_{12} \\ \ \ d\omega \equiv 0 \end{cases} \qquad \pmod{\varpi_0, \varpi_1, \varpi_2, \omega},$$

we have

$$\partial^2\mathcal{M} = \{\, \varpi_0 = \varpi_1 = \omega = 0 \,\}.$$

Consequently, we have obtained

$$\rho_*^{-1}(\mathcal{H}) = \{\, \rho^*\theta = \rho^*\omega = \rho^*\pi = 0 \,\} = \partial^2\mathcal{M}.$$

□

3.3 Characterization of Monge-Ampère equations

The results in the previous section guide us to consider the geometric characterization of Monge-Ampère equations. In fact, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be Monge characteristic systems of a hyperbolic equation (R, D). The differential systems $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ of corank 3 have the possibility to be Monge characteristic systems of a hyperbolic Monge-Ampère system. On the other hand, in parabolic case, the differential system $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ of corank 3 has the possibility to be the Monge characteristic system of a parabolic Monge-Ampère system.

3.3.1 Hyperbolic case

Let (R, D) be a hyperbolic equation and set $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote Monge characteristic systems of (R, D).

First, let us describe the structure equation of $\mathcal{M}_i$. We recall the structure equation of D

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\mathrm{mod}\ \varpi_0), \\ d\varpi_1 \equiv \omega^1 \wedge \pi_{11} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \qquad\qquad \omega^2 \wedge \pi_{22} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

and Monge characteristic systems $\mathcal{M}_i = \{\varpi_0 = \varpi_1 = \varpi_3 = \omega^i = \pi_{ii} = 0\}$ for $i = 1, 2$. Since $\mathcal{M}_1$ is of rank 2 and $d\varpi_0 \equiv d\varpi_1 \equiv 0$, $d\varpi_2 \equiv \omega^2 \wedge \pi_{22}$ $(\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1, \pi_{11})$, the first derived system $\partial\mathcal{M}_1$ is of constant rank 3. Similarly, $\partial\mathcal{M}_2$ is so. We can write

$$\begin{aligned} d\omega^1 &\equiv \omega^2 \wedge (h_1\,\pi_{11} + k_1\,\pi_{22}) \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1), \\ d\omega^2 &\equiv \omega^1 \wedge (h_2\,\pi_{11} + k_2\,\pi_{22}) \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2). \end{aligned}$$

Then since

$$\begin{aligned} 0 &= d^2\varpi_0 \\ &\equiv -\omega^1 \wedge (d\varpi_1 + \varpi_2 \wedge (h_2\,\pi_{11} + k_2\,\pi_{22})) \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^2), \\ 0 &= d^2\varpi_0 \\ &\equiv -\omega^2 \wedge (d\varpi_2 + \varpi_1 \wedge (h_1\,\pi_{11} + k_1\,\pi_{22})) \qquad (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1), \end{aligned}$$

we have

$$\begin{aligned} d\varpi_1 &\equiv \omega^1 \wedge \pi_{11} - \varpi_2 \wedge (h_2\,\pi_{11} + k_2\,\pi_{22}) \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2), \\ d\varpi_2 &\equiv \omega^2 \wedge \pi_{22} - \varpi_1 \wedge (h_1\,\pi_{11} + k_1\,\pi_{22}) \qquad (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2 \wedge \varpi_1). \end{aligned}$$

Furthermore, since

$$\begin{aligned} 0 &= d^2\varpi_1 \\ &\equiv (-k_1 + h_2)\,\omega^2 \wedge \pi_{11} \wedge \pi_{22} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1), \end{aligned}$$

we have $k_1 = h_2$. Replacing $\omega^1 - k_1\varpi_2$ and $\omega^2 - h_2\varpi_1$ with ω^1 and ω^2 respectively, we have

$$\begin{aligned} d\varpi_0 &\equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 && (\mathrm{mod}\ \varpi_0) \\ d\varpi_1 &\equiv \omega^1 \wedge \pi_{11} - k_2\,\varpi_2 \wedge \pi_{22} && (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2) \\ d\varpi_2 &\equiv \omega^2 \wedge \pi_{22} - h_1\,\varpi_1 \wedge \pi_{11} && (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2 \wedge \varpi_2) \\ d\omega^1 &\equiv h_1\,\omega^2 \wedge \pi_{11} && (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1) \\ d\omega^2 &\equiv k_2\,\omega^1 \wedge \pi_{22} && (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2) \end{aligned} \tag{3.3.16}$$

Then $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D) = \{\varpi_0 = \varpi_i = \omega^i = 0\}$ and we can write those structure equations as follows:

$$(3.3.17)\quad \begin{cases} d\varpi_0 \equiv \omega^2 \wedge \varpi_2 \\ d\varpi_1 \equiv -k_2\,\varpi_2 \wedge \pi_{22} \\ d\omega^1 \equiv h_1\,\omega^2 \wedge \pi_{11} + \varpi_2 \wedge (A_1\,\pi_{11} + B_1\,\pi_{22}) \end{cases} \quad (\mathrm{mod}\ \varpi_0,\, \varpi_1,\, \omega^1),$$

$$(3.3.18)\quad \begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 \\ d\varpi_2 \equiv -h_1\,\varpi_1 \wedge \pi_{11} \\ d\omega^2 \equiv k_2\,\omega^1 \wedge \pi_{22} + \varpi_2 \wedge (A_2\,\pi_{11} + B_2\,\pi_{22}) \end{cases} \quad (\mathrm{mod}\ \varpi_0,\, \varpi_2,\, \omega^2).$$

As it is seen in Corollary 3.4, for a hyperbolic Monge-Ampère system, pullbacks of Monge characteristic systems $\mathcal{H}_i$ coincide with $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$, where $\mathcal{M}_i$ is the corresponding Monge characteristic system of the corresponding Monge-Ampère equation (R, D). Conversely, we obtain the next theorem:

Theorem 3.12. *Let (R, D) be a hyperbolic equation and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic systems of (R, D). If $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ drops down to J, or equivalently, $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ drops down to J, then there exists a Monge-Ampère system $\mathcal{I}$ such that (R, D) coincides with the prolongation of $\mathcal{I}$ locally. Moreover, $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ and $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ are pullbacks of the Monge characteristic systems of the system $\mathcal{I}$.*

First, we prove the following lemma:

Lemma 3.13. *Let (R, D) be a hyperbolic equation and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic systems of (R, D). If $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ drops down to J for all $i = 1, 2$, there exists a Monge-Ampère system $\mathcal{I}$ such that (R, D) coincides with the prolongation of $\mathcal{I}$ locally. Moreover, $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ and $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ are pullbacks of the Monge characteristic systems of the system $\mathcal{I}$.*

Proof. For a point $v \in R$, we will construct a Monge-Ampère system $\mathcal{I}$ around $\rho(v) \in J$ and a neighborhood V of v such that (R, D) coincides with the prolongation of $\mathcal{I}$ on V.

Let us fix a point $v_o \in R$.

Since $\partial\mathcal{M}_i + Ch(\partial D)$ drops down to J for each $i = 1, 2$, there exists 1-forms $\hat{\pi}_1$, $\hat{\omega}^1$, $\hat{\pi}_2$, $\hat{\omega}^2$ around $u_o = \rho(v_o)$ such that $\hat{\omega}^1 \wedge \hat{\omega}^2\,|_{v_o} \neq 0$ and

$$\partial\mathcal{M}_i + \mathrm{Ch}(\partial D) = \left\{ \rho^*\theta = \rho^*\hat{\pi}_i = \rho^*\hat{\omega}^i = 0 \right\} \quad \text{for } i = 1, 2.$$

Let V be a neighborhood of v_o such that $\hat{\omega}^1 \wedge \hat{\omega}^2\,|_v \neq 0$ for all $v \in V$.

Let denote A_1, A_2 and B non-zero functions on V such that $\omega^i \wedge \varpi_i \equiv A_i \rho^*\hat{\omega}^i \wedge \rho^*\hat{\pi}_i$ $(\mathrm{mod}\ \varpi_0)$ and $\varpi_0 = B\rho^*\theta$. Then we have

$$\rho^* d\theta \equiv \frac{A_1}{B}\,\rho^*\hat{\omega}^1 \wedge \rho^*\hat{\pi}_1 + \frac{A_2}{B}\,\rho^*\hat{\omega}^2 \wedge \rho^*\hat{\pi}_2 \qquad (\mathrm{mod}\ \rho^*\theta).$$

This implies that there exists functions $\hat{K}_1, \hat{K}_2$ around u_o such that $\rho^*\hat{K}_i = \frac{A_i}{B}$ for $i = 1, 2$, and hence we have

$$d\theta \equiv (\hat{K}_1\,\hat{\omega}^1) \wedge \hat{\pi}_1 + (\hat{K}_2\,\hat{\omega}^2) \wedge \hat{\pi}_2 \qquad (\mathrm{mod}\ \theta).$$

Now let us consider the following hyperbolic Monge-Ampère system

$$\mathcal{I} = \left\{\theta,\, \hat{\omega}^1 \wedge \hat{\pi}_1\right\}_{\mathrm{diff}} = \left\{\theta,\, \hat{\omega}^2 \wedge \hat{\pi}_2\right\}_{\mathrm{diff}}$$

and its Monge characteristic systems are $\mathcal{H}_i = \{\theta = \hat{\pi}_i = \hat{\omega}^i = 0\}$. From the definition of $\mathcal{I}$, each point of V is an integral element of $\mathcal{I}$. Namely (R, D) coincides with the corresponding Monge-Ampère equation locally. □

Next, we show that, for a hyperbolic equation (R, D), $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$ if and only if $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$:

Lemma 3.14. *Let (R, D) be a hyperbolic equation and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic systems of (R, D). Then, $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$ if and only if $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$.*

Proof. From (3.3.17), if $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$, we have $h_1 = k_2 = 0$. From (3.3.18), if $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$, we have $h_1 = k_2 = 0$.

As we assume $h_1 = k_2 = 0$, it follows from (3.3.16), (3.3.17) and (3.3.18) that

$$\begin{aligned}
d\varpi_1 &\equiv \omega^1 \wedge \pi_{11} && (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2),\\
d\omega^1 &\equiv \varpi_2 \wedge (A_1\,\pi_{11} + B_1\,\pi_{22}) && (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1),\\
d\varpi_2 &\equiv \omega^2 \wedge \pi_{22} && (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2 \wedge \varpi_2),\\
d\omega^2 &\equiv \varpi_1 \wedge (A_2\,\pi_{11} + B_2\,\pi_{22}) && (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^2).
\end{aligned}$$

Since $d\varpi_0 \equiv d\varpi_1 \equiv d(\omega^1 \wedge \varpi_2) \equiv d(\omega^2 \wedge \varpi_2) \equiv 0 \ \ (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^2 \wedge \varpi_2, \omega^1)$,

$$\begin{aligned}
0 &= d^2\varpi_1\\
&\equiv -B_1\,\varpi_2 \wedge \pi_{11} \wedge \pi_{22} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^2 \wedge \varpi_2, \omega^1).
\end{aligned}$$

Since $d\varpi_0 \equiv d\varpi_1 \equiv d\omega^1 \equiv 0 \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^1}$,

$$\begin{aligned} 0 &= d^2\omega^1 \\ &\equiv -A_1\, \omega^2 \wedge \pi_{11} \wedge \pi_{22} \qquad \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^1}. \end{aligned}$$

Therefore, we have $A_1 = B_1 = 0$.

On the other hand, since $d\varpi_0 \equiv d\varpi_2 \equiv d(\omega^1 \wedge \varpi_1) \equiv d(\omega^2 \wedge \varpi_1) \equiv 0 \pmod{\varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2}$,

$$\begin{aligned} 0 &= d^2\omega^2 \\ &\equiv A_2\, \varpi_1 \wedge \pi_{11} \wedge \pi_{22} \qquad \pmod{\varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2}. \end{aligned}$$

Since $d\varpi_0 \equiv d\varpi_2 \equiv d\omega^2 \equiv 0 \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^2}$,

$$\begin{aligned} 0 &= d^2\omega^2 \\ &\equiv B_2\, \omega^1 \wedge \pi_{11} \wedge \pi_{22} \qquad \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^2}. \end{aligned}$$

Therefore, we have $A_2 = B_2 = 0$.

Consequently, our assertion follows. □

Proof of Theorem 3.12. From Equation (3.3.17), if $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ drops down to $J = R/\mathrm{Ch}(\partial D)$, $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ must coincides with $\mathrm{Ch}(\partial D)$. Similarly, from Equation (3.3.18), if $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ drops down to J, $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ must coincides with $\mathrm{Ch}(\partial D)$. Conversely, if $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$, or equivalently, if $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$, then $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ drops down to J for $i = 1, 2$. Thus $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ drops down to J if and only if $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ drops down to J. Consequently, our assertion follows from Lemma 3.13 and this argument. □

Remark 3.15. $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ and h_1, k_2 in the above proof are corresponding to the M_i-characteristic vector field systems $\mathrm{Char}(I_F, dM_i)$ and Monge-Ampère invariants introduced in [GK93]. They characterize Monge-Ampère equation by the invariants. On the other hand, we characterize Monge-Ampère equation by the property that $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ should satisfy and find that these differential systems coincides with pullbacks of the Monge characteristic systems of the corresponding Monge-Ampère system if (R, D) is a hyperbolic Monge-Ampère system.

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From Lemma 3.14, we can translate Theorem 3.12 into the following corollary:

Corollary 3.16. *Let (R, D) be a hyperbolic equation and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic systems of (R, D). If $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$, or equivalently, $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$, there exists a Monge-Ampère system $\mathcal{I}$ such that (R, D) coincides with the prolongation of $\mathcal{I}$ locally.*

3.3.2 Parabolic case

In parabolic case, we obtain similar results to hyperbolic case. Unlike hyperbolic case, as it is seen in Theorem 3.11, the regularity of the first derived system $\partial\mathcal{M}$ of the Monge characteristic system $\mathcal{M}$ of a parabolic equation does not follow from the regularity of $\mathcal{M}$. In order to obtain a similar result to Corollary 3.16, we need a further assumption of the regularity (see Theorem 3.18). The result for *Goursat equation*, i.e. a parabolic equation whose Monge characteristic system is completely integrable (hence the assumption is naturally satisfied) is particularly important.

First, let us describe the structure equation of $\mathcal{M}$. We recall the structure equation of D

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\mathrm{mod}\ \varpi_0), \\ d\varpi_1 \equiv \qquad\qquad\quad \omega^2 \wedge \pi_{12} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \omega^1 \wedge \pi_{12} + \omega^2 \wedge \pi_{22} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

and the Monge characteristic system $\mathcal{M} = \{\varpi_0 = \varpi_1 = \varpi_2 = \omega^2 = \pi_{12} = 0\}$. As we write $d\omega^2 \equiv \omega^1 \wedge (h\,\pi_{12} + k\,\pi_{22})$ $(\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2)$,

$$\begin{aligned} 0 &= d^2\varpi_0 \\ &\equiv -\omega^1 \wedge (d\varpi_1 + \varpi_2 \wedge (h\,\pi_{12} + k\,\pi_{22})) \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^2). \end{aligned}$$

Thus we have

$$d\varpi_1 \equiv \omega^2 \wedge \pi_{12} - \varpi_2 \wedge (h\,\pi_{12} + k\,\pi_{22}) \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2).$$

Since $d\varpi_0 \equiv d\varpi_1 \equiv d(\omega^1 \wedge \varpi_2) \equiv d(\omega^2 \wedge \varpi_2) \equiv 0$ $(\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1 \wedge \omega^2, \omega^2 \wedge \pi_{12})$, we have

$$\begin{aligned} 0 &= d^2\varpi_1 \\ &\equiv -2k\,\omega^1 \wedge \pi_{12} \wedge \pi_{22} - \omega^2 \wedge d\pi_{12} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1 \wedge \omega^2, \omega^2 \wedge \pi_{12}) \end{aligned}$$

and hence $k = 0$. Thus replacing $\omega^2 - h\,\varpi_2$ with ω^2, we have

$$\begin{aligned} d\varpi_0 &\equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 && (\mathrm{mod}\ \varpi_0), \\ d\varpi_1 &\equiv \omega^2 \wedge \pi_{12} && (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2), \\ d\omega^2 &\equiv 0 && (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2). \end{aligned}$$

Then $\partial(\mathcal{M} + \mathrm{Ch}(\partial D)) = \{\varpi_0 = \varpi_1 = \omega^2 = 0\}$ and we can write its structure equation as follows:

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv E\,\omega^1 \wedge \varpi_2 \\ d\omega^2 \equiv \varpi_2 \wedge (A\,\pi_{12} + B\,\pi_{22} + C\,\omega^1) \end{cases} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^2) \tag{3.3.19}$$

Furthermore, since

$$\begin{aligned} 0 &= d^2\varpi_1 \\ &\equiv -\omega^2 \wedge (d\pi_{12} - E\,\omega^1 \wedge \pi_{22}) \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2 \wedge \pi_{12}), \end{aligned}$$

we have

$$d\pi_{12} \equiv E\,\omega^1 \wedge \pi_{22} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2, \pi_{12}). \tag{3.3.20}$$

Therefore the regularity of $\partial\mathcal{M}$ correspond to the regularity of the function E.

As it is seen in Theorem 3.11, for a parabolic Monge-Ampère system, the pullback of the Monge characteristic system $\mathcal{H}$ coincides with $\partial(\mathcal{M}+\mathrm{Ch}(\partial D))$, where $\mathcal{M}$ is the corresponding Monge characteristic system of the corresponding Monge-Ampère equation (R, D). Conversely, we obtain the following theorem:

Theorem 3.17. *Let (R, D) be a parabolic equation. Let $\mathcal{M}$ denote the Monge characteristic system of (R, D). If $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ drops down to J, there exists a Monge-Ampère system $\mathcal{I}$ such that (R, D) coincides with the prolongation of $\mathcal{I}$ locally. Moreover, $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is the pullback of the Monge characteristic system of the system $\mathcal{I}$.*

Proof. For each $v \in R$, we will construct a Monge-Ampère system $\mathcal{I}$ around $\rho(v)$ and a neighborhood V of v such that (R, D) coincides with the prolongation of $\mathcal{I}$ on V.

Let us fix a point $v_o \in R$. Since $\partial(\mathcal{M}+\mathrm{Ch}(\partial D))$ drops down to J, there exists 1-forms $\hat{\pi}$, $\hat{\omega}$ around $u_o = \rho(v_o)$ such that $\hat{\omega}|_{v_o} \neq 0$ and

$$\partial(\mathcal{M} + \mathrm{Ch}(\partial D)) = \{\, \rho^*\theta = \rho^*\hat{\pi} = \rho^*\hat{\omega} = 0 \,\}\,.$$

Let denote A and B non-zero functions around v_o such that $\omega^2 \wedge \varpi_1 \equiv A\,\rho^*\hat{\omega} \wedge \rho^*\hat{\pi}$ $(\mathrm{mod}\ \varpi_0)$ and $\varpi_0 = B\rho^*\theta$, then we have $d\varpi_0 \wedge \omega^2 \wedge \varpi_1 \equiv AB\,\rho^*(d\theta \wedge \hat{\omega} \wedge \hat{\pi})$ $(\mathrm{mod}\ \varpi_0)$. Since $d\varpi_0 \wedge \omega^2 \wedge \varpi_1 \equiv 0$ $(\mathrm{mod}\ \varpi_0)$, we have $d\theta \wedge \hat{\omega} \wedge \hat{\pi} \equiv 0$ $(\mathrm{mod}\ \theta)$. Therefore, there exists 1-forms $\hat{\omega}^1$, $\hat{\pi}_2$ around u_o such that

$$d\theta \equiv \hat{\omega}^1 \wedge \hat{\pi} + \hat{\omega} \wedge \hat{\pi}_2 \qquad (\mathrm{mod}\ \theta).$$

Since θ is a contact form, 1-forms $\theta, \hat{\omega}^1, \hat{\omega}, \hat{\pi}, \hat{\pi}_2$ are linearly independent. We may assume that $\hat{\omega}^1 \wedge \hat{\omega}\,|_{v_o} \neq 0$.

Let V be a neighborhood of v_o such that $\hat{\omega}^1 \wedge \hat{\omega}\,|_v \neq 0$ for all $v \in V$.

Now let us consider the following parabolic Monge-Ampère system

$$\mathcal{I} = \{\, \theta, \hat{\omega} \wedge \hat{\pi} \,\}_{\mathrm{diff}}$$

and its Monge characteristic system is $\mathcal{H} = \{\theta = \hat{\pi} = \hat{\omega} = 0\}$. From the definition of $\mathcal{I}$, each point of V is an integral element of $\mathcal{I}$. Namely (R, D) coincides with the prolongation of $\mathcal{I}$ locally. □

Theorem 3.18. *Let (R, D) be a parabolic equation. Let $\mathcal{M}$ denote the Monge characteristic system of (R, D) and assume the first derived system $\partial\mathcal{M}$ of $\mathcal{M}$ is also a differential system. If $\mathrm{Ch}(\partial(\mathcal{M} + \mathrm{Ch}(\partial D)))(v)$ contains $\mathrm{Ch}(\partial D)(v)$ at each point $v \in R$, there exists a Monge-Ampère system $\mathcal{I}$ such that (R, D) coincides with the prolongation of $\mathcal{I}$ locally.*

Proof. It is sufficient to show that $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ drops down to J if $\mathrm{Ch}(\partial(\mathcal{M} + \mathrm{Ch}(\partial D)))(v)$ contains $\mathrm{Ch}(\partial D)(v)$ at each point $v \in R$.

From Equation (3.3.20) and the assumption of the regularity of $\partial\mathcal{M}$, E uniformly vanishes or is not zero at each point of R. In the former case, $\mathcal{M}$ is completely integrable, namely R is a Goursat equation. In the latter case, $\partial\mathcal{M}$ is of constant rank 3.

As E uniformly vanishes, since $d\varpi_1 \equiv \omega^2 \wedge \pi_{12} \pmod{\varpi_0, \varpi_1, \omega^2 \wedge \varpi_2}$,

$$\begin{aligned} 0 &= d^2\varpi_1 \\ &\equiv \varpi_2 \wedge (-B\,\pi_{12} \wedge \pi_{22} + C\,\omega^1 \wedge \pi_{12}) \qquad \pmod{\varpi_0, \varpi_1, \omega^2}. \end{aligned}$$

Therefore B and C vanish on R. Additionally, because the structure equation (3.3.19) is satisfied and $\mathrm{Ch}(\partial(\mathcal{M} + \mathrm{Ch}(\partial D)))(v)$ contains $\mathrm{Ch}(\partial D)(v)$ at each point $v \in R$, A vanishes on R. Consequently, $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is completely integrable and hence drops down to J.

As E is not zero at each point of R, because the structure equation (3.3.19) is satisfied and $\mathrm{Ch}(\partial(\mathcal{M}+\mathrm{Ch}(\partial D)))(v)$ contains $\mathrm{Ch}(\partial D)(v)$ at each point $v \in R$, we get $\mathrm{Ch}(\partial(\mathcal{M}+\mathrm{Ch}(\partial D))) = \mathrm{Ch}(\partial D)$. Consequently, $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ drops down to J. □

Particularly we note that

Corollary 3.19. *Let (R, D) be a Goursat equation and $\mathcal{M}$ the Monge characteristic system of (R, D). Namely, $\mathcal{M}$ is completely integrable. Then (R, D) is a Monge-Ampère equation if and only if $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is completely integrable.*

Proof. If a Goursat equation (R, D) is a Monge-Ampère equation, from Theorem 3.11, the Monge characteristic system $\mathcal{H}$ of the corresponding Monge-Ampère system $\mathcal{I}$ is completely integrable, and equivalently $\rho_*^{-1}(\mathcal{H}) = \partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is completely integrable. Conversely, if $\partial(\mathcal{M}+\mathrm{Ch}(\partial D))$ is completely integrable, $\mathrm{Ch}(\partial(\mathcal{M}+\mathrm{Ch}(\partial D))) = \partial(\mathcal{M} + \mathrm{Ch}(\partial D)) = \{\varpi_0 = \varpi_1 = \omega^2 = 0\}$ contains $\mathrm{Ch}(\partial D)$. Hence, from Theorem 3.18, (R, D) is a Monge-Ampère equation. □

4 Hyperbolic exterior differential systems

4.1 Preliminaries

Let (R, D) be a differential system. (R, D) is a *hyperbolic differential system of class $n+2$* ($n \geq 1$) if, for a point $u_o \in R$, there exists a coframe $\{\theta_1, \ldots, \theta_n, \xi_1, \xi_2, \omega^1, \omega^2, \pi_1, \pi_2\}$ around u_o such that

$$D = \{\, \theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = 0 \,\},$$

$$\text{(4.1.1)} \qquad \begin{cases} d\theta_1 \equiv 0 \\ \quad \vdots \\ d\theta_n \equiv 0 \\ d\xi_1 \equiv \omega^1 \wedge \pi_1 \\ d\xi_2 \equiv \qquad\quad \omega^2 \wedge \pi_2 \end{cases} \qquad (\operatorname{mod} \theta_1, \ldots, \theta_n, \xi_1, \xi_2).$$

By definition, the Cauchy characteristic system $\mathrm{Ch}(D)$ of a hyperbolic differential system is trivial. For a hyperbolic differential system (R, D) with the structure equation (4.1.1), *Monge characteristic systems* $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined as

$$\mathcal{M}_i = \left\{ \theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = \omega^i = \pi_i = 0 \right\},$$

which are well-defined differential systems of rank 2. They are invariant under diffeomorphisms of R preserving D.

An exterior differential system $\mathcal{I}$ on Σ is *a hyperbolic exterior differential system of class n* (≥ 1) ([BGH95a]) if, for each point $u_o \in \Sigma$, there exists a coframe $\{\theta_1, \ldots, \theta_n, \omega^1, \omega^2, \pi_1, \pi_2\}$ around u_o such that $\mathcal{I}$ is generated algebraically by 1-forms $\theta_1, \ldots, \theta_n$ and the decomposable 2-forms $\omega^1 \wedge \pi_1$ and $\omega^2 \wedge \pi_2$. *Monge characteristic systems* $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined as

$$\mathcal{M}_i = \left\{ \theta_1 = \cdots = \theta_n = \omega^i = \pi_i = 0 \right\},$$

which are well-defined differential systems of rank 2. They are invariant under diffeomorphisms of Σ preserving $\mathcal{I}$.

It is known that the prolongation of a hyperbolic differential system and exterior differential system is a hyperbolic differential system and therefore has the Monge characteristic systems:

Theorem 4.1 ([BGH95a]). *The prolongation of a hyperbolic differential system and exterior differential system of class n is a hyperbolic differential system of class $n+2$.*

We see easily that a hyperbolic Monge-Ampère system is a hyperbolic exterior differential system of class 1 and the hyperbolic Monge-Ampère equation is a hyperbolic differential system of class 3. Generally, so is a hyperbolic equation. Namely, the hyperbolic exterior differential system (resp. differential system) is a generalization of the Monge-Ampère system (resp. hyperbolic equation). Therefore we may naturally think a generalization of Theorem 3.3 and Corollary 3.4, which is discussed in the next section.

Finally, note that a hyperbolic, non-Pfaffian, exterior differential system has the following normal form (cf. (3.1.3)):

Theorem 4.2 ([SY]). *Let $\mathcal{I}$ be a hyperbolic exterior differential system on R and $(\hat{R}, \hat{D})$ its prolongation. Suppose $\mathcal{I}$ is not Pfaffian. Then there exists a coordinate system $x, y, z, p, q, t_1, \ldots, t_n$ in a neighborhood of such that*

$$\mathcal{I} = \{\, \theta,\, d\theta,\, \Psi,\, dt_1, \ldots, dt_n \,\}_{\mathrm{alg}},$$

where

$$\begin{aligned} \theta &= dz - pdx - qdy, \\ \Psi &= Adp \wedge dy - B(dq \wedge dy - dp \wedge dx) - Cdq \wedge dx + Ddx \wedge dy + Edp \wedge dq. \end{aligned}$$

Here, A, B, C, D, E are functions of variables $x, y, z, p, q, t_1, \ldots, t_n$.

From this theorem, we see clearly that a hyperbolic exterior differential system is a generalization of Monge-Ampère system and is a second order partial differential equation of one unknown function.

4.2 Properties and relations of the Monge characteristic systems of hyperbolic differential systems and exterior differential systems

The following theorem asserts the relations between Monge characteristic systems of a given hyperbolic differential system and its prolongation:

Theorem 4.3. *Let (R, D) be a hyperbolic differential system with $\dim R = n + 6$ $(n \geq 1)$ and $(\hat{R}, \hat{D})$ be the prolongation of (R, D), and $\rho : \hat{R} \longrightarrow R$ be the canonical projection. Let $\mathcal{M}_i$ (resp. $\hat{\mathcal{M}}_i$) for $i = 1, 2$ be the Monge characteristic systems of (R, D) (resp. $(\hat{R}, \hat{D})$). Then $\hat{\mathcal{M}}_i$, $\partial\hat{\mathcal{M}}_i$, $\partial^2\hat{\mathcal{M}}_i$ and $\partial\mathcal{M}_i$ are differential systems of rank 2, 3, 4 and 3 respectively, and satisfy*

$$\begin{aligned} \partial\hat{\mathcal{M}}_i &\subset \rho_*^{-1}(\mathcal{M}_i) \quad (\,\mathrm{rank}\,\rho_*^{-1}(\mathcal{M}_i) - \mathrm{rank}\,\partial\hat{\mathcal{M}}_i = 1\,), \\ \partial^2\hat{\mathcal{M}}_i &\subset \rho_*^{-1}(\partial\mathcal{M}_i) \quad (\,\mathrm{rank}\,\rho_*^{-1}(\partial\mathcal{M}_i) - \mathrm{rank}\,\partial^2\hat{\mathcal{M}}_i = 1\,). \end{aligned}$$

Moreover, we have

$$\rho_*^{-1}(\mathcal{M}_i) = \partial\hat{\mathcal{M}}_i + \mathrm{Ch}(\partial\hat{D}). \tag{4.2.2}$$

Proof. Fix a point $u_o \in R$. By definition, there exists a coframe $\{\theta_1, \ldots, \theta_n, \xi_1, \xi_2, \omega^1, \ldots, \omega^4\}$ around u_o such that $D = \{\theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = 0\}$ and

$$\begin{cases} d\theta_1 \equiv 0 \\ \quad\vdots \\ d\theta_n \equiv 0 \\ d\xi_1 \equiv \omega^1 \wedge \omega^3 \\ d\xi_2 \equiv \qquad\quad \omega^2 \wedge \omega^4 \end{cases} \pmod{\theta_1, \ldots, \theta_n, \xi_1, \xi_2}.$$

Now we will express the derived systems of Monge characteristic systems of $(\hat{R}, \hat{D})$ by using this coframe.

Let v_o be a 2-dimensional integral element of (R, D) at u_o. Then there are four cases: $\omega^1 \wedge \omega^2|_{v_o} \neq 0$, $\omega^1 \wedge \omega^4|_{v_o} \neq 0$, $\omega^2 \wedge \omega^3|_{v_o} \neq 0$, $\omega^3 \wedge \omega^4|_{v_o} \neq 0$. Without loss of generality, we can assume $\omega^1 \wedge \omega^2|_{v_o} \neq 0$ since the case $\omega^1 \wedge \omega^4|_{v_o} \neq 0$ results in the case $\omega^1 \wedge \omega^2|_{v_o} \neq 0$ if we replace ω^2 and ω^4 by ω^4 and $-\omega^2$ respectively in the argument below.

Assume $\omega^1 \wedge \omega^2|_{v_o} \neq 0$. Let U be a neighborhood of v_o such that $\omega^1 \wedge \omega^2|_v \neq 0$ for all $v \in U$. Then we can introduce functions α_1 and α_2 on U such that $\hat{D}$ is defined on U by the 1-forms $\theta_1, \ldots, \theta_n, \xi_1, \xi_2$, and

$$\eta_1 = \omega^3 - \alpha_1\,\omega^1, \qquad \eta_2 = \omega^4 - \alpha_2\,\omega^2.$$

For $1 \leq i \leq 4$, writing

$$\begin{aligned} d\omega^i \equiv\ & \omega^1 \wedge (J_i\,\omega^2 + K_i\,\omega^3 + L_i\,\omega^4) \\ & + \omega^2 \wedge (M_i\,\omega^3 + N_i\,\omega^4) + O_i\,\omega^3 \wedge \omega^4 \qquad \pmod{\theta_1, \ldots, \theta_n, \xi_1, \xi_2}, \end{aligned}$$

with functions $J_i, K_i, L_i, M_i, N_i, O_i$ on U, then we have

$$\left.\begin{aligned} d\eta_1 \equiv \omega^1 \wedge \pi_1 \\ d\eta_2 \equiv \omega^2 \wedge \pi_2 \end{aligned}\right\} \pmod{\theta_1, \ldots, \theta_n, \xi_1, \xi_2},$$

where

$$\pi_1 = d\alpha_1 + (J_3 - (M_3 + J_1)\alpha_1 + L_3\alpha_2 + M_1\alpha_1^2 + (O_3 - L_1)\alpha_1\alpha_2 - O_1\alpha_1^2\alpha_2)\,\omega^2,$$
$$\pi_2 = d\alpha_2 + (-J_4 + M_4\alpha_1 - (L_4 - J_2)\alpha_2 - (O_4 + M_2)\alpha_1\alpha_2 + L_2\alpha_2^2 + O_2\alpha_1\alpha_2^2)\,\omega^1.$$

Thus the Monge characteristic systems $\hat{\mathcal{M}}_i$ of $(\hat{R}, \hat{D})$ are defined as

$$\hat{\mathcal{M}}_i = \left\{ \theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = \eta_1 = \eta_2 = \omega^i = \pi_i = 0 \right\}.$$

Since $\mathrm{Ch}(D)$ is defined by the 1-forms $\theta_1, \ldots, \theta_n, \xi_1, \xi_2, \omega^1, \ldots, \omega^4$, we have $d\omega^i \equiv 0 \pmod{\hat{\mathcal{M}}_i^\perp}$ for $i = 1, 2$. By definition,

$$\begin{aligned} d\pi_1 &\equiv (L_3 + (O_3 - L_1)\alpha_1 - O_1\alpha_1^2)\,\pi_2 \wedge \omega^2 \qquad &&(\mathrm{mod}\ \hat{\mathcal{M}}_1^\perp),\\ d\pi_2 &\equiv (M_4 - (O_4 + M_2)\alpha_2 + O_2\alpha_2^2)\,\pi_1 \wedge \omega^1 \qquad &&(\mathrm{mod}\ \hat{\mathcal{M}}_2^\perp), \end{aligned}$$

which imply

$$\partial\hat{\mathcal{M}}_i = \left\{ \theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = \eta_i = \omega^i = \bar{\pi}_i = 0 \right\} \subset \rho_*^{-1}(\mathcal{M}_i),$$

where

$$\begin{aligned} \bar{\pi}_1 &= \pi_1 + (L_3 + (O_3 - L_1)\alpha_1 - O_1\alpha_1^2)\,\eta_2,\\ \bar{\pi}_2 &= \pi_2 + (M_4 - (O_4 + M_2)\alpha_2 + O_2\alpha_2^2)\,\eta_1. \end{aligned}$$

Since $\mathrm{Ch}(\partial\hat{D})$ is defined by the 1-forms $\theta_1, \ldots, \theta_n, \xi_1, \xi_2, \eta_1, \eta_2, \omega^1, \omega^2$, we obtain

$$\partial\hat{\mathcal{M}}_i + \mathrm{Ch}(\partial\hat{D}) = \left\{ \theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = \eta_i = \omega^i = 0 \right\} = \rho_*^{-1}(\mathcal{M}_i).$$

Since

$$\left.\begin{aligned} d\theta_1 &\equiv \cdots \equiv d\theta_n \equiv 0\\ d\xi_1 &\equiv 0\\ d\xi_2 &\equiv \omega^2 \wedge \eta_2\\ d\eta_1 &\equiv (N_3 - N_1\alpha_1)\,\omega^2 \wedge \eta_2\\ d\omega^1 &\equiv N_1\,\omega^2 \wedge \eta_2 \end{aligned}\right\} \qquad (\mathrm{mod}\ \partial\hat{\mathcal{M}}_1^\perp),$$

we have

$$\partial^2\hat{\mathcal{M}}_1 \subset \left\{ \theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = \bar{\eta}_1 = \bar{\omega}^1 = 0 \right\},$$

where $\bar{\eta}_1 = \eta_1 - (N_3 - N_1\alpha_1)\,\xi_2$, $\bar{\omega}^1 = \omega^1 - N_1\,\xi_2$.

The following formulas are useful to calculate to $d\bar{\pi}_1$: for the pullback F of a function around u_o, since $dF \equiv 0 \pmod{\theta_1, \ldots, \theta_n, \xi_1, \xi_2, \eta_1, \eta_2, \omega^1, \omega^2}$, we have

$$dF \wedge \omega^2 \equiv dF \wedge \eta_2 \equiv 0 \quad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \xi_2, \eta_1, \omega^1, \omega^2 \wedge \eta_2)$$

Moreover we have

$$\left.\begin{aligned} d\alpha_2 &\equiv \bar{\pi}_2 \\ d\alpha_1 \wedge \omega^2 &\equiv 0 \\ d\alpha_1 \wedge \eta_2 &\equiv 0 \\ d\eta_2 &\equiv -\bar{\pi}_2 \wedge \omega^2 \end{aligned}\right\} \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \xi_2, \eta_1, \omega^1, \bar{\pi}_1, \omega^2 \wedge \eta_2).$$

Therefore we calculate

$$\begin{aligned} d\bar{\pi}_1 &\equiv \left(L_3\, d\alpha_2 + (O_3 - L_1)\alpha_1\, d\alpha_2 - O_1\alpha_1^2\, d\alpha_2\right) \wedge \omega^2 + \left(L_3 + (O_3 - L_1)\alpha_1 - O_1\alpha_1^2\right) d\eta_2 \\ &\equiv \left(L_3\, \bar{\pi}_2 + (O_3 - L_1)\alpha_1\, \bar{\pi}_2 - O_1\alpha_1^2\, \bar{\pi}_2\right) \wedge \omega^2 + \left(L_3 + (O_3 - L_1)\alpha_1 - O_1\alpha_1^2\right) \omega^2 \wedge \bar{\pi}_2 \\ &\equiv 0 \qquad\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \xi_2, \eta_1, \omega^1, \bar{\pi}_1, \omega^2 \wedge \eta_2) \end{aligned}$$

Thus $\partial^2\hat{\mathcal{M}}_1$ is a subbundle of TU and $\mathrm{rank}\,\partial^2\hat{\mathcal{M}}_1 - \mathrm{rank}\,\partial\hat{\mathcal{M}}_1 = 1$.

On the other hand, since

$$\rho_*^{-1}(\mathcal{M}_1) = \left\{\theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = \omega^1 = \omega^3 = 0\right\},$$

$$\left.\begin{aligned} d\xi_2 &\equiv \omega^2 \wedge \omega^4 \\ d\omega^1 &\equiv N_1\, \omega^2 \wedge \omega^4 \\ d\omega^3 &\equiv N_3\, \omega^2 \wedge \omega^4 \end{aligned}\right\} \qquad (\mathrm{mod}\ \rho_*^{-1}(\mathcal{M}_1)^{\perp}),$$

we obtain $\rho_*^{-1}(\partial\mathcal{M}_1) = \{\theta_1 = \cdots = \theta_n = \xi_1 = \omega^1 - N_1\,\xi_2 = \omega^3 - N_3\,\xi_2 = 0\}$. By definition, we have

$$\begin{aligned} \bar{\omega}^1 &= \omega^1 - N_1\,\xi_2, \\ \bar{\eta}_1 &= \eta_1 - (N_3 - N_1\alpha_1)\,\xi_2 \\ &= (\omega^3 - N_3\,\xi_2) - \alpha_1(\omega^1 - N_1\,\xi_2), \end{aligned}$$

which imply

$$\rho_*^{-1}(\partial\mathcal{M}_1) = \left\{\theta_1 = \cdots = \theta_n = \xi_1 = \bar{\eta}_1 = \bar{\omega}^1 = 0\right\} \supset \partial^2\hat{\mathcal{M}}_1.$$

Similar arguments apply to the case $i = 2$. □

The same result holds for hyperbolic exterior differential system and is proven analogously to Theorem 4.3 (cf. Theorem 3.3 and Corollary 3.4):

Theorem 4.4. *Let $\mathcal{I}$ be a hyperbolic exterior differential system and $(\hat{R}, \hat{D})$ be the prolongation of $\mathcal{I}$, and $\rho : \hat{R} \longrightarrow R$ be the canonical projection. Let $\mathcal{M}_i$ (resp. $\hat{\mathcal{M}}_i$) for*

$i = 1, 2$ *be the Monge characteristic systems of* (R, D) *(resp.* $(\hat{R}, \hat{D})$*). Then* $\hat{\mathcal{M}}_i$, $\partial\hat{\mathcal{M}}_i$, $\partial^2\hat{\mathcal{M}}_i$ *and* $\partial\mathcal{M}_i$ *are differential systems of rank* 2, 3, 4 *and* 3 *respectively, and satisfy*

$$\begin{aligned} \partial\hat{\mathcal{M}}_i &\subset \rho_*^{-1}(\mathcal{M}_i) \quad (\operatorname{rank}\rho_*^{-1}(\mathcal{M}_i) - \operatorname{rank}\partial\hat{\mathcal{M}}_i = 1), \\ \partial^2\hat{\mathcal{M}}_i &\subset \rho_*^{-1}(\partial\mathcal{M}_i) \quad (\operatorname{rank}\rho_*^{-1}(\partial\mathcal{M}_i) - \operatorname{rank}\partial^2\hat{\mathcal{M}}_i = 1). \end{aligned}$$

Moreover, we have

$$\rho_*^{-1}(\mathcal{M}_i) = \partial\hat{\mathcal{M}}_i + \operatorname{Ch}(\partial\hat{D}). \tag{4.2.3}$$

4.3 Reduction theorem for hyperbolic differential systems

In previous section we have shown some properties that Monge characteristic systems of a hyperbolic differential system and exterior differential system should satisfy. Especially, Equation (4.2.2) and (4.2.3) are useful to construct a hyperbolic one or exterior differential system on a manifold of smaller dimension from a given hyperbolic differential system.

Theorem 4.5. *For the prolongation* $(\hat{R}, \hat{D})$ *of a hyperbolic differential system* (R, D) *with* $\dim R = n + 6$ $(n \geq 1)$*, the Cauchy characteristic system* $\operatorname{Ch}(\partial\hat{D})$ *of the differential system* $\partial\hat{D}$ *is a subbundle of* $\hat{D}$ *of rank* 2*, and* $\partial^2\hat{D}$ *is the differential system such that* $\operatorname{rank}\partial^2\hat{D} - \operatorname{rank}\partial\hat{D} = 2$*. Conversely, for a hyperbolic differential system* (R, D) *with* $\dim R = n + 6$ $(n \geq 3)$*, if* $\operatorname{Ch}(\partial D)$ *is a subbundle of* D *of rank* 2*, and if* $\partial^2 D$ *is the differential system such that* $\operatorname{rank}\partial^2 D - \operatorname{rank}\partial D = 2$*, then* (R, D) *coincides with the prolongation of the hyperbolic differential system* $(R/\operatorname{Ch}(\partial D), \partial D)$ *with an independence condition on a neighborhood.*

Proof. First we will show that differential systems $\partial\mathcal{M}_1 + \operatorname{Ch}(\partial D)$ and $\partial\mathcal{M}_2 + \operatorname{Ch}(\partial D)$ drop down to the quotient space $R/\operatorname{Ch}(\partial D)$ for the Monge characteristic systems $\mathcal{M}_i$ of (R, D). Using this, we will show that ∂D also drops down and $(R/\operatorname{Ch}(\partial D), \partial D)$ is the hyperbolic differential system we desire.

Fix a point $u_o \in R$. There exists a coframe $\{\theta_1, \ldots, \theta_n, \xi_1, \xi_2, \omega^1, \omega^2, \pi_1, \pi_2\}$ around u_o such that $D = \{\theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = 0\}$ and

$$\begin{cases} d\theta_1 \equiv 0 \\ \quad\vdots \\ d\theta_n \equiv 0 \\ d\xi_1 \equiv \omega^1 \wedge \pi_1 \\ d\xi_2 \equiv \qquad\quad \omega^2 \wedge \pi_2 \end{cases} \quad (\operatorname{mod} \theta_1, \ldots, \theta_n, \xi_1, \xi_2).$$

Writing

$$d\omega^i \equiv \xi_1 \wedge (A_i\,\omega^1 + B_i\,\pi_1 + C_i\,\omega^2 + D_i\,\pi_2) \\ + \xi_2 \wedge (E_i\,\omega^1 + F_i\,\pi_1 + G_i\,\omega^2 + H_i\,\pi_2) + I_i\,\xi_1 \wedge \xi_2 \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n),$$

where $A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i, I_i$ denote functions around u_o and calculating $d^2\theta_i = 0$ $(\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \xi_2)$, we obtain

(4.3.4)

$$d\theta_i \equiv \xi_1 \wedge (A_i\,\omega^1 + B_i\,\pi_1) + \xi_2 \wedge (G_i\,\omega^2 + H_i\,\pi_2) + I_i\,\xi_1 \wedge \xi_2 \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n)$$

By the hypothesis $\mathrm{Ch}(\partial D) \subset D$, we have $\mathrm{Ch}(\partial D) \subset \{\theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = 0\}$. Since

$$\begin{aligned} 0 &\equiv X \lrcorner\, d\theta_i \qquad\qquad (\mathrm{mod}\ (\theta_1)_{u_o}, \ldots, (\theta_n)_{u_o}) \\ &\equiv -\Big(X \lrcorner\, (A_i\,\omega^1 + B_i\,\pi_1)\Big)\,\xi_1 - \Big(X \lrcorner\, (G_i\,\omega^2 + H_i\,\pi_2)\Big)\,\xi_2 \end{aligned} \tag{4.3.5}$$

for $X \in \mathrm{Ch}(\partial D)(u_o)$, we have

(4.3.6)

$$\mathrm{Ch}(\partial D) = \Big\{\,\theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = A_i\,\omega^1 + B_i\,\pi_1 = G_i\,\omega^2 + H_i\,\pi_2 = 0\ (1 \le i \le n)\,\Big\}\,.$$

Since $\mathrm{Ch}(\partial D)$ is completely integrable and $\mathrm{Ch}(\partial D) \subset \{\xi_1 = \xi_2 = 0\}$, we have

$$\left.\begin{aligned} 0 &\equiv d\xi_1 \equiv \omega^1 \wedge \pi_1 \\ 0 &\equiv d\xi_2 \equiv \omega^2 \wedge \pi_2 \end{aligned}\right\} \qquad (\mathrm{mod}\ (\mathrm{Ch}(\partial D))^\perp). \tag{4.3.7}$$

From (4.3.5), (4.3.6), and (4.3.7), there exist i, j such that

$$\mathrm{Ch}(\partial D) \subset \Big\{\,\theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = A_i\,\omega^1 + B_i\,\pi_1 = G_j\,\omega^2 + H_j\,\pi_2 = 0\,\Big\}\,,$$

where $(A_i, B_i) \neq 0$ and $(G_j, H_j) \neq 0$ hold. Since $\mathrm{Ch}(\partial D)$ is of rank 2, this equality holds. Namely

$$\mathrm{Ch}(\partial D) = \Big\{\,\theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = A_i\,\omega^1 + B_i\,\pi_1 = G_j\,\omega^2 + H_j\,\pi_2 = 0\,\Big\}\,. \tag{4.3.8}$$

Then 1-forms

$$\bar\omega^1 = A_i\,\omega^1 + B_i\,\pi_1, \qquad \bar\pi_1 = -\frac{B_i}{A_i^2 + B_i^2}\,\omega^1 + \frac{A_i}{A_i^2 + B_i^2}\,\pi_1,$$

$$\bar\omega^2 = G_j\,\omega^2 + H_j\,\pi_2, \qquad \bar\pi_2 = -\frac{H_j}{H_j^2 + G_j^2}\,\omega^2 + \frac{G_j}{H_j^2 + G_j^2}\,\pi_2,$$

are linearly independent and satisfy, from (4.3.4) and (4.3.8),

$$d\theta_i \equiv \xi_1 \wedge (\alpha_i \,\bar{\omega}^1) + \xi_2 \wedge (\beta_i \,\bar{\omega}^2) \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1 \wedge \xi_2)$$

where α_i, β_i is functions around u_o. Then, since $\mathrm{rank}\,\partial^2 D - \mathrm{rank}\,\partial D = 2$, we may assume that $\theta_1, \ldots, \theta_n$ satisfy

$$\left.\begin{aligned} d\theta_1 \equiv \cdots \equiv d\theta_{n-2} &\equiv 0 \\ d\theta_{n-1} &\equiv \bar{\omega}^1 \wedge \xi_1 + \bar{I}_1\, \xi_1 \wedge \xi_2 \\ d\theta_n &\equiv \bar{\omega}^2 \wedge \xi_2 + \bar{I}_2\, \xi_1 \wedge \xi_2 \end{aligned}\right\} \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n)$$

for some functions $\bar{I}_i$ around u_o. Since $\mathrm{Ch}(\partial D) = \{\theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = \bar{\omega}^1 = \bar{\omega}^2 = 0\}$, we can write

$$\begin{aligned} d\xi_1 &\equiv \bar{\omega}^1 \wedge \bar{\pi}_1 + \xi_2 \wedge (\bar{A}_1\, \bar{\pi}_1 + \bar{B}_1\, \bar{\pi}_2 + \bar{C}_1\, \bar{\omega}^1 + \bar{D}_1\, \bar{\omega}^2) \\ &\qquad\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1), \\ d\bar{\omega}^1 &\equiv \xi_2 \wedge (\bar{E}_1\, \bar{\pi}_1 + \bar{F}_1\, \bar{\pi}_2) + \bar{\omega}^2 \wedge (h_1\, \bar{\pi}_1 + k_1\, \bar{\pi}_2) + \bar{G}_1\, \xi_2 \wedge \bar{\omega}^2 \\ &\qquad\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \bar{\omega}^1), \\ d\xi_2 &\equiv \bar{\omega}^2 \wedge \bar{\pi}_2 + \xi_1 \wedge (\bar{A}_2\, \bar{\pi}_1 + \bar{B}_2\, \bar{\pi}_2 + \bar{C}_2\, \bar{\omega}^1 + \bar{D}_2\, \bar{\omega}^2) \\ &\qquad\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_2), \\ d\bar{\omega}^2 &\equiv \xi_1 \wedge (\bar{E}_2\, \bar{\pi}_1 + \bar{F}_2\, \bar{\pi}_2) + \bar{\omega}^1 \wedge (h_2\, \bar{\pi}_1 + k_2\, \bar{\pi}_2) + \bar{G}_2\, \xi_1 \wedge \bar{\omega}^1 \\ &\qquad\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_2, \bar{\omega}^2), \end{aligned}$$

where $h_i, k_i, \bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i, \bar{E}_i, \bar{F}_i, \bar{G}_i$ are functions around u_o. Calculating $d^2\xi_i = 0$ under modulo $\theta_1, \ldots, \theta_n, \xi_1, \xi_2, \bar{\omega}^i$, we obtain $A_1 = -k_1$, $B_2 = -h_2$. Moreover, calculating $d^2\theta_{n-i+2} = 0$ under modulo $\theta_1, \ldots, \theta_n, \xi_1 \wedge \xi_2, \bar{\omega}^1 \wedge \xi_1, \bar{\omega}^2 \wedge \xi_2$, we obtain $h_1 = k_1 - \bar{I}_1 = \bar{B}_1 = 0$, $k_2 = h_2 + \bar{I}_2 = \bar{A}_2 = 0$. Thus we conclude

$$\left.\begin{aligned} d\theta_1 &\equiv 0 \\ &\ \vdots \\ d\theta_{n-2} &\equiv 0 \\ d\theta_{n-1} &\equiv (\bar{\omega}^1 - k_1\, \xi_2) \wedge \xi_1 \\ d\theta_n &\equiv (\bar{\omega}^2 - h_2\, \xi_1) \wedge \xi_2 \end{aligned}\right\} \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n)$$

$$
\begin{aligned}
d\xi_1 &\equiv (\bar{\omega}^1 - k_1\,\xi_2) \wedge (\bar{\pi}_1 - \bar{C}_1\,\xi_2) - \bar{D}_1\,\bar{\omega}^2 \wedge \xi_2 && (\mathrm{mod}\ \theta_1, \cdots, \theta_n, \xi_1),\\
d\bar{\omega}^1 &\equiv \xi_2 \wedge (\bar{E}_1\,\bar{\pi}_1 + \bar{F}_1\,\bar{\pi}_2) + k_1\,\bar{\omega}^2 \wedge \bar{\pi}_2 + \bar{G}_1\,\bar{\omega}^2 \wedge \xi_2 && (\mathrm{mod}\ \theta_1, \cdots, \theta_n, \xi_1, \bar{\omega}^1),\\
d\xi_2 &\equiv (\bar{\omega}^2 - h_2\,\xi_1) \wedge (\bar{\pi}_2 - \bar{D}_2\,\xi_1) - \bar{C}_2\,\bar{\omega}^1 \wedge \xi_1 && (\mathrm{mod}\ \theta_1, \cdots, \theta_n, \xi_2),\\
d\bar{\omega}^2 &\equiv \xi_1 \wedge (\bar{E}_2\,\bar{\pi}_1 + \bar{F}_2\,\bar{\pi}_2) + h_2\,\bar{\omega}^1 \wedge \bar{\pi}_1 - \bar{G}_2\,\bar{\omega}^1 \wedge \xi_1 && (\mathrm{mod}\ \theta_1, \cdots, \theta_n, \xi_2, \bar{\omega}^2).
\end{aligned}
$$

Then, for the Monge characteristic systems $\mathcal{M}_i$ of (R, D), we see that

$$
\partial\mathcal{M}_i + \mathrm{Ch}(\partial D) = \left\{ \theta_1 = \cdots = \theta_n = \xi_i = \tilde{\omega}^i = 0 \right\},
$$

where $\tilde{\omega}^1 = \bar{\omega}^1 - k_1\,\xi_2$, $\tilde{\omega}^2 = \bar{\omega}^2 - h_2\,\xi_1$.

Next we will show that the Cauchy characteristic system of $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ coincides with that of ∂D. Writing $\tilde{\pi}_1 = \bar{\pi}_1 - \bar{C}_1\,\xi_2$ and $\tilde{\pi}_2 = \bar{\pi}_2 - \bar{D}_2\,\xi_1$, we have

$$
\begin{aligned}
d\xi_1 &\equiv \tilde{\omega}^1 \wedge \tilde{\pi}_1 - \bar{D}_1\,\tilde{\omega}^2 \wedge \xi_2 && (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1),\\
d\xi_2 &\equiv \tilde{\omega}^2 \wedge \tilde{\pi}_2 - \bar{C}_2\,\tilde{\omega}^1 \wedge \xi_1 && (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_2).
\end{aligned}
$$

Since $d\tilde{\omega}^i \equiv 0 \ (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \xi_2, \tilde{\omega}^i)$, we can write

$$
\begin{aligned}
d\tilde{\omega}^1 &\equiv \xi_2 \wedge (\tilde{E}_1\,\tilde{\pi}_1 + \tilde{F}_1\,\tilde{\pi}_2 + \tilde{G}_1\,\tilde{\omega}^2) && (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \tilde{\omega}^1),\\
d\tilde{\omega}^2 &\equiv \xi_1 \wedge (\tilde{E}_2\,\tilde{\pi}_1 + \tilde{F}_2\,\tilde{\pi}_2 + \tilde{G}_2\,\tilde{\omega}^1) && (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_2, \tilde{\omega}^2).
\end{aligned}
$$

By calculation of the following integrability conditions:

$$
\begin{aligned}
d^2\xi_1 &= 0 \quad \text{under modulo } \theta_1, \ldots, \theta_n, \xi_1, \tilde{\omega}^1, \tilde{\omega}^2 \wedge \xi_2,\\
d^2\xi_2 &= 0 \quad \text{under modulo } \theta_1, \ldots, \theta_n, \xi_2, \tilde{\omega}^2, \tilde{\omega}^1 \wedge \xi_1,\\
d^2\tilde{\omega}^1 &= 0 \quad \text{under modulo } \theta_1, \ldots, \theta_n, \xi_1, \xi_2, \tilde{\omega}^1,\\
d^2\tilde{\omega}^2 &= 0 \quad \text{under modulo } \theta_1, \ldots, \theta_n, \xi_1, \xi_2, \tilde{\omega}^2.
\end{aligned}
$$

we have $\tilde{E}_1 = \tilde{E}_2 = \tilde{F}_1 = \tilde{F}_2 = 0$. Thus we achieve

$$
\left\{
\begin{aligned}
d\theta_1 &\equiv 0\\
&\ \vdots\\
d\theta_{n-2} &\equiv 0\\
d\theta_{n-1} &\equiv 0\\
d\theta_n &\equiv \tilde{\omega}^2 \wedge \xi_2\\
d\xi_1 &\equiv -\bar{D}_1\,\tilde{\omega}^2 \wedge \xi_2\\
d\tilde{\omega}^1 &\equiv -\tilde{G}_1\,\tilde{\omega}^2 \wedge \xi_2
\end{aligned}
\right.
\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \tilde{\omega}^1),
$$

$$\begin{cases} d\theta_1 \equiv 0 \\ \quad \vdots \\ d\theta_{n-2} \equiv 0 \\ d\theta_{n-1} \equiv \tilde{\omega}^1 \wedge \xi_1 \\ d\theta_n \equiv 0 \\ d\xi_2 \equiv -\bar{C}_2\, \tilde{\omega}^1 \wedge \xi_1 \\ d\tilde{\omega}^2 \equiv -\tilde{G}_2\, \tilde{\omega}^1 \wedge \xi_1 \end{cases} \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_2, \tilde{\omega}^2).$$

Thus we find $\mathrm{Ch}(\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$. Namely $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ drops down to $R/\mathrm{Ch}(\partial D)$ for all $i = 1, 2$.

Next we will show that the differential system ∂D drops down to $R/\mathrm{Ch}(\partial D)$ and will be a hyperbolic differential system on it.

Let ρ be the canonical projection from R onto $R/\mathrm{Ch}(\partial D)$. By definition, for $i = 1, 2$, there exist differential systems $\mathcal{H}_i$ and C on $R/\mathrm{Ch}(\partial D)$ of corank $n + 2$ and n such that $\rho_*^{-1}(\mathcal{H}_i) = \partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ and $\rho_*^{-1}(C) = \partial D$. Then they can be written as

$$\mathcal{H}_i = \left\{ \Theta_1 = \cdots = \Theta_n = \Xi_i = \Omega^i = 0 \right\},$$
$$C = \{ \rho^*\Theta_1 = \cdots = \rho^*\Theta_n = 0 \},$$

where $\{\Theta_1, \ldots, \Theta_n, \Omega^1, \Omega^2, \Xi_1, \Xi_2\}$ is a coframe around $\rho(u_o) \in R/\mathrm{Ch}(\partial D)$. These imply

$$d\rho^*\Theta_l \equiv 0 \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n)$$
$$\rho^*(\Omega^i \wedge \Xi_i) \equiv 0 \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \tilde{\omega}^i \wedge \xi_i)$$

for $1 \leq l \leq n-2$ and $i = 1, 2$. Since $d\theta_l \equiv 0 \pmod{\theta_1, \ldots, \theta_n}$, we may assume $d\Theta_l \equiv 0 \pmod{\Theta_1, \ldots, \Theta_n}$. By $d\theta_{n-i+2} \equiv \tilde{\omega}^1 \wedge \xi_1 + \tilde{\omega}^2 \wedge \xi_2 \pmod{\theta_1, \ldots, \theta_n}$, we have

$$d\rho^*\Theta_{n-i+2} \equiv A_i^1\, \rho^*(\Omega^1 \wedge \Xi_1) + A_i^2\, \rho^*(\Omega^2 \wedge \Xi_2) \qquad (\mathrm{mod}\ \rho^*\Theta_1, \ldots, \rho^*\Theta_n),$$

where A_i^1, A_i^2 are functions around u_o satisfying $\det(A_i^j) \neq 0$. This implies that the coefficients A_i^1 and A_i^2 in the right hand sides can be written as the pullbacks of some functions around $\rho(u_o)$. Thus C $(= \partial D)$ is a hyperbolic differential system on $R/\mathrm{Ch}(\partial D)$ and its Monge characteristic systems are $\mathcal{H}_i$.

Finally we will show that the prolongation of $(R/\mathrm{Ch}(\partial D), C)$ with an independence condition coincides locally with the given system (R, D). By the hypothesis that $\mathrm{Ch}(\partial D) \subset D$ is of corank 2, we can apply Realization Lemma to $\rho : R \longrightarrow R/\mathrm{Ch}(\partial D)$; we have a unique map

$$\psi : R \longrightarrow J(R/\mathrm{Ch}(\partial D), 2)\,; \quad v \longmapsto \rho_*(D(v))$$

such that $\Pi \circ \psi = \rho$ and $D = \psi_*^{-1}(C^*)$, where $\Pi : J(R/\mathrm{Ch}(\partial D), 2) \longrightarrow R/\mathrm{Ch}(\partial D)$ is the canonical projection and C^* is the canonical system on $J(R/\mathrm{Ch}(\partial D), 2)$. Furthermore ψ is an immersion since $\mathrm{Ker}\,(\psi_*)_v = \mathrm{Ker}\,(\rho_*)_v \cap \mathrm{Ch}(D) = \{0\}$.

Here, since $\mathcal{H}_i = \{\Theta_1 = \cdots = \Theta_n = \Xi_i = \Omega^i = 0\}$, we may assume $\rho^*(\Omega_1 \wedge \Omega_2)$ and $\omega^1 \wedge \omega^2$ are equivalent modulo $\theta_1, \ldots, \theta_n, \xi_1, \xi_2$, up to scale. Then let $(\hat{R}, \hat{D})$ denote the prolongation of $(R/\mathrm{Ch}(\partial D), C)$ with the independence condition $\Omega^1 \wedge \Omega^2$. For any $v \in R$, by definition, it is easy to see that $\Theta_i|_{\psi(v)} = d\Theta_i|_{\psi(v)} = 0$ and $\Omega^1 \wedge \Omega^2|_{\psi(v)} \neq 0$. Namely $\psi(v)$ is an element of $\hat{R}$. By the structure equation of C, we find $\dim \hat{R} = \dim(R/\mathrm{Ch}(\partial D)) + 2 = \dim R$. Therefore (R, D) coincides locally with the prolongation of $(R/\mathrm{Ch}(\partial D), C)$ with the independence condition $\Omega^1 \wedge \Omega^2$. □

Theorem 4.6. *Let (R, D) be a hyperbolic differential system with $\dim R = n + 6$ $(n \geq 1)$ and $\mathcal{M}_i$ the Monge characteristic systems of (R, D) for $i = 1, 2$. If (R, D) is the prolongation of a hyperbolic exterior differential system that is not Pfaffian, then $\mathrm{Ch}(\partial D)$ is a subbundle of D, $\partial^2 D$ is the differential system such that $\mathrm{rank}\,\partial^2 D - \mathrm{rank}\,\partial D = 1$, and $\mathrm{Ch}(\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$ for $i = 1, 2$. Conversely, suppose that $\mathrm{Ch}(\partial D)$ is a subbundle of D, and $\partial^2 D$ is the differential system such that $\mathrm{rank}\,\partial^2 D - \mathrm{rank}\,\partial D = 1$. If $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$, or equivalently, $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$, then (R, D) coincides with the prolongation of a hyperbolic exterior differential system with an independence condition on a neighborhood.*

Proof. First we will show that differential systems $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ drops down to the quotient space $R/\mathrm{Ch}(\partial D)$ for each $i = 1, 2$. Using these systems, we will construct a hyperbolic exterior differential system we desire.

Let fix a point $u_o \in R$. There exists a coframe $\{\theta_1, \ldots, \theta_n, \xi_1, \xi_2, \omega^1, \omega^2, \pi_1, \pi_2\}$ around u_o such that $D = \{\theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = 0\}$ and

$$\begin{cases} d\theta_1 \equiv 0 \\ \quad\vdots \\ d\theta_n \equiv 0 \\ d\xi_1 \equiv \omega^1 \wedge \pi_1 \\ d\xi_2 \equiv \qquad \omega^2 \wedge \pi_2 \end{cases} \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \xi_2).$$

From the hypothesis $\mathrm{Ch}(\partial D) \subset D$, analogously to the proof of Theorem 4.5, we obtain

(4.3.9)
$$d\theta_i \equiv \xi_1 \wedge (A_i\,\omega^1 + B_i\,\pi_1) + \xi_2 \wedge (G_i\,\omega^2 + H_i\,\pi_2) + I_i\,\xi_1 \wedge \xi_2 \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n),$$

where A_i, B_i, G_i, H_i, I_i are functions around u_o, and there exist i, j such that

$$\mathrm{Ch}(\partial D) \subset \left\{ \theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = A_i\,\omega^1 + B_i\,\pi_1 = G_j\,\omega^2 + H_j\,\pi_2 = 0 \right\}$$

with $(A_i, B_i) \neq 0$ and $(G_j, H_j) \neq 0$. Then, from the hypothesis $\operatorname{rank}\partial^2 D - \operatorname{rank}\partial D = 1$, it is easy to see that $\mathrm{Ch}(\partial D)$ is of rank 2, which implies

$$\mathrm{Ch}(\partial D) = \left\{ \theta_1 = \cdots = \theta_n = \xi_1 = \xi_2 = A_i\,\omega^1 + B_i\,\pi_1 = G_j\,\omega^2 + H_j\,\pi_2 = 0 \right\}. \tag{4.3.10}$$

Then 1-forms

$$\bar{\omega}^1 = A_i\,\omega^1 + B_i\,\pi_1, \qquad \bar{\pi}_1 = -\frac{B_i}{A_i^2 + B_i^2}\,\omega^1 + \frac{A_i}{A_i^2 + B_i^2}\,\pi_1,$$

$$\bar{\omega}^2 = G_j\,\omega^2 + H_j\,\pi_2, \qquad \bar{\pi}_2 = -\frac{H_j}{H_j^2 + G_j^2}\,\omega^2 + \frac{G_j}{H_j^2 + G_j^2}\,\pi_2,$$

are linearly independent and satisfy, from (4.3.9) and (4.3.10),

$$d\theta_i \equiv \xi_1 \wedge \alpha_i\,\bar{\omega}^1 + \xi_2 \wedge \beta_i\,\bar{\omega}^2 \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1 \wedge \xi_2), \tag{4.3.11}$$

where α_i, β_i are functions around u_o. Then, by the hypothesis $\operatorname{rank}\partial^2 D - \operatorname{rank}\partial D = 1$, we may assume that $\theta_1, \ldots, \theta_n$ satisfy

$$\left.\begin{aligned} &d\theta_1 \equiv \cdots \equiv d\theta_{n-1} \equiv 0 \\ &d\theta_n \equiv \bar{\omega}^1 \wedge \xi_1 + \bar{\omega}^2 \wedge \xi_2 + \bar{I}\,\xi_1 \wedge \xi_2 \end{aligned}\right\} \qquad (\mathrm{mod}\ \theta_1, \cdots, \theta_n),$$

where $\bar{I}$ is a function around u_o. From (4.3.10), we can write the following:

$$\begin{aligned}
d\xi_1 &\equiv \bar{\omega}^1 \wedge \bar{\pi}_1 + \xi_2 \wedge (\bar{A}_1\,\bar{\pi}_1 + \bar{B}_1\,\bar{\pi}_2 + \bar{C}_1\,\bar{\omega}^1 + \bar{D}_1\,\bar{\omega}^2) \\
&\qquad\qquad\qquad\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1), \\
d\bar{\omega}^1 &\equiv \xi_2 \wedge (\bar{E}_1\,\bar{\pi}_1 + \bar{F}_1\,\bar{\pi}_2) + \bar{\omega}^2 \wedge (h_1\,\bar{\pi}_1 + k_1\,\bar{\pi}_2) + \bar{G}_1\,\xi_2 \wedge \bar{\omega}^2 \\
&\qquad\qquad\qquad\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \bar{\omega}^1), \\
d\xi_2 &\equiv \bar{\omega}^2 \wedge \bar{\pi}_2 + \xi_1 \wedge (\bar{A}_2\,\bar{\pi}_1 + \bar{B}_2\,\bar{\pi}_2 + \bar{C}_2\,\bar{\omega}^1 + \bar{D}_2\,\bar{\omega}^2) \\
&\qquad\qquad\qquad\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_2), \\
d\bar{\omega}^2 &\equiv \xi_1 \wedge (\bar{E}_2\,\bar{\pi}_1 + \bar{F}_2\,\bar{\pi}_2) + \bar{\omega}^1 \wedge (h_2\,\bar{\pi}_1 + k_2\,\bar{\pi}_2) + \bar{G}_2\,\xi_1 \wedge \bar{\omega}^1 \\
&\qquad\qquad\qquad\qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_2, \bar{\omega}^2),
\end{aligned}$$

where h_i, k_i, $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, $\bar{D}_i$, $\bar{E}_i$, $\bar{F}_i$, $\bar{G}_i$ $(i = 1, 2)$ are functions around u_o. Calculating $d^2\xi_i = 0$ $(\mathrm{mod}\ \theta_1, \ldots, \theta_n, \xi_1, \xi_2, \bar{\omega}^i)$, we obtain $A_1 = -k_1$, $B_2 = -h_2$. Moreover, calculating $d^2\theta_n = 0 \pmod{\theta_1, \dots, \theta_n, \xi_1 \wedge \xi_2, \bar{\omega}^1 \wedge \xi_1, \bar{\omega}^2 \wedge \xi_2}$, we obtain $\bar{A}_2 = -h_1$, $\bar{I} = k_1 - h_2$, $\bar{B}_1 = -k_2$, and $\bar{C}_2 = \bar{D}_1 = 0$. Thus we conclude

$$\left.\begin{aligned} d\theta_1 &\equiv 0 \\ &\vdots \\ d\theta_{n-1} &\equiv 0 \\ d\theta_n &\equiv (\bar{\omega}^1 - k_1\,\xi_2) \wedge \xi_1 + (\bar{\omega}^2 - h_2\,\xi_1) \wedge \xi_2 \end{aligned}\right\} \pmod{\theta_1, \dots, \theta_n},$$

$$\begin{aligned} d\xi_1 &\equiv (\bar{\omega}^1 - k_1\,\xi_2) \wedge (\bar{\pi}_1 - \bar{C}_1\,\xi_2) - k_2\,\xi_2 \wedge \bar{\pi}_2 \qquad \pmod{\theta_1, \dots, \theta_n, \xi_1}, \\ d\bar{\omega}^1 &\equiv \xi_2 \wedge (\bar{E}_1\,\bar{\pi}_1 + \bar{F}_1\,\bar{\pi}_2) + \bar{\omega}^2 \wedge (h_1\,\bar{\pi}_1 + k_1\,\bar{\pi}_2) + \bar{G}_1\,\bar{\omega}^2 \wedge \xi_2 \\ &\qquad\qquad \pmod{\theta_1, \dots, \theta_n, \xi_1, \bar{\omega}^1}, \\ d\xi_2 &\equiv (\bar{\omega}^2 - h_2\,\xi_1) \wedge (\bar{\pi}_2 - \bar{D}_2\,\xi_1) - h_1\,\xi_1 \wedge \bar{\pi}_1 \qquad \pmod{\theta_1, \dots, \theta_n, \xi_2}, \\ d\bar{\omega}^2 &\equiv \xi_1 \wedge (\bar{E}_2\,\bar{\pi}_1 + \bar{F}_2\,\bar{\pi}_2) + \bar{\omega}^1 \wedge (h_2\,\bar{\pi}_1 + k_2\,\bar{\pi}_2) - \bar{G}_2\,\xi_1 \wedge \bar{\omega}^1 \\ &\qquad\qquad \pmod{\theta_1, \dots, \theta_n, \xi_2, \bar{\omega}^2}. \end{aligned}$$

Then, for the Monge characteristic systems $\mathcal{M}_i$ of (R, D), it is easy to see that

$$\partial\mathcal{M}_i + \mathrm{Ch}(\partial D) = \left\{ \theta_1 = \cdots = \theta_n = \xi_i = \tilde{\omega}^i = 0 \right\},$$

where $\tilde{\omega}^1 = \bar{\omega}^1 - k_1\,\xi_2$, $\tilde{\omega}^2 = \bar{\omega}^2 - h_2\,\xi_1$.

Next we show that h_1 and k_2 vanish around u_o, if and only if $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$, if and only if $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$.

Setting $\tilde{\pi}_1 = \bar{\pi}_1 - \bar{C}_1\,\xi_2$, $\tilde{\pi}_2 = \bar{\pi}_2 - \bar{D}_2\,\xi_1$, we have

$$\begin{aligned} d\xi_1 &\equiv \tilde{\omega}^1 \wedge \tilde{\pi}_1 - k_2\,\xi_2 \wedge \tilde{\pi}_2 \qquad \pmod{\theta_1, \dots, \theta_n, \xi_1}, \\ d\xi_2 &\equiv \tilde{\omega}^2 \wedge \tilde{\pi}_2 - h_1\,\xi_1 \wedge \tilde{\pi}_1 \qquad \pmod{\theta_1, \dots, \theta_n, \xi_2}. \end{aligned}$$

Since $d\Omega^i \equiv 0 \pmod{\theta_1, \dots, \theta_n, \xi_1, \xi_2, \tilde{\omega}^i}$ for $i = 1, 2$, we can write

$$\begin{aligned} d\tilde{\omega}^1 &\equiv h_1\,\tilde{\omega}^2 \wedge \tilde{\pi}_1 + \xi_2 \wedge (\tilde{E}_1\,\tilde{\pi}_1 + \tilde{F}_1\,\tilde{\pi}_2 + \tilde{G}_1\,\tilde{\omega}^2) \qquad \pmod{\theta_1, \dots, \theta_n, \xi_1, \tilde{\omega}^1}, \\ d\tilde{\omega}^2 &\equiv k_2\,\tilde{\omega}^1 \wedge \tilde{\pi}_2 + \xi_1 \wedge (\tilde{E}_2\,\tilde{\pi}_1 + \tilde{F}_2\,\tilde{\pi}_2 + \tilde{G}_2\,\tilde{\omega}^1) \qquad \pmod{\theta_1, \dots, \theta_n, \xi_2, \tilde{\omega}^2}. \end{aligned}$$

and we have

$$\begin{cases} d\theta_1 \equiv \cdots \equiv d\theta_{n-1} \equiv 0 \\ d\theta_n \equiv \tilde{\omega}^2 \wedge \xi_2 \\ d\xi_1 \equiv -k_2\,\xi_2 \wedge \tilde{\pi}_2 \\ d\tilde{\omega}^1 \equiv h_1\,\tilde{\omega}^2 \wedge \tilde{\pi}_1 + \xi_2 \wedge (\tilde{E}_1\,\tilde{\pi}_1 + \tilde{F}_1\,\tilde{\pi}_2 + \tilde{G}_1\,\tilde{\omega}^2) \end{cases} \pmod{\theta_1,\ldots,\theta_n,\xi_1,\tilde{\omega}^1},$$

$$\begin{cases} d\theta_1 \equiv \cdots \equiv d\theta_{n-1} \equiv 0 \\ d\theta_n \equiv \tilde{\omega}^1 \wedge \xi_1 \\ d\xi_1 \equiv -h_1\,\xi_1 \wedge \tilde{\pi}_1 \\ d\tilde{\omega}^1 \equiv k_2\,\tilde{\omega}^1 \wedge \tilde{\pi}_2 + \xi_1 \wedge (\tilde{E}_2\,\tilde{\pi}_1 + \tilde{F}_2\,\tilde{\pi}_2 + \tilde{G}_2\,\tilde{\omega}^2) \end{cases} \pmod{\theta_1,\ldots,\theta_n,\xi_2,\tilde{\omega}^2}.$$

If $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ or $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$, h_1 and k_2 should vanish. If h_1 and k_2 vanish, calculating $d^2\xi_1 = 0 \pmod{\theta_1,\ldots,\theta_n,\xi_1,\tilde{\omega}^1,\tilde{\omega}^2\wedge\xi_2}$; $d^2\omega^1 = 0 \pmod{\theta_1,\ldots,\theta_n,\xi_1,\tilde{\omega}^1,\xi_2}$; $d^2\xi_2 = 0 \pmod{\theta_1,\ldots,\theta_n,\xi_2,\tilde{\omega}^2,\tilde{\omega}^1\wedge\xi_1}$, and $d^2\omega^2 = 0 \pmod{\theta_1,\ldots,\theta_n,\xi_2,\tilde{\omega}^2,\xi_1}$, we obtain $\tilde{E}_1 = \tilde{F}_1 = \tilde{E}_2 = \tilde{F}_2 = 0$. Thus we have found that $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ and $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincide with $\mathrm{Ch}(\partial D)$.

Next we construct a hyperbolic exterior differential system whose prolongation coincides locally with the given system (R, D). Let ρ be the canonical projection from R to $R/\mathrm{Ch}(\partial D)$. There exist differential system $\mathcal{H}_i$ and C on $R/\mathrm{Ch}(\partial D)$ of corank $n+2$ and n such that $\rho_*^{-1}(\mathcal{H}_i) = \partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ and $\rho_*^{-1}(C) = \partial D$. Then they can be written by

$$\mathcal{H}_i = \left\{ \Theta_1 = \cdots = \Theta_n = \Xi_i = \Omega^i = 0 \right\},$$
$$C = \{ \rho^*\Theta_1 = \cdots = \rho^*\Theta_n = 0 \},$$

with a coframe $\{\Theta_1,\ldots,\Theta_n,\Omega^1,\Omega^2,\Xi_1,\Xi_2\}$ around $\rho(u_o) \in R/\mathrm{Ch}(\partial D)$. These imply

$$d\rho^*\Theta_l \equiv 0 \qquad \pmod{\theta_1,\ldots,\theta_n},$$
$$\rho^*(\Omega^i \wedge \Xi_i) \equiv 0 \qquad \pmod{\theta_1,\ldots,\theta_n,\tilde{\omega}^i\wedge\xi_i},$$

for $l = 1,\ldots,n-1$ and $i = 1, 2$. Since $d\theta_l \equiv 0 \pmod{\theta_1,\ldots,\theta_n}$, we may assume $d\Theta_l \equiv 0 \pmod{\Theta_1,\ldots,\Theta_n}$. By $d\theta_n \equiv \tilde{\omega}^1 \wedge \xi_1 + \tilde{\omega}^2 \wedge \xi_2 \pmod{\theta_1,\ldots,\theta_n}$, we have

$$d\rho^*\Theta_n \equiv A_1\,\rho^*(\Omega^1 \wedge \Xi_1) + A_2\,\rho^*(\Omega^2 \wedge \Xi_2) \qquad \pmod{\rho^*\Theta_1,\ldots,\rho^*\Theta_n},$$

where A_1 and A_2 are functions around u_o. This implies that the coefficients A_1 and A_2 in the right hand sides can be written as the pullbacks of some functions around $\rho(u_o)$. Putting

$$\mathcal{I} = \left\{ \Theta_1,\ldots,\Theta_n,\Omega^1\wedge\Xi_1,\Omega^2\wedge\Xi_2 \right\}_{\mathrm{alg}},$$

we can find that $\mathcal{I}$ is a hyperbolic exterior differential system on a neighborhood of $R/\mathrm{Ch}(\partial D)$ and its Monge characteristic systems are $\mathcal{H}_i$.

Finally we will show that the prolongation of $\mathcal{I}$ with a independence condition coincides locally with the given system (R, D). By the hypothesis that $\mathrm{Ch}(\partial D) \subset D$ is of corank 2, we can apply Realization Lemma to $\rho : R \longrightarrow R/\mathrm{Ch}(\partial D)$; we have a unique map

$$\psi : R \longrightarrow J(R/\mathrm{Ch}(\partial D), 2) \, ; \quad v \longmapsto \rho_*(D(v))$$

such that $\Pi \circ \psi = \rho$ and $D = \psi_*^{-1}(C^*)$, where $\Pi : J(R/\mathrm{Ch}(\partial D), 2) \longrightarrow R/\mathrm{Ch}(\partial D)$ is the canonical projection and C^* is the canonical system on $J(R/\mathrm{Ch}(\partial D), 2)$. Furthermore, since $\mathrm{Ker}\,(\psi_*)_v = \mathrm{Ker}\,(\rho_*)_v \cap \mathrm{Ch}(D) = \{0\}$, ψ is an immersion.

Here, since $\rho_*^{-1}(\mathcal{H}_i) = \partial \mathcal{M}_i + \mathrm{Ch}(\partial D)$, we may assume $\rho^*(\Omega_1 \wedge \Omega_2)$ and $\omega^1 \wedge \omega^2$ are equivalent modulo $\theta_1, \ldots, \theta_n$, ξ_1, ξ_2, up to scale. Then let $(\hat{R}, \hat{D})$ denote the prolongation of $(R/\mathrm{Ch}(\partial D), C)$ with the independence condition $\Omega^1 \wedge \Omega^2$. For any $v \in R$, by definition, it is easy to see that $\Theta_i|_{\psi(v)} = 0$, $d\Theta_i|_{\psi(v)} = 0$ and $\Omega^1 \wedge \Omega^2 \,|_{\psi(v)} \neq 0$. Namely $\psi(v)$ is an element of $\hat{R}$. By the structure equation of C, we find $\dim \hat{R} = \dim(R/\mathrm{Ch}(\partial D)) + 2 = \dim R$. Therefore (R, D) coincides locally with the prolongation of $(R/\mathrm{Ch}(\partial D), C)$ with the independence condition $\Omega^1 \wedge \Omega^2$. □

5 Second order partial differential equations of m (≥ 2) unknown functions

5.1 Preliminaries

5.1.1 Jet space $(J^2(M, n), C^2)$ of second order

First we will recall the jet space $(J^2(M, n), C^2)$ of second order in order to treat with second order partial differential equations of m (≥ 2) unknown functions ([Yam83]). For convention, we put $J^0(M, n) = M$ and $(J^1(M, n), C^1) = (J(M, n), C)$, and write the projection $\Pi : J^1(M, n) \longrightarrow M$ as Π^1_0. Let $Q^1 = \mathrm{Ker}\,(\Pi^1_0)_*$, which is the differential system of codimension mn. Each fiber $J^2(M, n)_x$ of $J^2(M, n)$ over $x \in J^1(M, n)$ consists of all n-dimensional integral elements u of C^1 at x that is transverse to $Q^1(x) \subset T_x J^1(M, n)$, namely $u \cap Q^1(x) = \{0\}$. $J^2(M, n)$ is the bundle of dimension $n + m + mn + m \cdot {}_n\mathrm{H}_2$, where ${}_m\mathrm{H}_n = \binom{m+n-1}{n}$. The canonical system C^2 is defined by

$$C^2(u) = (\Pi^2_1)_*^{-1}(u) \quad \text{for } u \in J^2(M, n),$$

where $\Pi^2_1 : J^2(M,n) \longrightarrow J^1(M,n)$ is the canonical projection. For a point $u_o \in J^2(M,n)$, we have a canonical coordinate $(x^1,\dots,x^n,z^1,\dots,z^m,p^1_1,\dots,p^m_n)$ on a neighborhood U of $\Pi^2_1(u_o)$ in $J^1(M,n)$. Let $\hat{U}$ be a neighborhood of u_o that consists of all points $u \in \Pi^{-1}(U)$ such that $dx^1 \wedge \cdots \wedge dx^n|_u \neq 0$. Let p^a_{ij} for $1 \leq a \leq m$ and $1 \leq i, j \leq n$ be functions on $\hat{U}$ so that $dp^b_j|_u - \sum_i p^b_{ji}(u)\, dx^i|_u = 0$ for $u \in \hat{U}$ and $1 \leq b \leq m$, $1 \leq j \leq n$. Since $d\varpi^b|_u = 0$ for $u \in \hat{U}$ and $1 \leq b \leq m$, we have $p^b_{ij} = p^b_{ji}$ for $1 \leq b \leq m$, $1 \leq i \leq j \leq n$. Thus $(x^i, z^a, p^a_i, p^a_{ij}\ (1 \leq a \leq m, 1 \leq i \leq j \leq n))$ forms a coordinate system on $\hat{U}$, which is called the canonical coordinate system of $J^2(M,n)$. The canonical system C^2 on $\hat{U}$ is given by

$$C^2 = \left\{ \varpi^a = \varpi^a_i = 0\ (1 \leq a \leq m,\ 1 \leq i \leq n) \right\}, \tag{5.1.1}$$

where $\varpi^a = dz^a - \sum_{i=1}^n p^a_i dx^i$, $\varpi^a_i = dp^a_i - \sum_{k=1}^n p^a_{ik} dx^k$.

5.1.2 Symbol algebras of differential systems

We will recall the symbol algebra $\mathfrak{m}(x)$ of a differential system (M,D) at $x \in M$, introduced by N. Tanaka ([Tan70]). Let (M,D) be a regular differential system such that $TM = D^{-\mu}$. We recall the *symbol algebra* $\mathfrak{m}(x)$ of (M,D) at $x \in M$. Let

$$\mathfrak{m}(x) = \bigoplus_{p=-\mu}^{-1} \mathfrak{g}_p(x), \qquad \mathfrak{g}_{-1}(x) = D^{-1}(x), \quad \mathfrak{g}_{-p}(x) = D^{-p}(x)/D^{-p+1}(x)\ (p > 1).$$

Let π_{-p} denote the projection of $D^{-p}(x)$ onto $\mathfrak{g}_{-p}(x)$. For $X \in \mathfrak{g}_{-p}(x)$, $Y \in \mathfrak{g}_{-q}(x)$, the bracket product $[X, Y] \in \mathfrak{g}_{-(p+q)}(x)$ is well-defined by

$$[X, Y] = \pi_{-(p+q)}([\hat{X}, \hat{Y}]_x),$$

where $\hat{X}$ and $\hat{Y}$ are vector fields taking values in $\mathcal{D}^{-p}$ and $\mathcal{D}^{-q}$ respectively such that $\pi_{-p}(\hat{X}_x) = X$ and $\pi_{-q}(\hat{Y}_x) = Y$. Then $\mathfrak{m}(x)$ is a nilpotent graded Lie algebra with this bracket operation, such that $\dim \mathfrak{m}(x) = \dim M$ and satisfies

$$\mathfrak{g}_{-p}(x) = [\mathfrak{g}_{-p+1}(x), \mathfrak{g}_{-1}(x)] \quad \text{for } p > 1.$$

The graded Lie algebra $\mathfrak{m}(x)$ is called the *symbol algebra of* (M,D) *at* x. Generally, $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a *fundamental graded Lie algebra of* μ*-th kind* if $\mathfrak{m}$ is a nilpotent graded Lie algebra such that $\mathfrak{g}_{-\mu} \neq 0$ and $\mathfrak{g}_{-k} = 0$ for all $k > \mu$, and

$$\mathfrak{g}_{-p} = [\mathfrak{g}_{-p+1}, \mathfrak{g}_{-1}] \quad \text{for } p > 1.$$

For a fundamental graded Lie algebra $\mathfrak{m}$, (M, D) is *of type* $\mathfrak{m}$ if the symbol algebra $\mathfrak{m}(x)$ of (M, D) is isomorphic to $\mathfrak{m}$ at each $x \in M$.

Conversely, given a fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p=-\mu}^{-1} \mathfrak{g}_p$ of μ-th kind, we can construct a regular differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$, which is called the *standard differential system of type* $\mathfrak{m}$: Let $M(\mathfrak{m})$ be the simply connected Lie group with Lie algebra $\mathfrak{m}$. Then we define a left invariant subbundle $D_{\mathfrak{m}}$ of $TM(\mathfrak{m})$ by $\mathfrak{g}_{-1}$. Then $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is a regular differential system of type $\mathfrak{m}$.

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded Lie algebra of μ-th kind. The *prolongation* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p(\mathfrak{m})$ of $\mathfrak{m}$ is defined inductively as follows ([Yam93]):

$$\begin{aligned}
\mathfrak{g}_{-p}(\mathfrak{m}) &= \mathfrak{g}_{-p} \quad \text{for } p > 0, \\
\mathfrak{g}_0(\mathfrak{m}) &= \left\{ u \in \bigoplus_{p<0} \mathfrak{g}_p \otimes \mathfrak{g}_p^* \;\middle|\; u([X, Y]) = [u(X), Y] + [X, u(Y)] \right\}, \\
\mathfrak{g}_k(\mathfrak{m}) &= \left\{ u \in \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^* \;\middle|\; u([X, Y]) = [u(X), Y] + [X, u(Y)] \right\} \quad \text{for } k > 0.
\end{aligned}$$

Now we will see that $\mathfrak{g}(\mathfrak{m})$ is a graded Lie algebra. The bracket operation of $\mathfrak{g}(\mathfrak{m})$ is given as follows: First, for $u_0, u_0' \in \mathfrak{g}_0$, we define $[u_0, u_0'] \in \mathfrak{g}_0$ by

$$[u_0, u_0'](X) = u_0(u_0'(X)) - u_0'(u_0(X)) \quad \text{for } X \in \mathfrak{m}.$$

Thus $\mathfrak{g}_0(\mathfrak{m})$ becomes a Lie algebra with this bracket operation. Moreover, putting

$$[u_0, X] = -[X, u_0] = u_0(X) \quad \text{for } u_0 \in \mathfrak{g}_0(\mathfrak{m}) \text{ and } X \in \mathfrak{m},$$

we see that $\bigoplus_{p\leq 0} \mathfrak{g}_p(\mathfrak{m})$ is a graded Lie algebra.

Similarly, for $u_k \in \mathfrak{g}_k(\mathfrak{m})$ $(k > 0)$ and $X \in \mathfrak{m}$, we put $[u_k, X] = -[X, u_k] = u_k(X)$. For $u_k \in \mathfrak{g}_k(\mathfrak{m})$ and $u_l \in \mathfrak{g}_l(\mathfrak{m})$ $(k, l \geq 0)$, by induction on the integer $k + l \geq 0$, we define $[u_k, u_l] \in \mathfrak{g}_{k+l}(\mathfrak{m})$ by

$$[u_k, u_l](X) = [u_k, [u_l, X]] - [u_l, [u_k, X]] \quad \text{for } X \in \mathfrak{m}.$$

Then it follows easily that $\mathfrak{g}(\mathfrak{m})$ is a graded Lie algebra with this bracket operation.

It is known that the structure of the Lie algebra $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ of all infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ can be described by $\mathfrak{g}(\mathfrak{m})$. Especially, $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ is isomorphic to $\mathfrak{g}(\mathfrak{m})$ when $\mathfrak{g}(\mathfrak{m})$ is finite dimensional. For detail, see [Tan70].

5.1.3 Symbol algebra $\mathfrak{C}^2(n,m)$ of $(J^2(M,n),C^2)$

We will recall the symbol algebra $\mathfrak{C}^2(n,m)$ of the canonical system $(J^2(M,n),C^2)$ ([Yam82]). Let M be a smooth manifold of dimension $m+n$ and $(J^2(M,n),C^2)$ the jet space of second order. Let us take the canonical coordinate system $(x^i,z^a,p_i^a,p_{ij}^a$ $(1 \leq a \leq m, 1 \leq i \leq j \leq n))$ on a neighborhood U as in Section 5.1.1. Then we have a local coframe

$$\left\{ \varpi^a,\, \varpi_i^a,\, dx^i,\, dp_{ij}^a \ (1 \leq a \leq m,\, 1 \leq i \leq j \leq n) \right\},$$

where $\varpi^a = dz^a - \sum_{i=1}^n p_i^a dx^i$, $\varpi_i^a = dp_i^a - \sum_{k=1}^n p_{ik}^a dx^k$. Let us take the dual frame of this coframe

$$\left\{ \frac{\partial}{\partial z^a}, \frac{\partial}{\partial p_i^a}, \frac{d}{dx^i}, \frac{\partial}{\partial p_{ij}^a} \ (1 \leq a \leq m,\, 1 \leq i \leq j \leq n) \right\},$$

where

$$\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_{a=1}^m p_i^a \frac{\partial}{\partial z^a} + \sum_{a=1}^m \sum_{k=1}^n p_{ik}^a \frac{\partial}{\partial p_k^a}.$$

Then we have

$$\left[\frac{\partial}{\partial p_{ij}^a}, \frac{d}{dx^k}\right] = \left(\delta_k^j - \frac{1}{2}\delta_i^j\right)\frac{\partial}{\partial p_i^a} + \left(\delta_k^i - \frac{1}{2}\delta_i^j\right)\frac{\partial}{\partial p_j^a}, \qquad \left[\frac{\partial}{\partial p_i^a}, \frac{d}{dx^k}\right] = \delta_k^i \frac{\partial}{\partial z^a}$$

and

$$\begin{aligned} C^2 &= \left\langle \frac{\partial}{\partial p_{ij}^a}, \frac{d}{dx^i} \ (1 \leq a \leq m,\, 1 \leq i \leq j \leq n) \right\rangle, \\ \partial^{(1)} C^2 &= \left\langle \frac{\partial}{\partial p_i^a}, \frac{\partial}{\partial p_{ij}^a}, \frac{d}{dx^i} \ (1 \leq a \leq m,\, 1 \leq i \leq j \leq n) \right\rangle, \\ \partial^{(2)} C^2 &= TJ^2(M,n). \end{aligned}$$

Thus we see that the symbol algebra of $(J^2(M,n),C^2)$ is isomorphic to $\mathfrak{C}^2(n,m)$, which is defined as follows([Yam82]): Let V and W be vector space of dimension n and m respectively. Let

$$\mathfrak{C}^2(V,W) = \mathfrak{C}^2_{-3} \oplus \mathfrak{C}^2_{-2} \oplus \mathfrak{C}^2_{-1}, \quad \mathfrak{C}^2_{-3} = W, \quad \mathfrak{C}^2_{-2} = W \otimes V, \quad \mathfrak{C}^2_{-1} = V \oplus W \otimes S^2(V^*).$$

The bracket operations of $\mathfrak{C}^2(n,m)$ is defined through the pairing between V and V^* such that V and $W \otimes S^2(V^*)$ are abelian subspaces of $\mathfrak{C}^2_{-1}$.

5.1.4 Graded simple Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let Φ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ and choose a simple root system Δ of Φ. Then we have the root decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \Big(\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{\alpha}\Big),$$

where Φ^+ denotes the set of positive roots and $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [h, X] = \alpha(h)X \text{ for } h \in \mathfrak{h}\}$ for $\alpha \in \Phi$. Let us take a non-empty subset Δ_1 of Δ. Then Δ_1 induces the following gradation:

$$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p, \quad \mathfrak{g}_{-p} = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{g}_0 = \Big(\bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_{\alpha}\Big), \quad \mathfrak{g}_p = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_{\alpha},$$

where

$$\Phi^+ = \bigcup_{p\geq 0} \Phi_p^+, \qquad \Phi_p^+ = \Big\{\alpha = \sum_{i=1}^{l} n_i\alpha_i \in \Phi^+ \,\Big|\, \sum_{\alpha_k\in\Delta_1} n_k = p\Big\}.$$

Moreover, the negative part $\mathfrak{m} = \bigoplus_{p\leq 0} \mathfrak{g}_p$ of $\mathfrak{g}$ is a fundamental graded Lie algebra, namely $\mathfrak{m}$ satisfies

$$\mathfrak{g}_{-(p+1)} = [\mathfrak{g}_{-p}, \mathfrak{g}_{-1}] \quad \text{for } p > 0.$$

Let θ be the highest root of Φ^+. Writing $\theta = \sum_{i=1}^{l} n_i(\theta)\alpha_i$ for some $n_i(\theta) \in \mathbb{Z}_{\geq 0}$, we have

$$\mu = \sum_{\alpha_k\in\Delta_1} n_k(\theta),$$

where μ denotes the lowest integer such that $\mathfrak{g}_{-\mu} \neq 0$ and $\mathfrak{g}_{-(\mu+1)} = 0$, namely $\mathfrak{m}$ is of μ-th kind.

When $\mathfrak{g}$ is a simple Lie algebra of type X_l, let (X_l, Δ_1) denote the simple Lie algebra $\mathfrak{g}$ with the gradation defined by Δ_1.

Conversely, it is known that the gradation of any simple graded Lie algebra over $\mathbb{C}$ is obtained from some $\Delta_1 \subset \Delta$:

Theorem 5.1 ([Yam93]). *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ satisfying $\mathfrak{g}_{-(p+1)} = [\mathfrak{g}_{-p}, \mathfrak{g}_{-1}]$ for $p > 0$. Let X_l be the Dynkin diagram of $\mathfrak{g}$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic to a graded Lie algebra (X_l, Δ_1) for some $\Delta_1 \subset \Delta$. Moreover (X_l, Δ_1) and (X_l, Δ_1') are isomorphic if and only if there exists a diagram automorphism ϕ of X_l such that $\phi(\Delta_1) = \Delta_1'$.*

In Section 5.4 we will seek a simple graded Lie algebra that is isomorphic to the prolongation of the symbol algebra of partial differential equations (or PD-manifolds $(R;\, D^1, D^2)$) of m (≥ 2) unknown functions.

5.2 Characterization of partial differential equations

Let M be a manifold of dimension $m + n$ (m, $n \geq 2$) Let R be a submanifold of $J^2(M,n)$ satisfying the condition

(**R.0**) $\rho : R \longrightarrow J^1(M,n)$ *is submersion*

where ρ is the restriction of the projection $\Pi : J^2(M,n) \longrightarrow J^1(M,n)$ to R. This condition implies that the system of second order differential equations R never contains equations of only first order. Let $\iota : R \longrightarrow J^2(M,n)$ be the inclusion. Let D^1 and D^2 be differential systems on R defined by the pullback by ι of ∂C^2 and C^2 respectively. Let $\varpi^1, \ldots, \varpi^m$ and $\varpi^1_1, \ldots, \varpi^m_n$ be a 1-forms on $J^2(M,n)$ such that $\partial C^2 = \{\varpi^a = 0\ (1 \leq a \leq m)\}$ and $C^2 = \{\varpi^a = \varpi^a_i = 0\ (1 \leq a \leq m,\, 1 \leq i \leq n)\}$. Then it follows from Condition (**R.0**) that these forms ϖ^a, ϖ^a_i are independent at each point on R and that

(5.2.2)

$$D^1 = \{\, \varpi^a = 0\ (1 \leq a \leq m)\,\}, \quad D^2 = \left\{\, \varpi^a = \varpi^a_i = 0\ (1 \leq a \leq m,\, 1 \leq i \leq n)\,\right\}.$$

Here, by our abuse of notation, we write $\iota^*\varpi$ as ϖ. Thus we see that

(**R.1**) D^1 *and* D^2*are differential systems of codimension* m *and* $m + mn$ *respectively.*

From (5.1.1), there exist 1-forms $\omega^1, \ldots, \omega^n$ on R such that the forms ϖ^a, ϖ^a_i, and ω^i are independent at each point and $d\varpi^a \equiv \sum_i \omega^i \wedge \varpi^a_i \pmod{\varpi^b\ (1 \leq b \leq m)}$. Therefore we have

(**R.2**) $\partial D^2 \subset D^1.$

Since $\mathrm{Ch}(D^1) = \{\varpi^a = \varpi^a_i = \omega^i = 0\ (1 \leq a \leq m,\, 1 \leq i \leq n)\}$,

(**R.3**) $\mathrm{Ch}(D^1)$ *is a subbundle of* D^2 *of codimension* n.

Since $d\varpi^a \wedge \omega^1 \wedge \cdots \wedge \omega^n \equiv 0 \pmod{\varpi^b\ (1 \leq b \leq m)}$ for $1 \leq a \leq m$, we have

(**R.4**) D^1 *is of Cartan rank* n.

Applying Realization Lemma to $\rho : R \longrightarrow J^1(M,n)$ and D^2, we obtain

(**R.5**) $$\mathrm{Ch}(D^1) \cap \mathrm{Ch}(D^2) = \{0\}.$$

In fact, from $\mathrm{Ker}\,\rho_* = \mathrm{Ch}(D^1) \subset D^2$, we have the unique map $\psi : R \longrightarrow J(J^1(M,n),n)$ such that $\rho = \Pi \circ \psi$ and $D^2 = \psi_*^{-1}(C)$, where $\Pi : J(J^1(M,n),n) \longrightarrow J^1(M,n)$ is the projection and C is the canonical system of $J(J^1(M,n))$. By definition, for $v \in R$, $\psi(v)$ is n-dimensional integral element of $(J^1(M,n), C^1)$ and transverse to $Q^1 = \mathrm{Ker}\,(\Pi^1_0)_*(\rho(v))$. Namely $\psi(v) \in J^2(M,n)$. By the uniqueness of ψ, we have $\psi = \iota$. Therefore $\mathrm{Ch}(D^1)(v) \cap \mathrm{Ch}(D^2)(v) = \mathrm{Ker}\,\rho_*(v) \cap \mathrm{Ch}(D^2)(v) = \{0\}$.

Furthermore we will see that there exists an additional differential system F in the following lemma:

Lemma 5.2. *Let R be a manifold and D^1, D^2 differential systems satisfying four conditions from (**R.1**) to (**R.4**). Then there exists a unique subbundle F of D^1 of corank n such that $\partial F \subset D^1$. Moreover, we have $F \cap D^2 = \mathrm{Ch}(D^1)$. Furthermore, if $m \geq 3$, F is completely integrable.*

Proof. D^1 and D^2 are locally expressed as follows:

$$D^1 = \left\{ \varpi^1 = \cdots = \varpi^m = 0 \right\}$$
$$D^2 = \left\{ \varpi^1 = \cdots = \varpi^m = \varpi^1_1 = \cdots = \varpi^m_n = 0 \right\}$$

with linearly independent 1-forms $\varpi^1, \ldots, \varpi^m, \varpi^1_1, \ldots, \varpi^m_n$. Condition (**R.2**) implies $d\varpi^a \equiv 0 \pmod{\varpi^b, \varpi^b_j\ (1 \leq b \leq m, 1 \leq j \leq n)}$ for $1 \leq a \leq m$, and thus they are expressed as $d\varpi^a \equiv \sum_{b,j} \pi^{aj}_b \wedge \varpi^b_j \pmod{\varpi^c_k\ (1 \leq c \leq m, 1 \leq k \leq n)}$. Since $\mathrm{Ch}(D^1) \subset D^2$, for each point x and $X \in \mathrm{Ch}(D^1)(x)$, we have

$$\begin{aligned} 0 &\equiv X \lrcorner\, d\varpi^a \qquad \pmod{\varpi^1_x, \ldots, \varpi^m_x} \\ &\equiv \sum_{b,j} \pi^{aj}_b(X)\, \varpi^b_j \end{aligned}$$

Therefore, $\mathrm{Ch}(D^1) = \{\varpi^a = \varpi^a_i = \pi^{aj}_b = 0\ (1 \leq a, b \leq m, 1 \leq i, j \leq n)\}$. On the other hand, from Condition (**R.4**), we can take 1-forms $\omega^1, \ldots, \omega^n$ so that $\varpi^1 \wedge \cdots \wedge \varpi^m \wedge \omega^1 \wedge \cdots \wedge \omega^n \neq 0$ and $d\varpi^a \wedge \omega^1 \wedge \cdots \wedge \omega^n \equiv 0 \pmod{\varpi^b\ (1 \leq b \leq m)}$. Substituting the expression of $d\varpi^a$ into the second equation, we obtain $\pi^{aj}_b \equiv 0 \pmod{\varpi^c, \varpi^c_k, \omega^k\ (1 \leq c \leq m, 1 \leq k \leq n)}$. Thus, from Condition (**R.3**), we achieve

$$\mathrm{Ch}(D^1) = \left\{ \varpi^a = \varpi^a_i = \omega^i = 0\ (1 \leq a \leq m, 1 \leq i \leq n) \right\}$$

and the ϖ^a, ϖ^a_i, ω^i are linearly independent. This allows us to write $d\varpi^a \equiv \sum_i \omega^i \wedge \pi^a_i$ $\pmod{\varpi^b\,(1 \leq b \leq m)}$. Since $d\varpi^a \equiv 0 \pmod{\varpi^b, \varpi^b_j\ (1 \leq b \leq m, 1 \leq j \leq n)}$, we have $\pi^a_i \equiv 0 \pmod{\varpi^b, \varpi^b_j, \omega^j\ (1 \leq b \leq m, 1 \leq j \leq n)}$, which implies $\mathrm{Ch}(D^1) \subset \{\varpi^a = \omega^i = 0\ (1 \leq a \leq m, 1 \leq i \leq n)\}$. Therefore, for each point x and $X \in \mathrm{Ch}(D^1)(x)$, we have $0 \equiv X \lrcorner\, d\varpi^a \equiv -\sum_i \pi^a_i(X)\,\omega^i_x \pmod{\varpi^b_x\ (1 \leq b \leq m)}$ and thus $\mathrm{Ch}(D^1) = \{\varpi^a = \pi^a_i = \omega^i = 0\ (1 \leq a \leq m, 1 \leq i \leq n)\}$. This follows us to write $\pi^a_i \equiv \sum_{b,j} A^{aj}_{ib}\,\varpi^b_j + \sum_j B^a_{ij}\,\omega^j \pmod{\varpi^c\ (1 \leq c \leq m)}$ with some functions A^{aj}_{ib}, B^a_{ij} and that the ϖ^a, π^a_i, ω^i are linearly independent. Substituting this into $\sum_i \omega^i \wedge \pi^a_i \equiv 0$ $\pmod{\varpi^b, \varpi^b_j\ (1 \leq b \leq m, 1 \leq j \leq n)}$, we obtain $B^a_{ji} = B^a_{ij}$. Replacing $A^{aj}_{ib}\,\varpi^b_j$ by ϖ^a_i, we achieve

$$d\varpi^a \equiv \sum_i \omega^i \wedge \varpi^a_i \qquad \pmod{\varpi^1, \ldots, \varpi^m} \tag{5.2.3}$$

and find that the ϖ^a, ϖ^a_i, ω^i are linearly independent.

Let F be a subbundle of D^1 of corank n defined by the ϖ^a and ω^i, which satisfies $\partial F \subset D^1$. Now, let $\hat{F}$ be an another subbundle of D^1 of corank n satisfying $\partial\hat{F} \subset D^1$. Write $\hat{F} = \{\varpi^a = \hat{\omega}^i = 0\ (1 \leq a \leq m, 1 \leq i \leq n)\}$ with some 1-forms $\hat{\omega}^i$. Since $\partial\hat{F} \subset D^1$, $d\varpi^a \equiv 0 \pmod{\varpi^b, \hat{\omega}^j\ (1 \leq b \leq m, 1 \leq j \leq n)}$. By Equation (5.2.3), we have $\omega^i \equiv 0 \pmod{\varpi^b, \varpi^b_j, \hat{\omega}^j\ (1 \leq b \leq m, 1 \leq j \leq n)}$. This follows us to write $\omega^i \equiv \sum_j A^i_j\,\hat{\omega}^j + \sum_{b,j} B^{ij}_b\,\varpi^b_j \pmod{\varpi^c, \varpi^c_k (1 \leq c \leq m, 1 \leq k \leq n)}$ with some functions A^i_j, B^{ij}_b. Substituting this into Equation (5.2.3), we have, for $1 \leq a \leq m$, $B^{ji}_a = B^{ij}_a$ and $B^{ij}_b = 0\,(b \neq a)$. It follows from $m \geq 2$ that $B^{ij}_a = 0$ for $1 \leq a \leq m$ and $1 \leq i, j \leq n$. Thus we achieve $F = \hat{F}$. From the expression of F, it easily follows that $F \cap D^2 = \mathrm{Ch}(D^1)$.

Assume $m \geq 3$. Since

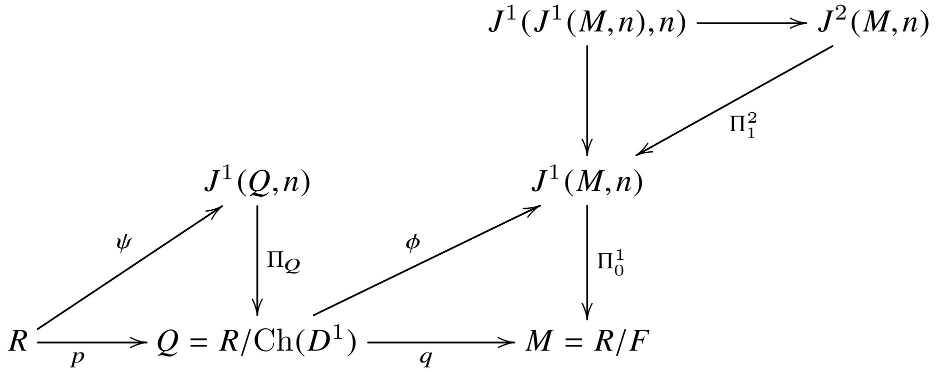
$$\begin{aligned} 0 &= d^2\varpi^a \\ &\equiv \sum_i d\omega^i \wedge \varpi^a_i \qquad \pmod{\varpi^b, \omega^j\ (1 \leq b \leq m, 1 \leq j \leq n)}, \end{aligned}$$

we have $d\omega^i \equiv 0 \pmod{\varpi^b, \varpi^a_j, \omega^j\ (1 \leq b \leq m, 1 \leq j \leq n)}$, for $1 \leq a \leq m$, $1 \leq i \leq n$. For $m \geq 3$, $d\omega^i \equiv 0 \pmod{\varpi^b, \omega^j\ (1 \leq b \leq m, 1 \leq j \leq n)}$, which implies that F is completely integrable. □

Under certain assumptions, converse of the above discussion is true: Let D^1, D^2 be differential systems on a manifold R satisfying the condition from (**R.1**) to (**R.5**) and the condition

(**R.6**) *F is completely integrable*, where the differential system F is defined in Lemma 5.2. Condition (**R.6**) is satisfied automatically from (**R.1**) to (**R.4**) unless $m = 2$. A triplet $(R;\ D^1, D^2)$ satisfying the conditions from (**R.1**) to (**R.6**) is called a *PD-manifold of second order*. Then the following theorem implies that a PD-manifold $(R;\ D^1, D^2)$ is locally embedded into $(J^2(M,n), C^2)$:

Theorem 5.3. *Let $(R;\ D^1, D^2)$ be a PD-manifold of second order. Let F be the differential system on R in Lemma 5.2. Assume R is regular with respect to F, namely the space $M = R/F$ of leaves of the foliation is a manifold of dimension $m + n$ such that each fiber of the projection $\rho : R \longrightarrow M$ is connected and ρ is submersion. Then there exists a local embedding $\iota : R \longrightarrow J^2(M,n)$ such that $\rho = \Pi \circ \iota$ and $D^2 = \iota_*^{-1}(C^2)$, where $\Pi = \Pi^2_1 \circ \Pi^1_0$ and Π^k_{k-1} is the canonical projection of $J^k(M,n)$ onto $J^{k-1}(M,n)$ for $k = 1, 2$.*



Proof. Let $Q = R/\mathrm{Ch}(D^1)$ be the space of leaves of the foliation. By the assumption on R, R is regular with respect to $\mathrm{Ch}(D^1)$ as well. Let p denote the canonical projection of R onto Q. Since $\mathrm{Ker}\, p_* = \mathrm{Ch}(D^1)$ and $\mathrm{Ch}(F) = F \supset \mathrm{Ker}\, p_*$, it follows that there exist differential systems C^1_Q and F_Q on Q such that $D^1 = p_*^{-1}(C^1_Q)$ and $F = p_*^{-1}(F_Q)$. F_Q is a subbundle of C^1_Q of codimension n and completely integrable. Let q denote the canonical projection of Q onto M. By applying Realization Lemma to q, we find that there exists a unique map $\phi : Q \longrightarrow J^1(M,n)$ such that $q = \Pi^1_0 \circ \phi$ and $C^1_Q = \phi_*^{-1}(C^1_M)$, and, moreover, that ϕ is immersion, where C^1_M is the canonical system on $J^1(M,n)$. Since $\dim J^1(M,n) = m + n + mn = \dim Q$, ϕ is a local diffeomorphism, namely $(J^1(M,n), C^1_M)$ and (Q, C^1_Q) are locally equivalent. By applying Realization Lemma to the map p, we find that there exists a unique map $\psi : R \longrightarrow J^1(Q,\, n)$ such that $\rho = \Pi_Q \circ \psi$ and $D^2 = \psi_*^{-1}(C^1)$, where $\Pi_Q : J^1(Q,n) \longrightarrow Q$ is the canonical projection. It follows from $\mathrm{Ker}\, \rho_* = \mathrm{Ch}(D^1)$ and

(**R.5**) that ψ is immersion. Finally we will show that $\psi(v)$ is a n-dimensional integral element of C^1_M and $\psi(v) \cap \mathrm{Ker}\,(\Pi^1_0)_*(\phi(p(v)))$ for each $v \in R$. Since $\psi(v) = p_*(D^2(v))$ and $\mathrm{Ch}(D^1)$ is a subbundle of D^2 of codimension n, $\psi(v)$ is a n-dimensional integral element of C_M. Since $D^2(v) \cap F(v) = \mathrm{Ch}(D^1)$ and $\mathrm{Ker}\,(\Pi^1_0)_*(\phi(p(v))) = F_Q(p(v))$, we have $p_*^{-1}(\psi(v) \cap \mathrm{Ker}\,(\Pi^1_0)_*) = \mathrm{Ch}(D^1)$. □

5.3 Partial differential equations of finite type

We will seek an example of partial differential equations of finite type by utilizing fundamental graded Lie algebras, simple Lie algebras and representation theory.

5.3.1 Symbol algebra of PD-manifold $(R;\, D^1, D^2)$

We will define the symbol algebra $\mathfrak{s}(x) = \mathfrak{s}_{-3}(x) \oplus \mathfrak{s}_{-2}(x) \oplus \mathfrak{s}_{-1}(x)$ of a PD-manifold $(R;\, D^1, D^2)$ at a point $x \in R$, following [Yam82]. Let us fix a point $x \in R$ and put $D^{-1} = D^2$, $D^{-2} = D^1$ and $D^{-3} = TR$. We set

$$\mathfrak{s}_{-3}(x) = D^{-3}(x)/D^{-2}(x), \qquad \mathfrak{s}_{-2}(x) = D^{-2}(x)/D^{-1}(x), \qquad \mathfrak{s}_{-1}(x) = D^{-1}(x).$$

The bracket operation of $\mathfrak{s}(x)$ is defined as follows: Let π_{-p} denote the projection of $D^{-p}(x)$ onto $\mathfrak{s}_{-p}(x)$ for $1 \leq p \leq 3$. For $X \in \mathfrak{s}_{-p}(x)$, $Y \in \mathfrak{s}_{-q}(x)$, the bracket product $[X, Y] \in \mathfrak{s}_{-(p+q)}(x)$ is well-defined by

$$[X, Y] = \pi_{-(p+q)}([\hat{X}, \hat{Y}]_x),$$

where $\hat{X}$ and $\hat{Y}$ denote vector fields taking values in $\mathcal{D}^{-p}$ and $\mathcal{D}^{-q}$ respectively such that $\pi_{-p}(\hat{X}_x) = X$ and $\pi_{-q}(\hat{Y}_x) = Y$. Let $\mathfrak{f}(x) = \mathrm{Ch}(D^1)(x)$. It follows from **(R.3)** that $\mathfrak{f}(x)$ is a subspace of $\mathfrak{s}_{-1}(x)$ of codimension n. For $X \in \mathfrak{s}_{-1}(x)$, since $d\varpi^a(X, Y) = 0$ for all $Y \in D^1(x)$ if and only if $[X, \mathfrak{s}_{-2}(x)] = 0$, we obtain

(5.3.4) $$\mathfrak{f}(x) = \{\, X \in \mathfrak{s}_{-1}(x) \mid [X, \mathfrak{s}_{-2}(x)] = 0 \,\}.$$

Let ϖ^a, ϖ^a_i $(1 \leq a \leq m, 1 \leq i \leq n)$ denote 1-forms defining D^1 and D^2 as in (5.2.2). Since they are the restriction of the defining 1-forms of C^2, we see that $\mathfrak{s}(x)$ is isomorphic to a graded Lie subalgebra of $\mathfrak{C}^2(n,m)$ satisfying $\mathfrak{s}_{-3}(x) \simeq \mathfrak{C}^2_{-3}$, $\mathfrak{s}_{-2}(x) \simeq \mathfrak{C}^2_{-2}$ and $\mathfrak{f}(x) = \mathrm{Ch}(\partial C^2)(x) \cap T_xR$.

We assume $\mathrm{Ch}(D^1) \neq \{0\}$, namely $\mathfrak{f}(x) \neq \{0\}$ at each point $x \in R$ in what follows. If $\mathrm{Ch}(D^1) = \{0\}$, applying Realization Lemma to the projection $\pi : R \longrightarrow M = R/F$ and D^1, we have a map $\psi : R \longrightarrow J^1(M,n)$ such that $\pi = \Pi^1_0 \circ \psi$ and $D^1 = \psi_*^{-1}(C^1)$.

Since $\operatorname{Ker}\psi_*(v) = F(v) \cap \operatorname{Ch}(D^1)(v) = \{0\}$ for $v \in R$ and $\dim R = \operatorname{codim}\operatorname{Ch}(D^1) = \dim J^1(M,n)$, R is locally diffeomorphic to $J^1(M,n)$. Therefore D^2 is completely integrable if R is integrable.

Now we assume that there exists a n-dimensional integral element V of (R, D^2) at each point $x \in R$ such that

$$\mathfrak{s}_{-1}(x) = V \oplus \mathfrak{f}(x),$$

where V is an abelian subalgebra in $\mathfrak{s}(x)$. By fixing a basis of $\mathfrak{s}_{-3}(x)$, we identify $\mathfrak{s}_{-3}(x)$ with a m-dimensional vector space W. It follows from $V \cap \mathfrak{f}(x) = \{0\}$ and (5.3.4) that $\mathfrak{s}_{-2}(x)$ is identified with $W \otimes V^*$ through the bracket product $[\ ,\ ] : \mathfrak{s}_{-2}(x) \times \mathfrak{s}_{-1}(x) \longrightarrow \mathfrak{s}_{-3}(x)$. Let $\mu : \mathfrak{f}(x) \longrightarrow W \otimes S^2(V^*)$ be a linear map defined by

$$\mu(f)(v_1, v_2) = [[f, v_1], v_2] \in \mathfrak{s}_{-3}(x) \simeq W \quad \text{for } f \in \mathfrak{f}(x) \text{ and } v_1, v_2 \in V,$$

which implies $\mu(f)(v_1, v_2) = \mu(f)(v_2, v_1)$. Moreover, we see easily that μ is injective.

Thus, we obtain

$$\mathfrak{s}_{-3}(x) \simeq W, \quad \mathfrak{s}_{-2}(x) \simeq W \otimes V^*, \quad \mathfrak{s}_{-1}(x) = V \oplus \mathfrak{f}(x), \quad \mathfrak{f}(x) \subset W \otimes S^2(V^*).$$

In consequent two sections we will seek a PD-manifold $(R;\, D^1, D^2)$ of type $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$ satisfying

$$(5.3.5) \qquad \mathfrak{s}_{-3} = W, \qquad \mathfrak{s}_{-2} = W \otimes V^*, \qquad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}$$

where W and V are vector spaces of dimension m and n, and $\mathfrak{f}$ is a non-zero subspace of $W \otimes S^2(V^*)$. Here, especially we have $\dim \mathfrak{s}_{-2} = \dim \mathfrak{s}_{-3} \cdot (\dim \mathfrak{s}_{-1} - \dim \mathfrak{f})$, which will be utilized in Section 5.4.

5.3.2 Example of partial differential equations of finite type

We will consider an example of partial differential equations $(R;\, D^1, D^2)$ of finite type and see that it has a pseudo-product structure of irreducible type $(\mathfrak{l}, S)$ ([Sa88], [YY02]):

Example 5.4. Let us consider the following system of second order partial differential equations of two unknown functions z^1, z^2 with three independent variables x_1, x_2, x_3 :

$$(5.3.6) \qquad \frac{\partial^2 z^a}{\partial x_1 \partial x_1} = \frac{\partial^2 z^a}{\partial x_2 \partial x_2} = \frac{\partial^2 z^a}{\partial x_2 \partial x_3} = \frac{\partial^2 z^a}{\partial x_3 \partial x_3} = 0 \quad \text{for } a = 1, 2.$$

Note that the system of equations of one unknown function

$$\frac{\partial^2 z}{\partial x_1 \partial x_1} = \frac{\partial^2 z}{\partial x_2 \partial x_2} = \frac{\partial^2 z}{\partial x_2 \partial x_3} = \frac{\partial^2 z}{\partial x_3 \partial x_3} = 0$$

is known as a model equation of type $(A_4, \{\alpha_1, \alpha_2, \alpha_4\})$ ([Yam09, Section 5.3]). The system of equations (5.3.6) defines the submanifold R of $J^2(\mathbb{R}^4, 2)$ and differential system D on R as follows:

$$\begin{aligned} R &= \left\{ p^a_{11} = p^a_{22} = p^a_{23} = p^a_{33} = 0 \ (a = 1, 2) \right\} \\ D &= \left\{ \varpi^1 = \varpi^2 = \varpi^1_1 = \varpi^1_2 = \varpi^1_3 = \varpi^2_1 = \varpi^2_2 = \varpi^2_3 = 0 \right\} \end{aligned}$$

where $(x^i, z^a, p^a_i, p^a_{ij}\ (1 \leq a \leq 2, 1 \leq i \leq j \leq 3))$ is the canonical coordinate system of $J^2(\mathbb{R}^4, 2)$ and

$$\begin{aligned} \varpi^1 &= dz^1 - p^1_1\, dx^1 - p^1_2\, dx^2 - p^1_3\, dx^3, \\ \varpi^2 &= dz^2 - p^2_1\, dx^1 - p^2_2\, dx^2 - p^2_3\, dx^3, \\ \varpi^1_1 &= dp^1_1 - p^1_{12}\, dx^2 - p^1_{13}\, dx^3, \\ \varpi^1_2 &= dp^1_2 - p^1_{12}\, dx^1, \\ \varpi^1_3 &= dp^1_3 - p^1_{13}\, dx^1, \\ \varpi^2_1 &= dp^2_1 - p^2_{12}\, dx^2 - p^2_{13}\, dx^3, \\ \varpi^2_2 &= dp^2_2 - p^2_{12}\, dx^1, \\ \varpi^2_3 &= dp^2_3 - p^2_{13}\, dx^1. \end{aligned}$$

We obtain the symbol algebra $\mathfrak{m}(x) = \mathfrak{m}$ of (R, D) at each point x as follows:

$$\begin{gathered} \mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \qquad \mathfrak{g}_{-3} = W, \quad \mathfrak{g}_{-2} = W \otimes V^*, \quad \mathfrak{g}_{-1} = V \oplus \mathfrak{f}, \\ \mathfrak{f} = W \otimes \langle e^1 \circledcirc e^2, e^1 \circledcirc e^3 \rangle \subset W \otimes S^2(V^*), \end{gathered}$$

where W and V is vector spaces of dimension 2 and 3 respectively, and $\{e^1, e^2, e^3\}$ is a basis of V^*. Now we will see that the prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$ is isomorphic to a pseudo-product GLA of irreducible type $(\mathfrak{l}, S)$ for some simple graded Lie algebra $\mathfrak{l}$ of depth 1 and irreducible $\mathfrak{l}$-module S.

Let $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{b} = \mathfrak{sl}(3, \mathbb{R})$. Let us fix a Cartan subalgebra $\mathfrak{h}^{\mathfrak{a}}$ of $\mathfrak{a}$ (resp. $\mathfrak{h}^{\mathfrak{b}}$ of $\mathfrak{b}$) and let $\Phi^{\mathfrak{a}}$ (resp. $\Phi^{\mathfrak{b}}$) be a root system of $\mathfrak{a}$ (resp. $\mathfrak{b}$) relative to $\mathfrak{h}^{\mathfrak{a}}$ (resp. $\mathfrak{h}^{\mathfrak{b}}$). Let us fix a simple root system $\Delta^{\mathfrak{a}} = \{\alpha_1\}$ of $\Phi^{\mathfrak{a}}$ (resp. $\Delta^{\mathfrak{b}} = \{\beta_1, \beta_2\}$ of $\Phi^{\mathfrak{b}}$). Then we have $\Phi^{\mathfrak{a}} = \{\pm\alpha_1\}$, $\Phi^{\mathfrak{b}} = \{\pm\beta_1, \pm\beta_2, \pm(\beta_1 + \beta_2)\}$ and the root decomposition of $\mathfrak{a}$ (resp. $\mathfrak{b}$) relative to $\Delta^{\mathfrak{a}}$ (resp. $\Delta^{\mathfrak{b}}$):

$$\mathfrak{a} = \mathfrak{h}^{\mathfrak{a}} \oplus \bigoplus_{\alpha \in \Phi^{\mathfrak{a}}} \mathfrak{g}^{\mathfrak{a}}_{\alpha}, \qquad \mathfrak{b} = \mathfrak{h}^{\mathfrak{b}} \oplus \bigoplus_{\beta \in \Phi^{\mathfrak{b}}} \mathfrak{g}^{\mathfrak{b}}_{\beta},$$

where $\mathfrak{g}^{\mathfrak{a}}_{\alpha} = \{X \in \mathfrak{a} \mid [H, X] = \alpha(H)X \ (H \in \mathfrak{h}^{\mathfrak{a}})\}$ (resp. $\mathfrak{g}^{\mathfrak{b}}_{\beta}$) is the root space for $\alpha \in \Phi^{\mathfrak{a}}$ (resp. $\beta \in \Phi^{\mathfrak{b}}$). Let $\Delta^{\mathfrak{a}}_1 = \{\alpha_1\} = \Delta^{\mathfrak{a}}$ and $\Delta^{\mathfrak{b}}_1 = \{\beta_1\} \subset \Delta^{\mathfrak{b}}$. They define gradations of $\mathfrak{a}$ and $\mathfrak{b}$ of depth 1 as follows:

$$\begin{aligned} &\mathfrak{a} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1, &&\mathfrak{a}_{\pm 1} = \mathfrak{g}^{\mathfrak{a}}_{\pm\alpha_1}, \quad \mathfrak{a}_0 = \mathfrak{h}^{\mathfrak{a}}, \\ &\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1, &&\mathfrak{b}_{\pm 1} = \mathfrak{g}^{\mathfrak{b}}_{\pm\beta_1} \oplus \mathfrak{g}^{\mathfrak{b}}_{\pm(\beta_1+\beta_2)}, \quad \mathfrak{b}_0 = \mathfrak{h}^{\mathfrak{b}} \oplus \mathfrak{g}^{\mathfrak{b}}_{-\beta_2} \oplus \mathfrak{g}^{\mathfrak{b}}_{\beta_2}. \end{aligned}$$

Let U be a vector space over $\mathbb{R}$ of dimension 2. Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be the reductive graded Lie algebra of depth 1 defined by

$$\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{gl}(U), \quad [\mathfrak{a}, \mathfrak{gl}(U)] = [\mathfrak{b}, \mathfrak{gl}(U)] = 0,$$
$$\mathfrak{l}_{\pm 1} = \mathfrak{a}_{\pm 1} \oplus \mathfrak{b}_{\pm 1}, \quad \mathfrak{l}_0 = \mathfrak{a}_0 \oplus \mathfrak{b}_0 \oplus \mathfrak{gl}(U).$$

Note that the semisimple ideal $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ of $\mathfrak{l}$ coincides with $\mathfrak{a} \oplus \mathfrak{b}$, which is not simple (cf. [Sa88], [YY02]).

Let $\{\varpi^{\mathfrak{a}}_1\}$ and $\{\varpi^{\mathfrak{b}}_1, \varpi^{\mathfrak{b}}_2\}$ be fundamental weights relative to $\Delta^{\mathfrak{a}}$ and $\Delta^{\mathfrak{b}}$. Let $T^{\mathfrak{a}}$ (resp. $T^{\mathfrak{b}}$) be the irreducible $\mathfrak{a}$-module (resp. $\mathfrak{b}$-module) with highest weight $\varpi^{\mathfrak{a}}_1$ (resp. $\varpi^{\mathfrak{b}}_2$). Then $S = T^{\mathfrak{a}} \otimes T^{\mathfrak{b}} \otimes U$ is a faithful irreducible $\mathfrak{l}$-module and decomposed as follows:

$$S = \bigoplus_{p=-3}^{-1} S_p, \qquad S_{-3} = V^{\mathfrak{a}}_1 \otimes V^{\mathfrak{b}}_1 \otimes U, \qquad S_{-2} = \left(V^{\mathfrak{a}}_1 \otimes V^{\mathfrak{b}}_0 \oplus V^{\mathfrak{a}}_0 \otimes V^{\mathfrak{b}}_1\right) \otimes U,$$
$$S_{-1} = V^{\mathfrak{a}}_0 \otimes V^{\mathfrak{b}}_0 \otimes U,$$

where $V^{\mathfrak{a}}_0 = V(\varpi^{\mathfrak{a}}_1)$, $V^{\mathfrak{a}}_1 = V(\varpi^{\mathfrak{a}}_1 - \alpha_1)$, $V^{\mathfrak{b}}_0 = V(\varpi^{\mathfrak{b}}_2) \oplus V(\varpi^{\mathfrak{b}}_2 - \beta_2)$, $V^{\mathfrak{b}}_1 = V(\varpi^{\mathfrak{b}}_2 - (\beta_1 + \beta_2))$ and $V(\lambda)$ is the weight space with weight λ. Since $\dim S_{-3} = 2$, $\dim S_{-2} = 6$ and $\dim \mathfrak{l}_{-1} = 3$, it follows from the property of S (see [Sa88, Proposition 4.3.1] or [YY02, Lemma 2.1 (4)]) that S_{-2} is isomorphic to $W \otimes V^*$, where $W = S_{-3}$ and $V = \mathfrak{l}_{-1}$. Namely $\mathfrak{l}_{-1} \oplus S$ is isomorphic to $\mathfrak{m}$. Moreover, by direct calculation, we can see that the prolongation of $\mathfrak{m}$ is isomorphic to $\mathfrak{l} \oplus S$.

5.4 Partial differential equations of simple type

We will seek a simple graded Lie algebra of type (X_l, Δ_1) that the negative part $\mathfrak{m}$ is isomorphic to the symbol algebra of PD-manifolds of m (≥ 2) unknown functions. A necessary condition for this is that $\mathfrak{m}$ is of third kind and $\dim \mathfrak{g}_{-3} \geq 2$. From extended Dynkin diagrams (see Figure 5.1), the simple graded Lie algebras of type (X_l, Δ_1) satisfying this condition are the followings: $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$ $(1 \leq i < j < k \leq l, (i, k) \neq (1, l))$, $(B_l, \{\alpha_1, \alpha_i\})$ $(3 \leq i \leq l)$, $(C_l, \{\alpha_i, \alpha_l\})$ $(2 \leq i \leq l-1)$, $(D_l, \{\alpha_1, \alpha_i\})$ $(3 \leq i \leq$

$l-2$), $(D_l, \{\alpha_i, \alpha_l\})$ $(3 \leq i \leq l-2)$, $(E_6, \{\alpha_4\})$, $(E_6, \{\alpha_1, \alpha_3\})$, $(E_7, \{\alpha_3\})$, $(E_7, \{\alpha_5\})$, $(E_7, \{\alpha_2, \alpha_7\})$, $(E_7, \{\alpha_6, \alpha_7\})$, $(E_8, \{\alpha_2\})$, $(E_8, \{\alpha_7\})$, $(F_4, \{\alpha_2\})$, $(G_2, \{\alpha_1\})$.

However, we will find that there exist no such simple graded Lie algebras. Precisely, we state as follows:

Theorem 5.5. *Let $\mathfrak{s} = \bigoplus_{p=-3}^{-1} \mathfrak{s}_p$ be a fundamental graded Lie algebra satisfying* (5.3.5). *Then, for any simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of type (X_l, Δ_1), $\mathfrak{s}$ is never isomorphic to the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$. In other words, there are no PD-manifolds of m (≥ 2) unknown functions of type $\mathfrak{s}$ that the prolongation of $\mathfrak{s}$ is isomorphic to some simple graded Lie algebra.*

Note that, among (X_l, Δ_1) listed above, $(C_l, \{\alpha_i, \alpha_l\})$ $(2 \leq i \leq l-1)$, $(D_l, \{\alpha_i, \alpha_l\})$ $(3 \leq i \leq l-2)$, $(E_6, \{\alpha_1, \alpha_3\})$ and $(E_7, \{\alpha_6, \alpha_7\})$ appeared in Theorem 2.3 (a) of [YY07]. That is, they are the prolongation of $\mathfrak{m} = \mathfrak{l}_{-1} \oplus S$ for some pseudo-product graded Lie algebra of type $(\mathfrak{l}, S)$. In the case of $(C_l, \{\alpha_i, \alpha_l\})$ $(2 \leq i \leq l-1)$ and $(D_l, \{\alpha_i, \alpha_l\})$ $(3 \leq i \leq l-2)$, according to Case (3) and (9) in Section 3 of [YY07], since $\dim \mathfrak{g}_{-2} = \dim \mathfrak{l}_{-1} (= \dim V)$ and $\mathfrak{f} = S_{-1}$, $\mathfrak{m}$ cannot be isomorphic to $\mathfrak{s}$ satisfying (5.3.5). In the case of $(E_6, \{\alpha_1, \alpha_3\})$, according to Case (2) in Section 4 of [YY07], since $\dim \mathfrak{g}_{-3} \cdot (\dim \mathfrak{g}_{-1} - \dim \mathfrak{f}) - \dim \mathfrak{g}_{-2} = |\Phi_3^+| \cdot (|\Phi_1^+| - |\Psi^1|) - |\Phi_2^+| > 0$, $\mathfrak{m}$ cannot be isomorphic to $\mathfrak{s}$ satisfying (5.3.5). In the case of $(E_7, \{\alpha_6, \alpha_7\})$, according to Case (4) in Section 4 of [YY07], since $\dim \mathfrak{g}_{-3} \cdot (\dim \mathfrak{g}_{-1} - \dim \mathfrak{f}) - \dim \mathfrak{g}_{-2} = |\Phi_3^+| \cdot (|\Phi_1^+| - |\Psi^7|) - |\Phi_2^+| > 0$, $\mathfrak{m}$ cannot be isomorphic to $\mathfrak{s}$ satisfying (5.3.5).

Thus it is enough to investigate the other types: $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$ $(1 \leq i < j < k \leq l, (i, k) \neq (1, l))$, $(B_l, \{\alpha_1, \alpha_i\})$ $(3 \leq i \leq l)$, $(D_l, \{\alpha_1, \alpha_i\})$ $(3 \leq i \leq l-2)$, $(E_6, \{\alpha_4\})$, $(E_7, \{\alpha_3\})$, $(E_7, \{\alpha_5\})$, $(E_7, \{\alpha_2, \alpha_7\})$, $(E_8, \{\alpha_2\})$, $(E_8, \{\alpha_7\})$, $(F_4, \{\alpha_2\})$, $(G_2, \{\alpha_1\})$.

Now we begin to prove Theorem 5.5 by contradiction. Let (X_l, Δ_1) be one of the other types. Suppose that the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies (5.3.5). Let $\Phi_{\mathfrak{f}} = \{\alpha \in \Phi^+ \mid \mathfrak{g}_{-\alpha} \subset \mathfrak{f}\}$. We divide each cases (X_l, Δ_1) into sequent subsections:

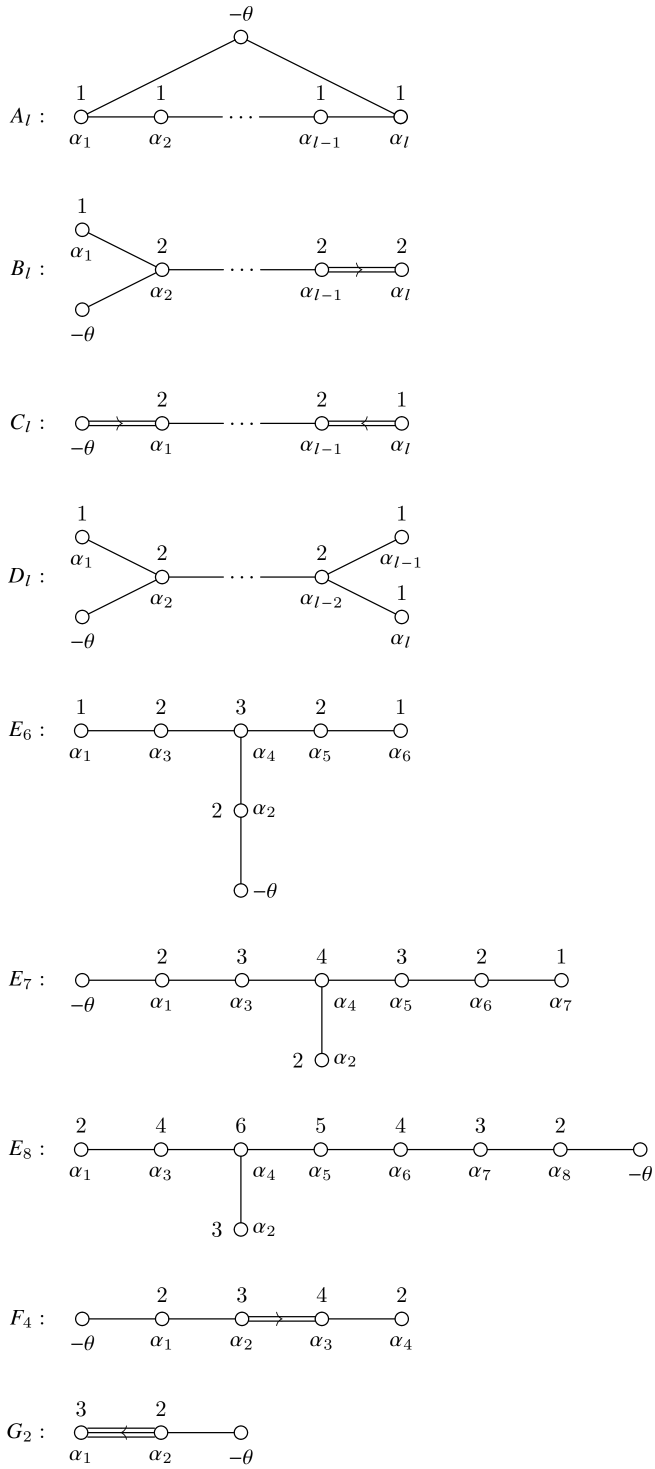


Figure 5.1: The Extended Dynkin Diagrams

5.4.1 $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$-type $(1 \leq i < j < k \leq l,\ (i,k) \neq (1,l))$

$$
\begin{aligned}
\Phi_3^+ &= \left\{ \alpha_p + \cdots + \alpha_i + \cdots + \alpha_j + \cdots + \alpha_k + \cdots + \alpha_q \mid 1 \leq p \leq i,\ k \leq q \leq l \right\}, \\
\Phi_2^+ &= \left\{ \alpha_p + \cdots + \alpha_i + \cdots + \alpha_j + \cdots + \alpha_q \mid 1 \leq p \leq i,\ j \leq q < k \right\} \\
&\quad \cup \left\{ \alpha_p + \cdots + \alpha_j + \cdots + \alpha_k + \cdots + \alpha_q \mid i < p \leq j,\ k \leq q \leq l \right\}, \\
\Phi_1^+ &= \left\{ \alpha_p + \cdots + \alpha_i + \cdots + \alpha_q \mid 1 \leq p \leq i \leq q < j \right\} \\
&\quad \cup \left\{ \alpha_p + \cdots + \alpha_j + \cdots + \alpha_q \mid i < p \leq j \leq q < k \right\} \\
&\quad \cup \left\{ \alpha_p + \cdots + \alpha_k + \cdots + \alpha_q \mid j < p \leq k \leq q \leq l \right\}, \\
\Phi_{\mathfrak{f}} &= \left\{ \alpha_p + \cdots + \alpha_j + \cdots + \alpha_q \mid i < p \leq j \leq q < k \right\}.
\end{aligned}
$$

Then we have $|\Phi_3^+| = i(l-k+1)$, $|\Phi_2^+| = i(k-j) + (j-i)(l-k+1)$, $|\Phi_1^+| = i(j-i) + (j-i)(k-j) + (k-j)(l-k+1)$ and $\dim \mathfrak{f} = (j-i)(k-j)$. Therefore,

$$
\begin{aligned}
&|\Phi_3^+| \cdot (|\Phi_1^+| - \dim \mathfrak{f}) - |\Phi_2^+| \\
=&(i-1)(i+1)(j-i)(l-k+1) + i(k-j)(l-k)(l-k+2) > 0,
\end{aligned}
$$

which implies that $\mathfrak{m}$ cannot satisfy (5.3.5).

Now we will describe a model equation of the PD-manifold of type $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$. We have the following matrix representation of a real form $\mathfrak{sl}(l+1, \mathbb{R}) = \bigoplus_{p=-3}^{3} \mathfrak{g}_p$ of $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$:

$$
\mathfrak{g}_{-3} = \left\{ \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline Z & 0 & 0 & 0 \end{array} \right) \middle| \ Z \in M(l-k+1, i) \right\},
$$

$$
\mathfrak{g}_{-2} = \left\{ \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline P_1 & 0 & 0 & 0 \\ \hline 0 & P_2 & 0 & 0 \end{array} \right) \middle| \ \begin{array}{l} P_1 \in M(k-j, i), \\ P_2 \in M(l-k+1, j-i) \end{array} \right\},
$$

$$
\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline X_1 & 0 & 0 & 0 \\ \hline 0 & F & 0 & 0 \\ \hline 0 & 0 & X_2 & 0 \end{array} \right) \middle| \ \begin{array}{l} X_1 \in M(j-i, i),\ X_2 \in M(l-k+1, k-j), \\ F \in M(k-j, j-i) \end{array} \right\},
$$

$$\mathfrak{g}_0 = \left\{ \left(\begin{array}{c|c|c|c} L_1 & 0 & 0 & 0 \\ \hline 0 & L_2 & 0 & 0 \\ \hline 0 & 0 & L_3 & 0 \\ \hline 0 & 0 & 0 & L_4 \end{array} \right) \middle| \begin{array}{l} L_1 \in M(i,i),\, L_2 \in M(j-i, j-i), \\ L_3 \in M(k-j, k-j),\, L_4 \in M(l-k+1, l-k+1), \\ \sum_{i=1}^4 \operatorname{tr} L_i = 0 \end{array} \right\},$$

$$\mathfrak{g}_i = \left\{ {}^tX \mid X \in \mathfrak{g}_{-i} \right\} \quad \text{for } 1 \leq i \leq 3,$$

where $M(a,b)$ denotes the space of all $a \times b$ matrices.

Now we recall the formula for the Maurer-Cartan form on $M(\mathfrak{m})$ by N. Tanaka in Section 2.3 of [Tan70]:

Proposition 5.6. *Let $\mathfrak{m} = \bigoplus_{p=-3}^{-1} \mathfrak{g}_p$ be a fundamental graded Lie algebra of third kind and $(M(\mathfrak{m}), D_{\mathfrak{m}})$ the standard differential system of type $\mathfrak{m}$. Let u_{-p} denote the projection of $\mathfrak{m}$ onto $\mathfrak{g}_{-p}$ for $p = 1, 2, 3$, which may be regarded as a $\mathfrak{g}_{-p}$-valued function on $\mathfrak{m}$. Let η_{-p} be the $\mathfrak{g}_{-p}$-component of the Maurer-Cartan form of $M(\mathfrak{m})$. Then η_{-p} is expressed as follows:*

$$\begin{aligned} \eta_{-3} &= du_{-3} - \frac{1}{3}[u_{-2}, du_{-1}] - \frac{2}{3}[u_{-1}, du_{-2}] + \frac{1}{6}[u_{-1}, [u_{-1}, du_{-1}]], \\ \eta_{-2} &= du_{-2} - \frac{1}{2}[u_{-1}, du_{-1}], \\ \eta_{-1} &= du_{-1}. \end{aligned} \tag{5.4.7}$$

Here, $M(\mathfrak{m})$ is identified with $\mathfrak{m}$ by $f = \rho \circ S$, where ρ denotes the projection of the affine transformation group $AF(\mathfrak{m})$ of $\mathfrak{m}$ onto $\mathfrak{m}$ and $S : M(\mathfrak{m}) \longrightarrow AF(\mathfrak{m})$ is induced by the injective homomorphism s of $\mathfrak{m}$ into the Lie algebra $\mathfrak{af}(\mathfrak{m})$ of all infinitesimal affine transformation of $\mathfrak{m}$ defined by

$$s(X)(Y) = X + \sum_{p,q<0} \frac{q}{p+q}[u_p(X), u_q(Y)] \in \mathfrak{m} \quad \textit{for } X, Y \in \mathfrak{m}.$$

By definition, we have the standard differential system $D_{\mathfrak{m}}$ of type $\mathfrak{m}$ as follows:

$$D_{\mathfrak{m}} = \{ \eta_{-3} = \eta_{-2} = 0 \}.$$

With respect to the matrix representation of $\mathfrak{sl}(l+1, \mathbb{R})$, we may write u_{-p} as

$$u_{-3} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline Z & 0 & 0 & 0 \end{array} \right), \quad u_{-2} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline P_1 & 0 & 0 & 0 \\ \hline 0 & P_2 & 0 & 0 \end{array} \right), \quad u_{-1} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline X_1 & 0 & 0 & 0 \\ \hline 0 & F & 0 & 0 \\ \hline 0 & 0 & X_2 & 0 \end{array} \right).$$

Substituting them for the formula, we have

$$\eta_{-3} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \Theta_0 & 0 & 0 & 0 \end{array}\right), \qquad \eta_{-2} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \Theta_1 & 0 & 0 & 0 \\ \hline 0 & \Theta_2 & 0 & 0 \end{array}\right),$$

and

$$D_{\mathfrak{m}} = \{\, \Theta_0 = \Theta_1 = \Theta_2 = 0 \,\},$$

where

$$\begin{aligned} \Theta_0 &= dZ - \frac{1}{3}P_2 dX_1 + \frac{1}{3}dX_2 P_1 - \frac{2}{3}X_2 dP_1 + \frac{2}{3}dP_2 X_1 \\ &\qquad + \frac{1}{6}X_2(F dX_1 - dF X_1) - \frac{1}{6}(X_2 dF - dX_2 F)X_1, \\ \Theta_1 &= dP_1 - \frac{1}{2}F dX_1 + \frac{1}{2}dF X_1, \\ \Theta_2 &= dP_2 - \frac{1}{2}X_2 dF + \frac{1}{2}dX_2 F. \end{aligned}$$

The exterior derivative of Θ_0 is

$$d\Theta_0 = -dX_2 \wedge d\left(P_1 + \frac{1}{2}F X_1\right) - d\left(P_2 - \frac{1}{2}X_2 F\right) \wedge dX_1.$$

Putting

$$\hat{P}_1 = P_1 + \frac{1}{2}F X_1, \qquad \hat{P}_2 = P_2 - \frac{1}{2}X_2 F, \qquad \hat{X}_1 = X_1, \qquad \hat{X}_2 = -X_2,$$

we have

$$\begin{aligned} \Theta_0 &= dZ - \frac{1}{3}(\hat{P}_2 + \hat{X}_2 F)\, d\hat{X}_1 - \frac{1}{3}d\hat{X}_2(\hat{P}_1 + F\hat{X}_1) - \frac{1}{3}\hat{X}_2 dF \hat{X}_1 + \frac{2}{3}\hat{X}_2 d\hat{P}_1 + \frac{2}{3}d\hat{P}_2 \hat{X}_1, \\ \Theta_1 &= d\hat{P}_1 - F d\hat{X}_1, \\ \Theta_2 &= d\hat{P}_2 - d\hat{X}_2 F, \end{aligned}$$

$$\begin{aligned} d\Theta_0 &= d\hat{X}_2 \wedge d\hat{P}_1 - d\hat{P}_2 \wedge d\hat{X}_1, \\ d\Theta_1 &= -\, dF \wedge d\hat{X}_1, \\ d\Theta_2 &= d\hat{X}_2 \wedge dF. \end{aligned}$$

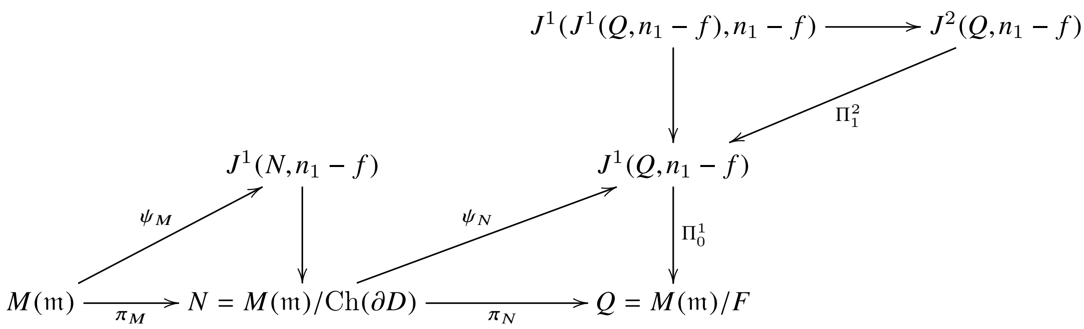
Digressing from determinating of the model equation, we now show theoretically that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into the 2-jet space $(J^2(Q,n), C^2_Q)$ over some manifold Q. From the structure equation of $D_{\mathfrak{m}}$, we have

$$\partial D_{\mathfrak{m}} = \{\, \Theta_0 = 0 \,\},$$
$$\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \left\{ \Theta_0 = \Theta_1 = \Theta_2 = d\hat{X}_1 = d\hat{X}_2 = 0 \right\}, \tag{5.4.8}$$

which are differential systems of codimension $n_3 + n_2$ and $n_3 + n_2 + (n_1 - f)$. Here, let $n_i = \dim \mathfrak{g}_{-i}$ for $1 \le i \le 3$ and $f = \dim \mathfrak{k}$. Putting

$$F = \left\{ \Theta_0 = d\hat{X}_1 = d\hat{X}_2 = 0 \right\},$$

we see that F is a completely integrable differential system of codimension $n_3 + (n_1 - f)$. Let $N = M(\mathfrak{m})/\mathrm{Ch}(\partial D_{\mathfrak{m}})$ and $Q = M(\mathfrak{m})/F$ be spaces of leaves of the foliation. Let $\pi_M : M(\mathfrak{m}) \longrightarrow N$ and $\pi_N : N \longrightarrow Q$ be the projections. From (5.4.8), $\partial D_{\mathfrak{m}}$ and F drop down to N. Since $\mathrm{Ker}\,(\pi_N)_* = F$ is a subbundle of $\partial D_{\mathfrak{m}}$ of codimension $n_1 - f$, applying Realization Lemma to π_N, we have $\psi_N : N \longrightarrow J^1(Q, n_1 - f)$ as in the lemma. Since $\mathrm{Ker}\,(\psi_N)_* = \mathrm{Ker}\,(\pi_M) \cap \mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$, ψ_N is immersion. Note that $\dim J^1(Q, n_1 - f) - \dim N = n_3(n_1 - f) - n_2 > 0$. Since $\mathrm{Ker}\,(\pi_M)_* = \mathrm{Ch}(\partial D_{\mathfrak{m}})$ is a subbundle of $D_{\mathfrak{m}}$ of codimension $n_1 - f$, applying Realization Lemma to π_M, we have $\psi_M : M(\mathfrak{m}) \longrightarrow J^1(N, n_1 - f)$ as in the lemma. Since $\mathrm{Ker}\,(\psi_M)_* = \mathrm{Ker}\,(\pi_M) \cap \mathrm{Ch}(D_{\mathfrak{m}}) = \{0\}$, ψ_M is immersion. Since $\psi_M(v) = (\pi_M)_*(D_{\mathfrak{m}}(v))$ for $v \in M$ and $(\psi_N)_*^{-1}(C^1_Q) = \partial D_{\mathfrak{m}}$, $\psi_M(v)$ is a $(n_1 - f)$-dimensional integral element of C^1_Q. Moreover, since $(\psi_N)_*^{-1}(Q^1) = F$, where $Q^1 = \mathrm{Ker}\,(\Pi^1_0)_*$ and $\Pi^1_0 : J^1(Q, n_1 - f) \longrightarrow Q$ the projection, we have $\psi_M(v) \cap Q^1(\pi_M(v)) = \{0\}$. Thus we have $\psi_M(v) \in J^2(Q, n_1 - f)$.



Now we return to the calculation of the model equation. From $F = \{\Theta_0 = d\hat{X}_1 = d\hat{X}_2 = 0\}$, we calculate

$$\Theta_0 \equiv d\left(Z - \frac{1}{3}\hat{X}_2 F \hat{X}_1 + \frac{2}{3}\hat{X}_2\hat{P}_1 + \frac{2}{3}\hat{P}_2\hat{X}_1\right) \qquad (\text{mod } F^{\perp}).$$

Putting $\hat{Z} = Z - \frac{1}{3}\hat{X}_2 F \hat{X}_1 + \frac{2}{3}\hat{X}_2\hat{P}_1 + \frac{2}{3}\hat{P}_2\hat{X}_1$, we have achieved a normal form of $D_{\mathfrak{m}}$:

$$\begin{aligned}
\Theta_0 &= d\hat{Z} - d\hat{X}_2\hat{P}_1 - \hat{P}_2 d\hat{X}_1,\\
\Theta_1 &= d\hat{P}_1 - F d\hat{X}_1,\\
\Theta_2 &= d\hat{P}_2 - d\hat{X}_2 F.
\end{aligned}$$

From now on, fix index ranges $1 \leq \alpha, \beta \leq l$, $i+1 \leq m, n \leq j$, $j+1 \leq s, t \leq k$, and $k+1 \leq a, b \leq l+1$. Setting

$$\hat{X}_1 = (y^m_\alpha), \quad \hat{X}_2 = (x^a_s), \quad F = (f^s_m), \quad \hat{Z} = (z^a_\alpha), \quad \hat{P}_1 = (p^s_\alpha), \quad \hat{P}_2 = (q^a_m),$$

we have

$$\begin{aligned}
\Theta_0 &= \left(dz^a_\alpha - \sum_t p^t_\alpha dx^a_t - \sum_n q^a_n dy^n_\alpha\right)_{a,\alpha},\\
\Theta_1 &= \left(dp^s_\alpha - \sum_n f^s_n dy^n_\alpha\right)_{s,\alpha},\\
\Theta_2 &= \left(dq^a_m - \sum_t f^t_m dy^a_t\right)_{a,m}.
\end{aligned}$$

Therefore we have a model equation of $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$ $(1 \leq i < j < k \leq l, (i,k) \neq (1,l))$ as follows:

$$\begin{cases}
\dfrac{\partial z^b_\alpha}{\partial x^a_s} = \dfrac{\partial z^a_\beta}{\partial y^m_\alpha} = 0 \quad \text{for } a \neq b, \alpha \neq \beta,\\[2ex]
\dfrac{\partial z^a_\alpha}{\partial x^a_s} = \dfrac{\partial z^b_\alpha}{\partial x^b_s}, \quad \dfrac{\partial z^a_\alpha}{\partial y^m_\alpha} = \dfrac{\partial z^a_\beta}{\partial y^m_\beta},\\[2ex]
\dfrac{\partial^2 z^a_\alpha}{\partial x^a_s \partial x^a_t} = 0, \quad \dfrac{\partial^2 z^a_\alpha}{\partial y^m_\alpha \partial y^n_\alpha} = 0.
\end{cases}$$

For example, a model equation of type $(A_4, \{\alpha_1, \alpha_2, \alpha_3\})$ is

$$\begin{cases} \dfrac{\partial z_1}{\partial x_2} = \dfrac{\partial z_2}{\partial x_3}, \quad \dfrac{\partial z_1}{\partial x_3} = \dfrac{\partial z_2}{\partial x_2} = 0, \\ \dfrac{\partial^2 z_1}{\partial x_1 \partial x_1} = \dfrac{\partial^2 z_1}{\partial x_2 \partial x_2} = \dfrac{\partial^2 z_2}{\partial x_1 \partial x_1} = 0, \end{cases}$$

where x_1, x_2, x_3 and z_1, z_2 are independent and dependent variables.

5.4.2 $(B_l, \{\alpha_1, \alpha_i\})$-type $(3 \leq i \leq l)$

$$\begin{aligned}
\Phi_3^+ &= \left\{ \alpha_1 + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_i + \cdots + 2\alpha_l \mid 1 \leq p < i \right\}, \\
\Phi_2^+ &= \left\{ \alpha_1 + \cdots + \alpha_i + \cdots + \alpha_p \mid i \leq p \leq l \right\} \\
&\quad \cup \left\{ \alpha_1 + \cdots + \alpha_i + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_l \mid i \leq p < l \right\} \\
&\quad \cup \left\{ \alpha_p + \cdots + \alpha_{q-1} + 2\alpha_q + \cdots + 2\alpha_i + \cdots + 2\alpha_l \mid 1 < p < q \leq i \right\}, \\
\Phi_1^+ &= \left\{ \alpha_1 + \cdots + \alpha_p \mid 1 \leq p < i \right\} \\
&\quad \cup \left\{ \alpha_p + \cdots + \alpha_i + \cdots + \alpha_q \mid 1 < p \leq i \leq q \leq l \right\} \\
&\quad \cup \left\{ \alpha_p + \cdots + \alpha_i + \cdots + \alpha_q + 2\alpha_{q+1} + \cdots + 2\alpha_l \mid 1 < p \leq q < l \right\}.
\end{aligned}$$

Then we see that $\Phi_{\mathfrak{f}} = \emptyset$. In fact, for any $\alpha \in \Phi_1^+$, there exists $\beta \in \Phi_2^+$ satisfying $\alpha + \beta \in \Phi_3^+$ according to the following list:

$\alpha \in \Phi_1^+$	$\beta \in \Phi_2^+$
$\alpha_1 + \cdots + \alpha_p$ $(1 \leq p < i-1)$	$\alpha_{p+1} + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_l$
$\alpha_1 + \cdots + \alpha_{i-1}$	$\alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_l$
$\alpha_p + \cdots + \alpha_i + \cdots + \alpha_q$ $(1 < p \leq i \leq q < l)$	$\alpha_{p+1} + \cdots + \alpha_q + 2\alpha_{q+1} + \cdots + 2\alpha_l$
$\alpha_p + \cdots + \alpha_i + \cdots + \alpha_l$ $(1 < p \leq i)$	$\alpha_1 + \cdots + \alpha_p + \cdots + \alpha_i + \cdots + \alpha_l$
$\alpha_p + \cdots + \alpha_i + \cdots + \alpha_q + 2\alpha_{q+1} + \cdots + 2\alpha_l$ $(1 < p \leq q < l)$	$\alpha_1 + \cdots + \alpha_p + \cdots + \alpha_i + \cdots + \alpha_q$

Therefore $[X, \mathfrak{g}_{-2}] \neq 0$ for all $X \in \mathfrak{g}_{-1}$. Namely $\mathfrak{f} = \{0\}$, which implies $\mathfrak{m}$ cannot satisfy (5.3.5).

Now we will describe a model equation of the PD-manifold of type $(B_l, \{\alpha_1, \alpha_i\})$ $(3 \leq i \leq l)$. Let $n = 2l + 1$ and let E_k be the identity matrix of size k. We have

$$\mathfrak{o}(n,\mathbb{R}) = \left\{ X \in \mathfrak{gl}(n,\mathbb{R}) \mid {}^t X J + J X = 0 \right\},$$

where

$$J = \left(\begin{array}{c|c|c|c|c} & & & & 1 \\ \hline & & & E_{i-1} & \\ \hline & & E_{n-2i} & & \\ \hline & E_{i-1} & & & \\ \hline 1 & & & & \end{array}\right).$$

Then we have the following matrix representation of a real form $\mathfrak{o}(n,\mathbb{R}) = \bigoplus_{p=-3}^{3} \mathfrak{g}_p$ of $(B_l, \{\alpha_1, \alpha_i\})$:

$$\mathfrak{g}_{-3} = \left\{ \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline Z & 0 & 0 & 0 & 0 \\ \hline 0 & -{}^tZ & 0 & 0 & 0 \end{array}\right) \middle| \; Z \in M(i-1,1) \right\},$$

$$\mathfrak{g}_{-2} = \left\{ \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline P_1 & 0 & 0 & 0 & 0 \\ \hline 0 & P_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^tP_1 & 0 & 0 \end{array}\right) \middle| \; \begin{array}{l} P_1 \in M(n-2i,1), \\ P_2 \in \mathfrak{o}(i-1,\mathbb{R}) \end{array} \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline X_1 & 0 & 0 & 0 & 0 \\ \hline 0 & X_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^tX_2 & 0 & 0 \\ \hline 0 & 0 & 0 & -{}^tX_1 & 0 \end{array}\right) \middle| \; \begin{array}{l} X_1 \in M(i-1,1), \\ X_2 \in M(n-2i,i-1) \end{array} \right\},$$

$$\mathfrak{g}_{0} = \left\{ \left(\begin{array}{c|c|c|c|c} L_1 & 0 & 0 & 0 & 0 \\ \hline 0 & L_2 & 0 & 0 & 0 \\ \hline 0 & 0 & L_3 & 0 & 0 \\ \hline 0 & 0 & 0 & -{}^tL_2 & 0 \\ \hline 0 & 0 & 0 & 0 & -{}^tL_1 \end{array}\right) \middle| \; \begin{array}{l} L_1,\, L_4 \in \mathbb{R},\, L_2 \in M(i-1,i-1), \\ L_3 \in \mathfrak{o}(n-2i,\mathbb{R}) \end{array} \right\},$$

$$\mathfrak{g}_i = \left\{ {}^tX \mid X \in \mathfrak{g}_{-i} \right\} \quad \text{for } 1 \leq i \leq 3.$$

With respect to the matrix representation of $\mathfrak{o}(n,\mathbb{R})$, we may write the projection u_{-p} :

$\mathfrak{m} \longrightarrow \mathfrak{g}_{-p}$ as

$$u_{-3} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline Z & 0 & 0 & 0 & 0 \\ \hline 0 & -{}^tZ & 0 & 0 & 0 \end{array}\right), \qquad u_{-2} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline P_1 & 0 & 0 & 0 & 0 \\ \hline 0 & P_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^tP_1 & 0 & 0 \end{array}\right) \; (P_2 = -{}^tP_2),$$

$$u_{-1} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline X_1 & 0 & 0 & 0 & 0 \\ \hline 0 & X_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^tX_2 & 0 & 0 \\ \hline 0 & 0 & 0 & -{}^tX_1 & 0 \end{array}\right).$$

Substituting them for Formula (5.4.7), we have

$$\eta_{-3} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \Theta_0 & 0 & 0 & 0 & 0 \\ \hline 0 & -{}^t\Theta_0 & 0 & 0 & 0 \end{array}\right), \qquad \eta_{-2} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \Theta_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \Theta_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^t\Theta_1 & 0 & 0 \end{array}\right),$$

and

$$D_{\mathfrak{m}} = \{\, \Theta_0 = \Theta_1 = \Theta_2 = 0 \,\},$$

where

$$\begin{aligned} \Theta_0 &= dZ - \frac{1}{3}P_2 dX_1 - \frac{1}{3}d{}^tX_2 P_1 + \frac{2}{3}{}^tX_2 dP_1 + \frac{2}{3}dP_2 X_1 \\ &\qquad - \frac{1}{6}{}^tX_2(X_2 dX_1 - dX_2 X_1) + \frac{1}{6}({}^tX_2 dX_2 - d{}^tX_2 X_2)X_1, \\ \Theta_1 &= dP_1 - \frac{1}{2}X_2 dX_1 + \frac{1}{2}dX_2 X_1, \\ \Theta_2 &= dP_2 + \frac{1}{2}{}^tX_2 dX_2 - \frac{1}{2}d{}^tX_2 X_2. \end{aligned}$$

The exterior derivative of Θ_0 is

$$d\Theta_0 = -d\left(P_2 + \frac{1}{2}{}^tX_2 X_2\right) \wedge dX_1 + d{}^tX_2 \wedge d\left(P_1 + \frac{1}{2}X_2 X_1\right).$$

Putting

$$\hat{P}_1 = P_1 + \frac{1}{2}X_2X_1, \qquad \hat{P}_2 = P_2 + \frac{1}{2}{}^tX_2X_2,$$

we have

$$\begin{aligned}
\Theta_0 &= dZ - \frac{1}{3}(\hat{P}_2 + {}^tX_2X_2)\,dX_1 - \frac{1}{3}d{}^tX_2(\hat{P}_1 + X_2X_1) - \frac{1}{3}{}^tX_2dX_2X_1 + \frac{2}{3}{}^tX_2d\hat{P}_1 + \frac{2}{3}d\hat{P}_2X_1,\\
\Theta_1 &= d\hat{P}_1 - X_2dX_1,\\
\Theta_2 &= d\hat{P}_2 - d{}^tX_2X_2,
\end{aligned}$$

$$\begin{aligned}
d\Theta_0 &= -d\hat{P}_2 \wedge dX_1 + d{}^tX_2 \wedge d\hat{P}_1,\\
d\Theta_1 &= -dX_2 \wedge dX_1,\\
d\Theta_2 &= d{}^tX_2 \wedge dX_2.
\end{aligned}$$

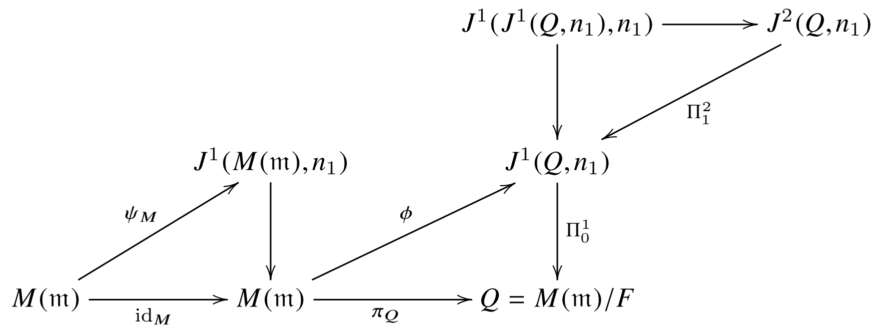
Digressing from determinating of the model equation, we now show theoretically that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into the 2-jet space $(J^2(Q,n), C^2_Q)$ over some manifold Q. From the structure equation of $D_{\mathfrak{m}}$, we have

$$\begin{aligned}
&\partial D_{\mathfrak{m}} = \{\, \Theta_0 = 0 \,\},\\
(5.4.9)\qquad &\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \left\{ \Theta_0 = \Theta_1 = \Theta_2 = dX_1 = d{}^tX_2 = 0 \right\} = \{0\},
\end{aligned}$$

which are differential systems of codimension $n_3 + n_2$ and $n_3 + n_2 + n_1$. Here, let $n_i = \dim \mathfrak{g}_{-i}$ for $1 \leq i \leq 3$. Putting

$$F = \left\{ \Theta_0 = dX_1 = d{}^tX_2 = 0 \right\} = \{\, \Theta_0 = dX_1 = dX_2 = 0 \,\},$$

we see that F is a completely integrable differential system of codimension $n_3 + n_1$. Let $Q = M(\mathfrak{m})/F$ be the space of leaves of the foliation. Let $\mathrm{id}_M : M(\mathfrak{m}) \longrightarrow M(\mathfrak{m})$ and $\pi_Q : N \longrightarrow Q$ be the identity map and the projection respectively.Since $\mathrm{Ker}\,(\pi_Q)_* = F$ is a subbundle of $\partial D_{\mathfrak{m}}$ of codimension n_1, applying Realization Lemma to π_Q, we have $\psi_N : N \longrightarrow J^1(Q,n_1)$ as in the lemma. Since $\mathrm{Ker}\,(\psi_N)_* = \mathrm{Ker}\,(\pi_N) \cap \mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$, ψ_N is immersion. Note that $\dim J^1(Q,n_1) - \dim M(\mathfrak{m}) = \dim \mathfrak{g}_{-3} \cdot \dim \mathfrak{g}_{-1} - \dim \mathfrak{g}_{-2} = i(i-2)(n-2i)+\frac{1}{2}i(i-1) > 0$. Since $\mathrm{Ker}\,(\mathrm{id}_M)_* = \{0\}$ is a subbundle of $D_{\mathfrak{m}}$ of codimension n_1, applying Realization Lemma to id_M, we have $\psi : M(\mathfrak{m}) \longrightarrow J^1(M(\mathfrak{m}),n_1)$ as in the lemma. Since $\mathrm{Ker}\,(\psi_M)_* = \mathrm{Ker}\,(\pi_M) \cap \mathrm{Ch}(D_{\mathfrak{m}}) = \{0\}$, ψ_M is immersion. Since $\psi_M(v) = (\pi_M)_*(D(v))$ for $v \in M(\mathfrak{m})$ and $(\phi)^{-1}_*(C^1_Q) = \partial D_{\mathfrak{m}}$, $\psi_M(v)$ is a n_1-dimensional integral element of C^1_Q. Moreover, since $\phi^{-1}_*(Q^1) = F$, where Q^1 denotes the kernel of $(\Pi^1_0)_*$ and $\Pi^1_0 : J^1(N,n_1) \longrightarrow N$ the projection, $\psi_M(v) \cap Q^1(\pi_M(v)) = \{0\}$. Thus we have $\psi_M(v) \in J^2(Q,n_1)$.



Now we return to the calculation of the model equation. From $F = \{\Theta_0 = dX_1 = dX_2 = 0\} = \{\Theta_0 = dX_1 = d^tX_2 = 0\}$, we calculate

$$\Theta_0 \equiv d\left(Z - \frac{1}{3}{}^tX_2X_2X_1 + \frac{2}{3}{}^tX_2\hat{P}_1 + \frac{2}{3}\hat{P}_2X_1\right) \qquad (\mathrm{mod}\ F^{\perp}).$$

Putting $\hat{Z} = Z - \frac{1}{3}{}^tX_2X_2X_1 + \frac{2}{3}{}^tX_2\hat{P}_1 + \frac{2}{3}\hat{P}_2X_1$, we have achieved a normal form of $D_{\mathfrak{m}}$:

$$\begin{aligned}\Theta_0 &= d\hat{Z} - d^tX_2\hat{P}_1 - \hat{P}_2dX_1,\\ \Theta_1 &= d\hat{P}_1 - X_2dX_1,\\ \Theta_2 &= d\hat{P}_2 - d^tX_2X_2.\end{aligned}$$

Since $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is embedded into the 2-jet space $J^2(Q,n_1)$, we should think n_1-dimensional integral element and manifold where dX_1 and dX_2 are independent. However, by $d\Theta_2 = dX_2 \wedge dX_1$, there are no integral elements and manifolds.

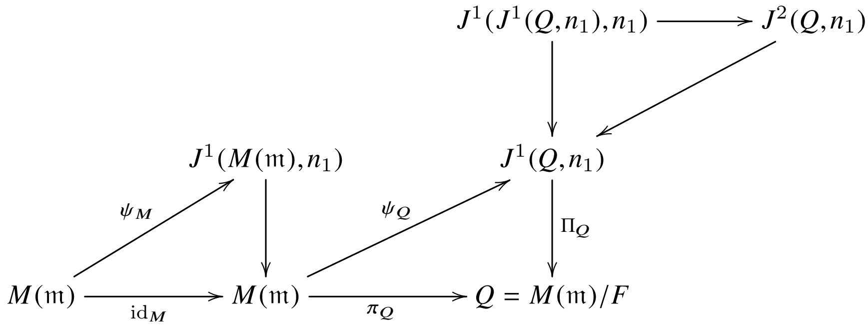
Now we generalize the above discussion as follows:

Proposition 5.7. *Let $\mathfrak{m}$ be a fundamental graded Lie algebra of third kind and $(M(\mathfrak{m}), D_{\mathfrak{m}})$ the standard differential system of type $\mathfrak{m}$. Assume $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Then $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into a 2-jet space $(J^2(Q,n_Q), C^2_Q)$ and furthermore $(M(\mathfrak{m}), D_{\mathfrak{m}})$ has no n_Q-dimensional integral elements and manifolds.*

Proof. Let η_{-p} be the $\mathfrak{g}_{-p}$-component of the Maurer-Cartan form of $M(\mathfrak{m})$. From Formula (5.4.7) in Section 5.4.1, we have the structure equation of $D_{\mathfrak{m}}$:

$$\text{(5.4.10)} \qquad \begin{cases} d\eta_{-3} = -[\eta_{-1}, \eta_{-2}],\\ d\eta_{-2} = -\dfrac{1}{2}[\eta_{-1}, \eta_{-1}],\end{cases}$$

Therefore we have $\partial D_{\mathfrak{m}} = \{\eta_{-3} = 0\}$. Let $F = \{\eta_{-3} = \eta_{-1} = 0\}$. It follows from the above structure equation that F is completely integrable. Let $Q = M(\mathfrak{m})/F$ be the space of leaves of the foliation and $\pi : M(\mathfrak{m}) \longrightarrow Q$ the projection. Applying Realization Lemma to π, since $\mathrm{Ker}\,\pi_* = F$ is a subbundle of $\partial D_{\mathfrak{m}}$ of codimension $n_1 = \dim \mathfrak{g}_{-1}$, we have the unique map $\psi_Q : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ such that $\pi = \Pi_Q \circ \psi_Q$ and $\partial D_{\mathfrak{m}} = (\psi_Q)_*^{-1}(C^1_Q)$, where $\Pi_Q : J^1(M(\mathfrak{m}), n_1) \longrightarrow Q$ is the projection and C_Q is the canonical system of $J^1(Q, n_1)$. Then ψ_Q is immersion since $\mathrm{Ker}\,(\psi_Q)_* = F \cap \mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Applying Realization Lemma to the identity map $\mathrm{id}_M : M(\mathfrak{m}) \longrightarrow M(\mathfrak{m})$, since $\mathrm{Ker}\,(\mathrm{id}_M)_*$ is a subbundle of $D_{\mathfrak{m}}$ of codimension n_1, we have a map $\psi_M : M(\mathfrak{m}) \longrightarrow J^1(M(\mathfrak{m}), n_1)$. Then ψ_M is immersion since $\mathrm{Ker}\,(\psi_M)_* = \mathrm{Ker}\,(\mathrm{id}_M)_* \cap \mathrm{Ch}(D_{\mathfrak{m}}) = \{0\}$. Since $\partial D_{\mathfrak{m}} = (\psi_Q)_*^{-1}(C^1_Q)$, $\psi_M(v)$ is a n_1-dimensional integral element of C^1_Q for $v \in M(\mathfrak{m})$. Since $F = (\psi_Q)_*^{-1}(\mathrm{Ker}\,(\Pi_Q)_*)$, we see that $\psi_M(v) \cap \mathrm{Ker}\,(\Pi_Q)_* = \{0\}$ for $v \in M(\mathfrak{m})$, which implies $\psi_M(v) \in J^2(Q, n_1)$. Namely $M(\mathfrak{m})$ is locally embedded into $(J^2(Q, n_1), C^2_Q)$. Regarding $M(\mathfrak{m})$ as a submanifold of $J^2(Q, n_1)$, we consider n_1-dimensional integral elements v with the independence condition $\eta_{-1}|_v \neq 0$. However, it follows from $d\eta_{-2} = -\frac{1}{2}[\eta_{-1}, \eta_{-1}]$ that $D_{\mathfrak{m}}$ has no such integral elements. □



5.4.3 $(D_l, \{\alpha_1, \alpha_i\})$-type $(3 \leq i \leq l-2)$

$$
\begin{aligned}
\Phi_3^+ = & \left\{ \alpha_1 + \cdots + \alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_i + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid 1 < p \leq i \right\}, \\
\Phi_2^+ = & \left\{ \alpha_1 + \cdots + \alpha_i + \cdots + \alpha_p \mid i \leq p \leq l \right\} \\
& \cup \{ \alpha_1 + \cdots + \alpha_i + \cdots + \alpha_{l-2} + \alpha_l \} \\
& \cup \left\{ \alpha_1 + \cdots + \alpha_i + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid i \leq p \leq l-3 \right\} \\
& \cup \left\{ \alpha_p + \cdots + \alpha_{q-1} + 2\alpha_q + \cdots + 2\alpha_i + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid 1 < p < q \leq i \right\}, \\
\Phi_1^+ = & \left\{ \alpha_1 + \cdots + \alpha_p \mid 1 \leq p < i \right\} \\
& \cup \left\{ \alpha_p + \cdots + \alpha_i + \cdots + \alpha_q \mid 1 < p \leq i \leq q \leq l \right\} \\
& \cup \left\{ \alpha_p + \cdots + \alpha_i + \cdots + \alpha_{l-2} + \alpha_l \mid 1 < p \leq i \right\} \\
& \cup \{ \alpha_p + \cdots + \alpha_i + \cdots + \alpha_{q-1} + 2\alpha_q + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid \\
& \qquad\qquad\qquad\qquad\qquad\qquad 1 < p \leq i < q \leq l-2 \},
\end{aligned}
$$

Then we see that $\Phi_{\mathfrak{f}} = \emptyset$, which implies that $\mathfrak{m}$ cannot satisfy (5.3.5).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Therefore, it follows from Propostion 5.7 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into a 2-jet space $(J^2(Q, n_1), C_Q^2)$ but $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds.

5.4.4 $(E_6, \{\alpha_4\})$-type

Let $\begin{pmatrix} c_1\, c_3\, c_4\, c_5\, c_6 \\ c_2 \end{pmatrix}$ denote the root $c_1\alpha_1 + \ldots + c_6\alpha_6$ of E_6.
Φ_3^+ consists of the following roots:

$$\begin{smallmatrix}1&2&3&2&1\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}1&2&3&2&1\\&&2&&\end{smallmatrix}$$

Φ_2^+ consists of the following roots:

$$\begin{smallmatrix}0&1&2&1&0\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}1&1&2&1&0\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}0&1&2&1&1\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}1&2&2&1&0\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}1&1&2&1&1\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}0&1&2&2&1\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}1&2&2&1&1\\&&1&&\end{smallmatrix}$$
$$\begin{smallmatrix}1&1&2&2&1\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}1&2&2&2&1\\&&1&&\end{smallmatrix}$$

Φ_1^+ consists of the following roots:

$$\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix} \quad \begin{smallmatrix}0&1&1&0&0\\&&0&&\end{smallmatrix} \quad \begin{smallmatrix}0&0&1&1&0\\&&0&&\end{smallmatrix} \quad \begin{smallmatrix}0&0&1&0&0\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix} \quad \begin{smallmatrix}0&1&1&1&0\\&&0&&\end{smallmatrix} \quad \begin{smallmatrix}0&1&1&0&0\\&&1&&\end{smallmatrix}$$
$$\begin{smallmatrix}0&0&1&1&1\\&&0&&\end{smallmatrix} \quad \begin{smallmatrix}0&0&1&1&0\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}1&1&1&1&0\\&&0&&\end{smallmatrix} \quad \begin{smallmatrix}1&1&1&0&0\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}0&1&1&1&1\\&&0&&\end{smallmatrix} \quad \begin{smallmatrix}0&1&1&1&0\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}0&0&1&1&1\\&&1&&\end{smallmatrix}$$
$$\begin{smallmatrix}1&1&1&1&1\\&&0&&\end{smallmatrix} \quad \begin{smallmatrix}1&1&1&1&0\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}0&1&1&1&1\\&&1&&\end{smallmatrix} \quad \begin{smallmatrix}1&1&1&1&1\\&&1&&\end{smallmatrix}$$

Then we see that $\Phi_{\mathfrak{f}} = \emptyset$, which implies that $\mathfrak{m}$ cannot satisfy (5.3.5).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Therefore, it follows from Propostion 5.7 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into a 2-jet space $(J^2(Q, n_1), C_Q^2)$ but $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds.

5.4.5 $(E_7, \{\alpha_3\})$-type

Let $\begin{pmatrix} c_1\, c_3\, c_4\, c_5\, c_6\, c_7 \\ c_2 \end{pmatrix}$ denote the root $c_1\alpha_1 + \cdots + c_7\alpha_7$ of E_7,
Φ_3^+ consists of the following roots:

1 3 4 3 2 1	2 3 4 3 2 1
2	2

Φ_2^+ consists of the following roots:

1 2 2 1 0 0	1 1 2 1 1 0	0 1 2 2 1 0	1 2 2 1 1 0	1 1 2 2 1 0	1 2 2 2 1 0
1	1	1	1	1	1
1 2 3 2 1 0	1 2 3 2 1 0	0 1 2 1 1 1	1 1 2 1 1 1	0 1 2 2 1 1	1 2 2 1 1 1
1	2	1	1	1	1
1 1 2 2 1 1	0 1 2 2 2 1	1 2 2 2 1 1	1 1 2 2 2 1	1 2 2 2 2 1	1 2 3 2 1 1
1	1	1	1	1	1
1 2 3 2 2 1	1 2 3 2 1 1	1 2 3 3 2 1	1 2 3 2 2 1	1 2 3 3 2 1	1 2 4 3 2 1
1	2	1	2	2	2

Φ_1^+ consists of the following roots:

0 1 0 0 0 0	1 1 0 0 0 0	0 1 1 0 0 0	1 1 1 0 0 0	0 1 1 1 0 0	0 1 1 0 0 0
0	0	0	0	0	1
1 1 1 1 0 0	1 1 1 0 0 0	0 1 1 1 1 0	0 1 1 1 0 0	1 1 1 1 1 0	1 1 1 1 0 0
0	1	0	1	0	1
0 1 1 1 1 1	0 1 1 1 1 0	1 1 1 1 1 1	1 1 1 1 1 0	0 1 1 1 1 1	1 1 1 1 1 1
0	1	0	1	1	1
0 1 2 1 0 0	1 1 2 1 0 0	0 1 2 1 1 0	1 1 2 1 1 0	0 1 2 2 1 0	1 1 2 2 1 0
1	1	1	1	1	1
0 1 2 1 1 1	1 1 2 1 1 1	0 1 2 2 1 1	1 2 2 1 1 1	1 1 2 2 1 1	0 1 2 2 2 1
1	1	1	1	1	1
1 1 2 2 2 1					
1					

Then we see that $\Phi_{\mathfrak{f}} = \emptyset$, which implies that $\mathfrak{m}$ cannot satisfy (5.3.5).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Therefore, it follows from Propostion 5.7 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into a 2-jet space $(J^2(Q, n_1), C_Q^2)$ but $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds.

5.4.6 $(E_7, \{\alpha_5\})$-type

Φ_3^+ consists of the following roots:

123321 1	123221 2	123321 2	124321 2	134321 2	234321 2

Φ_2^+ consists of the following roots:

012210 1	112210 1	122210 1	123210 1	123210 2	012211 1
112211 1	012221 1	122211 1	112221 1	122221 1	123211 1
123221 1	123211 2	123221 2			

Φ_1^+ consists of the following roots:

000100 0	001100 0	000110 0	011100 0	001100 1	001110 0
000111 0	111100 0	011110 0	011100 1	001110 1	001111 0
111110 0	111100 1	011111 0	011110 1	001111 1	111111 0
111110 1	011111 1	111111 1	012100 1	112100 1	012110 1
122100 1	112110 1	122110 1	012111 1	112111 1	122111 1

Then we have $\Phi_{\mathfrak{f}} = \emptyset$, which implies that $\mathfrak{m}$ cannot satisfy (5.3.5).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Therefore, it follows from Propostion 5.7 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into a 2-jet space $(J^2(Q, n_1), C_Q^2)$ but $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds.

5.4.7 $(E_7, \{\alpha_2, \alpha_7\})$-type

Φ_3^+ consists of the following roots:

1 2 3 2 1 1	1 2 3 3 2 1	1 2 3 2 2 1	1 2 3 3 2 1	1 2 4 3 2 1	1 3 4 3 2 1
2	1	2	2	2	2
2 3 4 3 2 1					
2					

Φ_2^+ consists of the following roots:

0 0 1 1 1 1	0 1 1 1 1 1	1 1 1 1 1 1	1 2 3 2 1 0	0 1 2 1 1 1	1 1 2 1 1 1
1	1	1	2	1	1
0 1 2 2 1 1	1 2 2 1 1 1	1 1 2 2 1 1	0 1 2 2 2 1	1 2 2 2 1 1	1 1 2 2 2 1
1	1	1	1	1	1
1 2 2 2 2 1	1 2 3 2 1 1	1 2 3 2 2 1	1 2 3 3 2 1		
1	1	1	1		

Φ_1^+ consists of the following roots:

0 0 0 0 0 1	0 0 0 0 0 0	0 0 1 0 0 0	0 0 0 0 1 1	0 1 1 0 0 0	0 0 1 1 0 0
0	1	1	0	1	1
0 0 0 1 1 1	1 1 1 0 0 0	0 1 1 1 0 0	0 0 1 1 1 0	0 0 1 1 1 1	1 1 1 1 0 0
0	1	1	1	0	1
0 1 1 1 1 1	0 1 1 1 1 0	1 1 1 1 1 1	1 1 1 1 1 0	0 1 2 1 0 0	1 1 2 1 0 0
0	1	0	1	1	1
0 1 2 1 1 0	1 2 2 1 0 0	1 1 2 1 1 0	0 1 2 2 1 0	1 2 2 1 1 0	1 1 2 2 1 0
1	1	1	1	1	1
1 2 2 2 1 0	1 2 3 2 1 0				
1	1				

Then we see that $\Phi_{\mathfrak{f}} = \emptyset$, which implies that $\mathfrak{m}$ cannot satisfy (5.3.5).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Therefore, it follows from Propostion 5.7 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into a 2-jet space $(J^2(Q, n_1), C^2_Q)$ but $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds.

5.4.8 $(E_8, \{\alpha_2\})$-type

Let $\begin{pmatrix} c_1\, c_3\, c_4\, c_5\, c_6\, c_7\, c_8 \\ c_2 \end{pmatrix}$ denote the root $c_1\alpha_1 + \cdots + c_8\alpha_8$ of E_8.
Φ_3^+ consists of the following roots:

1 3 5 4 3 2 1	2 3 5 4 3 2 1	2 4 5 4 3 2 1	2 4 6 4 3 2 1	2 4 6 5 3 2 1	2 4 6 5 4 2 1	2 4 6 5 4 3 1
3	3	3	3	3	3	3
2 4 6 5 4 3 2						
3						

Φ_2^+ consists of the following roots:

1 2 3 2 1 0 0	1 2 3 2 1 1 0	1 2 3 2 2 1 0	1 2 3 3 2 1 0	1 2 4 3 2 1 0	1 3 4 3 2 1 0	2 3 4 3 2 1 0
2	2	2	2	2	2	2
1 2 3 2 1 1 1	1 2 3 3 2 1 1	1 2 3 2 2 2 1	1 2 4 3 2 1 1	1 2 3 3 2 2 1	1 3 4 3 2 1 1	1 2 4 3 2 2 1
2	2	2	2	2	2	2
1 2 3 3 3 2 1	2 3 4 3 2 1 1	1 3 4 3 2 2 1	1 2 4 3 3 2 1	2 3 4 3 2 2 1	1 3 4 3 3 2 1	1 2 4 4 3 2 1
2	2	2	2	2	2	2
2 3 4 3 3 2 1	1 3 4 4 3 2 1	2 3 4 4 3 2 1	2 3 5 4 3 2 1	2 4 5 4 3 2 1		
2	2	2	2	2		

Φ_1^+ consists of the following roots:

0 0 0 0 0 0 0	0 0 1 0 0 0 0	0 1 1 0 0 0 0	0 0 1 1 0 0 0	1 1 1 0 0 0 0	0 1 1 1 0 0 0	0 0 1 1 1 0 0
1	1	1	1	1	1	1
1 1 1 1 0 0 0	0 1 1 1 1 0 0	0 0 1 1 1 1 0	1 1 1 1 1 0 0	0 1 1 1 1 1 0	0 0 1 1 1 1 1	1 1 1 1 1 1 0
1	1	1	1	1	1	1
0 1 1 1 1 1 1	1 1 1 1 1 1 1	0 1 2 1 0 0 0	1 1 2 1 0 0 0	0 1 2 1 1 0 0	1 2 2 1 0 0 0	1 1 2 1 1 0 0
1	1	1	1	1	1	1
0 1 2 2 1 0 0	1 2 2 1 1 0 0	1 1 2 2 1 0 0	1 2 2 2 1 0 0	1 2 3 2 1 0 0	0 1 2 1 1 1 0	1 1 2 1 1 1 0
1	1	1	1	1	1	1
0 1 2 2 1 1 0	1 2 2 1 1 1 0	1 1 2 2 1 1 0	0 1 2 2 2 1 0	1 2 2 2 1 1 0	1 1 2 2 2 1 0	1 2 2 2 2 1 0
1	1	1	1	1	1	1
1 2 3 2 1 1 0	1 2 3 2 2 1 0	1 2 3 3 2 1 0	0 1 2 1 1 1 1	0 1 2 2 1 1 1	1 1 2 1 1 1 1	0 1 2 2 2 1 1
1	1	1	1	1	1	1
1 2 2 1 1 1 1	1 1 2 2 1 1 1	1 2 2 2 1 1 1	1 1 2 2 2 1 1	0 1 2 2 2 2 1	1 2 3 2 1 1 1	1 2 2 2 2 1 1
1	1	1	1	1	1	1
1 1 2 2 2 2 1	1 2 3 2 2 1 1	1 2 2 2 2 2 1	1 2 3 3 2 1 1	1 2 3 2 2 2 1	1 2 3 3 2 2 1	1 2 3 3 3 2 1
1	1	1	1	1	1	1

Then we see that $\Phi_{\mathfrak{f}} = \emptyset$, which implies that $\mathfrak{m}$ cannot satisfy (5.3.5).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Therefore, it follows from Propostion 5.7 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into a 2-jet space $(J^2(Q, n_1), C_Q^2)$ but $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds.

5.4.9 $(E_8, \{\alpha_7\})$-type

Φ_3^+ consists of the following roots:

2465431 3	2465432 3

Φ_2^+ consists of the following roots:

0122221 1	1122221 1	1222221 1	1232221 1	1232221 2	1233221 1	1233221 2
1233321 1	1243221 2	1233321 2	1343221 2	1243321 2	2343221 2	1343321 2
1244321 2	2343321 2	1344321 2	1354321 2	2344321 2	1354321 3	2354321 2
2354321 3	2454321 2	2454321 3	2464321 3	2465321 3	2465421 3	

Φ_1^+ consists of the following roots:

0000010 0	0000110 0	0000011 0	0001110 0	0000111 0	0011110 0	0001111 0
0111110 0	0011111 0	0011110 1	1111110 0	0111111 0	0111110 1	0011111 1
1111111 0	1111110 1	0111111 1	1111111 1	0121110 1	1121110 1	0122110 1
1221110 1	1122110 1	0122210 1	1222110 1	1122210 1	1222210 1	1232110 1
1232210 1	1232110 2	1233210 1	1232210 2	1233210 2	1243210 2	1343210 2
2343210 2	0121111 1	0122111 1	1121111 1	0122211 1	1221111 1	1122111 1
1222111 1	1122211 1	1232111 1	1222211 1	1232111 2	1232211 1	1233211 1
1233211 2	1243211 2	1343211 2	2343211 2			

Then we see that $\Phi_{\mathfrak{f}} = \emptyset$, which implies that $\mathfrak{m}$ cannot satisfy (5.3.5).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Therefore, it follows from Propostion 5.7 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into a 2-jet space $(J^2(Q, n_1), C_Q^2)$ but $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds.

5.4.10 $(F_4, \{\alpha_2\})$-type

Let $(c_1\, c_2\, c_3\, c_4)$ denote the root $c_1\alpha_1 + \cdots + c_4\alpha_4$ of F_4.
Φ_3^+ consists of the following roots:

1342 2342

Φ_2^+ consists of the following roots:

1220 1221 1231 1222 1232 1242

Φ_1^+ consists of the following roots:

0100 1100 0110 1110 0111 1111
0120 1120 0121 1121 0122 1122

Then we see that $\Phi_{\mathfrak{f}} = \emptyset$, which implies that $\mathfrak{m}$ cannot satisfy (5.3.5).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Therefore, it follows from Propostion 5.7 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is locally embedded into a 2-jet space $(J^2(Q, n_1), C^2_Q)$ but $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds.

5.4.11 $(G_2, \{\alpha_1\})$-type

We have $\Phi_3^+ = \{3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2\}$, $\Phi_2^+ = \{2\alpha_1 + \alpha_2\}$, $\Phi_1^+ = \{\alpha_1, \alpha_1 + \alpha_2\}$. Then $\Phi_{\mathfrak{f}} = \emptyset$, which implies that $\mathfrak{m}$ cannot satisfy (5.3.5). For more detail, we refer to [Car10], [Tan70], and [Yam93].

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