**Ab initio** calculations of \( \Delta H_2 \) in type-II superconductors: Basic formalism and model calculations

Takafumi Kita  
Division of Physics, Hokkaido University, Sapporo 060-0810, Japan

Masao Arai  
National Institute for Materials Science, Namiki 1-1, Tsukuba, Ibaraki 305-0044, Japan

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Detailed Fermi-surface structures are essential to describe the upper critical field \( \Delta H_2 \) in type-II superconductors, as first noticed by and Hohenberg and Werthamer [Phys. Rev. 153, 493 (1967)] and shown explicitly by Butler for high-purity cubic Niobium [Phys. Rev. Lett. 44, 1516 (1980)]. We derive an \( \Delta H_2 \) equation for classic type-II superconductors that is applicable to systems with anisotropic Fermi surfaces and/or energy gaps under arbitrary field directions. It can be solved efficiently by using Fermi surfaces from \textit{ab initio} electronic-structure calculations. Thus, it is expected to enhance our quantitative understanding on the properties of type-II superconductors in finite magnetic fields, especially in the clean limit, as first recognized by Hohenberg and Werthamer. Their effort to include the Fermi-surface anisotropy in the \( \Delta H_2 \) equation was motivated by the fact that the Helfand-Werthamer theory\(^6\) using the spherical Fermi surface shows neither qualitative nor quantitative agreements with experiments on clean type-II superconductors, such as Nb\(^{46–48}\) and V\(^{49}\). Indeed, angular variation in \( \Delta H_2 \) by 10% was observed at low temperatures in high-quality Nb\(^{46,50,51}\) and V\(^{50,51}\) with cubic symmetry.\(^{52}\) Also, the reduced critical field

\[
h^{*\,\text{s}}(t) = \frac{\Delta H_2(t)}{d\Delta H_2(t)/dt}_{t=1} \quad (t = T/T_c),
\]

calculated by Helfand and Werthamer,\(^6\) has \( h^{*\,\text{s}}(0) = 0.727 \) in the clean limit, whereas a later experiment on high-purity Nb shows \( \langle h^{*\,\text{s}}(0) \rangle = 1.06 \) for the average overfield directions.\(^{51}\) Hohenberg and Werthamer\(^9\) carried out a perturbation expansion for cubic materials with respect to the nonlocal correction where the Fermi-surface anisotropy enters. They could thereby provide a qualitative understanding of the \( \Delta H_2 \) anisotropies.

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**I. INTRODUCTION**

The upper critical field \( \Delta H_2 \) is one of the most fundamental quantities in type-II superconductors. After the pioneering work by Abrikosov\(^1\) based on the Ginzburg-Landau (GL) equations,\(^2\) theoretical efforts have been made for its quantitative description at all temperatures.\(^3–41\) However, we still have a limited success when compared with those for the electronic structures in the normal state.\(^42\) The purpose of the present paper is to provide a theoretical framework that enables us \textit{ab initio} calculations of \( \Delta H_2 \) as accurate as electronic-structure calculations in the normal state.

Necessary ingredients to be included are (i) nonlocal effects effective at low temperatures, (ii) impurity scattering, (iii) Fermi-surface anisotropy, (iv) strong-coupling effects, (v) gap anisotropy, (vi) mixing of higher Landau levels in the spatial dependence of the pair potential, (vii) Landau-level quantization in the quasiparticle energy,\(^13,32,35–38\) and (viii) fluctuations beyond the mean-field theory.\(^43\) We here derive an \( \Delta H_2 \) equation, which is numerically tractable, including all the effects except (vii) and (viii).

An \( \Delta H_2 \) equation considering the effects (i) and (ii) was obtained by Helfand and Werthamer.\(^5\) It was extended by Hohenberg and Werthamer\(^9\) to take the Fermi-surface anisotropy (iii) into account. Equations with the strong-coupling effects (iv) were derived by Eilenberger and Ambegaokar\(^11\) using Matsubara frequencies and by Werthamer and McMillan\(^12\) on the real energy axis, which are equivalent to one another. Schossmann and Schachinger\(^27\) later incorporated Pauli paramagnetism into the strong-coupling equation. Although an equation including (i)–(iv) was presented by Langmann,\(^33\) it is still rather complicated for carrying out an actual numerical computation. On the other hand, Rieck and Scharnberg\(^40\) presented an efficient \( \Delta H_2 \) equation where the effects (i)–(iii) and (vi) were taken into account, and also (v) in the special case of the clean limit. See also the work by Rieck, Scharnberg, and Schopohl\(^31\) where the strong-coupling effects (v) have also been considered. Our study can be regarded as a direct extension of the Rieck-Scharnberg equation\(^40\) to incorporate (i)–(iv), simultaneously. To this end, we adopt a slightly different and (probably) more convenient procedure of using creation and annihilation operators. We will proceed with clarifying the connections with the Rieck-Scharnberg equation as explicitly as possible.
tropy and the enhancement of $\langle h^*(t) \rangle$ observed in Nb. They also derived an expression for $\langle h^*(0) \rangle$ applicable to anisotropic Fermi surfaces. It was later used by Mattheiss\textsuperscript{14} to estimate $\langle h^*(0) \rangle=0.989$ for Nb based on his detailed electronic-structure calculation. The strong dependence of $h^*(t)$ in the clean limit on detailed Fermi-surface structures can also be seen clearly in the numerical results from a model calculation by Rieck and Scharnberg,\textsuperscript{30} and from the difference $h^*(0)=0.727$ and 0.591 between spherical and cylindrical Fermi surfaces, respectively.\textsuperscript{41}

On the other hand, it was shown by Werthamer and McMillan\textsuperscript{12} that the strong-coupling effects change $h^*(t)$ by only $\leq 2\%$ for the spherical Fermi surface and cannot be the main reason for the enhancement of $h^*(0)$ in Nb.

The most complete calculation, including the effects (i)–(iv), was performed on pure Nb by Butler.\textsuperscript{22,23} He solved the strong-coupling equation by Eilenberger and Ambegaokar,\textsuperscript{11} taking full account of the Fermi-surface structure and the phonon spectra from his electronic-structure calculations. He could thereby obtain an excellent agreement with experiments by Williamson\textsuperscript{50} with $\langle h^*(0) \rangle=0.96$ and by Kerchner \textit{et al.}\textsuperscript{53} However, a later experiment by Sauerzopf \textit{et al.}\textsuperscript{51} on a high-purity Nb shows a larger value $\langle h^*(0) \rangle=1.06$, thereby suggesting that there may be some factors missing in Butler’s calculation.

Theoretical considerations on the effects (v) and (vi) started much later. It was Takanaka,\textsuperscript{18} Teichler,\textsuperscript{19} and Pohl and Teichler\textsuperscript{20} who included gap anisotropy ($\nu$) in the $H_{c2}$ equation. They considered the nonlocal effect perturbatively adopting a separable pair potential. Takanaka studied $H_{c2}$ anisotropy observed in uniaxial crystals, whereas Teichler applied his theory to the $H_{c2}$ anisotropy in cubic Nb. This approach by Teichler was extended by Prohammer and Schachinger\textsuperscript{28} to anisotropic polycrystals and used by Weber \textit{et al.}\textsuperscript{54} to analyze anisotropy effects in Nb.

The mixing of higher Landau levels (vi) was considered by Takanaka and Nagashima\textsuperscript{15} in extending the Hohenberg-Werthamer theory for cubic materials\textsuperscript{9} to higher orders in the nonlocal correction. It was also taken into account by Takanaka\textsuperscript{18} in the above-mentioned work, Youngner and Klemm\textsuperscript{24} in their perturbation expansion with respect to the nonlocal corrections; Schachinger and Klemm\textsuperscript{25} in studying $H_{c2}$ for $p$-wave superconductors, Rieck and Scharnberg\textsuperscript{30} for superconductors with nearly cylindrical model Fermi surfaces, and Prohammer and Carbott\textsuperscript{29} for $d$-wave superconductors. See also a recent work by Miranović, Machida,\textsuperscript{39} and Kogan on MgB$_2$. Although it plays an important role in the presence of gap anisotropy,\textsuperscript{25,28} this mixing was not considered by Teichler\textsuperscript{19} and Pohl and Teichler.\textsuperscript{20}

Now, one may be convinced that calculations including (i)–(vi) are still absent. Especially, many of the theoretical efforts have been focused only on the special case of cubic materials.\textsuperscript{9,15,19,20,22,23} For example, a detailed theory is still absent for the large positive (upward) curvature observed in $H_{c2}(T\lessgtr T_c)$ of layered superconductors,\textsuperscript{55,56} except a qualitative description by Takanaka\textsuperscript{19} and Dalrymple and Prober.\textsuperscript{57} Based on these observations, we here derive an $H_{c2}$ equation that is numerically tractable for arbitrary crystal structures and field directions by using Fermi surfaces from \textit{ab initio} electronic-structure calculations. This kind of calculations has been performed only for Nb by Butler so far.\textsuperscript{22,23} Making such calculations possible for other materials is expected to enhance our quantitative understanding on $H_{c2}$ substantially.

This paper is organized as follows. Section II considers the weak-coupling model with gap anisotropy and $s$-wave impurity scattering. We derive an $H_{c2}$ equation valid at all temperatures as well as an analytic expression for $H_{c2}(T\lessgtr T_c)$ up to second order in $1-T/T_c$. The main analytic results of Sec. II are listed in Table I for an easy reference. Section III extends the $H_{c2}$ equation so as to include $p$-wave impurity scattering, spin-orbit impurity scattering, and strong electron-phonon interactions. Section IV presents numerical examples for model Fermi surfaces of a three-dimensional tight-binding model with cubic symmetry. Section V summarizes the paper. We use $k_B=1$ throughout.

### II. WEAK-COUPLING $H_{c2}$ EQUATION

#### A. Fermi-surface harmonics and gap anisotropy

We first specify the gap anisotropy in our consideration with respect to the Fermi-surface harmonics. The Fermi-surface harmonics were introduced by Allen\textsuperscript{55} as convenient polynomials in solving the Boltzmann and Eliashberg equations. They were later used by Langmann\textsuperscript{33} to derive an $H_{c2}$ equation applicable to anisotropic Fermi surfaces and anisotropic pairing interactions. However, the polynomials constructed by Allen based on the Gram-Schmidt orthonormalization are not very convenient for treating the gap anisotropy. We here adopt an alternative construction starting from the pairing interaction $V(k_F, k'_F)$ on the Fermi surface,\textsuperscript{59} where $k_F$ denotes the Fermi wave vector. Evidently $V(k_F, k'_F)$ is Hermitian $V^*(k_F, k'_F)=V(k'_F, k_F)$, and invariant under every symmetry operation $R$ of the group $G$ for the relevant crystal as $RV(k_F, k'_F)R^{-1}=V(k_F, k'_F)$. We hence consider the following eigenvalue problem:

$$
\int dS_F \rho(k_F) V(k_F, k'_F) \phi^{(f)}(k_F) = V^{(f)} \phi^{(f)}(k_F).
$$

Here $dS_F$ denotes an infinitesimal area on the Fermi surface and $\rho(k_F)=[(2\pi)^3 N(0)|v_F|]^{-1}$ with $v_F$ the Fermi velocity and $N(0)$ the density of states per one spin and per unit volume at $T=0$.

### Table I

| $\langle \cdots \rangle \phi(k_F)$ | $B$ | $\Phi_0$ | $T_c$ | $l_c$ | $\bar{v}_{F+}$ | $c_{12}$ | $\chi_{ij}$ | $\bar{v}_{n_{i2}}$ | $\bar{\beta}$ | $A_{NN'}$ | $K_{NN'}$ | $\eta(x)$ | $H_{c2}(T \lessgtr T_c)$ | $B_1$ | $B_2$ | $R$ | $w_{\nu, \mu}$ |
the Fermi energy in the normal state. The superscript $\Gamma$ denotes an irreducible representation of $G$, $j$ distinguishes different eigenvalues belonging to $\Gamma$, and $\gamma$ specifies an eigenvector in $(\Gamma, j)$. This eigenvalue problem was also considered by Pokrovskii\textsuperscript{60} without specifying the symmetry. The basis functions thereby obtained naturally have all the properties of Fermi-surface harmonics introduced by Allen.\textsuperscript{58} Especially, they satisfy the orthonormality and completeness
\begin{equation}
\langle \phi_{\gamma}^{(1)} \phi_{\gamma'}^{(1') \dagger} \rangle = \delta_{\Gamma \Gamma'} \delta_{jj'} \delta_{\gamma \gamma'}, \left(3a\right)
\end{equation}
\begin{equation}
\sum_{\Gamma j} \phi_{\gamma j}^{*(1)}(k_F) \phi_{\gamma j}^{(1')} = \frac{\delta}{\rho(k_F)} \left( k_F - k_F' \right), \quad \left(3b\right)
\end{equation}
where $\langle \cdot \cdot \cdot \rangle$ denotes the Fermi-surface average
\begin{equation}
\langle A \rangle = \int ds_F \rho(k_F) A(k_F). \quad \left(4\right)
\end{equation}
Using Eqs. (2) and (3), we obtain an alternative expression for the dimensionless pairing interaction $\lambda(k_F, k_F') = -N(0)V(k_F, k_F')$ as
\begin{equation}
\lambda(k_F, k_F') = \sum_{\Gamma j} \phi_{\gamma j}^{*(1)}(k_F) \phi_{\gamma j}^{(1')} \left( k_F' \right). \quad \left(5\right)
\end{equation}
Thus, it is always possible to express a pairing interaction as a sum of separable interactions. Notice that the above procedure is applicable also to multiband superconductors. Indeed, we only have to extend the integration over $k_F$ to all the Fermi surfaces.

The Fermi-surface harmonics can be constructed also from the coupling function $\lambda(k_F, k_F', e_n) = \mu^*(k_F, k_F')$ in the strong-coupling Eliashberg theory,\textsuperscript{61,62} where $e_n = (2n + 1)\pi T$ is the Matsubara energy. Indeed, we only have to specify an appropriate bosonic Matsubara energy $\omega_0 = 2\pi T$ and set $V(k_F, k_F') = -\left[ \lambda(k_F, k_F', \omega_0) - \mu^*(k_F, k_F') / N(0) \right]$ in Eqs. (2) and (3). We thereby obtain an alternative expression for the coupling function as
\begin{equation}
\lambda(k_F, k_F', e_n) = \mu^* \left( k_F, k_F' \right) \sum_{\Gamma j} \left[ \phi_{\gamma j}^{*(1)}(k_F) - \mu^*(k_F, k_F') \right] \phi_{\gamma j}^{(1')} \left( k_F' \right). \quad \left(6\right)
\end{equation}
We expect that this construction does not depend on the choice of $\omega_0$ substantially. It is worth noting that \textit{ab initio} calculations of the coupling function are now possible for phonon-mediated superconductors, as performed recently for MgB\textsubscript{2}.\textsuperscript{63} Hence \textit{ab initio} constructions of the Fermi-surface harmonics by Eq. (2) can be carried out in principle.

From now on we consider the cases where (i) the system has inversion symmetry and (ii) a single $\lambda^{(1)}$ is relevant that belongs to an even-parity one-dimensional representation $\Gamma$. Indeed, these conditions are met for most superconductors. Hereafter we will drop all the indices as $\phi_{\gamma j}^{(1)}(k_F) \rightarrow \phi(k_F)$, for example, and choose $\phi(k_F)$ as a real function.

### B. Eilenberger equations

Now, let us derive an $H_{c2}$ equation for the second-order transition in the weak-coupling model with $s$-wave impurity scattering based on the quasiclassical Eilenberger equations.\textsuperscript{64-66} The Eilenberger equations are derived from the Gor’kov equations by assuming a constant density of states near the Fermi energy in the normal state and integrating out an irrelevant energy variable.\textsuperscript{64-66} Thus, phenomena closely connected with either the energy dependence of the density of states\textsuperscript{26} or the discreteness in the quasiparticle energy levels\textsuperscript{13,32,35-38} are beyond the scope of the present consideration. We also do not consider Josephson vortices appearing in very anisotropic layered superconductors.\textsuperscript{67}

Within the limitations, however, the Eilenberger equations provide one of the most convenient starting points for deriving an $H_{c2}$ equation, as seen below. This approach was also adopted by Rieck and Scharnberg\textsuperscript{68} and Rieck et al.\textsuperscript{31}

We take the external magnetic field $\mathbf{H}$ along the $z$-axis. In the presence of Pauli paramagnetism, the average flux density $B$ in the bulk is connected with $H$ as $H = B - 4\pi \chi_\alpha B$, where $\chi_\alpha$ is the normal-state spin susceptibility. The fact that $\chi_\alpha$ is multiplied by $B$ rather than $H$ corresponds to the fact that the spins respond to the true magnetic field in the bulk. It, hence, follows that $B$ is enhanced over $H$ as
\begin{equation}
B = H/(1 - 4\pi \chi_\alpha). \quad \left(7\right)
\end{equation}
The vector potential in the bulk at $H = H_{c2}$ can be written accordingly as
\begin{equation}
A(r) = (0, B x, 0). \quad \left(8\right)
\end{equation}
The field $\mathbf{H}$ is supposed to be along the direction $(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ in the crystallographic coordinates $(X, Y, Z)$. The two coordinate systems are connected by the rotation matrix
\begin{equation}
R = \begin{bmatrix}
\cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\
\sin \varphi & \cos \varphi & 0 \\
-\sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta
\end{bmatrix}, \quad \left(9\right)
\end{equation}
as $\mathbf{RH} = (0, 0, H)^T$, where $T$ denotes transpose. We assume that the vortex lattice is uniform along $z$.

With the gap anisotropy specified by $\phi(k_F)$ and in the presence of Pauli paramagnetism, the Eilenberger equations read
\begin{equation}
\left( e_n - i\mu_B B + \frac{\hbar}{2\tau} \right) g + \frac{1}{2} \hbar v_F \cdot \partial f = \left( \phi \Delta + \frac{\hbar}{2\tau} \right) g, \quad \left(10a\right)
\end{equation}
\begin{equation}
\Delta(r) \frac{\tau_{c0}}{T} = \pi T \sum_{n \rightarrow \infty} \left[ \frac{\Delta(r)}{|e_n|} - \langle \phi(k_F) f(e_n, k_F, r) \rangle \right]. \quad \left(10b\right)
\end{equation}
Here $\mu_B$ is the Bohr magneton, $\tau$ is the relaxation time by nonmagnetic impurity scattering in the second-Born approximation, $\Delta(r)$ is the pair potential, and $\partial$ is defined by
\begin{equation}
\partial = \nabla - i\frac{2\pi}{\Phi_0} \mathbf{A}, \quad \left(11\right)
\end{equation}
with $\Phi_0 = hc/2e$ the flux quantum. We will consider positively charged particles following the convention; the case of
electrons can be obtained directly by $A ightarrow -A$, i.e., reversing the magnetic-field direction. The quasiclassical Green’s functions $f$ and $g$ are connected by $g = (1 - f^2)^{1/2} \text{sgn}(e_n)$ with $f^0(e_n, \mathbf{k}_F, \mathbf{r}) = f^0 (-e_n, \mathbf{k}_F, \mathbf{r})$, and $T_{c0}$ denotes the transition temperature in the clean limit $\tau = \infty$.

To obtain $B_{c2}$, we formally expand the quasiclassical Green’s functions up to the first order in $\Delta$ as $f = f^{(1)}$ and $g = \text{sgn}(e_n)$. Substituting the expressions into Eqs. (10a) and (10b), we obtain the linearized self-consistency equations as

$$\left[ \frac{\text{sgn}(e_n)}{2} \hbar \mathbf{v}_F \cdot \mathbf{\hat{\partial}} \right] f^{(1)} = \phi \Delta + \frac{\hbar}{2\tau} (f^{(1)}), \quad (12a)$$

$$\Delta \ln \frac{T_{c0}}{T} = -\pi T \sum_{n = -\infty}^{\infty} \left( \langle \phi f^{(1)} \rangle - \frac{\Delta}{|e_n|} \right), \quad (12b)$$

with

$$\bar{e}_n = e_n - i\mu_B B \text{sgn}(e_n), \quad \bar{e}_n = |e_n| + \frac{\hbar}{2\tau}. \quad (13)$$

C. Operators and basis functions

It is useful to transform the gradient operator in Eq. (10a) as

$$\mathbf{v}_F \cdot \mathbf{\hat{\partial}} = (\mathbf{\hat{a}} \mathbf{v}_F a - \mathbf{\hat{a}}^\dagger \mathbf{v}_F a^\dagger) / \sqrt{2} l_c. \quad (14)$$

Here $l_c$ denotes $1/\sqrt{2}$ times the magnetic length as

$$l_c = \sqrt{\Phi_0 / 2 \pi B} = \sqrt{\hbar c / 2eB}. \quad (15)$$

The operators $a$ and $a^\dagger$ are defined by

$$\begin{bmatrix} a \\ a^\dagger \end{bmatrix} = \frac{l_c}{\sqrt{2}} \begin{bmatrix} c_1 \text{i} c_2 \\ -c_1^* \text{i} c_2^* \end{bmatrix} \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix}, \quad (16)$$

where the constants $c_1$ and $c_2$ are constrained by

$$c_1 c_2 + c_1^* c_2^* = 2, \quad (17)$$

so that $[a, a^\dagger] = 1$. Finally, $\mathbf{\tilde{v}}_{F+}$ is defined by

$$\mathbf{\tilde{v}}_{F+} = c_2 \mathbf{v}_{F+} + \text{i} c_1 \mathbf{v}_{F+}. \quad (18)$$

The constants $(c_1, c_2)$ can be fixed conveniently by requiring that the gradient term in the Ginzburg-Landau equation be expressed in terms of $a^\dagger a$ without using $aa$ and $a^\dagger a^\dagger$, i.e., the pair potential near $T_c$ be described in terms of the lowest Landau level only. As shown in Appendix I, this condition yields

$$c_1 = \left( \frac{\chi_{\alpha\alpha}^2}{\chi_{\alpha\alpha} \chi_{\alpha\gamma} - \chi_{\gamma\gamma}^2} \right)^{1/4} \quad (19a)$$

$$c_2 = \left( \frac{\chi_{\gamma\gamma}^2}{\chi_{\alpha\alpha} \chi_{\alpha\gamma} - \chi_{\gamma\gamma}^2} \right)^{1/4} \exp \left[ \text{i} \tan^{-1} \frac{-\chi_{\gamma\gamma}}{\sqrt{\chi_{\alpha\alpha} \chi_{\alpha\gamma} - \chi_{\gamma\gamma}^2}} \right], \quad (19b)$$

where $\chi_{ij} = \chi_{ij}(T_c)$ is defined by

$$\chi_{ij} = \frac{24(\pi T_c)^3}{7\zeta(3)(\nu^2)} \sum_{n = 0}^{\infty} \frac{1}{\nu_n^2} \left[ \langle \phi (\delta \mathbf{v}_F \mathbf{v}_F) + \frac{\langle \phi (\delta \mathbf{v}_F \mathbf{v}_F) \rangle}{2\tau_n} \right] + \frac{\langle \phi (\delta \mathbf{v}_F \mathbf{v}_F) \rangle}{2\tau_n^2}, \quad (20)$$

with $\zeta$ the Riemann zeta function. Note that $X_{ij}$ is dimensionless, approaching to $\delta_i$ as $\tau \rightarrow \infty$ for the spherical Fermi surface. It is a direct generalization of the $\chi$ function introduced by Gor’kov \cite{69} to anisotropic systems.

The operators in Eq. (16) extend $(a_\alpha, a_\alpha^\dagger)$ introduced by Helfand and Werthamer \cite{6} for anisotropic crystals. For uniaxial crystals, they reduce to the operators used by Takanaka \cite{18}.

Using Eq. (16), we can also make up a set of basis functions to describe vortex-lattice structures as

$$\psi_{Nq}(\mathbf{r}) = \sqrt{\frac{2\pi l_c}{c_1 a_2 / \pi V_{m-Nq^2+1}}} \exp \left[ i q_y (y + \frac{\bar{q}_x}{2}) \right] \times \exp \left[ i \frac{\bar{q}_x}{2} - \frac{c_1 c_2 (x - \bar{q}_x)^2}{c_1 l_c} \right] \times \frac{1}{\sqrt{2N!}} \delta H_N \left( x - \frac{\bar{q}_x}{2} - \frac{c_1 c_2}{c_1 l_c} \right), \quad (21)$$

Here $N = 0, 1, 2, \cdots$ denotes the Landau level, $\mathbf{q}$ is an arbitrary chosen magnetic Bloch vector characterizing the broken translational symmetry of the vortex lattice and specifying the core locations, and $V$ is the volume of the system. The quantities $a_1$, $a_2$, and $c_2$ are the components of the basis vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ in the $xy$ plane, respectively, with $\mathbf{a}_1$, $\mathbf{a}_2$, and $c_2$ being the basis functions of the flux quantum in the system, and $H_N(x) = e^{\text{c}^2 -(d/dx)^2} e^{-x^2}$ is the Hermite polynomial. The basis functions are both orthonormal and complete, satisfying $a_\psi_{Nq} = \delta_N \delta_{Nq}$ and $c_1^* \psi_{Nq} = \delta_N \delta_{Nq}$.

The function (21) is a direct generalization of the Eilenberger function \cite{10} $\psi_\ell(\mathbf{r}, T_\ell)$ with $c_1 = c_2 = 1$ to anisotropic Fermi surfaces and energy gaps. For $\mathbf{q} = 0$ in the clean limit, Eq. (21) reduces to the function obtained by Rieck and Scharnberg \cite{30, 71} and Riecket al. \cite{31}. However, they derived it without recourse to the creation and annihilation operators of Eq. (16). These operators have simplified the derivation of the basis functions and will also make the whole calculations below much easier and transparent.

D. Analytic expression of $H_{c2}$ near $T_c$

Using Eq. (16), it is also possible to obtain an analytic expression for $B_{c2} = H_{c2} / (1 - 4\pi \chi_0)$ near $T_c$. Let us express it as

$$B_{c2} = B_1 (1 - t) + B_2 (1 - t)^2, \quad (22)$$

with $t = T / T_c$. The coefficients $B_1$ and $B_2$ determine the initial slope and the curvature, respectively.

It is shown in Appendix I that $B_1$ is obtained as
The factor $h(v_F^2)^{1/2}/T_c$ in the denominator of Eq. (23) is essentially the BCS coherence length. Also, $R$ is dimensionless and approaches unity for $\tau \to \infty$. Equation (23) is a direct generalization of the result by Rieck and Schamberg for $\phi(k_F) = 1$ to the cases with gap anisotropy and for arbitrary strength of the impurity scattering.

It is convenient to express $\langle v_F v_F \rangle$ in Eq. (20) with respect to the crystallographic coordinates $(X, Y, Z)$ to see the anisotropy in $B_1$ manifestly. Using Eq. (9), $v_{Fx}$ and $v_{Fy}$ are rewritten as

$$v_{Fx} = v_{Fy} \cos \theta \varphi + v_{Fz} \sin \theta \sin \varphi - v_{Fz} \sin \theta$$

$$v_{Fy} = -v_{Fx} \sin \varphi + v_{Fz} \cos \varphi$$

(25)

so that

$$\langle v_{Fx}^2 \rangle = \langle v_{Fy}^2 \rangle \cos^2 \varphi + \langle v_{Fz}^2 \rangle \sin^2 \theta + \langle v_{Fz}^2 \rangle \sin^2 \theta$$

$$\langle v_{Fy}^2 \rangle = \langle v_{Fy}^2 \rangle \sin^2 \varphi + \langle v_{Fz}^2 \rangle \cos^2 \varphi$$

$$\langle v_{Fz}^2 \rangle = \langle v_{Fz}^2 \rangle$$

(26)

The quantities $\langle \phi \bar{\phi} v_F v_F \rangle$ and $\langle \bar{\phi}^2 v_F v_F \rangle$ can be expressed similarly in the crystallographic coordinates once $\phi(k_F)$ is given explicitly. In particular, when $\phi(k_F)$ belongs to $A_{1g}$ representation, the expressions for the two averages are essentially the same as Eq. (26). From Eqs. (23), (20), and (26), we realize immediately that the initial slope is isotropic when $\phi(k_F)$ belongs to $A_{1g}$ and the crystal has cubic symmetry.

The expression for $B_2$ is more complicated as given explicitly by Eq. (A9a). It includes Fermi-surface averages of $v_{Fx}^4, v_{Fy}^4, v_{Fz}^4$, etc., and enables us to estimate the initial curvature of $H_{c2}$ given the Fermi-surface structure.

**E. $H_{c2}$ equation**

We now derive an $H_{c2}$ equation that can be solved efficiently at all temperatures. To this end, we transform Eqs. (12a) and (12b) into algebraic equations by expanding $\Delta$ and $f^{(1)}$ in the basis functions of Eq. (21) as

$$\Delta(r) = \sqrt{\sum_{N=0}^{\infty} \Delta_N \psi_{NQ}(r)}$$

(27a)

$$f^{(1)}(e_n, k_F, r) = \sqrt{\sum_{N=0}^{\infty} f^{(1)}_N(e_n, k_F) \psi_{NQ}(r)}$$

(27b)

Let us substitute Eqs. (14) and (27) into Eqs. (12a) and (12b), multiply them by $\psi_{Q}(r)$, and perform integrations over $r$. Equations (12a) and (12b) are thereby transformed into

$$\sum_{N'} M_{NN'} f^{(1)}_{N'} = \phi \Delta_N + \frac{\hbar}{2\tau} \langle f^{(1)}_N \rangle$$

(28a)

$$\Delta_N \ln \frac{T_c}{T} = -\tau T \sum_{n=0}^{\infty} \left( \langle \phi \bar{\phi}^{(1)}_n \rangle - \Delta_N \right) |e_n|$$

(28b)

where the matrix $M$ is tridiagonal as

$$M_{NN'} = \bar{\epsilon}_n \delta_{NN'} + \sqrt{N + 1} \bar{\beta} \delta_{N,N'-1} - \bar{\Delta}_N \delta_{N,N'+1}$$

(29)

with

$$\bar{\beta} = \frac{\hbar v_F \text{sgn}(e_n)}{2\sqrt{2l_c}}$$

(30)

We first focus on Eq. (28a) and introduce the matrix $K$ by

$$K_{NN'} = (M^{-1})_{NN'}$$

(31)

which necessarily has the same symmetry as $M$.

$$K_{NN'}(e_n, \bar{\beta}) = K_{NN'}(e_n, -\bar{\beta}^*)$$

(32)

Using $K$, Eq. (28a) is solved formally as

$$f^{(1)}_N = \sum_{N'} K_{NN'} \left( \phi \Delta_{N'} + \frac{\hbar}{2\tau} \langle f^{(1)}_{N'} \rangle \right)$$

(33)

Taking the Fermi-surface average to obtain $\langle f^{(1)}_N \rangle$ and substituting it back into Eq. (33), we arrive at an expression for the vector $f^{(1)} = (f^{(1)}_1, f^{(1)}_2, \cdots)^T$ as

$$f^{(1)} = \left[ K \phi + \frac{\hbar}{2\tau} (I - \frac{\hbar}{2\tau} K)^{-1} \langle K \phi \rangle \right] \Delta$$

(34)

with $I$ the unit matrix in the Landau-level indices and $\Delta = (\Delta_0, \Delta_1, \Delta_2, \cdots)^T$.

We next substitute Eq. (34) into Eq. (28b). We thereby obtain the condition that Eq. (28b) has a nontrivial solution for $\Delta$ as

$$\det A = 0,$$

(35)

where the matrix $A$ is defined by

$$A = I \ln \frac{T}{T_c} + \pi T \sum_{n=0}^{\infty} \left[ I - \langle K \phi \rangle \right]^{-1} \langle K \phi \rangle$$

(36)

with $I$ the unit matrix in the Landau-level indices. The upper critical field $B_{c2}$ corresponds to the highest field where Eq. (35) is satisfied, with $B$ and $H$ connected by Eq. (7). Put another way, $B_{c2}$ is determined by requiring that the smallest eigenvalue of $A$ be zero. Notice that $A$ is Hermitian, as can
be shown by using Eq. (32), so that it can be diagonalized easily.

Equation (36) tells us that central to determining $B_{c2}$ lies the calculation of $K_{NN'}$ defined by Eqs. (29) and (31). An efficient algorithm for it was already developed in Sec. IIF of Ref. 41, which is summarized as follows. Let us define $R_s(N=0,1,2,\cdots)$ and $\bar{R}_s(N=1,2,\cdots)$ by

$$R_{N-1} = (1 + N_x^2 R_N)^{-1},$$

$$\bar{R}_{N-1} = (1 + N_x^2 \bar{R}_N)^{-1}, \quad \bar{R}_1 = 1,$$

respectively, with

$$\chi = |\bar{\beta}/\bar{E}_n^s|.$$  

Then $K_{NN'}$ for $N \geq N'$ can be obtained by

$$K_{NN'} = \frac{1}{\bar{E}_n^s} \bar{\eta}_N(x) \bar{\eta}_N(x)^{-N-N'},$$

with

$$\bar{\eta}_N = \left\{ \begin{array}{cc} 1 & (N = 0) \\ \frac{1}{\sqrt{N!}} \prod_{k=1}^{N} 1 & (N \geq 1) \end{array} \right.$$  

The expression of $K_{NN'}$ for $N < N'$ follows immediately by Eq. (32).

As shown in Appendix II, Eqs. (40a) and (40b) can be written alternatively as

$$\bar{\eta}_N(x) = \frac{1}{\sqrt{N!}} \left( \frac{\chi}{\sqrt{2}x} \right)^N H_N \left( \frac{i}{\sqrt{2}x} \right),$$

$$= \frac{1}{\sqrt{N!}} \left( \frac{\chi}{\sqrt{2}x} \right)^N H_N \left( \frac{i}{\sqrt{2}x} \right) \left( \frac{d}{dy} \right)^N e^{y^2/2} \bigg|_{y=1/x}.$$  

respectively, where $z = 1 / \sqrt{2}x$ and $i^N \text{erf}(z)$ denotes the repeated integral of the error function.\(^{73}\) The latter function $\bar{\eta}_N(x)$ is an $(N/2)$th-order $[(N-1/2)$th-order] polynomial of $x^2$ for $N=$even (odd).

Thus, the key quantity $K_{NN'}$ is given here in a compact separable form with respect to $N$ and $N'$. This is a plausible feature for performing numerical calculations, which may be considered as one of the main advantages of the present formalism over that of Langmann.\(^{33}\) Our $K_{00}$ in Eq. (39) is more convenient than Eq. (26) of Hohenberg and Werthamer\(^{6b}\) in that $H_{c2}$ near $T_c$ is described in terms of the lowest Landau level for arbitrary crystal structures.

Equations (35) and (36) with Eqs. (13), (15), (18)-(20), (30), and (39) are one of the main results of the paper (see also Table I). They enable us efficient calculations of $H_{c2}$ at all temperatures based on the Fermi surfaces from $ab\text{ initio}$ electronic-structure calculations. They form a direct extension of the Rieck-Scharnberg equation\(^{30}\) to the cases with gap anisotropy and arbitrary strength of the impurity scattering. Indeed, Eq. (41b) is written alternatively as

$$\bar{\eta}_{2N}(x) = \frac{1}{\sqrt{(2N)!}} z^{2N} P_N(2z^2),$$

with $z = 1 / \sqrt{2}x$, where $P_N$ is the polynomial defined below Eq. (6) of Rieck and Scharnberg.\(^{30}\) Substituting this result and the last expression of Eq. (41a) into Eq. (39), it can be checked directly that $\bar{\eta}_N = N'$ for $N' = N$ is equal to $M_{2N'/2N}$ in Eq. (6) of Rieck and Scharnberg.\(^{30}\) Using this fact, one can show that the matrix $A$ in Eq. (36) reduces to the corresponding matrix in Eq. (5) of Rieck and Scharnberg either (i) for the isotropic gap with arbitrary impurity scattering or (ii) in the clean limit with an arbitrary gap structure. Here we have adopted $\chi$ in Eq. (38) as a variable instead of $z$ because $x$ remains finite at finite temperatures.

From Eq. (39) and the symmetry $\bar{\eta} \rightarrow -\bar{\eta}$ for $v_T \rightarrow -v_T$, we realize that $\langle K_{2N,2N'} \rangle$, $\langle K_{2N,2N'} \rangle$, $\langle K_{2N,2N'} \rangle$, and $\langle K_{2N,2N'} \rangle$ all vanish in the present case where the system has inversion symmetry and $\phi(k_F)$ belongs to an even-parity representation. It, hence, follows that we only have to consider $N=$even Landau levels in the calculation of Eq. (36). To obtain a matrix element of Eq. (36), we have to perform a Fermi-surface integral for each $n$ and perform the summation over $n$, which is well within the capacity of modern computers, however. Actual calculations of the smallest eigenvalue may be performed by taking only $N \leq N_{\text{cut}}$ Landau levels into account, and the convergence can be checked by increasing $N_{\text{cut}}$. We can put $N_{\text{cut}}=0$ near $T_c$ due to Eq. (19), and must increase $N_{\text{cut}}$ as the temperature is lowered. However, excellent convergence is expected at all temperatures by choosing $N_{\text{cut}} \leq 20$.

III. EXTENSIONS OF THE $H_{c2}$ EQUATION

We extend the $H_{c2}$ equation of Sec. II in several directions.

A. $p$-wave impurity scattering

We first take $p$-wave impurity scattering into account. In this case, Eq. (10a) is replaced by

$$f = \left( e_n - \frac{i}{2} \xi B + \frac{\hbar}{2\tau} \right) \langle g \rangle + \frac{3\hbar}{2\tau_1} \hat{k}_F \cdot \langle \hat{k}_F g \rangle + \frac{1}{2} \hbar v_F \cdot \hat{\partial} \langle \hat{k}_F g \rangle,$$

where $\langle \hat{k}_F \rangle = (\langle \hat{k}_F \rangle_{\text{c}} (e_n, \hat{k}_F, \mathbf{r}) \rangle$, for example, and $\hat{k}_F = k_F/|k_F|^{1/2}$. Notice that $\hat{k}_F$ is not a unit vector in general.
AB INITIO CALCULATIONS OF $H_{12}$ IN TYPE-II…

Linearizing Eq. (43) with respect to $\Delta$, we obtain

$$ \left( \tilde{\varepsilon}_n' + \frac{\text{sgn}(\varepsilon_n)}{2} \hbar v_F \cdot \theta \right) f^{(1)}(t) = \phi \Delta + \frac{\hbar}{2\tau} f^{(1)} + \frac{3\hbar}{2\tau_t} \vec{k}_{F} \cdot (\vec{k}^{(1)}_{F})' \cdot \vec{k}^{(1)}_{F}, $$

(44)

with $\tilde{\varepsilon}_n'$ defined by Eq. (13).

First of all, we derive expressions for $T_c$ at $H=0$, the coefficients $(c_1, c_2)$ in Eq. (16), and $B_{12}$ near $T_c$ up to the first order in $1 - t$, based on Eq. (44) and following the procedure in Sec. A. It turns out that we only need a change of the definition of $\chi_{ij}$ from Eq. (20) into

$$ \chi_{ij} = \frac{24(\pi T_c)^3}{7\xi(3)\nu_F} \sum_{n \neq 0} \frac{1}{\varepsilon_n} \left[ \left\langle v_F v_j \right| \phi + \frac{\langle \phi \rangle}{2\tau_e \hbar} \right|^2 \right] + \frac{3}{2\tau_s \varepsilon_n \hbar} \left( \mathcal{P}^1 \mathcal{Q}^{-1} \mathcal{P} \right)_{ij}, $$

(45)

where the matrices $\mathcal{P}$ and $\mathcal{Q}$ are defined by

$$ \mathcal{P}_{ij} = \left( \phi + \frac{\langle \phi \rangle}{2\tau_e \hbar} \right) \left( \vec{k}_{F} \cdot \vec{v}_F \right) \gamma_{ij}. $$

(46a)

Then $T_c$, $(c_1, c_2)$, and $B_{12}$ in Eq. (22) are given by the same equations, i.e., Eqs. (A5), (19), and (23), respectively.

Using Eqs. (14) and (27), we next transform Eq. (44) into an algebraic equation. The resulting equation can solved in the same way as Eq. (33) to yield

$$ f^{(1)} = \mathcal{K} \left( \phi \Delta + \frac{\hbar}{2\tau} f^{(1)} + \frac{3\hbar}{2\tau_t} \vec{k}_{F} \cdot (\vec{k}^{(1)}_{F})' \cdot \vec{k}^{(1)}_{F} \right), $$

(47)

where $\mathcal{K}$ is given by Eq. (39). It is convenient to introduce the quantities

$$ p_0 = \sqrt{\frac{\hbar}{2\tau_t}}, \quad p_j = \sqrt{\frac{3\hbar}{2\tau_t}} \vec{k}_{Fj} \quad (j=x,y,z). $$

(48)

Then from Eq. (47), we obtain self-consistent equations for $\langle p_0 f^{(1)} \rangle$ and $\langle p_j f^{(1)} \rangle$ as

$$ \begin{pmatrix} \langle p_0 f^{(1)} \rangle \\ \langle p_j f^{(1)} \rangle \end{pmatrix} = \mathcal{W} \begin{pmatrix} \langle p_0 \mathcal{K} \phi \Delta \rangle \\ \langle p_j \mathcal{K} \phi \Delta \rangle \end{pmatrix}. $$

(49)

where the matrix $\mathcal{W}$ is defined by

$$ \mathcal{W} = \begin{bmatrix} \mathcal{I} - \langle |p_0|^2 \rangle \mathcal{K} & -\langle p_0^* p_j \mathcal{K} \rangle & -\langle p_0^* p_j \mathcal{K} \rangle \\ -\langle p_j^* p_0 \mathcal{K} \rangle & \mathcal{I} - \langle |p_j|^2 \rangle \mathcal{K} & -\langle p_j^* p_j \mathcal{K} \rangle \\ -\langle p_j^* p_0 \mathcal{K} \rangle & -\langle p_j^* p_j \mathcal{K} \rangle & \mathcal{I} - \langle |p_j|^2 \rangle \mathcal{K} \end{bmatrix}^{-1}. $$

(50)

The complex conjugations $^*$ in Eqs. (49) and (50) are not necessary here but for a later convenience. Not the symmetry $\mathcal{W}_{\nu\nu}(\varepsilon_n, \vec{p}) = \mathcal{W}_{\nu\nu}(-\varepsilon_n, \vec{p})$ in the matrix elements of $\mathcal{W}$, as seen from Eq. (32). Using Eq. (49) in Eq. (47), we obtain an explicit expression for $f^{(1)}$ as

$$ f^{(1)} = \mathcal{K} \phi \Delta + [p_0 \mathcal{K} p_0^* \mathcal{K} p_j] \mathcal{W} \begin{pmatrix} \langle p_0 \mathcal{K} \phi \rangle \\ \langle p_j^* \mathcal{K} \phi \rangle \\ \langle p_j \mathcal{K} \phi \rangle \end{pmatrix}. $$

(51)

Finally, let us substitute Eq. (51) into Eq. (28b). We thereby find that Eq. (36) is replaced by

$$ A = \frac{T}{T_c 0} + \pi T \sum_{n=\infty}^{\infty} \frac{T}{|\varepsilon_n|} \langle \mathcal{K} \phi \rangle^2 - \left[ \langle p_0 \mathcal{K} \phi \rangle \langle p_0 \mathcal{K} \phi \rangle \right] \mathcal{W} \begin{pmatrix} \langle p_0 \mathcal{K} \phi \rangle \\ \langle p_j^* \mathcal{K} \phi \rangle \\ \langle p_j \mathcal{K} \phi \rangle \end{pmatrix}. $$

(52)

As before, $H_{12}$ is determined by requiring that the smallest eigenvalue of Eq. (52) be zero. This $A$ is Hermitian, as can be shown by using Eq. (32) and $\mathcal{W}^{*\nu\nu}(\varepsilon_n, \vec{p}) = \mathcal{W}_{\nu\nu}(\varepsilon_n, \vec{p})$. Thus, Eq. (52) can be diagonalized easily.
It is straightforward to extend Eq. (52) to a more general impurity scattering with the $k_F$-dependent relaxation time $\tau(k_F, k_F')$. To this end, we apply the procedure of Eqs. (2)-(5) to $1/\tau(k_F, k_F')$ to expand it as

$$
\frac{1}{\tau(k_F, k_F')} = \sum_{I \gamma} \eta_I^{(F)}(k_F) \eta_{I \gamma}^{(F)*}(k_F),
$$

(53)

where $1/\tau(I \gamma)$ and $\eta_I^{(F)}(k_F)$ denote an eigenvalue and its eigenfunction, respectively. We then realize that

$$
p_I^{(F)} = \sqrt{\frac{\hbar}{2m_F}} \eta_I^{(F)}(k_F)
$$

(54)

substitutes for $p_0$ and $p_j$ in Eq. (52).

B. Spin-orbit impurity scattering

It was noticed by Werthamer et al.\textsuperscript{7} and Maki\textsuperscript{8} that, for high-field superconducting alloys with short mean-free paths, Pauli paramagnetism has to be incorporated simultaneously with spin-orbit impurity scattering. They presented a theory valid for $\tau \ll \tau_s$, where $\tau_s$ is spin-orbit scattering time. It was later generalized by Rieck et al.\textsuperscript{31} for an arbitrary value of $\tau_s$. This effect can also be taken into account easily in the formulation.

In the presence of spin-orbit impurity scattering, Eq. (10a) is replaced by

$$
\left( \epsilon_n - i\mu_B B + \frac{\hbar}{2\tau_s} \eta_{I \gamma}(k_F) \eta_{I \gamma}(k_F) \right) \chi_{I \gamma} = \langle fD_k^{(s)} \rangle, \langle \eta_{I \gamma}(k_F) \eta_{I \gamma}(k_F) \rangle, \frac{1}{2} \eta_{I \gamma}(k_F) \eta_{I \gamma}(k_F),
$$

(55)

with $c_{so} = 1/\langle \langle (k_F \times \hat{k}_F)^{(s)} \rangle \rangle$. To simplify the notations and make the argument transparent, it is useful to introduce the quantities

$$
p_0 = \sqrt{\frac{\hbar}{2\tau_s}}, \quad p_{12} = \sqrt{\frac{\hbar c_{so}}{2\tau_s} (\hat{k}_F \times \hat{k}_F)},
$$

(56a)

$$
q_0 = \sqrt{\frac{\hbar}{2\tau_s}}, \quad q_{12} = \sqrt{\frac{\hbar c_{so}}{2\tau_s} (\hat{k}_F \times \hat{k}_F)},
$$

(56b)

and the vectors

$$
p = (p_0, p_{12}, p_{12}, p_{12}, p_{12}, p_{12}, p_{12})^T,
$$

(57a)

$$
q = (q_0, q_{12}, q_{12}, q_{12}, q_{12}, q_{12}, q_{12})^T.
$$

(57b)

Then Eq. (55) linearized with respect to $\Delta$ is written in terms of Eq. (57) as

$$
\left( \epsilon_n + \frac{\text{sgn}(\epsilon_n)}{2} \hbar \bf{v}_F \cdot \bf{\vartheta} \right) f^{(1)} = \Delta + p \cdot (q f^{(1)}),
$$

(58)

where $\bar{\epsilon}_n$ is defined by

$$
\bar{\epsilon}_n = \epsilon_n - i\mu_B B \text{sgn}(\epsilon_n), \quad \bar{\epsilon}_n = |\epsilon_n| + p \cdot (q).
$$

(59)

Note $p \cdot (q) = (p) \cdot (q)$.

It follows from the procedure in Sec. III A that $T_c$ at $H=0$ satisfies

$$
\ln \frac{T_c}{T_c} = 2\pi T_c \sum_{n=0}^{\infty} \left[ \frac{1}{\tilde{\epsilon}_n} - \frac{\langle \phi^2 \rangle}{\tilde{\epsilon}_n^2} - \frac{\langle p^T \phi \rangle}{\tilde{\epsilon}_n} Q^{-1} \left( \frac{\langle \phi \phi \rangle}{\tilde{\epsilon}_n} \right) \right],
$$

(60)

where the matrix $Q$ is defined by $(r,s=0,xx, \cdots, zz)$

$$
Q_{rs} = \delta_{rs} - \frac{\langle q_r p_s \rangle}{\tilde{\epsilon}_n}.
$$

(61)

Also, $\chi_{ij}$ in Eq. (20) should be modified into

$$
\chi_{ij} = \frac{24(\pi T_c)^3}{\tilde{\epsilon}_n^2} \sum_{n=0}^{\infty} \left[ \langle \bf{v}_{F,r} \bf{v}_{F,s} \rangle \langle \phi + \langle p^T \phi \rangle Q^{-1} q \rangle \right.
$$

$$
\left. \times \left[ \phi + \langle p^T Q^{-1} q \phi \rangle \right] \right],
$$

(62)

Finally, $R$ in Eq. (24) is replaced by

$$
R = 1 - 2\pi T_c \sum_{n=0}^{\infty} \left[ \langle \phi^2 \rangle + \langle p \cdot (q) p^T \phi \rangle Q^{-1} \frac{\langle \phi \phi \rangle}{\tilde{\epsilon}_n} \right],
$$

(63)

With the above modifications, $T_c$, $(c_1, c_2)$, and $B_1$ in Eq. (22) are given by Eqs. (A5), (19), and (23), respectively.

We now transform Eq. (58) into an algebraic equation by using Eqs. (14) and (27). The resulting equation can be solved in the same way as Eq. (33). We thereby obtain

$$
f^{(1)} = \mathcal{K} \phi \Delta + \sum_r p_r \mathcal{K} (q_r f^{(1)}),
$$

(64)

where $\mathcal{K}$ is given by Eq. (39) with $\bar{\epsilon}_n$ replaced by Eq. (59). From Eq. (64), we obtain self-consistent equations for $\langle q_0 f^{(1)} \rangle$ and $\langle q_{12} f^{(1)} \rangle$ as

$$
\begin{bmatrix}
\langle q_0 f^{(1)} \rangle \\
\langle q_{12} f^{(1)} \rangle \\
\langle q_{12} f^{(1)} \rangle \\
\langle q_{12} f^{(1)} \rangle \\
\langle q_{12} f^{(1)} \rangle \\
\langle q_{12} f^{(1)} \rangle \\
\langle q_{12} f^{(1)} \rangle \\
\end{bmatrix} = \mathcal{W} \begin{bmatrix}
\langle q_0 \phi \rangle \\
\langle q_{12} \phi \rangle \\
\langle q_{12} \phi \rangle \\
\langle q_{12} \phi \rangle \\
\langle q_{12} \phi \rangle \\
\langle q_{12} \phi \rangle \\
\langle q_{12} \phi \rangle \\
\end{bmatrix},
$$

(65)

where the matrix $\mathcal{W}$ is defined by

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Using Eq. (65) in Eq. (64), we obtain an explicit expression for \( f^{(1)} \) as

\[
f^{(1)} = K \phi \Delta + \left[ p_0 K p_{xx} K p_{yy} \cdots \right] \mathcal{W}
\]

(67)

with \( \mathcal{W} \) defined by \( [\mathcal{W}(\epsilon_n, \tilde{\mu})]_{nm} = \mathcal{W}_{mn}^*(-\epsilon_n, \tilde{\mu}) \). The latter expression originates from the self-consistency equations for \( \langle p_0 f^{(1)} \rangle \) and \( \langle p_1 f^{(1)} \rangle \) similar to Eq. (65). Finally, let us substitute Eq. (67) into Eq. (28b). We thereby find that Eq. (36) is replaced by

\[
\mathcal{A} = \mathcal{I} \ln \frac{T}{T_c} + \pi T \sum_{n=-\infty}^{\infty} \left[ \frac{\mathcal{I}}{\epsilon_n} - \langle K \phi \rangle^2 \right] - \left[ \langle p_0 K \phi \rangle \langle p_{xx} K \phi \rangle \langle p_{yy} K \phi \rangle \cdots \right] \mathcal{W}
\]

(68)

As before, \( H_{c2} \) is determined by requiring that the smallest eigenvalue of Eq. (68) be zero. This \( \mathcal{A} \) is Hermitian, as can be shown by using Eq. (32) and \( [\mathcal{W}_n^*(\epsilon_m, \tilde{\mu})]_{nm} = \mathcal{W}_{mn} \).

\[
\text{C. Strong electron-phonon interactions}
\]

We finally consider the effects of strong electron-phonon interactions within the framework of the Eliashberg theory.\(^{61,62}\) We adopt the notations used by Allen and Mitrovic,\(^{62}\) except for the replacement \( Z \Delta \rightarrow \Delta \).

The Eilenberger equations were extended by Teichler\(^ {74}\) to include the strong-coupling effects. They can also be derived directly from the equations given by Allen and B. Mitrovic\(^ {62}\) by carrying out the “\( \xi \) integration”\(^ {66}\) as

\[
\left( Z_{\epsilon_n} - i \mu_B B \frac{\hbar}{2T} \right) \mathcal{W}_n = \mathcal{W}_m^*(-\epsilon_n, \tilde{\mu})
\]

(69a)

\[
\Delta(\epsilon_n, \cdot \cdot) = \pi T \sum_{n'=-n_c}^{n_c} [\lambda(\epsilon_n - \epsilon_{n'}) - \mu^*] \langle \phi(k_F) f(\epsilon_{n'}, k_F, \cdot \cdot) \rangle,
\]

(69b)

\[
Z(\epsilon_n, k_F) = 1 + \frac{\pi T}{\epsilon_n} \sum_{n'=n_c}^{n_c} \langle \lambda(k_F, k_F', \epsilon_n - \epsilon_{n'}) g(\epsilon_{n'}, k_F', \cdot \cdot) \rangle',
\]

(69c)

where \( n_c \) corresponds to the Matsubara frequency about five times as large as the Debye frequency.\(^ {62,63}\) We have retained full \( k_F \) dependence of \( \lambda \) in Eq. (69c) because the contribution from other pairing channels, which may be negligible for the pair potential, can be substantial for the renormalization factor \( Z \).

We linearize Eqs. (69) with respect to \( \Delta \) and repeat the procedure in Sec. A up to the zeroth order in \( 1 - \tau \). It then follows that \( T_c \) at \( H=0 \) is determined by the condition that the smallest eigenvalue of the following matrix be zero:

\[
\mathcal{A}^{(0)}_{nm} = \delta_{nm} - \pi T [\lambda(\epsilon_n - \epsilon_{n'}) - \mu^*] \left[ \frac{\phi^2}{\tilde{\epsilon}_{n'}} \right] + \frac{\hbar}{2T} \left[ \frac{\phi}{\tilde{\epsilon}_{n'}} \right]^2 \left[ \frac{\tilde{\epsilon}_{n'}}{Z_{\epsilon_n}^0 |\tilde{\epsilon}_{n'}|} \right],
\]

(70)

where \( Z^{(0)} \) is given by

\[
Z^{(0)}(\epsilon_n, k_F) = 1 + \frac{\pi T}{\epsilon_n} \sum_{n'=n_c}^{n_c} [\lambda(k_F, k_F', \epsilon_n - \epsilon_{n'}) g(\epsilon_{n'}, k_F', \cdot \cdot)'] \text{sgn}(\epsilon_{n'}),
\]

(71)

and \( \tilde{\epsilon}_{n} \) is defined together with \( \tilde{\epsilon}_{n'} \) by

\[
\tilde{\epsilon}_{n} = Z^{(0)}(\epsilon_n|\tilde{\epsilon}_{n}| + \frac{\hbar}{2T}, \tilde{\epsilon}_{n'} = \tilde{\epsilon}_{n} - i\mu_B B \text{sgn}(\epsilon_n).
\]

(72)

We next fix \((c_1, c_2)\) in Eq. (16) conveniently. For the weak-coupling model, we have fixed it by using Eq. (A6) near \( T_c \).
so that the coefficient of $aa$ vanishes, i.e., there is no mixing of higher Landau levels in the $H_{c2}$ equation near $T_c$. However, the coefficient of $aa$ in the corresponding strong-coupling equation becomes frequency dependent. It, hence, follows that even near $T_c$ there is no choice for $(\epsilon_1, \epsilon_2)$, which prevents mixing of higher Landau levels from the $H_{c2}$ equation. We here adopt the weak-coupling expression in Eq. (19).

We now consider the $H_{c2}$ equation and repeat the same calculations as those in Sec. II B. We thereby find that Eq. (36) is replaced by

$$
A_{nNn'N'} = \delta_{nn'}\delta_{NN'} - \pi T[\lambda(\epsilon_n - \epsilon_{n'}) - \mu^*] \left\langle K' \phi^2 \right\rangle 
+ \frac{\hbar}{2\tau} \left\langle K' \phi \right\rangle (I - \frac{\hbar(K')}{2\tau})^{-1} \left\langle K' \phi \right\rangle \right|_{NN'},
$$

(73)

where $K' = K(\epsilon_n, \beta)$ which also has $k_F$ dependence through $Z(0) = Z(0)(\epsilon_n, \beta)$. As before, $H_{c2}$ is determined by requiring that the smallest eigenvalue of Eq. (73) be zero.

We may alternatively use, instead of Eq. (73), the matrix

$$
A'_{nNn'N'} = (\lambda - \mu^*)^{-1} \delta_{nn'} - \delta_{nn'} \pi T \left\langle K' \phi^2 \right\rangle 
+ \frac{\hbar}{2\tau} \left\langle K' \phi \right\rangle (I - \frac{\hbar(K')}{2\tau})^{-1} \left\langle K' \phi \right\rangle \right|_{NN'},
$$

(74)

where $(\lambda - \mu^*)^{-1}$ denotes inverse matrix of $\lambda - \mu^*$. It is Hermitian for $\mu_B \beta = 0$, and also acquire the property by combining $n > 0$ and $n < 0$ elements.

IV. MODEL CALCULATIONS

We now present results of a model calculation based on the formalism developed above. We restrict ourselves to the weak-coupling model of Sec. II with an isotropic gap, no impurities, and no Pauli paramagnetism. As for the energy-band structure, we adopt a tight-binding model in the simple cubic lattice whose dispersion is given by

$$
\epsilon_k = -2t(\cos \beta + \cos \alpha + \cos \gamma).
$$

(75)

Here $a$ denotes lattice spacing of the cubic unit cell and $t$ is the nearest-neighbor transfer integral. We set $t = a = 1$ in the following. The corresponding Fermi surfaces are plotted in Fig. 1 for various values of the Fermi energy $\epsilon_F$. For $t = a = 6$, i.e., near the bottom of the band, the Fermi surface is almost spherical with slight distortion due to the cubic symmetry. As $\epsilon_k$ increases, the cubic distortion is gradually enhanced. Then at $\epsilon_k \approx -2$, the Fermi surface touches the Brillouin-zone boundary at $k_b = (0,0, \pm \pi), (0, \pm \pi, 0), (\pm \pi, 0, 0)$. Above this critical Fermi energy, the topology of the Fermi surface changes, as shown in Fig. 1(c). It is interesting to see how such a topological change of the Fermi surface affects $H_{c2}$.

We computed $H_{c2}$ based on Eq. (35) in the clean limit without Pauli paramagnetism. The Fermi-surface average in Eq. (36) was performed by two different methods. For general values of $\epsilon_F$, we used the linear tetrahedron method, which is applicable to any structure of the Fermi surface. In this method, the irreducible Brillouin zone is divided into a collection of small tetrahedra. From each tetrahedron that intersects the Fermi surface, a segment of the Fermi surface is obtained as a polygon by a linear interpolation of the energy band. Numerical integrations over the Fermi surface were then performed as a sum over those polygons. Another description of the Fermi surface is possible for $\epsilon_F \approx -2$, where we can adopt the polar coordinate $k = (k \sin \theta \cos \phi, k \sin \theta \sin \phi, k \cos \theta)$, and the Fermi surface $k_f = k_f(\theta, \phi)$ is obtained by solving the equation $\epsilon_k = \epsilon_F$ numerically for each $(\theta, \phi)$. An integration over the Fermi surface is then performed by using the variables $(\theta, \phi)$. We performed both types of calculations to check the numerical convergence of the tetrahedron method. Excellent agreements were achieved generally by using 3375 tetrahedrons. An exception is the region $\epsilon_F \approx -2$, where a larger number of tetrahedrons was necessary due to the singularity around $k_b$.

The infinite matrix $A_{nN'}$ in Eq. (36) was approximated by a finite matrix of $N_s N'' \approx N_{cut}$, and the convergence was checked by increasing $N_{cut}$. The choice $N_{cut} = 0$ is sufficient for $T \approx T_c$, and it was found numerically that $N_{cut} = 8$ yields enough convergence for all field directions at the lowest temperature. It was also found that higher Landau levels of $N > 1$ contribute to $H_{c2}$ by only 4% even at $T/T_c = 0.05$. Thus, the lowest-Landau-level approximation to the pair potential is excellent for this cubic lattice. This is not generally the case, however, and the contribution of higher Landau levels can be considerable for low-symmetry crystals, as will be reported elsewhere.75

Before presenting any detailed results, it is worth noting that the GL equations,12 where the anisotropy enters only through the effective-mass tensor, cannot explain possible anisotropy of $H_{c2}$ in cubic symmetry, as already pointed out by Hohenberg and Werthamer.9 This GL theory is valid near $T_c$ so that the upper critical field for $T \approx T_c$ should be isoto-
are defined in Eq.
below. The field directions are \( d = [111], [110], \) and [100] from top
to bottom in each case.

describe the difference in \( h^* \) for \( t \leq 1 \) among field
directions. The upward curvature is more and more pronounced as
the Fermi surface approaches the Brillouin zone boundary, as
can be seen clearly in Fig. 2 for \( e_F = -2.02 \). The correspon-
ding ratio \( B_2 / B_1 \) for the [110] and [111] directions are about
three times larger than those for \( e_F = -3 \). Thus, the present
calculation clearly indicates that the Fermi surface aniso-
tropy can be a main source of the upward curvature in \( H_c \)
near \( T_c \).

In Fig. 3, we plot \( h^*_d \) at \( t = 0.05 \) as a function of \( e_F \). As
\( e_F \rightarrow -6 \), the angle dependence of \( h^*_d \) vanishes and it
converges to the value for the spherical Fermi surface. As \( e_F \) is
increased from \(-6 \), cubic distortion is gradually introduced to
the Fermi surface as shown in Fig. 1, and \( h^*_d \) gradually
devolves anisotropy as a consequence. For \(-6 < e_F < -2.5 \),
curves of \( h^*_{[100]} \) fall below that for the spherical Fermi
surface, whereas \( h^*_{[110]} \) and \( h^*_{[111]} \) are enhanced over it.
As \( e_F \) approaches to \(-2 \), \( h^*_{d} \) is enhanced significantly irre-
spective of the field direction. Indeed, \( h^*_{d} \) for every field
direction shows a singularity at \( e_F = -2 \) where the Fermi sur-
f ace touches the Brillouin zone at \( k_X \) with vanishing Fermi
velocity \( v_F \) at these points. As a result, the contribution
around these points becomes important in the integration
\( \langle K_{NN} \rangle \) over the Fermi surface at low temperatures. This is
the origin of the enhancement of \( h^*_d \) around \( e_F = -2 \). For
\( e_F > -2 \), the difference between \( h^*_{[110]} \) and \( h^*_{[111]} \) is larger
than that for \( e_F \leq -2.5 \). This may be attributed to the topological
difference of the Fermi surface. At \( e_F = 0 \), the tight-binding
band is half filled and the Fermi-surface nesting occurs.
However, \( h^*_{d} \) does not show any singularity around this
energy.

Finally, we present results on the higher Landau-level
contributions to the pair potential \( \Delta(r) \), which is expanded as
Eq. (27a). In general, when the system has \( n \)-fold symmetry
around the field direction, mixing of higher Landau levels
with multiples of \( n \) develops as the temperature is lowered.\(^70\)
Figure 4 shows the ratio \( \Delta_{N}/\Delta_0 \) as a function of \( T/T_c \)
for \( e_F = -3 \) (solid lines) and \( e_F = -2.02 \) (dotted lines) with (a)
\( H_{[100]} \) (\( N = 4, 8 \) from bottom to top lines), (b) \( H_{[110]} \)
(\( N = 2, 4, 6 \) from bottom to top lines), and (c) \( H_{[111]} \)
(\( N = 6 \)). One can clearly observe a general tendency that the
mixing is more pronounced as the symmetry around \( H \) be-
comes lower as well as \( e_F \) approaches closer to \(-2 \). Espe-

---

**Table II.** The ratio \( B_2 / B_1 \) for the field directions [100], [110],
and [111] in the cases \( e_F = -3 \) and \(-2.02 \). The quantities \( B_1 \) and \( B_2 \)
are defined in Eq. (22). The values should be compared with 0.13
for the spherical Fermi surface.

<table>
<thead>
<tr>
<th>( e_F )</th>
<th>[100]</th>
<th>[110]</th>
<th>[111]</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0.08</td>
<td>0.27</td>
<td>0.33</td>
</tr>
<tr>
<td>-2.02</td>
<td>0.44</td>
<td>0.78</td>
<td>0.90</td>
</tr>
</tbody>
</table>
impurity scattering, and strong electron-phonon interactions.

Finally, we have presented numerical examples in Sec. IV performed for model Fermi surfaces from the three-dimensional tight-binding model. The results clearly demonstrate crucial importance of including detailed Fermi-surface structures in the calculation of \( H_{c2} \). It has been found that, as the Fermi surface approaches the Brillouin zone boundary, the reduced critical field \( h^\#(t) \) in Eq. (1) is much enhanced over the value for the isotropic model with a significant upward curvature near \( T_c \).

It is very interesting to see to what degree the upper critical field of classic type-II superconductors can be described quantitatively by calculations using realistic Fermi surfaces. The result by Butler^{22,23} on high-purity Niobium provides promise to this issue. We have performed detailed evaluations of \( H_{c2} \) for various materials based on Eq. (35) by using Fermi surfaces from density-functional electronic-structure calculations as an input. The results are reported elsewhere.\(^{75}\)

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**APPENDIX A: DETERMINATION OF \( (c_1, c_2) \) AND ANALYTIC EXPRESSION OF \( H_{c2} \) NEAR \( T_c \)**

We here fix the constants \((c_1, c_2)\) in Eqs. (16)–(18) conveniently so that \( H_{c2} \) near \( T_c \) can be described in terms of the lowest Landau level only. We also derive analytic expressions for \( B_1 \) and \( B_2 \) in Eq. (22) so that one can calculate them once the relevant Fermi-surface structure is given.

In the region \( T \leq T_c \) where \( l_c \to \infty \) in Eq. (14), one can perform a perturbation expansion with respect to the gradient operator \( \mathbf{v}_F \cdot \mathbf{\nabla} \). The equation for the \( \nu \)-th order solution \( f^{(\nu)}_v \) (\( \nu = 0,1, \cdots \)) is obtained from Eq. (12a) as

\[
\begin{align*}
 f^{(\nu)}_v &= \delta_{\nu 0} \frac{\phi \Delta}{\varepsilon_n} + \frac{\hbar \langle f^{(1)}_v \rangle}{2 \varepsilon_n} - \frac{\text{sgn}(\varepsilon_n)}{2 \varepsilon_n} \hbar \mathbf{v}_F \cdot \mathbf{\nabla} f^{(1)}_{v-1}, \\
 & \text{with } f^{(1)}_v = 0. \quad \text{Noting } \phi(-k_F) = \phi(k_F), \text{ we solve Eq. (A1) self-consistently for } f^{(1)}_v, \text{ put the resulting expression back into Eq. (A1) to express } f^{(1)}_v \text{ explicitly, and finally take the Fermi-surface average } \langle \phi f^{(1)}_v \rangle. \quad \text{(A1)}
\end{align*}
\]

This procedure yields

\[
\begin{align*}
 \langle \phi f^{(1)}_0 \rangle &= \frac{1}{\varepsilon_n} \langle \phi^2 \rangle + \frac{\hbar \langle \phi^2 \rangle}{2 \varepsilon_n} \Delta, \\
 \langle \phi f^{(1)}_2 \rangle &= \frac{1}{4 \varepsilon_n^3} \left( \langle \phi^2 \rangle + \frac{\hbar \langle \phi \rangle}{2 \varepsilon_n^2} \right) \left( \hbar \mathbf{v}_F \cdot \mathbf{\nabla} \right) \Delta. \quad \text{(A2a)}
\end{align*}
\]
\[\langle \phi f_4^{(1)} \rangle = \frac{1}{16\hbar^2} \left[ \left( \phi + \frac{\hbar \langle \phi \rangle}{2\tau e_n} \right)^2 \langle \hbar v_F \cdot \nabla \phi \rangle^2 \right] \Delta, \]

with \( |e_n| = |e_n| - i\mu_B B \text{sgn}(e_n) \), and \( \langle \phi f_4^{(1)} \rangle = \langle \phi f_3^{(1)} \rangle = 0 \).

Let us substitute Eq. (A2) into Eq. (12b), replace the gradient operator by the right-hand side of Eq. (14), put \( B = B_{c2} \) in \( l \), of Eq. (15), and expand \( |e_n|' \) with respect to \( \mu_B B_{c2}/|e_n| \). We thereby obtain the self-consistency equation near \( T_c \) as

\[ w_{0,0} \Delta + \frac{B_{c2}}{B_1} [w_{2,0} a^4 + w_{2,2} a^2 a^2 + w_{4,0} (a a^2 + a^4)] \Delta \]

\[ + \left( \frac{B_{c2}}{B_1} \right)^2 [w_{4,2,0} a^4 + w_{4,4,0} a^4 + w_{6,0,0} a^4 + w_{6,2,0} a^4 + w_{6,4,0} a^4 + w_{8,0,0} a^4 + w_{8,2,0} a^4 + w_{8,4,0} a^4] \Delta = 0. \]  

(A3)

Here \( B_1 \) is given in Eq. (22), which is incorporated into the denominator for convenience. The functions \( w_{r,s} \) and \( w_r \) are dimensionless and defined by

\[ w_{0,0}(T) = \ln \frac{T_c}{T} - (1 - \langle \phi \rangle^2) 2 \pi T \sum_{n=0}^{\infty} \left( \frac{1}{e_n} - \frac{1}{E_n} \right), \]

(A4a)

\[ w_{2,2}(T) = \frac{B_1 \hbar^2 \pi T}{2 \Phi_0} \sum_{n=0}^{\infty} \frac{1}{e_n} \left( \phi + \frac{\hbar \langle \phi \rangle}{2\tau e_n} \right)^2 \left[ \langle \hbar v_F \rangle^2 \right], \]

(A4b)

\[ w_{2,0}(T) = \frac{B_1 \hbar^2 \pi T}{2 \Phi_0} \sum_{n=0}^{\infty} \frac{1}{e_n} \left( \phi + \frac{\hbar \langle \phi \rangle}{2\tau e_n} \right)^2 \left[ \langle \hbar v_F \rangle^2 \right], \]

(A4c)

\[ w_{4,4}(T) = \frac{B_1^2 \hbar^4 \pi T}{8 \Phi_0} \sum_{n=0}^{\infty} \frac{1}{e_n} \left( \phi + \frac{\hbar \langle \phi \rangle}{2\tau e_n} \right)^2 \left[ \langle \hbar v_F \rangle^2 \right], \]

(A4d)

\[ w_{4,2}(T) = \frac{B_1^2 \hbar^4 \pi T}{8 \Phi_0} \sum_{n=0}^{\infty} \frac{1}{e_n} \left( \phi + \frac{\hbar \langle \phi \rangle}{2\tau e_n} \right)^2 \left[ \langle \hbar v_F \rangle^2 \right], \]

(A4e)

We next substitute Eq. (22) into Eq. (A3) and expand \( w_{r,s} \) in Eq. (A3) up to the \((4-1)/2\)th order in \( t-r \). We also put \( w_p = w_p(T_c) \). This procedure yields three equations corresponding to order 1, 1-\( t \), and \((1-t)^2 \). The equation of order 1 is given by \( w_{0,0}(T_c) \Delta = 0 \). It determines \( T_c \) at \( H = 0 \) by

\[ \ln \frac{T_c}{T} = (1 - \langle \phi \rangle^2) \left[ \phi \left( \frac{1}{2} + \frac{\hbar}{4\pi T} \right) - \phi \left( \frac{1}{2} \right) \right], \]

(A5)

with \( \phi(x) \) the digamma function.

The equation of order \((1-t)^2 \) in Eq. (A3) is obtained as

\[ [-T_c w_{0,0}(T_c) - w_{2,0}(T_c) (2a^2 + 1) + w_{2,2}(T_c) a^2 + w_{2,2}(T_c) a^2 = 0], \]

(A6)

To solve it, we use the arbitrariness in \( (c_1, c_2) \) and impose \( w_{2,2}(T_c) = 0 \). Noting Eqs. (A4b) and (18), this condition is transformed into a dimensionless form as

\[ \chi_{c_2} c_2^2 + 2 \chi_{c_1} c_1 c_2 - \chi_{c_1} c_1^2 = 0, \]

(A7)

where \( \chi_{c_i} = \chi_{c_i}(T_c) \) is defined by Eq. (20). Equation (A7) can be solved easily in terms of \( c_2 \). Substituting the resulting expression into Eq. (17) and choosing \( c_1 \) real, we obtain Eq. (19).

Now that \( w_{2,2}(T_c) = 0 \) in Eq. (A6), the highest field for a nontrivial solution corresponds to the lowest Landau level where \( w_{2,0}(T_c) = -w_{0,0}'(T_c) \). Introducing \( R = -T_c w_{0,0}'(T_c) \), which is given explicitly as Eq. (24), and using Eqs. (A4a), (A4c), (18), and (19), we obtain the expression for \( B_1 \) as Eq. (23).

Finally, we consider the equation of order \((1-t)^2 \) in Eq. (A3) and expand the pair potential as

\[ \Delta(r) = \Delta_0 \psi_{0,q}(r) + (1-t) [r_2 \psi_{2,q}(r) + r_4 \psi_{4,q}(r)], \]

(A8)

where \( \psi_{0,q}(r) \) is defined by Eq. (21), and \( \Delta_0, r_2, r_4 \) are the expansion coefficients with \( r_2, r_4 \) describing relative mixing of higher Landau levels in the pair potential. Let us substitute Eq. (A8) into Eq. (A3), multiply the equation of order \((1-t)^2 \) by \( \psi_{0,q}(r) \), and perform integration over \( r \). The resulting equations for \( N = 0, 2, 4 \) yield
respectively. The functions in Eqs. (A9a) and (A9b) are defined by Eqs. (A4) and (24) and should be evaluated at $T_c$. In the clean limit $\tau \to \infty$, these functions acquire simple expressions as

$$R = T_c^2 w_{0,0}^{(2)} = 1, \quad T_c w_{2,0}^{2} = -2, \quad T_c w_{2,2}^{2} = 0,$$  \hspace{1cm} (A10a)

$$w_{4,\mu} = \frac{31(5) \langle \phi^2 | \psi_{E_{\mu}} | \phi^2 \psi_{E_{\mu}} \rangle}{[7 \zeta(3)]^2 \langle \phi^2 | \psi_{E_{\mu}} | 2 \phi^2 \psi_{E_{\mu}} \rangle},$$  \hspace{1cm} (A10b)

$$w_p = -\frac{7 \zeta(3) (\mu_B B_r)^2}{4(\pi T_c)^2},$$  \hspace{1cm} (A10c)

with $\mu=0,2,4$ and $w_{4,0}=w_{4,0a}=w_{4,0b}$. Equation (A9) with Eq. (A10), includes the result by Hohenberg and Werthamer\(^9\) for cubic materials and also the one by Takanaka\(^18\) for uniaxial materials in the relevant order, both except the Pauli term $w_p$. Thus, we have extended the results by Hohenberg and Werthamer\(^9\) and Takanaka\(^18\) to arbitrary crystal structures and impurity-scattering time, including also Pauli paramagnetism.

Equation (A9) reveals a close connection of both the curvature in $H_m(T \leq T_c)$ and the mixing of higher Landau levels in $\Delta(r)$ with the Fermi-surface structure. For example, we realize from Eq. (A9b) with Eqs. (A10), (18), and (26) that the mixing of $N=2$ Landau level is absent for cubic materials where $c_1=c_2=1$. This is not the case for low-symmetry crystals, however. Equation (A9) enables us to estimate the curvature and the mixing based on Fermi-surface structures from detailed electronic-structure calculations.

**APPENDIX B: PROOF OF EQ. (41)**

The first expression in Eq. (41a) can be proved by induction as follows. First of all, $\eta_0 = R_0$ is transformed from Eq. (37a) as\(^7\)

$$\eta_0 = \frac{1}{1 + \frac{x^2}{2} - \frac{1}{1 + \cdots}} = \int_0^\infty \frac{e^{-x^2}}{\sqrt{\pi x}} dx$$

Thus, Eq. (41a) holds for $N=0$. The last expression in Eq. (B1) is the same integral which appears in Eq. (26) of Hohenberg and Werthamer.\(^9\) We next rewrite Eq. (37a) with respect to $\eta_N$ in Eq. (40a) as

$$\eta_N = \begin{cases} \frac{(-\eta_0 + 1)/x^2}{(-\eta_{N-1} + \sqrt{N-1} \eta_{N-2})/\sqrt{Nx^2}} & (N=1) \\
\end{cases},$$  \hspace{1cm} (B2)

Using Eqs. (B1) and (B2), it is easy to see that Eq. (41a) holds for $N=1$. Proceeding to the general case, we assume that Eq. (41a) is valid for $N=M-1$. We also remember the following properties of the Hermite polynomials:

$$H_N(s) = 2sH_{N-1}(s) + 2(N-1)H_{N-2}(s),$$  \hspace{1cm} (B3a)

Then $\eta_M$ is obtained explicitly by using Eq. (B2) as

$$\eta_M = \frac{1}{\sqrt{M} \tilde{x}} \left[ -\eta_{M-1} + \sqrt{M-1} \eta_{M-2} \right]$$

Thus, we have established the first expression in Eq. (41a). The proof for the second expression proceeds in the same way by using partial integrations for $\eta_0$ and $\eta_{N-2}$ in Eq. (B2). Equation (41b) can be proved similarly by induction, starting from $\tilde{R}_1=1$ and using Eq. (37b).
AB INITIO CALCULATIONS OF $H_{c2}$ IN TYPE-II...
It should be noted that $\beta$ is not independent of $\varepsilon_n$ but depends on the sign of $\varepsilon_n$. Thus Eq. (32) may be somewhat misleading, but it allows us a compact notation as seen from Eq. (29).


