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CRITICAL TWO-POINT FUNCTIONS FOR LONG-RANGE STATISTICAL-MECHANICAL MODELS IN HIGH DIMENSIONS

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We consider long-range self-avoiding walk, percolation and the Ising model on $\mathbb{Z}^d$ that are defined by power-law decaying pair potentials of the form $D(x) \asymp |x|^{-d-\alpha}$ with $\alpha > 0$. The upper-critical dimension d_c is $2(\alpha \wedge 2)$ for self-avoiding walk and the Ising model, and $3(\alpha \wedge 2)$ for percolation. Let $\alpha \neq 2$ and assume certain heat-kernel bounds on the n-step distribution of the underlying random walk. We prove that, for $d > d_c$ (and the spread-out parameter sufficiently large), the critical two-point function $G_{p_c}(x)$ for each model is asymptotically $C|x|^{\alpha\wedge 2-d}$, where the constant $C \in (0, \infty)$ is expressed in terms of the model-dependent lace-expansion coefficients and exhibits crossover between $\alpha < 2$ and $\alpha > 2$. We also provide a class of random walks that satisfy those heat-kernel bounds.

1. Introduction. The two-point function is one of the key observables to understand phase transitions and critical behavior. For example, the two-point function for the Ising model indicates how likely the spins located at those two sites point in the same direction. If it decays fast enough to be summable, then there is no macroscopic order. The summability of the two-point function is lost as soon as the model parameter (e.g., temperature) is above the critical point and, therefore, it is naturally hard to investigate critical behavior.

The lace expansion is a powerful tool to rigorously prove mean-field behavior above the model-dependent critical dimension. The mean-field behavior here is for the two-point function at the critical point to exhibit similar behavior to the underlying random walk. It has been successful to prove such behavior for various statistical-mechanical models, such as self-avoiding walk, percolation, lattice trees/animals and the Ising model. The best lace-expansion result obtained so far is to identify an asymptotic expression (= the Newtonian potential times a model-dependent constant) of the critical two-point function for finite-range models, such as the nearest-neighbor model. However, this ultimate goal has not been achieved

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before this paper for long-range models, especially when the 1-step distribution for the underlying random walk decays in powers of distance; only the infrared bound on the Fourier transform of the two-point function was available. This was partly because of our poor understanding of the long-range models in the x-space, not in the Fourier space. For example, the random-walk Green's function is known to be asymptotically Newtonian/Riesz depending on the power of the aforementioned power-law decaying 1-step distribution, but we were unable to find optimal error estimates in the literature. Also, the subcritical two-point function is known to decay exponentially for the finite-range models, but this is not the case for the power-law decaying long-range models; as is shown in this paper, the decay rate of the subcritical two-point function is the same as the 1-step distribution of the underlying random walk.

Therefore, the goal of this paper is to overcome those difficulties and derive an asymptotic expression of the critical two-point function for the power-law decaying long-range models above the critical dimension, using the lace expansion. We would also like to investigate crossover in the asymptotic expression when the power of the 1-step distribution of the underlying random walk changes.

1.1. *Models and known results.* Self-avoiding walk (SAW) is a model for linear polymers. We define the two-point function for SAW on $\mathbb{Z}^d$ as

$$G_p^{\mathrm{SAW}}(x) = \sum_{\omega : o \to x} p^{|\omega|} \prod_{j=1}^{|\omega|} D(\omega_j - \omega_{j-1}) \prod_{s<t} (1 - \delta_{\omega_s, \omega_t}), \tag{1.1}$$

where $p \geq 0$ is the fugacity, $|\omega|$ is the length of a path $\omega = (\omega_0, \omega_1, \dots, \omega_{|\omega|})$ and $D : \mathbb{Z}^d \to [0, 1]$ is the $\mathbb{Z}^d$-symmetric nondegenerate [i.e., $D(o) \neq 1$] 1-step distribution for the underlying random walk (RW); the contribution from the 0-step walk is considered to be $\delta_{o,x}$ by convention. If the indicator function $\prod_{s<t}(1 - \delta_{\omega_s,\omega_t})$ is replaced by 1, then $G_p^{\mathrm{SAW}}(x)$ turns into the RW Green's function $G_p^{\mathrm{RW}}(x)$, whose radius of convergence $p_{\mathrm{c}}^{\mathrm{RW}}$ is 1, as $\chi_p^{\mathrm{RW}} \equiv \sum_{x \in \mathbb{Z}^d} G_p^{\mathrm{RW}}(x) = (1 - p)^{-1}$ for $p < 1$ and $\chi_p^{\mathrm{RW}} = \infty$ for $p \geq 1$. Therefore, the radius of convergence $p_{\mathrm{c}}^{\mathrm{SAW}}$ for $G_p^{\mathrm{SAW}}(x)$ is not less than 1. It is known that $\chi_p^{\mathrm{SAW}} \equiv \sum_{x \in \mathbb{Z}^d} G_p^{\mathrm{SAW}}(x) < \infty$ if and only if $p < p_{\mathrm{c}}^{\mathrm{SAW}}$ and diverges as $p \uparrow p_{\mathrm{c}}^{\mathrm{SAW}}$. Here, and in the remainder of the paper, we often use "$\equiv$" for definition.

Percolation is a model for random media. Each bond $\{u, v\}$, which is a pair of vertices in $\mathbb{Z}^d$, is either occupied or vacant independently of the other bonds. The probability that $\{u, v\}$ is occupied is defined to be $pD(v - u)$, where $p \geq 0$ is the percolation parameter. Since D is a probability distribution, the expected number of occupied bonds per vertex equals $p \sum_{x \neq o} D(x) = p(1 - D(o))$. The percolation two-point function $G_p^{\mathrm{perc}}(x)$ is defined to be the probability that there is a self-avoiding path of occupied bonds from o to x; again by convention, $G_p^{\mathrm{perc}}(o) = 1$.

The Ising model is a model for magnets. For $\Lambda \subset \mathbb{Z}^d$ and $\varphi = \{\varphi_v\}_{v\in\Lambda} \in \{\pm 1\}^\Lambda$, we define the Hamiltonian (under the free-boundary condition) as

$$H_\Lambda(\varphi) = -\sum_{\{u,v\}\subset\Lambda} J_{u,v}\varphi_u\varphi_v, \tag{1.2}$$

where $J_{u,v} = J_{o,v-u} \geq 0$ is the ferromagnetic pair potential and inherits the properties of the given D, as explained below. The finite-volume two-point function at the inverse temperature $\beta \geq 0$ is defined as

$$\langle\varphi_o\varphi_x\rangle_{\beta,\Lambda} = \sum_{\varphi\in\{\pm1\}^\Lambda} \varphi_o\varphi_x e^{-\beta H_\Lambda(\varphi)} \Big/ \sum_{\varphi\in\{\pm1\}^\Lambda} e^{-\beta H_\Lambda(\varphi)}. \tag{1.3}$$

It is known that $\langle\varphi_o\varphi_x\rangle_{\beta,\Lambda}$ is increasing in $\Lambda \uparrow \mathbb{Z}^d$. Let $p = \sum_{x\in\mathbb{Z}^d} \tanh(\beta J_{o,x})$. The Ising two-point function $G_p^{\mathrm{Ising}}(x)$ is defined to be the increasing-volume limit of $\langle\varphi_o\varphi_x\rangle_{\beta,\Lambda}$:

$$G_p^{\mathrm{Ising}}(x) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \langle\varphi_o\varphi_x\rangle_{\beta,\Lambda}. \tag{1.4}$$

Let $D(x) = p^{-1}\tanh(\beta J_{o,x})$.

For percolation and the Ising model, there is a model-dependent critical point $p_\mathrm{c} \geq 1$ (from now on, we omit the superscript, unless it causes any confusion) such that

$$\begin{aligned} \chi_p &\equiv \sum_{x\in\mathbb{Z}^d} G_p(x) \begin{cases} <\infty, & [p<p_\mathrm{c}], \\ =\infty, & [p\geq p_\mathrm{c}], \end{cases} \\ \theta_p &\equiv \sqrt{\lim_{|x|\to\infty} G_p(x)} \begin{cases} =0, & [p<p_\mathrm{c}], \\ >0, & [p>p_\mathrm{c}]. \end{cases} \end{aligned} \tag{1.5}$$

The order parameter θ_p^{perc} is the probability that the occupied cluster of the origin is unbounded, while $\theta_p^{\mathrm{Ising}}$ is the spontaneous magnetization, which is the infinite-volume limit of the finite-volume single-spin expectation $\langle\varphi_o\rangle_{\beta,\Lambda}^+$ under the plus-boundary condition. The continuity of θ_p at $p = p_\mathrm{c}$ in a general setting is still a remaining issue.

We are interested in asymptotic behavior of $G_{p_\mathrm{c}}(x)$ as $|x| \to \infty$. For the "uniformly spread-out" finite-range models, for example, $D(x) = \mathbb{1}_{\{|x|=1\}}/(2d)$ or $D(x) = \mathbb{1}_{\{\|x\|_\infty \leq L\}}/(2L+1)^d$ for some $L \in [1,\infty)$, it has been proved [15, 18, 24] that, if $d > 4$ for SAW and the Ising model and $d > 6$ for percolation, and if d or L is sufficiently large (depending on the models), then there is a model-dependent constant A ($= 1$ for RW) such that

$$G_{p_\mathrm{c}}(x) \underset{|x|\to\infty}{\sim} \frac{a_d/\sigma^2}{A|x|^{d-2}}, \tag{1.6}$$

where "$\sim$" means that the asymptotic ratio of the left-hand side to the right-hand side is 1, and

$$a_d = \frac{d\Gamma((d-2)/2)}{2\pi^{d/2}}, \qquad \sigma^2 \equiv \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) = O(L^2). \tag{1.7}$$

This is a sufficient condition for the following mean-field behavior [1–3, 5, 22]:

$$\chi_p \underset{p\uparrow p_c}{\asymp} (p_c - p)^{-1}, \qquad \theta_p \underset{p\downarrow p_c}{\asymp} \begin{cases} \sqrt{p - p_c}, & \text{[Ising]}, \\ p - p_c, & \text{[percolation]}, \end{cases} \tag{1.8}$$

where "$\asymp$" means that the asymptotic ratio of the left-hand side to the right-hand side is bounded away from zero and infinity.

The proof of the above result is based on the lace expansion (e.g., [17, 22, 24, 25]). The core concept of the lace expansion is to systematically isolate interaction among individuals (e.g., mutual avoidance between distinct vertices for SAW or between distinct occupied pivotal bonds for percolation) and derive macroscopic recursive structure that yields the random-walk like behavior (1.6). When $d > d_c$ and $d \vee L \gg 1$ (i.e., d or L sufficiently large depending on the models), there is enough room for those individuals to be away from each other, and the lace expansion converges [17, 22, 24, 25]. The resultant recursion equation for G_p is the following:

$$G_p(x) = \begin{cases} \delta_{o,x} + \sum_{v\in\mathbb{Z}^d} pD(v)G_p(x-v), & \text{[RW]}, \\ \delta_{o,x} + \sum_{v\in\mathbb{Z}^d} \big(pD(v) + \pi_p(v)\big)G_p(x-v), & \\ & \text{[SAW]}, \\ \pi_p(x) + \sum_{\substack{u,v\in\mathbb{Z}^d \\ (u\neq v)}} \pi_p(u)pD(v-u)G_p(x-v), & \\ & \text{[Ising and percolation]}, \end{cases} \tag{1.9}$$

where π_p is the lace-expansion coefficient. To treat all models simultaneously, we introduce the notation $f * g$ to denote the convolution of functions f and g in $\mathbb{Z}^d$:

$$(f * g)(x) = \sum_{v\in\mathbb{Z}^d} f(v)g(x-v). \tag{1.10}$$

Then the above identities can be simplified as (the spatial variables are omitted)

$$G_p = \begin{cases} \delta + pD * G_p, & \text{[RW]}, \\ \delta + (pD + \pi_p) * G_p, & \text{[SAW]}, \\ \pi_p + \pi_p * p\big(D - D(o)\delta\big) * G_p, & \text{[Ising and percolation]}. \end{cases} \tag{1.11}$$

Repeated use of these identities yields[3]

$$G_p = 1_p + \mathit{\Pi}_p * pD * G_p, \tag{1.12}$$

where

$$\mathit{\Pi}_p(x) = \begin{cases} \delta_{o,x}, & \text{[RW]}, \\ \displaystyle\sum_{n=0}^{\infty} \pi_p^{*n}(x) \equiv \sum_{n=0}^{\infty} (\underbrace{\pi_p * \cdots * \pi_p}_{n\text{-fold}})(x), & \text{[SAW]}, \\ \displaystyle\sum_{n=1}^{\infty} \bigl(-pD(o)\bigr)^{n-1} \pi_p^{*n}(x), & \text{[Ising and percolation]} \end{cases} \tag{1.13}$$

with the convention $f^{*0}(x) \equiv \delta_{o,x}$ for general f. When $d > d_c$ and $d \vee L \gg 1$, there is a $\rho > 0$ such that $|\mathit{\Pi}_{p_c}(x)|$ is summable and decays as $|x|^{-d-2-\rho}$ [15, 18, 24]. The multiplicative constant A in (1.6) and p_c can be represented in terms of $\mathit{\Pi}_{p_c}(x)$ as

$$p_c = \biggl(\sum_{x \in \mathbb{Z}^d} \mathit{\Pi}_{p_c}(x)\biggr)^{-1}, \qquad A = p_c\biggl(1 + \frac{p_c}{\sigma^2} \sum_{x \in \mathbb{Z}^d} |x|^2 \mathit{\Pi}_{p_c}(x)\biggr). \tag{1.14}$$

In this paper, we investigate long-range SAW, percolation and the Ising model on $\mathbb{Z}^d$ defined by power-law decaying pair potentials of the form $D(x) \asymp |x|^{-d-\alpha}$

[3]For SAW, since $\|\pi_p\|_1 = o(1)$ as $d \vee L \to \infty$ and $\|G_p\|_\infty < \infty$ for every $p \le p_c$ [15, 18],

$$\begin{aligned} G_p &= \delta + pD * G_p + \pi_p * \underbrace{G_p}_{\text{replace}} \\ &= \delta + pD * G_p + \pi_p * (\delta + pD * G_p + \pi_p * G_p) \\ &= (\delta + \pi_p) + (\delta + \pi_p) * pD * G_p + \pi_p^{*2} * \underbrace{G_p}_{\text{replace}} = \cdots \to (1.12). \end{aligned}$$

For percolation and the Ising model, since $D(o) = o(1)$ and $p\|\pi_p\|_1 = 1 + o(1)$ as $d \vee L \to \infty$ and $\|G_p\|_\infty \le 1$ for every $p \le p_c$ [15, 18, 24],

$$\begin{aligned} G_p &= \pi_p + \pi_p * pD * G_p - pD(o)\pi_p * \underbrace{G_p}_{\text{replace}} \\ &= \pi_p + \pi_p * pD * G_p - pD(o)\pi_p * \bigl(\pi_p + \pi_p * pD * G_p - pD(o)\pi_p * G_p\bigr) \\ &= \bigl(\pi_p - pD(o)\pi_p^{*2}\bigr) + \bigl(\pi_p - pD(o)\pi_p^{*2}\bigr) * pD * G_p + \bigl(-pD(o)\bigr)^2 \pi_p^{*2} * \underbrace{G_p}_{\text{replace}} \\ &= \cdots \to (1.12). \end{aligned}$$

with $\alpha > 0$. For example, as in [9, 10], we can consider the following uniformly spread-out long-range D with parameter $L \in [1, \infty)$:

$$D(x) = \frac{|||x/L|||_1^{-d-\alpha}}{\sum_{y\in\mathbb{Z}^d} |||y/L|||_1^{-d-\alpha}}, \tag{1.15}$$

where $|||x|||_\ell = |x| \vee \ell$. As a result,

$$D(x) = O\big(L^\alpha\big)|||x|||_L^{-d-\alpha}, \tag{1.16}$$

which we require throughout the paper (cf., Assumption 1.1 below). The goal is to see how the asymptotic expression (1.6) of $G_{p_c}(x)$ changes depending on the value of α. We note that (1.6) and (1.14) are invalid for $\alpha \le 2$ because then $\sigma^2 = \infty$.

Let

$$d_c = \begin{cases} 2(\alpha \wedge 2), & \text{[SAW and Ising]}, \\ 3(\alpha \wedge 2), & \text{[percolation]}. \end{cases} \tag{1.17}$$

It has been proved [20] that, for $d > d_c$ and $L \gg 1$, the Fourier transform $\hat{G}_p(k) \equiv \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} G_p(x)$ for the long-range models is bounded above and below by a multiple of $\hat{G}_{\hat{p}}^{\mathrm{RW}}(k) \equiv (1 - \hat{p}\hat{D}(k))^{-1}$ with $\hat{p} = p/p_c$, uniformly in $p < p_c$. Although this gives an impression of the similarity between $G_{p_c}(x)$ and $G_1^{\mathrm{RW}}(x)$, it is still too weak to identify the asymptotic expression of $G_{p_c}(x)$. The proof of the above Fourier-space result makes use of the following properties of D that we make use of here as well: there are $v_\alpha = O(L^{\alpha\wedge 2})$ and $\varepsilon > 0$ such that

$$\begin{aligned} \hat{D}(k) &\equiv \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} D(x) \\ &= 1 - v_\alpha |k|^{\alpha\wedge 2} \times \begin{cases} 1 + O\big((L|k|)^\varepsilon\big), & [\alpha \ne 2], \\ \log \dfrac{1}{L|k|} + O(1), & [\alpha = 2]. \end{cases} \end{aligned} \tag{1.18}$$

If $\alpha > 2$, then $v_\alpha = \sigma^2/(2d)$. Moreover, if $L \gg 1$, then there is a constant $\Delta \in (0, 1)$ such that[4]

$$\begin{aligned} &\|D^{*n}\|_\infty \le O\big(L^{-d}\big)n^{-d/(\alpha\wedge 2)} \qquad [n \ge 1], \\ &1 - \hat{D}(k) \begin{cases} < 2 - \Delta, & \big[k \in [-\pi, \pi]^d\big], \\ > \Delta, & \big[\|k\|_\infty \ge L^{-1}\big]. \end{cases} \end{aligned} \tag{1.20}$$

All those properties hold for D in (1.15) (cf., [9–11]).

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[4]In the proof of the bound on $\|D^{*n}\|_\infty$, we simply bounded the factor $\log \frac{\pi}{2r}$ in [9], (A.4), by a positive constant. If we make the most of that factor instead, we can readily improve the bound for $\alpha = 2$ as

$$\|D^{*n}\|_\infty \le O\big(L^{-d}\big)(n \log n)^{-d/2}. \tag{1.19}$$

1.2. *Main result.* In addition to the above properties, the n-step transition probability obeys the following bound:

$$D^{*n}(x) \le \frac{O(L^{\alpha\wedge 2})}{|||x|||_L^{d+\alpha\wedge 2}} n \times \begin{cases} 1, & [\alpha \ne 2], \\ \log |||x|||_L, & [\alpha = 2]. \end{cases} \tag{1.21}$$

This is due to the following two facts: (i) the contribution from the walks that have at least one step which is longer than $c|||x|||_L$ for a given $c > 0$ is bounded by $O(L^\alpha)n/|||x|||_L^{d+\alpha}$; (ii) the contribution from the walks whose n steps are all shorter than $c|||x|||_L$ is bounded, due to the local CLT, by $O(\tilde{v}n)^{-d/2}e^{-|x|^2/O(\tilde{v}n)} \le O(\tilde{v}n)/|||x|||_L^{d+2}$ (times an exponentially small normalization constant), where $\tilde{v}$ is the variance of the truncated 1-step distribution $\tilde{D}(y) \equiv D(y)\mathbb{1}_{\{|y|\le c|x|\}}$ and equals

$$\tilde{v} = \sum_{y\in\mathbb{Z}^d} |y|^2 \tilde{D}(y) \le O(L^{\alpha\wedge 2}) \times \begin{cases} |||x|||_L^{2-\alpha}, & [\alpha < 2], \\ \log |||x|||_L, & [\alpha = 2], \\ 1, & [\alpha > 2]. \end{cases} \tag{1.22}$$

For $\alpha \ne 2$, inequality (1.21) is a discrete space–time version of the heat-kernel bound on the transition density $p_s(x)$ of an α-stable/Gaussian process:

$$p_s(x) \equiv \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - s|k|^{\alpha\wedge 2}} \le \frac{O(s)}{|x|^{d+\alpha\wedge 2}}. \tag{1.23}$$

In Section 2.1, we will show that the properties (1.16), (1.18) and (1.21) are sufficient to obtain an asymptotic expression of $G_1^{\mathrm{RW}}(x)$. However, these properties are not good enough to fully control error terms arising from convolutions of $D^{*n}(x)$ and $\Pi_p(x)$ in (1.13). To overcome this difficulty, we assume the following bound on the discrete derivative of the n-step transition probability:

$$\left| D^{*n}(x) - \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2} \right| \le \frac{O(L^{\alpha\wedge 2})|||y|||_L^2}{|||x|||_L^{d+\alpha\wedge 2+2}} n \qquad \left[|y| \le \frac{1}{3}|x|\right]. \tag{1.24}$$

Here is the summary of the properties of D that we use throughout the paper.

ASSUMPTION 1.1. The $\mathbb{Z}^d$-symmetric 1-step distribution D satisfies the properties (1.16), (1.18), (1.20), (1.21) and (1.24).

In Appendix, we will show that the following D satisfies all properties in the above assumption:

$$D(x) = \sum_{t\in\mathbb{N}} U_L^{*t}(x) T_\alpha(t), \tag{1.25}$$

where U_L is in a class of $\mathbb{Z}^d$-symmetric distributions on $\mathbb{Z}^d \cap [-L, L]^d$, and T_α is the stable distribution on $\mathbb{N}$ with parameter $\alpha/2 \ne 1$.

Under the above assumption on D, we can prove the following theorem.

THEOREM 1.2. *Let $\alpha > 0$, $\alpha \neq 2$ and*

$$\gamma_\alpha = \frac{\Gamma((d-\alpha \wedge 2)/2)}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma((\alpha\wedge 2)/2)} \tag{1.26}$$

and assume all properties of D in Assumption 1.1. Then, for RW with $d > \alpha \wedge 2$ and any $L \geq 1$, and for SAW, percolation and the Ising model with $d > d_c$ and $L \gg 1$, there are $\mu \in (0, \alpha \wedge 2)$ and $A = A(\alpha, d, L) \in (0, \infty)$ ($A \equiv 1$ for random walk) such that, as $|x| \to \infty$,

$$G_{p_c}(x) = \frac{\gamma_\alpha/v_\alpha}{A|x|^{d-\alpha\wedge 2}} + \frac{O(L^{-\alpha\wedge 2+\mu})}{|x|^{d-\alpha\wedge 2+\mu}}. \tag{1.27}$$

As a result, by [20], χ_p and θ_p exhibit the mean-field behavior (1.8). Moreover, p_c and A can be expressed in term of Π_p in (1.13) as

$$p_c = \hat{\Pi}_{p_c}(0)^{-1}, \qquad A = p_c + \begin{cases} 0, & [\alpha < 2], \\ \dfrac{p_c^2}{\sigma^2}\displaystyle\sum_x |x|^2 \Pi_{p_c}(x), & [\alpha > 2]. \end{cases} \tag{1.28}$$

REMARK 1.3. (a) The finite-range models are formally considered as the $\alpha = \infty$ model. Indeed, the leading term in (1.27) for $\alpha > 2$ is identical to (1.6).

(b) Following the argument in [15, 24], we can "almost" prove Theorem 1.2 for $\alpha > 2$ without assuming the bounds on $D^{*n}(x)$. The shortcoming is the restriction $d > 10$, not $d > 6$, for percolation. This is due to the peculiar diagrammatic estimate in [15], which we do not use in this paper.

(c) The asymptotic behavior of $G_{p_c}(x)$ in (1.6) or (1.27) is a key element for the so-called 1-arm exponent to take on its mean-field value [16, 19, 21, 23]. For finite-range critical percolation, for example, the probability that $o \in \mathbb{Z}^d$ is connected to the surface of the d-dimensional ball of radius r centered at o is bounded above and below by a multiple of r^{-2} in high dimensions [21]. The value of the exponent may change in a peculiar way depending on the value of α [19].

(d) As described in (1.28), the constant A exhibits crossover between $\alpha < 2$ and $\alpha > 2$; in particular, $A = p_c$ for $\alpha < 2$ [cf., (3.6) below]. According to some rough computations, it seems that the asymptotic expression of $G_{p_c}(x)$ for $\alpha = 2$ is a mixture of those for $\alpha < 2$ and $\alpha > 2$, with a logarithmic correction:

$$G_{p_c}(x) \underset{|x|\to\infty}{\sim} \frac{\gamma_2/v_2}{p_c|x|^{d-2}\log|x|}. \tag{1.29}$$

One of the obstacles to prove this conjecture is a lack of good control on convolutions of the RW Green's function and the lace-expansion coefficients for $\alpha = 2$. As hinted in the above expression, we may have to deal with logarithmic factors more actively than ever. We are currently working in this direction.

1.3. *Notation and the organization.* From now on, we distinguish G_p^{RW} from G_p for the other three models, and define

$$S_p = G_p^{\mathrm{RW}}. \tag{1.30}$$

Here, and in the remainder of the paper, the spatial variables are sometimes omitted. For example,

$$S_p = \delta + pD * S_p \tag{1.31}$$

is the abbreviated version of the convolution equation

$$S_p(x) = \delta_{o,x} + (pD * S_p)(x) = \delta_{o,x} + \sum_{y\in\mathbb{Z}^d} pD(y)S_p(x-y). \tag{1.32}$$

We also recall the notation

$$|\!|\!|x|\!|\!|_\ell = |x| \vee \ell. \tag{1.33}$$

The remainder of the paper is organized as follows. In Section 2, we prove the asymptotic expression (1.27) for S_1, as well as bounds on S_p for $p \le 1$ and some basic properties of G_p for $p \le p_{\mathrm{c}}$. Then, by using these facts and the diagrammatic bounds on the lace-expansion coefficients in [18, 24], we prove (1.27) for $G_{p_{\mathrm{c}}}$ in Section 3.

2. Preliminaries. In this section, we derive the asymptotic expression (1.27) for S_1, which will be restated as Proposition 2.1, and prove some properties of G_p that will be used to prove Theorem 1.2 in Section 3.

2.1. *Asymptotics of S_p.*

PROPOSITION 2.1. *Let $\alpha > 0, \alpha \neq 2$ and $d > \alpha \wedge 2$, and assume all properties but* (1.24) *in Assumption* 1.1. *Then there is a $\mu \in (0, \alpha \wedge 2)$ such that, for any $L \ge 1$, $p \le 1$ and $\kappa > 0$,*

$$\delta_{o,x} \le S_p(x) \le \delta_{o,x} + \frac{O(L^{-\alpha\wedge 2})}{|\!|\!|x|\!|\!|_L^{d-\alpha\wedge 2}} \qquad [\forall x \in \mathbb{Z}^d], \tag{2.1}$$

$$S_1(x) = \frac{\gamma_\alpha/v_\alpha}{|x|^{d-\alpha\wedge 2}} + \frac{O(L^{-\alpha\wedge 2+\mu})}{|x|^{d-\alpha\wedge 2+\mu}} \qquad [|x| > L^{1+\kappa}], \tag{2.2}$$

where a constant in the $O(L^{-\alpha\wedge 2+\mu})$ term depends on κ.

PROOF. Inequality (2.1) is an immediate result of (1.31), $p \le 1$ and (1.20)–(1.21) as[5]

$$0 \le S_p(x) - \delta_{o,x} \le \sum_{n=1}^{\infty} D^{*n}(x) \le \frac{O(L^{\alpha\wedge 2})}{|||x|||_L^{d+\alpha\wedge 2}} \sum_{n=1}^{(|||x|||_L/L)^{\alpha\wedge 2}} n + O(L^{-d}) \sum_{n=(|||x|||_L/L)^{\alpha\wedge 2}}^{\infty} n^{-d/(\alpha\wedge 2)} = \frac{O(L^{-\alpha\wedge 2})}{|||x|||_L^{d-\alpha\wedge 2}}. \tag{2.3}$$

To prove the asymptotic expression (2.2), we first rewrite $S_1(x)$ for $d > \alpha \wedge 2$ as

$$\begin{aligned} S_1(x) &= \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-ik\cdot x}}{1-\hat{D}(k)} \\ &= \int_0^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - t(1-\hat{D}(k))} \\ &= \int_T^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - t(1-\hat{D}(k))} + I_1 \end{aligned} \tag{2.4}$$

for any $T \in (0, \infty)$, where

$$\begin{aligned} I_1 &= \int_0^T \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - t(1-\hat{D}(k))} \\ &= \int_0^T \mathrm{d}t\, e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} D^{*n}(x). \end{aligned} \tag{2.5}$$

Next, we rewrite the large-t integral as

$$\int_T^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - t(1-\hat{D}(k))} = \int_0^\infty \mathrm{d}t\, p_{v_\alpha t}(x) + \sum_{j=2}^{5} I_j, \tag{2.6}$$

[5]For $\alpha = 2$, we can readily bound $S_p(x) - \delta_{o,x}$ by using (1.19) for $n \ge N_x \equiv |||x|||_L^2/(L^2 \log |||x|||_L)$ and (1.21) for $n < N_x$ as

$$S_p(x) - \delta_{o,x} \le \sum_{n=1}^{N_x - 1} D^{*n}(x) + \sum_{n=N_x}^{\infty} D^{*n}(x) \le \frac{O(L^{-2})}{|||x|||_L^{d-2} \log |||x|||_L}.$$

where $p_s(x)$ is the transition density of an α-stable/Gaussian process [cf., (1.23)], and for any $R \in (0, \pi)$,

$$I_2 = -\int_0^T \mathrm{d}t\; p_{v_\alpha t}(x) \equiv -\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - v_\alpha t|k|^{\alpha\wedge 2}}, \tag{2.7}$$

$$I_3 = \int_T^\infty \mathrm{d}t \int_{|k|\le R} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x}\big(e^{-t(1-\hat{D}(k))} - e^{-v_\alpha t|k|^{\alpha\wedge 2}}\big), \tag{2.8}$$

$$I_4 = \int_T^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - t(1-\hat{D}(k))} \mathbb{1}_{\{|k|>R\}}, \tag{2.9}$$

$$I_5 = -\int_T^\infty \mathrm{d}t \int_{|k|>R} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - v_\alpha t|k|^{\alpha\wedge 2}}. \tag{2.10}$$

By using the identity

$$\begin{aligned} &\int_0^\infty \mathrm{d}t\, e^{-v_\alpha t|k|^{\alpha\wedge 2}} \\ &\qquad = \frac{1}{v_\alpha |k|^{\alpha\wedge 2}} = \frac{1}{v_\alpha \Gamma((\alpha\wedge 2)/2)} \int_0^\infty \mathrm{d}t\, t^{((\alpha\wedge 2)/2)-1} e^{-|k|^2 t}, \end{aligned} \tag{2.11}$$

we obtain

$$\begin{aligned} &\int_0^\infty \mathrm{d}t\; p_{v_\alpha t}(x) \\ &\qquad = \frac{1}{v_\alpha \Gamma((\alpha\wedge 2)/2)} \int_0^\infty \mathrm{d}t\, t^{((\alpha\wedge 2)/2)-1} \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-|k|^2 t - ik\cdot x} \\ &\qquad = \frac{\gamma_\alpha / v_\alpha}{|x|^{d-\alpha\wedge 2}}. \end{aligned} \tag{2.12}$$

As a result, we arrive at

$$S_1(x) = \frac{\gamma_\alpha / v_\alpha}{|x|^{d-\alpha\wedge 2}} + \sum_{j=1}^5 I_j. \tag{2.13}$$

It remains to estimate $\sum_{j=1}^5 I_j$. First, by (1.21) and (1.23), we can estimate $I_1 + I_2$ for $|x| > L$ as

$$|I_1 + I_2| \le \frac{O(L^{\alpha\wedge 2})}{|x|^{d+\alpha\wedge 2}} \int_0^T \mathrm{d}t\, t \le \frac{O(L^{\alpha\wedge 2}) T^2}{|x|^{d+\alpha\wedge 2}}. \tag{2.14}$$

Let

$$\mu = \frac{2(\alpha\wedge 2)\varepsilon}{d + \alpha\wedge 2 + \varepsilon}, \qquad T = \left(\frac{|x|}{L}\right)^{\alpha\wedge 2 - \mu/2}. \tag{2.15}$$

Then we obtain

$$|I_1 + I_2| \le \frac{O(L^{-\alpha\wedge 2+\mu})}{|x|^{d-\alpha\wedge 2+\mu}}. \tag{2.16}$$

Next, we estimate I_3. For small R, whose value will be determined shortly, we use (1.18) to obtain

$$\bigl|e^{-t(1-\hat{D}(k))} - e^{-v_\alpha t|k|^{\alpha\wedge 2}}\bigr| \le O\bigl(L^{\alpha\wedge 2+\varepsilon}\bigr)t|k|^{\alpha\wedge 2+\varepsilon}e^{-v_\alpha t|k|^{\alpha\wedge 2}}. \tag{2.17}$$

Therefore, by (2.15),

$$\begin{aligned}
|I_3| &\le O\bigl(L^{\alpha\wedge 2+\varepsilon}\bigr)\int_T^\infty \mathrm{d}t\, t\int_{|k|\le R} \mathrm{d}^d k|k|^{\alpha\wedge 2+\varepsilon}e^{-v_\alpha t|k|^{\alpha\wedge 2}} \\
&= O\bigl(L^{\alpha\wedge 2+\varepsilon}\bigr)\int_T^\infty \mathrm{d}t\, t\int_0^{v_\alpha t R^{\alpha\wedge 2}} \frac{\mathrm{d}r}{r}\biggl(\frac{r}{v_\alpha t}\biggr)^{(d+\alpha\wedge 2+\varepsilon)/(\alpha\wedge 2)} e^{-r} \\
&\le O\bigl(L^{\alpha\wedge 2+\varepsilon}\bigr)\int_T^\infty \mathrm{d}t\, t\,(v_\alpha t)^{-(d+\alpha\wedge 2+\varepsilon)/(\alpha\wedge 2)} \\
&\le O\bigl(L^{-d}\bigr)T^{1-((d+\varepsilon)/(\alpha\wedge 2))} = \frac{O(L^{-\alpha\wedge 2+\mu})}{|x|^{d-\alpha\wedge 2+\mu}}.
\end{aligned} \tag{2.18}$$

Finally we estimate $I_4 + I_5$ and determine the value of R during the course. First, by (1.18)–(1.20), we have

$$\begin{aligned}
|I_4| &\le \int_T^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-t(1-\hat{D}(k))}\mathbb{1}_{\{|k|>R\}} \\
&\qquad\qquad \times \bigl(\mathbb{1}_{\{\|k\|_\infty<L^{-1}\}} + \mathbb{1}_{\{\|k\|_\infty\ge L^{-1}\}}\bigr) \\
&\le \int_T^\infty \mathrm{d}t\biggl(\int_{|k|>R} \frac{\mathrm{d}^d k}{(2\pi)^d}e^{-tc(L|k|)^{\alpha\wedge 2}} + O(1)e^{-t\Delta}\biggr) \\
&\le O\bigl(L^{-d}\bigr)\int_T^\infty \mathrm{d}t\, t^{-d/(\alpha\wedge 2)}\Gamma\biggl(\frac{d}{\alpha\wedge 2}; tc(LR)^{\alpha\wedge 2}\biggr) + O(1)e^{-T\Delta},
\end{aligned} \tag{2.19}$$

where $\Gamma(a;x) \equiv \int_x^\infty \mathrm{d}t\, t^{a-1}e^{-t}$ is the incomplete gamma function, which is bounded by $O(x^{a-1})e^{-x}$ for large x. Here, we choose R to satisfy

$$tc(LR)^{\alpha\wedge 2} = \frac{2\varepsilon}{\alpha\wedge 2}\log t. \tag{2.20}$$

Then, for large t,

$$\begin{aligned}
&\Gamma\biggl(\frac{d}{\alpha\wedge 2}; tc(LR)^{\alpha\wedge 2}\biggr) \\
&\quad\le O\bigl(\bigl(tc(LR)^{\alpha\wedge 2}\bigr)^{(d/(\alpha\wedge 2))-1}\bigr)e^{-tc(LR)^{\alpha\wedge 2}} \\
&\quad= O\bigl((\log t)^{(d/(\alpha\wedge 2))-1}\bigr)t^{-(2\varepsilon)/(\alpha\wedge 2)} \le O\bigl(t^{-\varepsilon/(\alpha\wedge 2)}\bigr).
\end{aligned} \tag{2.21}$$

Therefore, again by (2.15) [cf., (2.18)],

$$
\begin{aligned}
&O(L^{-d})\int_T^\infty \mathrm{d}t\, t^{-d/(\alpha\wedge 2)}\Gamma\biggl(\frac{d}{\alpha\wedge 2}; tc(LR)^{\alpha\wedge 2}\biggr) \\
&\qquad \le O(L^{-d})T^{1-((d+\varepsilon)/(\alpha\wedge 2))} = \frac{O(L^{-\alpha\wedge 2+\mu})}{|x|^{d-\alpha\wedge 2+\mu}}.
\end{aligned}
\tag{2.22}
$$

We can estimate I_5 in exactly the same way. The exponentially decaying term in (2.19) obeys the same bound, since, for sufficiently large N (depending on κ),

$$
\begin{aligned}
e^{-T\Delta} &\le \frac{{}^{\exists}c_N}{T^N} = c_N L^{-d}T^{1-((d+\varepsilon)/(\alpha\wedge 2))}L^d T^{-(N+1-((d+\varepsilon)/(\alpha\wedge 2)))} \\
&\le c_N L^{-d}T^{1-((d+\varepsilon)/(\alpha\wedge 2))}L^{d-(N+1-((d+\varepsilon)/(\alpha\wedge 2)))(\alpha\wedge 2-\mu/2)\kappa} \\
&\quad \bigl[\because |x| > L^{1+\kappa} \Rightarrow T > L^{(\alpha\wedge 2-\mu/2)\kappa}\bigr] \\
&\le c_N L^{-d}T^{1-((d+\varepsilon)/(\alpha\wedge 2))} = \frac{O(L^{-\alpha\wedge 2+\mu})}{|x|^{d-\alpha\wedge 2+\mu}}.
\end{aligned}
\tag{2.23}
$$

Summarizing the above, we obtain that, for $|x| > L^{1+\kappa}$,

$$
\Biggl|\sum_{j=1}^5 I_j\Biggr| \le \frac{O(L^{-\alpha\wedge 2+\mu})}{|x|^{d-\alpha\wedge 2+\mu}}.
\tag{2.24}
$$

This together with (2.13) completes the proof of Proposition 2.1. □

2.2. *Basic properties of G_p.* In this subsection, we summarize some basic properties of G_p. Roughly speaking, those properties are the continuity up to $p = p_c$ (Lemma 2.2), the RW bound that is optimal for $p \le 1$ (Lemma 2.3) and the *a priori* bound that is not sharp but finite as long as $p < p_c$ (Lemma 2.4). We will use them in the next section (especially in Section 3.2) to prove Theorem 1.2.

LEMMA 2.2. *For every $x \in \mathbb{Z}^d$, $G_p(x)$ is nondecreasing and continuous in $p < p_c$ for* SAW, *and in $p \le p_c$ for percolation and the Ising model. The continuity up to $p = p_c^{\mathrm{SAW}}$ for* SAW *is also valid if $G_p^{\mathrm{SAW}}(x)$ is uniformly bounded in $p < p_c^{\mathrm{SAW}}$.*

PROOF. For SAW, since $G_p^{\mathrm{SAW}}(x)$ is a power series of $p \ge 0$ with nonnegative coefficients, it is nondecreasing and continuous in $p < p_c^{\mathrm{SAW}}$. The continuity up to $p = p_c^{\mathrm{SAW}}$ under the hypothesis is due to monotone convergence.

For the Ising model, we first note that, by Griffiths' inequality [12], $\langle\varphi_o\varphi_x\rangle_{\beta,\Lambda}$ is nondecreasing and continuous in $\beta \ge 0$ and nondecreasing in $\Lambda \subset \mathbb{Z}^d$. Therefore, the infinite-volume limit $G_p^{\mathrm{Ising}}(x)$ is nondecreasing and left-continuous in $p \ge 0$. The continuity in $p \le p_c^{\mathrm{Ising}}$ follows from the fact that, for $p < p_c^{\mathrm{Ising}}$, $G_p^{\mathrm{Ising}}(x)$

coincides with the decreasing limit of the finite-volume two-point function under the "plus-boundary" condition, which is right-continuous in $p \geq 0$.

For percolation, $G_p^{\text{perc}}(x)$ is nondecreasing in $p \geq 0$ because the event that there is a path of occupied bonds from o to x is an increasing event. The continuity in $p \geq 0$ is obtained by following the same strategy as explained above for the Ising model and using the fact that there is at most one infinite occupied cluster for all $p \geq 0$. This completes the proof of Lemma 2.2. □

LEMMA 2.3. *For every $p < p_c$ and $x \in \mathbb{Z}^d$,*

$$G_p(x) \leq S_p(x), \qquad pD(x)(1-\delta_{o,x}) \leq G_p(x) - \delta_{o,x} \leq (pD * G_p)(x). \tag{2.25}$$

PROOF. The first inequality for $p > 1 \equiv p_c^{\text{RW}}$ is trivial since $S_p(x) = \infty$ for every $x \in \mathbb{Z}^d$. On the other hand, the first inequality for $p \leq 1$ is obtained by using the second inequality N times and then using (1.20), as

$$\begin{aligned} G_p(x) &\leq \sum_{n=0}^{N-1} p^n D^{*n}(x) + p^N \big(D^{*N} * G_p\big)(x) \\ &\leq S_p(x) + \|D^{*N}\|_\infty \chi_p \underset{N\uparrow\infty}{\to} S_p(x). \end{aligned} \tag{2.26}$$

It remains to prove the second inequality in (2.25). In fact, it suffices to prove the inequality only for $x \neq o$, since $G_p(o) = 1$ for all three models and therefore the inequality is trivial for $x = o$. For SAW and percolation, the inequality is obtained by specifying the first step pD and then using subadditivity for SAW or the BK inequality for percolation [26]. For the Ising model, we use the following random-current representation [1, 13] (see also [24], Section 2.1):

$$\langle \varphi_o \varphi_x \rangle_{\beta,\Lambda} = \frac{\sum_{\partial \mathbf{n} = \{o\} \triangle \{x\}} w_\Lambda(\mathbf{n})}{\sum_{\partial \mathbf{n} = \varnothing} w_\Lambda(\mathbf{n})}, \qquad w_\Lambda(\mathbf{n}) = \prod_{\{u,v\} \subset \Lambda} \frac{(\beta J_{u,v})^{n_{u,v}}}{n_{u,v}!}, \tag{2.27}$$

where $\mathbf{n} \equiv \{n_{u,v}\}$ is a collection of $\mathbb{Z}_+$-valued undirected bond variables (i.e., $n_{u,v} = n_{v,u} \in \mathbb{Z}_+ \equiv \{0\} \cup \mathbb{N}$ for each bond $\{u, v\} \subset \Lambda$), $\partial \mathbf{n}$ is the set of vertices y such that $\sum_{z \in \Lambda} n_{y,z}$ is an odd number, and "$\triangle$" represents symmetric difference (i.e., $\{o\} \triangle \{x\} = \varnothing$ if $x = o$, otherwise $\{o\} \triangle \{x\} = \{o, x\}$). Using this representation, we prove below that, for $x \neq o$,

$$pD(x) \leq \langle \varphi_o \varphi_x \rangle_{\beta,\Lambda} \leq \sum_{y \in \Lambda} pD(y) \langle \varphi_y \varphi_x \rangle_{\beta,\Lambda}, \tag{2.28}$$

where $pD(x) = \tanh(\beta J_{o,x})$. The second inequality in (2.25) for the Ising model is the infinite-volume limit of the above inequality.

To prove the lower bound of (2.28), we first specify the parity of $n_{o,x}$ to obtain that, for $x \neq o$ (so that $\{o\} \triangle \{x\} = \{o, x\}$),

$$\langle \varphi_o \varphi_x \rangle_{\beta,\Lambda} = \frac{\sum_{\partial \mathbf{n} = \{o,x\}, (n_{o,x} \text{ odd})} w_\Lambda(\mathbf{n}) + \sum_{\partial \mathbf{n} = \{o,x\}, (n_{o,x} \text{ even})} w_\Lambda(\mathbf{n})}{\sum_{\partial \mathbf{n} = \varnothing, (n_{o,x} \text{ odd})} w_\Lambda(\mathbf{n}) + \sum_{\partial \mathbf{n} = \varnothing, (n_{o,x} \text{ even})} w_\Lambda(\mathbf{n})}. \tag{2.29}$$

Let

$$\tilde{Y}_y(z,x) \equiv \sum_{\substack{\partial\mathbf{n}=\{z\}\triangle\{x\} \\ (n_{o,y}\text{ even})}} w_\Lambda(\mathbf{n}), \qquad \tilde{Z}_y \equiv \sum_{\substack{\partial\mathbf{n}=\varnothing \\ (n_{o,y}\text{ even})}} w_\Lambda(\mathbf{n}). \tag{2.30}$$

Then, by changing the parity of $n_{o,x}$ (and the constraint on $\partial\mathbf{n}$ accordingly) and recalling $\tanh(\beta J_{o,x}) = pD(x)$, we obtain

$$\sum_{\substack{\partial\mathbf{n}=\{o,x\} \\ (n_{o,x}\text{ odd})}} w_\Lambda(\mathbf{n}) = pD(x) \sum_{\substack{\partial\mathbf{n}=\varnothing \\ (n_{o,x}\text{ even})}} w_\Lambda(\mathbf{n}) = pD(x)\tilde{Z}_x, \tag{2.31}$$

$$\sum_{\substack{\partial\mathbf{n}=\varnothing \\ (n_{o,x}\text{ odd})}} w_\Lambda(\mathbf{n}) = pD(x) \sum_{\substack{\partial\mathbf{n}=\{o,x\} \\ (n_{o,x}\text{ even})}} w_\Lambda(\mathbf{n}) = pD(x)\tilde{Y}_x(o,x), \tag{2.32}$$

hence

$$\begin{aligned} \langle\varphi_o\varphi_x\rangle_{\beta,\Lambda} &= \frac{pD(x)\tilde{Z}_x + \tilde{Y}_x(o,x)}{pD(x)\tilde{Y}_x(o,x) + \tilde{Z}_x} \\ &= pD(x) + \frac{(1-p^2D(x)^2)\tilde{Y}_x(o,x)}{pD(x)\tilde{Y}_x(o,x) + \tilde{Z}_x} \geq pD(x). \end{aligned} \tag{2.33}$$

To prove the upper bound in (2.28), we first note that, if $\partial\mathbf{n} = \{o,x\}$, then there must be at least one $y \in \Lambda$ such that $n_{o,y}$ is an odd number. By similar computation to (2.31), we obtain that, for $x \neq o$,

$$\begin{aligned} \sum_{\partial\mathbf{n}=\{o,x\}} w_\Lambda(\mathbf{n}) &\leq \sum_{y\in\Lambda} \sum_{\substack{\partial\mathbf{n}=\{o,x\} \\ (n_{o,y}\text{ odd})}} w_\Lambda(\mathbf{n}) \\ &= \sum_{y\in\Lambda} pD(y) \underbrace{\sum_{\substack{\partial\mathbf{n}=\{y\}\triangle\{x\} \\ (n_{o,y}\text{ even})}} w_\Lambda(\mathbf{n})}_{\tilde{Y}_y(y,x)}. \end{aligned} \tag{2.34}$$

Moreover, $\tilde{Y}_y(y,x) \leq \sum_{\partial\mathbf{n}=\{y\}\triangle\{x\}} w_\Lambda(\mathbf{n})$ for any $y \in \Lambda$. Therefore, for $x \neq o$,

$$\begin{aligned} \langle\varphi_o\varphi_x\rangle_{\beta,\Lambda} &\equiv \frac{\sum_{\partial\mathbf{n}=\{o,x\}} w_\Lambda(\mathbf{n})}{\sum_{\partial\mathbf{n}=\varnothing} w_\Lambda(\mathbf{n})} \\ &\leq \sum_{y\in\Lambda} \frac{pD(y)\tilde{Y}_y(y,x)}{\sum_{\partial\mathbf{n}=\varnothing} w_\Lambda(\mathbf{n})} \\ &\leq \sum_{y\in\Lambda} pD(y)\langle\varphi_y\varphi_x\rangle_{\beta,\Lambda}. \end{aligned} \tag{2.35}$$

This completes the proof of (2.28), hence the proof of Lemma 2.3. $\square$

LEMMA 2.4. *Assume the property* (1.16) *in Assumption* 1.1. *Then, for every* $\alpha > 0$ *and* $p < p_c$, *there is a* $K_p = K_p(\alpha, d, L) < \infty$ *such that, for any* $x \in \mathbb{Z}^d$,

$$G_p(x) \le K_p |||x|||_L^{-d-\alpha}. \tag{2.36}$$

REMARK 2.5. This together with the lower bound in (2.25) implies that, for every $p < p_c$, $G_p(x)$ is bounded above and below by a p-dependent multiple of $|||x|||_L^{-d-\alpha}$. This shows sharp contrast to the exponential decay of $G_p(x)$ for the finite-range models.

PROOF OF LEMMA 2.4. Since $G_p(o) \le \chi_p < \infty$ for $p < p_c$, it suffices to prove (2.36) for $x \ne o$. We follow the idea of the proof of [4], Lemma 5.2, for one-dimensional long-range percolation and extend it to those three models in general dimensions. The key ingredient is the following Simon–Lieb type inequality: for $0 < \ell < |x|$,

$$G_p(x) \le \sum_{\substack{\{u,v\} \subset \mathbb{Z}^d \\ (|u| \le \ell < |v|)}} G_p(u) p D(v-u) G_p(x-v). \tag{2.37}$$

For SAW and percolation, this is a result of subadditivity or the BK inequality (cf., e.g., [14, 22]). For the Ising model, this is obtained by using the random-current representation (2.27) and a restricted version of the source-switching lemma [24], Lemma 2.3, as follows. Let $Z_\Lambda = \sum_{\partial \mathbf{n} = \varnothing} w_\Lambda(\mathbf{n})$ such that, for $x \ne o$,

$$\langle \varphi_o \varphi_x \rangle_{\beta,\Lambda} = \sum_{\partial \mathbf{n} = \{o,x\}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}. \tag{2.38}$$

We note that, if $\partial \mathbf{n} = \{o, x\}$, then there is a path $\omega = (\omega_0, \omega_1, \dots, \omega_t) \subset \Lambda$ from $\omega_0 = o$ to $\omega_t = x$ such that $n_{\omega_{s-1},\omega_s}$ is odd for every $s \in \{1, \dots, t\}$; moreover, there is a unique $\tau \in \{1, \dots, t\}$ such that $|\omega_{\tau-1}| \le \ell < |\omega_\tau|$ (i.e., τ is the first time when ω crosses the surface of the ball B_ℓ of radius ℓ centered at the origin). This can be restated as follows: if $\partial \mathbf{n} = \{o, x\}$, then there is a bond $\{u, v\} \subset \Lambda$ such that $n_{u,v}$ is odd and that u is connected from o with a path of bonds $\subset B_\ell$ with odd numbers. Therefore,

$$\langle \varphi_o \varphi_x \rangle_{\beta,\Lambda} \le \sum_{\substack{\{u,v\} \subset \Lambda \\ (|u| \le \ell < |v|)}} \sum_{\partial \mathbf{n} = \{o,x\}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{n_{u,v} \text{ odd}\}} \mathbb{1}_{\{o \underset{\mathbf{n}}{\longleftrightarrow} u \text{ in } B_\ell\}}, \tag{2.39}$$

where $\{o \underset{\mathbf{n}}{\longleftrightarrow} u \text{ in } B_\ell\}$ is the event that o is connected to u with a path of bonds $b \subset B_\ell$ satisfying $n_b > 0$. Multiplying $Z_{B_\ell}/Z_{B_\ell} \equiv 1$ to both sides of (2.39) and using the identity $Z_{B_\ell} = \sum_{\partial \mathbf{m} = \varnothing} w_{B_\ell}(\mathbf{m})$, we obtain

$$\langle \varphi_o \varphi_x \rangle_{\beta,\Lambda} \le \sum_{\substack{\{u,v\} \subset \Lambda \\ (|u| \le \ell < |v|)}} \sum_{\substack{\partial \mathbf{m} = \varnothing \\ \partial \mathbf{n} = \{o,x\}}} \frac{w_{B_\ell}(\mathbf{m})}{Z_{B_\ell}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{n_{u,v} \text{ odd}\}} \mathbb{1}_{\{o \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} u \text{ in } B_\ell\}}, \tag{2.40}$$

where we have used the trivial inequality $\mathbb{1}_{\{o \underset{\mathbf{n}}{\longleftrightarrow} u \text{ in } B_\ell\}} \leq \mathbb{1}_{\{o \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} u \text{ in } B_\ell\}}$. Then, by using the source-switching lemma [24], Lemma 2.3, we obtain

$$\begin{aligned}
\langle \varphi_o \varphi_x \rangle_{\beta,\Lambda} &\leq \sum_{\substack{\{u,v\}\subset\Lambda \\ (|u|\leq\ell<|v|)}} \sum_{\substack{\partial\mathbf{m}=\{o\}\triangle\{u\} \\ \partial\mathbf{n}=\{u,x\}}} \frac{w_{B_\ell}(\mathbf{m})}{Z_{B_\ell}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{n_{u,v} \text{ odd}\}} \mathbb{1}_{\{o \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} u \text{ in } B_\ell\}} \\
&= \sum_{\substack{\{u,v\}\subset\Lambda \\ (|u|\leq\ell<|v|)}} \langle \varphi_o \varphi_u \rangle_{\beta,B_\ell} \sum_{\substack{\partial\mathbf{n}=\{u,x\} \\ (n_{u,v} \text{ odd})}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda},
\end{aligned} \tag{2.41}$$

where we have used the identity $\mathbb{1}_{\{o \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} u \text{ in } B_\ell\}} = 1$ given $\partial\mathbf{m} = \{o\}\triangle\{u\}$ and then used (2.27). Finally, by following the same argument as in (2.34)–(2.35) and then taking the infinite-volume limit, we obtain (2.37) for the Ising model.

Now we prove (2.36) by using (2.37) with $\ell = \frac{1}{3}|x|$ (the factor $\frac{1}{3}$ is unimportant as long as it is less than $\frac{1}{2}$). Let

$$c_x = \sum_{\substack{\{u,v\}\subset\mathbb{Z}^d \\ (|u|\leq(1/3)|x|<|v|)}} G_p(u) D(v-u). \tag{2.42}$$

We note that $c_x \to 0$ as $|x| \to \infty$, because

$$\begin{aligned}
c_x &= \sum_{\substack{\{u,v\}\subset\mathbb{Z}^d \\ (|u|\leq(1/4)|x|,(1/3)|x|<|v|)}} G_p(u) p D(v-u) \\
&\quad + \sum_{\substack{\{u,v\}\subset\mathbb{Z}^d \\ ((1/4)|x|<|u|\leq(1/3)|x|<|v|)}} G_p(u) p D(v-u) \\
&\leq \chi_p p \underbrace{\sup_{u:|u|\leq(1/4)|x|} \sum_{v:|v|>(1/3)|x|} D(v-u)}_{O(|x|^{-\alpha})} + p \underbrace{\sum_{u:|u|>(1/4)|x|} G_p(u)}_{\text{Tail of } \chi_p<\infty}.
\end{aligned} \tag{2.43}$$

Therefore, for any $\varepsilon \in (0, 1)$, there is an $\tilde{\ell} \in [L, \infty)$ such that $2^{d+\alpha} c_x p \leq \varepsilon$ for all $|x| \geq \tilde{\ell}$. Then, for $|x| \geq \tilde{\ell}$, (2.37) implies

$$\begin{aligned}
G_p(x) &\leq \sum_{\substack{\{u,v\}\subset\mathbb{Z}^d \\ (|u|\leq(1/3)|x|<|v|\leq(1/2)|x|)}} G_p(u) p D(v-u) G_p(x-v) \\
&\quad + \sum_{\substack{\{u,v\}\subset\mathbb{Z}^d \\ (|u|\leq(1/3)|x|,|v|>(1/2)|x|)}} G_p(u) p D(v-u) G_p(x-v)
\end{aligned} \tag{2.44}$$

$$\leq c_x p \sup_{v\,:\,|v|\leq(1/2)|x|} G_p(x-v) + \chi_p^2 p \sup_{\substack{\{u,v\}\subset\mathbb{Z}^d \\ (|u|\leq(1/3)|x|,|v|>(1/2)|x|)}} D(v-u)$$

$$\leq 2^{-d-\alpha}\varepsilon \sup_{v\,:\,|v|>(1/2)|x|} G_p(v) + \frac{C_p}{|||x|||_L^{d+\alpha}}$$

for some $C_p = O(\chi_p^2)$. If $2\tilde{\ell} \leq |x| < 4\tilde{\ell}$, then we use (2.44) twice to obtain

$$\begin{aligned} G_p(x) &\leq \big(2^{-d-\alpha}\varepsilon\big)^2 \sup_{v\,:\,|v|>(1/4)|x|} G_p(v) \\ &\quad + 2^{-d-\alpha}\varepsilon \frac{C_p}{|||x/2|||_L^{d+\alpha}} + \frac{C_p}{|||x|||_L^{d+\alpha}} \\ &= \big(2^{-d-\alpha}\varepsilon\big)^2 \sup_{v\,:\,|v|>(1/4)|x|} G_p(v) + (1+\varepsilon)\frac{C_p}{|||x|||_L^{d+\alpha}}. \end{aligned} \tag{2.45}$$

In general, if $2^{n-1}\tilde{\ell} \leq |x| < 2^n\tilde{\ell}$ for some $n \in \mathbb{N}$, then we repeatedly use (2.44) to obtain

$$\begin{aligned} G_p(x) &\leq \big(2^{-d-\alpha}\varepsilon\big)^n \sup_{v\,:\,|v|>(1/2^n)|x|} G_p(v) \\ &\quad + \big(1+\varepsilon+\cdots+\varepsilon^{n-1}\big)\frac{C_p}{|||x|||_L^{d+\alpha}} \\ &\leq \frac{\tilde{\ell}^{d+\alpha}}{|||x|||_L^{d+\alpha}}\chi_p + \frac{C_p}{(1-\varepsilon)|||x|||_L^{d+\alpha}}. \end{aligned} \tag{2.46}$$

For $|x| < \tilde{\ell}$, we use the trivial inequality $G_p(x) \leq \chi_p \leq \tilde{\ell}^{d+\alpha}\chi_p/|||x|||_L^{d+\alpha}$. This completes the proof of (2.36), where $K_p = \tilde{\ell}^{d+\alpha}\chi_p + C_p/(1-\varepsilon)$. □

3. Proof of the main result. In this section, we prove the asymptotic behavior (1.27) of G_{p_c} in high dimensions. To do so, we show in Section 3.2 that, if $d > d_c$ and $L \gg 1$, then G_p for $p \leq p_c$ obeys the same bound as in (2.1) on S_p for $p \leq 1$. Then, in Section 3.3, we show that the obtained infrared bound on G_{p_c} implies its asymptotic expression (1.27). The proofs rely on the lace expansion (1.12) for G_p.

3.1. *Bounds on Π_p assuming the infrared bound on G_p.* In this subsection, we assume the infrared bound on G_p and prove bounds on Π_p and related quantities, such as its sum $\hat{\Pi}_p(0) \equiv \sum_x \Pi_p(x)$, in high dimensions. Before stating this more precisely, we need introduce the following parameter for $\alpha > 0$, $\alpha \neq 2$ and $d > \alpha \wedge 2$ [cf., (2.1)]:

$$\lambda = \sup_{x\neq o} \frac{S_1(x)}{|||x|||_L^{\alpha\wedge 2-d}} = O\big(L^{-\alpha\wedge 2}\big). \tag{3.1}$$

PROPOSITION 3.1. *Let $\alpha > 0$, $\alpha \neq 2$ and $d > d_c$, and assume the properties* (1.16) *and* (1.18) *in Assumption* 1.1. *Suppose that*

$$p \leq 3, \qquad G_p(x) \leq 3\lambda |||x|||_L^{\alpha\wedge 2-d} \qquad [x \neq o]. \tag{3.2}$$

If $\lambda \ll 1$ (i.e., $L \gg 1$), then, for any $x \in \mathbb{Z}^d$,

$$(pD * G_p)(x) \leq O(\lambda)|||x|||_L^{\alpha\wedge 2-d}, \tag{3.3}$$

$$\left|\Pi_p(x) - \delta_{o,x}\right| \leq O\left(L^{-d}\right)\delta_{o,x} + O\left(\lambda^\ell\right)|||x|||_L^{(\alpha\wedge 2-d)\ell}, \tag{3.4}$$

where $\ell = 2$ for percolation and $\ell = 3$ for SAW and the Ising model. As a result,

$$\hat{\Pi}_p(0) = 1 + O\left(L^{-d}\right), \tag{3.5}$$

$$\begin{aligned} \bar{\nabla}^{\alpha\wedge 2}\hat{\Pi}_p(0) &\equiv \lim_{|k|\to 0} \frac{\hat{\Pi}_p(0) - \hat{\Pi}_p(k)}{1 - \hat{D}(k)} \\ &= \begin{cases} 0, & [\alpha < 2], \\ \dfrac{1}{\sigma^2}\displaystyle\sum_x |x|^2 \Pi_p(x) = O\left(L^{-d(\ell-1)}\right), & [\alpha > 2]. \end{cases} \end{aligned} \tag{3.6}$$

We prove this proposition by using the following lemma, which is an improved version of [18], Proposition 1.7.

LEMMA 3.2. (i) *For any $a \geq b > 0$ with $a + b > d$, there is an L-independent constant $C = C(a, b, d) < \infty$ such that*

$$\sum_{y\in\mathbb{Z}^d} |||x - y|||_L^{-a} |||y|||_L^{-b} \leq \begin{cases} CL^{d-a}|||x|||_L^{-b}, & [a > d], \\ C|||x|||_L^{d-a-b}, & [a < d]. \end{cases} \tag{3.7}$$

(ii) *Let f and g be functions on $\mathbb{Z}^d$, with g being $\mathbb{Z}^d$-symmetric. Suppose that there are $C_1, C_2, C_3 > 0$ and $\rho > 0$ such that*

$$f(x) = C_1|||x|||_L^{\alpha\wedge 2-d}, \qquad \left|g(x)\right| \leq C_2\delta_{o,x} + C_3|||x|||_L^{-d-\rho}. \tag{3.8}$$

Then there is a $\rho' \in (0, \rho \wedge 2)$ such that, for $d > \alpha \wedge 2$,

$$(f * g)(x) = \frac{C_1\|g\|_1}{|||x|||_L^{d-\alpha\wedge 2}} + \frac{O(C_1C_3)}{|||x|||_L^{d-\alpha\wedge 2+\rho'}}. \tag{3.9}$$

PROOF OF PROPOSITION 3.1. First, we note that

$$D(x) = \frac{O(L^\alpha)}{|||x|||_L^{d+\alpha}} = \frac{O(L^\alpha)|||x|||_L^{-\alpha-\alpha\wedge 2}}{|||x|||_L^{d-\alpha\wedge 2}} \leq \frac{O(\lambda)}{|||x|||_L^{d-\alpha\wedge 2}}. \tag{3.10}$$

We also note that the identity $G_p(y) = \delta_{o,y} + G_p(y)\mathbb{1}_{\{y\neq o\}}$ holds for all three models. Therefore, by using the assumed bound (3.2) and Lemma 3.2(i), we ob-

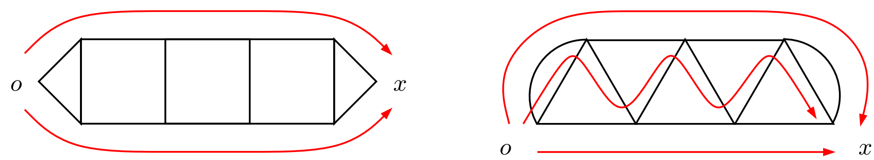

FIG. 1. *The left figure is an example of the lace-expansion diagrams for percolation, and the right one is for SAW and the Ising model. The number ℓ of disjoint paths from o to x using different sets of line segments is* 2 *in the left figure and* 3 *in the right figure.*

tain (3.3) as

$$
\begin{aligned}
(D * G_p)(x) &= D(x) + \sum_{y \neq o} D(x-y) G_p(y) \\
&\le \frac{O(\lambda)}{|||x|||_L^{d-\alpha\wedge 2}} + \sum_{y \in \mathbb{Z}^d} \frac{O(L^\alpha)}{|||x-y|||_L^{d+\alpha}} \frac{3\lambda}{|||y|||_L^{d-\alpha\wedge 2}} \qquad (3.11)\\
&\le \frac{O(\lambda)}{|||x|||_L^{d-\alpha\wedge 2}}.
\end{aligned}
$$

Inequality (3.4) is obtained by repeatedly applying (3.2)–(3.3) and Lemma 3.2(i) to the diagrammatic bounds on $\Pi_p(x)$ in [18, 24] ($\Pi_p(x)$ in this paper equals $\delta_{o,x} + \Pi_z(x)$ in [18], Proposition 1.8), where ℓ is the number of disjoint paths in the diagrams from o to x (cf., Figure 1). The proof is quite similar to [18], Proposition 1.8 and [24], Proposition 3.1; the only difference is the use of $|||\cdot|||_L$ instead of $|||\cdot|||_1$ and Lemma 3.2(i). Because of this, we gain the factor $O(L^{-d})(= O(\lambda)|||o|||_L^{\alpha\wedge 2-d})$ in (3.4), which is much smaller than $O(\lambda)$ as claimed in [18, 24].

It remains to prove (3.5)–(3.6). By (3.4), we readily obtain (3.5) as

$$
\hat{\Pi}_p(0) \equiv \sum_{x \in \mathbb{Z}^d} \Pi_p(x) = 1 + O(L^{-d}) + O(L^{-d(\ell-1)}) = 1 + O(L^{-d}). \qquad (3.12)
$$

Moreover,

$$
\begin{aligned}
&\left|\hat{\Pi}_p(0) - \hat{\Pi}_p(k)\right| \\
&\quad \equiv \left| \sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) \Pi_p(x) \right| \le O(\lambda^\ell) \sum_{x \in \mathbb{Z}^d} \frac{1 - \cos(k \cdot x)}{|||x|||_L^{(d-\alpha\wedge 2)\ell}}. \qquad (3.13)
\end{aligned}
$$

If $\alpha < 2$, then there is a $\delta \in (0, (2-\alpha) \wedge ((\ell-1)(d-d_c)))$ such that $1 - \cos(k \cdot x) \le O(|k \cdot x|^{\alpha+\delta})$, hence

$$
\begin{aligned}
&\left|\hat{\Pi}_p(0) - \hat{\Pi}_p(k)\right| \\
&\quad \le O(|k|^{\alpha+\delta}) \Biggl(L^{-d\ell} \sum_{x : |x| \le L} |x|^{\alpha+\delta} + L^{-\alpha\ell} \sum_{x : |x| > L} \frac{|x|^{\alpha+\delta}}{|x|^{(d-\alpha)\ell}} \Biggr) \qquad (3.14)\\
&\quad = O(L^{-d(\ell-1)+\alpha+\delta}) |k|^{\alpha+\delta}.
\end{aligned}
$$

If $\alpha > 2$, then there is a $\delta \in (0, 2 \wedge ((\ell-1)(d-d_c)))$ such that $1-\cos(k\cdot x) = \frac{1}{2}|k\cdot x|^2 + O(|k\cdot x|^{2+\delta})$ and, therefore,

$$\begin{aligned}
&\hat{\Pi}_p(0) - \hat{\Pi}_p(k) \\
&\quad = \frac{1}{2}\sum_{x\in\mathbb{Z}^d} |k\cdot x|^2 \Pi_p(x) + O\big(L^{-2\ell}\big)|k|^{2+\delta} \sum_{x\in\mathbb{Z}^d} \frac{|x|^{2+\delta}}{|||x|||_L^{(d-2)\ell}} \\
&\quad = \frac{|k|^2}{2d}\sum_{x\in\mathbb{Z}^d} |x|^2 \Pi_p(x) + O\big(L^{-d(\ell-1)+2+\delta}\big)|k|^{2+\delta}.
\end{aligned} \tag{3.15}$$

Then, by the above estimates and (1.18), we obtain

$$\begin{aligned}
&\frac{\hat{\Pi}_p(0) - \hat{\Pi}_p(k)}{1-\hat{D}(k)} \\
&\quad = \begin{cases} O\big(L^{-d(\ell-1)+\delta}\big)|k|^{\delta}, & [\alpha < 2], \\ \dfrac{1}{\sigma^2}\displaystyle\sum_x |x|^2 \Pi_p(x) + O\big(L^{-d(\ell-1)+\delta}\big)|k|^{\delta}, & [\alpha > 2], \end{cases}
\end{aligned} \tag{3.16}$$

hence (3.6) by taking $|k| \to 0$. This completes the proof of Proposition 3.1. □

PROOF OF LEMMA 3.2. The proof of (3.7) is almost identical to that of [18], Proposition 1.7(i). However, since we are using $|||\cdot|||_L$ rather than $|||\cdot|||_1$ as in [18], we can gain the extra factor L^{d-a} for $a > d$ in (3.7). To clarify this, we include the proof here. First of all, since $a \geq b$, we have

$$\begin{aligned}
&\sum_{y\in\mathbb{Z}^d} |||x-y|||_L^{-a} |||y|||_L^{-b} \\
&\quad \leq \sum_{y:|x-y|\leq|y|} |||x-y|||_L^{-a} |||y|||_L^{-b} + \sum_{y:|x-y|>|y|} |||x-y|||_L^{-a} |||y|||_L^{-b} \\
&\quad \leq 2 \sum_{y:|x-y|\leq|y|} |||x-y|||_L^{-a} |||y|||_L^{-b}.
\end{aligned} \tag{3.17}$$

Since $|x-y| \leq |y|$ implies $|y| \geq \frac{1}{2}|x|$, we obtain that, for $a > d$,

$$\begin{aligned}
&\sum_{y:|x-y|\leq|y|} |||x-y|||_L^{-a} |||y|||_L^{-b} \\
&\quad \leq 2^b |||x|||_L^{-b} \sum_{y\in\mathbb{Z}^d} |||x-y|||_L^{-a} \\
&\quad = \frac{C}{2} L^{d-a} |||x|||_L^{-b}.
\end{aligned} \tag{3.18}$$

For $a < d$, on the other hand, we use the identity $1 = \mathbb{1}_{\{|y| \le (3/2)|x|\}} + \mathbb{1}_{\{|y| > (3/2)|x|\}}$ and the fact that $|y| > \frac{3}{2}|x|$ implies $|x-y| \ge \frac{1}{3}|y|$. Then we obtain

$$\begin{aligned}
&\sum_{y\,:\,|x-y|\le|y|} |||x-y|||_L^{-a} |||y|||_L^{-b} \\
&\quad\le 2^b |||x|||_L^{-b} \sum_{y\,:\,|x-y|\le|y|} |||x-y|||_L^{-a} \mathbb{1}_{\{|y|\le(3/2)|x|\}} \\
&\qquad + 3^a \sum_{y\,:\,|x-y|\le|y|} |||y|||_L^{-a-b} \mathbb{1}_{\{|y|>(3/2)|x|\}} \\
&\quad\le 2^b |||x|||_L^{-b} \sum_{y\,:\,|x-y|\le(3/2)|x|} |||x-y|||_L^{-a} \\
&\qquad + 3^a \sum_{y\,:\,|y|>(3/2)|x|} |||y|||_L^{-a-b} \\
&\quad\le \frac{C}{2} |||x|||_L^{d-a-b}.
\end{aligned} \tag{3.19}$$

This completes the proof of (3.7).

The proof of (3.9) is also quite similar to that of [18], Proposition 1.7(ii), where [18], (5.8), is used. However, [18], (5.8), is valid only for $d > 4$, not $d > 2$ as claimed in [18], Proposition 1.7(ii). In fact, it is not difficult to avoid this problem, and we include the proof here to clarify this. First, we note that

$$(f * g)(x) = \|g\|_1 f(x) + \sum_{y\in\mathbb{Z}^d} g(y)\big(f(x-y) - f(x)\big). \tag{3.20}$$

To prove (3.9), it suffices to show that the sum in the right-hand side is the error term in (3.9). For that, we split the sum into the following three sums:

$$\begin{aligned}
\sum_{y\in\mathbb{Z}^d} &= \sum_{y\,:\,|y|\le(1/3)|x|} + \sum_{y\,:\,|x-y|\le(1/3)|x|} + \sum_{y\,:\,|y|\wedge|x-y|>(1/3)|x|} \\
&\equiv \sum_y{}' + \sum_y{}'' + \sum_y{}'''.
\end{aligned} \tag{3.21}$$

It is not difficult to estimate the last two sums, as

$$\begin{aligned}
&\left|\sum_y{}'' g(y)\big(f(x-y) - f(x)\big)\right| \\
&\quad\le \frac{O(C_3)}{|||x|||_L^{d+\rho}} \sum_{y\,:\,|x-y|\le(1/3)|x|} \big(f(x-y) + f(x)\big) \\
&\quad\le \frac{O(C_1 C_3)}{|||x|||_L^{d-\alpha\wedge 2+\rho}}
\end{aligned} \tag{3.22}$$

and

$$
\begin{aligned}
&\Big|\sum_y{}''' g(y)\big(f(x-y)-f(x)\big)\Big| \\
&\quad \le \frac{O(C_1)}{|||x|||_L^{d-\alpha\wedge 2}} \sum_{y:\,|y|>(1/3)|x|} \frac{C_3}{|||y|||_L^{d+\rho}} \\
&\quad \le \frac{O(C_1C_3)}{|||x|||_L^{d-\alpha\wedge 2+\rho}}.
\end{aligned}
\tag{3.23}
$$

To estimate the sum $\sum_y'$, we use the $\mathbb{Z}^d$-symmetry of g to obtain

$$
\begin{aligned}
&\sum_y{}' g(y)\big(f(x-y)-f(x)\big) \\
&\quad = \sum_{y:\,0<|y|\le(1/3)|x|} g(y)\bigg(\frac{f(x+y)+f(x-y)}{2}-f(x)\bigg).
\end{aligned}
\tag{3.24}
$$

Notice that

$$
\begin{aligned}
&\bigg|\frac{f(x+y)+f(x-y)}{2}-f(x)\bigg| \\
&\quad \le \frac{O(C_1)}{|||x|||_L^{d-\alpha\wedge 2}} \times \begin{cases} 1, & \Big[|x|\le \frac{3}{2}L\Big], \\ |y|^2/|x|^2, & \Big[|x|\ge \frac{3}{2}L\Big]. \end{cases}
\end{aligned}
\tag{3.25}
$$

To verify this for $|x|\le\frac{3}{2}L$, we simply bound each f by $O(C_1)|||x|||_L^{\alpha\wedge 2-d}$. For $|x|\ge\frac{3}{2}L$, since $|x\pm y|\ge|x|-|y|\ge\frac{2}{3}|x|\ge L$, we have $f(x\pm y)=C_1|x\pm y|^{\alpha\wedge 2-d}$. Then, by Taylor's theorem, since $|\pm 2\frac{x\cdot y}{|x|^2}+\frac{|y|^4}{|x|^4}|\le\frac{7}{9}<1$, we have

$$
\begin{aligned}
|x\pm y|^{\alpha\wedge 2-d} &= |x|^{\alpha\wedge 2-d}\bigg(1\pm 2\frac{x\cdot y}{|x|^2}+\frac{|y|^4}{|x|^4}\bigg)^{(\alpha\wedge 2-d)/2} \\
&= |x|^{\alpha\wedge 2-d}\bigg(1\mp(d-\alpha\wedge 2)\frac{x\cdot y}{|x|^2}+O\bigg(\frac{|y|^2}{|x|^2}\bigg)\bigg)
\end{aligned}
\tag{3.26}
$$

and (3.25) follows. Therefore, if $|x|\le\frac{3}{2}L$, then $|y|\le\frac{1}{2}L$ and we obtain

$$
\begin{aligned}
\Big|\sum_y{}' g(y)\big(f(x-y)-f(x)\big)\Big| &\le \frac{O(C_1)}{|||x|||_L^{d-\alpha\wedge 2}} \sum_{y:\,0<|y|\le(1/2)L} \frac{C_3}{L^{d+\rho}} \\
&\le \frac{O(C_1C_3)}{|||x|||_L^{d-\alpha\wedge 2+\rho}}.
\end{aligned}
\tag{3.27}
$$

If $|x| \geq \frac{3}{2}L$, then $|||x|||_L = |x|$ and we obtain

$$\begin{aligned}
&\Bigl|\sum_y{}' g(y)\bigl(f(x-y)-f(x)\bigr)\Bigr| \\
&\quad \leq \frac{O(C_1)}{|||x|||_L^{d-\alpha\wedge 2+2}} \sum_{y\,:\,0<|y|\leq(1/3)|x|} |y|^2 \biggl(\frac{C_3 \mathbb{1}_{\{|y|\leq L\}}}{L^{d+\rho}} + \frac{C_3 \mathbb{1}_{\{|y|>L\}}}{|y|^{d+\rho}}\biggr) \qquad (3.28)\\
&\quad \leq \frac{O(C_1 C_3)}{|||x|||_L^{d-\alpha\wedge 2+2}} \times \begin{cases} L^{-\rho+2}, & [\rho > 2], \\ \log|x|, & [\rho = 2], \\ |x|^{2-\rho}, & [\rho < 2]. \end{cases}
\end{aligned}$$

Summarizing the above yields (3.9). This completes the proof of Lemma 3.2. □

3.2. *Proof of the infrared bound on G_p.* In this subsection, we prove that the hypothesis of Proposition 3.1 indeed holds for $p \leq p_c$ in high dimensions. The precise statement is the following.

THEOREM 3.3. *Let $\alpha > 0$, $\alpha \neq 2$ and $d > d_c$, and assume the properties* (1.16), (1.18) *and* (1.24) *in Assumption* 1.1. *Then, for $L \gg 1$ and $p \leq p_c$,*

$$G_p(x) \leq O\bigl(L^{-\alpha\wedge 2}\bigr)|||x|||_L^{\alpha\wedge 2-d} \qquad [x \neq o]. \tag{3.29}$$

PROOF. Let

$$g_p = p \vee \sup_{x\neq o} \frac{G_p(x)}{\lambda |||x|||_L^{\alpha\wedge 2-d}}, \tag{3.30}$$

where we recall the definition (3.1) of λ. Suppose that the following properties hold:

(i) g_p is continuous (and nondecreasing) in $p \in [1, p_c)$.
(ii) $g_1 \leq 1$.
(iii) If $\lambda \ll 1$ (i.e., $L \gg 1$), then $g_p \leq 3$ implies $g_p \leq 2$ for every $p \in (1, p_c)$.

If the above properties hold, then in fact $g_p \leq 2$ for all $p < p_c$, as long as $d > d_c$ and $\lambda \ll 1$. In particular, $G_p(x) \leq 2\lambda |||x|||_L^{\alpha\wedge 2-d}$ for all $x \neq o$ and $p < p_c$ (≤ 2). By Lemma 2.2, we can extend this bound up to $p = p_c$, hence the proof completed.

Now we verify those properties (i)–(iii).

Verification of (i). It suffices to show that, for every $p_0 \in (1, p_c)$, $\sup_{x\neq o} G_p(x)/|||x|||_L^{\alpha\wedge 2-d}$ is continuous in $p \in [1, p_0]$. By the monotonicity of $G_p(x)$ in $p \leq p_0$ and using Lemma 2.4, we have

$$\frac{G_p(x)}{|||x|||_L^{\alpha\wedge 2-d}} \leq \frac{G_{p_0}(x)}{|||x|||_L^{\alpha\wedge 2-d}} \leq \frac{K_{p_0}|||x|||_L^{-d-\alpha}}{|||x|||_L^{\alpha\wedge 2-d}} = \frac{K_{p_0}}{|||x|||_L^{\alpha+\alpha\wedge 2}}. \tag{3.31}$$

On the other hand, for any $x_0 \neq o$ with $D(x_0) > 0$, there exists an $R = R(p_0, x_0) < \infty$ such that, for all $|x| \geq R$,

$$\frac{K_{p_0}}{|||x|||_L^{\alpha+\alpha\wedge 2}} \leq \frac{D(x_0)}{|||x_0|||_L^{\alpha\wedge 2-d}}. \tag{3.32}$$

Moreover, by using $p \geq 1$ and the lower bound of the second inequality in (2.25), we have

$$\frac{D(x_0)}{|||x_0|||_L^{\alpha\wedge 2-d}} \leq \frac{pD(x_0)}{|||x_0|||_L^{\alpha\wedge 2-d}} \leq \frac{G_p(x_0)}{|||x_0|||_L^{\alpha\wedge 2-d}}. \tag{3.33}$$

As a result, for any $p \in [1, p_0]$, we obtain

$$\sup_{x\neq o} \frac{G_p(x)}{|||x|||_L^{\alpha\wedge 2-d}} = \frac{G_p(x_0)}{|||x_0|||_L^{\alpha\wedge 2-d}} \vee \max_{x\,:\,0<|x|<R} \frac{G_p(x)}{|||x|||_L^{\alpha\wedge 2-d}}. \tag{3.34}$$

Since $G_p(x)$ is continuous in p (cf., Lemma 2.2) and the maximum of finitely many continuous functions is continuous, we can conclude that g_p is continuous in $p \in [1, p_0]$, as required.

Verification of (ii). By the first inequality in (2.25) and the definition (3.1) of λ, we readily obtain

$$g_1 = 1 \vee \sup_{x\neq o} \frac{G_1(x)}{\lambda |||x|||_L^{\alpha\wedge 2-d}} \leq 1 \vee \sup_{x\neq o} \frac{S_1(x)}{\lambda |||x|||_L^{\alpha\wedge 2-d}} = 1 \tag{3.35}$$

as required.

Verification of (iii). If $d > d_{\mathrm{c}}$, $\lambda \ll 1$ and $g_p \leq 3$, then, by Proposition 3.1, Π_p satisfies (3.4)–(3.6) as well as (3.16). We use these estimates and the lace expansion to prove $g_p \leq 2$ as follows.

First, we recall (1.12) and (1.31):

$$G_p = \Pi_p + \Pi_p * pD * G_p, \qquad S_p = \delta + pD * S_p, \tag{3.36}$$

or equivalently

$$\Pi_p = G_p * (\delta - \Pi_p * pD), \qquad \delta = (\delta - pD) * S_p. \tag{3.37}$$

Inspired by the similarity of the above identities, we approximate G_p to $r\Pi_p * S_q$ with some constant $r \in (0, \infty)$ and the parameter change $q \in [0, 1]$. Rewrite G_p as follows:

$$\begin{aligned} G_p &= r\Pi_p * S_q + G_p * \delta - r\Pi_p * S_q \\ &= r\Pi_p * S_q + G_p * (\delta - qD) * S_q - rG_p * (\delta - \Pi_p * pD) * S_q \\ &= r\Pi_p * S_q + G_p * E_{p,q,r} * S_q, \end{aligned} \tag{3.38}$$

where

$$E_{p,q,r} = \delta - qD - r(\delta - \Pi_p * pD). \tag{3.39}$$

We choose q, r to satisfy

$$\hat{E}_{p,q,r}(0) = \bar{\nabla}^{\alpha\wedge 2}\hat{E}_{p,q,r}(0) = 0, \tag{3.40}$$

or equivalently

$$\begin{cases} 1 - q - r\big(1 - \hat{\Pi}_p(0)p\big) = 0, \\ -q + r\big(\hat{\Pi}_p(0) + \bar{\nabla}^{\alpha\wedge 2}\hat{\Pi}_p(0)\big)p = 0. \end{cases} \tag{3.41}$$

Solving these simultaneous equations for r and using (3.6), we obtain

$$r = \big(1 + p\bar{\nabla}^{\alpha\wedge 2}\hat{\Pi}_p(0)\big)^{-1} = 1 + \begin{cases} 0, & [\alpha < 2], \\ O\big(L^{-d(\ell-1)}\big), & [\alpha > 2]. \end{cases} \tag{3.42}$$

On the other hand, by taking the Fourier transform of (3.36) and setting $k = 0$, we obtain

$$\chi_p = \hat{\Pi}_p(0) + \hat{\Pi}_p(0)p\chi_p, \tag{3.43}$$

or equivalently $\hat{\Pi}_p(0) = \chi_p/(1 + p\chi_p)$ and, therefore,

$$q = 1 - r\big(1 - \hat{\Pi}_p(0)p\big) = 1 - \frac{r}{1 + p\chi_p} \in (0, 1], \tag{3.44}$$

where we have used $p \geq 1$, $\chi_p \geq 1$ and (3.42) to guarantee the positivity (by taking $L \gg 1$ if $\alpha > 2$).

In addition, by solving (3.43) for χ_p and using (3.5), we have

$$\chi_p = \frac{\hat{\Pi}_p(0)}{1 - \hat{\Pi}_p(0)p} = \frac{1 + O(L^{-d})}{1 - \hat{\Pi}_p(0)p}, \tag{3.45}$$

hence $1 - \hat{\Pi}_p(0)p \geq 0$. In particular, $p \leq \hat{\Pi}_p(0)^{-1} = 1 + O(L^{-d}) \leq 2$, as required.

It remains to prove $G_p(x) \leq 2\lambda |||x|||_L^{\alpha\wedge 2-d}$. To do so, we use the following property of $E_{p,q,r}$.

PROPOSITION 3.4. *Let q, r be defined as in (3.42)–(3.44). Under the hypothesis of Proposition 3.1, there is a $\rho \in (0, \alpha \wedge 2)$ such that*

$$\big|(E_{p,q,r} * S_q)(x)\big| \leq O\big(L^{-d(\ell-1)}\big)\bigg(\mathbb{1}_{\{\alpha>2\}}\delta_{o,x} + \frac{L^\rho}{|||x|||_L^{d+\rho}}\bigg). \tag{3.46}$$

For now, we assume this proposition and complete verifying the property (iii). First, by rearranging (3.38) and using $S_q \le S_1$ as well as (3.5) and (3.42) for $L \gg 1$, we obtain

$$\begin{aligned} G_p &= r\Pi_p * S_q + G_p * E_{p,q,r} * S_q \\ &= r\hat{\Pi}_p(0)S_q - r\big(\hat{\Pi}_p(0)\delta - \Pi_p\big) * S_q + G_p * E_{p,q,r} * S_q \\ &\le \big(1 + O\big(L^{-d}\big)\big)S_1 - r\big(\hat{\Pi}_p(0)\delta - \Pi_p\big) * S_q + G_p * E_{p,q,r} * S_q. \end{aligned} \tag{3.47}$$

Then, by Proposition 3.4 and Lemma 3.2(i), the third term is bounded as

$$\begin{aligned} &\big|(G_p * E_{p,q,r} * S_q)(x)\big| \\ &\quad\le O\big(L^{-d(\ell-1)}\big) \sum_{y\in\mathbb{Z}^d} \frac{3\lambda}{\|\!|y|\!\|_L^{d-\alpha\wedge 2}} \bigg(\delta_{y,x} + \frac{L^\rho}{\|\!|x-y|\!\|_L^{d+\rho}}\bigg) \\ &\quad\le \frac{O(L^{-d(\ell-1)})\lambda}{\|\!|x|\!\|_L^{d-\alpha\wedge 2}}. \end{aligned} \tag{3.48}$$

Also, by (3.4) and Lemma 3.2(i), the second term in (3.47) is bounded as

$$\begin{aligned} &\big|\big(\big(\hat{\Pi}_p(0)\delta - \Pi_p\big) * S_q\big)(x)\big| \\ &\quad= \bigg|\sum_{y\ne o} \Pi_p(y)\big(S_q(x) - S_q(x-y)\big)\bigg| \\ &\quad\le \sum_{y\ne o}\big|\Pi_p(y)\big|S_q(x) + \sum_{y\ne o}\big|\Pi_p(y)\big|S_q(x-y) \\ &\quad\le \frac{O(L^{-d(\ell-1)})\lambda}{\|\!|x|\!\|_L^{d-\alpha\wedge 2}}. \end{aligned} \tag{3.49}$$

Putting these estimates back into (3.47), we obtain that, for $L \gg 1$,

$$G_p(x) \le \big(1 + O\big(L^{-d}\big)\big)\frac{\lambda}{\|\!|x|\!\|_L^{d-\alpha\wedge 2}} + \frac{O(L^{-d(\ell-1)})\lambda}{\|\!|x|\!\|_L^{d-\alpha\wedge 2}} \le \frac{2\lambda}{\|\!|x|\!\|_L^{d-\alpha\wedge 2}} \tag{3.50}$$

as required. This completes the proof of Theorem 3.3 assuming Proposition 3.4. $\square$

PROOF OF PROPOSITION 3.4. First, by substituting $q = 1 - r(1 - \hat{\Pi}_p(0)p)$ [cf., (3.44)] into (3.39) and using $1 - r = pr\bar{\nabla}^{\alpha\wedge 2}\hat{\Pi}_p(0)$ [cf., (3.42)], we obtain

$$E_{p,q,r} = pr\big(\bar{\nabla}^{\alpha\wedge 2}\hat{\Pi}_p(0)(\delta - D) - \big(\hat{\Pi}_p(0)\delta - \Pi_p\big) * D\big). \tag{3.51}$$

Using this representation, we prove (3.46) for $|x| \le 2L$ and $|x| > 2L$, separately.

For $|x| \le 2L$, we simply use (2.1) to bound $|(E_{p,q,r} * S_q)(x)|$ by

$$\big|(E_{p,q,r} * S_q)(x)\big| \le \big|E_{p,q,r}(x)\big| + O\big(L^{-d}\big) \sum_{y\in\mathbb{Z}^d} \big|E_{p,q,r}(y)\big|. \tag{3.52}$$

By (3.51), we have

$$\begin{aligned} &\frac{|E_{p,q,r}(x)|}{pr} \\ &\quad \le \big|\bar\nabla^{\alpha\wedge 2}\hat\Pi_p(0)\big|\big(\delta_{o,x} + D(x)\big) + \big|\big(\big(\hat\Pi_p(0)\delta - \Pi_p\big) * D\big)(x)\big| \\ &\quad = \big|\bar\nabla^{\alpha\wedge 2}\hat\Pi_p(0)\big|\big(\delta_{o,x} + D(x)\big) + \bigg|\sum_{z\ne o} \Pi_p(z)\big(D(x) - D(x-z)\big)\bigg| \\ &\quad \le \big|\bar\nabla^{\alpha\wedge 2}\hat\Pi_p(0)\big|\big(\delta_{o,x} + D(x)\big) + \sum_{z\ne o}\big|\Pi_p(z)\big|\big(D(x) + D(x-z)\big). \end{aligned} \tag{3.53}$$

Using (3.4)–(3.6) and (3.42), we obtain that

$$\begin{aligned} &\big|E_{p,q,r}(x)\big| \\ &\quad \le O\big(L^{-d(\ell-1)}\big)\mathbb{1}_{\{\alpha>2\}}\big(\delta_{o,x} + O\big(L^{-d}\big)\big) + O\big(L^{-d}\big)\sum_{z\ne o}\big|\Pi_p(z)\big| \\ &\quad \le O\big(L^{-d(\ell-1)}\big)\mathbb{1}_{\{\alpha>2\}}\delta_{o,x} + O\big(L^{-d\ell}\big) \end{aligned} \tag{3.54}$$

and that, by summing (3.53) over $x \in \mathbb{Z}^d$,

$$\begin{aligned} &O\big(L^{-d}\big) \sum_{x\in\mathbb{Z}^d} \big|E_{p,q,r}(x)\big| \\ &\quad \le O\big(L^{-d}\big)\bigg(2\big|\bar\nabla^{\alpha\wedge 2}\hat\Pi_p(0)\big| + 2\sum_{z\ne o}\big|\Pi_p(z)\big|\bigg) \\ &\quad \le O\big(L^{-d\ell}\big). \end{aligned} \tag{3.55}$$

Therefore, for $|x| \le 2L$,

$$\begin{aligned} \big|(E_{p,q,r} * S_q)(x)\big| &\le O\big(L^{-d(\ell-1)}\big)\mathbb{1}_{\{\alpha>2\}}\delta_{o,x} + O\big(L^{-d\ell}\big) \\ &\le O\big(L^{-d(\ell-1)}\big)\mathbb{1}_{\{\alpha>2\}}\delta_{o,x} + \frac{O(L^{-d(\ell-1)+\rho})}{|\!|\!|x|\!|\!|_L^{d+\rho}}. \end{aligned} \tag{3.56}$$

It remains to prove (3.46) for $|x| > 2L$. To do so, we first rewrite $(E_{p,q,r} * S_q)(x)$ as

$$\begin{aligned} (E_{p,q,r} * S_q)(x) &= \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d}\,\hat E_{p,q,r}(k)\frac{e^{-ik\cdot x}}{1 - q\hat D(k)} \\ &= \int_0^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d}\,\hat E_{p,q,r}(k)e^{-t(1-q\hat D(k))-ik\cdot x}. \end{aligned} \tag{3.57}$$

Then we split the integral with respect to t into $\int_0^T$ and $\int_T^\infty$, where T is arbitrary for now, but it will be determined shortly. For the latter integral, we use the Fourier transform of (3.51), which is

$$\hat{E}_{p,q,r}(k) = pr\big(1-\hat{D}(k)\big)\bigg(\bar{\nabla}^{\alpha\wedge 2}\hat{\Pi}_p(0) - \frac{\hat{\Pi}_p(0)-\hat{\Pi}_p(k)}{1-\hat{D}(k)}\hat{D}(k)\bigg). \tag{3.58}$$

Because of (1.18), (3.6) and (3.16), there is a $\delta > 0$ such that

$$\hat{E}_{p,q,r}(k) = O\big(L^{-d(\ell-1)+\alpha\wedge 2+\delta}\big)|k|^{\alpha\wedge 2+\delta} \qquad [\alpha \neq 2]. \tag{3.59}$$

Since $1-q\hat{D}(k) \geq q(1-\hat{D}(k))$, the contribution to (3.57) from the large-t integral is bounded as

$$\begin{aligned} &\bigg|\int_T^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d}\hat{E}_{p,q,r}(k)e^{-t(1-q\hat{D}(k))-ik\cdot x}\bigg| \\ &\qquad \leq O\big(L^{-d(\ell-1)+\alpha\wedge 2+\delta}\big)\int_T^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d}|k|^{\alpha\wedge 2+\delta}e^{-tq(1-\hat{D}(k))}. \end{aligned} \tag{3.60}$$

Since $p \geq 1$, we have $q \geq 1 - r/(1+\chi_1) \geq 1 - r/2$ [cf., (3.44)], which is bounded away from zero when $L \gg 1$. Therefore, by using (1.18), we obtain

$$\int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d}|k|^{\alpha\wedge 2+\delta}e^{-tq(1-\hat{D}(k))} = O\big(L^{\alpha\wedge 2}t\big)^{-1-((d+\delta)/(\alpha\wedge 2))}, \tag{3.61}$$

hence

$$\begin{aligned} &\bigg|\int_T^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d}\hat{E}_{p,q,r}(k)e^{-t(1-q\hat{D}(k))-ik\cdot x}\bigg| \\ &\qquad \leq O\big(L^{-d\ell}\big)T^{-(d+\delta)/(\alpha\wedge 2)}. \end{aligned} \tag{3.62}$$

Let

$$\rho = \frac{(\alpha\wedge 2)\delta}{d+\alpha\wedge 2+\delta}, \qquad T = \bigg(\frac{|x|}{L}\bigg)^{\alpha\wedge 2-\rho}. \tag{3.63}$$

Then, since $|x| > 2L$,

$$O\big(L^{-d\ell}\big)T^{-(d+\delta)/(\alpha\wedge 2)} = \frac{O(L^{-d(\ell-1)+\rho})}{|||x|||_L^{d+\rho}}. \tag{3.64}$$

To estimate the contribution to (3.57) from the small-t integral, we use the identity

$$\begin{aligned} &\int_0^T \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d}\hat{E}_{p,q,r}(k)e^{-t(1-q\hat{D}(k))-ik\cdot x} \\ &\qquad = \int_0^T \mathrm{d}t\, e^{-t}\sum_{n=0}^\infty \frac{(tq)^n}{n!}\big(E_{p,q,r} * D^{*n}\big)(x), \end{aligned} \tag{3.65}$$

where, by (3.51) and (3.6),

$$\begin{aligned}(E_{p,q,r} * D^{*n})(x) = {} & \underbrace{pr\bar{\nabla}^{\alpha\wedge 2}\hat{\varPi}_p(0)}_{O(L^{-d(\ell-1)})\mathbb{1}_{\{\alpha>2\}}} \sum_{y\in\mathbb{Z}^d} D(y)\big(D^{*n}(x) - D^{*n}(x-y)\big) \\ & - pr \sum_{y\in\mathbb{Z}^d} \varPi_p(y)\big(D^{*(n+1)}(x) - D^{*(n+1)}(x-y)\big).\end{aligned} \tag{3.66}$$

In the following, we use the decomposition (3.21) of $\sum_y$ and estimate the contribution to (3.65) from $\sum'_y$, $\sum''_y$ and $\sum'''_y$, separately.

First, we estimate the contribution from $\sum''_y \equiv \sum_{y:|x-y|\le(1/3)|x|}$. Since $|y| \ge |x| - |x-y| \ge \frac{2}{3}|x|$ in this domain of summation, we bound $|\varPi_p(y)|$ by $O(\lambda^\ell)|||x|||_L^{(\alpha\wedge 2-d)\ell}$ [cf., (3.4)] and then use (1.21), $\sum''_y 1 \le O(|||x|||_L^d)$ and $\sum''_y D^{*(n+1)}(x-y) \le 1$. As a result,

$$\begin{aligned}\bigg|\sum_y{}'' \varPi_p(y)\big(D^{*(n+1)}(x) - D^{*(n+1)}(x-y)\big)\bigg| & \le \frac{O(\lambda^\ell)}{|||x|||_L^{(d-\alpha\wedge 2)\ell}}\bigg(\frac{O(L^{\alpha\wedge 2})n}{|||x|||_L^{\alpha\wedge 2}} + 1\bigg) \\ & \le \frac{O(L^{-d(\ell-1)+\alpha\wedge 2})}{|||x|||_L^{d+\alpha\wedge 2}}\bigg(\frac{O(L^{\alpha\wedge 2})n}{|||x|||_L^{\alpha\wedge 2}} + 1\bigg).\end{aligned} \tag{3.67}$$

Similarly, for $\alpha > 2$,

$$\begin{aligned}O\big(L^{-d(\ell-1)}\big)\bigg|\sum_y{}'' D(y)\big(D^{*n}(x) - D^{*n}(x-y)\big)\bigg| & \le \frac{O(L^{-d(\ell-1)+\alpha})}{|||x|||_L^{d+\alpha}}\bigg(\frac{O(L^2)n}{|||x|||_L^2} + 1\bigg) \\ & \le \frac{O(L^{-d(\ell-1)+2})}{|||x|||_L^{d+2}}\bigg(\frac{O(L^2)n}{|||x|||_L^2} + 1\bigg).\end{aligned} \tag{3.68}$$

To estimate the contribution to (3.65) from $\sum'''_y \equiv \sum_{y:|y|\wedge|x-y|>(1/3)|x|}$ in (3.66), we bound $D^{*(n+1)}(x)$ and $D^{*(n+1)}(x-y)$ by $O(L^{\alpha\wedge 2})n/|||x|||_L^{d+\alpha\wedge 2}$ and then use (3.4) to bound $|\varPi_p(y)|$. The result is

$$\begin{aligned}\bigg|\sum_y{}''' \varPi_p(y)\big(D^{*(n+1)}(x) - D^{*(n+1)}(x-y)\big)\bigg| & \le \frac{O(L^{\alpha\wedge 2})n}{|||x|||_L^{d+\alpha\wedge 2}} \sum_{y:|y|>(1/3)|x|} \big|\varPi_p(y)\big| \\ & \le \frac{O(L^{-(\ell-1)(\alpha\wedge 2)})n}{|||x|||_L^{d+2(\alpha\wedge 2)+(\ell-1)(d-d_c)}} \le \frac{O(L^{-d(\ell-1)+2(\alpha\wedge 2)})n}{|||x|||_L^{d+2(\alpha\wedge 2)}}.\end{aligned} \tag{3.69}$$

Similarly, for $\alpha > 2$,

$$\begin{aligned}
&O\big(L^{-d(\ell-1)}\big)\bigg|\sum_y{}''' D(y)\big(D^{*n}(x) - D^{*n}(x-y)\big)\bigg| \\
&\quad\le \frac{O(L^{-d(\ell-1)+2})n}{|||x|||_L^{d+2}} \sum_{y:|y|>(1/3)|x|} D(y) \\
&\quad\le \frac{O(L^{-d(\ell-1)+4})n}{|||x|||_L^{d+4}}.
\end{aligned}\tag{3.70}$$

Finally, we estimate the contribution to (3.65) from $\sum_y' \equiv \sum_{y:|y|\le(1/3)|x|}$ in (3.66). By the $\mathbb{Z}^d$-symmetry of Π_p and using (3.4) and the assumption (1.24), we obtain

$$\begin{aligned}
&\bigg|\sum_y{}' \Pi_p(y)\big(D^{*(n+1)}(x) - D^{*(n+1)}(x-y)\big)\bigg| \\
&\quad= \bigg|\sum_y{}' \Pi_p(y)\bigg(D^{*(n+1)}(x) - \frac{D^{*(n+1)}(x+y) + D^{*(n+1)}(x-y)}{2}\bigg)\bigg| \\
&\quad\le \frac{O(L^{\alpha\wedge 2})n}{|||x|||_L^{d+\alpha\wedge 2+2}} \sum_{y:|y|\le(1/3)|x|} \frac{O(\lambda^\ell)|||y|||_L^2}{|||y|||_L^{(d-\alpha\wedge 2)\ell}} \\
&\quad\le \frac{O(L^{-(\ell-1)(\alpha\wedge 2)})n}{|||x|||_L^{d+\alpha\wedge 2+2}} \\
&\qquad\times \begin{cases} |||x|||_L^{d+2-(d-\alpha\wedge 2)\ell}, & [d+2 > (d-\alpha\wedge 2)\ell], \\ 1+\log(|||x|||_L/L), & [d+2 = (d-\alpha\wedge 2)\ell], \\ L^{d+2-(d-\alpha\wedge 2)\ell}, & [d+2 < (d-\alpha\wedge 2)\ell], \end{cases} \\
&\quad\le \frac{O(L^{-d(\ell-1)+2(\alpha\wedge 2)})n}{|||x|||_L^{d+2(\alpha\wedge 2)}},
\end{aligned}\tag{3.71}$$

where, to obtain the last inequality for $d+2 = (d-\alpha\wedge 2)\ell$, which implies $\alpha < 2$, we have used fact that $(|||x|||_L/L)^{\alpha-2}(1+\log(|||x|||_L/L))$ is bounded. Similarly, for $\alpha > 2$,

$$\begin{aligned}
&O\big(L^{-d(\ell-1)}\big)\bigg|\sum_y{}' D(y)\big(D^{*n}(x) - D^{*n}(x-y)\big)\bigg| \\
&\quad= O\big(L^{-d(\ell-1)}\big)\bigg|\sum_y{}' D(y)\bigg(D^{*n}(x) - \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2}\bigg)\bigg|
\end{aligned}\tag{3.72}$$

$$\leq \frac{O(L^{-d(\ell-1)+2})n}{|||x|||_L^{d+4}} \underbrace{\sum_{y:|y|\leq(1/3)|x|} \frac{O(L^{\alpha})|||y|||_L^2}{|||y|||_L^{d+\alpha}}}_{O(L^2)}$$

$$= \frac{O(L^{-d(\ell-1)+4})n}{|||x|||_L^{d+4}}.$$

Now, by putting these estimates back into (3.66), we obtain

$$\left|(E_{p,q,r} * D^{*n})(x)\right| \leq \frac{O(L^{-d(\ell-1)+\alpha\wedge 2})}{|||x|||_L^{d+\alpha\wedge 2}}\left(\frac{O(L^{\alpha\wedge 2})}{|||x|||_L^{\alpha\wedge 2}}n+1\right), \tag{3.73}$$

hence, by (3.63),

$$\begin{aligned} &\left|\int_0^T \mathrm{d}t\, e^{-t} \sum_{n=0}^{\infty} \frac{(tq)^n}{n!}(E_{p,q,r} * D^{*n})(x)\right| \\ &\qquad \leq \frac{O(L^{-d(\ell-1)+\alpha\wedge 2})}{|||x|||_L^{d+\alpha\wedge 2}}\left(\frac{O(L^{\alpha\wedge 2})}{|||x|||_L^{\alpha\wedge 2}}T^2+T\right) \\ &\qquad \leq \frac{O(L^{-d(\ell-1)+\alpha\wedge 2})}{|||x|||_L^{d+\alpha\wedge 2}}T = \frac{O(L^{-d(\ell-1)+\rho})}{|||x|||_L^{d+\rho}}. \end{aligned} \tag{3.74}$$

This completes the proof of Proposition 3.4. $\square$

3.3. *Derivation of the asymptotics of G_{p_c}.* Finally, we derive the asymptotic expression (1.27) for G_{p_c}. First, by repeatedly applying (3.38), we obtain

$$\begin{aligned} G_p &= r\Pi_p * S_q + G_p * E_{p,q,r} * S_q \\ &= r\Pi_p * S_q + (r\Pi_p * S_q + G_p * E_{p,q,r} * S_q) * E_{p,q,r} * S_q \\ &= r\Pi_p * S_q * (\delta + E_{p,q,r} * S_q) + G_p * (E_{p,q,r} * S_q)^{*2} \\ &\ \ \vdots \\ &= r\Pi_p * S_q * \sum_{n=0}^{N-1}(E_{p,q,r} * S_q)^{*n} + G_p * (E_{p,q,r} * S_q)^{*N}. \end{aligned} \tag{3.75}$$

By Proposition 3.4 and Lemma 3.2(i), we have that, for $p \leq p_c$,

$$\left|(E_{p,q,r} * S_q)^{*n}(x)\right| \leq O\left(L^{-d(\ell-1)n}\right)\left(\mathbb{1}_{\{\alpha>2\}}\delta_{o,x} + \frac{L^{\rho}}{|||x|||_L^{d+\rho}}\right), \tag{3.76}$$

hence, for any $N \in \mathbb{N}$,

$$\sum_{n=0}^{N-1}\left|(E_{p,q,r} * S_q)^{*n}(x)\right| \leq \left(1 + O\left(L^{-d(\ell-1)}\right)\right)\delta_{o,x} + \frac{O(L^{-d(\ell-1)+\rho})}{|||x|||_L^{d+\rho}}. \tag{3.77}$$

Therefore, we can take $N \to \infty$ to obtain that, for $p \le p_c$,

$$G_p = r\Pi_p * S_q * \sum_{n=0}^{\infty}(E_{p,q,r} * S_q)^{*n} \equiv H_p * S_q, \tag{3.78}$$

where, by (3.4) and (3.77),

$$\begin{aligned} H_p(x) &= r\left(\Pi_p * \sum_{n=0}^{\infty}(E_{p,q,r} * S_q)^{*n}\right)(x) \\ &= r\sum_{y\in\mathbb{Z}^d}\left(\big(1+O(L^{-d})\big)\delta_{o,y} + \frac{O(L^{-(\alpha\wedge 2)\ell})}{|||y|||_L^{(d-\alpha\wedge 2)\ell}}\right) \\ &\qquad \times \left(\big(1+O(L^{-d(\ell-1)})\big)\delta_{y,x} + \frac{O(L^{-d(\ell-1)+\rho})}{|||x-y|||_L^{d+\rho}}\right). \end{aligned} \tag{3.79}$$

Notice that, by Lemma 3.2(i) and using (3.42) and $d+\rho < (d-\alpha\wedge 2)\ell$,

$$H_p(x) = \big(r + O(L^{-d})\big)\delta_{o,x} + \frac{O(L^{-d(\ell-1)+\rho})}{|||x|||_L^{d+\rho}}. \tag{3.80}$$

Now we set $p = p_c$, so, by (3.44), $q = 1$. By Proposition 2.1 and Lemma 3.2(ii), we obtain the asymptotic expression

$$G_{p_c}(x) = \hat{H}_{p_c}(0)\frac{\gamma_\alpha/v_\alpha}{|||x|||_L^{d-\alpha\wedge 2}} + \frac{O(L^{-\alpha\wedge 2+\mu})}{|x|^{d-\alpha\wedge 2+\mu}} + \frac{O(L^{-d(\ell-1)-\alpha\wedge 2+\rho})}{|||x|||_L^{d-\alpha\wedge 2+\rho'}}. \tag{3.81}$$

Since H_{p_c} is absolutely summable, we can change the order of the limit and the sum as

$$\begin{aligned} \hat{H}_{p_c}(0) &= \lim_{|k|\to 0}\hat{H}_{p_c}(k) \\ &= \lim_{|k|\to 0} r\hat{\Pi}_{p_c}(k)\sum_{n=0}^{\infty}\big(\hat{E}_{p_c,1,r}(k)\hat{S}_1(k)\big)^n \\ &= r\hat{\Pi}_{p_c}(0) + r\hat{\Pi}_{p_c}(0)\sum_{n=1}^{\infty}\Big(\lim_{|k|\to 0}\hat{E}_{p_c,1,r}(k)\hat{S}_1(k)\Big)^n. \end{aligned} \tag{3.82}$$

By (3.45) and the fact that χ_p diverges as $p \uparrow p_c$, we have $\hat{\Pi}_{p_c}(0) = p_c^{-1}$. Moreover, by (3.58) and (3.16),

$$\hat{E}_{p_c,1,r}(k)\hat{S}_1(k) = \hat{E}_{p_c,1,r}(k)\big(1-\hat{D}(k)\big)^{-1} \underset{|k|\to 0}{\to} 0. \tag{3.83}$$

Therefore,

$$A = \hat{H}_{p_c}(0)^{-1} = \frac{p_c}{r} \equiv p_c\big(1 + p_c\bar{\nabla}^{\alpha\wedge 2}\hat{\Pi}_{p_c}(0)\big). \tag{3.84}$$

This completes the proof of Theorem 1.2.

APPENDIX: VERIFICATION OF ASSUMPTION 1.1

In this appendix, we show that the $\mathbb{Z}^d$-symmetric 1-step distribution D in (1.25), defined more precisely below, satisfies the properties (1.16), (1.18), (1.20), (1.21) and (1.24) in Assumption 1.1.

First, for $\alpha > 0$ and $\alpha \neq 2$, we define

$$T_\alpha(t) = \frac{t^{-1-\alpha/2}}{\sum_{s\in\mathbb{N}} s^{-1-\alpha/2}} \qquad [t \in \mathbb{N}]. \tag{A.1}$$

Next, let h be a nonnegative bounded function on $\mathbb{R}^d$ that is piecewise continuous, $\mathbb{Z}^d$-symmetric, supported in $[-1,1]^d$ and normalized [i.e., $\int_{[-1,1]^d} h(x)\,\mathrm{d}^d x = 1$]; for example, $h(x) = 2^{-d}\mathbb{1}_{\{\|x\|_\infty \le 1\}}$. Then, for large L (to ensure positivity of the denominator), we define

$$U_L(x) = \frac{h(x/L)}{\sum_{y\in\mathbb{Z}^d} h(y/L)} \qquad \bigl[x \in \mathbb{Z}^d\bigr], \tag{A.2}$$

where (cf., [11, 27])

$$\sigma_L^2 \equiv \sum_{x\in\mathbb{Z}^d} |x|^2 U_L(x) = O\bigl(L^2\bigr), \tag{A.3}$$

$$\hat{U}_L(k) \begin{cases} = 1 - \dfrac{\sigma_L^2}{2d}|k|^2 + O\bigl((L|k|)^{2+\zeta}\bigr), & \bigl[|k| \to 0\bigr], \\ \in (-1+\Delta, 1-\Delta), & \bigl[|k| \ge \sigma_L^{-1}\bigr] \end{cases} \tag{A.4}$$

for some $\zeta \in (0,2)$ and $\Delta \in (0,1)$. (The assumption $|\hat{U}_L(k)| < 1-\Delta$ is used only to get exponential decay of $\mathcal{I}_2$ in (A.28) below.) Combining these distributions, we define D as

$$D(x) = \sum_{t\in\mathbb{N}} U_L^{*t}(x) T_\alpha(t). \tag{A.5}$$

We note that the above definition is a discrete version of the transition kernel for the so-called subordinate process (e.g., [7]). Just like (A.5), the transition kernel for the subordinate process is given by an integral of the Gaussian density with respect to the 1-dimensional $\alpha/2$-stable distribution. Bogdan and Jakubowski [8] make the most of this integral representation to estimate derivatives of the transition kernel. This is close to what we want: to prove (1.24). However, in the current discrete space–time setting, we cannot simply adopt their proof to show (1.24). To overcome this difficulty, we will approximate the lattice distribution U_L^{*t} in (A.5) by a Gaussian density (multiplied by a polynomial) by using a discrete version of the Cramér–Edgeworth expansion [6], Corollary 22.3.

Before doing so, we first show that the above D satisfies (1.18) and (1.20).

VERIFICATION OF (1.18) AND (1.20). Due to the above definition of U_L, we can follow the same argument as in [27], Appendix A, to verify the bound on

$1-\hat{D}$ in (1.20). Moreover, if (1.18) is also verified, then we can follow the same argument as in [9], Appendix A, to confirm the bound on $\|D^{*n}\|_\infty$ in (1.20) as well.

It remains to verify (1.18) for small k. First, we note that

$$1-\hat{D}(k)=\sum_{t\in\mathbb{N}}(1-\hat{U}^t)T_\alpha(t)=(1-\hat{U})\sum_{t\in\mathbb{N}}T_\alpha(t)\sum_{s=1}^{t}\hat{U}^{s-1}, \tag{A.6}$$

where $\hat{U}$ is an abbreviation for $\hat{U}_L(k)$. If $\alpha>2$, we can take any $\xi\in(0,\alpha/2-1)$ to obtain

$$\begin{aligned}1-\hat{D}(k)&\\ &=(1-\hat{U})\sum_{t\in\mathbb{N}}T_\alpha(t)\sum_{s=1}^{t}1-(1-\hat{U})\sum_{t\in\mathbb{N}}T_\alpha(t)\sum_{s=1}^{t}(1-\hat{U}^{s-1})\\ &=(1-\hat{U})\sum_{t\in\mathbb{N}}tT_\alpha(t)+O\big((1-\hat{U})^{1+\xi}\big),\end{aligned} \tag{A.7}$$

where we have used the inequality

$$\begin{aligned}&\sum_{t\in\mathbb{N}}T_\alpha(t)\sum_{s=1}^{t}(1-\hat{U}^{s-1})\\ &\quad=(1-\hat{U})^\xi\sum_{t\in\mathbb{N}}T_\alpha(t)\sum_{s=1}^{t}\bigg(\frac{1-\hat{U}^{s-1}}{1-\hat{U}}\bigg)^\xi(1-\hat{U}^{s-1})^{1-\xi}\\ &\quad\le 2^{1-\xi}(1-\hat{U})^\xi\sum_{t\in\mathbb{N}}t^{1+\xi}T_\alpha(t)=O\big((1-\hat{U})^\xi\big).\end{aligned} \tag{A.8}$$

This together with (A.3)–(A.4) implies (1.18) for $\alpha>2$, with $\varepsilon=\zeta\wedge(2\xi)$ and

$$v_\alpha=\frac{\sigma_L^2}{2d}\sum_{t\in\mathbb{N}}tT_\alpha(t)=O\big(L^2\big). \tag{A.9}$$

If $\alpha\in(0,2)$, on the other hand, we first rewrite (A.6) for small k by setting $\hat{u}\equiv\log 1/\hat{U}$ and changing the order of summations as

$$\begin{aligned}1-\hat{D}(k)&=\frac{1-\hat{U}}{\hat{U}}\sum_{t\in\mathbb{N}}T_\alpha(t)\sum_{s=1}^{t}e^{-\hat{u}s}\\ &=\frac{1-\hat{U}}{\hat{U}}\frac{\sum_{s\in\mathbb{N}}e^{-\hat{u}s}\sum_{t=s}^\infty t^{-1-\alpha/2}}{\sum_{s\in\mathbb{N}}s^{-1-\alpha/2}}.\end{aligned} \tag{A.10}$$

We note that, for small k,

$$\frac{1-\hat{U}}{\hat{U}}=1-\hat{U}+O\big((1-\hat{U})^2\big),\qquad \hat{u}=1-\hat{U}+O\big((1-\hat{U})^2\big). \tag{A.11}$$

Therefore, by a Riemann-sum approximation, we can estimate the numerator in (A.10) as

$$\begin{aligned}
&\sum_{s\in\mathbb{N}} e^{-\hat{u}s}\sum_{t=s}^{\infty} t^{-1-\alpha/2}\\
&\quad=\sum_{s\in\hat{u}\mathbb{N}} e^{-s}\sum_{\substack{t\in\hat{u}\mathbb{N}\\(t\geq s)}}\left(\frac{t}{\hat{u}}\right)^{-1-\alpha/2}\\
&\quad=\hat{u}^{\alpha/2-1}\big(1+O(\hat{u})\big)\int_0^{\infty}\mathrm{d}s\, e^{-s}\int_s^{\infty}\mathrm{d}t\, t^{-1-\alpha/2}\\
&\quad=\frac{2}{\alpha}\Gamma(1-\alpha/2)\hat{u}^{\alpha/2-1}\big(1+O(\hat{u})\big).
\end{aligned}\tag{A.12}$$

This together with (A.3)–(A.4) and (A.9)–(A.12) implies (1.18) for $\alpha\in(0,2)$, with $\varepsilon=\zeta$ and

$$v_\alpha=\frac{2}{\alpha}\frac{\Gamma(1-\alpha/2)}{\sum_{s\in\mathbb{N}}s^{-1-\alpha/2}}\left(\frac{\sigma_L^2}{2d}\right)^{\alpha/2}=O\big(L^\alpha\big).\tag{A.13}$$

This verifies that D in (A.5) satisfies both (1.18) and (1.20).

VERIFICATION OF (1.16), (1.21) AND (1.24). To verify these x-space bounds on the transition probability D^{*n} and its discrete derivative, we use the Cramér–Edgeworth expansion to approximate the lattice distribution $U_L^{*t}(x)$ in (A.5) to the Gaussian density $\nu_{\sigma_L^2 t}(x)$ (multiplied by a polynomial of $x/\sqrt{\sigma_L^2 t}$), where

$$\nu_c(x)=\left(\frac{d}{2\pi c}\right)^{d/2}\exp\left(-\frac{d|x|^2}{2c}\right).\tag{A.14}$$

Before showing a precise statement (cf., Theorem A.1 below), we explain the formal expansion (A.21) of $U_L^{*t}(x)$. First, we note that $\hat{U}_L(k)$ is a generating function of cumulants $Q_{\vec{n}}$ for $\vec{n}\in\mathbb{Z}_+^d$:

$$\log\hat{U}_L(k)=\sum_{\substack{\vec{n}\in\mathbb{Z}_+^d\\(\|\vec{n}\|_1\geq 1)}} Q_{\vec{n}}\prod_{s=1}^{d}\frac{(ik_s)^{n_s}}{n_s!}.\tag{A.15}$$

Since U_L is $\mathbb{Z}^d$-symmetric, we have $Q_{\vec{n}}=0$ if $\|\vec{n}\|_1$ is odd, and $Q_{(2,0,\dots,0)}=\cdots=Q_{(0,\dots,0,2)}=\sigma_L^2/d$. Therefore,

$$\log\hat{U}_L(k)=-\frac{\sigma_L^2}{2d}|k|^2+\sum_{l=4}^{\infty}\sum_{\substack{\vec{n}\in\mathbb{Z}_+^d\\(\|\vec{n}\|_1=l)}} Q_{\vec{n}}\prod_{s=1}^{d}\frac{(ik_s)^{n_s}}{n_s!}.\tag{A.16}$$

By the Fourier inversion theorem, we may rewrite $U_L^{*t}(x)$ as

$$\begin{aligned}
U_L^{*t}(x) &= \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \hat{U}_L(k)^t e^{-ik\cdot x} \\
&= \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-(\sigma_L^2/(2d))t|k|^2 - ik\cdot x} \\
&\qquad \times \exp\Biggl(t \sum_{l=4}^{\infty} \sum_{\substack{\vec{n}\in\mathbb{Z}_+^d \\ (\|\vec{n}\|_1=l)}} Q_{\vec{n}} \prod_{s=1}^{d} \frac{(ik_s)^{n_s}}{n_s!}\Biggr) \qquad \text{(A.17)}\\
&= (\sigma_L^2 t)^{-d/2} \int_{\sqrt{\sigma_L^2 t}[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-(1/(2d))|k|^2 - ik\cdot\tilde{x}} \\
&\qquad \times \exp\Biggl(\sum_{l=4}^{\infty} t^{1-l/2} \tilde{Q}_l(ik)\Biggr),
\end{aligned}$$

where, in the third equality, we have replaced k by $k/\sqrt{\sigma_L^2 t}$ and used the abbreviations

$$\text{(A.18)} \qquad \tilde{x} = \frac{x}{\sqrt{\sigma_L^2 t}}, \qquad \tilde{Q}_l(ik) = \sum_{\substack{\vec{n}\in\mathbb{Z}_+^d \\ (\|\vec{n}\|_1=l)}} \frac{Q_{\vec{n}}}{\sigma_L^l} \prod_{s=1}^{d} \frac{(ik_s)^{n_s}}{n_s!}.$$

Notice that, since U_L is supported in $[-L, L]^d$, the coefficients $Q_{\vec{n}}/\sigma_L^l$ for $\|\vec{n}\|_1 = l$ are uniformly bounded in L. Then the exponential factor involving higher-order cumulants in (A.17) may be expanded as

$$\begin{aligned}
&\exp\Biggl(\sum_{l=2}^{\infty} t^{-l/2} \tilde{Q}_{l+2}(ik)\Biggr) \\
&\quad = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{l_1,\dots,l_m\ge 2} \prod_{r=1}^{m} \bigl(t^{-l_r/2} \tilde{Q}_{l_r+2}(ik)\bigr) \qquad \text{(A.19)}\\
&\quad = 1 + \sum_{j=2}^{\infty} t^{-j/2} \sum_{m=1}^{\lfloor j/2 \rfloor} \frac{1}{m!} \sum_{\substack{l_1,\dots,l_m\ge 2 \\ (l_1+\cdots+l_m=j)}} \prod_{r=1}^{m} \tilde{Q}_{l_r+2}(ik).
\end{aligned}$$

Let

$$\begin{aligned}
&P_0(ik) = 1, \qquad P_1(ik) = 0, \\
&P_j(ik) = \sum_{m=1}^{\lfloor j/2 \rfloor} \frac{1}{m!} \sum_{\substack{l_1,\dots,l_m\ge 2 \\ (l_1+\cdots+l_m=j)}} \prod_{r=1}^{m} \tilde{Q}_{l_r+2}(ik) \qquad [j \ge 2]. \qquad \text{(A.20)}
\end{aligned}$$

Then, by (A.17) and (A.19), we arrive at the formal Cramér–Edgeworth expansion

$$U_L^{*t}(x) = (\sigma_L^2 t)^{-d/2} \times \int_{\sqrt{\sigma_L^2 t}[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-(1/(2d))|k|^2 - ik\cdot\tilde{x}} \sum_{j=0}^{\infty} t^{-j/2} P_j(ik). \tag{A.21}$$

Now we note that, if $\sqrt{\sigma_L^2 t}[-\pi,\pi]^d$ is replaced by $\mathbb{R}^d$, if $\sum_{j=0}^{\infty}$ is replaced by $\sum_{j=0}^{\ell}$ for some $\ell < \infty$, and if x is considered to be an element of $\mathbb{R}^d$ instead of $\mathbb{Z}^d$, then we obtain

$$\begin{aligned}
&(\sigma_L^2 t)^{-d/2} \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-(1/((2d))|k|^2 - ik\cdot\tilde{x}} \sum_{j=0}^{\ell} t^{-j/2} P_j(ik) \\
&\quad = (\sigma_L^2 t)^{-d/2} \sum_{j=0}^{\ell} t^{-j/2} \tilde{P}_j \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-(1/(2d))|k|^2 - ik\cdot\tilde{x}} \\
&\quad = (\sigma_L^2 t)^{-d/2} \sum_{j=0}^{\ell} t^{-j/2} \tilde{P}_j \nu_1(\tilde{x}),
\end{aligned} \tag{A.22}$$

where $\tilde{P}_j$ is the differential operator defined by replacing each ik_s of $P_j(ik)$ in (A.20) by $-\partial/\partial\tilde{x}_s$:

$$\tilde{P}_0 = 1, \qquad \tilde{P}_1 = 0, \qquad \tilde{P}_j = P_j\biggl(\frac{-\partial}{\partial\tilde{x}_1}, \dots, \frac{-\partial}{\partial\tilde{x}_d}\biggr) \qquad [j \geq 2]. \tag{A.23}$$

Notice that, by (A.18) and (A.20),

$$(\sigma_L^2 t)^{-d/2} \tilde{P}_j \nu_1(\tilde{x}) = H_{j+2}^{2j}\biggl(\frac{x}{\sqrt{\sigma_L^2 t}}\biggr) \nu_{\sigma_L^2 t}(x), \tag{A.24}$$

where H_{j+2}^{2j} is a polynomial of degree at least $j+2$ and at most $2j$ (due to the symmetry of U_L). The coefficients of the polynomial are uniformly bounded in L, as explained below (A.18).

The following theorem is a version of [6], Corollary 22.3, for symmetric distributions, which gives a bound on the difference between $U_L^{*t}(x)$ and (A.22).

THEOREM A.1. *For any $x \in \mathbb{Z}^d$, $t \in \mathbb{N}$ and $\ell \in \mathbb{Z}_+$,*

$$\bigl(1 + |\tilde{x}|^{\ell+2}\bigr) \Biggl| U_L^{*t}(x) - (\sigma_L^2 t)^{-d/2} \sum_{j=0}^{\ell} t^{-j/2} \tilde{P}_j \nu_1(\tilde{x}) \Biggr| \leq \frac{O(L^{-d})}{t^{(d+\ell)/2}}, \tag{A.25}$$

where $\tilde{x}$ and $\tilde{P}_j$ are defined in (A.18) *and* (A.23), *respectively.*

Before using this theorem to verify (1.16), (1.21) and (1.24), we briefly explain how to prove that the contribution which comes from 1 on the left-hand side of (A.25) is bounded by $O(L^{-d})t^{-(d+\ell)/2}$, as in (A.25). (To investigate the contribution that comes from $|\tilde{x}|^{\ell+2}$ on the left-hand side of (A.25), we also use identities such as

$$\begin{aligned}&\tilde{x}_1^{\ell+2}U_L^{*t}(x)\\ &\quad=(\sigma_L^2 t)^{-d/2}\int_{\sqrt{\sigma_L^2 t}[-\pi,\pi]^d}\frac{\mathrm{d}^d k}{(2\pi)^d}e^{-ik\cdot\tilde{x}}\frac{\partial^{\ell+2}}{\partial(ik_1)^{\ell+2}}\hat{U}_L\biggl(\frac{k}{\sqrt{\sigma_L^2 t}}\biggr)^t,\end{aligned}\tag{A.26}$$

which is a result of integration by parts.) First, we split the domain of integration in Fourier space into $E_1=\{k\in\mathbb{R}^d:|k|\le\sqrt{t}\}$, $E_2=\sqrt{\sigma_L^2 t}[-\pi,\pi]^d\setminus E_1$ and $E_3=\mathbb{R}^d\setminus E_1$. Then the difference between $U_L^{*t}(x)$ and (A.22) is equal to $\mathcal{I}_1+\mathcal{I}_2-\mathcal{I}_3$, where

$$\begin{aligned}\mathcal{I}_1=(\sigma_L^2 t)^{-d/2}&\int_{E_1}\frac{\mathrm{d}^d k}{(2\pi)^d}e^{-ik\cdot\tilde{x}}\\ &\times\Biggl(\hat{U}_L\biggl(\frac{k}{\sqrt{\sigma_L^2 t}}\biggr)^t-e^{-(1/(2d))|k|^2}\sum_{j=0}^{\ell}t^{-j/2}P_j(ik)\Biggr),\end{aligned}\tag{A.27}$$

$$\mathcal{I}_2=(\sigma_L^2 t)^{-d/2}\int_{E_2}\frac{\mathrm{d}^d k}{(2\pi)^d}e^{-ik\cdot\tilde{x}}\hat{U}_L\biggl(\frac{k}{\sqrt{\sigma_L^2 t}}\biggr)^t,\tag{A.28}$$

$$\mathcal{I}_3=(\sigma_L^2 t)^{-d/2}\int_{E_3}\frac{\mathrm{d}^d k}{(2\pi)^d}e^{-(1/(2d))|k|^2-ik\cdot\tilde{x}}\sum_{j=0}^{\ell}t^{-j/2}P_j(ik).\tag{A.29}$$

Since (A.25) for $t=1$ is trivial, we can assume $t\ge 2$ with no loss of generality. Then it is not difficult to prove that $\mathcal{I}_2$ and $\mathcal{I}_3$ are both bounded by $O(L^{-d})t^{-(d+\ell)/2}$, due to direct computation for $\mathcal{I}_3$, and due to (A.4) and similar computation to [9], (A.2), for $\mathcal{I}_2$. For $\mathcal{I}_1$, we can bound the integrand by $Ct^{-\ell/2}(|k|^{\ell+2}+|k|^{2\ell})e^{-c|k|^2}$ for some L-independent constants $C,c\in(0,\infty)$, due to a version of [6], Theorem 9.12, for symmetric distributions. Then, by direct computation, we can prove that $\mathcal{I}_1$ is also bounded by $O(L^{-d})t^{-(d+\ell)/2}$.

Now we apply (A.25) to verify the x-space bounds (1.16), (1.21) and (1.24). In particular, by (A.5) and (A.23)–(A.25),

$$\begin{aligned}D(x)=&\sum_{t=1}^{\infty}\nu_{\sigma_L^2 t}(x)T_\alpha(t)\\ &+\sum_{t=1}^{\infty}\sum_{j=2}^{\ell}t^{-j/2}H_{j+2}^{2j}\biggl(\frac{x}{\sqrt{\sigma_L^2 t}}\biggr)\nu_{\sigma_L^2 t}(x)T_\alpha(t)\\ &+\sum_{t=1}^{\infty}\frac{O(L^{-d})}{t^{(d+\ell)/2}}\biggl(1\wedge\biggl(\frac{\sqrt{\sigma_L^2 t}}{|x|}\biggr)^{\ell+2}\biggr)T_\alpha(t).\end{aligned}\tag{A.30}$$

The leading term is bounded as

$$\sum_{t=1}^{\infty} \nu_{\sigma_L^2 t}(x) T_\alpha(t) \le O\big(L^{-d}\big) \sum_{1\le t<\|\!|x/\sigma_L|\!\|_1^2} \frac{\exp(-(d|x|^2)/(2\sigma_L^2 t))}{t^{1+(d+\alpha)/2}}$$
$$+ O\big(L^{-d}\big) \sum_{t\ge \|\!|x/\sigma_L|\!\|_1^2} t^{-1-(d+\alpha)/2}$$
$$\le O\big(L^{-d}\big) \sum_{1\le t<\|\!|x/\sigma_L|\!\|_1^2} \frac{O((|x|^2/(\sigma_L^2 t))^{-1-(d+\alpha)/2})}{t^{1+(d+\alpha)/2}} \tag{A.31}$$
$$+ O\big(L^{-d}\big)\|\!|x/\sigma_L|\!\|_1^{-(d+\alpha)}$$
$$= O\big(L^{\alpha}\big)\|\!|x|\!\|_L^{-d-\alpha}.$$

The second term on the right-hand side of (A.30) is bounded, due to (A.24), as follows: for any $j \in \{2,\dots,\ell\}$ and $h \in \{j+2,\dots,2j\}$,

$$\sum_{t=1}^{\infty} t^{-j/2}\bigg(\frac{|x|}{\sqrt{\sigma_L^2 t}}\bigg)^h \nu_{\sigma_L^2 t}(x) T_\alpha(t)$$
$$\le O\big(L^{-d-h}\big)|x|^h \sum_{1\le t<\|\!|x/\sigma_L|\!\|_1^2} \frac{\exp(-(d|x|^2)/(2\sigma_L^2 t))}{t^{1+(d+h+j+\alpha)/2}}$$
$$+ O\big(L^{-d-h}\big)|x|^h \sum_{t\ge \|\!|x/\sigma_L|\!\|_1^2} t^{-1-(d+h+j+\alpha)/2} \tag{A.32}$$
$$\le O\big(L^{-d-h}\big)|x|^h \frac{O(L^{d+h+j+\alpha})}{\|\!|x|\!\|_L^{d+h+j+\alpha}} = \frac{O(L^{j+\alpha})}{\|\!|x|\!\|_L^{d+j+\alpha}} \le \frac{O(L^{2+\alpha})}{\|\!|x|\!\|_L^{d+2+\alpha}}.$$

Therefore,

$$\sum_{t=1}^{\infty}\sum_{j=2}^{\ell} t^{-j/2} H_{j+2}^{2j}\bigg(\frac{x}{\sqrt{\sigma_L^2 t}}\bigg)\nu_{\sigma_L^2 t}(x) T_\alpha(t) \le \frac{O(L^{\alpha+2})}{\|\!|x|\!\|_L^{d+\alpha+2}}. \tag{A.33}$$

Similarly, the third term on the right-hand side of (A.30) is bounded as

$$\sum_{t=1}^{\infty} \frac{O(L^{-d})}{t^{(d+\ell)/2}}\bigg(1 \wedge \bigg(\frac{\sqrt{\sigma_L^2 t}}{|x|}\bigg)^{\ell+2}\bigg) T_\alpha(t)$$
$$\le O\big(L^{-d+\ell+2}\big)|x|^{-\ell-2} \sum_{1\le t<\|\!|x/\sigma_L|\!\|_1^2} t^{-(d+\alpha)/2}$$
$$+ O\big(L^{-d}\big) \sum_{t\ge \|\!|x/\sigma_L|\!\|_1^2} t^{-1-(d+\ell+\alpha)/2} \tag{A.34}$$

$$= \begin{cases} O(L^{-d+\ell+2})|||x|||_L^{-\ell-2}, & [d+\alpha>2], \\ O(L^{-d+\ell+2})|||x|||_L^{-\ell-2}\log|||x/\sigma_L|||_1, & [d+\alpha=2], \\ O(L^{\alpha+\ell})|||x|||_L^{-d-\alpha-\ell}, & [d+\alpha<2], \end{cases}$$

which is further bounded by $O(L^{\alpha+2})|||x|||_L^{-d-\alpha-2}$ for sufficiently large ℓ. Summarizing the above estimates, we can conclude (1.16):

$$D(x) = \sum_{t=1}^{\infty} \nu_{\sigma_L^2 t}(x) T_\alpha(t) + \frac{O(L^{\alpha+2})}{|||x|||_L^{d+\alpha+2}} \le \frac{O(L^\alpha)}{|||x|||_L^{d+\alpha}}. \tag{A.35}$$

The bound (1.21) on the n-step transition probability is then automatically verified, due to the argument below (1.21). Heuristically, since

$$D^{*n}(x) = \sum_{t=n}^{\infty} U_L^{*t}(x) T_\alpha^{*n}(t), \tag{A.36}$$

this suggests that

$$T_\alpha^{*n}(t) \le O(n) T_{\alpha\wedge 2}(t). \tag{A.37}$$

In fact, we can verify this (or a stronger version) by following the same argument as given below (1.21), but we omit the details here.

Finally, we verify (1.24) by using (A.25) with sufficiently large ℓ and (A.35)–(A.37). For $|y| \le \frac{1}{3}|x|$ (so that $|x \pm y| \ge \frac{2}{3}|x|$), we obtain

$$\begin{aligned} D^{*n}(x) &- \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2} \\ &= \sum_{t=1}^{\infty} \left(\frac{\eta}{\pi t}\right)^{d/2} \left(e^{-\eta|x|^2/t} - \frac{e^{-\eta|x+y|^2/t} + e^{-\eta|x-y|^2/t}}{2}\right) T_\alpha^{*n}(t) \\ &\quad + \frac{O(L^{\alpha\wedge 2+2})}{|||x|||_L^{d+\alpha\wedge 2+2}} n, \end{aligned} \tag{A.38}$$

where we have set $\eta = d/(2\sigma_L^2) = O(L^{-2})$ for convenience. By a Taylor expansion,

$$e^{-\eta|x|^2/t} - \frac{e^{-\eta|x+y|^2/t} + e^{-\eta|x-y|^2/t}}{2} = \frac{O(\eta|y|^2)}{t} e^{-\eta|x|^2/t}. \tag{A.39}$$

Using this and (A.37) and following the same analysis as in (A.31)–(A.32), we can bound the sum in (A.38) by

$$O(\eta^{1+d/2})|y|^2 n \sum_{t=1}^{\infty} \frac{e^{-\eta|x|^2/t}}{t^{2+(d+\alpha\wedge 2)/2}} = \frac{O(\eta^{-(\alpha\wedge 2)/2})|y|^2}{|||x|||_{1/\sqrt{\eta}}^{d+\alpha\wedge 2+2}} n. \tag{A.40}$$

This together with (A.38) and $|||y|||_L = |y| \vee L$ yields (1.24).

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