ON THE COMMUTATIVE FAMILY OF SUBNORMAL OPERATORS

By

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Introduction. Halmos has given in [3] the definition of a subnormal operator and the characteristic property of it. A bounded operator $A$ defined on a Hilbert space $\mathcal{H}$ is said to be subnormal if there exist a Hilbert space $\mathcal{H}'$ containing $\mathcal{H}$ and a bounded normal operator $N$ on $\mathcal{H}'$ such that $Ax=Nx$ for every $x$ in $\mathcal{H}$. Recently in [1] Bram has made Halmos' characterization simpler ([1], Theorem 1) and given another characteristic property ([1], Theorem 2) and some results about subnormal operators (for example, [1], Theorems 4, 7, 8, 9).

In this paper first we shall study the problem under what conditions it is possible to extend the commutative family of subnormal operators acting on a Hilbert space $\mathcal{H}$ to the commutative family of normal operators on a Hilbert space $\mathcal{H}'$ containing $\mathcal{H}$. Theorem 1 answers to this question. Then we shall give a generalization of Bram's theorems (for example Theorem 6 and Theorem 7) and another simpler proof of Bram's theorem about the spectrum of subnormal operators (Theorem 8). Theorem 3 is a generalization of Cooper's result in [2] (cf. [9], p. 393). Theorem 5 gives a new characterization of subnormal operators.

1. An abelian semi-group of subnormal operators. Throughout the paper, a Hilbert space is a vector space over the complex numbers, an operator is a bounded linear transformation unless denoted explicitly. For an operator $A$ we denote by $A^*$ an adjoint operator of $A$.

Lemma 1. Let $A_i (l=1,2,\cdots,n)$ be $n$ commutative operators on a Hilbert space $\mathcal{H}$. If for every non-negative integer $M$ and element $x_{i_1,i_2,\cdots,i_n}$ in $\mathcal{H}$ ($0\leq i_l \leq M$, $l=1,2,\cdots,n$)

\[ \sum_{i_1,i_2,\cdots,i_n}^{M} (A_{i_1}x_{j_1,i_2,\cdots,i_n}A_{i_2}x_{j_2,i_3,\cdots,i_n}\cdots A_{i_n}x_{i_1,i_2,\cdots,i_n}) \geq 0, \]

then we have the inequality such that for every $M$, $x_{i_1,i_2,\cdots,i_n}$ in $\mathcal{H}$ ($0\leq i_l \leq M$, $l=1,2,\cdots,n$) and non-negative integer $\nu_l (l=1,2,\cdots,n)$
\[
\sum_{\ell_i, j_i \geq 0}^{M} (A_\ell^{\ell_i+\nu_i} A_{j_i}^{\nu_j} \cdots A_n^{\nu_n} x_{j_1, \ldots, j_n})
\]

\[
\leq \| A_\ell \|^{2\nu_i} \| A_{j_i} \|^{2\nu_j} \cdots \| A_n \|^{2\nu_n} \sum_{\ell_i, j_i \geq 0}^{M} (A_\ell^{\ell_i+\nu_i} A_{j_i}^{\nu_j} x_{j_1, \ldots, j_n})
\]

**Proof.** Essentially the proof is the same as that of [1] Theorem 1. Heinz's theorem ([5]) is essential.

Let $\mathfrak{H}_{i_1, i_2, \ldots, i_n}$ be spaces isomorphic to $\mathfrak{H}$, $\mathfrak{R}$ be the direct sum of $\mathfrak{H}_{i_1, i_2, \ldots, i_n}$, that is, $\mathfrak{R} = \sum \oplus \mathfrak{H}_{i_1, i_2, \ldots, i_n}$. We denote the element of $\mathfrak{R}$ by $\bar{x} = \{x_{i_1, i_2, \ldots, i_n}\}$, where $x_{i_1, i_2, \ldots, i_n}$ is $i_1, i_2, \ldots, i_n$-component of $\bar{x}$. For a positive number $\varepsilon > 0$ we put $B_l = (\|A_l\| + \varepsilon)^{-1} A_l$ ($l = 1, 2, \ldots, n$), then $\|B_l\| < 1$ ($l = 1, 2, \ldots, n$).

We can define a linear transformation $S$ on $\mathfrak{R}$ such that

\[
S \bar{x} = \bar{y} = \{y_{i_1, i_2, \ldots, i_n}\}, \quad \bar{x} = \{x_{i_1, i_2, \ldots, i_n}\},
\]

As $\|B_l\| < 1$ ($l = 1, 2, \ldots, n$), the right hand of (1.3) is convergent and $S$ is a bounded operator. Because

\[
\|S \bar{x}\|^2 = \sum_{l=1,2,\ldots,n} \| \sum_{j \geq 0} B_{l}^{j} x_{j_1, \ldots, j_n} \|^2
\]

\[
\leq \sum_{l=1,2,\ldots,n} \left\{ \sum_{j \geq 0} \| B_{l}^{j} \|^2 \right\}\cdot \sum_{j \geq 0} \| x_{j_1, \ldots, j_n} \|^2
\]

\[
= (1 - \| B_{l} \|^2)^{-2} \cdots (1 - \| B_{n} \|^2)^{-2} \| \bar{x} \|^2.
\]

So we have

\[
\|S \bar{x}\| \leq (1 - \| B_{l} \|^2)^{-1} \cdots (1 - \| B_{n} \|^2)^{-1} \| \bar{x} \| \quad (\bar{x} \in \mathfrak{R}).
\]

On the other hand for $\bar{x} = \{x_{i_1, i_2, \ldots, i_n}\}$ whose components equal to zero except for finite number of $x_{i_1, i_2, \ldots, i_n}$ we have by assumption (1.1)

\[
(S \bar{x}, \bar{x}) = \sum_{l=1,2,\ldots,n}^{M} (B_l^{j_1} \cdots B_n^{j_n} x_{j_1, \ldots, j_n}, B_l^{j_1} \cdots B_n^{j_n} x_{j_1, \ldots, j_n})
\]

\[
= \sum_{l=1,2,\ldots,n}^{M} (A_l^{j_1} \cdots A_n^{j_n} z_{j_1, j_2, \ldots, j_n}, A_l^{j_1} \cdots A_n^{j_n} z_{j_1, j_2, \ldots, j_n}) \geq 0,
\]

where $z_{i_1, i_2, \ldots, i_n} = (\| A_l \| + \varepsilon)^{-1} \cdots (\| A_n \| + \varepsilon)^{-1} x_{i_1, i_2, \ldots, i_n}$.
The whale of such $\overline{x}$ is dense in $\Re$ evidently. Therefore $S$ is a bounded positive symmetric operator on $\Re$.

In the same way we define a linear transformation $T$ on $\Re$ such that

$$
T\overline{x} = \overline{\overline{z}} = \{z_{i_{1},i_{2},\ldots,i_{n}}\}, \quad \overline{x} = \{x_{i_{1},i_{2},\ldots,i_{n}}\},
$$

$$
z_{i_{1},i_{2},\ldots,i_{n}} = \sum_{j_{l} \geq 0} B_{n}^{*j_{n}}\cdots B_{1}^{*j_{1}}x_{j_{1},j_{2},\ldots,j_{n}}.
$$

(1.6)

It is proved like $S$ that $T$ is a bounded positive symmetric operator on $\Re$.

Next we see

$$
\|T\overline{x}\| \leq \|S\overline{x}\| \quad (\overline{x} \in \Re).
$$

Because

$$
\|T\overline{x}\|^{2} \leq \sum_{l=1,2,\ldots,n} \|\sum_{j_{l} \geq 0} B_{n}^{*j_{n}}\cdots B_{1}^{*j_{1}}x_{j_{1},j_{2},\ldots,j_{n}}\|^{2}
$$

$$
\leq \sum_{l=1,2,\ldots,n} \|\sum_{j_{l} \geq 0} B_{n}^{*j_{n}}\cdots B_{1}^{*j_{1}}x_{j_{1},j_{2},\ldots,j_{n}}\|^{2},
$$

as $B_{l} (l=1,2,\ldots,n)$ are commutative we have

$$
\leq \sum_{l=1,2,\ldots,n} \|B_{n}^{*j_{n}}\cdots B_{1}^{*j_{1}}\|^{2} \cdot \|S\overline{x}\|^{2} \leq \|S\overline{x}\|^{2}.
$$

Owing to Heinrz's theorem (cf. [5] or Kato [7]) we obtain from (1.7)

$$
(T\overline{x}, \overline{x}) \leq (S\overline{x}, \overline{x}) \quad (\overline{x} \in \Re).
$$

Hence

$$
\sum_{l=1,2,\ldots,n} \|B_{n}^{*j_{n}}\cdots B_{1}^{*j_{1}}x_{j_{1},\ldots,j_{n}}\|^{2}
$$

$$
\leq \sum_{l=1,2,\ldots,n} \|B_{n}^{*j_{n}}\cdots B_{1}^{*j_{1}}x_{j_{1},\ldots,j_{n}}\|^{2},
$$

(1.8)

Therefore we have

$$
\sum_{l=1,2,\ldots,n} \|A_{n}^{*j_{n}}\cdots A_{1}^{*j_{1}}x_{j_{1},\ldots,j_{n}}\|^{2}
$$

$$
\leq \sum_{l=1,2,\ldots,n} \|A_{n}^{*j_{n}}\cdots A_{1}^{*j_{1}}x_{j_{1},\ldots,j_{n}}\|^{2},
$$

(1.9)

$$
\leq (\|A_{n}\| + \varepsilon)^{\nu_{n}} \cdots (\|A_{1}\| + \varepsilon)^{\nu_{1}} \sum_{l=1,2,\ldots,n} \|A_{n}^{*j_{n}}\cdots A_{1}^{*j_{1}}x_{j_{1},\ldots,j_{n}}\|^{2},
$$
Rememering $\varepsilon$ was an arbitrary positive number we obtain the inequality (1.2) from (1.9).

Let $\Gamma$ be an abelian semi-group having at least one zero element 0. The function $A_r (r \in \Gamma)$ from $\Gamma$ into the algebra of bounded operators on a Hilbert space $\mathfrak{F}$ is called an operator representation of $\Gamma$ if

\[
\begin{cases}
A_r A_r = A_{r_1 + r_2} (r_1, r_2 \in \Gamma) \\
A_0 = I \quad \text{(an identity operator on $\mathfrak{F}$)}
\end{cases}
\]

(1.10)

Through the paper such an operator representation of $\Gamma$ will be denoted by $A_r (r \in \Gamma, \mathfrak{F})$.

**Definition 1.** An operator representation $A_r (r \in \Gamma, \mathfrak{F})$ will be called positive definite if

\[
\sum_{i,j} (A_{i}x_j, A_{r_j}x_i) \geq 0
\]

for every finite number of $x_i$ in $\mathfrak{F}$ and $r_i$ in $\Gamma$.

From Lemma 1 following lemma is proved.

**Lemma 2.** Let an operator representation $A_r (r \in \Gamma, \mathfrak{F})$ be positive definite. Then we have for every finite number of $x_i$ in $\mathfrak{F}$, $r_i$ in $\Gamma$ and an arbitrary $\rho$ in $\Gamma$

\[
\sum_{i,j} \langle A_{i}x_j, A_{r_j}x_i \rangle \leq \|A_{\rho}\|^2 \sum_{i,j} \langle A_{i}x_j, A_{r_j}x_i \rangle.
\]

(1.12)

**Proof.** Assuming that $i$ and $j$ run from 1 to $n-1$ we put $A_i = A_{r_i} (1 \leq i \leq n-1)$ and $A_n = A_\rho$. By the fact that $A_r (r \in \Gamma, \mathfrak{F})$ is positive definite we can see easily $A_l (l=1,2,\ldots,n)$ satisfy the assumption of Lemma 1 namely the inequality (1.1). Therefore putting in (1.2)

\[
\begin{cases}
x_{i_1, i_2, \ldots, i_n} = x_l & \text{if } i_1 = 1, \ i_m = 0 \ (m \neq l), \ l=1,2,\ldots,n-1, \\
x_{i_1, i_2, \ldots, i_n} = 0 & \text{besides,} \\
\nu_l = 0 & \text{for } l=1,2,\ldots,n-1, \ \nu_n = 1,
\end{cases}
\]

we have (1.12).

**Definition 2.** For two operator representations of $\Gamma A_r (r \in \Gamma, \mathfrak{F})$ and $B_r (r \in \Gamma, \mathfrak{F})$ it is defined that $B_r (r \in \Gamma, \mathfrak{F})$ is an extension of $A_r (r \in \Gamma, \mathfrak{F})$ if following conditions are satisfied

\[
\mathfrak{F} \supset \mathfrak{S} \quad \text{and} \quad B_r x = A_r x (x \in \mathfrak{S}) \quad \text{for all } r \in \Gamma.
\]

(1.13)

If all $B_r$ are normal operators on $\mathfrak{S}$ we call $B_r (r \in \Gamma, \mathfrak{S})$ the normal extension of $A_r (r \in \Gamma, \mathfrak{F})$.

We obtain the following theorem which is a generalization of
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Halmos' theorem ([3], Theorem 3).

Theorem 1. An operator representation $A_{\gamma}(\gamma \in \Gamma, \mathfrak{H})$ of an abelian semigroup $\Gamma$ has a normal extension $N_{\tau}(\tau \in \Gamma, \mathfrak{H})$ if and only if $A_{\gamma}(\gamma \in \Gamma, \mathfrak{H})$ is positive definite.

Proof. Necessity. For every finite number of $x_{i}$ in $\mathfrak{H}$ and $\gamma_{i}$ in $\Gamma$ we have

$$\sum_{i,j}(A_{\gamma_{i}}x_{i}, A_{\gamma_{j}}x_{i}) = \sum_{i,j}(N_{\gamma_{i}}x_{i}, N_{\gamma_{i}}x_{i}) = \sum_{i,j}(N_{\gamma_{i}}^{*}x_{i}, N_{\gamma_{i}}^{*}x_{i}) = \|\sum_{i}N_{\gamma_{i}}^{*}x_{i}\|^{2} \geq 0.$$  

Sufficiency. The construction of $\mathfrak{K}$ and $N_{\tau}(\tau \in \Gamma)$ are obtained by generalizing Halmos' method ([3]) to the case of semi-groups.

Putting $\mathfrak{K}$ the Cartesian product of $\mathfrak{K}_{\gamma}(\gamma \in \Gamma)$, namely $\mathfrak{K} = \prod_{\tau \in \Gamma} \mathfrak{K}_{\gamma}$, here every $\mathfrak{K}_{\gamma}$ is isomorphic to $\mathfrak{K}$. We shall denote the element of $\mathfrak{K}$ by $\bar{x} = \{x_{\gamma}\}$ whose $\tau$-component is $x_{\tau}$. Let $\mathfrak{D} = \{\bar{x}; \bar{x} = \{x_{\gamma}\}, x_{\gamma} \neq 0 \text{ at most finite number of } \gamma\}$, then $\mathfrak{D}$ is a linear manifold in $\mathfrak{K}$. We shall introduce onto $\mathfrak{D}$ a bilinear functional such that

$$(1.15) \quad \langle \bar{x}, \bar{y} \rangle = \sum_{\tau, \tau' \in \Gamma} (A_{\tau}x_{\tau'}, A_{\tau'}y_{\tau}) \quad (\bar{x}, \bar{y} \in \mathfrak{D}),$$

for brevity we identify all $\mathfrak{K}_{\gamma}$ with $\mathfrak{K}$. Since $A_{\gamma}(\gamma \in \Gamma, \mathfrak{H})$ is positive definite, $\langle \bar{x}, \bar{y} \rangle$ is a positive symmetric bilinear functional. Putting $\mathfrak{Z} = \{\bar{x}; \langle \bar{x}, \bar{x} \rangle = 0\}$, then naturally the quotient space $\mathfrak{D}/\mathfrak{Z}$ is an inner product space. The completion $\mathfrak{K}$ of $\mathfrak{D}/\mathfrak{Z}$ by this inner product is a Hilbert space. Evidently the correspondence $\mathfrak{K} \ni x \rightarrow \bar{x} = \{x_{\gamma}\}$, where $x_{\gamma} = x$ and $x_{\tau} = 0 (\tau \neq 0)$, is an isomorphism from $\mathfrak{K}$ into $\mathfrak{D}$. Thus $\mathfrak{K}$ is imbeded into $\mathfrak{K}$.

Next we shall define linear transformations $N_{\rho}(\rho \in \Gamma)$ on $\mathfrak{D}$ such that

$$N_{\rho} \bar{x} = \bar{y} = \{y_{\gamma}\}, \quad \bar{x} = \{x_{\gamma}\},$$

$$y_{\tau} = A_{\tau}x_{\tau}, \quad (x \in \Gamma).$$

Then we have from Lemma 2

$$\langle N_{\rho} \bar{x}, N_{\rho} \bar{x} \rangle = \sum_{\tau, \tau' \in \Gamma} (A_{\tau + \rho}x_{\tau'}, A_{\tau' + \rho}x_{\tau'}) \leq \|A_{\rho}\|^{2} \sum_{\tau, \tau' \in \Gamma} (A_{\tau}x_{\tau'}, A_{\tau'}x_{\tau}) = \|A_{\rho}\|^{2} \langle \bar{x}, \bar{x} \rangle.$$ 

Therefore $N_{\rho}$ is regarded as a bounded operator on $\mathfrak{K}$. We shall denote this operator on $\mathfrak{K}$ by the same notation $N_{\rho}$.

We shall show $N_{\rho}$ is a normal operator on $\mathfrak{K}$. For every $\rho, \gamma \in \Gamma$, putting $\Gamma_{\gamma - \rho} = \{\delta; \delta + \rho = \gamma\}$, and we introduce linear transformations
$L_\rho (\rho \in \Gamma)$ on $\mathfrak{D}$ such that

\begin{equation}
\begin{aligned}
L_\rho \bar{x} = \bar{z} = \{z_\gamma, x_\gamma \}, \\
z_\gamma = \sum_{\delta \in \Gamma \setminus \gamma} x_\delta \text{ or } = 0 \text{ if } \Gamma \setminus \gamma = \phi,
\end{aligned}
\end{equation}

generally $\Gamma \setminus \gamma$ is an infinite set, but for $\bar{x}$, $x_\delta = 0$ except for finite number of $\delta$, so $\sum x_\delta$ has a meaning. Thus

\begin{equation}
\begin{aligned}
\langle L_\rho \bar{x}, L_\rho \bar{z} \rangle &= \sum_{\gamma, \gamma' \in \Gamma} \langle A_{\gamma} \sum_{\delta \in \Gamma \setminus \gamma} x_\delta, A_{\gamma'} \sum_{\delta \in \Gamma \setminus \gamma} x_\delta \rangle \\
&= \sum_{\delta, \delta' \in \Gamma} \langle A_{\delta + \rho} x_{\delta'}, A_{\delta' + \rho} x_{\delta'} \rangle = \langle N_\rho \bar{x}, N_\rho \bar{z} \rangle \leq ||A_\rho||^2 \langle \bar{x}, \bar{z} \rangle.
\end{aligned}
\end{equation}

Therefore $L_\rho$ defines a bounded operator on $\mathfrak{D}$, we shall denote that operator by the same notation $L_\rho$. Then likewise we have

\begin{equation}
\langle L_\rho \bar{x}, \bar{y} \rangle = \langle \bar{x}, N_\rho \bar{y} \rangle \quad (\bar{x}, \bar{y} \in \mathfrak{D}),
\end{equation}

therefore $L_\rho^* = N_\rho (\rho \in \Gamma)$ on $\mathfrak{D}$. From (1.19) and (1.20) $N_\rho (\rho \in \Gamma)$ are normal operators on $\mathfrak{D}$. And evidently $N_\rho = A_\rho$ on $\mathfrak{D}$. Furthermore by (1.16) $N_0 = I$ and $N_{\gamma_1} N_{\gamma_2} = N_{\gamma_1 \gamma_2} (\gamma_1, \gamma_2 \in \Gamma)$. The proof is complete.

Definition 3. Let $N_\gamma (\gamma \in \Gamma, \mathfrak{D})$ be a normal extension of $A_\gamma (\gamma \in \Gamma, \mathfrak{D})$. If for any subspace $\mathfrak{G}_\gamma$ such that $\mathfrak{G} \supset \mathfrak{G}_\rho \supset \mathfrak{D}$ and every $N_\gamma$, is reduced by $\mathfrak{G}_\gamma$ we have $\mathfrak{G}_\gamma = \mathfrak{D}$, then $N_\gamma (\gamma \in \Gamma, \mathfrak{D})$ is called a minimal normal extension of $A_\gamma (\gamma \in \Gamma, \mathfrak{D})$.

Putting $\mathcal{Q} = \{ \sum N_{\gamma_i}^* x_i ;$ for every finite number of $x_i$ in $\mathcal{Q}$ and $\gamma_i \in \Gamma \}$, then evidently the closure of $\mathcal{Q}$ in $\mathfrak{D}$ is a subspace containing $\mathfrak{D}$ and invariant under every $N_\gamma$ and $N_{\gamma_i}$. Therefore the necessary and sufficient condition that $N_{\gamma_i} (\gamma \in \Gamma, \mathfrak{D})$ be a minimal normal extension of $A_\gamma (\gamma \in \Gamma, \mathfrak{D})$ is that a linear manifold $\mathcal{Q}$ be dense in $\mathfrak{D}$. It is noted that the normal extension $N_{\gamma_i} (\gamma_i \in \Gamma, \mathfrak{D})$ which was obtained in Theorem 1 is a minimal normal extension of $A_\gamma (\gamma \in \Gamma, \mathfrak{D})$.

Theorem 2. A minimal normal extension $N_{\gamma_i} (\gamma_i \in \Gamma, \mathfrak{D})$ of $A_\gamma (\gamma \in \Gamma, \mathfrak{D})$ is unique except for unitary isomorphism and $||N_{\gamma_i}||_{\mathcal{Q}} = ||A_{\gamma_i}||_{\mathcal{Q}} (\gamma \in \Gamma)$, where $||N_{\gamma_i}||_{\mathcal{Q}}$ and $||A_{\gamma_i}||_{\mathcal{Q}}$ are respectively the operator norms on $\mathfrak{D}$ and $\mathfrak{D}$.

Proof. Let $N_{\gamma_i} (\gamma_i \in \Gamma, \mathfrak{D}_1)$ and $M_\gamma (\gamma \in \Gamma, \mathfrak{D}_2)$ be two minimal normal extensions of $A_\gamma (\gamma \in \Gamma, \mathfrak{D})$ and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be respectively linear manifolds defined above (cf. after Definition 3). Then we have

\begin{align*}
|| \sum_{i} N_{\gamma_i}^* x_i ||_{\mathfrak{D}}^2 &= \sum_{i,j} (N_{\gamma_i} x_i, N_{\gamma_j} x_j) = \sum_{i,j} (A_{\gamma_i} x_i, A_{\gamma_j} x_j) \\
&= \sum_{i,j} (M_{\gamma_i} x_i, M_{\gamma_j} x_j) = || \sum_{i} M_{\gamma_i}^* x_i ||_{\mathfrak{D}}^2.
\end{align*}
From (1.17) we have \( \|N_\rho\|_\mathfrak{H} \leq \|A_\rho\|_\mathfrak{H} \), on the other hand from \( \mathfrak{D} \subset \mathfrak{R} \) we have \( \|A_\rho\|_\mathfrak{H} \leq \|N_\rho\|_\mathfrak{H} \), therefore we obtain \( \|N_\rho\|_\mathfrak{H} = \|A_\rho\|_\mathfrak{H} \) \((\rho \in \Gamma)\).

Remark. The fact \( \|N_\tau\|_\mathfrak{H} = \|A_\tau\|_\mathfrak{H} \) \((\tau \in \Gamma)\) is a generalization of Bram [1] Lemma 2, but remarked that Halmos' theorem about the spectrum of subnormal operators is not necessary.

2. A commutative family of isometric operators. In this section the partially isometric operator \( V \) such that \( V^*V=I \) will be called isometric simply. By the application of Theorem 1 we can show the following Theorem 3. This is a generalization of Cooper's result ([2] or cf. [9] p. 393) about the continuous one parameter semi-group \( V_t(t \geq 0) \) consisting of isometric operators. In our proof any assumption about the parameter is not necessary.

**Theorem 3.** Let \( V_\tau (\tau \in \Gamma, \mathfrak{D}) \) be an operator representation consisting of isometric operators. Then it can be extended to an unitary operator representation \( U_\tau (\tau \in \Gamma, \mathfrak{D}) \).

Let \( \mathfrak{B} = \{V\} \) be a commutative family of isometric operators on \( \mathfrak{D} \). As the semi-group generated by \( \mathfrak{B} \) consists of isometric operators, from Theorem 3 we can extend \( \mathfrak{B} \) to a commutative family \( \{U = U\} \) of unitary operators.

Before the proof we shall show the following Lemmas.

**Lemma 3.** Let \( A_\tau (\tau \in \Gamma, \mathfrak{D}) \) be positive definite, \( N_\tau (\tau \in \Gamma, \mathfrak{R}) \) be a minimal extension of \( A_\tau (\tau \in \Gamma, \mathfrak{D}) \) and \( B \) be a bounded operator on \( \mathfrak{D} \).

a) The necessary and sufficient conditions that \( B \) can be extended to an operator \( L \) on \( \mathfrak{R} \) being commutative with all \( N_\tau (\tau \in \Gamma) \) is that

\[
(i) \quad BA_\tau = A_\tau B \quad (\tau \in \Gamma)
\]

and some positive number \( C > 0 \) exist such that

\[
(ii) \quad \sum_{i,j}(A_\tau Bx_j, A_\tau Bx_i) \leq C \sum_{i,j}(A_\tau x_j, A_\tau x_i)
\]

for every finite number \( x_i \) in \( \mathfrak{R} \) and \( \tau \) in \( \Gamma \). And such \( L \) is unique.

b) Let \( B_1 \) and \( B_2 \) be bounded operators on \( \mathfrak{D} \) and satisfy conditions (i) and (ii) in a) and \( L_1 \) and \( L_2 \) be the extensions on \( \mathfrak{R} \) of \( B_1 \) and \( B_2 \) respectively. Then if \( B_1 \) and \( B_2 \) are commutative, \( L_1 \) and \( L_2 \) are commutative also.

c) If adding to (i), (ii) of a) \( B \) is a normal operator, then \( L \) is also a
normal operator on $\mathfrak{R}$.

Proof.

a). Necessity. (i) is evident. By observing that

$$
\sum_{i,j} (A_{r_i} B x_j, A_{r_j} B x_i) = \sum_{i,j} (N^*_{r_i} L x_i, N^*_{r_j} L x_j) = \| L(\sum N^*_{r_i} x_i) \|^2
$$

$$
\leq \| L \|^2 \sum_{i,j} N^*_{r_i} x_i \|^2 = \| L \|^2 \sum_{i,j} (A_{r_i} x_j, A_{r_j} x_i),
$$

(ii) is obtained.

Sufficiency. First we define a linear transformation $L$ for the element of $\mathfrak{L}$ such that $L(\sum N^*_{r_i} x_i) = \sum N^*_{r_i} B x_i$. Then

$$
\| L(\sum N^*_{r_i} x_i) \|^2 = \sum_{i,j} (N^*_{r_i} B x_j, N^*_{r_j} B x_i)
$$

$$
\leq C \sum_{i,j} (A_{r_i} B x_j, A_{r_j} B x_i) = C \| \sum N^*_{r_i} x_i \|^2.
$$

Hence $L$ is a bounded operator on $\mathfrak{L}$, and $L$ can be extended onto $\mathfrak{S}$ uniquely. We see easily $LN = NL (r \in \Gamma)$ on $\mathfrak{L}$ and $L$ is unique on $\mathfrak{L}$. Thus we have conclusion.

b). $L_1 L_2 (\sum N^*_{r_i} x_i) = \sum N^*_{r_i} B_1 B_2 x_i
$$
$$
= \sum N^*_{r_i} B_2 B_1 x_i = L_2 L_1 (\sum N^*_{r_i} x_i),
$$

hence $L_1 L_2 = L_2 L_1$.

c). As $B$ is normal and commutative to all $A_r$, $B^*$ commutes with all $A_r, r \in \Gamma$. And

$$
\sum_{i,j} (A_{r_i} B^* x_j, A_{r_j} B^* x_i) = \sum_{i,j} (A_{r_i} B x_j, A_{r_j} B x_i)
$$

$$
\leq C \sum_{i,j} (A_{r_i} x_j, A_{r_j} x_i).\quad\cdot
$$

Therefore from a) $B^*$ has an extension $M$ on $\mathfrak{S}$ uniquely. From b) $L$ and $M$ are commutative and

$$
(L(\sum N^*_{r_i} x_i), \sum N^*_{r_i} y_i) = \sum_{i,j} (A_{r_i} B x_i, A_{r_j} y_j)
$$

$$
= \sum_{i,j} (A_{r_j} x_i, A_{r_i} B^* y_j) = (\sum N^*_{r_i} x_i, M(\sum N^*_{r_i} y_i)).
$$

Thus we obtain $L^* = M$, consequently $L$ is a normal operator on $\mathfrak{S}$.

Lemma 4. Let $V_i (l=1,2,\cdots,n)$ be $n$ commutative isometric operators on $\mathfrak{S}$, then we can extend them to $n$ commutative unitary operators $U_i (l=1,2,\cdots,n)$ on $\mathfrak{S}$ containing $\mathfrak{S}$.

Proof. In the case $n=1$, it is evident that $V_i$ can be extended to the unitary operator. Therefore the minimal normal extension of $V_i$
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is an unitary operator. Let a semi-group generated by $V_1, V_2, \cdots, V_{\nu} (\nu < n)$ have a minimal normal extension consisting of unitary operators on $\mathcal{H}_0$ containing $\mathcal{H}$ and $W_l (l=1,2,\cdots,\nu)$ be extensions of $V_l (l=1,2,\cdots,\nu)$. By Lemma 3, a) $V_{\nu+1}$ can be extended to the operator $W_{\nu+1}$ on $\mathcal{H}_0$. It is easily proved $W_{\nu+1}$ is an isometric operator on $\mathcal{H}$. Hence putting $U_{\nu+1}$ a minimal normal extension of $W_{\nu+1}$ on $\mathcal{H}_0$ containing $\mathcal{H}'$ and $W_l (l=1,2,\cdots,\nu)$ be extensions of $V_l (l=1,2,\cdots,\nu)$.

By Lemma 3, a) $V_{\nu+1}$ can be extended to the operator $W_{\nu+1}$ on $\mathcal{H}_0$. It is easily proved $W_{\nu+1}$ is an isometric operator on $\mathcal{H}$. Hence putting $U_{\nu+1}$ a minimal normal extension of $W_{\nu+1}$ on $\mathcal{H}_0$ containing $\mathcal{H}'$ and $W_l (l=1,2,\cdots,\nu)$ be extensions of $V_l (l=1,2,\cdots,\nu)$.

By the induction the conclusion is obtained.

Proof of Theorem 3. From Lemma 4 $V_\gamma (\gamma \in \Gamma, \mathcal{H})$ is positive definite. Therefore from Theorem 1 $V_\gamma (\gamma \in \Gamma, \mathcal{H})$ has a minimal normal extension $U_\gamma (\gamma \in \Gamma, \mathcal{H})$. If we replace $A_\gamma$ with $V_\gamma$ and $N_\gamma$ with $U_\gamma$ in the inequality (1.17), we have $\langle U_\rho \overline{x}, U_\rho \overline{x} \rangle = \langle \overline{x}, \overline{x} \rangle$. Therefore $U_\gamma (\gamma \in \Gamma)$ are unitary operators on $\mathcal{H}$.

Remark. In Lemma 3 and Lemma 4, Theorem 1 is not used essentially. And by Maximal theorem (or transfinite induction) and Lemma 4, Theorem 3 is proved independently of Theorem 1.

3. A continuous one parameter semi-group. In this section we shall study a continuous one parameter semi-group consisting of subnormal operators and give two types of characterization of subnormal operators. One parameter family of bounded operators $A_t (t \geq 0)$ on $\mathcal{H}$ is called continuous one parameter semi-group when

\begin{align}
(i) & \quad A_{t_1} A_{t_2} = A_{t_1 + t_2}, \quad (t_1 \geq 0, t_2 \geq 0), \quad A_0 = I, \\
(ii) & \quad \text{weakly continuous on } t \geq 0.
\end{align}

Lemma 5. Continuous one parameter semi-group $A_t (t \geq 0, \mathcal{H})$ of subnormal operators is positive definite.

Proof. For an arbitrary finite number of $t_i \geq 0 (i = 1,2,\cdots,n)$ we find sequences of positive rational numbers $r_{\nu,i} (\nu = 1,2,\cdots; i = 1,2,\cdots,n)$ such that $\lim_{\nu \to \infty} r_{\nu,i} = t_i (i = 1,2,\cdots,n)$. We can put $r_{\nu,i} = b_{\nu,i}/a_{\nu} (\nu = 1,2,\cdots; i = 1,2,\cdots,n)$, where $b_{\nu,i}, a_{\nu}$ are positive integers. Since one parameter semi-group is weakly continuous if and only if strongly continuous ([6]), we have

\begin{align}
\sum_{i,j} (A_{t_i} x_j, A_{t_j} x_i) = \lim_{\nu \to \infty} \sum_{i,j} (A_{t_{\nu,i}} x_j, A_{t_{\nu,j}} x_i) \\
= \lim_{\nu \to \infty} \sum_{i,j} ((A_{1/a_{\nu}})^{b_{\nu,i}} x_j, (A_{1/a_{\nu}})^{b_{\nu,j}} x_i)
\end{align}
On the other hand every $A_{1/a_{\nu}}$ is subnormal by assumption, hence
\[ \sum_{i,j} (A_{1/a_{\nu}})^{\nu+i} x_j, (A_{1/a_{\nu}})^{\nu+j} x_i \geq 0. \]
Therefore $A_{t}(t \geq 0, \mathfrak{H})$ is positive definite.

**Theorem 4.** Continuous one parameter semi-group $A_{t}(t \geq 0, \mathfrak{H})$ of subnormal operators can be extended to continuous one parameter semi-group $N_{t}(t \geq 0, \mathfrak{H})$ consisting of normal operators on $\mathfrak{H}$ containing $\mathfrak{D}$.

**Proof.** From Lemma 5 $A_{t}(t \geq 0, \mathfrak{H})$ has a minimal normal extension $N_{t}(t \geq 0, \mathfrak{H})$. We shall show the continuity of $N_{t}$ about the parameter. It is evident $N_{t}$ is continuous about $t \geq 0$ on the linear manifold $\mathcal{Q}$ (cf. Def. 3) of $\mathfrak{H}$, and for any $t_{0} \geq 0$ and a sequence of rational numbers $r_{\nu} (\nu = 1, 2, \cdots)$ such that $\lim_{\nu \to \infty} r_{\nu} = t_{0}$, $\{ N_{r_{\nu}} ; \nu = 1, 2, \cdots \}$ is uniformly bounded, because $\| N_{r_{\nu}} \| = \| N_{t_{0}} \|^{r_{\nu}} (\nu = 1, 2, \cdots)$. Therefore easily we can see $\lim_{\nu \to \infty} N_{r_{\nu}} = N_{t_{0}}$ strongly by observing that $\mathcal{Q}$ is dense in $\mathfrak{H}$ and $\{ \| N_{r_{\nu}} \| ; \nu = 1, 2, \cdots \}$ is uniformly bounded. Thus $N_{t}$ is strongly continuous about $t \geq 0$.

Remark. (i) If in Theorem 4 $\mathfrak{H}$ is separable, the space $\mathfrak{H}$ of the minimal normal extension $N_{t}(t \geq 0, \mathfrak{H})$ is also separable.

(ii) From Theorem 2 we have $\| A_{t} \| = \| A \|^{t} (t \geq 0)$ for every continuous one parameter semi-group $A_{t}(t \geq 0)$ of subnormal operators.

**Theorem 5.** A bounded operator $A$ on $\mathfrak{H}$ is subnormal if and only if one parameter semi-group $exp(tA) (t \geq 0, \mathfrak{H})$ is positive definite.

**Proof.** Necessity. Let $N$ be a minimal normal extension of $A$. Then we have

\[ \sum (\exp(t_{i}A)x_{j}, \exp(t_{j}A)x_{i}) = \sum (\exp(t_{j}N)x_{j}, \exp(t_{i}N)x_{i}) = \| \sum \exp(t_{i}N)x_{i} \|^{2} \geq 0. \]

Sufficiency. Let $N_{t}(t \geq 0, \mathfrak{H})$ be a minimal normal extension of $\exp(tA) (t \geq 0, \mathfrak{H})$. Then $N_{t}$ is continuous about $t \geq 0$ from Theorem 4. If we put $N = \frac{dN_{t}}{dt} \big|_{t=0}$, that is, the infinitesimal operator of $N_{t}(t \geq 0)$ ([6]), $N$ is regular normal operator on $\mathfrak{H}$ ([8]) (generally non-bounded). Since $\frac{d \exp(tA)}{dt} \big|_{t=0} = A$ on $\mathfrak{H}$, we have $\cap_{n=1}^{\infty} D_{N^{n}} \supseteq \mathfrak{H}$, where $D_{N^{n}}$ denotes the domain of $N^{n}$. Therefore

\[ \sum (A^{n}x_{m}, A^{n}x_{n}) = \sum (N^{n}x_{m}, N^{n}x_{n}) = \sum (N^{*n}x_{m}, N^{*n}x_{n}) = \| \sum N^{*n}x_{n} \|^{2} \geq 0. \]
Hence $A$ is subnormal.

Remark. Naturally $A$ is subnormal if and only if $\exp (tA)$ \((-\infty < t < +\infty)\) is positive definite.

Let $\Phi$ be a group and $e$ be an identity element of $\Phi$. An operator valued function $\phi(\tau)$ from $\Phi$ into bounded operators on a Hilbert space $\mathfrak{H}$ is called a positive definite function in Nagy's sense ([11]) if $\phi(e)=I$; identity operator on $\mathfrak{H}$, and $\sum_{i,j} (x_i, \phi(\tau_i^{-1}\tau_j) x_j) \geq 0$ for every finite number of $x_i$ in $\mathfrak{H}$ and $\tau_i$ in $\Phi$.

Lemma 6. Let $\Phi$ be a group, $\phi(\gamma)$ \((\gamma \in \Phi, \mathfrak{H})\) an operator representation of $\Phi$ and $\phi(\gamma) = \phi(\gamma^{-1})^* \phi(\gamma)$. Then $\phi(\gamma)(\gamma \in \Phi, \mathfrak{H})$ is positive definite in the sense of Definition 1 if and only if $\phi(\gamma)$ is a positive definite function on $\Phi$ in Nagy's sense.

Proof. Because

$$\sum_{i,j} (x_i, \phi(\tau_i^{-1}\tau_j) x_j) = \sum_{i,j} (\phi(\tau_j^{-1}\tau_i) x_i, \phi(\tau_i^{-1}\tau_j) x_j)$$

(3.5)

$$= \sum_{i,j} (\phi(\delta_i) y_i, \phi(\delta_i) y_j)$$

where $\delta_i = \gamma_i^{-1}$ and $y_i = \phi(\gamma_i) x_i$ for all $i$. And hence the conclusion is clear.

From Theorem 5 and Lemma 6 we obtain the following theorem which is a generalization of Bram's theorem ([1] Theorem 2).

Theorem 6. A bounded operator $A$ on a Hilbert space $\mathfrak{H}$ is subnormal if and only if $\exp (-tA^*) \exp (tA)$ \((-\infty < t < +\infty)\) is a positive definite function in Nagy's sense.

Remark. If for an arbitrary positive definite function on an abelian group $\phi(\gamma)$ in Nagy's sense it is possible to find an operator representation $\phi(\gamma)$ such that $\phi(\gamma) = \phi(-\gamma)^* \phi(\gamma)$, we shall obtain from Theorem 1 Nagy's result ([11] Theorem III) in the case of abelian groups. But in general it is impossible. For example for continuous one parameter semi-group $T_t$ \((t \geq 0)\) of contractions; $\|T_t\| \leq 1$, if we put $\phi(t) = T_t$ for $t \geq 0$ and $\phi(t) = T_t^*$ for $t \leq 0$, then $\phi(t)$ exists for such $\phi(t)$ if and only if all $T_t$ are unitary operators.

4. A weak closure of $A_\tau (\tau \in \Phi, \mathfrak{H})$. Let $A_\tau$ \((\tau \in \Phi, \mathfrak{H})\) be a positive definite operator representation and $A_\omega$ \((\omega \in \Omega, \mathfrak{H})\) be the weakly closed algebra (not necessary self-adjoint) generated by $A_\tau$ \((\tau \in \Phi)\). In this section we shall give a theorem which shows the relation between the minimal normal extension $N_\tau$ \((\tau \in \Phi, \mathfrak{H})\) of $A_\tau$ \((\tau \in \Phi, \mathfrak{H})\) and that of $A_\omega$ \((\omega \in \Omega, \mathfrak{H})\). This theorem is a generalization of [1] Theorem 9 but our
proof seems to be simpler than that of [1].

Theorem 7. If \( A_\tau (\tau \in \Gamma, \mathfrak{S}) \) is positive definite, then the weak closed algebra \( A_\omega (\omega \in \Omega, \mathfrak{S}) \) is positive definite also. And let \( N_\tau (\tau \in \Gamma, \mathfrak{S}) \) and \( L_\omega (\omega \in \Omega, \mathfrak{M}) \) be respectively the minimal normal extensions of \( A_\tau (\tau \in \Gamma, \mathfrak{S}) \) and \( A_\omega (\omega \in \Omega, \mathfrak{S}) \). Then we may consider \( \mathfrak{M} = \mathfrak{S} \) and \( L_\omega \in \mathfrak{M} \{ N_\tau (\tau \in \Gamma) \} (\omega \in \Omega) \), where \( \mathfrak{M} \{ N_\tau (\tau \in \Gamma) \} \) is the operator ring (weakly closed self-adjoint algebra) generated by \( N_\tau (\tau \in \Gamma) \) on \( \mathfrak{S} \).

Proof. Since \( A_\tau (\tau \in \Gamma, \mathfrak{S}) \) is positive definite, the algebra \( \mathfrak{S} \) generated by \( A_\tau (\tau \in \Gamma) \) is evidently positive definite. By the definition of positive definite naturally the strong closure of \( \mathfrak{S} \) is also positive definite. Therefore \( A_\omega (\omega \in \Omega, \mathfrak{S}) \) is positive definite, because the strong closure of a linear set of operators is the same as its weak closure.

Let \( L_\omega (\omega \in \Omega, \mathfrak{M}) \) be the minimal normal extension of \( A_\omega (\omega \in \Omega, \mathfrak{S}) \), \( L_\tau (\tau \in \Gamma) \) be its part which is an extension of \( A_\tau (\tau \in \Gamma) \) onto \( \mathfrak{M} \) and \( \mathfrak{M} \{ L_\tau (\tau \in \Gamma) \} \) be the operator ring on \( \mathfrak{M} \) generated by \( L_\tau (\tau \in \Gamma) \).

First we shall show that for any \( L_\omega (\omega \in \Omega) \) (fixed) and for any \( \epsilon > 0 \) and \( x_i \in \mathfrak{S} (i = 1, 2, \cdots, n) \) there exists an operator \( L \) such that

\[
\begin{align*}
L & \in \mathfrak{M} \{ L_\tau (\tau \in \Gamma) \}, \\
\| L \| & \leq \sqrt{2} \| L_\omega \|, \\
\| L \| & \leq \epsilon \| L_\omega \|, \\
\| L x_i - L x_i \| & \leq \epsilon \quad (i = 1, 2, \cdots, n).
\end{align*}
\]

Because for \( A_\omega \) there exists \( B \in \mathfrak{S} \) such that \( \| A_\omega x_i - B x_i \| \leq \epsilon (i = 1, 2, \cdots, n) \), putting \( M \) the extension of \( B \) onto \( \mathfrak{M} \), then \( M \in \mathfrak{M} \{ L_\tau (\tau \in \Gamma) \} \) and \( \| L \| \leq \epsilon \) \( (i = 1, 2, \cdots, n) \), it follows that \( \| R(L_\omega) x_i - R(M) x_i \| \leq \epsilon \), \( \| I(L_\omega) x_i - I(M) x_i \| \leq \epsilon \) \( (i = 1, 2, \cdots, n) \), where \( R(T) \) and \( I(T) \) denote respectively the real part and the imaginary part of \( T \), evidently \( R(M), I(M) \in \mathfrak{M} \{ L_\tau (\tau \in \Gamma) \} \), we can find a polynomial \( P(t) \) (cf. von Neumann [9] p. 399) such that

\[
\begin{align*}
\| P(R(M)) \| & \leq \| L_\omega \|, \\
\| P(I(M)) \| & \leq \| L_\omega \|, \\
\| P(R(M)) x_i - R(L_\omega) x_i \| & \leq \epsilon, \\
\| P(I(M)) x_i - I(L_\omega) x_i \| & \leq \epsilon \quad (i = 1, 2, \cdots, n)
\end{align*}
\]

If we put \( L = P(R(M)) + i P(I(M)) \), then \( L \) satisfies (4.1).

Putting \( \mathfrak{S} = \{ \sum \xi^* L \xi x_i \}; \) for every finite number of \( x_i \in \mathfrak{S} \) and \( \xi \in \Gamma \) and \( \mathfrak{S} = \mathfrak{S} \) (closure of \( \mathfrak{S} \) in \( \mathfrak{M} \)). Then \( \mathfrak{S} \) reduces every \( L_\tau (\tau \in \Gamma) \) and therefore \( \mathfrak{S} \) reduces \( \mathfrak{M} \{ L_\tau (\tau \in \Gamma) \} \). We can see by using (4.1) \( \mathfrak{S} \) reduces also every \( L_\omega (\omega \in \Omega) \). Because, for any \( f \in \mathfrak{S} \) and \( \epsilon > 0 \), we can find \( \sum \xi^* L \xi x_i \in \mathfrak{S} \) and \( L \in \mathfrak{S} \{ L_\tau (\tau \in \Gamma) \} \) such that \( \| f - \sum \xi^* L \xi x_i \| \leq \epsilon / \sqrt{2} \| L_\omega \| \),
\[\|L\| \leq \sqrt{2} \|L_{\omega}\|, \quad \|L_{\omega} x_{i} - L x_{i}\| \leq \varepsilon \frac{\sum_{i=1}^{n} \|L_{r_{i}}\|}{\|L_{\omega}\|}. \] Hence \[\|L_{\omega} f - L f\| \leq \|L_{\omega} f - L_{\omega}(\sum_{i=1}^{n} L_{r_{i}} x_{i})\| + \|L_{\omega}(\sum_{i=1}^{n} L_{r_{i}} x_{i}) - L(\sum_{i=1}^{n} L_{r_{i}} x_{i})\| + \|L(\sum_{i=1}^{n} L_{r_{i}} x_{i}) - L f\| \leq 3\varepsilon,\] that is, we have \(\|L_{\omega} f - L f\| \leq 3\varepsilon\). Since \(L_{\omega}\) and \(L\) are commutative we have \(\|L_{\omega} f - L f\| \leq 3\varepsilon\) also. As \(L f, L f \in \mathfrak{S}\) and \(\varepsilon\) is arbitrary, it follows that \(L_{\omega} f, L_{\omega} f \in \mathfrak{S}\) namely \(\mathfrak{S}\) reduces \(L_{\omega}\). Since \(L_{\omega}(\omega \in \Omega, \mathfrak{N})\) is a minimal normal extension, we must have \(\mathfrak{S} = \mathfrak{R}\). Likewise above we can see for arbitrary \(L_{\omega}, f_{i} \in \mathfrak{S}(i = 1, 2, \ldots, n)\) and \(\epsilon > 0\) there exists \(L \in \mathfrak{R}\{L_{r}(\gamma \in \Gamma)\}\) such that \(\|L_{\omega} f_{i} - L f_{i}\| \leq 3\varepsilon\) \((i = 1, 2, \ldots, n)\). Therefore \(L_{\omega}\) belongs to the strong closure of \(\mathfrak{R}\{L_{r}(\gamma \in \Gamma)\}\) on \(\mathfrak{S}\), that is, \(L_{\omega} \in \mathfrak{R}\{L_{r}(\gamma \in \Gamma)\}\). \(L_{r}(\gamma \in \Gamma, \mathfrak{S})\) is evidently the minimal normal extension of \(A_{r}(\gamma \in \Gamma, \mathfrak{S})\). Thus the proof is complete.

Remark. As \(A_{\omega}(\omega \in \Omega, \mathfrak{S})\) is positive definite, by using Lemma 3 a) \(A_{\omega}\) can be extended uniquely to an operator \(L_{\omega}\) acting on \(\mathfrak{S}\) which is the space of the minimal normal extension \(N_{r}(\gamma \in \Gamma, \mathfrak{S})\) of \(A_{r}(\gamma \in \Gamma, \mathfrak{S})\). In other words Theorem 7 is as follows \(A_{\omega}\), an element of the weakly closed algebra generated by \(A_{r}(\gamma \in \Gamma)\), can be extended uniquely to an operator \(L_{\omega}\) on \(\mathfrak{S}\) such that \(L_{\omega} \in \mathfrak{R}\{N_{r}(\gamma \in \Gamma)\}\) and \(\|L_{\omega}\| = \|A_{\omega}\|\).

5. A spectrum of subnormal operators. HALMOS [4] has shown that if \(N\) is the minimal normal extension of the subnormal operator \(A\), then the resolvent set \(\rho(A)\) of \(A\) is contained in the resolvent set \(\rho(N)\) of \(N\). Let \(\rho_{n}(N)\) \((n = 1, 2, \ldots)\) be all connected components of \(\rho(N)\). Then BRAM [1] has shown \(\rho(A) = \sum_{n \in J} \rho_{n}(N)\), where \(J\) is a subset of the positive integers. In this section we shall show simpler another proof of this theorem, in our proof the theory of complex variable functions is not necessary.

Denoting \(\rho(A)\) the resolvent set of a operator \(A\), \(\mathfrak{B}\) the whole of polynomials \(P(\lambda)\) on the complex plane, \(\mathfrak{F}_{A}\) the whole of rational functions \(f(\lambda) = P_{1}(\lambda)/P_{2}(\lambda)\) which are regular on the spectrum of \(A\). We can define \(P(A)\) for \(P(\lambda) \in \mathfrak{B}\) and \(f(A) = P_{1}(A)/P_{2}(A)^{-1}\) for \(f(\lambda) \in \mathfrak{F}_{A}\).

**Lemma 7.** The following conditions are equivalent each other

a) \(A\) is subnormal,

b) \(P(A) (P(\lambda) \in \mathfrak{B}, \mathfrak{S})\) is positive definite,

c) \(f(A) (f(\lambda) \in \mathfrak{F}_{A}, \mathfrak{S})\) is positive definite,

d) \((A-\lambda)^{-1}\) for some \(\lambda \in \rho(A)\) is subnormal.

**Proof.** It is evident that a) implies b) and c) implies d). We shall prove b) implies c). Because for arbitrary finite number of \(f_{i}(\lambda) \in \mathfrak{F}_{A}\)
\((i=1,2,\ldots,n)\), \(f_{t}{}^{(A)}=P_{t}{}^{(A)}/Q_{t}{}^{(A)}\), we have

\[
(5.1) \quad \sum_{t,j=1}^{n} (f_{t}{}^{(A)}x_{j}, f_{j}{}^{(A)}x_{j}) = \sum_{t,j=1}^{n} (P_{t}{}^{(A)}Q_{t}{}^{(A)}-x_{j}, P_{j}{}^{(A)}Q_{j}{}^{(A)}-x_{j})
\]

\[
= \sum_{t,j=1}^{n} (P_{t}{}^{(A)}R_{t}{}^{(A)}y_{j}, P_{j}{}^{(A)}R_{j}{}^{(A)}y_{j}) \geq 0 ,
\]

where \(y_{t}=Q_{1}{}^{(A)}-Q_{1}{}^{(A)}-\ldots Q_{n}{}^{(A)}-x_{i}(i=1,2,\ldots,n)\) and \(R_{t}{}^{(A)}=Q_{t}{}^{(A)}Q_{i}{}^{(A)}-\ldots Q_{n}{}^{(A)}Q_{i}{}^{(A)}(i=1,2,\ldots,n)\). And likewise \((d)\) implies \((a)\).

Remark. Let \(f(A)(f \in \mathfrak{F}_{A}, \mathfrak{F})\) be positive definite and \(N_{f}(f \in \mathfrak{F}_{A}, \mathfrak{F})\) be its minimal extension. Then by rembering the proof of Theorem 1 we obtain \(N_{f_{1}}, N_{f_{2}}=N_{f_{1}f_{2}}\) and \(N_{f_{1}f_{2}}=N_{f_{1}}+N_{f_{2}}\). Therefore we have from Theorem 2 \(\|f_{1}(A)+f_{2}(A)\|_{\mathfrak{F}}=\|N_{f_{1}}+N_{f_{2}}\|_{\mathfrak{F}}\) for every \(f_{1}, f_{2} \in \mathfrak{F}_{A}\) (this fact will be used in the proof of Theorem 8).

Theorem 8. (Halmos, Bram). Let \(A\) be subnormal and \(N\) be the minimal normal extension of \(A\). Then we have

\[
(5.2) \quad \rho(A) = \sum_{n \in J} \rho_{n}(N)
\]

where \(\rho_{n}(N) (n=1,2,\ldots)\) are all connected components of \(\rho(N)\) and \(J\) is a suitable subset of the positive integers.

Proof. From Lemma 9 \(f(A)(f \in \mathfrak{F}_{A}, \mathfrak{F})\) is positive definite, hence it has a minimal normal extension \(N_{f}(f \in \mathfrak{F}_{A}, \mathfrak{F})\). We shall denote by \(N\) the extension of \(A\) onto \(\mathfrak{F}\). For any \(\lambda_{0} \in \rho(A)\) if we put \(P(\lambda)=\lambda-\lambda_{0}\), then \(P(\lambda)^{-1} \in \mathfrak{F}_{A}\) and \(P(\lambda)P(\lambda)^{-1}=1\), hence \(N_{p}N_{p-1}=N_{p}N_{p-1}N_{p}=I\). Therefore \(N_{p}N_{p-1}=N_{p-1}N_{p}\), namely \(\lambda_{0} \in \rho(N)\) and \(\rho_{n}(N)_{\cap} \rho(A) \neq \phi\). Furthermore we obtain \(N_{p}f(A)(f(A) \in \mathfrak{F}_{A}, \mathfrak{F})\) for every \(f \in \mathfrak{F}_{A}\). On the other hand \(N\) is a minimal normal extension of \(A\). Because, if a subspace \(\mathfrak{J}_{0}\) of \(\mathfrak{J}\) contains \(\mathfrak{J}\) and reduces \(N\), then evidently \(\mathfrak{J}_{0}\) reduces every \(f(N)\) for \(f \in \mathfrak{F}_{A}\). Thus we obtain Halmos' theorem, that is, \(\rho(A) \subseteq \rho(N)\).

Next we shall prove that \(\rho_{n}(N) \cap \rho(A) = \phi\) or \(\rho_{n}(N) = \rho(A)\) for every \(n\). If \(\rho_{n}(N) \cap \rho(A) \neq \phi\) for some \(n\), then \(\rho_{n}(N) \cap \rho(A)\) is non-empty open set. On the other hand \(\rho_{n}(N) \cap \rho(A)\) is a closed set in \(\rho_{n}(N)\). Because, for every sequence \(\lambda_{t} \in \rho_{n}(N) \cap \rho(A)\) such that \(\lambda_{t} \in \rho_{n}(N) \cap \rho(A)\) we have

\[
\lim_{t \to \infty} \|(N-\lambda_{t})^{-1}-(N-\lambda_{0})^{-1}\|_{\mathfrak{J}} = 0, \text{by the Remark after Lemma 9,} \quad (A-\lambda_{t})^{-1}-(A-\lambda_{0})^{-1} \|_{\mathfrak{J}} = (A-\lambda_{t})^{-1}-(N-\lambda_{0})^{-1} \|_{\mathfrak{J}},
\]

therefore we find an operator \(B\) on \(\mathfrak{J}\) such that \(\lim_{t \to \infty} \|(A-\lambda_{t})^{-1}-B\|_{\mathfrak{J}} = 0\), and hence for every \(\lambda \in \rho(A)\) we have

\[
(A-\lambda)^{-1}-B = \lim_{t \to \infty} (A-\lambda)^{-1}-(A-\lambda_{t})^{-1} = \lim_{t \to \infty} (\lambda-\lambda_{t})(A-\lambda)^{-1} = (\lambda-\lambda_{0})(A-\lambda)^{-1}B, \text{ that is,} \quad (A-\lambda)^{-1}-B = (\lambda-\lambda_{0})(A-\lambda)^{-1}B.
\]
follows that $\lambda_0 \in \rho(A)$ and $B = (A - \lambda_0)^{-1}$. Consequently $\rho_n(N) \cap \rho(A)$ is open and closed in $\rho_n(N)$, hence $\rho_n(N) \cap \rho(A) = \rho_n(N)$. The proof is complete.

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