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Braid groups in complex Grassmannians

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Abstract

We describe the fundamental group and second homotopy group of ordered $k$–point sets in $Gr(k, n)$ generating a subspace of fixed dimension.

Keywords:

complex space, configuration spaces, braid groups.


1 Introduction

Let $M$ be a manifold and $\Sigma_h$ be the symmetric group on $h$ elements. The ordered and unordered configuration spaces of $h$ distinct points in $M$, $\mathcal{F}_h(M) = \{(x_1, \ldots, x_h) \in M^h | x_i \neq x_j, \ i \neq j\}$ and $\mathcal{C}_h(M) = \mathcal{F}_h(M)/\Sigma_h$, have been widely studied. In recent papers ([BP, MPS, MS]), new configuration spaces were introduced when $M$ is, respectively, the projective space $\mathbb{C}P^n$, the affine space $\mathbb{C}^n$ and the Grassmannian manifold $Gr(k, n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$, by stratifying the configuration spaces $\mathcal{F}_h(M)$ (resp. $\mathcal{C}_h(M)$) with complex submanifolds $\mathcal{F}_h^i(M)$ (resp. $\mathcal{C}_h^i(M)$) defined as the ordered (resp. unordered) configuration spaces of all $h$ points in $M$.

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generating a subspace of dimension $i$. The homotopy groups of those configuration spaces are interesting as they are strongly related to the homotopy groups of the Grassmannian manifolds, i.e. of spheres.

In [BP] (resp. [MPS]), the fundamental groups $\pi_1(F^i_h(\mathbb{P}^n))$ and $\pi_1(C^i_h(\mathbb{P}^n))$ (resp. $\pi_1(F^i_h(\mathbb{C}^n))$ and $\pi_1(C^i_h(\mathbb{C}^n))$) are computed, proving that the former are trivial and the latter are isomorphic to the symmetric group $\Sigma_h$ except when $i = 1$ (resp. $i = 1$ and $i = n = h - 1$) providing, in this last case, a presentation for both $\pi_1(F^1_h(\mathbb{P}^n))$ and $\pi_1(C^1_h(\mathbb{P}^n))$ (resp. $\pi_1(F^1_h(\mathbb{C}^n))$ and $\pi_1(C^1_h(\mathbb{C}^n))$) which is similar to those of the braid groups of the sphere.

In this paper we generalize the results obtained in [BP] when $M$ is the projective space $\mathbb{P}^{n-1} = Gr(1, n)$, to the case of Grassmannian manifold $Gr(k, n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$. We prove that if $F^i_h(k, n)$ is the $i$-th ordered configuration space of all distinct points $H_1, \ldots, H_h$ in the Grassmannian manifold $Gr(k, n)$ whose sum is a subspace of dimension $i$, then the following result holds.

**Theorem 1.1.** The non-empty ordered configuration spaces $F^i_h(k, n)$ are all simply connected if $k > 1$.

From this, we immediately obtain that the fundamental group of the $i$-th unordered configuration space $F^i_h(k, n)/\Sigma_h$ is isomorphic to $\Sigma_h$.

These results are stated in Section 2. In Section 3 we compute the second homotopy group of the $i$-th configuration spaces in two special cases: the case in which the subspaces are in direct sum and the case of two subspaces.

**Theorem 1.2.** If $hk < n$, $\pi_2(F^{hk}_h(k, n)) = \mathbb{Z}^h$, while $\pi_2(F^{hk}_h(k, h)) = \mathbb{Z}^{h-1}$. If $k < i < n$, $\pi_2(F^i_2(k, n)) = \mathbb{Z}^3$, while $\pi_2(F^i_2(k, n)) = \mathbb{Z}^2$.

## 2 The first homotopy group of $F^i_h(k, n)$

Let $Gr(k, n)$ be the Grassmannian manifold parametrizing $k$-dimensional subspaces of $\mathbb{C}^n$, $0 < k < n$. In [MS] authors define the space $F^i_h(k, n)$ as the ordered configuration space of all $h$ distinct points $H_1, \ldots, H_h$ in $Gr(k, n)$ such that the dimension of the sum $\dim(H_1 + \cdots + H_h)$ equals $i$.

**Remark 2.1.** The following easy facts hold:

1. if $h = 1$, $F^i_h(k, n)$ is empty except for $i = k$ and $F^k_1(k, n) = Gr(k, n)$;
2. if $i = 1$, $\mathcal{F}_h^i(k,n)$ is empty except for $k,h = 1$ and $\mathcal{F}_1^1(1,n) = \text{Gr}(1,n) = \mathbb{CP}^{n-1}$;

3. if $h \geq 2$ and $k = n - 1$ then $\mathcal{F}_h^i(k,n)$ is empty except for $i = n$, and, since the sum of two (different) hyperplanes is $\mathbb{C}^n$, $\mathcal{F}_h^n(n-1,n) = \mathcal{F}_h(\text{Gr}(n-1,n)) = \mathcal{F}_h(\mathbb{CP}^{n-1})$;

4. if $h \geq 2$ then $\mathcal{F}_h^i(k,n) \neq \emptyset$ if and only if $k + 1 \leq i \leq \min(hk,n)$;

5. if $h \geq 2$ then $\mathcal{F}_h^i(\text{Gr}(k,n)) = \bigcup_{i=k+1}^{\min(hk,n)} \mathcal{F}_h^i(k,n)$, with the open stratum given by the case of maximum dimension $i = \min(hk,n)$;

6. if $h \geq 2$ then the adjacency of the non-empty strata is given by

$$\overline{\mathcal{F}_h^i(k,n)} = \mathcal{F}_h^{k+1}(k,n) \prod \ldots \prod \mathcal{F}_h^i(k,n).$$

As the case $k = 1$ has been treated in [BP] and, by the above remarks, the case $h = 1$ is trivial, in this paper we will consider $h,k > 1$ (and hence $i > k$).

In [MS], authors proved that $\mathcal{F}_h^i(k,n)$ is (when non empty) a complex submanifold of $\text{Gr}(k,n)^h$ of dimension $i(n-i) + hk(i-k)$, and that if $i = \min(n,hk)$ and $n \neq hk$ then the open strata $\mathcal{F}_h^i(k,n)$ are simply connected except for $n = 2$ (and $k = 1$), i.e.

$$
\pi_1(\mathcal{F}_h^{\min(n,hk)}(k,n)) = \begin{cases} 
0 & \text{if } n \neq hk \\
\mathcal{PB}_h(S^2) & \text{if } n = 2, \ k = 1
\end{cases}
$$

(1)

where $\mathcal{PB}_h(S^2)$ is the pure braid group on $h$ strings of the sphere $S^2$.

In order to complete this result and compute fundamental groups in all cases we need two Lemmas.

**Lemma 2.2.** Let $V = (H_1, \ldots, H_h)$ be an element in the space $\mathcal{F}_h^i(k,n)$ and denote the sum $H_1 + \cdots + H_h \in \text{Gr}(i,n)$ by $\gamma(V)$, then the map

$$
\gamma : \mathcal{F}_h^i(k,n) \to \text{Gr}(i,n)
$$

(2)

is a locally trivial fibration with fiber $\mathcal{F}_h^i(k,i)$.
Proof. Let $V_0$ be an element in the Grassmannian manifold $Gr(i, n)$. Fix $L_0 \in Gr(n - i, n)$ such that $L_0 \cap V_0 = \{0\}$ and let $\varphi : \mathbb{C}^n \to V_0$ be the linear projection on $V_0$ given by the direct sum decomposition $L_0 + V_0 = \mathbb{C}^n$. If $F_h^i(k, V_0)$ is the ordered configuration space of $h$ distinct $k$-dimensional spaces in $V_0$ whose sum is an $i$-dimensional subspace, then $F_h^i(k, V_0)$ coincides with $F_h^i(k, i)$ when a basis in $V_0$ is fixed.

Let $U_{L_0}$ be the open neighborhood of $V_0$ in $Gr(i, n)$ defined as

$$U_{L_0} = \{V \in Gr(i, n) \mid L_0 \cap V = \{0\}\}.$$ 

The restriction of the projection $\varphi$ to an element $V$ in $U_{L_0}$ is a linear isomorphism $\varphi_V : V \to V_0$ and a local trivialization for $\gamma$ is given by the homeomorphism

$$f : \gamma^{-1}(U_{L_0}) \to U_{L_0} \times F_h^i(k, V_0)$$

$$y = (H_1, \ldots, H_h) \mapsto (\gamma(y), (\varphi_{\gamma(y)}(H_1), \ldots, \varphi_{\gamma(y)}(H_h)))$$

which makes the following diagram commute.

\[ \begin{array}{ccc}
\gamma^{-1}(U_{L_0}) & \xrightarrow{f} & U_{L_0} \times F_h^i(k, i) \\
\gamma \downarrow & & \downarrow \text{pr}_1 \\
U_{L_0} & & \end{array} \]

This completes the proof. \qed

Lemma 2.3. The projection map on the first $h - 1$ entries

$$pr : F_h^{kh}(k, n) \to F_{h-1}^{kh-1}(k, n)$$

$$(H_1, \ldots, H_h) \mapsto (H_1, \ldots, H_{h-1})$$

is a locally trivial fibration for any $n \geq kh$. Moreover, if $n = kh$, the fiber is $\mathbb{C}^{k(kh-k)}$.

Proof. Let $V_0$ be an element in $F_{h-1}^{kh-1}(k, n)$. Fix $L_0 \in Gr(n - k(h - 1), n)$ such that $L_0 \cap \gamma(V_0) = \{0\}$ and let $\varphi : \mathbb{C}^n \to \gamma(V_0)$ be the linear projection
on $\gamma(V_0)$ given by the direct sum decomposition $L_0 + \gamma(V_0) = \mathbb{C}^n$.

The fiber of the projection map $pr$ over $V_0$ is the open set

$$U_{\gamma(V_0)} = \{ H \in Gr(k, n) | H \cap \gamma(V_0) = \{0\} \}.$$ 

Let $U_{L_0}$ be the open neighborhood of $V_0$ in $F_{h-1}^{k} (k, n)$ defined as

$$U_{L_0} = \{ V \in F_{h-1}^{k} (k, n) | L_0 \cap \gamma(V) = \{0\} \}.$$ 

If $V$ is a point in $U_{L_0}$, the restriction of the map $\varphi$ to $\gamma(V)$ is a linear isomorphism $\tilde{\varphi}_{V} : \gamma(V) \to \gamma(V_0)$ that can be extended to an isomorphism $\varphi_V$ of $\mathbb{C}^n$ by requiring it to be the identity on $L_0$.

A local trivialization for the projection $pr$ is given by the homeomorphism

$$f : pr^{-1}(U_{L_0}) \to U_{L_0} \times U_{\gamma(V_0)}$$

$$y = (H_1, \ldots, H_h) \mapsto (pr(y), \varphi_{\gamma(pr(y))}(H_h))$$

which makes the following diagram commute.

$$\xymatrix{ pr^{-1}(U_{L_0}) \ar[rr]^f \ar[dr]_{pr} & & U_{L_0} \times U_{\gamma(V_0)} \ar[dl]^{pr_1} \\
& U_{L_0} & }$$

Remark that if $n = kh$, then $U_{\gamma(V_0)} = \{ H \in Gr(k, n) | H + \gamma(V_0) = \mathbb{C}^n \}$ is a single coordinate chart of the Grassmannian manifold $Gr(k, kh)$, that is it is homeomorphic to $\mathbb{C}^{kh-k}$. This completes the proof. \qed

Let us remark that if $V = (H_1, \ldots, H_h)$ is a point in the space $F_{h}^{kh}(k, n)$, then the $h$ subspaces $H_1, \ldots, H_h$ are in direct sum and the map

$$pr : F_{h}^{kh}(k, n) \to F_{h-1}^{k(h-1)}(k, n)$$

$$(H_1, \ldots, H_h) \mapsto (H_1, \ldots, H_{h-1})$$

is well defined.

We have, from the homotopy long exact sequence of the fibration $pr$ with $n = kh$, that

$$\pi_j(F_{h}^{kh}(k, kh)) = \pi_j(F_{h-1}^{k(h-1)}(k, kh))$$

(4)
for all \( j \) and, by equation (1), that
\[
\pi_1(\mathcal{F}_h^{kh}(k, kh)) = \pi_1(\mathcal{F}_{h-1}^{k(h-1)}(k, kh)) = 0.
\]
It follows that the open stratum \( \mathcal{F}_h^{kh}(k, kh) \) is simply connected, hence all open strata are simply connected.

Moreover, from the homotopy long exact sequence of the fibration \( \gamma \), we have that
\[
\pi_1(\mathcal{F}_i^i(k, i)) \to \pi_1(\mathcal{F}_i^i(k, n)) \to \pi_1(Gr(i, n)) = 0.
\]
As \( \mathcal{F}_i^i(k, i) \) is an open stratum, it is simply connected and hence \( \pi_1(\mathcal{F}_i^i(k, n)) = 0 \).

That is, all our configuration spaces are simply connected and Theorem 1.1 is proved.

## 3 The second homotopy group

In this section we compute the second homotopy group \( \pi_2(\mathcal{F}_i^i(k, n)) \) when \( i = hk \), i.e. subspaces in direct sum, and when \( h = 2 \), i.e. the case of two subspaces. In order to compute those homotopy groups, we need to know that the third homotopy group for Grassmannian manifolds is trivial if \( k > 1 \). Even if it should be a classical result we didn’t find references and we decided to give a proof here.

Let \( V_{k,n} \) be the space parametrizing the (ordered) \( k \)-uples of orthonormal vectors in \( \mathbb{C}^n \), \( 1 \leq k \leq n \). It is an easy remark that \( V_{1,n} = S^{2n-1} \) and \( V_{n,n} = U(n) \). It’s well known that the function that maps an element of \( V_{k,n} \) to the subspace generated by its entries is a locally trivial fibration:
\[
V_{k,k} \hookrightarrow V_{k,n} \to Gr(k, n) \quad (k < n),
\]
while the projection on the last entry is the locally trivial fibration:
\[
V_{k-1,n-1} \hookrightarrow V_{k,n} \to S^{2n-1} \quad (k > 1).
\]

Using the long exact sequence in homotopy induced by fibration (6), it’s easy to see (crf. [St]) that \( \pi_1(V_{k,n}) = \pi_2(V_{k,n}) = \pi_3(V_{k,n}) = 0 \), except for \( \pi_1(V_{n,n}) = \pi_3(V_{n,n}) = \pi_3(V_{n-1,n}) = \mathbb{Z} \).
The exact sequence of homotopy groups associated to fibration (5) for $k < n - 1$ then becomes

$$
\mathbb{Z} \rightarrow 0 \rightarrow \pi_3(Gr(k, n)) \rightarrow 0 \rightarrow 0 \rightarrow \pi_2(Gr(k, n)) \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \pi_1(Gr(k, n)) \rightarrow 0
$$

that is $\pi_1(Gr(k, n)) = 0$, $\pi_2(Gr(k, n)) = \mathbb{Z}$ and $\pi_3(Gr(k, n)) = 0$ if $k < n - 1$.

If $k = n - 1$ then $Gr(n - 1, n) = \mathbb{P}^{n-1}$ and $\pi_3(Gr(n - 1, n)) = 0$ except if $n = 2$ in which case $Gr(1, 2) = S^2$ and $\pi_3(Gr(1, 2)) = \mathbb{Z}$. That is the third homotopy group of the Grassmannian manifold $Gr(k, n)$ is trivial if $k > 1$.

Since the third homotopy group of the Grassmannian manifold $Gr(k, n)$ is trivial if $k > 1$ then for $i < n$ the homotopy long exact sequence of the fibration $\gamma$ defined in equation (2) gives:

$$0 = \pi_3(Gr(i, n)) \rightarrow \pi_2(F^i(k, i)) \rightarrow \pi_2(F^i_h(k, n)) \rightarrow \mathbb{Z} = \pi_2(Gr(i, n)) \rightarrow 0.
$$

As the second homotopy groups are abelian and the above short exact sequence splits, we have

$$\pi_2(F^i_h(k, n)) = \pi_2(F^i(k, i)) \times \mathbb{Z}.
$$

**The case $i = hk$.** If $i = hk$, by equation (4), $\pi_2(F^hk_h(k, hk)) = \pi_2(F^{k(h-1)}_h(k, h))$ and the following equalities hold:

$$
\pi_2(F^{hk}_h(k, h)) = \pi_2(F^{k(h-1)}_h(k, k(h-1))) \times \mathbb{Z} = \\
= \pi_2(F^{k(h-2)}_h(k, k(h-1))) \times \mathbb{Z} = \\
= \pi_2(F^{k(h-2)}_h(k, k(h-2))) \times \mathbb{Z} = \\
= \pi_2(F^{2k}_h(k, 2k)) \times \mathbb{Z}^{h-2} = \\
= \pi_2(F^1_h(k, 2k)) \times \mathbb{Z}^{h-2} = \\
= \pi_2(Gr(k, 2k)) \times \mathbb{Z}^{h-2} = \\
= \mathbb{Z}^{h-1}
$$

while, if $hk < n$, $\pi_2(F^{hk}_h(k, n)) = \mathbb{Z}^h$. 

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The case $h = 2$. If $h = 2$ a point $(H_1, H_2)$ is in the space $\mathcal{F}_2^h(k, n)$ if and only if the dimension of intersection $\dim(H_1 \cap H_2) = 2k - i$. If $i = 2k$ (which includes the cases $k = 1$ and $n = 2$) $H_1$ and $H_2$ are in direct sum otherwise the following Lemma holds.

**Lemma 3.1.** If $k < i < 2k$, the map

$$
\eta : \mathcal{F}_2^i(k, n) \to Gr(2k - i, n)
$$

$$(H_1, H_2) \mapsto H_1 \cap H_2$$

is a locally trivial fibration with fiber $\mathcal{F}_2^{2i-2k}(i - k, n - 2k + i)$.

**Proof.** Let $V_0$ be a point in the Grassmannian manifold $Gr(2k - i, n)$. Fix

$L_0 \in Gr(n - 2k + i, n)$ such that $L_0 \cap V_0 = \{0\}$ and let $\varphi : \mathbb{C}^n \to V_0$ be the linear projection given by the direct sum decomposition $L_0 + V_0 = \mathbb{C}^n$.

The fiber $\eta^{-1}(V_0)$ is the set of all pairs $(H_1, H_2)$ of $k$-dimensional subspaces of $\mathbb{C}^n$ such that $H_1 \cap H_2 = V_0$. That is, a pair $(H_1, H_2)$ is in $\eta^{-1}(V_0)$ if and only if it corresponds to a pair of $(i - k)$-dimensional subspaces of $\mathbb{C}^n/V_0$ are in direct sum, i.e. a point in $\mathcal{F}_2^{2(i-k)}(i - k, n - 2k + i)$.

Let $U_{L_0}$ be the open neighborhood of $V_0$ in $Gr(2k - i, n)$, defined as

$$U_{L_0} = \{V \in Gr(2k - i, n) | L_0 \cap V = \{0\}\}.$$

If $V$ is a point in $U_{L_0}$, the restriction of $\varphi$ to $\gamma(V)$ is a linear isomorphism $\tilde{\varphi}_V : V \to V_0$ that can be extended to an isomorphism $\varphi_V$ of $\mathbb{C}^n$ by requiring it to be the identity on $L_0$.

A local trivialization for $\eta$ is the homeomorphism

$$f : \eta^{-1}(U_{L_0}) \to U_{L_0} \times \eta^{-1}(V_0)$$

$$(H_1, H_2) \mapsto (\eta(y), (\varphi_{\eta(y)}(H_1), \varphi_{\eta(y)}(H_2)))$$

This completes the proof.

By the homotopy long exact sequence of the map $\eta$, we get:

$$0 \to \pi_2(\mathcal{F}_2^{2i-2k}(i - k, n - 2k + i)) \to \pi_2(\mathcal{F}_2^i(k, n)) \to \mathbb{Z} \to 0$$

and hence $\pi_2(\mathcal{F}_2^i(k, n)) = \mathbb{Z} \times \pi_2(\mathcal{F}_2^{2(i-k)}(i - k, n - 2k + i))$. By the previous case, $\pi_2(\mathcal{F}_2^{2(i-k)}(i - k, n - 2k + i))$ is equal to $\mathbb{Z}$ if $2(i - k) = n - 2k + i$, that is if $i = n$, and is equal to $\mathbb{Z}^2$ otherwise. So, we get $\pi_2(\mathcal{F}_2^n(k, n)) = \mathbb{Z}^2$ and $\pi_2(\mathcal{F}_2^i(k, n)) = \mathbb{Z}^3$ if $i < n$.  

8
References


