INVARIANT SUBSPACES OF FINITE CODIMENSION AND UNIFORM ALGEBRAS

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Abstract. Let \( A \) be a uniform algebra on a compact Hausdorff space \( X \) and \( m \) a probability measure on \( X \). Let \( H^p(m) \) be the norm closure of \( A \) in \( L^p(m) \) with \( 1 \leq p < \infty \) and \( H^\infty(m) \) the weak * closure of \( A \) in \( L^\infty(m) \). In this paper, we describe a closed ideal of \( A \) and exhibit a closed invariant subspace of \( H^p(m) \) for \( A \) that is of finite codimension.

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1. Introduction. Let \( A \) be a uniform algebra on a compact Hausdorff space \( X \). \( M(A) \) denotes the maximal ideal space of \( A \). Let \( m \) be a probability measure on \( X \). \( H^p(m) \) denotes the norm closure of \( A \) in \( L^p(m) \) with \( 1 \leq p < \infty \) and \( H^\infty(m) \) denotes the weak * closure of \( A \) in \( L^\infty(m) \). \( H^p(m) \) is called an abstract Hardy space. When \( A \) is a disc algebra, if \( m \) is the normalized Lebesgue measure on the unit circle, \( H^p(m) \) is the usual Hardy space and if \( m \) is the normalized area measure on the unit disc, \( H^p(m) \) is the usual Bergman space.

Let \( I \) be a closed ideal of \( A \). In this paper, we are interested in \( I \) with \( \dim A/I < \infty \). Then \( A/I \) is called a Q-algebra. Two dimensional Q-algebras can be described easily; that is, \( I = \{ f \in A; \phi_1(f) = \phi_2(f) = 0 \} \), where \( \phi_j \in M(A) \) \( (j = 1, 2) \), or \( I = \{ f \in A; \phi(f) = D_\phi(f) = 0 \} \), where \( \phi \in M(A) \) and \( D_\phi \) is a bounded point derivation at \( \phi \). One of the authors [3] showed that a two dimensional operator algebra on a Hilbert space is a Q-algebra. It seems to be worthwhile to describe a finite dimensional Q-algebra. In Section 2, we describe an ideal \( I \) with \( \dim A/I < \infty \) using a theorem of T. W. Gamelin [2]. As a result, a finite dimensional Q-algebra is described.

When \( M \) is a closed subspace of \( H^p(m) \) and \( AM \subset M \), \( M \) is called an invariant subspace for \( A \). In this paper, we are interested in \( M \) with \( \dim H^p(m)/M < \infty \). When \( A \) is the polydisc algebra on \( T^n \) and \( m \) is the normalized Lebesgue measure on \( T^n \), a finite codimensional invariant subspace \( M \) in \( H^p(m) \) was described by P. Ahern and D. N. Clark [1] using the ideals in the polynomial ring \( C[z_1, \ldots, z_n] \) of finite codimension whose zero sets are contained in the polydisc \( D^n \). In Section 3, for an

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arbitrary uniform algebra \( A \) we describe a finite codimensional invariant subspace \( M \) in \( H^p(m) \) using the result in Section 2.

2. Finite codimensional ideal. Let \( \phi \in M(A) \). A closed subalgebra \( H \) of \( A \) is a \((\phi, k)\)-subalgebra if there is a sequence of closed subalgebras \( A = A_0 \supset A_1 \supset \cdots \supset A_k = H \) such that \( A_j \) is the kernel of a continuous point derivation \( D_j \) of \( A_{j-1} \) at \( \phi \). If \( H \) is a \((\phi, k)\)-subalgebra of \( A \), then \( H \) has finite codimension in \( A \) and \( M(H) = M(A) \) by [2, Lemma 9.1].

If \( I \) is a closed ideal of \( A \) and \( A/I \) is of finite dimension, \( B = \mathcal{C} + I \) is a closed subalgebra of \( A \), and \( A/B \) is of finite dimension. By a theorem of T. W. Gamelin [2, Theorem 9.8], we can describe \( B \) and so also \( I \). Since \( B \) is a special closed subalgebra of \( A \) we can describe \( I \) more explicitly.

**Theorem 1.** If \( I \) is a closed ideal of \( A \) and \( A/I \) is of finite dimension, then there exists a closed subalgebra \( E = E(I) \) of \( A \) such that \( E = \{ f \in A : \phi_1(f) = \cdots = \phi_n(f) \} \), \( 1 \leq n < \infty \), \( \{ \phi_j \} \subset M(A) \) and

\[
I = H^E \cap \ker \phi,
\]

where \( \phi = \phi_j/E \), \( 1 \leq j \leq n \) and \( H^E \) is a \((\phi, k)\)-subalgebra with respect to \( E \) for some \( k \).

**Proof.** Put \( H = I + \mathcal{C} \); then \( A/H \) is of finite dimension. By a theorem of T. W. Gamelin [2, Theorem 9.8], \( H \) can be obtained from \( A \) in two steps.

(i) There exist pairs of points \( \psi_j, \psi_j' \), \( 1 \leq j \leq \ell \), in \( M(A) \) such that if \( E \) consists of the \( f \in A \) such that \( \psi_j(f) = \psi_j'(f) \), \( 1 \leq j \leq \ell \), then \( H \subset E \subset A \).

(ii) There exist distinct points \( \theta_j \in M(E) \) and \( \theta_j \)-subalgebras \( H_j \) of \( E \), \( 1 \leq j \leq k \), such that \( H = H_1 \cap \cdots \cap H_k \).

Put \( \tilde{\psi}_j = \phi_j/E = \psi_j'/E \) for \( 1 \leq j \leq \ell \); then \( \tilde{\psi}_j \) belongs to \( M(E) \). Since \( I \) is an ideal of \( A \), \( \tilde{\psi}_j \in \bigcap_{\ell = 1}^k \ker \tilde{\psi}_j \). To see this, let \( f \in A \) such that \( \psi_j(f) \neq \psi_j'(f) \). If \( g \in I \), then \( f g \in I \) but \( \psi_j(f g) \neq \psi_j'(f g) \) when \( \tilde{\psi}_j(g) 
eq 0 \). This contradicts the fact that \( f g \in E \). Thus \( \tilde{\psi}_j(g) = 0 \). Hence \( I \subset \bigcap_{\ell = 1}^k \ker \tilde{\psi}_j \) and so \( H \subset \bigcap_{\ell = 1}^k \ker \tilde{\psi}_j + \mathcal{C} \). By the definition of \( E \), \( \tilde{\psi}_1 = \cdots = \tilde{\psi}_\ell \). Therefore \( E \) has the form \( E = \{ f \in A : \phi_1(f) = \cdots = \phi_n(f) \} \), \( 1 \leq n < \infty \), and \( \{ \phi_j \} \subset M(A) \).

For each \( j \) with \( 1 \leq j \leq k \), \( H_j \) is a \( \theta_j \)-subalgebra of \( A \) for \( \theta_j \in M(E) \). Hence there is a sequence of closed subalgebras \( E = E_{j_0} \supset E_{j_1} \supset \cdots \supset E_{j_\ell} = H_j \) such that \( E_{j_\ell} \) is the kernel of a continuous point derivation \( D_{\ell_j} \) of \( E_{j_\ell-1} \) at \( \theta_j \). We shall write \( E_{j_\ell} = \ker D_{\theta_j} \), where \( D_{\theta_j} \) is a derivation on \( E_{j_\ell-1} \). Then \( H = \bigcap_{\ell = 1}^k \ker D_{\theta_j} \) and so \( I = \bigcap_{\ell = 1}^k \ker D_{\theta_j} \cap \ker \theta_j \) for some \( \theta_j \in M(H) \). Suppose that \( g \) is an arbitrary function in \( I \). For any \( j(1 \leq j \leq k) \), there exists a function \( f \in E_{j_\ell-1} \) such that \( f \notin E_{j_\ell} = \ker D_{\theta_j} \). Since \( f g \in E \) and \( D_{\theta_j}(g) = 0 \), \( D_{\theta_j}(f g) = \theta_j(g) D_{\theta_j}(f) = 0 \) because \( D_{\theta_j} \) is a derivation on \( E_{j_\ell-1} \). This implies that \( \theta_j(g) = 0 \). Hence \( I \subset \bigcap_{\ell = 1}^k \ker \theta_j \). Therefore by the definition of \( E \), \( \theta_1 = \cdots = \theta_k \in M(E) \), and so \( H_1 = \cdots = H_k \). Thus \( \theta_1 | H = \theta \) and \( I = (\ker D_{\theta_1}) \cap \ker \theta_1 \).

Corollary 1. If \( I \) is a closed ideal of \( A \) and \( A/I \) is of finite dimension 2, then \( I = \{ f \in A : \phi_1(f) = \phi_2(f) = 0 \} \), where \( \phi_j \in M(A) \) (\( j = 1, 2 \)) and \( \phi_1 \neq \phi_2 \), or \( I = \{ f \in A : \phi(f) = D_\phi(f) = 0 \} \), where \( \phi \in M(A) \), and \( D_\phi \) is a bounded point derivation at \( \phi \).
proof. When dim \( A/I = 2 \), by Theorem 1, \( E = A \) or \( E = \{ f \in A; \phi_1(f) = \phi_2(f) \} \). If \( E = A \), then \( H^E_\phi = \{ f \in A; D_\phi(f) = 0 \} \) and if \( E = \{ f \in A; \phi_1(f) = \phi_2(f) \} \), then \( H^E_\phi = E \), since \( dim A/H^E_\phi = 1 \) because \( H^E_\phi = I + C \). This implies the corollary.

Corollary 2. If \( B \) is a finite dimensional \( Q \)-algebra and \( B_0 = \text{rad} \, B \) is its radical, then there exist subalgebras \( B_1, B_2, \ldots, B_{k+1} \) in \( B_0 \), such that \( B_{k+1} = \{ 0 \} \), \( dim B_j/B_{j+1} = 1 \) and \( B_{j+1} \) is an ideal of \( B_j \) for \( j = 0, 1, \ldots, k \). Hence \( rad \, B \) has a basis \( \{ f_0, f_1, \ldots, f_k \} \) such that \( (f_j)^{2(k+1)-j} = 0 \) for \( j = 0, 1, \ldots, k \).

Proof. Since \( B \) is a \( Q \)-algebra, \( B = A/I \) for some uniform algebra \( A \) and some closed ideal \( I \) of \( A \). Also, since \( B \) is of finite dimension, we can apply Theorem 1 to \( A \) and \( I \). In the notation of Theorem 1, \( rad \, B = \{ f \in E; \phi(f) = 0 \}/I \). Since \( H^E_\phi \) is a \( \phi \)-subalgebra with respect to \( E \), there exists a sequence of closed subalgebras \( E = E_0 \supset E_2 \supset \cdots \supset E_{k+1} = H^E_\phi \) such that \( E_j \) is the kernel of a continuous point derivation \( D_j \) of \( E_{j-1} \) at \( \phi \). Hence \( E_j \cap \ker \phi \) is an ideal of \( E_j \cap \ker \phi \) and \( dim \{ E_j \cap \ker \phi \}, \ker \phi \} = 1 \). Put \( B_j = (E_j \cap \ker \phi)/I \). Then \( dim B_j/B_{j+1} = 1 \) and \( B_{j+1} \) is an ideal of \( B_j \), for \( j = 0, 1, \ldots, k \), and \( B_{k+1} = \{ 0 \} \). For each \( j \), there exists \( f_j \) such that \( B_j = \langle f_j \rangle + B_{j+1} \) and then \( \{ f_0, f_1, \ldots, f_k \} \) is a basis of \( rad \, B = B_0 \). Observe that \( f_j^2 \) belongs to \( B_{j+1} \) because \( E_{j+1} \cap \ker D_{j+1} \). Thus \( (f_j)^{2(k+1)-j} = 0 \).

3. Finite codimensional invariant subspace. For a subset \( S \) of \( H^p(m) \), \([S]_p \) denotes the closure of \( S \) in \( H^p(m) \).

Theorem 2. If \( M \) is an invariant subspace of \( H^p(m) \) with \( dim H^p/M = n < \infty \), then there exists a closed ideal \( A \) such that \( dim A/I = n \), \([I]_p = M \) and \( I = M \cap A \). If \( H^E_\phi \) is a \( \phi \)-subalgebra with respect to \( E = E(I) \), then \( [E]_p \supset [E_{j+1}]_p \) for any \( 0 \leq j \leq k-1 \) and \( dim H^p/[E]_p = dim A/E \). Conversely, if \( dim A/I = n < \infty \), then \( dim H^p/[I]_p \leq n \).

Proof. Suppose that \( M \) is an invariant subspace of \( H^p(m) \) and \( dim H^p(m)/M = n < \infty \). Then there exist \( n \) linearly independent linear functionals \( \psi_1, \psi_2, \ldots, \psi_n \) in \( (H^p)^* \) such that \( \psi_j = 0 \) on \( M \) for \( 1 \leq j \leq n \). Put \( \phi_j = \psi_j | A \) for \( 1 \leq j \leq n \) and \( I = M \cap A \). Then \( I = \bigcap_{j=1}^n \ker \phi_j \) and so \( dim A/I = n \). For \( \phi_1, \ldots, \phi_n \) are independent linear functionals in \( A^* \) because \( A \) is dense in \( H^p(m) \). If \( M \supset [I]_p \), then there exists \( \psi_{n+1} \in H^p(m)^* \) such that \( \psi_{n+1} = 0 \) on \([I]_p \) and \( \psi_1, \ldots, \psi_n, \psi_{n+1} \) are independent linear functionals in \( H^p(m)^* \). If we put \( \phi_{n+1} = \psi_{n+1} | A \), then \( \phi_1, \ldots, \phi_n, \phi_{n+1} \) are independent linear functionals in \( A^* \) because \( A \cap [I]_p \cap \ker \phi \) and \( I \subseteq \bigcap_{j=1}^{n+1} \ker \phi_j \). This contradicts implies that \( M = [I]_p \). Note that \( dim H^p/[E_k] = dim H^p/[I]_p = dim A/I = dim A/E_k \). If \( dim H^p/[E_0] < dim A/E_0 \) where \( E_0 = E \) or \([E_j]_p = [E_{j+1}]_p \), for some \( j \), then this contradicts the fact that \( dim H^p/[E_k] = dim A/E_k \). The converse is clear.

Corollary 1. If \( M \) is an invariant subspace of \( H^p \) with \( dim H^p/M = 2 \), then \( M = \{ f \in H^p; \Phi_1(f) = \Phi_2(f) = 0 \} \), where \( \Phi_j \in (H^p)^* \), and \( \Phi_j(f) = \Phi(f) \Phi_j(g) \) for \( f \in H^p \) and \( g \in A \), or \( M = \{ f \in H^p; \Phi(f) = D_\phi(f) = 0 \} \), where \( \Phi, D_\phi \in (H^p)^* \), \( \Phi(f) = \Phi(f) \Phi(g) \) and \( D_\phi(f) = \Phi(f) D_\phi(g) + \Phi(g) D_\phi(f) \) for \( f \in H^p \) and \( g \in A \).

Proof. This follows from Corollary 1 and Theorem 2.
Corollary 4. If $M$ is an invariant subspace of $H^p$ with $\dim H^p / M = n < \infty$, then there exist $f_1, \ldots, f_n$ in $A$ such that $\{f_j + M\}_{j=1}^n$ is a basis in $H^p / M$.

Proof. By Theorem 2, if $I = M \cap A$, then $\dim A/I = n$ and $M = [I]_p$. Hence there exist $f_1, \ldots, f_n$ in $A$ such that $\{f_j + I\}_{j=1}^n$ is a basis in $A/I$. If $f_j$ belongs to $M$, then $f_j$ also belongs to $M \cap A = I$ and so $f_j$ does not belong to $M$. This proves the corollary.

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