A REMARK ON A CLOSED ORIENTABLE HYPERSURFACE WITH CONSTANT REDUCED MEAN CURVATURE

By

Yoshihiko TAZAWA

Introduction. The theorem concerning homothety given by A. Aeppli (Theorem I in [3]), which brings a relation between the global form and reduced mean curvature rH_1 (with respect to a fixed point O) of a closed orientable hypersurface in $(n+1)$-dimensional Euclidean space R^{n+1}, suggests us to investigate the global property of a closed orientable hypersurface with constant reduced mean curvature $rH_1=c$. A star-shaped hypersurface (with respect to O) with $rH_1=c$ is a hypersphere around O (cf. footnotes 10), 11) in [3]). So we can confine ourselves to the case of non-star-shaped closed hypersurfaces. In this paper we are going to show that under a certain condition ("radial convexity", defined in § 1) such hypersurfaces cannot have constant reduced mean curvature, and that any ovaloid with $rH_1=c$ is a hypersphere around O:

Theorem. *Let F be a closed orientable hypersurface of class 2 which is radially convex with respect to O in the strict sense. For $p\in F$ we denote by p^* the second point on F which lies on the ray with p. Then there exists at least one point $p\in F$, for which $rH_1(p)\neq rH_1(p^*)$.*

Corollary. *Let F be a closed orientable hypersurface of class 2 which is star-shaped or radially convex in the strict sense with respect to O. Then, if $rH_1(p)=c$ for all $p\in F$, F is a hypersphere around O. Especially any ovaloid with $rH_1=c$ is a hypersphere around O.*

In § 1 we show the existence of a centralprojection from a radially convex hypersurface onto itself, which reverses the orientation. In § 2 we prove a lemma, which is closely related to the theorem concerning homothety. Thereby the theorem and the corollary are proved immediately in § 3.

The auther wishes to express here his sincere thanks to Professor Yoshie Katsurada for her kind guidance.

§ 1. Radially Convex Hypersurfaces.

We follow the notations in [2] and [3]. Let O be a fixed point in $(n+1)$-dimensional Euclidean space R^{n+1}, $n\geqq 2$.

Definition. *We call the hypersurface F radially convex (with respect to O), if $O \notin F$ and any half straight line with its endpoint O (i.e., a ray from O) has either two intersecting points with F, only one contact point or no common point. We call F radially convex in the strict sense, if F is radially convex and has an additional condition such that the direction of any ray which is tangent to F is not an asymptotic direction of F at the contact point.*

If we put the point O infinitely far away in a certain fixed direction $\mathfrak{r}$, radial convexity becomes "convexity in the direction $\mathfrak{r}$" (cf. [1] p. 187, [2] p. 192). We may consider that O is the origin of R^{n+1}. Throughout this section let F be a closed orientable hypersurface of class 2, which is radially convex in the strict sense. We denote by F_0 a subset of F given by

$$F_0 := \{p \in F \mid (\mathfrak{n}\mathfrak{x})(p) = 0\} \qquad \text{(the "}\textit{Schattengrenze}\text{" of } F\text{)},$$

where $\mathfrak{x} = (x^1, \cdots, x^{n+1})$ is the position vector of F and $\mathfrak{n}$ is the normal unit vector of F.

We obtain a mapping $p^* = \varphi(p)$ from F onto itself, if for any point p on F we take the ray from O through p and choose the second intersecting point p^* on F (if the ray is tangent to F at p, i.e., $p \in F_0$, let $\varphi(p) = p$). We are to show that φ is a regular centralprojection of class 2, which reverses the orientation.

It is clear that φ is bijective and $\varphi \circ \varphi = id$, that is, $\varphi^{-1} = \varphi$. We have to prove only the continuity in oder to show that φ is a homeomorphism (and therefore a centralprojection). For this purpose we consider first a local mapping π_p for $p \in F$.

Let S^n be the unit n-sphere around O and be oriented with the outer normal vector. F and S^n are given by

$$\mathfrak{x} = \mathfrak{x}(u^1, \cdots, u^n) = (x^1(u^i), \cdots, x^{n+1}(u^i)),$$
$$\tilde{\mathfrak{x}} = \tilde{\mathfrak{x}}(\tilde{u}^1, \cdots, \tilde{u}^n) = (\tilde{x}^1(\tilde{u}^i), \cdots, \tilde{x}^{n+1}(\tilde{u}^i))$$

respectively. Let π be a mapping $\pi : F \to S^n$ expressed by

$$\pi((x^1, \cdots, x^{n+1})) := (x^1, \cdots, x^{n+1})/\rho, \qquad \text{i.e.,} \quad \tilde{x}^A = \rho^{-1} x^A,$$

where ρ is the function of class 2 on F given by

$$\rho(p) := \|\mathfrak{x}(p)\| = \left\{\sum_{A=1}^{n+1} (x^A)^2\right\}^{\frac{1}{2}},$$

i.e., the distance between p and O. $\rho(p) \neq 0$ for all $p \in F$, so π is well defined all over F and of class 2. Since $x^A(p) \neq 0$ for at least one A, we can assume now $x^1(p)>0$ without losing generality (after a suitable rotation of the coordinate axes, if necessary). It follows that also $\tilde{x}^1(\pi(p))>0$, so we can take $(\tilde{x}^2, \cdots, \tilde{x}^{n+1})$ as a local coordinate of S^n at $\pi(p)$, preserving the orientation, that is, $\tilde{u}^i=\tilde{x}^{i+1}$ for $i=1, \cdots, n$. Let J be the functionaldeterminant of π.

$$J := \frac{\partial(\tilde{u})}{\partial(u)} = det\left|\frac{\partial \tilde{x}^{i+1}}{\partial u^j}\right| = |(\rho^{-1}x^{i+1})_j| = \rho^{-n-2}x^1\sqrt{g}\,(\mathfrak{n}\mathfrak{x})\,.$$

Therefore the sign of $J(p)$ depends only upon $(\mathfrak{n}\mathfrak{x})(p)$ and, if $(\mathfrak{n}\mathfrak{x})(p)\neq 0$, there exists a neighbourhood U_p of p, such that $\pi_p := \pi_{|U_p}$ is diffeomorphic.

If $p \notin F_0$, then it follows that $p^* = \varphi(p) \notin F_0$, accordingly there are two local diffeomorphic mappings $\pi_p : U_p \to S^n$ and $\pi_{p^*} : U_{p^*} \to S^n$, for which

$$\pi_p(p) = \pi_{p^*}(p^*)\,, \qquad \pi_p(U_p) \cap \pi_{p^*}(U_{p^*}) \neq \emptyset\,,$$
$$\varphi_{|W} = \pi_{p^*}^{-1} \circ \pi_{p|W}\,, \qquad \text{where } W := \pi_p^{-1}(\pi_p(U_p) \cap \pi_{p^*}(U_{p^*}))\,.$$

Therefore φ is continuous at p.

If $p \in F_0$, let $\{p_n\}$ be an arbitrary sequence of points on F with $\lim_{n\to\infty} p_n = p$. The sequence $\{\varphi(p_n)\}$ on F has at least one convergent point p_0, because F is compact. Accordingly there exists a subsequence $\{p_{i_n}\}$ of $\{p_n\}$ such that $\lim_{n\to\infty} \varphi(p_{i_n}) = p_0$. Then, because of the continuity of π, the two sequences on S^n $\{\pi(p_{i_n})\}$ and $\{\pi(\varphi(p_{i_n}))\}$ converge to $\pi(p)$ and $\pi(p_0)$ respectively. Since $\pi \circ \varphi = \pi$, these two sequences coincide completely, that means $\pi(p_0) = \pi(p)$, in other words, p_0 and p lie on the same ray. On the other hand since $p \in F_0$, it follows that $p_0 = p$, that is, all convergent points of $\{\varphi(p_n)\}$ coincide with p and $\{\varphi(p_n)\}$ converge to $p = \varphi(p)$. So φ is continuous at p.

The differentiability and the regularity of φ comes out now directly from the radial convexity of F in the strict sense (cf. footnote 2) in [3]). Therefore φ is a regular centralprojection of class 2 from F onto itself. In oder to show that φ reverses the orientation of F, we consider next two connected subsets F_+ and F_- of F, such that $F = F_+ \cup F_0 \cup F_-$.

We construct a continuous function ψ on F by

$$\psi(p) := \rho(p) - \rho(\varphi(p))\,.$$

$\psi(p) = 0$ if and only if $p \in F_0$. We put

$$F_+ := \left\{p \in F \,|\, \psi(p) > 0\right\}, \qquad F_- := \left\{p \in F \,|\, \psi(p) < 0\right\},$$

and so $F = F_+ \cup F_0 \cup F_-$, $F_0 \cap F_+ = F_0 \cap F_- = F_+ \cap F_- = \emptyset$. Because of the continuity of ψ, F_+ and F_- are open and mapped by φ mutually onto themselves. And F_0 is not empty. Because, if $F_+ = \emptyset$ (and so $F_- = \emptyset$), it follows that $F_0 = F \neq \emptyset$; if $F_+ \neq \emptyset$ (and so $F_- \neq \emptyset$), ψ has both positive and negative values and so because of the continuity of ψ, $F_0=\psi^{-1}(\{0\})$ is not empty.

The gradient of the function $\mathfrak{n}\mathfrak{x}$ of class 1 on F is not equal to zero at $p\in F_0$. Because the vector $\mathfrak{x}(p)$ lies in the tangent plane of F at p, i.e., $\mathfrak{x}(p)=\alpha^i\mathfrak{x}_i(p)$, and differs from any asymptotic direction at p, i.e., $l_{jk}(p)\alpha^j\alpha^k\neq 0$. Since the i-th component of $\mathrm{Grad}(\mathfrak{n}\mathfrak{x})$ is $(\mathfrak{n}\mathfrak{x})_i=-l_{ki}\alpha^k$, so $(\mathfrak{n}\mathfrak{x})_i(p)\neq 0$ for at least one i, i.e., $\mathrm{Grad}(\mathfrak{n}\mathfrak{x})(p)$ is not equal to zero. Accordingly any neighbourhood of p posesses a point, at which $\mathfrak{n}\mathfrak{x}\neq 0$, namely the point of $F-F_0$, that is, F_0 has no inner point (and so $F-F_0\neq\emptyset$ and F_+ and F_- are not empty). We can therefore choose a sequence $\{p_n\}$ on $F-F_0$, that converges to p. Since φ is continuous, the sequence $\{\varphi(p_n)\}$ converges to $\varphi(p)=p$. If we construct a sequence $\{q_n\}$ by $q_{2n-1}:=p_n$, $q_{2n}:=\varphi(p_n)$, it is convergent to p, and because of $\varphi(q_{2n-1})=q_{2n}$, one of q_{2n} and q_{2n-1} belongs to F_+ and the other to F_-. Therefore p is a boundary point of both F_+ and F_-, that is, $F_0\subset\partial F_+\cap\partial F_-$. Since $\overline{F}_+\cap F_-=\emptyset$, $\partial F_+=\overline{F}_+-F_+\subset F-F_+-F_-=F_0$. In the same way $\partial F_-\subset F_0$. Therefore F_0 is the common boundary of F_+ and F_-.

Let $u=(u^i)$ be the local coordinate at $p\in F_0$. Since $\mathrm{Grad}(\mathfrak{n}\mathfrak{x})(p)\neq 0$, we assume now $(\mathfrak{n}\mathfrak{x})_n\neq 0$ at p. Then F_0 is given by $\mathfrak{n}\mathfrak{x}(u^i)=0$, and because of $(\mathfrak{n}\mathfrak{x})_n(p)\neq 0$, this is equivalent to $u^n=h(u^1,\cdots,u^{n-1})$ in a suitable neighbourhood U of p, where h is a function of class 1. We introduce a new parameter $v=(v^i)$ by

$$\begin{aligned} v^n &:= u^n-h(u^1,\cdots,u^{n-1}), \\ v^i &:= u^i-u^i(p) \qquad \text{for} \quad i=1,\cdots,n-1. \end{aligned}$$

Although v is not a local coordinate since it is not of class 2, v maps U C^1-diffeomorphic onto a neighbourhood $v(U)$ of the origin of the $(v^1,\cdots,v^n)$-space, $F_0\cap U$ to the $(v^1,\cdots,v^{n-1})$-plane and p to the origin. The neighbourhood $v(U)$ includes an open ball B around the origin. We put

$$\begin{aligned} &B_+ := \left\{(v^i)\in B \mid v^n>0\right\}, \qquad B_- := \left\{(v^i)\in B \mid v^n<0\right\}, \\ &B_0 := \left\{(v^i)\in B \mid v^n=0\right\}=v(F_0\cap U)\cap B \\ &V := v^{-1}(B). \end{aligned}$$

B_+ does not include points of both $v(F_+\cap V)$ and $v(F_-\cap V)$ at the same time. Because, if there exist two points $p_0\in B_+\cap v(F_+\cap V)$ and $p_1\in B_+\cap v(F_-\cap V)$, the segment s from $p_0=s(0)$ to $p_1=s(1)$ is included completely in B_+. We construct a curve in $v^{-1}(B_+)$ by $c:=v^{-1}\circ s$. If we put $\lambda(t):=\psi(c(t))$ for $t\in[0,1]$, it follows that $\lambda(0)>0$, $\lambda(1)<0$. Since λ is continuous, there exists a number $t_0\in[0,1]$ with $\lambda(t_0)=0$, i.e., $\psi(c(t_0))=0$ hence $c(t_0)\in F_0$. This is contradictory to the fact that $v^{-1}(B_+)$ includes no point of F_0 and that $c([0,1])\subset$

$v^{-1}(B_-)$. Therefore either $B_+=v(F_+\cap V)$ (accordingly $B_-=v(F_-\cap V)$) or $B_+=v(F_-\cap V)$ (accordingly $B_-=v(F_+\cap V)$). Because of the archwise connectedness of B_+ and B_-, $v^{-1}(B_+)$ and $v^{-1}(B_-)$ are archwise connected. Therefore any point p of F_0 posesses a neighbourhood V such that $V_+:=F_+\cap V$ and $V_-:=F_-\cap V$ are archwise connected.

Let p_0 and p_1 be arbitrary points of F_+. We show that there is a curve in F_+ from p_0 to p_1 and so F_+ is archwise connected (therefore connected). We can choose a curve $\tilde{c}:[0,1]\to F$ with $\tilde{c}(0)=p_0$ and $\tilde{c}(1)=p_1$, because F is archwise connected. We consider here a continuous mapping $\tau: F\to F_+\cup F_0$ defined by

$$\tau|_{F_+\cup F_0}:=id_{F_+\cup F_0}, \qquad \tau|_{F_-}:=\varphi|_{F_-}.$$

Then $c:=\tau\circ\tilde{c}$ is a curve in $F_+\cup F_0$ from p_0 to p_1.

We construct an open covering of $c([0,1])$ as follows; for $p=c(t)\in F_+$ let U_p be a neighbourhood of p, which is archwise connected and included in F_+; for $q\in F_0$ let V_q be a neighbourhood of q, for which $V_{q+}:=V_q\cap F_+$ is archwise connected, as seen above. Being compact, $c([0,1])$ is covered by a finite number of U_{p_i} and V_{q_j}, i.e., $\left(\bigcup_{i=1}^{\mu} U_{p_i}\right)\cup\left(\bigcup_{j=1}^{\nu} V_{q_j}\right)\supset c([0,1])$. We put $A:=\left(\bigcup_{i=1}^{\mu} U_{p_i}\right)\cup\left(\bigcup_{j=1}^{\nu} V_{q_j+}\right)$, and so $A\subset F_+$ and $p_0, p_1\in A$. We show that A is archwise connected. If $\nu=0$, it is clear, because $c([0,1])\subset A$ and $U_{p_i}\cap c([0,1])\neq\emptyset$ for all i. If $\nu\neq 0$, among U_{p_i} and V_{q_j} there exists at least one neighbourhood which posesses a common point of $c([0,1])$ with V_{q_1}. In the case that U_{p_i} posesses a common point $c(t_i)$ with V_{q_1}, $U_{p_i}\cap V_{q_1+}\neq\emptyset$, and so $U_{p_i}\cup V_{q_1+}$ is archwise connected. In the case that V_{q_j} posesses a common point $c(t_j)$ with V_{q_1}, we choose a neighbourhood W of $c(t_j)$, which is included in $V_{q_j}\cap V_{q_1}$. Since $c(t_j)\in F_+\cup F_0=\bar{F}_+$, W posesses a point q of F_+ and $q\in V_{q_j+}\cap V_{q_1+}$ and so $V_{q_j+}\cup V_{q_1+}$ is archwise connected. If we continue this process finite times, A becomes at last archwise connected. So we can choose a curve $\bar{c}:[0,1]\to A\subset F_+$ with $\bar{c}(0)=p_0$ and $\bar{c}(1)=p_1$. Hence F_+ is connected. Likewise F_- is connected.

Because of the connectedness of F_+ and F_-, and because $(\mathfrak{n}\mathfrak{x})(p)=0$ if and only if $p\in F_0$, the continuous function $\mathfrak{n}\mathfrak{x}$ has a fixed sign on F_+ and F_- respectively. Since $\mathrm{Grad}(\mathfrak{n}\mathfrak{x})(p)\neq 0$ for $p\in F_0$, $\mathfrak{n}\mathfrak{x}$ exchanges the sign in a neighbourhood of $p\in F_0$. Therefore it follows that $\mathfrak{n}\mathfrak{x}>0$ on one of F_+ and F_-, and $\mathfrak{n}\mathfrak{x}<0$ on the other.

Let $p\in F-F_0$ and $p^*=\varphi(p)$. φ is given in a suitable neighbourhood W of p by $\varphi=\pi_{p^*}^{-1}\circ\pi_p$. Since φ maps F_+ and F_- mutually onto themselves, one of p and p^* belongs to F_+ and the other to F_-, and so $(\mathfrak{n}\mathfrak{x})(p)$ and $(\mathfrak{n}\mathfrak{x})(p^*)$

have different signs. Accordingly the functionaldeterminants $J(p)$ and $J(p^*)$ of π_p and π_{p^*} at p and p^* respectively have different signs, because the sign of $J(q)$ depends only upon $(\mathfrak{n}\mathfrak{x})(q)$ for $q\in F-F_0$. Therefore the functionaldeterminant of $\pi_{p^*}^{-1}\circ\pi_p$ has the negative sign, that is, the sign of the functionaldeterminant of φ is negative on $F-F_0$.

Since φ is everywhere regular, i.e., the functionaldeterminant of φ is nowhere zero, it is negative all over F because of the continuity. Hence φ reverses the orientation of F. So we have proved the following lemma:

Lemma 1. *Let F be a closed orientable hypersurface of class 2 which is radially convex in the strict sense. Then there exists a regular centralprojection of class 2 from F onto itself, which reverses the orientation.*

§ 2. A Lemma on a Centralprojection.

Lemma 2. *Let F and $\bar{F}$ be closed orientable hypersurfaces of class 2 which hold a centralprojection $T: F\to\bar{F}$. Let T be of class 2 and regular and preserve the orientation. Let the "Schattengrenze" of F ($F_0:=\{p\in F\mid(\mathfrak{n}\mathfrak{x})(p)=0\}$) include no inner point. Then there exists at least one point $p\in F$ with $\bar{r}\bar{H}_1(T_p)\neq -rH_1(p)$, where rH_1 and $\bar{r}\bar{H}_1$ are reduced mean curvatures with respect to O of F and $\bar{F}$ respectively.*

Proof. We prove following the proof of Theorem I in [3]. F and $\bar{F}$ are given by

$$\mathfrak{x}=\mathfrak{x}(u^1,\cdots,u^n),$$
$$\bar{\mathfrak{x}}=\bar{\mathfrak{x}}(u^1,\cdots,u^n)=f(u^1,\cdots,u^n)\,\mathfrak{x}(u^1,\cdots,u^n)\quad(f>0)\qquad\text{(a)}$$

respectively. If we exchange "$\mathfrak{n}-\bar{\mathfrak{n}}$" of the formula (1.5) in [3] for "$\mathfrak{n}+\bar{\mathfrak{n}}$", we obtain

$$d(\mathfrak{n}+\bar{\mathfrak{n}},\mathfrak{x},d\mathfrak{x},\cdots,d\mathfrak{x})=(\mathfrak{n}+\bar{\mathfrak{n}},d\mathfrak{x},\cdots,d\mathfrak{x})-(\mathfrak{x},d\mathfrak{n}+d\bar{\mathfrak{n}},d\mathfrak{x},\cdots,d\mathfrak{x})\qquad\text{(b)}$$

Then the formulae (1.6) and (1.7) in [3] are exchanged for

$$(\mathfrak{n}+\bar{\mathfrak{n}},d\mathfrak{x},\cdots,d\mathfrak{x})=\frac{1}{2}n!(\mathfrak{n}+\bar{\mathfrak{n}})^2dA\qquad\text{(c)}$$

$$-(\mathfrak{x},d\mathfrak{n}+d\bar{\mathfrak{n}},d\mathfrak{x},\cdots,d\mathfrak{x})=n!(H_1+f\bar{H}_1)(\mathfrak{n}\mathfrak{x})\,dA\qquad\text{(d)}$$

respectively. From (b), (c), (d) and the Formula of Stokes we have

$$\frac{1}{n!}\int_{\partial F}(\mathfrak{n}+\bar{\mathfrak{n}},\mathfrak{x},d\mathfrak{x},\cdots,d\mathfrak{x})=\frac{1}{2}\int_F(\mathfrak{n}+\bar{\mathfrak{n}})^2dA+\int_F(rH_1+\bar{r}\bar{H}_1)(\mathfrak{n}\mathfrak{r})dA\qquad\text{(e)}$$

where $\mathfrak{r}=\dfrac{\mathfrak{x}}{r}$. If we assume here that $rH_1(p)=-\bar{r}\bar{H}_1(T_p)$ for all $p\in F$, then

$$\int_F (\mathfrak{n}+\bar{\mathfrak{n}})^2 dA = 0\,,$$

so that $\mathfrak{n} = -\bar{\mathfrak{n}}$. Therefore

$$0 = \bar{\mathfrak{x}}_i \bar{\mathfrak{n}} = (f\mathfrak{x})_i(-\mathfrak{n}) = -f_i(\mathfrak{x}\mathfrak{n}) - f(\mathfrak{x}_i\mathfrak{n}) = -f_i(\mathfrak{x}\mathfrak{n})$$

for $i = 1, \cdots, n$, and the "*Schattengrenze*" including no inner point, it follows that $f_i = 0$, i.e., $f = \text{const}$. Accordingly the hypersurface $\bar{F}$ is given by $f\mathfrak{x}$ with positive constant f, and so $\mathfrak{n} = \bar{\mathfrak{n}}$, which is contradictory to $\mathfrak{n} = -\bar{\mathfrak{n}}$. Q.E.D.

In the case of parallelmappings this conclusion is not true. Because, if we exchange "$\bar{\mathfrak{n}} - \mathfrak{n}$" of the formula (9.5) in [2] for "$\bar{\mathfrak{n}} + \mathfrak{n}$", we obtain

$$n\int_F (\bar{H}_1 + H_1)(\mathfrak{w}\mathfrak{n})\, dA + \frac{1}{2}\int_F (\bar{\mathfrak{n}}+\mathfrak{n})^2 (d\bar{A} - dA)$$
$$= \frac{1}{(n-1)!}\int_{\partial F} (\bar{\mathfrak{n}}+\mathfrak{n}, \mathfrak{w}, d\mathfrak{x}, \cdots, d\mathfrak{x})\,.$$

Assuming $-H_1 = \bar{H}_1$, we obtain

$$\int_F (\bar{\mathfrak{n}}+\mathfrak{n})^2 (d\bar{A} - dA) = 0\,.$$

Since $d\bar{A} - dA$ does not hold a fixed sign, we cannot conclude $\mathfrak{n} = -\mathfrak{n}$.

§ 3. A Closed Hypersurface with $rH_1 = c$.

Theorem. *Let F be a closed orientable hypersurface of class 2 which is radially convex with respect to O in the strict sense. For $p \in F$ we denote by p^* the second point on F, which lies on the ray with p. Then there exists at least one point $p \in F$, for which $rH_1(p) \neq rH_1(p^*)$.*

Proof. According to Lemma 1 there exists a regular centralprojection φ of class 2 from F onto itself, which reverses the orientation. Let $\tilde{F}$ be a hypersurface which is identical to F except the orientation, so that the mapping $\varphi: F \to \tilde{F}$ preserves the orientation. By exchanging the orientation of F $H_1(p) = -\tilde{H}_1(p)$, and so it follows that $rH_1(p) = -\tilde{r}\tilde{H}_1(p)$. Since $\varphi: F \to \tilde{F}$ satisfies the assumption of Lemma 2, there exists at least one point $p \in F$ such that $rH_1(p) \neq -\tilde{r}\tilde{H}_1(\varphi(p)) = rH_1(\varphi(p)) = rH_1(p^*)$. Q.E.D.

Corollary. *Let F be a closed orientable hypersurface of class 2 which is star-shaped or radially convex in the strict sense with respect to O. Then, if $rH_1(p) = c$ for all $p \in F$, F is a hypersphere around O. Especially any ovaloid with $rH_1 = c$ is a hypersphere around O.*

Proof. The conclusion comes out immediately from Theorem above and the footnotes 10) and 11) in [3]. Since an n-dimensional ovaloid is a closed hypersurface with the positive (or negative) definite second fundamental form, which is the boundary of a bounded convex set in R^{n+1}, the following is verified without difficulty; if an ovaloid has O outer side, it is radially convex with respect to O in the strict sense; if it has O inner side, it is star-shaped with respect to O. If O lies on an ovaloid, then $rH_1=0$ at O and it cannot have constant reduced mean curvature. Q.E.D.

In the case of a closed curve C in R^2, the analogous definition of radial convexity in the strict sense (*we call C radially convex in the strict sense, if C is radially convex and further* $Grad(\mathfrak{n}\mathfrak{x})(p)\neq 0$ *for* $(\mathfrak{n}\mathfrak{x})(p)=0$ $p\in C$) and the integral formula

$$(\mathfrak{t}+\bar{\mathfrak{t}})\,\mathfrak{x}\Big|_{\partial C}=\frac{1}{2}\int_C(\mathfrak{t}+\bar{\mathfrak{t}})^2 ds+\int_C(rk+\bar{r}\bar{k})(\Re\,\mathfrak{n})\,ds$$

bring the analogous conclusions.

References

[1] H. Hopf und K. Voss: Ein Satz aus der Flächentheorie im Großen, Archiv der Math. 3 (1952), 187–192.

[2] K. Voss: Einige differentialgeometrische Kongruenzsätze für geschlossene Flächen und Hyperflächen, Math. Ann. 131 (1956), 180–218.

[3] A. Aeppli: Einige Ähnlichkeits- und Symmetriesätze für differenzierbare Flächen im Raum, Comment. Math. Helv. 33 (1959), 174–195.

Department of Mathematics,
Hokkaido University.

(Received May 15, 1968)