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On the commutativity of fundamental groups of complements to plane curves

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Introduction

In this paper, we prove the following:

THEOREM. *Let $C \subset \mathbb{P}^3$ be a complex reduced irreducible space curve which is non-degenerate (i.e., C is not contained in any plane). Let $p: C \rightarrow \mathbb{P}^2$ be a general projection. Then the topological fundamental group $\pi_1(\mathbb{P}^2 \setminus p(C))$ is abelian.*

When C is non-singular, the image of the general projection has only nodes as its singularities, and hence the complement has an abelian fundamental group thanks to Fulton–Deligne’s theorem on Zariski conjecture ([1, 2]). The point of our theorem is that we make no assumptions on the singularities of the space curve C .

We apply this theorem to the problem of commutativity of fundamental groups of complements to plane curves of a given degree with prescribed numbers of nodes and cusps. Here a cusp means a germ of curve singularity which is analytically isomorphic to a small neighborhood of the singular point of the affine curve $x^2 + y^3 = 0$. The strongest result concerned with this problem is Nori’s result [5, proposition 6.5]. (Nori considered not only plane curves with only nodes and cusps but curves on arbitrary surfaces with any kind of singularities.) Let $C \subset \mathbb{P}^2$ be a reduced irreducible plane curve of degree d whose singular locus consists of n nodes and k cusps.

THEOREM (Nori). *Suppose that $2n + 6k < d^2$. Then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.*

We will prove a proposition which enlarges slightly the region of (d, n, k) on which the commutativity of $\pi_1(\mathbb{P}^2 \setminus C)$ is guaranteed.

PROPOSITION 1. *Suppose that $2n \geq d^2 - 5d + 8$. Then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.*

1. Proof of Theorem

Let G denote the Grassmannian variety $\text{Grass}(\mathbb{P}^1, \mathbb{P}^3)$ of all lines in $\mathbb{P}^3$, and let $(\mathbb{P}^3)^*$ be the dual projective space $\text{Grass}(\mathbb{P}^2, \mathbb{P}^3)$ of $\mathbb{P}^3$. For a plane $\Pi \in (\mathbb{P}^3)^*$, we put

$$\Sigma(\Pi) := \{l \in G; l \subset \Pi\}.$$

We also put

$$U := \{l \in G; C \cap l = \emptyset\}.$$

When $\Pi \in (\mathbb{P}^3)^*$ is general, no three points among $C \cap \Pi$ are co-linear. Indeed, let $B \subset (\mathbb{P}^3)^*$ be the Zariski open dense subset consisting of the planes which intersect C

at its non-singular points transversely. Let $\Pi_0 \in B$ be a plane which serves as a base point of B . Since C is irreducible, the monodromy group of the action of $\pi_1(B, \Pi_0)$ on the set $C \cap \Pi_0$ is the full symmetric group, because the action is 2-transitive and the monodromy group contains a simple transposition (cf. [4, Uniform Position Lemma]). Hence if there exist co-linear three points among $C \cap \Pi$ for a general $\Pi \in (\mathbb{P}^3)^*$, then all points in $C \cap \Pi$ are on a line for a general Π . Note that this condition is closed for the choice of Π in $(\mathbb{P}^3)^*$ because C is non-degenerate. Thus all points in $C \cap \Pi$ are on a line for every Π , which is absurd. Consequently, $\Sigma(\Pi) \setminus U$ is a union of lines on $\Sigma(\Pi) \cong \mathbb{P}^2$, which is a normal crossing divisor when the plane Π is generally chosen. Hence $\pi_1(\Sigma(\Pi) \cap U)$ is abelian for a general Π because of Fulton–Deligne’s theorem ([1–3]). Let $I \subset (\mathbb{P}^3)^* \times G$ be the closed subvariety $\{(\Pi, l); \Pi \supset l\}$. The second projection $\rho_2: I \rightarrow G$ is a fibre bundle with fibres isomorphic to $\mathbb{P}^1$. Hence ρ_2 induces an isomorphism $\pi_1(\rho_2^{-1}(U)) \cong \pi_1(U)$. Consider the projection $\rho_1: \rho_2^{-1}(U) \rightarrow (\mathbb{P}^3)^*$. The fibre $\rho_1^{-1}(\Pi) \subset \rho_2^{-1}(U)$ over $\Pi \in (\mathbb{P}^3)^*$ is isomorphic to $\Sigma(\Pi) \cap U$, which is non-empty and non-singular for every Π . We apply Nori’s Lemma [5, lemma 1.5 (C)] to conclude that the inclusion of the fibre induces a surjection $\pi_1(\Sigma(\Pi) \cap U) \rightarrow \pi_1(\rho_2^{-1}(U))$ for a general Π . Hence $\pi_1(\rho_2^{-1}(U))$ is abelian, and thus $\pi_1(U)$ is also abelian.

We fix a plane $\Pi_0 \subset \mathbb{P}^3$ and denote by A the complement $\mathbb{P}^3 \setminus \Pi_0$. Let $J_A \subset A \times G$ be the closed subvariety $\{(P, l); P \in l\}$. Consider the projection $\tau_2: J_A \rightarrow G$. If $l \not\subset \Pi_0$, then $\tau_2^{-1}(l)$ is isomorphic to $\mathbb{A}^1$, while if $l \subset \Pi_0$, then $\tau_2^{-1}(l)$ is empty. Thus τ_2 induces an isomorphism $\pi_1(\tau_2^{-1}(U)) \cong \pi_1(U \setminus \Sigma(\Pi_0))$. Since $\Sigma(\Pi_0)$ is of codimension 2 in G , we have $\pi_1(U \setminus \Sigma(\Pi_0)) \cong \pi_1(U)$. Therefore, $\pi_1(\tau_2^{-1}(U))$ is isomorphic to $\pi_1(U)$. We have an isomorphism $J_A \cong A \times \Pi_0$ given by $(P, l) \mapsto (P, l \cap \Pi_0)$. Let $\tilde{\tau}_2: A \times \Pi_0 \rightarrow G$ be the morphism corresponding to $\tau_2: J_A \rightarrow G$ via this isomorphism. Let $D \subset G$ be the Zariski closed subset $G \setminus U$. We provide it with the reduced structure. For $P \in A$, consider the scheme theoretic intersection

$$D_P := (\{P\} \times \Pi_0) \cap \tilde{\tau}_2^{-1}(D),$$

which is considered as a sub-scheme of Π_0 . If $P \in C$, then $D_P = \Pi_0$. Suppose that $P \notin C$. Let $p\langle P \rangle: C \rightarrow \Pi_0$ denote the projection with the centre P . Then we see that the reduced part $(D_P)_{\text{red}}$ of D_P coincides with the reduced part $p\langle P \rangle(C)_{\text{red}}$ of the image of the projection $p\langle P \rangle$. Hence D_P is a reduced divisor of Π_0 if $\deg p\langle P \rangle(C) = \deg C$.

Claim. There are at most finitely many points $P \in A \setminus C$ such that the degree of $p\langle P \rangle(C)_{\text{red}}$ is less than the degree of C .

Since $C \cap A \subset A$ is of codimension 2, Claim implies that the locus of all points $P \notin A$ such that the scheme theoretic intersection D_P is *not* a reduced divisor of Π_0 is of codimension 2 in A . Let $P \in A$ be generally chosen. Applying [6, theorem 1], we get $\pi_1(\Pi_0 \setminus p\langle P \rangle(C)) \cong \pi_1(\tilde{\tau}_2^{-1}(U)) \cong \pi_1(U)$. Therefore $\pi_1(\Pi_0 \setminus p\langle P \rangle(C))$ is abelian, and Theorem is proved.

Proof of Claim. Suppose that there exists an irreducible curve $\Xi \subset A \setminus C$ such that $\deg p\langle P \rangle(C)_{\text{red}} < \deg C$ for all $P \in \Xi$. Let P_0 be a general point of Ξ and Q_0 a general point of C . Since P_0 is not on C , the projection $p\langle P_0 \rangle: C \rightarrow \Pi_0$ with the centre P_0 must be of mapping degree ≥ 2 onto its image. Thus there exists a point $R_0 \in C$ such

that $Q_0 \neq R_0$ and $p\langle P_0 \rangle(Q_0) = p\langle P_0 \rangle(R_0)$. Since Q_0 is chosen generally, $p\langle P_0 \rangle(Q_0)$ is a non-singular point of $p\langle P_0 \rangle(C)_{\text{red}}$, and the morphism $p\langle P_0 \rangle: C \rightarrow p\langle P_0 \rangle(C)_{\text{red}}$ is étale at Q_0 and at R_0 . This implies that the tangent lines $l(Q_0)$ and $l(R_0)$ of C at Q_0 and at R_0 , respectively, intersect each other in $\mathbb{P}^3$. Since P_0 is also chosen generally on Ξ , there exist small open neighbourhoods (in the sense of complex analytic geometry) $V \subset \Xi$ of P_0 in Ξ and $W \subset C$ of R_0 in C , and an isomorphism $f: V \xrightarrow{\sim} W$ such that $p\langle P \rangle(Q_0) = p\langle P \rangle(f(P))$ for all $P \in V$. This implies that the tangent line $l(R)$ of C at R intersects $l(Q_0)$ for all $R \in W$. Therefore $l(R) \cap l(Q_0) \neq \emptyset$ for every non-singular point R of C . However, consider the projection $\lambda: C \rightarrow \mathbb{P}^1$ with the centre $l(Q_0)$. Since C is non-degenerate, λ is surjective, and hence, by Sard's theorem, λ is smooth at a general point $R \in C$. This implies that $l(R) \cap l(Q_0) = \emptyset$ for a general $R \in C$, and we get a contradiction.

2. Proof of Proposition 1

First note that if $2n \geq d^2 - 5d + 8$ and $2n + 2k < d^2 - 4d + 3$, then $2n + 6k$ is automatically less than d^2 . Therefore, taking Nori's theorem into account, we may assume that the inequality $2n + 2k \geq d^2 - 4d + 3$ also holds.

Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C . The genus of $\tilde{C}$ is given by

$$g = (d-1)(d-2)/2 - n - k.$$

The assumed inequalities imply that

$$d \geq \max(2g+1, g+3), \quad \text{and} \quad k \leq d - g - 3. \quad (2.1)$$

Let L be the pull-back of $\mathcal{O}_C(1)$ by ν , which is an invertible sheaf of degree d on $\tilde{C}$. By the first inequality of (2.1), L is very ample and $\dim |L| = d - g$. Let C' be the image of the embedding of $\tilde{C}$ in $\mathbb{P}^{d-g}$ by L . Then C is obtained by a certain projection $C' \rightarrow \mathbb{P}^2$. Let $\Gamma \subset \mathbb{P}^{d-g}$ be the centre of this projection, which is a linear subspace of codimension 3. Let $T \subset \mathbb{P}^{d-g}$ be the union of all lines tangent to C' . Then Γ intersects T at distinct k points $P_1, \dots, P_k$, each of which corresponds to a cusp of C bijectively. By the second inequality of (2.1), there exists a linear subspace $\Lambda \subset \Gamma$ of codimension 1 which contains $P_1, \dots, P_k$. Let $p\langle \Lambda \rangle: C' \rightarrow \mathbb{P}^3$ be the projection with the centre Λ . The image $C'' \subset \mathbb{P}^3$ is a non-degenerate curve, which has at least k singular points $Q_1, \dots, Q_k$ corresponding to $P_1, \dots, P_k \in \Lambda \cap T$. The projection $p\langle \Gamma \rangle: C' \rightarrow C$ with the centre Γ factors as $C' \rightarrow C'' \rightarrow C$, and $C'' \rightarrow C$ maps $Q_1, \dots, Q_k$ to the cusps of C bijectively. Therefore, C'' has k cusps and some nodes (possibly none) as its only singularities. Thus the image $p(C'')$ of the general projection of C'' to $\mathbb{P}^2$ has the same numbers and types of singular points as the original curve C . Note that C and $p(C'')$ are contained in an irreducible equisingular family of plane curves, every member of which is obtained as an image of a projection from $C'' \subset \mathbb{P}^3$ to $\mathbb{P}^2$. Thus our main theorem implies that $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.

3. Examples

Let $\mathcal{A}$ denote the set of triples (d, n, k) with $d \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$ for which the following hold:

- (i) There exists a reduced irreducible plane curve $C \subset \mathbb{P}^2$ of degree d with n nodes and k cusps as its only singularities.

(ii) For any plane curve C as in (i), $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.

Nori's result says that if (d, n, k) satisfies (i) and $2n + 6k < d^2$, then $(d, n, k) \in \mathcal{A}$. Using Proposition 1, we shall enlarge the known region of $\mathcal{A}$.

Let $C \subset \mathbb{P}^n$ be a non-degenerate irreducible curve of degree d whose singular locus consists of only nodes and cusps. Let k be the number of cusps. We write by $\text{Tan}(C) \subset \mathbb{P}^n$ the union of all lines in $\mathbb{P}^n$ tangent to C at its non-singular points. The following proposition is elementary:

PROPOSITION 2. *If P is a general point of $\text{Tan}(C)$, then the image of the projection $C \rightarrow \mathbb{P}^{n-1}$ with the centre P is of degree d , has only nodes and cusps as its singularities, and the number of cusps is $k + 1$.*

Let C be a non-singular curve of genus g , and L a line bundle on C of degree $d \geq \max(2g + 1, g + 3)$. We embed C into $\mathbb{P}^{d-g}$ by $|L|$. Applying Proposition 2 to $C \subset \mathbb{P}^{d-g}$ repeatedly, we obtain a non-degenerate space curve $C' \subset \mathbb{P}^3$ of degree d with certain number of nodes (possibly none) and $d - g - 3$ cusps as its only singularities. Let $C'' \subset \mathbb{P}^2$ be the image of a general projection $C' \rightarrow \mathbb{P}^2$. Then C'' is an irreducible plane curve of degree d with

$$n = (d^2 - 5d + 8)/2 \text{ nodes} \quad \text{and} \quad k = d - g - 3 \text{ cusps}$$

as its only singularities. Since $2n \geq d^2 - 5d + 8$, Proposition 1 implies that $(d, n, k) \in \mathcal{A}$. Thus, choosing a line bundle of degree $\geq 6g + 10$, we obtain a region of $\mathcal{A}$ not covered by Nori's theorem.

Note that there is another known region of $\mathcal{A}$, which is given by the following theorem due to Zariski [7]. Let $R(d, k) \subset \mathbb{P}_*(\Gamma(\mathbb{P}^2, \mathcal{O}(d)))$ be the locus of all *rational* plane curves C of degree d with n nodes and k cusps as its only singularities. Since the genus of the normalization of C is 0, we have $n = (d - 1)(d - 2)/2 - k$.

THEOREM (Zariski). (1) *The locus $R(d, k)$ is non-empty if and only if $k \leq 3(d - 2)/2$.*
 (2) *For a member C of $R(d, k)$, $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian unless d is even and $k = 3(d - 2)/2$.*

Thus, if $n + k = (d - 1)(d - 2)/2$ and $k \leq (3d - 7)/2$, we have $(d, n, k) \in \mathcal{A}$.

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