Siegel Eisenstein series of degree n and Λ-adic Eisenstein series

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Abstract

Let F be a totally real field and χ a primitive narrow ray class character of F. We prove a formula for the Fourier coefficients of the Siegel Eisenstein series of degree n, weight k and character χ under certain conditions on χ. We also prove the existence of Λ-adic Siegel Eisenstein series of degree n.

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1. Introduction

In [15], we calculated the Fourier coefficients of Siegel Eisenstein series of $\mathrm{Sp}_2(\mathbb{Q})$ for an arbitrary primitive Dirichlet character. As an application, we constructed the p-adic analytic family of Siegel modular forms that interpolates Siegel Eisenstein series of $\mathrm{Sp}_2(\mathbb{Q})$, that is Λ-adic Siegel Eisenstein series. In this paper, we generalize our result to the case of Siegel Eisenstein series $E_{k,\chi}^{(n)}$ of $\mathrm{Sp}_n(F)$ for a primitive narrow ray class character χ of F, where F is a totally real field and n is a positive integer. We calculate the Fourier coefficients of $E_{k,\chi}^{(n)}$ under the following conditions on χ:

(i) The conductor of χ is relatively prime to 2.

(ii) If v is a finite place of F and χ_v is ramified, then χ_v^2 is also ramified.

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If F is the rational field and χ is a primitive Dirichlet character modulo N, then the conditions mean N is odd and χ_p is not the quadratic character modulo p for every prime $p \mid N$. Though we cannot explicitly compute Fourier coefficients of $E_{k,\chi}^{(n)}$ for an arbitrary χ, we can construct Λ-adic Siegel Eisenstein series that interpolates $E_{k,\chi}^{(n)}$ with χ satisfying the conditions above. The key ingredient of the proof of our main results is the functional equation of Whittaker functions (Theorem 5.1), which was proved by Ikeda. We assume the conditions above χ not only because we use the functional equation. If χ does not satisfy the conditions above, Fourier coefficients of $E_{k,\chi}^{(n)}$ would be complicated (see [9], [15] for the case where $n = 2$). In this case, The difficulty of computing the Fourier coefficients of Siegel Eisenstein series lies in the fact that the space defined in Lemma 8.2 is not one dimensional.

We state our main results. Let F be a totally real field with $[F : \mathbb{Q}] = m$ and $\mathcal{O}_F$ the integer ring of F. For a place v of F, we denote by F_v the completion of F with respect to v and by $\mathcal{O}_v$ the integer ring of F_v if v is a finite place. Let χ be a narrow ray class character of F modulo $\mathfrak{N}$. For an infinite place v of F, let r_v be an element of $\mathbb{Z}/2\mathbb{Z}$ satisfying the following condition.

$$\chi\left((a)\right) = \prod_{v|\infty} \operatorname{sgn}\left(\iota_v(a)\right)^{r_v} \text{ for } a \equiv 1 \bmod \mathfrak{N}.$$

Here v runs over the set of infinite places of F and ι_v is the real embedding corresponding to v. For $a \in F^\times$, we put

$$\operatorname{sgn}_\chi(a) = \prod_{v|\infty} \operatorname{sgn}^{r_v}\left(\iota_v(a)\right).$$

We define a character $\chi_f : (\mathcal{O}_F/\mathfrak{N})^\times \to \mathbb{C}^\times$ by

$$\chi_f(a) = \operatorname{sgn}_\chi(a)\chi\left((a)\right).$$

We denote by $\mathfrak{H}_n$ the Siegel upper half space of degree n:

$$\mathfrak{H}_n = \left\{ z = x + iy \;\middle|\; x, y \in \operatorname{Sym}_n(\mathbb{R}), y \text{ is positive definite} \right\}.$$

Let n be a positive integer. For $0 \le i \le n$, we denote by $w_{n,i}$ the matrix given as follows.

$$w_{n,i} = \left(\begin{array}{cc|cc} 0_i & 0 & -1_i & 0 \\ 0 & 1_{n-i} & 0 & 0_{n-i} \\ \hline 1_i & 0 & 0_i & 0 \\ 0 & 0_{n-i} & 0 & 1_{n-i} \end{array}\right).$$

We put $w_n = w_{n,n}$. We define the symplectic group of degree n by

$$\mathrm{Sp}_n(R) = \left\{ g \in \mathrm{GL}_{2n}(R) \;\middle|\; {}^t g w_n g = w_n \right\},$$

for any commutative ring R. For $g \in \mathrm{Sp}_n$, we denote $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$ with $a_g, b_g, c_g, d_g \in M_n$. Define the Siegel parabolic subgroup P_n by

$$P_n(R) = \left\{ g \in \mathrm{Sp}_n(R) \;\middle|\; c_g = 0 \right\}.$$

We define a congruence subgroup $\Gamma_0^{(n)}(\mathfrak{N})$ by

$$\Gamma_0^{(n)}(\mathfrak{N}) = \left\{ g \in \mathrm{Sp}_n(\mathcal{O}_F) \;\middle|\; c_g \equiv 0 \bmod \mathfrak{N} \right\}. \tag{1.1}$$

Let k be positive integer. We assume that $\chi_f(-1) = (-1)^k$ and $k > n+1$. We define the Siegel Eisenstein series $E_{k,\chi}^{(n)}$ of degree n on $\prod_{v|\infty} \mathfrak{H}_n$ as follows.

$$E_{k,\chi}^{(n)}(z) = \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P_n(\mathcal{O}_F) \cap \Gamma_0^{(n)}(\mathfrak{N}) \backslash \Gamma_0^{(n)}(\mathfrak{N})} \chi_f^{-1}(\det d) \det(cz+d)^{-k}. \tag{1.2}$$

Here $\det(cz+d)^{-k}$ is give by

$$\prod_{v|\infty} \det(\iota_v(c) z_v + \iota_v(d))^{-k}, \qquad \text{for } z = (z_v)_v \in \prod_{v|\infty} \mathfrak{H}_n.$$

The right hand side of (1.2) is absolutely convergent and defines a Siegel-Hilbert modular form of degree n, weight k, character χ and level $\mathfrak{N}$. For $R = \mathcal{O}_F$ or $\mathcal{O}_v$ with $v < \infty$, we define the set of half integral matrices by

$$\mathrm{Sym}_n^{(*)}(R) = \left\{ B \in \mathrm{Sym}_n(K) \;\middle|\; \mathrm{Tr}_{K/L}(\mathrm{Tr}(AB)) \in S, \text{ for all } A \in \mathrm{Sym}_n(R) \right\}.$$

Here we put

$$S = \begin{cases} \mathbb{Z} & \text{if } R = \mathcal{O}_F, \\ \mathbb{Z}_p & \text{if } R = \mathcal{O}_v \text{ and } p \text{ is the residual characteristic of } R, \end{cases}$$

$K = \mathrm{Frac}\, R$ and $L = \mathrm{Frac}\, S$. Let

$$E_{k,\chi}^{(n)}(z) = \sum_{0 \le B \in \mathrm{Sym}_n^{(*)}(\mathcal{O}_F)} a(B, E_{k,\chi}^{(n)}) \exp(2\pi i \,\mathrm{Tr}\, Bz),$$

be the Fourier expansion of $E^{(n)}_{k,\chi}$. Here we put

$$\mathbf{e}(Bz) = \exp\left(2\pi i \sum_{v|\infty} \operatorname{Tr} \iota_v(B) z_v\right),$$

for $B \in \mathrm{Sym}^{(*)}_n(\mathcal{O}_F)$ and $z = (z_v)_v \in \prod_{v|\infty} \mathfrak{H}_n$. The inequality $0 \leq B$ means B is totally positive semi-definite. The first main result of this paper is the formula for $a(B, E^{(n)}_{k,\chi})$. We prepare some notation for our main result. Let $B \in \mathrm{Sym}^{(*)}_n(\mathcal{O}_F)$ and put $r = \operatorname{rank}_F B$. There exists a matrix $A \in \mathrm{GL}_n(F)$ such that

$${}^tABA = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix},$$

with $B' \in \mathrm{Sym}_r(F)$. Then $\det B' \in F^\times/F^{\times 2}$ does not depend on the choice of A. If r is even, then we denote by χ_B the primitive narrow class character of F associated with the extension $F(\sqrt{(-1)^{r/2}\det B'})/F$ by the global class field theory. For a prime $\mathfrak{p}$ of F with $\mathfrak{p} \nmid \mathfrak{N}$, there exists a matrix $U \in \mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$ that satisfies

$${}^tUBU = \begin{pmatrix} B'_\mathfrak{p} & 0 \\ 0 & 0 \end{pmatrix},$$

with $B'_\mathfrak{p} \in \mathrm{Sym}^*_r(\mathcal{O}_\mathfrak{p})$. The matrix $B'_\mathfrak{p}$ is unique up to unimodular equivalence. Therefore $\Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p}; T)$ does not depend on the choice of U, where $\Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p}; T)$ is the polynomial obtained by the Siegel series (see (6.1) for the definition). We put $\Phi^{(r)}_\mathfrak{p}(B, T) = \Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p}, T)$.

Theorem 1.1. *Let $0 \leq B \in \mathrm{Sym}^{(*)}_n(\mathcal{O}_F)$ be a half integral totally positive semi-definite matrix of size n and $k > n+1$ an integer. We have $a(0_n, E^{(n)}_{k,\chi}) = 1$. Put $r = \operatorname{rank} B$ and assume $r > 0$. Let χ be a primitive narrow class character of F of conductor $\mathfrak{N}$ with $\chi_f(-1) = (-1)^k$. Assume that $\mathfrak{N}$ is relatively prime to 2 and $\chi^2_\mathfrak{p}$ is ramified for all $\mathfrak{p} \mid \mathfrak{N}$. If r is even, then $a(B, E^{(n)}_{k,\chi})$ is given by*

$$2^{rm/2} L(1-k,\chi)^{-1} L^{(\mathfrak{N})}(1+r/2-k, \chi_B\chi) \prod_{i=1}^{r/2} L^{(\mathfrak{N})}(1+2i-2k, \chi^2)^{-1} \times \prod_{\mathfrak{p}\nmid\mathfrak{N}} \Phi^{(r)}_\mathfrak{p}(B; \chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-r-1}).$$

If r is odd, then $a(B, E_{k,\chi}^{(n)})$ is given by

$$2^{(r+1)m/2} L(1-k,\chi)^{-1} \prod_{i=1}^{(r-1)/2} L^{(\mathfrak{N})}(1+2i-2k,\chi^2)^{-1} \prod_{\mathfrak{p}\nmid\mathfrak{N}} \Phi_{\mathfrak{p}}^{(r)}(B;\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-r-1}).$$

Here for a Hecke L-function $L(s,\chi)$ and an ideal $\mathfrak{N}$, we denote

$$L^{(\mathfrak{N})}(s,\chi) = \prod_{\mathfrak{p}\nmid\mathfrak{N}} (1-\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1},$$

where $\mathfrak{p}$ runs over the set of primes of F relatively prime to the conductor of χ and the ideal $\mathfrak{N}$.

Remark 1.2. The product $\prod_{\mathfrak{p}\nmid\mathfrak{N}} \Phi_{\mathfrak{p}}^{(r)}(B;\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-r-1})$ in the statement of the theorem is essentially a finite product. If $F=\mathbb{Q}$ and $n=1$, this factor is equal to the divisor sum $\sigma_{k-1,\chi}(B) = \sum_{d|B} \chi_f(d) d^{k-1}$.

Remark 1.3. Let $F=\mathbb{Q}$. We regard χ as a Dirichlet character modulo $N=\prod_{p|N} p^{n_p}$. We decompose $\chi = \prod_{p|N} \chi_p$, where χ_p is a Dirichlet character modulo p^{n_p}. In this case, the assumption of the theorem is equivalent to N is odd and χ_p is not equal to the quadratic character modulo p for all $p \mid N$. For $F=\mathbb{Q}$, Katsurada [6] proved an explicit formula for the polynomial $\Phi_{\mathfrak{p}}^{(n)}(B;T)$ for an arbitrary n. Thus by the theorem, we obtain an explicit formula for $a(B, E_{k,\chi}^{(n)})$ in this case. If $p \mid N$ and χ_p is the quadratic character modulo p, then p-Euler factor of $a(B, E_{k,\chi}^{(n)})$ does not seem to be simple (cf. [9], [15]).

In the next section, we shall define another Siegel Eisenstein $G_{k,\chi}^{(n)}$ and state the formula for the Fourier coefficients of $G_{k,\chi}^{(n)}$ (see Theorem 2.3). In Theorem 2.3, we do not assume $\mathfrak{N}$ is relatively prime to 2 and $\chi_{\mathfrak{p}}^2$ is ramified for all $\mathfrak{p} \mid \mathfrak{N}$. Theorem 1.1 is a consequence of Theorem 2.3 and the fact that $G_{k,\chi}^{(n)}$ coincides with $E_{k,\chi}^{(n)}$ if $\mathfrak{N}$ is relatively prime to 2 and $\chi_{\mathfrak{p}}^2$ is ramified for all $\mathfrak{p} \mid \mathfrak{N}$. The key ingredient for the proof of Theorem 2.3 is the explicit functional equation of Whitaker functions proved by Ikeda.

The second main result of this paper is the existence of Λ-adic Siegel Eisenstein series. By Theorem 2.3 and the existence of the p-adic Hecke L-functions [1] for totally real fields, we can prove the existence of Λ-adic Siegel Eisenstein series.

Before we state the next theorem, we recall the p-adic Hecke L-functions for the totally real field F. We follow Hida's book [3, §3.9]. For an integral ideal $\mathfrak{N}$, we denote the group of fractional ideals of F relatively prime to $\mathfrak{N}$ by $I_{\mathfrak{N}}$. We define the narrow ray class group of F of conductor $\mathfrak{N}$ by $\mathrm{Cl}_F(\mathfrak{N}) = I_{\mathfrak{N}}/P_+(\mathfrak{N})$. For a rational prime p, we consider the projective limit

$$\mathrm{Cl}_F(\mathfrak{N}p^\infty) = \varprojlim_{\alpha} \mathrm{Cl}_F(\mathfrak{N}p^\alpha).$$

We put $G(\mathfrak{N}) = \mathrm{Cl}_F(\mathfrak{N}p^\infty)$. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ and $\mathcal{O}_{\mathbb{C}_p}$ the integer ring of $\mathbb{C}_p$. We fix the embedding of $\overline{\mathbb{Q}}$ to $\mathbb{C}$ and $\mathbb{C}_p$. We denote $\mathrm{Meas}(G(\mathfrak{N}), \mathcal{O}_{\mathbb{C}_p})$ by the bounded p-adic measure on $G(\mathfrak{N})$ with values in $\mathcal{O}_{\mathbb{C}_p}$. Since $I_{\mathfrak{N}p}$ can be considered as a dense subgroup of $G(\mathfrak{N})$ and the norm map $N : I_{\mathfrak{N}p} \to \mathbb{Z}_p^\times$ is continuous, we can extend N to $G(\mathfrak{N})$. We denote the extended character by the same letter. Then for $a \in G(\mathfrak{N})$, there exists a p-adic measure $\zeta_a \in \mathrm{Meas}(G(\mathfrak{N}), \mathcal{O}_{\mathbb{C}_p})$ such that for any narrow ray class character $\chi : \mathrm{Cl}_F(\mathfrak{N}p^\alpha)$ and any $n \in \mathbb{Z}_{\geq 0}$, we have

$$\int \chi(x)N(x)^n d\zeta_a(x) = (1 - \chi(a)N(a)^{n+1})L^{(\mathfrak{N}p)}(-n, \chi).$$

Theorem 1.4. *Let $\mathfrak{N}$ be an ideal of $\mathcal{O}_F$ and p a rational prime. We assume $(\mathfrak{N}, p) = 1$. For $B \in \mathrm{Sym}_n^{(*)}(\mathbb{Z})$, we denote by $\mathfrak{f}(\chi_B)$ the conductor of χ_B. Then for an integer $k > n+1$ and a primitive narrow ray class character χ of conductor $\mathfrak{N}p^\alpha$ with $\alpha \geq 1$, there exists a non-zero constant $c_{k,\chi} \in \overline{\mathbb{Q}}^\times$ that satisfies the following statement. For each B, there exits a p-adic measure $\mu_B \in \mathrm{Meas}(G(\mathfrak{N}\ \mathfrak{f}(\chi_B)), \mathcal{O}_{\mathbb{C}_p})$ that satisfies*

$$\int \chi(x)N(x)^k d\mu_B(x) = a(B, H_{k,\chi}),$$

Here we put $H_{k,\chi} = c_{k,\chi}G_{k,\chi}^{(n)}$ and $G_{k,\chi}^{(n)}$ is the Siegel Eisenstein series defined in §2.

We can prove this theorem easily by Theorem 2.3, the existence of the p-adic measure which interpolates the valued of Hecke L function and the same argument in the Proposition 1 of chapter 7 [3]. We omit the proof.

We recall recent results on the explicit formula for the Fourier coefficients of $E_{k,\chi}^{(n)}$ and Λ-adic Siegel Eisenstein series in the case where $F = \mathbb{Q}$. Katsurada [6] proved the explicit formula for the polynomial $\Phi_{\mathfrak{p}}^{(n)}(B;T)$ for an arbitrary degree n. If the degree n is equal to 2 and the conductor of χ

is square free odd integer, Mizuno [9] proved the explicit formula for $E^{(2)}_{k,\chi}$. Gunji [2] also computed local factor of Fourier coefficients of $E^{(2)}_{k,\chi}$ at the odd prime. For a general primitive character χ, the author [15] computed the Fourier coefficients of $E^{(2)}_{k,\chi}$. H. Kawamura [8] proved the existence Λ-adic Siegel Eisenstein series that interpolates p-adic stabilization of Siegel Eisenstein series of degree n, level 1. Recently, he [7] also proved the existence Λ-adic Siegel Eisenstein series that interpolates Eisenstein series of degree n with a character of conductor $p^{\nu+1}$, where p is an odd prime and $\nu \geq 1$. His method uses the control theorem proved by V. Pilloni.

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Notation. For a commutative ring R, we denote by $M_{n,m}(R)$ the set of n-by-m matrices with entries in R. We denote by $M_n(R)$ the set of n-by-n matrices with entries in R. For $\alpha \in M_{n,m}(R)$, we denote by ${}^t\alpha$ the transpose of α. We denote by $\mathrm{Sym}_n(R)$ the set of symmetric matrices of size n with entries in R and by $\mathrm{Sp}_n(R)$ the symplectic group of size $2n$. For $\alpha \in \mathrm{Sym}_n(R)$ and $\beta \in \mathrm{GL}_n(R)$, we put $\nu_n(\alpha) = \begin{pmatrix} 1_n & \alpha \\ & 1_n \end{pmatrix}$ and $\mu_n(\beta) = \begin{pmatrix} \beta & \\ & {}^t\beta^{-1} \end{pmatrix}$. Then $\nu(\alpha)$ and $\mu(\beta)$ are elements of $\mathrm{Sp}_n(R)$. For $x \in M_n$ and $y \in M_{n,m}$, we define $x[y] \in M_m$ by ${}^tyxy$. For $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix} \in \mathrm{Sp}_r$ and $h = \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix} \in \mathrm{Sp}_s$, we put

$$\iota(g \times h) = \left(\begin{array}{cc|cc} a_g & & b_g & \\ & a_h & & b_h \\ \hline c_g & & d_g & \\ & c_h & & d_h \end{array}\right) \in \mathrm{Sp}_{r+s}.$$

We put

$$\iota_{r+s}(g) = \left(\begin{array}{cc|cc} a_g & & b_g & \\ & 1_s & & 0_s \\ \hline c_g & & d_g & \\ & 0_s & & 1_s \end{array}\right).$$

The organization of this paper is as follows. In §2, we define another Siegel Eisenstein series $G^{(n)}_{k,\chi}$ and state result for the Fourier coefficients of $G^{(n)}_{k,\chi}$. In §3, we introduce basic notions of Siegel modular forms. In §4, we introduce general results of Euler factor of Fourier coefficients of Siegel Eisenstein series.

In §5, we introduce the functional of Whittaker functions associated with $\mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^{s})$, that is a key result for our main theorem. In §6, we introduce a result for Euler factors of Siegel Eisenstein series at unramified places. In §7, we prove Theorem 2.3 and in §8 we prove Proposition 2.2.

2. The definition of $G_{k,\chi}^{(n)}$

In this section, we define $G_{k,\chi}^{(n)}$ that appeared in Theorem 1.4. Let χ be a narrow class character modulo $\mathfrak{N}$ and k a positive integer with $\chi_f(-1) = (-1)^k$. We denote by $\mathbb{A}_F$ the adele ring of F and by $\mathbb{A}_F^\times$ the idele group of F. Denote the character of finite order of $\mathbb{A}_F^\times/F^\times$ corresponding to χ by $\widetilde{\chi}$.

For a place v of F, we define a maximal compact subgroup of $\mathrm{Sp}_n(F_v)$ by

$$C_v = \begin{cases} \left\{ g \in \mathrm{Sp}_n(\mathbb{R}) \;\middle|\; a_g = d_g \text{ and } b_g = -c_g \right\} & \text{if } v \text{ is real,} \\ \mathrm{Sp}_n(\mathcal{O}_v) & \text{if } v \text{ is finite.} \end{cases}$$

We denote the space for the normalized induction by $\mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_v^{-1}|\cdot|_v^{s})$, that is, the set of $\mathbb{C}$-valued functions on $\mathrm{Sp}_n(F_v)$ satisfying the following two conditions.

(i) For $\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \in P_n(F_v)$ and $g \in \mathrm{Sp}_n(F_v)$,

$$f\left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} g \right) = \widetilde{\chi}_{\mathfrak{p}}^{-1}(\det a)|\det a|_v^{s+(n+1)/2} f(g).$$

(ii) f is right C_v-finite.

Here $|\cdot|_v$ is the absolute value of F_v. If v is a finite place, then we normalized $|\cdot|_v$ so that

$$|\varpi|_v = \#\left(\mathcal{O}_v/\varpi\mathcal{O}_v\right)^{-1}$$

for a uniformizer ϖ of F_v. We define the intertwining operator

$$M_{w_n}^{(s)} : \mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_v^{-1}|\cdot|_v^{s}) \to \mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_v|\cdot|_v^{-s}) \tag{2.1}$$

by

$$M_{w_n}^{(s)}(f)(g) = \int_{\mathrm{Sym}_n(F_v)} f\left(w_n \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx,$$

for $g \in \mathrm{Sp}_n(F_v)$. Here we take a Haar measure of $\mathrm{Sym}_n(F_v)$ so that we have $\int_{\mathrm{Sym}_n(\mathcal{O}_v)} dx = 1$ for a finite place v. The integral is convergent if $\mathrm{Re}\, s$ is sufficiently large and has meromorphic continuation to the whole complex plane.

We define a compact subgroup $C_{0,v}$ of C_v as follows.

(i) If v is real or $\widetilde{\chi}_v$ is unramified, then we define $C_{0,v} = C_v$.

(ii) If $v = \mathfrak{p}$ is a finite place and $\widetilde{\chi}_{\mathfrak{p}}$ is ramified, then we define

$$C_{0,v} = \left\{ g \in \mathrm{Sp}_n(\mathcal{O}_v) \;\middle|\; c_g \equiv 0 \bmod \mathfrak{p}^{\nu} \right\}.$$

Here $\mathfrak{p}^{\nu}$ is the conductor of $\widetilde{\chi}_v$.

We define a character κ_v of $C_{0,v}$ as follows.

(i) If v is real, then we define

$$\kappa_v\left(\begin{pmatrix} u & -v \\ v & u \end{pmatrix}\right) = \det(u+iv)^{-k}.$$

(ii) If v is finite and $\widetilde{\chi}_v$ is unramified, then we define $\kappa_v = 1$.

(iii) If v is finite and $\widetilde{\chi}_v$ is ramified, then we define

$$\kappa_v(\gamma) = \widetilde{\chi}_v(\det d_\gamma).$$

We denote by $\phi_v(s,\cdot)$ the element of $\mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_v^{-1}|\cdot|_v^s)$ satisfying the following conditions.

$$\begin{aligned} &\operatorname{supp} \phi_v(s,\cdot) = P_n(F_v)C_{0,v}, \\ &\phi_v(s,g\gamma) = \kappa_v(\gamma)\phi_v(s,g) \quad \text{for all } \gamma \in C_{0,v}, \\ &\phi_v(s,1) = 1. \end{aligned} \tag{2.2}$$

Here $\operatorname{supp} \phi_v(s,\cdot)$ is the support of the function $g \mapsto \phi_v(s,g)$. We put $\phi(s,g) = \prod_v \phi_v(s,g)$. Then by (3.5), the Eisenstein series $E_{k,\chi}^{(n)}$ corresponds to an Eisenstein series on $\mathrm{Sp}_n(\mathbb{A}_F)$ defined by

$$\sum_{\gamma \in P_n(F)\backslash \mathrm{Sp}_n(F)} \phi\left(k - \frac{n+1}{2}, \gamma g\right).$$

We denote by $\phi'_v(-s,\cdot)$ the element of $\mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_v|\cdot|_v^{-s})$ satisfying the following conditions.

$$\begin{aligned}
&\operatorname{supp}\phi'_v(-s,\cdot) = P_n(F_v)w_nC_{0,v},\\
&\phi'_v(-s,g\gamma) = \kappa_v(\gamma)\phi'_v(-s,g) \quad \text{for all } \gamma\in C_{0,v}.\\
&\phi'_v(-s,w_n) = 1.
\end{aligned} \tag{2.3}$$

For a place v of F, we define $\varphi_{n,v}(s,\cdot)=\varphi_v(s,\cdot)\in \mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_v^{-1}|\cdot|_v^{s})$ as follows.

$$\varphi_v(s,g) = \begin{cases} \phi_v(s,g) & \text{if } v \text{ is real or } \widetilde{\chi}_v \text{ is unramified.}\\ M_{w_n}^{(-s)}(\phi'_v(-s,\cdot))(g) & \text{if } v \text{ is finite and } \widetilde{\chi}_v \text{ is ramified.}\end{cases} \tag{2.4}$$

Definition 2.1. Let $\varphi_v(s,g)$ be a function defined in (2.4). For $g=(g_v)_v\in \mathrm{Sp}_n(\mathbb{A}_F)$, we put

$$\varphi(s,g)=\prod_v \varphi_v(s,g_v),$$

where v runs over the set of the all places of F. We define an Eisenstein series on $\mathrm{Sp}_n(\mathbb{A}_F)$ by

$$\mathcal{G}_{s,\chi}^{(n)}(g)=\sum_{\gamma\in P_n(F)\backslash \mathrm{Sp}_n(F)}\varphi(s,\gamma g).$$

We define $G_{k,\chi}^{(n)}$ by the function on $\prod_{v|\infty}\mathfrak{H}_n$ corresponding to $\mathcal{G}_{k-(n+1)/2,\chi}^{(n)}$ by (3.5).

We explain why we choose $\varphi_v(s,g)$ as above. To compute the Fourier coefficients of $E_{k,\chi}^{(n)}$, we have to compute the following local integral at a place v of F

$$\int_{u\in \mathrm{Sym}_n(F_v)} f_v\left(w_n\nu_n(u)g\right)\psi_v(-Bu)du \tag{2.5}$$

for $f_v=\phi_v(s,\cdot)$ (see Proposition 4.2). Here ψ_v is a character of F_v and $B\in \mathrm{Sym}_n(F)$. But it is difficult to compute the integral (2.5) for $f_v=\phi_v(s,\cdot)$ in general. As we shall see in Proposition 8.1, ϕ_v is equal to φ_v if χ satisfies the condition in the introduction. Thus we compute the integral (2.5) for $f_v=\varphi_v(s,\cdot)$ instead. By the functional equation (Theorem 5.1), it is enough to compute the integral (2.5) for $f_v=\phi'_v(-s,\cdot)$ to compute the integral for $f_v=\varphi_v(s,\cdot)$ at a ramified place v. The reason for the choice of φ_v is that the computation of the integral (2.5) for $f_v=\phi'_v(-s,\cdot)$ is easy.

Put

$$\mathcal{P} = \left\{ \mathfrak{p} : \text{a prime of } F \;\middle|\; \mathfrak{p} \mid \mathfrak{N} \text{ and } \widetilde{\chi}_{\mathfrak{p}}^2 \text{ is unramified} \right\}.$$

Let $g \in \mathrm{Sp}_n(\mathcal{O}_F)$ and assume $c_g \in \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{N})}$ if $\mathfrak{p} \notin \mathcal{P}$ and $\mathrm{rank}_{\mathcal{O}_F/\mathfrak{p}}(c_g \bmod \mathfrak{p}) = i_{\mathfrak{p}}$ with $0 \leq i_{\mathfrak{p}} \leq n$ if $\mathfrak{p} \in \mathcal{P}$. For $\mathfrak{p} \in \mathcal{P}$, the assumption for g implies $g \bmod \mathfrak{p} \in P_n(\mathcal{O}_F/\mathfrak{p}) w_{i_{\mathfrak{p}}} P_n(\mathcal{O}_F/\mathfrak{p})$. Therefore if $\mathfrak{p} \in \mathcal{P}$, there exist elements $x_{\mathfrak{p}},\ y_{\mathfrak{p}} \in \mathrm{GL}(\mathcal{O}_F/\mathfrak{p})$ satisfying

$$g \bmod \mathfrak{p} = \begin{pmatrix} x_{\mathfrak{p}} & * \\ 0 & {}^t x_{\mathfrak{p}}^{-1} \end{pmatrix} w_{i_{\mathfrak{p}}} \begin{pmatrix} y_{\mathfrak{p}} & * \\ 0 & {}^t y_{\mathfrak{p}}^{-1} \end{pmatrix}.$$

We put

$$\chi(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,; g) = \prod_{\substack{\mathfrak{p}\mid\mathfrak{N} \\ \mathfrak{p}\notin\mathcal{P}}} (\chi_f)_{\mathfrak{p}} (\det d_g) \prod_{\mathfrak{p}\in\mathcal{P}} (\chi_f)_{\mathfrak{p}} (\det x_{\mathfrak{p}} \det y_{\mathfrak{p}}).$$

Here $(\chi_f)_{\mathfrak{p}}$ is the $\mathfrak{p}$-component of χ_f. If $(\mathfrak{N}, 2) = 1$, then we define an Eisenstein series $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,; z)$ as follows.

$$E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,; z) = \sum_{g} \chi(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,; g)^{-1} \det(c_g z + d_g), \tag{2.6}$$

where g runs over the set $P_n(F) \cap \mathrm{Sp}_n(\mathcal{O}_F) \backslash \mathrm{Sp}_n(\mathcal{O}_F)$ satisfying the property $c_g \in \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{N})}$ if $\mathfrak{p} \notin \mathcal{P}$ and $\mathrm{rank}_{\mathcal{O}_F/\mathfrak{p}}(c_g \bmod \mathfrak{p}) = i_{\mathfrak{p}}$ if $\mathfrak{p} \in \mathcal{P}$. By the definition, we have $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,; z) = E^{(n)}_{k,\chi}$ if $i_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathcal{P}$.

Let $\mathfrak{p}$ be a prime of F and assume $\mathfrak{p} \in \mathcal{P}$ and $(\mathfrak{p}, 2) = 1$. For $0 \leq i \leq n$, we set

$$m_i(k, \chi, \mathfrak{p}) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \widetilde{\chi}_{\mathfrak{p}}(-1)^{i/2} q^{-i/2} \left(q^{-1}; q^{-2}\right)_{i/2} & \\ \quad \times \left(\widetilde{\chi}_{\mathfrak{p}}^2(\mathfrak{p}) q^{-i-2k+2n+1}; q^2\right)_{i/2}^{-1} & \text{if } i \text{ is even.} \end{cases} \tag{2.7}$$

Here we put $q = \mathrm{N}\mathfrak{p}$ and for a positive integer r, we denote by $(x; y)_r$ the following product:

$$(x; y)_r = \prod_{m=0}^{r-1} (1 - xy^m).$$

We prove the following proposition in §8.

Proposition 2.2. *Assume that $\mathfrak{N}$ is relatively prime to 2. If $\mathcal{P} \neq \emptyset$, we have*

$$G_{k,\chi}^{(n)} = \sum_{\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}} \in \{0,\dots,n\}^{|\mathcal{P}|}} \left(\prod_{\mathfrak{p}\in\mathcal{P}} m_{i_{\mathfrak{p}}}(k,\chi,\mathfrak{p}) \right) E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,; z).$$

If $\mathcal{P} = \emptyset$, we have

$$G_{k,\chi}^{(n)} = E_{k,\chi}^{(n)}.$$

The following theorem gives a formula for Fourier coefficients of $G_{k,\chi}^{(n)}$. We prove the theorem in §7.

Theorem 2.3. *Let $0 \leq B \in \mathrm{Sym}_n^{(*)}(\mathcal{O}_F)$ be a half integral totally positive semi-definite matrix of size n and $k > n+1$ an integer. Let χ be a primitive narrow class character of F of conductor $\mathfrak{N}$. We have $a(0_n, G_{k,\chi}^{(n)}) = 1$. Put $r = \operatorname{rank} B$ and assume $r > 0$. Then the following assertions hold. If r is even, then $a(B, G_{k,\chi}^{(n)})$ is given by*

$$2^{rm/2} \prod_{\mathfrak{p}\nmid\mathfrak{N}} \Phi_{\mathfrak{p}}^{(r)}(B; \chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-r-1})$$
$$\times L(1-k,\chi)^{-1} L^{(\mathfrak{N})}(1+r/2-k, \chi_B\chi) \prod_{i=1}^{r/2} L^{(\mathfrak{N})}(1+2i-2k, \chi^2)^{-1}.$$

If r is odd, then $a(B, G_{k,\chi}^{(n)})$ is given by

$$2^{(r+1)m/2} \prod_{\mathfrak{p}\nmid\mathfrak{N}} \Phi_{\mathfrak{p}}^{(r)}(B; \chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-r-1}) L(1-k,\chi)^{-1} \prod_{i=1}^{(r-1)/2} L^{(\mathfrak{N})}(1+2i-2k, \chi^2)^{-1}.$$

Remark 2.4. Theorem 1.1 follows from Proposition 2.2 and Theorem 2.3.

3. Classical and adelic Siegel-Hilbert modular forms

In this section, we define classical and adelic Siegel-Hilbert modular forms and recall the relation between them.

The symmetric group $\mathrm{Sp}_n(\mathbb{R})$ acts on $\mathfrak{H}_n$ by

$$g \cdot z = (a_g z + b_g)(c_g z + d_g)^{-1},$$

for $g \in \mathrm{Sp}_n(\mathbb{R})$ and $z \in \mathfrak{H}_n$. We define a factor of automorphy $j(g,z)$ by

$$j(g,z) = \det(c_g z + d_g).$$

The group $\mathrm{Sp}_n(F \otimes_{\mathbb{Q}} \mathbb{R}) \cong \prod_{v|\infty} \mathrm{Sp}_n(\mathbb{R})$ acts on $\prod_{v|\infty} \mathfrak{H}_n$ diagonally. Thus as a subgroup of $\mathrm{Sp}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$, $\mathrm{Sp}_n(F)$ also acts on $\prod_{v|\infty} \mathfrak{H}_n$. For $g = (g_v) \in \prod_{v|\infty} \mathrm{Sp}_n(\mathbb{R})$ and $z = (z_v) \in \prod_{v|\infty} \mathfrak{H}_n$, we put

$$j(g, z) = \prod_{v|\infty} j(g_v, z_v).$$

Let $\mathfrak{N}$ be an ideal of F. Let $\Gamma_0^{(n)}(\mathfrak{N})$ be the congruence subgroup defined by (1.1). Let χ be a narrow ray class character modulo $\mathfrak{N}$ and k be an integer. We define the space $M_k(\Gamma_0^{(n)}(\mathfrak{N}), \chi)$ of Siegel-Hilbert modular forms of weight k, of character χ, of level $\mathfrak{N}$ by the set of holomorphic $\mathbb{C}$-valued functions f on $\prod_{v|\infty} \mathfrak{H}_n$ satisfying the following condition:

$$f(\gamma \cdot z) = \chi_f(\det \gamma_d) j(\gamma, z)^k f(z), \qquad \text{for all } \gamma \in \Gamma_0^{(n)}(\mathfrak{N}). \tag{3.1}$$

If $m = n = 1$, then we add the cusp condition. By the condition (3.1), we have the Fourier expansion

$$f(z) = \sum_{B \in \mathrm{Sym}_n^{(*)}(\mathcal{O}_F)} a(B, f) \mathbf{e}(Bz).$$

By the Koecher principle (or the cusp condition), we have $a(B, f) = 0$ unless B is totally positive semi-definite.

Next we consider modular forms on the adele group $\mathrm{Sp}_n(\mathbb{A}_F)$. Let C_v and $C_{0,v}$ be the compact subgroups of $\mathrm{Sp}_n(F_v)$ defined in §2. We put $C_0(\mathfrak{N}) = \prod_{v<\infty} C_{0,v}$. Since the strong approximation theorem holds for Sp_n, we have

$$\mathrm{Sp}_n(\mathbb{A}_F) = \mathrm{Sp}_n(F) C_0(\mathfrak{N}) \prod_{v|\infty} \mathrm{Sp}_n(F_v).$$

For a $\mathbb{C}$-valued function f on $\prod_{v|\infty} \mathfrak{H}_n$ satisfying (3.1), we define a $\mathbb{C}$-valued function $\widetilde{f}$ on $\mathrm{Sp}_n(\mathbb{A}_F)$ by

$$\widetilde{f}(g) = f(\xi \cdot i1_n) j(\xi, i1_n)^{-k} \prod_{v|\mathfrak{N}} \widetilde{\chi}_v(\det\ (d_\gamma)),$$

where $\alpha \in \mathrm{Sp}_n(F)$, $\gamma \in C_0(\mathfrak{N})$, $\xi \in \prod_{v|\infty} \mathrm{Sp}_n(F_v)$ and $g = \alpha\gamma\xi$. Then $\widetilde{f}(g)$ does not depend on the choice of α, γ and ξ.

By definition $\widetilde{f}$ satisfies the following conditions:

$$\widetilde{f}(\alpha g) = \widetilde{f}(g) \text{ for } \alpha \in \mathrm{Sp}_n(F), \tag{3.2}$$

$$\widetilde{f}(g\theta) = \widetilde{f}(g) j(\theta, i1_n)^{-k} \text{ for } \theta \in \prod_{v|\infty} C_v, \tag{3.3}$$

$$\widetilde{f}(g\gamma) = \widetilde{f}(g) \prod_{v|\mathfrak{N}} \widetilde{\chi}_v(\det\,(d_\gamma)) \text{ for } \gamma \in C_0(\mathfrak{N}). \tag{3.4}$$

Conversely, given a function $\widetilde{f}$ on $\mathrm{Sp}_n(\mathbb{A}_F)$ satisfying (3.2), (3.3) and (3.4) we define a function f on $\prod_{v|\infty} \mathfrak{H}_n$ satisfying (3.1) by

$$f(z) = \widetilde{f}(g_\infty) j(g_\infty, i1_n)^k, \tag{3.5}$$

where $z = g_\infty \cdot i$ and $g_\infty \in \prod_{v|\infty} \mathrm{Sp}_n(F_v)$. Then this gives the inverse map to the map $f \mapsto \widetilde{f}$.

For a place v of F, we define a character ψ_v of F_v as follows. If v is real, then we put $\psi_v(x) = \mathbf{e}(\iota_v(x))$. Let v is a finite place and p be the residual characteristic of F_v. We define a character ψ_p of $\mathbb{Q}_p$ by the unique continuous character such that $\psi_p(x) = \mathbf{e}(-x)$ for $x \in \mathbb{Z}[p^{-1}]$. We put $\psi_v(x) = \psi_p(\mathrm{Tr}_{F_v/\mathbb{Q}_p}\, x)$. We define a character ψ of $\mathbb{A}_F/F$ by $\psi = \prod_{v\,:\,\text{place of } F} \psi_v$.

For $B \in \mathrm{Sym}_n(F)$ and a function $\widetilde{f}$ on $\mathrm{Sp}_n(\mathbb{A}_F)$ satisfying the conditions (3.2), (3.3) and (3.4), we define a function $\mathrm{Wh}_B(f)$ on $\mathrm{Sp}_n(\mathbb{A}_F)$ as follows.

$$\mathrm{Wh}_B(f)(g) = \int_{\mathrm{Sym}_n(F)\backslash \mathrm{Sym}_n(\mathbb{A}_F)} \widetilde{f}(\nu(x)g)\psi^{-1}(Bx)dx.$$

Here we take a Haar measure of $\mathrm{Sym}_n(\mathbb{A}_F)$ so that

$$\int_{\mathrm{Sym}_n(F)\backslash \mathrm{Sym}_n(\mathbb{A}_F)} dx = 1.$$

For $f \in M_k(\Gamma_0(\mathfrak{N}), \chi)$ and $B \in \mathrm{Sym}_n^{(*)}(\mathcal{O}_F)$, we have

$$a(B, f)\mathbf{e}(iBy) = \det y^{-k/2}\, \mathrm{Wh}_B(f)\left(\mu(y^{1/2})\right). \tag{3.6}$$

Here for $y = (y_v)_{v|\infty} \in \prod_{v|\infty} \mathrm{Sym}_n(\mathbb{R})$ with $y_v > 0$, we put

$$\det y^{-k/2} = \prod_{v|\infty} \det y_v^{-k/2}, \qquad \mu(y^{1/2}) = \prod_{v|\infty} \mu(y_v^{1/2}).$$

4. Siegel Eisenstein series and the Euler factor of Fourier coefficients

Let k be an integer and f an element of $\mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}^{-1}|\cdot|^{k-(n+1)/2})$ and suppose $f=\prod_v f_v$, where $f_v \in \mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_v^{-1}|\cdot|_v^{k-(n+1)/2})$. We define an Eisenstein series $\mathcal{E}_f$ on $\mathrm{Sp}_n(\mathbb{A}_F)$ associated with f by

$$\mathcal{E}_f(g) = \sum_{\gamma \in P_n(F)\backslash \mathrm{Sp}_n(F)} f(\gamma g), \quad g \in \mathrm{Sp}_n(\mathbb{A}_F). \tag{4.1}$$

We assume that the right hand side is absolutely convergent.

In Proposition 4.2, we shall show that the Fourier coefficients of $\mathcal{E}_f$ has the Euler product expression when $\mathcal{E}_f$ corresponds to holomorphic Siegel Eisenstein series. For the proof, we need some results of the infinite factor of the Fourier coefficients of Siegel Eisenstein series.

Let $B, y \in \mathrm{Sym}_n(\mathbb{R})$ be symmetric matrices and assume y is positive definite. For $\alpha, \beta \in \mathbb{C}$, we define a function ξ by

$$\xi(y, B; \alpha, \beta) = \int_{\mathrm{Sym}_n(\mathbb{R})} \det(x+iy)^{-\alpha} \det(x-iy)^{-\beta} \mathbf{e}(-Bx)dx. \tag{4.2}$$

Theorem 4.1 ([12] (7.11),(7.12); [11] (4.34K),(4.35K))**.** *Suppose $y, B \in \mathrm{Sym}_n(\mathbb{R})$ be symmetric matrices and y is positive definite. Let p and q be the number of positive and negative eigenvalue of B respectively and put $t = n-p-q$. We denote by $\delta_+(By)$ (resp. $\delta_-(By)$) the product of all positive (resp. negative) eigenvalues of $y^{1/2}By^{1/2}$. For $l \in \mathbb{Z}_{\geq 0}$, we define a function Γ_l by*

$$\Gamma_l(s) = \begin{cases} 1 & \text{if } l = 0, \\ \pi^{l(l-1)/4} \prod_{i=0}^{l-1} \Gamma\left(s - \frac{i}{2}\right) & \text{if } l \geq 1. \end{cases}$$

Then, there exists a function $\omega(y, B; \alpha, \beta)$ holomorphic with respect to α and β and satisfies the following equation.

$$\begin{aligned} \xi(y, B; \alpha, \beta) &= \mathbf{e}(n(\alpha-\beta)/4) 2^{\tau} \pi^{\theta} \Gamma_t\Big(\alpha + \beta - \frac{n+1}{2}\Big) \Gamma_{n-q}(\alpha)^{-1} \Gamma_{n-p}(\beta)^{-1} \\ &\times \det(y)^{(n+1)/2-\alpha-\beta} \delta_+(By)^{\alpha-(n+1)/2+q/4} \delta_-(By)^{\beta-(n+1)/2+p/4} \omega(y, B; \alpha, \beta). \end{aligned}$$

Here τ, θ is

$$\begin{aligned} \tau &= (2p-n)\alpha + (2q-n)\beta + \frac{(n+t)(n+1)}{2} + \frac{pq}{2}, \\ \theta &= p\alpha + q\beta + t + \frac{1}{2}\{t(t-1) - pq\}. \end{aligned}$$

If B is positive definite, then the following holds.

$$\xi(y, B; \alpha, 0) = 2^{n(1-n)/2} i^{-n\alpha} (2\pi)^{n\alpha} \Gamma_n(\alpha)^{-1} \det(B)^{\alpha-(n+1)/2} \mathbf{e}(iyB).$$

For $g \in \mathrm{Sp}_n(\mathbb{A}_F)$ and $0 \leq i \leq n$, we put

$$\mathrm{Wh}_{B,i}(\mathcal{E}_f)(g) = \int_{\mathrm{Sym}_n(F\backslash\mathbb{A}_F)} \sum_{\gamma} f\left(\gamma \nu_n(u) g\right) \psi(-Bu) du.$$

Here γ runs over the set $P_n(F)\backslash P_n(F) w_{n,i} P_n(F)$. We set

$$S_{i,n-i}(F) = \left\{ \begin{pmatrix} 0_{i,i} & * \\ * & * \end{pmatrix} \in \mathrm{Sym}_n(F) \right\},$$
$$P_{i,n-i}(F) = \left\{ \begin{pmatrix} * & * \\ 0_{i,n-i} & * \end{pmatrix} \in \mathrm{GL}_n(F) \right\},$$

where $0_{i,j}$ is the zero matrix of size $i \times j$. If $i = 0$, we understand $S_{0,n}(F) = \{0_n\}$ and $P_{0,n}(F) = \mathrm{GL}_n(F)$. Then $\mathrm{Wh}_{B,i}(\mathcal{E}_f)(g)$ becomes

$$\int_{\mathrm{Sym}_n(F\backslash\mathbb{A}_F)} \sum_{\beta,\alpha} f\left(w_{n,i} \nu_n(\beta) \mu_n\left({}^t\alpha^{-1}\right) \nu_n(u) g\right) \psi(-Bu) du.$$

Here β and α run over $S_{i,n-i}(F)\backslash\mathrm{Sym}_n(F)$ and $P_{i,n-i}(F)\backslash\mathrm{GL}_n(F)$ respectively. For $\alpha \in P_{i,n-i}(F)\backslash\mathrm{GL}_n(F)$, we put

$$\mathrm{Wh}_{B,\alpha,i}(\mathcal{E}_f)(g) = \int_{\mathrm{Sym}_n(F\backslash\mathbb{A}_F)} \sum_{\beta \in S_{i,n-i}(F)\backslash\mathrm{Sym}_n(F)} f\left(w_{n,i} \nu_n(\beta) \mu_n\left({}^t\alpha^{-1}\right) \nu_n(u) g\right) \psi(-Bu) du.$$

Proposition 4.2. *Let $0 \leq B \in \mathrm{Sym}_n(F)$ be a totally positive semi-definite symmetric matrix, $f = \prod_v f_v$ an element of $\mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}^{-1}|\cdot|^{k-(n+1)/2})$. We assume that there exits an infinite place of F such that $f_w(g) = \phi_w(k - \frac{k+1}{2}, g)$. Suppose $k > n+1$. Then $\mathrm{Wh}_{B,i}(\mathcal{E}_f)(g) = 0$ unless $\mathrm{rank}\, B = i$. If $\mathrm{rank}\, B = i$ and $B = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}$ with $B' \in \mathrm{Sym}_i(F)$ then $\mathrm{Wh}_B(\mathcal{E}_f)(g)$ is given by*

$$\prod_v \int_{u \in \mathrm{Sym}_i(F_v)} f_v\left(w_{n,i} \nu_n\left(\begin{pmatrix} u & 0 \\ 0 & 0_{n-i} \end{pmatrix}\right) g\right) \psi_v\left(-B'u\right) du.$$

Here v runs through all the places of F.

Proof. By definition, we have

$$\mathrm{Wh}_{B,\alpha,i}(\mathcal{E}_f)(g) = \mathrm{Wh}_{B[{}^t\alpha],1,i}(\mathcal{E}_f)\left(\mu_n\left({}^t\alpha^{-1}\right)g\right).$$

Put

$$B[{}^t\alpha] = \begin{pmatrix} B_{11}(\alpha) & B_{12}(\alpha) \\ {}^tB_{12}(\alpha) & B_{22}(\alpha) \end{pmatrix}, \quad u = \begin{pmatrix} u_{11} & u_{12} \\ {}^tu_{12} & u_{22} \end{pmatrix}.$$

Then the function $f\left(w_{n,i}\nu_n(\beta)\nu_n(u)g\right)$ does not depend on u_{12} and u_{22}. Therefore unless $B_{12}(\alpha) = B_{22}(\alpha) = 0$, $\mathrm{Wh}_{B,\alpha,i}(\mathcal{E}_f)(\mu_n({}^t\alpha^{-1})g) = 0$. If $B_{12}(\alpha) = B_{22}(\alpha) = 0$, then we have

$$\begin{aligned}
&\mathrm{Wh}_{B[{}^t\alpha],1}(\mathcal{E}_f)\left(\mu_n\left({}^t\alpha^{-1}\right)g\right) \\
&\quad = \prod_{v:\ \text{place of } F} \int_{u\in\mathrm{Sym}_i(F_v)} f_v\left(w_{n,i}\nu_n\left(\begin{pmatrix} u & 0 \\ 0 & 0_{n-i} \end{pmatrix}\right)\mu_n\left({}^t\alpha^{-1}\right)g\right) \\
&\qquad\qquad \times \psi_v\left(-B_{11}(\alpha)u\right)du.
\end{aligned}$$

Fix an infinite place w of F satisfying the assumption in the theorem. Let

$$\mu_n\left({}^t\alpha^{-1}\right)g = \nu_n(\beta)\mu_n(\gamma), \quad \beta\in\mathrm{Sym}_n(F_w),\ \gamma\in\mathrm{GL}_n(F_w)$$

be the Iwasawa decomposition in $\mathrm{Sp}_n(F_w)$. Then we have

$$\begin{aligned}
&f\left(w_{n,i}\nu_n\left(\begin{pmatrix} u & 0 \\ 0 & 0_{n-i} \end{pmatrix}\right)\mu_n\left({}^t\alpha^{-1}\right)g\right) \\
&= \det\gamma^{-1}\det\left(\begin{pmatrix} u+\beta' & \\ & 1_{n-i} \end{pmatrix} + \begin{pmatrix} 1_i & \\ & 0_{n-i} \end{pmatrix}\gamma\,{}^t\gamma\mathbf{i}\right)^{-k} \\
&= \det\gamma^{-1}\det\left(u+\beta'+y'\mathbf{i}\right)^{-k}.
\end{aligned}$$

Here β' and y' are the upper left $i\times i$ block of β and $\gamma\,{}^t\gamma$ respectively. Then we have

$$\begin{aligned}
&\int_{\mathrm{Sym}_n(F_w)} f\left(w_{n,i}\nu_n\left(\begin{pmatrix} u & 0 \\ 0 & 0_{n-i} \end{pmatrix}\right)\mu_n\left({}^t\alpha^{-1}\right)g\right) \\
&\qquad\qquad \times \psi_w\left(-B_{11}(\alpha)u\right)du. \\
&= \psi_w\left(B_{11}(\alpha)\beta'\right)\det\gamma^{-1}\int_{\mathrm{Sym}_n(F_w)}\det\left(u+y'\mathbf{i}\right)^{-k}\psi_w\left(-B_{11}(\alpha)u\right)du \\
&= \psi_w\left(B_{11}(\alpha)\beta'\right)\det\gamma^{-1}\xi\left(y',B_{11}(\alpha);k,0\right).
\end{aligned}$$

Here $\xi\left(y', B_{11}(\alpha); k, 0\right)$ is defined by (4.2). By Theorem 4.1, this value vanishes unless $\det B_{11}(\alpha) \neq 0$. Thus

$$\mathrm{Wh}_{B,\alpha,i}(\mathcal{E}_f)(g) = 0$$

unless $\operatorname{rank} B = i$. Suppose $\operatorname{rank} B = i$ and $B = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}$ with $B' \in \mathrm{Sym}_i(F)$. Then $\operatorname{rank} B_{11}(\alpha) = i$ if and only if $\alpha \in P_{i,n-i}(F)$. Thus we have

$$\begin{aligned} \mathrm{Wh}_{B,i}(\mathcal{E}_f)(g) &= \sum_{\alpha \in P_{i,n-i}(F)\backslash \mathrm{GL}_n(F)} \mathrm{Wh}_{B,\alpha,i}(\mathcal{E}_f)(g) = \mathrm{Wh}_{B,1,i}(\mathcal{E}_f)(g) \\ &= \prod_v \int_{\mathrm{Sym}_i(F_v)} f_v\left(w_{n,i}\nu_n\left(\begin{pmatrix} u & 0 \\ 0 & 0_{n-i} \end{pmatrix}\right) g\right) \psi_v\left(-B'u\right) du. \end{aligned}$$

□

Before proving the next proposition, we introduce a lemma.

Lemma 4.3 (Proposition 3.6 of [13])**.** *We define a set $\mathfrak{M}_n$ by*

$$\mathfrak{M}_n = \left\{(c,d) \in M_{n,2n}(\mathcal{O}_v) \;\middle|\; \det c \neq 0 \text{ and } (c,d) \text{ is symmetric and co-prime}\right\}.$$

Here a pair (c,d) is said to be symmetric if $c\,{}^t d = d\,{}^t c$ and said to be co-prime if there exist matrices $u \in \mathrm{GL}_n(\mathbb{Z})$ and $v \in \mathrm{GL}_{2n}(\mathbb{Z})$ such that $u(c,d)v = (1_n, 0_n)$. Then $\mathrm{GL}_n(\mathcal{O}_v)$ acts on $\mathfrak{M}_n$ by the left diagonally. Then, the map

$$\mathrm{GL}_n(\mathcal{O}_v)\backslash\mathfrak{M}_n \ni (c,d) \mapsto c^{-1}d \in \mathrm{Sym}_n(F_v)$$

is bijective.

Proposition 4.4. *Let v be a finite place of F. Let $\phi'_{n,v}(-s,g) = \phi'_v(-s,g)$ be the function on $\mathrm{Sp}_n(F_v)$ defined by (2.3) and $\varphi_{n,v}(s,g) = \varphi(s,g)$ the function on $\mathrm{Sp}_n(F_v)$ defined by (2.4). Then, for $1 \leq r \leq n$ and $g' \in \mathrm{Sp}_r(F_v)$, we have*

$$\varphi_{n,v}(s, \iota_n(g')) = \varphi_{r,v}(s + \frac{n-r}{2}, g').$$

Proof. By the Iwasawa decomposition $\mathrm{Sp}_r(F_v) = P_r(F_v)\mathrm{Sp}_r(\mathcal{O}_v)$, we may assume $g' \in \mathrm{Sp}_r(\mathcal{O}_v)$. By definition, we have

$$\varphi_{n,v}(s, \iota_n(g')) = \int_{\mathrm{Sym}_n(F_v)} \phi'_{n,v}\left(-s, w_n\nu_n(x)\iota_n(g')\right) dx.$$

Here we take a Haar measure of $\mathrm{Sym}_n(F_v)$ so that $\int_{\mathrm{Sym}_n(\mathcal{O}_v)} dx = 1$. Let L be a lattice of $\mathrm{Sym}_r(\mathcal{O}_v)$ such that

$$\phi'_{n,v}\left(-s, g\,\iota_n(\nu_r(l)g')\right) = \phi'_{n,v}\left(-s, g\,\iota_n(g')\right),$$

for any $g \in \mathrm{Sp}_n(F_v)$ and $l \in L$. Put $x = \begin{pmatrix} x_{11} & x_{12} \\ {}^t x_{12} & x_{22} \end{pmatrix}$ with $x_{11} \in M_r(F_v)$, $x_{12} \in M_{r,n-r}(F_v)$ and $x_{22} \in M_{n-r}(F_v)$. Then we have

$$\begin{aligned}\iota_n(g')^{-1}\nu_n(x)\iota_n(g') = \iota_n(g'^{-1}\nu_r(x_{11})g')\mu_n\left(\begin{pmatrix} 1_r & 0 \\ {}^t x_{12} c_{g'} & 1_{n-r} \end{pmatrix}\right) \\ \times \nu_n\left(\begin{pmatrix} 0_r & {}^t d_{g'} x_{12} \\ {}^t x_{12} d_{g'} & x_{22} - (c_{g'}{}^t d_{g'})[x_{12}] \end{pmatrix}\right).\end{aligned}$$

By the definition of $\phi'_{n,v}(-s, g)$, we have

$$\phi'_{n,v}(-s, g\gamma) = \widetilde{\chi}_v(\det d_\gamma)\phi'_{n,v}(-s, g)$$

for $\gamma \in C_{0,v}$. Thus $\phi'_{n,v}\left(w_n\nu_n(x)\iota_n(g')\right)$ is determined by x_{11} mod L, x_{12} mod $M_{r,n-r}(\mathcal{O}_v)$ and x_{22} mod $\mathrm{Sym}_{n-r}(\mathcal{O}_v)$.

Let $\mathfrak{M}_n$ be the set defined in Lemma 4.3. For $(c,d), (c',d') \in \mathfrak{M}_n$ and a lattice M of $\mathrm{Sym}_n(F_v)$, we denote $(c,d) \sim_M (c',d')$ if $c^{-1}d \equiv c'^{-1}d'$ mod M. We denote the equivalence class of $\mathfrak{M}_n$ by $\mathfrak{M}_n/\sim_M$. We define a lattice $M_0 \subset \mathrm{Sym}_n(F_v)$ by

$$M_0 = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ {}^t x_{12} & x_{22} \end{pmatrix} \,\middle|\, x_{11} \in L, x_{12} \in M_{r,n-r}(\mathcal{O}_v), x_{22} \in \mathrm{Sym}_{n-r}(\mathcal{O}_v) \right\}.$$

If $\mathrm{Re}(-s)$ is sufficiently large, then by Lemma 4.3 we have

$$\varphi_{n,v}(s, \iota_n(g')) = \int_L dx \sum_{(c,d)} \phi'_{n,v}\left(-s, w_n\nu_n(c^{-1}d)\iota_n(g')\right),$$

where the summation index runs over $\mathrm{GL}_n(\mathcal{O}_v)\backslash\mathfrak{M}_n/\sim_{M_0}$. The Iwasawa decomposition of $w_n\nu_n(c^{-1}d)\iota_n(g')$ is given by

$$\begin{pmatrix} {}^t c & * \\ 0 & c^{-1} \end{pmatrix} \left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline * & c_{12} & * & d_{12} \\ * & c_{22} & * & d_{22} \end{array}\right).$$

Here $c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ and $d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$ with $c_{11}, d_{11} \in M_r(F_v)$ and $c_{22}, d_{22} \in M_{n-r}(F_v)$. By the definition of $\phi'_{n,v}(-s, g)$, we have $\phi'_{n,v}\left(w_n \nu_n(c^{-1}d)\iota_n(g')\right) = 0$ unless

$$\operatorname{rank}_{\mathbf{F}_v}\left(\begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} \bmod \mathfrak{p}_v\right) = n - r \tag{4.3}$$

Here $\mathfrak{p}_v$ is the maximal ideal of $\mathcal{O}_v$ and $\mathbf{F}_v$ is the residue field of $\mathcal{O}_v$. We assume

$$\phi'_{n,v}\left(-s, w_n \nu_n(c^{-1}d)\iota_n(g')\right)$$

does not vanish. Put

$$\mathfrak{M}_{n,r} = \left\{(c, d) \in \mathfrak{M}_n \;\middle|\; c \text{ satisfies (4.3)}\right\}.$$

Let $\widetilde{\mathfrak{M}}_r$ be a complete set of representable of $\mathrm{GL}_r(\mathcal{O}_v)\backslash\mathfrak{M}_r/\sim_L$. For $c' \in \mathrm{GL}_r(F_v) \cap M_r(\mathcal{O}_v)$, we denote by $\widetilde{U}(c')$ a complete set of representable of $M_{n-r,r}(\mathcal{O}_v)/M_{n-r,r}(\mathcal{O}_v)c'$. For $(c', d') \in \widetilde{\mathfrak{M}}_r$ and $u \in \widetilde{U}(c')$, we put

$$\alpha(c', u) = \begin{pmatrix} c' & \\ u & 1_{n-r} \end{pmatrix}, \quad \beta(d', u) = \begin{pmatrix} d' & -d'\,{}^t u \\ & 0_{n-r} \end{pmatrix}.$$

Then the set

$$\widetilde{\mathfrak{M}}_{n,r} = \left\{(\alpha(c', u), \beta(d', u)) \;\middle|\; (c', d') \in \widetilde{\mathfrak{M}}_r,\ u \in \widetilde{U}(c')\right\}$$

gives a complete set of representable of $\mathrm{GL}_n(\mathcal{O}_v)\backslash\mathfrak{M}_{n,r}/\sim_{M_0}$. We can prove this as follows. It is easy to see that every element $(c, d) \in \mathfrak{M}_{n,r}$ is equivalent to some element of $\widetilde{\mathfrak{M}}_{n,r}$. Suppose that two elements $(\alpha(c'_1, u_1), \beta(d'_1, u_1))$ and $(\alpha(c'_2, u_2), \beta(d'_2, u_2))$ in the set $\widetilde{\mathfrak{M}}_{n,r}$ are equivalent. Then by

$$\alpha(c', u)^{-1}\beta(d', u) = \begin{pmatrix} c'^{-1}d' & -c'^{-1}d'\,{}^t u \\ -uc'^{-1}d' & uc'^{-1}d'\,{}^t u \end{pmatrix},$$

we have $c'_1 = c'_2,\ d'_1 = d'_2$ and $uc'^{-1}d' \in M_{n-r,r}(\mathcal{O}_v)$. Here we put $u = u_1 - u_2,\ c' = c'_1$ and $d' = d'_1$. Since the pair $(c',\ d')$ is co-prime and

$$uc'^{-1}\begin{pmatrix} c' & d' \end{pmatrix} = \begin{pmatrix} u' & uc'^{-1}d' \end{pmatrix} \in M_{n-r,2r}(\mathcal{O}_v),$$

we have $uc'^{-1} \in M_{n-r,r}(\mathcal{O}_v)$ (see [13, Lemma 3.3]). Thus $u_1 = u_2$ and the set $\widetilde{\mathfrak{M}}_{n,r}$ is a complete set of representable of $\mathrm{GL}_n(\mathcal{O}_v)\backslash\mathfrak{M}_{n,r}/\sim_{M_0}$.

For $(\alpha(c',u),\beta(d',u)) \in \widetilde{\mathfrak{M}}_{n,r}$, we have

$$\begin{aligned}
&w_n\nu_n(\alpha(c',u)^{-1}\beta(d',u)) \\
&= \mu_n\left({}^t\alpha(c',u)\right)\nu_n\left(\begin{pmatrix} & a_{12} \\ {}^t a_{12} & a_{22}\end{pmatrix}\right)\iota\left(\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix}\times w_{n-r}\right)\mu_n\left(\begin{pmatrix} 1_r & \\ u & 1_{n-r}\end{pmatrix}\right),
\end{aligned}$$

with $a_{12} \in M_{r,n-r}(\mathcal{O}_v)$, $a_{22} \in M_{n-r}(\mathcal{O}_v)$ and $\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix} \in \mathrm{Sp}_r(\mathcal{O}_v)$. Therefore by

$$\iota_n(g')^{-1}\mu_n\left(\begin{pmatrix} 1_r & \\ u & 1_{n-r}\end{pmatrix}\right)\iota_n(g') = \left(\begin{array}{cc|cc} 1_r & & & {}^t b_{g'}{}^t u \\ ua_{g'} & 1_{n-r} & ub_{g'} & \\ \hline & & 1_r & -a_{g'}{}^t u \\ & & & 1_{n-r}\end{array}\right),$$

and

$$\phi'_{n,v}(-s,\iota(g\times w_{n-r})) = \phi'_{r,v}(-s+\frac{n-r}{2},g) \quad \text{for } g\in \mathrm{Sp}_r(F_v),$$

we have

$$\phi'_{n,v}\left(-s, w_n\nu_n\left(\alpha(c',u)^{-1}\beta(d',u)\right)\iota_n(g')\right) = \phi'_{r,v}\left(-s+\frac{n-r}{2}, w_r\nu_r(c'^{-1}d')g'\right).$$

Thus we have

$$\begin{aligned}
&\varphi_{n,v}(s,\iota_n(g')) \\
&= \int_L dx \sum_{(c',d')\in\widetilde{\mathfrak{M}}_r}\sum_{u\in\widetilde{U}(c')} \phi'_{r,v}\left(-s+\frac{n-r}{2}, w_r\nu_r(c'^{-1}d')g'\right), \\
&= \int_L dx \sum_{(c',d')\in\widetilde{\mathfrak{M}}_r} \phi'_{r,v}\left(-s+\frac{-n+r}{2}, w_r\nu_r(c'^{-1}d')g'\right), \\
&= \varphi_{r,v}\left(s+\frac{n-r}{2}, g'\right).
\end{aligned}$$

□

5. Functional equation of Whittaker functions

In this section, we introduce the explicit functional equation of Whittaker functions proved by Ikeda.

Let K be a local field. Let ψ be a nontrivial additive character of K and χ be a character of $K^\times$. Suppose $B \in \mathrm{Sym}_n(K)$ and $\det B \neq 0$. For $f \in \mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\chi|\cdot|^s)$, we define a function $\mathrm{Wh}_B^{(s)}(f)$ on $\mathrm{Sp}_n(K)$ by

$$\mathrm{Wh}_B^{(s)}(f)(g) = \int_{\mathrm{Sym}_n(K)} f(w_n \nu_n(x) g)\psi(-Bx)dx, \qquad \text{for } g \in \mathrm{Sp}_n(K).$$

For a character ω of $K^\times$, we denote by $\varepsilon(s,\omega,\psi)$ the local factor from Tate's thesis and put $\gamma(s,\omega,\psi) = \varepsilon(s,\omega,\psi)\frac{L(1-s,\omega^{-1})}{L(s,\omega)}$. If n is even, we define D_B by $(-1)^{n/2}\det B$. We denote by χ_B the character of $K^\times$ corresponding to the extension $K(\sqrt{D_B})/K$ by the local class field theory. For $a \in K^\times$, we denote by $\alpha(a)$ the Weil index of the character of second degree $x \mapsto \psi(ax^2)$. By [14, lemma 2.4], we have

$$\frac{\alpha(D_B)}{\alpha(1)} = \varepsilon\left(\frac{1}{2}, \chi_B, \psi\right). \tag{5.1}$$

If n is odd, we define η_B by

$$\eta_B = \langle (-1)^{(n-1)/2}, \det B\rangle \langle -1,-1\rangle^{(n^2-1)/8} h(B).$$

Here $\langle\cdot,\cdot\rangle$ is the Hilbert symbol of K and $h(B)$ is the Hasse invariant defined by

$$h(B) = \prod_{1\le i<j\le n} \langle a_i, a_j\rangle,$$

where $B[U] = \mathrm{diag}(a_1,\dots,a_n)$ for a matrix $U \in \mathrm{GL}_n(K)$.

Theorem 5.1 (Ikeda)**.** *The following equation holds.*

$$\mathrm{Wh}_B^{(-s)} \circ M_w = \chi(\det B)^{-1}|\det B|^{-s} c(s,B)\,\mathrm{Wh}_B^{(s)}.$$

The function $c(s,B)$ is given as follows.

(i) Let n be even. Then we have

$$c(s,B) = |2|^{-ns}\frac{\alpha(D_B)}{\alpha(1)}\chi(2)^{-n}\gamma(s+\frac{1}{2},\chi\chi_B,\psi) \\ \times \gamma(s-\frac{n-1}{2},\chi,\psi)^{-1}\prod_{r=1}^{n/2}\gamma(2s-n+2r,\chi^2,\psi)^{-1}.$$

(ii) Let n be odd. Then we have

$$c(s,B) = |2|^{-(n-1)s}\chi(2)^{-(n-1)}\eta_B\gamma(s-\frac{n-1}{2},\chi,\psi)^{-1}\prod_{r=1}^{(n-1)/2} \times \gamma(2s-n+2r,\chi^2,\psi)^{-1}.$$

Remark 5.2. To prove this theorem, it is enough to calculate the coefficients appearing in the local functional equation of prehomogenous vector space $\mathrm{Sym}_n(K)$ (cf. [4, Lemma 4.3]). Sweet [14, Theorem 1.1] calculated such coefficients in the case where K is non-archimedean and χ is unramified. In [5], Ikeda generalized Sweet's result for an arbitrary character χ.

6. Euler factor at unramified places

Let v be a finite place of F such that $\widetilde{\chi}_v$ is unramified and $B \in \mathrm{Sym}_n^{(*)}(\mathcal{O}_F)$ with $\det B \neq 0$. By equation (3.6) and Proposition 4.2, $a(B, G_{k,\chi}^{(n)})$ has an Euler factor

$$\mathrm{Wh}_B^{(k-(n+1)/2)}(\varphi_v)(1) = \int_{\mathrm{Sym}_n(F_v)} \varphi_v\left(k-\frac{n+1}{2}, w_n\nu_n(x)\right)\psi_v(-B)dx$$

at v.

For $x \in \mathrm{Sym}_n(F_v)$, we put $x = c^{-1}d$, $(c,d) \in \mathfrak{M}_n$. Here $\mathfrak{M}_n$ the set defined in Lemma 4.3. We put

$$\kappa(x) = \mathrm{ord}_v \det(c),$$

where ord_v is the normalized additive valuation of F_v. Then, we have

$$\int_{\mathrm{Sym}_n(F_v)} \varphi_v\left(k-\frac{n+1}{2}, w_n\nu_n(x)\right)\psi_v(-B)dx = \sum_{x\in\mathrm{Sym}_n(F_v)/\mathrm{Sym}_n(\mathcal{O}_v)} \left(\widetilde{\chi}_v^{-1}(\varpi_v)q_v^{-k}\right)^{\kappa(x)}\psi_v(-Bx).$$

Here q_v is the order of the residue field $\mathcal{O}_v/(\varpi_v)$. This series is called the Siegel series and has been examined by several authors. We introduce some of the known results.

Let $A_B^{(n)}(T)$ be the formal power series such that

$$A_B^{(n)}(\widetilde{\chi}_v^{-1}(\varpi_v)q_v^{-k}) = \int_{\mathrm{Sym}_n(F_v)} \varphi_v\left(k-\frac{n+1}{2}, w_n\nu_n(x)\right)\psi_v(-B)dx.$$

Proposition 6.1 ([13] Proposition 14.9)**.** *Let $B \in \mathrm{Sym}_n^*(\mathcal{O}_v)$ and assume $\det B \neq 0$. Then $A_B^{(n)}(T)$ is $\mathbb{Z}$-coefficient polynomial with constant term 1 and divisible by a polynomial $\gamma(B;T)$ defined as follows.*

$$\gamma(B;T) = \begin{cases} \frac{1-T}{1-\lambda(B)q_v^{n/2}T} \prod_{i=1}^{n/2}(1-q_v^{2i}T^2) & \text{if } n \text{ is even,} \\ (1-T) \prod_{i=1}^{(n-1)/2} (1-q_v^{2i}T^2) & \text{if } n \text{ is odd.} \end{cases}$$

Here, when n is even, we define $d = (-1)^{n/2}\det(B)$, $K_B = F_v(\sqrt{d})$ and

$$\lambda(B) = \begin{cases} 1 & \text{if } K_B = F_v, \\ -1 & \text{if } K_B/F_v \text{ is unramified quadratic extension,} \\ 0 & \text{if } K_B/F_v \text{ is ramified extension.} \end{cases}$$

Moreover if B satisfies the following condition,

$$\begin{cases} \det(2B) \in \mathcal{O}_v^\times & \text{if } n \text{ is even,} \\ \det(2B) \in 2\mathcal{O}_v^\times & \text{if } n \text{ is odd,} \end{cases}$$

then

$$A_B^{(n)}(T) = \gamma(B;T).$$

We put

$$\Phi_v^{(n)}(B;T) = A_B^{(n)}(T)/\gamma(B;T). \tag{6.1}$$

Then $\Phi_v^{(n)}(B;T)$ is a $\mathbb{Z}$-coefficient polynomial by Proposition 6.1.

By the functional equation of Whittaker functions 5.1, the following theorem holds.

Theorem 6.2 (Ikeda)**.** *Let $B \in \mathrm{Sym}_n^{(*)}(\mathcal{O}_v) \cap \mathrm{GL}_n(F_v)$. Let ρ_B be the character of $F_v^\times$ that corresponds to the extension K_B/F_v by the local class field theory. Here K_B is defined in Proposition 6.1. Let $\mathfrak{f}(\rho_B)$ denote the conductor of ρ_B. Define an ideal f_B by $(\det(2B)) = \mathfrak{f}(\rho_B)f_B^2$. Then f_B is an integral ideal. Put*

$$e_B = \begin{cases} 2\mathrm{ord}_v(f_B) & \text{if } n \text{ is even,} \\ \mathrm{ord}_v(2^{n-1}\det B) & \text{if } n \text{ is odd,} \end{cases}$$

$$\zeta_B = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \eta_B & \text{if } n \text{ is odd,} \end{cases}$$

Here η_B is defined in §5. Then the following functional equation holds.

$$\Phi_B^{(n)}(B; q_v^{-n-1}T^{-1}) = \zeta_B \left(q_v^{(n+1)/2}T\right)^{-e_B} \Phi_B^{(n)}(B;T).$$

Remark 6.3. The theorem was first proved by Katsurada [6] in the case where p is a rational prime and $F_v = \mathbb{Q}_p$.

7. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. Let $B \in \mathrm{Sym}_n^{(*)}(\mathcal{O}_F)$ and put $r = \mathrm{rank}_F B$. We assume that r is even. Since the computation is similar, we omit the proof when r is odd.

Choose $U \in \mathrm{GL}_n(F)$ so that $B[U] = \begin{pmatrix} B' & 0 \\ 0 & 0_{n-r} \end{pmatrix}$. For a place v of F, choose U_v so that $B[U_v] = \begin{pmatrix} B'_v & 0 \\ 0 & 0_{n-r} \end{pmatrix}$ and

$$U_v \in \begin{cases} \mathrm{O}_n(F_v) & \text{if } v \mid \infty, \\ \mathrm{GL}_n(\mathcal{O}_v) & \text{if } v \nmid \infty. \end{cases}$$

Put

$$k_n = \frac{k-(n+1)}{2}.$$

From (3.6), we have

$$a(B, G_{k,\chi}^{(n)}) = \mathbf{e}(-iBy)(\det y)^{-k/2}\, \mathrm{Wh}_B\left(\mathcal{G}_{\varphi,\chi}^{(n)}\right)(\mu_n(y^{1/2}))\Big|_{s=k_n}.$$

By replacing x by $x[{}^tU]$, we have

$$a(B, G_{k,\chi}^{(n)}) = \mathbf{e}(-iBy)(\det y)^{-k/2}\, \mathrm{Wh}_{\mathrm{diag}(B',0_{n-r})}\left(\mathcal{G}_{\varphi,\chi}^{(n)}\right)(\mu_n(U^{-1})\mu_n(y^{1/2}))\Big|_{s=k_n}.$$

By Proposition 4.2, we have

$$\begin{aligned} &a(B, G_{k,\chi}^{(n)}) = \mathbf{e}(-iBy)(\det y)^{-k/2} \\ &\times \prod_v \int_{\mathrm{Sym}_r(F_v)} \varphi_v\left(k_n, w_{n,r}\nu_n\left(\mathrm{diag}(x, 0_{n-r})\right)\mu_n(U^{-1})\mu_n(g_v)\right)\psi_v(-B'x)dx, \end{aligned}$$

where

$$g_v = \begin{cases} y_v^{1/2} & \text{if } v \mid \infty, \\ 1 & \text{if } v \nmid \infty. \end{cases}$$

Since $\begin{pmatrix} B' & \\ & 0 \end{pmatrix}[U^{-1}U_v] = \begin{pmatrix} B'_v & \\ & 0 \end{pmatrix}$, the upper right block of $U^{-1}U_v$ is the zero matrix. We put

$$U^{-1}U_v = \begin{pmatrix} \alpha_v & 0 \\ * & \beta_v \end{pmatrix}.$$

Let v be an infinite place of F and let us consider the factor at v. Replacing x by $x[{}^t\alpha_v]$ and noting that

$$\mu_n(U)\mu_n(y_v^{1/2}) = \mu_n(UU_v^{-1})\mu_n(y_v^{1/2}[U_v])\mu_n(U_v^{-1}),$$

we have

$$\begin{aligned}
&\int_{\mathrm{Sym}_r(F_v)} \varphi_v\left(k_n, w_{n,r}\nu_n\left(\mathrm{diag}(x, 0_{n-r})\right)\mu_n(U^{-1})\mu_n(y_v^{1/2})\right)\psi_v(-B'x)dx \\
&= \widetilde{\chi}_v(\det U_v \det \alpha_v \det \beta_v^{-1})|\det \alpha_v|_v^{-k+r+1}\,|\det \beta_v|_v^k \\
&\times \int_{\mathrm{Sym}_r(F_v)} \varphi_v\left(k_n, w_{n,r}\nu_n\left(\mathrm{diag}(x, 0_{n-r})\right)\mu_n(y_v^{1/2}[U_v])\right)\psi_v(-B'_v x)dx.
\end{aligned}$$

By a similar computation at a finite place and the fact that $\det U \in F^\times$, we have

$$\begin{aligned}
a(B, G_{k,\chi}^{(n)}) &= \mathbf{e}(-iBy)(\det y)^{-k/2}\prod_v \widetilde{\chi}_v(\det \alpha_v^2)\,|\det \alpha_v|_v^{-2k+(r+1)} \\
&\times \prod_v \int_{\mathrm{Sym}_r(F_v)} \varphi_v\left(k_n, w_{n,r}\nu_n\left(\mathrm{diag}(x, 0_{n-r})\right)\mu_n(g_v^{U_v})\right)\psi_v(-B'x)dx, \quad (7.1)
\end{aligned}$$

where $g_v^{U_v} = U_v^{-1}g_vU_v$.

Let χ_B denote the narrow class character corresponding to the extension

$$F(\sqrt{(-1)^{r/2}\det B'})/F$$

by the global class field theory. By Proposition 4.4, we have

$$\begin{aligned}
&\prod_{v\nmid\infty}\int_{\mathrm{Sym}_r(F_v)} \varphi_v\left(k_n, w_{n,r}\nu_n\left(\mathrm{diag}(x, 0_{n-r})\right)\right)\psi_v(-B'x)dx \\
&= \prod_{v\nmid\infty}\int_{\mathrm{Sym}_r(F_v)} \varphi_v\left(k_r, w_r\nu_r(x)\right)\psi_v(-B'x)dx
\end{aligned}$$

By Proposition 6.1, we have

$$\begin{aligned}
&\prod_{v\nmid\infty}\int_{\mathrm{Sym}_r(F_v)}\varphi_v\left(k_r, w_r\nu_r(x)\right)\psi_v(-B'x)dx\\
&=\prod_{\substack{v\nmid\infty\\ v\nmid\mathfrak{N}}}\Phi_v^{(r)}(B'_v,\widetilde{\chi}_v(\varpi_v)^{-1}q_v^{-k})\\
&\quad\times L^{(\mathfrak{N})}(k-\frac{r}{2},\chi_B\chi^{-1})L(k,\chi^{-1})^{-1}\prod_{i=1}^{r/2}L^{(\mathfrak{N})}(2k-2i,\chi^{-2})^{-1}\\
&\quad\times\prod_{\substack{v\nmid\infty\\ v|\mathfrak{N}}}\int_{\mathrm{Sym}_r(F_v)}\varphi_v\left(k_r, w_r\nu_r(x)\right)\psi_v(-B'_vx)dx.
\end{aligned}$$

Let v be an infinite place and $\kappa_v^{(r)}$ the upper left $r\times r$ block of $y_v[U_v]$. Then by the definition of φ_v, we have

$$\begin{aligned}
&\int_{\mathrm{Sym}_r(F_v)}\varphi_v\left(k_n, w_{n,r}\nu_n\left(\mathrm{diag}(x,0_{n-r})\right)\mu_n(y_v^{1/2}[U_v])\right)\psi_v(-B'_vx)dx\\
&=|\det y_v|_v^{k/2}\,\xi(\kappa_v^{(r)},B'_v;k,0).
\end{aligned}$$

Here ξ is the function defined by (4.2).

By (7.1) and equations above, we have

$$\begin{aligned}
a(B,G_{k,\chi}^{(n)})&=\mathbf{e}(-iBy)\prod_v\widetilde{\chi}_v(\det\alpha_v^2)\,|\det\alpha_v|_v^{-2k+(r+1)}\\
&\quad\times\prod_{\substack{v\nmid\infty\\ v\nmid\mathfrak{N}}}\Phi_v^{(r)}(B'_v,\widetilde{\chi}_v(\varpi_v)^{-1}q_v^{-k})\\
&\quad\times L^{(\mathfrak{N})}(k-\frac{r}{2},\chi_B\chi^{-1})L(k,\chi^{-1})^{-1}\prod_{i=1}^{r/2}L^{(\mathfrak{N})}(2k-2i,\chi^{-2})^{-1}\\
&\quad\times\prod_{\substack{v\nmid\infty\\ v|\mathfrak{N}}}\mathrm{Wh}_{B'_v}^{(k_r)}(\varphi_v)(1).
\end{aligned}\tag{7.2}$$

By Theorem 4.1, we have

$$\begin{aligned}
\xi(\kappa_v^{(r)},B'_v;k,0)&=2^{r/2}i^{r^2/4}\left(\det 2B'_v\right)^{k-(r+1)/2}\mathbf{e}(i\kappa_v^{(r)}B'_v)\\
&\quad\times\frac{\gamma_v\left(k,\widetilde{\chi}_v^{-1}\right)}{\gamma_v\left(k-r/2,\widetilde{\chi}_{B,v}\widetilde{\chi}_v^{-1}\right)}\prod_{i=1}^{r/2}\gamma_v\left(2k-2i,\widetilde{\chi}_v^{-1}\right).
\end{aligned}\tag{7.3}$$

By equations (7.2), (7.3), Theorem 5.1, Theorem 6.2 and the functional equation of Heck L-function, the following equation holds.

$$\begin{aligned} a(B, G_{k,\chi}^{(n)}) = A \prod_{v|\infty} i^{r^2/4} \left(\det B'_v\right)^{k-(r+1)/2} \prod_{v} \widetilde{\chi}_v(\det \alpha_v^2) \left|\det \alpha_v\right|_v^{-2k+(r+1)} \\ \times \prod_{\substack{v\nmid\infty \\ v\nmid\mathfrak{N}}} \widetilde{\chi}_v(\varpi_v)^{-e_{B'_v}} q_v^{((r+1)/2-k)e_{B'_v}} \prod_{\substack{v|\mathfrak{f}(\chi_B) \\ v\nmid\mathfrak{N}}} \gamma_v\left(k-r/2, \widetilde{\chi}_{B,v}\widetilde{\chi}_v^{-1}\right) \\ \times \prod_{\substack{v\nmid\infty \\ v|\mathfrak{N}}} \widetilde{\chi}_v(2\det B'_v)^{-1} \left|2\det B'_v\right|^{k-(r+1)/2} \mathrm{Wh}_{B'_v}^{(-k_r)}(\phi'_v)(1), \end{aligned}$$

Here ϕ'_v is defined by (2.3) and A is given by

$$\begin{aligned} A = 2^{rm/2} \prod_{\substack{v\nmid\infty \\ v\nmid\mathfrak{N}}} \Phi_{\mathfrak{p}}^{(r)}(B; \widetilde{\chi}_v(\varpi_v) q_v^{k-r-1}) \\ \times L(1-k,\chi)^{-1} L^{(\mathfrak{N})}(1+r/2-k, \chi_B\chi) \prod_{i=1}^{r/2} L^{(\mathfrak{N})}(1+2i-2k, \chi^2)^{-1}. \end{aligned}$$

By the definition of ϕ'_v, we have

$$\mathrm{Wh}_{B'_v}^{(-k_r)}(\phi'_v)(1) = 1. \tag{7.4}$$

Let v be a finite place such that $\widetilde{\chi}_v$ is unramified. Then

$$\varepsilon_v\left(k-r/2, \widetilde{\chi}_{B,v}\widetilde{\chi}_v^{-1}\right) = \left(\widetilde{\chi}_v^{-1}|\cdot|_v^{k-(r+1)/2}\right)(\mathfrak{f}(\widetilde{\chi}_{B,v}))\, \varepsilon_v\left(1/2, \widetilde{\chi}_{B,v}\right).$$

By (7.4) and the equation above, we have

$$\begin{aligned} a(B, G_{k,\chi}^{(n)}) = & A \prod_{v|\infty} i^{r^2/4} \prod_{v\nmid\infty} \varepsilon_v\left(1/2, \widetilde{\chi}_{B,v}\right) \\ & \times \prod_{v} \widetilde{\chi}_v^{-1}\left(\det \alpha_v^{-2} \det(2B_{v'})\right) \left|\det \alpha_v^{-2} \det 2B'_v\right|_v^{k-(r+1)/2}. \end{aligned}$$

By $B'_v[\alpha_v^{-1}] = B' \in \mathrm{GL}_r(F)$, we have

$$a(B, G_{k,\chi}^{(n)}) = A \prod_{v|\infty} i^{r^2/4} \prod_{v\nmid\infty} \varepsilon_v\left(1/2, \widetilde{\chi}_{B,v}\right).$$

By equation (5.1) and the product formula for the Weil index ([16, chapter 2 proposition 5]), $a(B, G_{k,\chi}^{(n)})$ is as stated in Theorem 2.3 if r is even. In a similar way, we can prove the theorem for an odd r.

8. The value of intertwining operators at odd primes

In this section, we prove Proposition 2.2. Since the Fourier coefficients of $E^{(n)}_{k,\chi}$ and $G^{(n)}_{k,\chi}$ have Euler product expression by (4.2), we may assume $\mathfrak{N} = \mathfrak{p}^\nu$ where $\mathfrak{p}$ is a prime of F such that $(\mathfrak{p}, 2) = 1$.

When $\widetilde{\chi}_\mathfrak{p}$ is a ramified character of $F_\mathfrak{p}^\times$ such that $\widetilde{\chi}_\mathfrak{p}^2$ is unramified, we define a function f_r of $\mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_\mathfrak{p}^{-1}|\cdot|_\mathfrak{p}^s)$ as follows.

$$\begin{aligned} \operatorname{supp} f_r &= P_n(F_\mathfrak{p}) w_{n,r} C_0(\mathfrak{p}), \\ f_r(w_{n,r}) &= 1. \end{aligned}$$

Here, for an ideal $\mathfrak{p}^\nu$ of $\mathcal{O}_\mathfrak{p}$, $C_0(\mathfrak{p}^\nu)$ is the compact open subgroup of $\mathrm{Sp}_n(F_\mathfrak{p})$ defined by

$$C_0(\mathfrak{p}^\nu) = \left\{ g \in \mathrm{Sp}_n(\mathcal{O}_\mathfrak{p}) \;\middle|\; c_g \equiv 0 \mod \mathfrak{p}^\nu \right\}.$$

Let $E'_{k,\chi}(\{r\}\,; z)$ be the Eisenstein series defined by (2.6). Then Eisenstein series $E'_{k,\chi}(\{r\}\,; z)$ corresponds to $\mathcal{E}_{f_r}$ by (3.5), where $\mathcal{E}_{f_r}$ is the Eisenstein series on $\mathrm{Sp}_n(\mathbb{A}_F)$ defined by (4.1).

Thus, to prove Proposition 2.2, it is enough to prove the following proposition.

Proposition 8.1. *Let $\phi_\mathfrak{p}$ and $\varphi_\mathfrak{p}$ be the functions defined by* (2.2) *and* (2.4). *If $\widetilde{\chi}_\mathfrak{p}^2$ is ramified, then $\varphi_\mathfrak{p} = \phi_\mathfrak{p}$. If $\widetilde{\chi}_\mathfrak{p}^2$ is unramified, then*

$$\varphi_\mathfrak{p} = \sum_{r=0}^{n} m_r(s + (n+1)/2, \chi, \mathfrak{p}) f_r,$$

where $m_r(k, \chi, \mathfrak{p})$ is defined by (2.7).

When $\widetilde{\chi}_\mathfrak{p}^2$ is ramified, then this proposition follows from the lemma below and (7.4). When $\widetilde{\chi}_\mathfrak{p}^2$ is unramified, we need some computation of the value of intertwining operators.

Lemma 8.2. *Let $\mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_\mathfrak{p}^{-1}|\cdot|_\mathfrak{p}^s)^{C_0(\mathfrak{p}^\nu)}$ denote the space defined below.*

$$\left\{ f \in \mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_\mathfrak{p}^{-1}|\cdot|_\mathfrak{p}^s) \;\middle|\; \begin{array}{c} f(g\gamma) = \widetilde{\chi}_\mathfrak{p}(\det d_\gamma) f(g), \\ \textit{for all } g \in \mathrm{Sp}_n(F_\mathfrak{p}) \textit{ and } \gamma \in C_0(\mathfrak{p}^\nu) \end{array} \right\}.$$

Then we have

$$\mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_\mathfrak{p}^{-1}|\cdot|_\mathfrak{p}^s)^{C_0(\mathfrak{p}^\nu)} = \begin{cases} \mathbb{C}\phi_\mathfrak{p} & \textit{if } \widetilde{\chi}_\mathfrak{p}^2 \textit{ is ramified,} \\ \bigoplus_{r=0}^{n} \mathbb{C} f_r & \textit{if } \widetilde{\chi}_\mathfrak{p}^2 \textit{ is unramified.} \end{cases}$$

Proof. By Bruhat decomposition, we have

$$\mathrm{Sp}_n(F_\mathfrak{p}) = \bigcup_{\substack{0\le r\le n \\ c\in\mathfrak{p}\mathrm{Sym}_{n-r}(\mathcal{O}_\mathfrak{p})}} P_n(F_\mathfrak{p})\iota\left(w_r \times {}^t\nu_{n-r}(c)\right) C_0(\mathfrak{p}^\nu).$$

If $\nu = 1$, then we also have

$$\mathrm{Sp}_n(F_\mathfrak{p}) = \bigcup_{r=0}^{n} P_n(F_\mathfrak{p}) w_{n,r} C_0(\mathfrak{p}).$$

When $\widetilde{\chi}_\mathfrak{p}^2$ is unramified, then the assertion of the lemma follows from this. Next, assume that $\widetilde{\chi}_\mathfrak{p}^2$ is ramified. Let $f \in \mathrm{Ind}_{P_n}^{\mathrm{Sp_n}}(\widetilde{\chi}_\mathfrak{p}^{-1}|\cdot|_\mathfrak{p}^s)^{C_0(\mathfrak{p}^\nu)}$ and $r > 0$, then for $a \in \mathrm{GL}_r(\mathcal{O}_\mathfrak{p})$, we have

$$\begin{aligned}\widetilde{\chi}_\mathfrak{p}^{-1}(\det a) f(g) &= f\left(\iota\left(\mu_r(a) \times 1_{2(n-r)}\right) g\right) \\ &= f(g\iota\left(\mu_r({}^t a^{-1}) \times 1_{2(n-r)}\right)) = \widetilde{\chi}_\mathfrak{p}(\det a) f(g).\end{aligned}$$

Here $g = \iota\left(w_r \times {}^t\nu_{n-r}(c)\right)$. Thus if $\widetilde{\chi}_\mathfrak{p}^2$ is ramified, then we have $\operatorname{supp} f = P_n(F_\mathfrak{p}) C_0(\mathfrak{p}^\nu)$. Therefore, we have the assertion of the lemma. □

In the rest of this section, we assume $\widetilde{\chi}_\mathfrak{p}^2$ is unramified. By Lemma 8.2, we have

$$\varphi_\mathfrak{p} = \sum_{r=0}^{n} M_{rn}(s) f_r.$$

Here, we put

$$M_{rr'}(s) = M_{w_n}^{(s)}(f_{r'})(w_{n,r}),$$

where $M_{w_n}^{(s)}$ is the intertwining operator defined by (2.1). Thus, to prove Proposition 8.1, it is enough to prove proposition below.

Proposition 8.3. *If r is odd, then $M_{rn}(s) = 0$. If r is even, then*

$$M_{rn}(s) = \widetilde{\chi}_\mathfrak{p}(-1)^{r/2} q^{-r/2} \left(q^{-1}; q^{-2}\right)_{r/2} \left(\widetilde{\chi}_\mathfrak{p}(\varpi)^2 q^{n-r-2s}; q^2\right)_{r/2}^{-1}.$$

Here we denote ϖ by the uniformizer of $\mathcal{O}_\mathfrak{p}$ and q by the order of the residue field $\mathbb{F}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p}$.

Proof. Define $\sigma = (\sigma_{ij})_{ij} \in \mathrm{GL}_n$ by $\sigma_{ij} = 1$ if $i + j = n + 1$ and $\sigma_{ij} = 0$ if $i + j \neq n + 1$. Since $\mu_n(\sigma)\, w_{n,r} \mu_n(\sigma) = 1_{2(n-r)} \times w_r$, we have

$$M_{rn}(s) = \int_{\mathrm{Sym}_n(F_\mathfrak{p})} f_n(\alpha(x))\, dx,$$

where

$$\alpha(x) = w_n \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \iota \left(1_{2(n-r)} \times w_r\right).$$

Put $x = \begin{pmatrix} x_{11} & x_{12} \\ {}^t x_{12} & x_{22} \end{pmatrix}$, where $x_{11} \in \mathrm{Sym}_{n-r}(F_\mathfrak{p})$. We define a lattice $L \subset \mathrm{Sym}_n(F_\mathfrak{p})$ by

$$L = \left\{ x \in \mathrm{Sym}_n(F_\mathfrak{p}) \;\middle|\; \begin{array}{c} x_{11} \in \mathrm{Sym}_{n-r}(\mathcal{O}_\mathfrak{p}),\ x_{12} \in M_{n-r,r}(\mathcal{O}_\mathfrak{p}), \\ x_{22} \in \mathfrak{p}\mathrm{Sym}_r(\mathcal{O}_\mathfrak{p}) \end{array} \right\}.$$

Since the value of the function $f_n(\alpha(x))$ is determined by $x \bmod L$, we have

$$M_{rn}(s) = q^{-r(r+1)/2} \sum_{x \bmod L} f_n(\alpha(x)).$$

Let $\mathfrak{M}_n$ be the set defined in Lemma 4.3. Put $x = c^{-1}d$, where $(c, d) \in \mathfrak{M}_n$. Then by Lemma 4.3, we have

$$M_{rn}(s) = q^{-r(r+1)/2} \sum_{(c,d) \in \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}) \backslash \mathfrak{M}_n / \sim_L} f_n\left(\alpha(c^{-1}d)\right).$$

Here, for (c, d), $(c', d') \in \mathfrak{M}_n$, we denote $(c, d) \sim_L (c', d')$ if and only if $c^{-1}d \equiv c'^{-1}d' \mod L$. Put

$$c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \qquad d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

Here upper left blocks are $n - r \times n - r$ matrices. Then the Iwasawa decomposition of $\alpha(x)$ is given by

$$\alpha(x) = \begin{pmatrix} {}^t c & * \\ 0 & c^{-1} \end{pmatrix} \left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline c_{11} & d_{12} & d_{11} & -c_{12} \\ c_{21} & d_{22} & d_{21} & -c_{22} \end{array} \right).$$

By the definition of f_n, $f_n(\alpha(x))$ does not vanish if and only if

$$\operatorname{rank}_{\mathbb{F}_\mathfrak{p}} \begin{pmatrix} c_{11} & d_{12} \\ c_{21} & d_{22} \end{pmatrix} \bmod \mathfrak{p} = n. \tag{8.1}$$

When x satisfies the condition above, we have

$$f_n\left(\alpha(x)\right) = \widetilde{\chi}_\mathfrak{p}(\det c)\,|\det c|_\mathfrak{p}^{s+(n+1)/2}\,\widetilde{\chi}_\mathfrak{p}\left(\det \begin{pmatrix} c_{11} & d_{12} \\ c_{21} & d_{22} \end{pmatrix}\right).$$

Assume c, d satisfies the equation (8.1). Put $\operatorname{rank}_{\mathbf{F}_\mathfrak{p}} c \bmod \mathfrak{p} = n - r + r_0$, then $r_0 \geq 0$. Let

$$c = v' \operatorname{diag}(1_{n-r}, 1_{r_0}, \varpi^{e_1} 1_{r_1}, \ldots, \varpi^{e_k} 1_{r_k}) v, \quad v, v' \in \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}),$$

be the elementary divisor decomposition. Here ϖ is the uniformizer of the integer ring $\mathcal{O}_\mathfrak{p}$. Put

$$E_r = \left\{ c' \;\middle|\; \begin{array}{l} c' = \operatorname{diag}(\varpi^{e_0} 1_{r_0}, \varpi^{e_1} 1_{r_1}, \ldots, \varpi^{e_k} 1_{r_k}), \\ 0 = e_0 < e_1 < \cdots < e_k,\ \sum_{a=0}^{k} r_a = r \end{array} \right\}.$$

Let $[\mathrm{Sym}_r(\mathcal{O}_\mathfrak{p})/\mathfrak{p}\mathrm{Sym}_r(\mathcal{O}_\mathfrak{p})]$ be a set of complete representatives of the set

$$\mathrm{Sym}_r(\mathcal{O}_\mathfrak{p})/\mathfrak{p}\mathrm{Sym}_r(\mathcal{O}_\mathfrak{p}).$$

We set

$$\mathfrak{D}_r(c') = \left\{ d' \in M_r(\mathcal{O}_\mathfrak{p}) \;\middle|\; c'^{-1} d' \in [\mathrm{Sym}_r(\mathcal{O}_\mathfrak{p})/\mathfrak{p}\mathrm{Sym}_r(\mathcal{O}_\mathfrak{p})] \right\}.$$

We put

$$\Gamma^0(c') = \begin{pmatrix} 1_{n-r} & \\ & c' \end{pmatrix}^{-1} \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}) \begin{pmatrix} 1_{n-r} & \\ & c' \end{pmatrix} \cap \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}),$$

and denote by $\mathfrak{V}(c')$ a set of complete representatives of

$$\Gamma^0(c') \backslash \left\{ v \in \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}) \;\middle|\; v = \begin{pmatrix} 1_{n-r} & * \\ 0 & * \end{pmatrix} \right\}.$$

With the notation above, we can take a set of complete representatives of

$$\mathrm{GL}_n(\mathcal{O}_\mathfrak{p}) \backslash \left\{ (c, d) \in \mathfrak{M}_n \;\middle|\; (c, d) \text{ satisfies (8.1)} \right\} / \sim_L$$

as the following set:

$$\left\{ \left(\begin{pmatrix} 1_{n-r} & \\ & c' \end{pmatrix} v,\ \begin{pmatrix} 0_{n-r} & \\ & d' \end{pmatrix} {}^t v^{-1} \right) \,\middle|\, c' \in E_r,\ d' \in \mathfrak{D}_r(c') \cap \mathrm{GL}_r(\mathcal{O}_\mathfrak{p}),\ v \in \mathfrak{V}(c') \right\}.$$

Thus $M_{rn}(s)$ becomes

$$M_{rn}(s) = q^{-r(r+1)/2} \sum_{c' \in E_r} \widetilde{\chi}_\mathfrak{p}(\det c')\, |\det c'|_\mathfrak{p}^{s+(n+1)/2} \sum_{\substack{v \in \mathfrak{V}(c') \\ d' \in \mathfrak{D}_r(c') \cap \mathrm{GL}_r(\mathcal{O}_\mathfrak{p})}} \widetilde{\chi}_\mathfrak{p}\left(\det d'\right). \tag{8.2}$$

Take $c' \in E_r$ and fix $c' = \mathrm{diag}(\varpi^{e_0} 1_{r_0}, \ldots, \varpi^{e_k} 1_{r_k}) \in E_r$. By [13, Lemma 15.2], the cardinality of $\mathfrak{V}(c')$ is given by

$$|\mathfrak{V}(c')| = b_r(q) \prod_{a=0}^{k} q^{(n-r)e_a r_a} b_{r_a}(q)^{-1} \prod_{0 \le a < b \le k} q^{(e_b - e_a) r_a r_b}, \tag{8.3}$$

Here $b_r(q)$ is defined by

$$b_r(q) = \prod_{a=1}^{r} (1 - q^{-a}).$$

We define $W_m(0, \widetilde{\chi}_\mathfrak{p})$ by

$$W_m(0, \widetilde{\chi}_\mathfrak{p}) = \sum_{s \in \mathrm{Sym}_m(\mathbb{F}_\mathfrak{p}) \cap \mathrm{GL}_m(\mathbb{F}_\mathfrak{p})} \widetilde{\chi}_\mathfrak{p}(\det s).$$

Then we have

$$\sum_{d' \in \mathfrak{D}_r(c') \cap \mathrm{GL}_r(\mathcal{O}_\mathfrak{p})} \widetilde{\chi}_\mathfrak{p}(\det d') = \prod_{a=0}^{k} q^{e_a r_a (r_a+1)/2} W_{r_a}(0, \widetilde{\chi}_\mathfrak{p}) \prod_{0 \le a < b \le k} q^{(e_a+1) r_a r_b}. \tag{8.4}$$

By (8.2), (8.3) and (8.4), we have

$$M_{rn}(s) = q^{-r(r+1)/2} b_r(q) \sum_{c' \in E_r} q^{\alpha(c')} \prod_{a=0}^{k} \widetilde{\chi}_\mathfrak{p}(\varpi^{e_a r_a}) b_{r_a}(q)^{-1} W_{r_a}(0, \widetilde{\chi}_\mathfrak{p}). \tag{8.5}$$

Here for $c' = \mathrm{diag}(\varpi^{e_0} 1_{r_0}, \ldots, \varpi^{e_k} 1_{r_k}) \in E_r$, $\alpha(c')$ is given by

$$\alpha(c') = \sum_{a=0}^{k} \left(e_a r_a \left(-s + \frac{n}{2} - r \right) + \frac{1}{2} e_a r_a^2 \right) + \sum_{0 \le a < b \le k} (e_b + 1) r_a r_b.$$

By [10, Theorem 1.3], we have

$$W_m(0,\widetilde{\chi}_{\mathfrak{p}}) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \widetilde{\chi}_{\mathfrak{p}}(-1)^{m/2} q^{m^2/2} \dfrac{b_m(q)}{b_{m/2}(q^2)} & \text{if } m \text{ is even.} \end{cases} \tag{8.6}$$

Thus $M_{rn}(s) = 0$ unless $r \equiv 0 \mod 2$. Assume $r \equiv 0 \mod 2$. For $c' = \mathrm{diag}(\varpi^{e_0} 1_{2r_0}, \ldots, \varpi^{e_k} 1_{2r_k}) \in E_r$, we put $\xi = \xi(c') = \mathrm{diag}(\varpi^{e_0} 1_{r_0}, \ldots, \varpi^{e_k} 1_{r_k}) \in E_{r/2}$. Substituting r_a by $2r_a$ in (8.5) and by (8.6), we have

$$M_{rn}(s) = q^{-r(r+1)/2} \widetilde{\chi}_{\mathfrak{p}}(-1)^{r/2} b_r(q) \sum_{\xi \in E_{r/2}} q^{\beta(\xi)} \prod_{a=0}^{k} \widetilde{\chi}_{\mathfrak{p}}(\varpi^{2e_a r_a}) b_{r_a}(q^2)^{-1}.$$

Here $\beta(\xi)$ is define by

$$\beta(\xi) = \sum_{a=0}^{k} e_a r_a(-2s+n-2r) + \sum_{a=0}^{k} 2e_a r_a^2 + \sum_{0 \le a < b \le k} 4e_b r_a r_b + r^2/2,$$

where $\xi = \mathrm{diag}(\varpi^{e_0} 1_{r_0}, \ldots, \varpi^{e_k} 1_{r_k}) \in E_{r/2}$. Therefore we have

$$M_{rn}(s) = q^{-r/2} \widetilde{\chi}_{\mathfrak{p}}(-1)^{r/2} \frac{b_r(q)}{b_{r/2}(q^2)} \sum_{\xi \in E_{r/2}} \varphi_{r/2}(\xi; \widetilde{\chi}_{\mathfrak{p}}(\varpi)^2 q^{n-r} X). \tag{8.7}$$

Here $X = q^{-2s}$ and $\varphi_m(\xi; X)$ is defined by

$$\varphi_m(\xi; X) = \frac{b_m(q^2)}{\prod_{a=0}^{k} b_{r_a}(q^2)} \prod_{a=0}^{k} (q^{-2m} X)^{e_a r_a} q^{2e_a r_a^2} \prod_{0 \le a < b \le k} q^{4e_b r_a r_b},$$

for $\xi = \mathrm{diag}(\varpi^{e_0} 1_{r_0}, \ldots, \varpi^{e_k} 1_{r_k}) \in E_m$. We define formal power series as follows.

$$\Phi_m(X) = \sum_{\xi \in E_m} \varphi_m(\xi; X), \qquad \Psi_m(X) = \sum_{\xi \in \varpi E_m} \varphi_m(\xi; X).$$

When $m = 0$, we put $\Phi_0(X) = \Psi_0(X) = 1$. We shall prove that

$$\Phi_m(X) = \prod_{a=0}^{m-1} (1 - q^{2a} X)^{-1}, \qquad \Psi_m(X) = X^m \Phi_m(X). \tag{8.8}$$

The assertion of the proposition follows from (8.7) and (8.8). Let $\xi = \mathrm{diag}(1_r, \xi') \in E_m$, where $\xi' \in \varpi E_{m-r}$. Then we have

$$\varphi_m(\xi; X) = c(m,r)\varphi_{m-r}(\xi', q^{2r}X),$$

where $c(m,r) = \frac{b_m(q^2)}{b_r(q^2)b_{m-r}(q^2)}$. Thus we obtain

$$\Phi_m(X) = \sum_{r=0}^{m} c(m,r)\Psi_{m-r}(q^{2r}X). \tag{8.9}$$

In a similar way, we have

$$\Psi_m(X) = \sum_{r=0}^{m} c(m,r)q^{2r(m-r)}X^{2m-r}\Phi_{m-r}(q^{2r}X). \tag{8.10}$$

Recursive equations (8.9), (8.10) and $\Phi_0(X) = \Psi_0(X) = 1$ determine $\Phi_m(X)$ and $\Psi_m(X)$. By these equations, we have $\Phi_m(X) = X^m\Psi_m(X)$. Therefore the following recursive equation holds

$$\Phi_m(X) = \sum_{r=0}^{m} c(m,r)q^{2r(m-r)}X^{m-r}\Phi_{m-r}(q^{2r}X).$$

By [13, Lemma 15.3], the recursive equation above is satisfied for $\Phi_m(X) = \prod_{a=0}^{m-1}(1-q^{2a}X)^{-1}$. Thus equation (8.8) holds and we obtain our proposition. □

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