Backstepping observer design for parabolic PDEs with measurement of weighted spatial averages[☆]

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Abstract

This paper is concerned with the observer design for one-dimensional linear parabolic partial differential equations whose output is a weighted spatial average of the state over the entire spatial domain. We focus on the backstepping approach, which provides a systematic procedure to design an observer gain for systems with boundary measurement. If the output is not a boundary value of the state, the backstepping approach is not directly applicable to obtaining an observer gain that stabilizes the error dynamics. Therefore, we attempt to convert the error system into another system to which backstepping is applicable. The conversion is successfully achieved for a class of weighting functions, and the resultant observer realizes exponential convergence of the estimation error with an arbitrary decay rate in terms of the L^2 norm. In addition, an explicit expression of the observer gain is available in a special case. The effectiveness of the proposed observer is also confirmed by numerical simulations.

Keywords: Distributed-parameter systems, Observers, Backstepping, Parabolic partial differential equations, Spatial averages

1. Introduction

The observer design for systems modeled by partial differential equations (PDEs) is a classical but still important problem in control engineering. The estimated state can be used not only to implement state feedback controllers but also to monitor an invisible state distribution such as the concentration of some chemical species in process engineering (Delattre et al., 2004). The theory of the Luenberger observer for linear infinite dimensional systems was established by replacing matrices with linear operators (Curtain and Zwart, 1995; Lasiecka and Triggiani, 2000), see also the recent survey paper Hidayat et al. (2011). Hence, the observer design is reduced to determining a gain operator that stabilizes the associated error dynamics. Unlike finite dimensional systems, it is not easy to find such a gain even numerically because operators are not generally represented with a finite number of parameters. A well-known systematic approach to designing a stabilizing gain is the infinite dimensional optimal filtering theory (Curtain, 1978), where a stabilizing gain is constructed by using a solution of the operator Riccati equation (Bensoussan et al., 2007). However, solving the Riccati equation is generally difficult. Besides, numerical methods require a solution of a very high order matrix Riccati equation (Lasiecka and Triggiani, 1991). Therefore, we need to develop a computationally light design method that also guarantees some prescribed performance.

Recently, another framework was proposed in Smyshlyaev and Krstic (2005, 2010) for systems described by a one-dimensional parabolic PDE whose output is a boundary value of the state. The proposed framework is based on the infinite dimensional backstepping approach (Balogh and Krstic, 2002; Liu, 2003; Smyshlyaev and Krstic, 2004), which is a systematic design tool for state feedback gains. The observer gain is determined so that the error system is converted into an exponentially stable target system by a state transformation called the backstepping transformation. The resulting observer gain stabilizes the error system exponentially with a given decay rate, and it is characterized by the solution of a linear hyperbolic PDE. Since this equation is linear, a symbolic or numerical approximate solution is easily obtained. In particular, explicit solutions can be obtained in some special cases. The backstepping observer has been extended to systems described by other types of PDEs (Krstic et al., 2008b,a; Vazquez and Krstic, 2010; Krstic et al., 2011).

These practical advantages are attractive enough to expect that the backstepping approach can be applied to systems with other kinds of observation. An important class of measurement for the distributed state is the weighted spatial average. Strictly speaking, all sensors measure some averaged value of the state around them, because there is no infinitesimal sensor. This paper, therefore, considers observer design based on the backstepping approach when the output is a weighted spatial average of the state. As the first study on this issue, we restrict the scope to the systems described by a one-dimensional parabolic PDE. Moreover, the output is assumed to be a spatial average of the state over the entire spatial domain. In other words, the output is an integral of the product of a weighting function and the state over the spatial domain. Such sensing can be approximately

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realized by distributing a number of sensors and applying consensus algorithms (Olfati-Saber et al., 2007).

Contrary to our expectations, the backstepping approach is not directly applicable if the output is not a boundary value. This is due to the spatial structure of the error dynamics. The backstepping transformation exploits a structure of PDEs. However, the structure of the output error feedback term is not compatible with that desired in the backstepping framework. We introduce an auxiliary transformation to circumvent this problem. It will be shown that, under a certain condition for the weighting function, the proposed transformation converts the error system into a system for which the backstepping method provides an exponentially stabilizing gain. Once a gain for the transformed system is obtained, we can construct gains for the original error system by using the inverse transformation. In addition, the original error system inherits the exponential stability with a given decay rate from the transformed system. Noted that the proposed transformation is completely different from the backstepping transformation. In particular, its inverse is a discontinuous map.

The idea and the approach presented in this paper are the same as those in our conference paper Tsubakino and Hara (2011). However, there are substantial differences. The derivation of the observer is simplified by assembling the transformations used in the previous paper. Moreover, we succeed in deriving an observer that estimates the original state directly in this paper, whereas the previous observer estimated the transformed state. Although this seems a minor change, a new difficulty regarding the regularity arises, because the inverse of our transformation is discontinuous. Explicit observer gains are obtained in a more general case. The omitted proofs are fully included in the present paper.

The paper is organized as follows. In Section 2, we formulate the system and problem to be considered. Section 3 presents our approach using an additional transformation to resolve the problem and an analysis of the properties of the transformation as a linear operator. Section 4 deals with the design of observer gains based on backstepping. The convergence property of the estimation error is revealed. Section 5 explains the design procedure of the proposed framework. We also show that explicit observer gains can be obtained in a special case. We demonstrate the performance of the proposed observer by a numerical simulation in Section 6. Finally, we conclude the paper in Section 7.

Notation. Throughout this paper, we write I for the open interval $(0, 1) \subset \mathbb{R}$. Its closure in $\mathbb{R}$, that is, the closed interval $[0, 1]$, is denoted by $\bar{I}$. Let $L^2(I)$ be a set of (equivalent classes of) square integrable real-valued functions over I with respect to the Lebesgue measure. For $k \in \mathbb{N}$, $H^k(I)$ stands for the kth order Sobolev space, in other words, a vector space consisting of elements in $L^2(I)$ whose distributional derivative up to order k can be identified with an element of $L^2(I)$. We always assume that $L^2(I)$ and $H^k(I)$ are Hilbert spaces equipped with the inner products

$$(f, g)_{L^2(I)} = \int_0^1 f(x)g(x)dx, \quad f, g \in L^2(I),$$

$$(f, g)_{H^k(I)} = \sum_{i=0}^{k} \left(f^{(i)}, g^{(i)}\right)_{L^2(I)}, \quad f, g \in H^k(I),$$

where $f^{(i)}$ is the ith order (distributional) derivative of f and $f^{(0)} = f$. In the remaining sections, the notations $(\cdot)'$, $(\cdot)''$, and $(\cdot)'''$ are used instead of $(\cdot)^{(1)}$, $(\cdot)^{(2)}$, and $(\cdot)^{(3)}$, respectively. The associated norms with the above inner products are denoted by $\|f\|_X = \sqrt{(f, f)_X}$ for each $f \in X$, where X is $L^2(I)$ or $H^k(I)$.

2. Problem setting

Consider a system described by the parabolic PDE equation

$$u_t(x, t) = au_{xx}(x, t) + \lambda(x)u(x, t) \tag{1}$$

with boundary conditions

$$u_x(0, t) + \alpha u(0, t) = 0, \tag{2}$$

$$u(1, t) = U(t), \tag{3}$$

where $u : I \times [0, +\infty) \to \mathbb{R}$ is the state, $U(t) \in \mathbb{R}$ is the control input, and the coefficients are assumed to be $a > 0$, $\lambda \in C^1(\bar{I})$, and $\alpha \in \mathbb{R}$. Although the control input acts at the right end-point, the place of the input is not important to the observer design. More general parabolic equations that contain a term proportional to the spatial derivative of the state, such as $b(x)u_x(x, t)$, can be transformed into (1) as shown in Smyshlyaev and Krstic (2004, 2005).

We assume that a weighted average of the state over the spatial domain I is measured. Namely, the output is given by

$$Y(t) = \int_0^1 h(\xi)u(\xi, t)d\xi, \tag{4}$$

where h is a positive spatially weighting function[2]. In practice, the function h is determined by the sensor properties. However, as a first step toward general weighting functions, we restrict the class of weighting functions to solutions of the following ordinary differential equation (ODE) with the parameter $\gamma \in \mathbb{R}$:

$$ah''(x) + \lambda(x)h(x) = \gamma h(x), \quad x \in I \tag{5}$$

under the single initial condition

$$h'(0) + \alpha h(0) = 0. \tag{6}$$

The parameter γ and initial value $h(0)$ are not specified. For appropriate γ, there always exist positive functions that satisfies the initial value problem (5)–(6). We call such a solution *positive*. Positive solutions to (5)–(6) do not cause a lack of observability. These topics are discussed in Appendix A.

[2]In this paper, we call a real-valued function f *positive* if the range of f is contained in $(0, +\infty)$.

The main purpose of this paper is to develop a systematic design procedure of the state observer for the system (1)–(4). To this end, we focus on the backstepping observer (Smyshlyaev and Krstic, 2005). In the backstepping framework, we first construct a standard Luenberger-type observer and then apply backstepping in order to obtain an observer gain that stabilizes the resulting error system. Such an observer for (1)–(4) can be written as

$$\begin{aligned}\hat{u}_t(x,t) &= a\hat{u}_{xx}(x,t) + \lambda(x)\hat{u}(x,t) \\ &\quad - l(x)\left(Y(t) - \int_0^1 h(\xi)\hat{u}(\xi,t)d\xi\right)\end{aligned} \tag{7}$$

$$\hat{u}_x(0,t) + \alpha\hat{u}(0,t) = 0, \tag{8}$$

$$\hat{u}(1,t) - l_b\left(Y(t) - \int_0^1 h(\xi)\hat{u}(\xi,t)d\xi\right) = U(t) \tag{9}$$

where $l : \overline{I} \to \mathbb{R}$ and $l_b \in \mathbb{R}$ are observer gains. Subtracting (7)–(9) from the system equation (1)–(3), we obtain the following error system:

$$\begin{aligned}\tilde{u}_t(x,t) &= a\tilde{u}_{xx}(x,t) + \lambda(x)\tilde{u}(x,t) \\ &\quad + l(x)\int_0^1 h(\xi)\tilde{u}(\xi,t)d\xi,\end{aligned} \tag{10}$$

$$\tilde{u}_x(0,t) + \alpha\tilde{u}(0,t) = 0, \tag{11}$$

$$\tilde{u}(1,t) + l_b\int_0^1 h(\xi)\tilde{u}(\xi,t)d\xi = 0, \tag{12}$$

where $\tilde{u}$ is the estimation error defined by $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$. For any $x \in I$, the terms containing the observer gains depend on the value of $\tilde{u}$ at (almost) all points in the spatial domain I. This fact prevents us from directly applying the backstepping method because, in the backstepping observer design, the error system is required to have *triangular terms* only. Namely, all the terms in the equation must depend only on the value of $\tilde{u}$ or its derivatives at some points greater than or equal to x for the upper-triangular case and less than or equal to x for the lower-triangular case. This is the most crucial problem that we need to solve.

Remark 1. We can restrict the output error feedback to the right boundary value only, that is, $l(x) \equiv 0$ as in Vries et al. (2010). Then, the observer gain to be designed is a scalar constant. However, to analyze the stability and the convergence rate of the error system, we must calculate an eigenfunction many times for different l_b, which is generally not obtained in a closed form. In addition, for certain h and λ, there is no l_b such that the error system with $l(x) \equiv 0$ is stable.

3. Approach using auxiliary transformation

In this section, we introduce the key idea to design an observer for (1)–(4) based on the backstepping approach. As discussed in the previous section, the main difficulty with applying backstepping is the presence of the non-triangular terms caused by the dependence of the output Y on values of the state u at (almost) all $x \in I$. Hence, we will attempt to convert the error system (10)–(12) into a system to which backstepping is applicable.

$$\begin{tikzcd}
\text{system: } \tilde{u}\text{-system} \arrow[r, "\text{transformation (13)}"] & \tilde{v}\text{-system} \arrow[d, "\text{backstepping design}"] \\
\text{gains: } l,\ l_b & m,\ m_b \arrow[l, "\text{inverse transformation}"']
\end{tikzcd}$$

Figure 1: Diagram of proposed framework.

3.1. Integral transform

We introduce the new variable $\tilde{v}$ defined by

$$\tilde{v}(x,t) = \frac{1}{h(x)}\int_0^x h(\xi)\tilde{u}(\xi,t)d\xi. \tag{13}$$

The motivation for introducing this transformation comes from a quite simple fact. If the tilde is dropped in (13), we can regard the resulting equation as a state transformation from u to v. Then, the output equation for this new state v becomes

$$Y(t) = \int_0^1 h(\xi)u(\xi,t)d\xi = h(1)v(1,t). \tag{14}$$

This means that the output is a boundary value of v. Hence, it is expected that backstepping is applicable to the transformed system. This expectation holds true for our class of weighting functions. The transformation (13) maps a solution $\tilde{u}$ of (10)–(12) into a solution of

$$\tilde{v}_t(x,t) = a\tilde{v}_{xx}(x,t) + \mu(x)\tilde{v}(x,t) + m(x)\tilde{v}(1,t), \tag{15}$$

$$\tilde{v}(0,t) = 0, \tag{16}$$

$$\tilde{v}_x(1,t) + (\beta + m_b)\,\tilde{v}(1,t) = 0, \tag{17}$$

where we set $\beta = h'(1)/h(1)$ and

$$\mu(x) = \lambda(x) + 2a\frac{d}{dx}\left(\frac{h'(x)}{h(x)}\right), \tag{18}$$

respectively. Note that $\mu \in C^1(\overline{I})$ whenever h is a positive solution to (5)–(6). The transformed observer gains m and m_b are defined as

$$m(x) = \frac{h(1)}{h(x)}\int_0^x h(\xi)l(\xi)d\xi, \tag{19}$$

$$m_b = h(1)l_b. \tag{20}$$

Since (15) is a parabolic PDE that contains triangular terms only, the backstepping method provides the observer gains m and m_b that stabilize (15)–(17) exponentially as in Smyshlyaev and Krstic (2005). Then, we can obtain the observer gains l and l_b for the original system through inverse transformation. This is our strategy (see Fig. 1). Of course, this is possible only if the designed interior gain m is compatible with (19). Namely, m is differentiable and satisfies $m(0) = 0$.

Let us derive (15)–(17). Suppose that h is a positive solution of (5)–(6). The left boundary condition (16) easily follows from the definition (13). Differentiating both sides of (13) with respect to the spatial variable x yields

$$\begin{aligned}\tilde{v}_x(x,t) &= -\frac{h'(x)}{h(x)^2}\int_0^x h(\xi)\tilde{u}(\xi,t)d\xi + \tilde{u}(x,t)\\ &= -\frac{h'(x)}{h(x)}\tilde{v}(x,t) + \tilde{u}(x,t).\end{aligned}$$

By substituting $x = 1$, we see that $\tilde{v}$ must satisfy the right boundary condition (17). It also follows from the above relation that

$$\tilde{u}(x,t) = \tilde{v}_x(x,t) + \frac{h'(x)}{h(x)}\tilde{v}(x,t), \tag{21}$$

which gives an explicit formula for the inverse transformation. Differentiating (21) with respect to x leads to the expression of $\tilde{u}_x$ in terms of $\tilde{v}$:

$$\begin{aligned}\tilde{u}_x(x,t) = \tilde{v}_{xx}(x,t) &+ \frac{h'(x)}{h(x)}\tilde{v}_x(x,t)\\ &+ \frac{d}{dx}\left(\frac{h'(x)}{h(x)}\right)\tilde{v}(x,t).\end{aligned} \tag{22}$$

The temporal derivative of $\tilde{v}$ is computed as

$$\begin{aligned}\tilde{v}_t(x,t) &= \frac{1}{h(x)}\int_0^x h(\xi)\left(a\tilde{u}_{\xi\xi}(\xi,t) + \lambda(\xi)\tilde{u}(\xi,t)\right)d\xi\\ &\quad + \frac{1}{h(x)}\int_0^x h(\xi)l(\xi)d\xi\int_0^1 h(\xi)\tilde{u}(\xi,t)d\xi\\ &= a\tilde{u}_x(x,t) - a\frac{h'(x)}{h(x)}\tilde{u}(x,t)\\ &\quad + a\frac{1}{h(x)}\left(h'(0) + \alpha h(0)\right)\tilde{u}(0,t)\\ &\quad + \frac{1}{h(x)}\int_0^x \left(ah''(\xi) + \lambda(\xi)h(\xi)\right)\tilde{u}(\xi,t)d\xi\\ &\quad + m(x)\tilde{v}(1,t),\end{aligned}$$

where we use integration by parts twice. Then, substituting (5), (6), (21), and (22) into the right hand side of the above equation gives (15).

Remark 2. The left boundary condition (11) for the original error variable $\tilde{u}$ seems to be lost. However, it can be recovered from (15) and (16) if, for each $t > 0$, $\tilde{v}(\cdot,t)$ can be continuously extended to a function on $\bar{I}$ up to the second partial derivative with respect to x. In this case, it follows from (21) and (22) that

$$\begin{aligned}&\tilde{u}_x(0,t) + \alpha\tilde{u}(0,t)\\ &\quad = \tilde{v}_{xx}(0,t) + \frac{h'(0)}{h(0)}\tilde{v}_x(0,t) + \alpha\tilde{v}_x(0,t) = \tilde{v}_{xx}(0,t),\end{aligned}$$

where $\alpha = -h'(0)/h(0)$ is used. To evaluate $\tilde{v}_{xx}(0,t)$, note that $\tilde{v}_t(0,t) = 0$ because $\tilde{v}(0,t) = 0$. Substituting $x = 0$ into (15) gives

$$\begin{aligned}0 = \tilde{v}_t(0,t) &= a\tilde{v}_{xx}(0,t) + \mu(0)\tilde{v}(0,t) + m(0)\tilde{v}(1,t)\\ &= a\tilde{v}_{xx}(0,t).\end{aligned}$$

Thus, we have $\tilde{v}_{xx}(0,t) = 0$, and (11) holds. The extension of $\tilde{v}$ is possible if $\tilde{v}(\cdot,t) \in H^3(I)$ due to the fact that every element in $H^3(I)$ has a representative in $C^2(\bar{I})$. This regularity is also necessary to guarantee that $\tilde{u}(\cdot,t) \in H^2(I)$ because the right hand side of (21) contains the spatial derivative of $\tilde{v}$. The regularity of $\tilde{v}$ will be justified later.

3.2. Continuity and invertibility

In this subsection, we consider the continuity and invertibility of the proposed transformation (13) as a linear operator on $L^2(I)$. Both play an important role in the analysis of the convergence property of the error system (10)–(12). Proofs of all the results in this subsection are given in Appendix B.

Define a closed subspace V of $H^1(I)$ by

$$V = \{f \in H^1(I) \mid f(0) = 0\}, \tag{23}$$

where the boundary value of an element in $H^1(I)$ indicates that of its absolutely continuous representative as usual. This convention is used throughout the paper. We equip V with an inner product. Set, for $f, g \in V$,

$$(f,g)_V = (f',g')_{L^2(I)} = \int_0^1 f'(x)g'(x)dx.$$

This gives an inner product for V that induces the norm $\|f\|_V = \sqrt{(f,f)_V} = \|f'\|_{L^2(I)}$ that is equivalent to the H^1 norm by virtue of the Poincaré-type inequality (Hardy et al., 1952)

$$\|f\|_{L^2(I)} \le \frac{2}{\pi}\|f'\|_{L^2(I)}$$

for all $f \in V$. Since V is a closed subspace of $H^1(I)$, the inner product $(\cdot,\cdot)_V$ turns V into a Hilbert space.

Lemma 1. *Consider the linear operator T on $L^2(I)$ defined by*

$$(Tf)(x) = \frac{1}{h(x)}\int_0^x h(\xi)f(\xi)d\xi, \tag{24}$$

where $h \in C^1(\bar{I})$ and $h(x) > 0$ for all $x \in \bar{I}$. Then, the range of T is contained in V, and there exists a constant $C > 0$ such that, for all $f \in L^2(I)$,

$$\|Tf\|_V \le C\|f\|_{L^2(I)}$$

The next lemma deals with the inverse of (24) and its continuity.

Lemma 2. *Consider the linear operator T on $L^2(I)$ defined by (24) for some $h \in C^1(\bar{I})$ that satisfies $h(x) > 0$ for all $x \in \bar{I}$. Then, T is a bijection from $L^2(I)$ to V, and the inverse operator T^{-1} is given by*

$$(T^{-1}g)(x) = \frac{1}{h(x)}\frac{d}{dx}\Big(h(x)g(x)\Big) = g'(x) + \frac{h'(x)}{h(x)}g(x)$$

with the domain $D(T^{-1}) = V \subset L^2(I)$. Furthermore, there exists a constant $C > 0$ such that, for any $g \in V$,

$$\|T^{-1}g\|_{L^2(I)} \le C\|g\|_V.$$

We emphasize that T^{-1} is a discontinuous operator on $L^2(I)$. Therefore, the inequality in Lemma 2 no longer holds if $\|\cdot\|_V$ is replaced by the L^2 norm $\|\cdot\|_{L^2(I)}$. The situation is summarized in Fig. 2. This complicates the discussion in a later section.

$$\begin{array}{c}
L^2(I) \to L^2(I) \quad \text{continuous} \\
L^2(I) \to V \qquad \text{continuous} \\
L^2(I) \underset{T^{-1}}{\overset{T}{\rightleftarrows}} V \subset L^2(I) \\
L^2(I) \leftarrow L^2(I) \quad \text{discontinuous} \\
L^2(I) \leftarrow V \qquad \text{continuous}
\end{array}$$

Figure 2: Continuity of proposed transformation

4. Backstepping observer design

In this section, we design the observer gains l and l_b for (10)–(12) based on the backstepping method. We also prove the exponential stability under the obtained gains.

4.1. Observer gains

As alluded to earlier, we apply backstepping to the $\tilde{v}$-system (15)–(17). In accordance with Smyshlyaev and Krstic (2005), we can find a state transformation of the form

$$\tilde{v}(x,t) = \tilde{w}(x,t) - \int_x^1 p(x,y)\tilde{w}(y,t)dy \tag{25}$$

that, with suitably selected observer gains, converts the $\tilde{v}$-system (15)–(17) into the exponentially stable target system

$$\tilde{w}_t(x,t) = a\tilde{w}_{xx}(x,t) - c\tilde{w}(x,t), \tag{26}$$

$$\tilde{w}(0,t) = 0, \tag{27}$$

$$\tilde{w}_x(1,t) = 0, \tag{28}$$

where $c > 0$ is a design parameter that determines the convergence rate. The exponential stability will be clarified later. The state transformation (25) is called the backstepping transformation. The conversion is possible if the observer gains m and m_b satisfy

$$m(x) = ap_y(x,1), \tag{29}$$

$$m_b = -(p(1,1) + \beta) \tag{30}$$

and the integral kernel p is a solution of

$$ap_{yy}(x,y) = ap_{xx}(x,y) + (\mu(x) + c)\, p(x,y), \tag{31}$$

$$p(0,y) = 0, \tag{32}$$

$$p(x,x) = -\frac{1}{2a}\int_0^x (\mu(\xi) + c)\, d\xi. \tag{33}$$

The transformation (25) with the integral kernel satisfying (31)–(33) is continuously invertible on $L^2(I)$ and $H^1(I)$. Thus, the error system (15)–(17) inherits the exponential stability with respect to such norms from the target system (26)–(28). This is the essence of the backstepping method. We need to keep in mind that the boundary value problem (31)–(33) is well-posed for any $\mu \in C^1(\bar{I})$. Namely, there exists a unique solution p to (31)–(33) that is twice continuously differentiable on the closed domain $\mathcal{T} := \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le y \le 1\}$. See Smyshlyaev and Krstic (2005, 2010), for details on the derivation of (29)–(33). The well-posedness of (31)–(33) is also proved there through the conversion of (31)–(33) into an integral equation and the application of the successive approximation.

Remark 3. Recalling that μ is defined by (18), we can rewrite the boundary value $p(x,x)$ in (33) as

$$\begin{aligned}
p(x,x) &= -\frac{1}{2a}\int_0^x (\lambda(\xi) + c)\, d\xi - \left[\frac{h'(\xi)}{h(\xi)}\right]_0^x \\
&= -\frac{1}{2a}\int_0^x (\lambda(\xi) + c)\, d\xi - \frac{h'(x)}{h(x)} - \alpha.
\end{aligned}$$

Since $\beta = h'(1)/h(1)$, the boundary gain m_b becomes

$$m_b = -\left(p(1,1) + \frac{h'(1)}{h(1)}\right) = \frac{1}{2a}\int_0^1 (\lambda(\xi) + c)\, d\xi + \alpha,$$

which means that m_b does not depend on h.

Once the observer gains that stabilize (15)–(17) exponentially are obtained, the ones for the original error system (10)–(12) are determined by the relation (19)–(20). Indeed, the boundary condition (32) gives $p_y(0,y) = 0$ for all $y \in \bar{I}$. Hence, we have $m(0) = ap_y(0,1) = 0$. This fact allows us to calculate l and l_b as

$$l(x) = \frac{a}{h(1)}\left(\frac{h'(x)}{h(x)}p_y(x,1) + p_{xy}(x,1)\right), \tag{34}$$

$$l_b = \frac{1}{h(1)}\left(\frac{1}{2a}\int_0^1 (\lambda(\xi) + c)\, d\xi + \alpha\right). \tag{35}$$

Interestingly, except for $1/h(1)$, the boundary gain l_b is the same as the boundary gain of the backstepping observer for the system (1)–(3) with the boundary measurement $Y(t) = u(1,t)$ rather than (4).

4.2. Convergence of error

We discuss the convergence of the estimation error in this subsection. The main result is summarized as follows:

Theorem 4. *Let $a > 0$, $\lambda \in C^1(\bar{I})$, $c > 0$ and let h be a positive solution of (5)–(6) for some $\gamma \in \mathbb{R}$. Assume that l and l_b are given by (34)–(35) for the solution p of (31)–(33). Then, for any initial error $\tilde{u}_0 \in L^2(I)$, there exists a unique solution $\tilde{u} \in C([0,+\infty); L^2(I)) \cap C^1((0,+\infty); L^2(I))$ to the error system (10)–(12) with $\tilde{u}(\cdot,0) = \tilde{u}_0$. Furthermore, for all $t \ge 0$, the following estimate holds*

$$\|\tilde{u}(\cdot,t)\|_{L^2(I)} \le Me^{-\left(\frac{\pi^2}{4}a + c\right)t}\|\tilde{u}_0\|_{L^2(I)}, \tag{36}$$

where $M \ge 1$ is a constant independent of $\tilde{u}_0$.

To prove the theorem, careful attention should be paid to the state space. In Theorem 4, $L^2(I)$ is regarded as the state space for the first error system (10)–(12). Recall that the linear operator (24) corresponding to the proposed transformation (13) is a continuous and invertible map from $L^2(I)$ onto V and that its inverse is not a continuous operator defined everywhere on $L^2(I)$. Consequently, if we employ $L^2(I)$ as the state space for the $\tilde{v}$- and $\tilde{w}$-systems, a continuous relationship to $\tilde{u}$ can not be established. For this reason, we lift up the state space for $\tilde{v}$ and $\tilde{w}$ to the Hilbert space V endowed with the inner product $(\cdot,\cdot)_V$.

We begin by analyzing the $\tilde{v}$- and $\tilde{w}$-systems. Define a linear operator A_w on V by

$$(A_w f)(x) = af''(x) - cf(x), \tag{37}$$

with the domain

$$D(A_w) = \{f \in H^3(I) \,|\, f(0) = 0,\ f'(1) = 0,\ f''(0) = 0\}.$$

Of course, we assume that $a, c > 0$. Obviously, A_w is the system operator of the target system (26)–(28). The condition on the second derivative is necessary to ensure that $A_w f \in V$ whenever $f \in D(A_w)$. It is not difficult to show that A_w is a self-adjoint maximal dissipative operator[3] on V. In other words, the operator A_w satisfies the following three conditions:

1. $(A_w f, f)_V \le 0$ for all $f \in D(A_w)$,
2. for each $g \in V$, there exists $f \in D(A_w)$ such that $f - A_w f = g$, and
3. $(A_w f, g)_V = (f, A_w g)_V$ for all $f,\ g \in D(A_w)$.

Indeed, for any $f \in D(A_w)$, we get

$$\begin{aligned}(A_w f, f)_V &= \int_0^1 (af'''(x) - cf'(x))\, f'(x)dx \\ &= [af''(x)f'(x)]_0^1 - a\|f''\|_{L^2(I)}^2 - c\|f'\|_{L^2(I)}^2 \\ &\le -\left(\frac{\pi^2}{4}a + c\right)\|f\|_V^2 \le 0, \end{aligned} \tag{38}$$

where we utilize integration by parts and the Poincaré-type inequality $\|f'\|_{L^2(I)} \le (2/\pi)\|f''\|_{L^2(I)}$. In addition, for a given $g \in V$, the boundary value problem

$$\begin{aligned} f - A_w f = -af'' + (c+1)f = g,& \\ f(0) = 0,& \\ f'(1) = 0& \end{aligned}$$

has a solution in $H^2(I)$. The second derivative of such a solution f satisfies $f'' = ((c+1)/a)f - (1/a)g$. Since $f,\ g \in V$, we can deduce that $f'' \in V$, which implies $f \in D(A_w)$. Finally, the third condition can be checked directly by using integration by parts twice.

Based on the maximal dissipativity of A_w, we can show the well-posedness of the $\tilde{v}$-system (15)–(17) as well as the target system (26)–(28). The following lemma is almost the same as Theorem 3 in Smyshlyaev and Krstic (2005), but the state space is different, and we also mention the higher order regularity of a solution to (15)–(17). The regularity is necessary in the proposed framework, as described in Remark 2.

Lemma 3. *Assume that $a > 0$, $\beta \in \mathbb{R}$, $\mu \in C^1(\bar{I})$ and that p satisfies (31)–(33). Let $m \in C(\bar{I})$ and $m_b \in \mathbb{R}$ be given by (29) and (30), respectively. Then, for any initial data $\tilde{v}_0 \in V$, there exists a unique solution $\tilde{v} \in C([0, +\infty); V) \cap C^1((0, +\infty); V)$ to (15)–(17) such that $\tilde{v}(\cdot, 0) = \tilde{v}_0$ and $\tilde{v}(\cdot, t) \in H^3(I)$ for any $t > 0$.*

[3] Equivalently, $-A_w$ is a self-adjoint maximal monotone operator.

We prove the lemma in Appendix B.

Once the existence of a solution $\tilde{v}$ that belongs to an appropriate solution space is clarified, we can show the exponential stability of the $\tilde{v}$-system (15)–(17) with respect to the V norm $\|\cdot\|_V$ in a similar manner to Smyshlyaev and Krstic (2010); Liu (2003). Indeed, the temporal derivative of $(1/2)\|\tilde{w}(\cdot, t)\|_V^2$ is given by $(A_w\tilde{w}(\cdot, t), \tilde{w}(\cdot, t))_V$ for all $t > 0$. Then, the inequality (38) implies the exponential stability of the $\tilde{w}$-system along with Gronwall's inequality. The continuous invertibility of the backstepping transformation gives

$$\|\tilde{v}(\cdot, t)\|_V \le M_v e^{-\left(\frac{\pi^2}{4}a + c\right)t}\|\tilde{v}_0\|_V \tag{39}$$

for all $t \ge 0$, where the constant $M_v \ge 1$ depends only on p, that is, a, c, β, and μ.

We prove Theorem 4 based on the foregoing discussion.

Proof (Theorem 4). Given $\tilde{u}_0 \in L^2(I)$, we set $\tilde{v}_0 = T\tilde{u}_0$, where T is the operator defined by (24). From Lemma 1, $\tilde{v}_0 \in V$. Then, Lemma 3 guarantees the existence of a unique solution $\tilde{v} \in C([0, +\infty); V) \cap C^1((0, +\infty); V)$ for the initial value $\tilde{v}_0$. Moreover, $\tilde{v}(\cdot, t) \in H^3(I)$ for all $t > 0$. Hence, the unique solution $\tilde{u}$ to the error system (10)–(12) is constructed as $\tilde{u}(\cdot, t) = T^{-1}\tilde{v}(\cdot, t)$. The continuity of T^{-1} as a map from V to $L^2(I)$, which is proved in Lemma 2, confirms that $\tilde{u}$ belongs to $C([0, +\infty); L^2(I)) \cap C^1((0, +\infty); L^2(I))$.

The exponential convergence of $\tilde{u}$ is easily follows from that of $\tilde{v}$. From Lemmas 1 and 2, there exist constants C_1, $C_2 > 0$ such that

$$\begin{aligned} \|\tilde{v}_0\|_V &\le C_1\|\tilde{u}_0\|_{L^2(I)}, \\ \|\tilde{u}(\cdot, t)\|_{L^2(I)} &\le C_2\|\tilde{v}(\cdot, t)\|_V \end{aligned}$$

for all $t \ge 0$. These constants depend on h, that is, a, α, γ, and λ. Combining these inequalities and (39) leads to (36), and the theorem follows. □

In our conference paper Tsubakino and Hara (2011), we had not succeeded in proving the higher order regularity of the transformed error $\tilde{v}$. Thus, we could only conclude the convergence of the image of $\tilde{v}$ under T^{-1}. Neither the fact that $T^{-1}\tilde{v}$ is definitely the original error $\tilde{u}$ nor the well-posedness of the original error system are direct consequences of our previous result.

5. Design procedure

We summarize the design procedure in the proposed framework. Explicit observer gains are also provided for a special class of systems.

5.1. General cases

For the system (1)–(3), suppose that the measurement can be modeled by (4) with a positive function h satisfying (5)–(6) for some γ. Then, the design procedure consists of three steps:

1. Set the parameter c based on the desired rate of convergence.

2. Solve the resulting kernel PDE (31)–(33) by some method.
3. Calculate the observer gains l and l_b for (7)–(9) by using (34) and (35), respectively.

The obtained gains ensure the exponential convergence of the estimation error with the decay rate $\pi^2 a/4 + c$ in terms of the L^2 norm. Although we introduce two transformations and the associated systems, neither is necessary in the actual design procedure.

5.2. *Explicit observer gains*

In the second step, we need to solve the kernel PDE (31)–(33) to compute the observer gains. Generally, this step requires numerical or symbolic computation. If the coefficient $\lambda(x)$ is a constant function, an explicit solution is available.

Let $\lambda(x) = \lambda_0 \in \mathbb{R}$ for all $x \in \bar{I}$. For a given initial value $h_0 > 0$, the solution of the ODE (5)-(6) can be written as

$$h(x) = h_0 \left(\cosh\left(\omega_\gamma x\right) - \frac{\alpha}{\omega_\gamma} \sinh\left(\omega_\gamma x\right) \right), \tag{40}$$

where $\omega_\gamma := \left((\gamma - \lambda_0)/a \right)^{1/2}$. If $\gamma - \lambda_0 < 0$, then ω_γ is a purely imaginary number. The lower bound of the possible γ is given by $\lambda_0 + a\omega^2$, where ω is the largest[4] real or purely imaginary root of the nonlinear equation

$$\cosh \omega = \frac{\alpha}{\omega} \sinh \omega.$$

For example, $\omega = \mathrm{i}\pi/2$ for $\alpha = 0$ and $\omega = 0$ for $\alpha = 1$. In general, ω^2 is greater than $-\pi^2$ and increases monotonically as α increases.

With the aid of the method explained in Smyshlyaev and Krstic (2010), we can obtain the solution to the kernel PDE (31)–(33) for λ_0 and h given by (40) as

$$\begin{aligned} p(x, y) = &-\bar{\lambda} x \frac{I_1(\phi(x, y))}{\phi(x, y)} - \left(\frac{h'(x)}{h(x)} + \alpha \right) I_0(\phi(x, y)) \\ &- \alpha \left(\frac{h'(x)}{h(x)} + \alpha \right) \int_x^y e^{\alpha(\xi - x)} I_0\left(\phi(\xi, y)\right) d\xi, \end{aligned}$$

where $\bar{\lambda} := (\lambda_0 + c)/a$, $\phi(x, y) = (\bar{\lambda}(y^2 - x^2))^{1/2}$, and I_k is the kth order modified Bessel function of the first kind. Then, from (34) and (35), the observer gains can be calculated as

$$\begin{aligned} l(x) = &\frac{a}{h(1)} \Bigg(\bar{\lambda}^2 \left(\alpha x - 1\right) \frac{I_2(\phi(x, 1))}{\phi(x, 1)^2} + \bar{\lambda}^3 x^2 \frac{I_3(\phi(x, 1))}{\phi(x, 1)^3} \\ &+ \left(\alpha^2 - \omega_\gamma^2\right) \Bigg(\bar{\lambda} \frac{I_1(\phi(x, 1))}{\phi(x, 1)} + \alpha e^{\alpha(1 - x)} \\ &+ \alpha \bar{\lambda} \int_x^1 e^{\alpha(\xi - x)} \frac{I_1(\phi(\xi, 1))}{\phi(\xi, 1)} d\xi \Bigg) \Bigg), \\ l_b = &\frac{1}{h(1)} \left(\frac{\bar{\lambda}}{2} + \alpha \right), \end{aligned}$$

where $h(1) = h_0 \left(\cosh \omega_\gamma - (\alpha/\omega_\gamma) \sinh \omega_\gamma \right)$.

[4]Here, the word *largest* means the squared value is maximum.

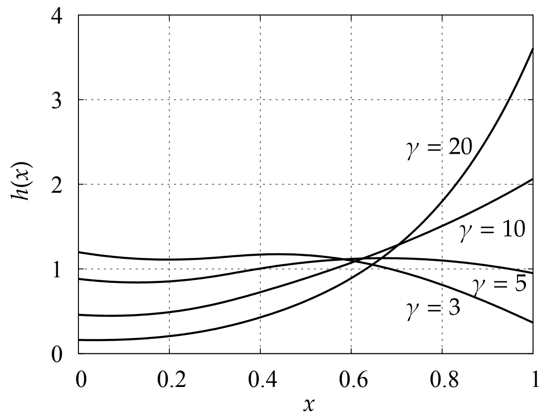

Figure 3: Weighting functions for some γ's.

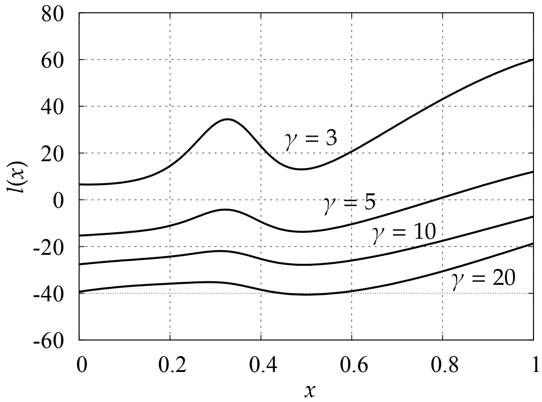

Figure 4: Distributed observer gains for some γ's.

These explicit expressions tell us that the weighting function h itself is involved with the gains $l(x)$ and l_b merely as a multiplier. In particular, only the value taken by h at the right end-point $x = 1$ is important. When γ approaches γ_0, $h(1)$ tends to 0. Accordingly, the gains drastically increase. We can also observe that the spatial shape of $l(x)$ is essentially determined by the three parameters $\bar{\lambda}$, α, and ω_γ. If $\omega_\gamma^2 = \alpha^2$ or $\alpha = 0$, the resultant interior gain $l(x)$ has a comparatively simple form.

6. Numerical simulation

We confirm the effectiveness of the proposed observer by numerical simulation. Let the system parameters be given by $a = 1$, $\alpha = 3/4$, and

$$\lambda(x) = \frac{9}{2} \tanh\left(20\left(x - \frac{1}{3}\right)\right) + \frac{7}{2}.$$

The system (1)–(3) with $U(t) \equiv 0$ is unstable under these conditions. To begin with, we find out weighting functions that are admissible in the proposed framework. The lower bound γ_0 is located between 0 and 1. The numerical solutions of the ODE (5)–(6) can be computed as shown in Fig. 3. The initial value $h(0)$ is determined so that the L^1 norm of the resultant solution is 1. This is a natural choice since h is a weighting function.

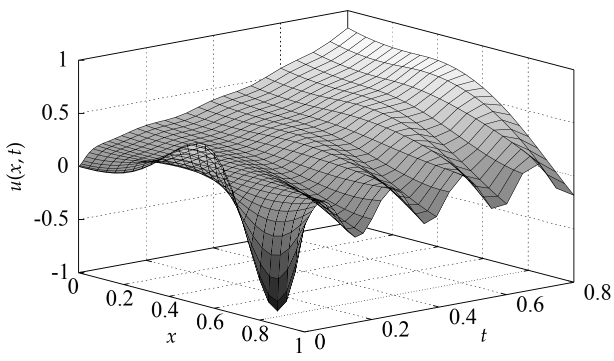

Figure 5: State response of system.

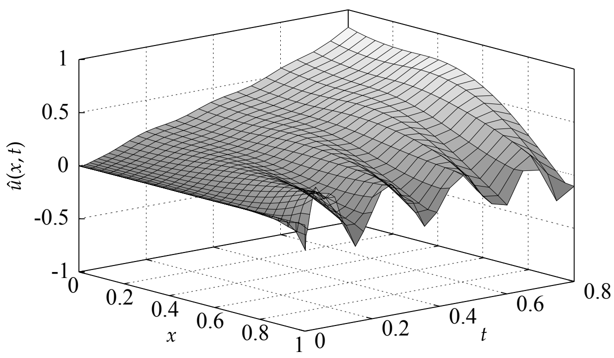

Figure 6: Estimated state $\hat{u}$.

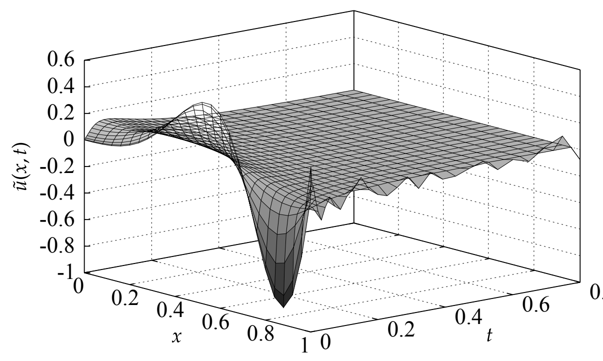

Figure 7: Estimation error $\tilde{u}$.

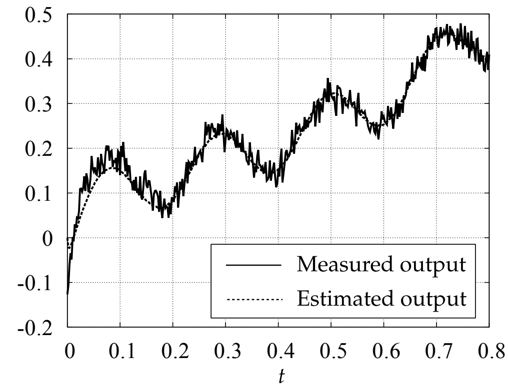

Figure 8: Measured output and estimated outputs.

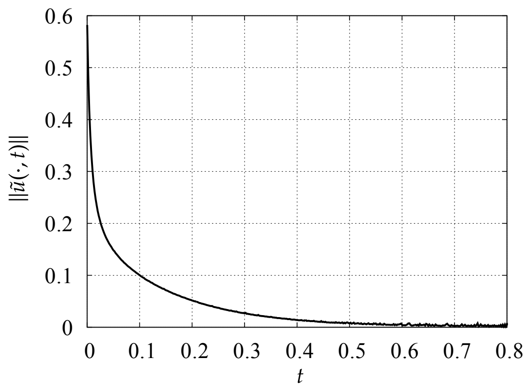

Figure 9: The L^2 norm of the error $\tilde{u}$.

If we set the design parameter $c = 4$, the corresponding observer gains are calculated as those shown in Fig. 4. As γ approaches γ_0, the value of $l(x)$ increases. This can be easily understood from the definition (34) because $l(x)$ contains $1/h(1)$ and $h(1)$ tends to 0 as γ goes to γ_0. On the other hand, the absolute value of l at each point in $\overline{I}$ also increases gradually when γ increases, even though $h(1)$ takes large values. It can be inferred that the terms in the parentheses in (34) grow more rapidly than $h(1)$. Here we present an intuitive understanding. The output contains much information about the state u around the right end-point $x = 1$ as γ increases. However, the value of u at the right end-point is determined by the boundary condition, and we assume that it is a known quantity. Hence, such an output is less informative from the viewpoint of state estimation, and more gain would be required.

We then perform the simulation. In order to imitate practical situations, we disturb the output with additive noise and discretize the the system PDE (1) and observer PDE (7) by different methods. More precisely, the fourth order explicit Runge-Kutta method and sixth order compact finite difference scheme (Lele, 1992) are applied to (1) for temporal and spatial discretization, respectively, while (7) is discretized through a simple second order method consisting of the midpoint method in time and the central difference in space. The latter simple scheme is preferable in the implementation stage because of its lower computational cost.

Fig. 5 shows the state response of the system (1)–(3) to the input $U(t) = (1/5)\sin(30t)$ under the initial condition $u(x, 0) = x\sin(2\pi x^2)$. The observer state $\hat{u}$ and error variable $\tilde{u}$ are plotted in Figs. 6 and 7, respectively. The initial estimate is set to $\hat{u}(x, 0) \equiv 0$. We can see that the error distribution immediately converges to 0 except at the right end-point $x = 1$. The behavior at the right end-point arises from the presence of observation noise. In the simulation, the observer generates the estimate $\hat{u}$ of u based on the disturbed output in Fig. 8. The effect of the noise on the internal state is mitigated by time integration. At the right end-point, however, the estimate $\hat{u}$ is directly affected by the noise through the output error feedback under the right boundary condition (9). Hence, we need to take care of the value of the boundary gain to avoid the resultant observer being sensitive to the noise. Nonetheless, Fig. 9 indicates that the L^2 norm of the estimation error $\tilde{u}$ still decays exponentially except for tiny perturbations due to the observation noise. Therefore, the results demonstrate the effectiveness of the proposed observer.

7. Conclusion

We have developed a design method of the observer for systems modeled by a one-dimensional PDE when the output is a weighted spatial average of the state over the spatial domain. We proposed a novel state transformation to exploit the backstepping method. This successfully results in a systematic design procedure that ensures a given performance regarding the convergence of the estimation error for a class of weighting functions. The proposed transformation has a discontinuous inverse. Hence, it is also interesting from a system theoretic point of view.

In future work, we will extend the class of weighting functions. In particular, functions with small support are important in practice.

Appendix A. Weighting functions and observability

We begin by showing the existence of a positive solution to (5)–(6).

Proposition 1. *Let $a > 0$, $\lambda \in C(\bar{I})$, and $\alpha \in \mathbb{R}$. Then, there exists a constant $\gamma_0 \in \mathbb{R}$ such that, for any $h_0 > 0$, a solution to (5)-(6) with $h(0) = h_0$ is a positive function on $\bar{I}$ whenever $\gamma > \gamma_0$.*

Proof. By adding the homogeneous Dirichlet boundary condition at the right end-point $x = 1$ to (5)–(6), we define the following boundary value problem:

$$af''(x) + \lambda(x)f(x) = \gamma f(x), \quad x \in I \tag{A.1}$$

$$f'(0) + \alpha f(0) = 0, \tag{A.2}$$

$$f(1) = 0. \tag{A.3}$$

There is a major difference between (5)–(6) and (A.1)–(A.3). The original problem (5)–(6) has a non-trivial solution for any $\gamma \in \mathbb{R}$. However, the problem (A.1)–(A.3) has a non-trivial solution if and only if γ is an eigenvalue. According to the Sturm-Liouville theory (Zettl, 2005; Coddington and Levinson, 1955), there exist countably many eigenvalues $\{\gamma_i\}_{i=0}^{\infty} \subset \mathbb{R}$ such that $\gamma_0 < +\infty$ and $\gamma_i > \gamma_{i+1}$ for any $i \in \mathbb{N} \cup \{0\}$. Moreover, eigenfunctions associated with the largest eigenvalue γ_0 have no zero in I.

Let f be an eigenfunction associated with γ_0 and let h be a solution of (5)–(6) with $h(0) = h_0$ for some $\gamma \in \mathbb{R}$. Without loss of generality, we can let $f(0) = h_0$. In this case, we have $f(x) > 0$ for all $x \in \bar{I} \setminus \{1\}$ and $f'(0) = -\alpha h_0 = h'(0)$. We prove the positivity of h by showing that $h(x) > f(x)$ for all $x \in \bar{I} \setminus \{0\}$ whenever $\gamma > \gamma_0$.

Since γ_0 is the largest eigenvalue, the condition $\gamma > \gamma_0$ implies $h(1) \neq 0 = f(1)$. Hence, it suffices to show the inequality for $x \in I$. Suppose, contrary to our claim, that $h(x) \leq f(x)$ for some $x \in I$. If $\gamma > \gamma_0$, we obtain

$$h''(0) = \frac{\gamma - \lambda(0)}{a} h(0) > \frac{\gamma_0 - \lambda(0)}{a} f(0) = f''(0).$$

Thus, there exists $x_0 \in I$ such that

$$h(x) > f(x), \quad h'(x) > f'(x) \quad \text{for all } x \in (0, x_0)$$

and $h(x_0) = f(x_0)$. We use an argument similar to the one used in the proof of the Sturm comparison theorem (Coddington and Levinson, 1955) to obtain a contradiction. Simple computation gives

$$\begin{aligned}\frac{d}{dx}\left(h'(x)f(x) - h(x)g'(x)\right) &= h''(x)f(x) - h(x)f''(x) \\ &= \frac{\gamma - \gamma_0}{a} h(x)f(x).\end{aligned}$$

Integrating the above from 0 to x_0 yields

$$\big(h'(x_0) - f'(x_0)\big)h(x_0) = \frac{\gamma - \gamma_0}{a} \int_0^{x_0} h(x)f(x)dx > 0.$$

Since $h(x_0) > 0$, we have $h'(x_0) > f'(x_0)$. On the other hand, it follows from the inequality $h(x) > f(x)$ that

$$\frac{h(x_0) - h(x)}{x_0 - x} < \frac{f(x_0) - f(x)}{x_0 - x} \quad \text{for all } x \in (0, x_0).$$

Letting $x \to x_0-$ gives $h'(x_0) \leq f'(x_0)$, which is impossible. Therefore, $h(x) > f(x)$ for all $x \in I$, which is the desired conclusion. □

Non-trivial solutions to the boundary value problem (A.1)–(A.3) are nothing but eigenfunctions of the operator characterizing the original system (1)–(3) with $U(t) \equiv 0$. Therefore, if there exists a non-trivial solution f_i to (A.1)–(A.3) for some eigenvalue γ_i that is orthogonal to the weighting function h in the sense of the $L^2(I)$ inner product, the observability is lost. We can show that this is not the case for a positive solution of (5)–(6).

Assume that an eigenfunction f_i associated with an eigenvalue γ_i is orthogonal to h, that is,

$$\int_0^1 f_i(\xi)h(\xi)d\xi = 0.$$

Then, it follows from (5)–(6) and (A.1)–(A.3) that

$$\begin{aligned}0 &= \gamma \int_0^1 f_i(\xi)h(\xi)d\xi \\ &= \int_0^1 f_i(\xi)\left(ah''(\xi) + \lambda(\xi)h(\xi)\right)d\xi \\ &= -af_i'(1)h(1) + \int_0^1 \left(af_i''(\xi) + \lambda(\xi)f_i(\xi)\right)h(\xi)d\xi \\ &= -af_i'(1)h(1) + \gamma_i \int_0^1 f_i(\xi)h(\xi)d\xi = -af_i'(1)h(1).\end{aligned}$$

Since $ah(1) > 0$, f_i must satisfy $f_i'(1) = 0$ in addition to (A.1)–(A.3). This implies that $f_i(x) \equiv 0$, which contradicts the fact that f_i is an eigenfunction. Therefore, h is not orthogonal to any eigenfunctions of the system operator.

Appendix B. Proofs of lemmas

Proof (Lemma 1). It is evident that $Tf \in V$ for any $f \in L^2(I)$ whenever $h \in C^1(\bar{I})$ and $h(x) > 0$ in $\bar{I}$. Hence, we show the latter assertion. We check at once that T is continuous with respect to the L^2 norm. Namely, there exists a constant $C' > 0$ such that

$$\|Tf\|_{L^2(I)} \leq C'\|f\|_{L^2(I)}$$

for all $f \in L^2(I)$. Let $g := Tf$ for a fixed $f \in L^2(I)$. Since the derivative of g' satisfies

$$g'(x) = -\frac{h'(x)}{h(x)}g(x) + f(x),$$

we have the following estimate:

$$\begin{aligned}\|g\|_V = \|g'\|_{L^2(I)} &\leq \left\|\frac{h'}{h}Tf\right\|_{L^2(I)} + \|f\|_{L^2(I)}\\ &\leq \left(C' \max_{x\in\bar{I}}\left|\frac{h'(x)}{h(x)}\right| + 1\right)\|f\|_{L^2(I)},\end{aligned}$$

which completes the proof. □

Proof (Lemma 2). Let $f_1, f_2 \in L^2(I)$ be such that $Tf_1 = Tf_2$. This is equivalent to stating that, for almost all $x \in I$,

$$\int_0^x h(\xi)\left(f_1(\xi) - f_2(\xi)\right)d\xi = 0.$$

As an immediate consequence, we have $h(x)(f_1(x) - f_2(x)) = 0$ for almost all $x \in I$. Then, it follows that $f_1(x) = f_2(x)$ for almost all $x \in I$ because $h(x) > 0$ in $\bar{I}$. Thus, T is injective. Recall that Lemma 1 states that the range of T is contained in V. Hence, we only need to show that, for any $g \in V$, there exists $f \in L^2(I)$ such that $Tf = g$. Take arbitrary $g \in V$ and define $f = (hg)'/h = g' + (h'/h)g$. It is obvious that $Tf = g$. Since $g \in V \subset H^1(I)$, we see that $g' \in L^2(I)$. This together with the fact that $h'/h \in C(\bar{I})$ ensures that $f \in L^2(I)$, which is our claim. Note that we have also proved a formula of the inverse transformation.

Our next task is to estimate the L^2 norm of $f = T^{-1}g$. Since $f = T^{-1}f = g' + (h'/h)g$,

$$\begin{aligned}\|f\|_{L^2(I)} &\leq \|g'\|_{L^2(I)} + \left\|\frac{h'}{h}g\right\|_{L^2(I)}\\ &\leq \left(1 + \frac{2}{\pi}\max_{x\in\bar{I}}\left|\frac{h'(x)}{h(x)}\right|\right)\|g\|_V.\end{aligned}$$

This completes the proof. □

Proof (Lemma 3). The proof is divided into three steps. We first prove the existence and the uniqueness of $\tilde{v}$ by a standard argument used in the backstepping approach. Then, the regularity of $\tilde{v}$ at each instant of time is shown.

Let $\tilde{w}_0$ be the image of $\tilde{v}_0$ under the backstepping transformation (25). Since $\tilde{v}_0 \in V$ and the integral kernel p satisfies (32), we have $\tilde{w}_0 \in V$. Hence, owing to the fact that A_w is a self-adjoint maximal dissipative operator, we can conclude that there exists a unique solution $\tilde{w} \in C([0,+\infty); V) \cap C^1((0,+\infty); V) \cap C((0,+\infty); D(A_w))$ that satisfies $\tilde{w}(\cdot,0) = \tilde{w}_0$ to the target system (26)–(28) (Brezis, 2010). We can now define the inverse image $\tilde{v}$ of $\tilde{w}$ under the backstepping transformation. Then, it is a solution to (15)–(17) with $\tilde{v}(\cdot,0) = \tilde{v}_0$. The continuity of the inverse backstepping transformation with respect to the H^1 norm guarantees $\tilde{v} \in C([0,+\infty); V) \cap C^1((0,+\infty); V)$. The uniqueness follows from that of $\tilde{w}$.

We next show the regularity of $\tilde{v}$. Rearranging (15) yields

$$\tilde{v}_{xx}(x,t) = \frac{1}{a}\left(\tilde{v}_t(x,t) - \mu(x)\tilde{v}(x,t) - m(x)\tilde{v}(1,t)\right),$$

for any $t > 0$. It should be noted that $\tilde{v}_t(t) \in V \subset H^1(I)$ if $t > 0$, because we have shown that $\tilde{v} \in C^1((0,+\infty); V)$. In view of the fact that $\mu, m \in C^1(\bar{I})$, the second partial derivative $\tilde{v}_{xx}(\cdot,t)$ must belong to $H^1(I)$. Therefore, we conclude that $\tilde{v}(\cdot,t)$ is in at least $H^3(I)$ for any $t > 0$. This completes the proof. □

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