ON THE REDUCTION OF MINKOWSKI SPACE

By

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Introduction. Recently Prof. A. KAWAGUCHI succeeded in constructing a new theory of FINSLER space which does not based on the CARTAN's concept of line element space but on the one of non-linear connection. The most part of his paper is sacrificed to the study of the geometrical property of the indicatrix in tangent MINKOWSKI spaces at every point of the FINSLER space [2][1], [3]. Then he showed making use of this property that there exists a natural relation between CARTAN's theory of FINSLER space [4] and the classical results in affine differential geometry in the sense of W. BLASCHKE [5]. And from this point of view A. KAWAGUCHI found several conditions under which a FINSLER space can be regarded essentially as a RIEMANNIAN space. Especially a new proof of the result by DEICKE was given with beautiful geometric interpretations.

The present author develops the theory of tangential MINKOWSKI space from KAWAGUCHI's scheme in more detail on consideration of some topological properties of the indicatrix such as compactness and simply connectivity. Some sufficient conditions for the reduction problems are obtained. The first part of §1 is the brief summary of KAWAGUCHI's view point.

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§ **1.** A MINKOWSKI space M_n is an n-dimensional vectorspace associated with a positive function $F(X)$ which is positively homogeneous of degree 1. A FINSLER space F_n is an n-dimensional differentiable manifold at any point of which an n-dimensional MINKOWSKI space with metric function $F(x \cdot X) \equiv F_x(X)$ attaches as a tangent-space. An indicatrix of FINSLER space at any point x is a subspace of tangent MINKOWSKI space at the point defined by $F_x(X)=1$. We assume in the present paper

1) Numbers in brackets refer to the references at the end of the paper.

that the indicatrix has no singular point of sufficiently many times differentiability. A MINKOWSKI space with a hyperquadric as its indicatrix is essentially a EUCLIDean space. According to the fact we are enough to study only the indicatrix of each M_n as far as concerned with the reduction problems, indeed all the properties of MINKOWSKI spaces are completely determined by that of the indicatrix.

Let us consider $(n-1)$-dimensional coordinate system on the indicatrix. Then we have an N-ple $(X^i X^i_\alpha)$ at any point of the indicatrix, where $X^i_\alpha = \frac{\partial X^i}{\partial u^\alpha}$. Then we can express any quantity of MINKOWSKI space referred to this N-ple [2], e.g.

$$g_{ij} = g_{\alpha\beta} X^\alpha_i X^\beta_j + X_i X_j \tag{1.1}$$

$$A_{ijk} = A_{\alpha\beta\gamma} X^\alpha_i X^\beta_j X^\gamma_k \,. \tag{1.2}$$

The indicatrix is a RIEMANNian manifold of $(n-1)$-dimensions, because the quardratic form $g_{\alpha\beta} = g_{ij} X^i_\alpha X^j_\beta$ is defined on it. We have as in [2]

$$A_{\alpha\beta\gamma.\delta} = A_{\alpha\beta\delta.\gamma} \tag{1.3}$$

$$R_{\alpha\beta\gamma\delta} = A^\sigma_{\alpha\gamma} A_{\beta\delta\sigma} - A^\sigma_{\alpha\delta} A_{\beta\gamma\sigma} + g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \tag{1.4}$$

$$A_\alpha \equiv A_{\alpha\beta\gamma} g^{\beta\gamma} = \frac{\partial \log p}{\partial u^\gamma} \tag{1.5}$$

where $A_{\alpha\beta\gamma.\delta}$ denotes a component of covariant derivative of $A_{\alpha\beta\gamma}$ by means of u^α and P denotes the affine distance of the point X from the origin. Identity (1.5) shows that a MINKOWSKI space with an indicatrix satisfying $A_\alpha = 0$ must be essentially a EUCLIDean space. As from (1.2) we can easily verify that our condition $A_i = 0$ is equivalent to $A_\alpha = 0$ from which follows DEICKE's result immediately. We can state this theorem in a formally generalized form as follows

Theorem 1.1. *If we have a formally condition that the vector field A_α generates an isometrie of RIEMANNian metric on the indicatrix, M_n must be E_n, $A_\alpha = 0$ in truth.*

Proof. The theorem is almost evident. The KILLING equation on the indicatrix [2]$L\, g_{\alpha\beta} = 0$ means that the affine distance p satisfies an elliptic differential equation $(\log p)_{.\alpha.\beta} g^{\alpha\beta} = 0$. As the indicatrix in consideration is compact, E. HOPF's theorem [8] shows us that p must be constant, i.e. our indicatrix is essentially a hyper quadric.

2) Operator L means so-called Lie derivative [11].

We can further generalize the theorem as follows, when M_n is of odd dimensions.

Theorem 1.2. *If the vector field A_α of indicatrix in M_{2r+1} is orthogonal to the KILLING tensor field on it generated by A_α, then A_α must be zero vector field.*

Proof. It is a well known fact that a closed differentiable manifold admits a continuous field of every where non-zero tangent vector field if and only if its EULER number is zero, and also that the EULER number of the surface of n-dimensional topological sphere is $1+(-1)^n$. As the indicatrix is a closed hyper surface of even dimensions, its EULER number is 2. This fact shows the existence of at least one point at which A_α vanish [10]. On the other hand the assumption $(Lg_{\alpha\beta})A^\alpha=0$ means $A_\alpha A^\alpha=$const. Hence this constant must be zero, accordingly $A_\alpha=0$ at every point on the indicatrix. Furthermore we obtain the following

Theorem 1.3. *In M_{2r+1} the RIEMANNian metric on the indicatrix can not have the scalar curvature of everywhere non-positive value.*

Proof. Contracting (1.4) with $g^{\alpha\gamma}$ and $g^{\beta\delta}$ we obtain

$$R=g^{\alpha\beta}R_{\alpha\beta}=A^{\alpha\beta\gamma}A_{\alpha\beta\gamma}-A^\alpha A_\alpha+(n-1)(n-2) \tag{1.6}$$

As there exists at least one point at which A_α vanishes, then from (1.6) the value of the scalar curvature at that point is strictly positive. By virtue of results of these theorems we have.

Theorem 1.4. *If $A_{\alpha\beta\gamma\cdot\delta}=A_{\alpha\beta\gamma}T_\delta$ holds at every point on the indicatrix, where T_δ denote components of a vector field on the indicatrix, then $A_{\alpha\beta\gamma}$ must be zero tensor field, i.e. the space is EUCLIDean.* $(n\geqq 3)$.

Proof. $I_n(x)=D_1\cup D_2\cup D_3$ where $I_n(x)$ denotes indicatrix,

$$D_1=\{X:\ T_\delta=0\},\quad D_2=\{X:\ A_\delta=0\},\quad D_3=\{X:\ T_\delta\neq 0\ \ A_\delta\neq 0\}.$$

Making use of (1.3) and the assumption, we have at any point of D_3

$$A_{\alpha\beta\gamma}T_\delta=A_{\alpha\beta\delta}T_\gamma \tag{1.7}$$

contracted by $g^{\alpha\beta}$ (1.7) gives us $A_\gamma T_\delta=A_\delta T_\gamma$, i.e.

$$T_\alpha=\phi A_\alpha \qquad \phi\neq 0. \tag{1.8}$$

Substituting (1.8) into (1.7) we get

$$A_{\alpha\beta\gamma}A_\delta=A_{\alpha\beta\delta}A_\gamma \tag{1.9}$$

Contracting (1.9) with A^δ or $g^{\delta\beta}$ we have

(1. 10) $\quad A_{\alpha\beta\gamma}\cdot A_\delta A^\delta = A_{\alpha\beta\delta}A^\delta\cdot A_\gamma$ or $A_{\alpha\beta\delta}A^\delta = A_\alpha\cdot A_\beta$ respectively.

Again substituting the second equation of (1.10) into the first ones, we obtain

(1. 11) $$A_{\alpha\beta\gamma}\cdot A_\delta A^\delta = A_\alpha\cdot A_\beta\cdot A_\gamma\ .$$

As $A_\delta\neq 0$ in D_3, from (1.11) we have $S_{\alpha\beta\gamma\delta}=0$.

We proceed to take D_1 into consideration, where $A_{\alpha\beta\gamma\cdot\delta}=0$ hence

(1. 12) $$(S_{\alpha\beta\gamma\delta}S^{\alpha\beta\gamma\delta})_{.\varepsilon}=0 \quad (A_\alpha\cdot A^\alpha)_{.\varepsilon}=0.$$

Furthermore we can regard D_1 as a sum of finite numbers of connected closed domains D_{1k} and isolated points and we have two constants in each D_{1k} satisfying

(1. 13) $$S_{\alpha\beta\gamma\delta}S^{\alpha\beta\gamma\delta} = C_k \quad A_\alpha A^\alpha = C'_k.$$

Evidently, D_1 and D_2 are closed. Let us consider the point set $D'_3=\{X\colon S_{\alpha\beta\gamma\delta}=0\}$ then $D'_3\supset\overline{D}_3\supset D_3$ and D'_3 is closed. If D_{1k} does not cover all the indicatrix then D_{1k} must have a common point with at least one of D_2 and D'_3. Therefore in each D_{1k} one of the following equations hold good:

(1. 14) $$S_{\alpha\beta\gamma\delta}S^{\alpha\beta\gamma\delta} = 0 \text{ or } A_\alpha A^\alpha = 0\ .$$

As isolated points of D_1 must be limit points of D_2 or D'_3, these points must belong to $D_2\cup D'_3$, i.e. we have $S_{\alpha\beta\gamma\delta}=0$ or $A_\alpha=0$ at these points. These considerations show that the following two case only can occur 1) D_1 covers all the indicatrix. 2) D_2 and D'_3 together cover the indicatrix. In the first case the condition is equivalent with that of Theorem 1.1. The second case remains to be proved. Since each connected subdomain of D_2 can be regarded as a part of a hyper quadric, $S_{\alpha\beta\gamma\delta}=0$ holds at points of any connected subdomains, then $S_{\alpha\beta\gamma\delta}=0$ at every point of the indicatrix even at the isolated point of D_2. Again after KAWAGUCHI's result, our MINKOWSKI space must be EUCLIDean. This completes the proof.

Remark. The set D_3 in this proof is in fact vanity.

§ **2.** As we can see Theorem 1. 3 in an odd dimensional MINKOWSKI space there exists at least one point at which the scalar curvature has the value not less than $(n-1)(n-2)$. By this reason MINKOWSKI space with the RIEMANNian metric of which scalar curvature is constant will be most interest. On the other hand, as KAWAGUCHI has shown in [2], the vanishing of PICK's invariant $\mathfrak{P}=\mathfrak{a}_{\alpha\beta\gamma}\mathfrak{a}_{\delta\varepsilon\varphi}\mathfrak{g}^{\alpha\delta}\mathfrak{g}^{\beta\varepsilon}\mathfrak{g}^{\gamma\varphi}$ means that the space is EUCLIDean, the space with constant PICK's invariant is also important. Where $\mathfrak{g}_{\alpha\beta}=p\cdot g_{\alpha\beta}$ is the affine metric in affine differential geometry (cf. [2] p.179).

We shall proceed to prove.

Theorem 2.1. *If scalar curvature of RIEMANNian metric and PICK's invariant of affine metric on the indicatrix of a MINKOWSKI space are both constant, then the trajectories of the one parameter transformation group generated by the vectorfield on the indicatrix are geodesics.*

Proof. As in [2].

$$R = Q-A^2+(n-1)(n-2) \qquad Q=A_{\alpha\beta\gamma}A^{\alpha\beta\gamma},\ A^2=A_\alpha A^\alpha . \tag{2.1}$$

$$Q = \frac{3}{n+1}A^2+\mathfrak{P}p^3 \text{ where } \mathfrak{P} \text{ is pick's invariant}. \tag{2.2}$$

Let us substitute (2.2) into (2.1) then we obtain

$$R = \frac{2-n}{n+1}A^2+p^3\mathfrak{P}+(n-1)(n-2), \tag{2.3}$$

and differentiating the above equation by u^δ,

$$0 = \frac{2(n-2)}{n+1}A_{\alpha\,\delta}A^\alpha+3p^2p._\delta\mathfrak{P} . \tag{2.4}$$

On the other hand we have $-\dfrac{2}{n+1}A_\delta=p._\delta$ (cf. [2] p.181), hence from (2.4) follows

$$A_{\alpha.\delta}A^\alpha = \frac{3p^3}{n-2}\mathfrak{P}\cdot A_\delta . \tag{2.5}$$

Q.E.D.

Especially for the space of which scalar curvature has the value $(n-1)(n-2)$, we have moreover.

Theorem 2.2. *If a MINKOWSKI space is of odd dimensions, scalar curvature of its RIEMANNian metric is $(n-1)(n-2)$ and its PICK's invariant is constant, the space must be EUCLIDean.*

Proof. As $R=(n-1)(n-2)$, then (2.1) turns into.

$$Q = A^2, \tag{2.6}$$

and (2.2) into

$$\frac{n-2}{n+1}A^2 = p^3\cdot\mathfrak{P} . \tag{2.7}$$

Since the MINKOWSKI space is of odd dimensions, there exists one point where all the components of A_α vanish. Accordingly we can see that PICK's invariant is zero at every point of the indicatrix. This proves the theorem.

Theorem 2.3. *If a MINKOWSKI space with RIEMANNian metric of which scalar curvature has the constant value and PICK's invariant is also constant, and the one parameter transformation group generated by A_α is conformal ones then the space must be EUCLIDean.* $(n \geqq 3)$.

Proof. By the assumption.

$$L(g_{\alpha\beta}) = Kg_{\alpha\beta} \qquad K = \frac{2}{n-1} A^{\alpha}_{\cdot\alpha} . \tag{2.8}$$

Contracting with A^α, (2.8) becomes

$$A_{\alpha\cdot\beta}A^\alpha = \frac{1}{n-1} A^{\alpha}_{\cdot\alpha} A_\beta . \tag{2.9}$$

On the other hand, from (2.5) we have $A_{\alpha\cdot\beta}A^\alpha = \frac{3p^3}{n-2}\cdot\mathfrak{P}\cdot A_\beta$, from which and (2.9) we obtain

$$\left(\frac{A^{\tau}_{\cdot\tau}}{(n-1)} - \frac{3p^3\cdot\mathfrak{P}}{(n-2)}\right)\cdot A_\beta = 0 . \tag{2.10}$$

From (2.10) we can regard the indicatrix as a union of two closed point sets S_1 and S_2 where $S_1 = \left\{X\colon \frac{A^{\tau}_{\tau}}{(n-1)} = \frac{3p^3\cdot\mathfrak{P}}{n-2}\right\}$, $S_2 = \{X\colon A_\delta = 0\}$. Isolated point of one of S_1 and S_2 belong to the other, hence we can divide the indicatrix into two parts S_1' S_2' of points not isolated in S_1 and S_2 respectively. We can easily see that inequality $A^{\tau}_{\cdot\tau} \geqq 0$ holds in both of S_1' and S_2', i.e. all over the indicatrix which is compact. Hence from HOPF's theorem it follows that the indicatrix must be hyper-quadric essentially.

In the special case $n=3$ the following theorem hold good:

Theorem 2.4. *In M_3 the condition that A_α generates conformal transformations means that the space be EUCLIDean. i.e. $A_\alpha = 0$ in fact.*

Proof. Differentiation of the relation $a_{\alpha\beta\gamma} = -A_{\alpha\beta\gamma} + \frac{1}{n+1}(g_{\alpha\beta}A_\gamma + g_{\alpha\gamma}A_\beta + g_{\beta\gamma}A_{\alpha\delta})$ offers us

$$a_{\alpha\beta\gamma\cdot\delta} = -A_{\alpha\beta\gamma\cdot\delta} + \frac{1}{n+1}(g_{\alpha\beta}A_{\gamma\cdot\delta} + g_{\alpha\gamma}A_{\beta\cdot\delta} + g_{\beta\gamma}A_{\alpha\cdot\delta}) \tag{2.11}$$

of which the asymmetric part with respect to γ, δ is by account of (1.3)

$$(2.12) \qquad \mathfrak{a}_{\alpha\beta\gamma\cdot\delta}-\mathfrak{a}_{\alpha\beta\delta\cdot\gamma}=\frac{1}{n+1}(g_{\alpha\gamma}A_{\beta\cdot\delta}+g_{\beta\gamma}A_{\alpha\cdot\delta}-g_{\alpha\delta}A_{\beta\cdot\gamma}-A_{\alpha\cdot\gamma}g_{\beta\delta}).$$

Contracting with $g^{\alpha\delta}$, (2.12) becomes

$$(2.13) \qquad \mathfrak{a}_{\alpha\gamma\beta}{}^{\cdot\beta}=\frac{1}{n+1}\left(g_{\alpha\gamma}A^{\cdot\tau}_{\cdot\tau}-(n-1)A_{\alpha\cdot\gamma}\right)$$

from which and (2.8) we have $a_{\alpha\gamma\beta}{}^{\cdot\beta}=0$. But as shown in [2], there exists a relation $\mathfrak{a}_{\alpha\beta\gamma}{}^{,\gamma}$ 3) $=\mathfrak{a}_{\alpha\beta\gamma}{}^{\cdot\gamma}+(n-3)\mathfrak{a}_{\alpha\beta\gamma}\sigma^{\gamma}$ from which we have $\mathfrak{a}_{\alpha\gamma\beta}{}^{,\beta}=0$. Again making use of the result of KAWAGUCHI [2] we can conclude that the statement is true.

From the last theorem follows

Theorem 2.5. *If the vector field generates transformations preserving the RICCI tensor made of the RIEMANNian metric of the indicatrix, the space must be EUCLIDean.* $(n=3)$

Proof. The RICCI tensor made of the RIEMANNian metric satisfies the identity $R_{\alpha\beta}=\frac{R}{2}g_{\alpha\beta}$. Then $L(R_{\alpha\beta})=0$ means $(LR)g_{\alpha\beta}+R\cdot(Lg_{\alpha\beta})=0$, from which we can conclude that the transformations preserving RICCI tensor are conformal at the point where the scalar curvature is not zero. We can divide the indicatrix into two parts D_1 and D_2 such that $D_1=\{X\colon R=0\}$, $D_2=\{X\colon R\neq 0\}$. There exists no isolated point in D_2, and D_2 alone cannot cover all the indicatrix, as follows from the existence of the point at which all the components of A_α vanish. While the point $A_\alpha=0$ must be in a connected subdomain D_2^* of D_2. Making use of the equation in [2] we see that A_i is zero in D_2^*. As D_2^* has at least one point belonging to D_1 unless D_2 cover all the indicatrix, and we have on the other hand $A_iA^i>1$ in D_1, at these point the function A_i must be singular. This contradicts to the assumption of regularity. We can find that there exists no point of D_1. Hence our hypothese in the theorem is equivalent to the preceding theorem, we have established the proof of the theorem.

§ 3. A theorem of affine differential geometry.

In the affine differential geometry in the sense of W. BLASCHKE, the fundamental equations are following [2]

$$(3.1) \qquad X^i_{\alpha,\beta}=\mathfrak{a}_{\alpha}{}^{\gamma}{}_{\beta}X^i_\gamma+\mathfrak{g}_{\alpha\beta}Y^i \quad \text{where} \quad \mathfrak{B}^\beta_\alpha=\mathfrak{D}^\beta_\alpha-(n-1)\mathfrak{K}\delta^\beta_\alpha$$

3) „ ' " denote covariant derivatve by means of $\mathfrak{g}_{\alpha\beta}$.

$$Y^i_\alpha = \mathfrak{B}^\beta_\alpha X^i_\beta \qquad \mathfrak{Q} = \frac{1}{(n-1)(n-2)}\mathfrak{D}^\alpha_\alpha = \mathfrak{R} - \mathfrak{B} = -\frac{1}{n-1}\mathfrak{B}^\alpha_\alpha$$

$$\mathfrak{D}^\beta_\alpha = (\mathfrak{R}_{\alpha\gamma\delta\varepsilon}\mathfrak{g}^{\gamma\varepsilon} + \mathfrak{a}_{\alpha}{}^{\gamma}_{\delta,\gamma} - \mathfrak{a}_{\alpha}{}^{\varepsilon}_{\gamma}\mathfrak{a}_{\delta}{}^{\gamma}_{\varepsilon})\mathfrak{g}^{\delta\beta}$$

$$= \mathfrak{B}^\beta_\alpha - \mathfrak{B}^\gamma_\gamma \delta^\beta_\alpha .$$

We shall prove the following.

Theorem 3.1. *If the tensor field $\mathfrak{B}_{\alpha\beta}$ is covariant constant, MINKOWSKI space must be EUCLIDean.*

Proof. As the condition of the integrability of partial differential equations of (3.1) we have

$$\text{(3.2)} \qquad \mathfrak{R}_{\alpha\beta\gamma\delta} = \mathfrak{a}_{\alpha\beta\gamma,\delta} - \mathfrak{a}_{\alpha\beta\delta,\gamma} + \mathfrak{a}_{\alpha}{}^{\varepsilon}_{\gamma}\mathfrak{a}_{\beta\delta,\varepsilon} - \mathfrak{a}_{\alpha}{}^{\varepsilon}_{\delta}\mathfrak{a}_{\beta\gamma\varepsilon} + \mathfrak{g}_{\alpha\gamma}\mathfrak{B}_{\beta\delta} - \mathfrak{g}_{\alpha\delta}\mathfrak{B}_{\beta\gamma}$$

$$\mathfrak{B}_{\alpha\beta,\gamma} - \mathfrak{B}_{\gamma\beta,\alpha} + \mathfrak{B}_{\alpha\delta}\mathfrak{a}_{\beta}{}^{\delta}_{\gamma} - \mathfrak{B}_{\gamma\delta}\mathfrak{a}_{\beta}{}^{\delta}_{\alpha} = 0 .$$

The second equation of (3.2) is by hypothesis reduced to

$$\text{(3.3)} \qquad \mathfrak{B}_{\alpha\delta}\mathfrak{a}_{\beta}{}^{\delta}_{\gamma} - \mathfrak{B}_{\gamma\delta}\mathfrak{a}_{\beta}{}^{\delta}_{\alpha} = 0 .$$

Contracting with $\mathfrak{g}^{\alpha\beta}$ we have from the last equation

$$\text{(3.4)} \qquad \mathfrak{B}^{\alpha\delta}\mathfrak{a}_{\alpha\delta\gamma} = 0 .$$

On the other hand taking symmetric part of the first equation of (3.2) with respect to α, δ and contracting with $\mathfrak{g}^{\beta\delta}$ follows

$$\text{(3.5)} \qquad 2\mathfrak{a}_{\alpha}{}^{\beta}_{\gamma,\beta} + \mathfrak{g}_{\alpha\gamma}\mathfrak{B}^\beta_\beta - (n-1)\mathfrak{B}_{\alpha\gamma} = 0$$

from which contracting with $\mathfrak{g}^{\alpha\gamma}$ we obtain

$$\text{(3.6)} \qquad (\mathfrak{B}^\beta_\beta)^2 = (n-1)\mathfrak{B}_{\alpha\gamma}\mathfrak{B}^{\alpha\gamma}$$

$$\text{or} \qquad \left(\mathfrak{B}_{\alpha\gamma} - \frac{\mathfrak{B}^\beta_\beta}{n-1}\mathfrak{g}_{\alpha\gamma}\right)\left(\mathfrak{B}^{\alpha\gamma} - \frac{\mathfrak{B}^\varepsilon_\varepsilon}{n-1}\mathfrak{g}^{\alpha\gamma}\right) = 0$$

this is equivalent to

$$\text{(3.7)} \qquad \mathfrak{B}_{\alpha\beta} = \frac{\mathfrak{B}^\gamma_\gamma}{n-1}\mathfrak{g}_{\alpha\beta} .$$

On account of the last relation and (3.5) we have

$$\text{(3.8)} \qquad \mathfrak{a}_{\alpha}{}^{\beta}_{\gamma,\beta} = 0$$

which shows that M_n must be E_n.

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