ON A CERTAIN PROPERTY OF A CLOSED HYPERSURFACE IN A RIEMANN SPACE

By

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Introduction. Recently the properties of closed hypersurfaces in a Riemann space have been investigated by Y. Katsurada [1][1], [2], K. Yano [3], T. Ōtsuki [4] and others. First Y. Katsurada proved the following theorem with respect to the ν-th mean curvature $H_\nu=$constant $(\nu=1, 2, \cdots, m-1)$:

Theorem 0.1. *Let R^{m+1} be a Riemann space with constant Riemann curvature, V^m a closed orientable hypersurface in R^{m+1}. If there exists a one-parameter group G of conformal transformations of R^{m+1} such that the scalar product $n^i\xi_i$ of the normal vector n of V^m and the generating vector ξ of G does not change the sign on V^m and is not identically zero, and if the principal curvatures $k_1, k_2, \cdots, k_m$ on V^m are positive and H_ν is constant for any $\nu\,(1\leqq\nu\leqq m-1)$, then every point of V^m is umbilic.* ([1], p. 291)

The purpose of the present paper, especially in the only case of $\nu=2$, is to investigate an analoguous property in a more general Riemann space admitting a one-parameter group of conformal transformations. In § 1 the integral formula of Minkowski type given by Y. Katsurada [2] which holds in arbitrary Riemann space is explained. In § 2, we give an integral formula for a closed orientable hypersurface with certain special second fundamental tensor in a Riemann space admitting a one-parameter group of conformal transformations. And in § 3 a property of a closed orientable hypersurface with such the second fundamental tensor is discussed.

The author wishes to express to Prof. Y. Katsurada his very sincere thanks for her kind guidance.

§ 1. The generalized Minkowski formula (I). We consider a Riemann space $R^{m+1}(m\geqq 3)$ of class $C^\nu(\nu\geqq 3)$ which admits a one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\,\delta\tau \tag{1.1}$$

(where x^i are local coordinates in R^{m+1} and ξ^i are the components of a con-

1) Numbers in brackets refer to the references at the end of paper.

travariant vector ξ). We suppose that the paths of these transformations cover R^{m+1} simly and that ξ is everywhere continuous and $\neq 0$. If ξ is a conformal Killing vector, then the group G is called conformal.

We now consider a closed orientable hypersurface V^m of class C^3 imbedded in R^{m+1}, locally given by

$$x^i = x^i(u^\alpha);$$

here and henceforth, Latin indices run from 1 to $m+1$ and Greek indices from 1 to m.

To the vector ξ introduced above, there is a covariant vector $\bar{\xi}$ of V^m with the components

$$\bar{\xi}_\alpha = B^i_\alpha \xi_i,$$

where B^i_α mean $\partial x^i/\partial u^\alpha$ and ξ_i the covariant components of ξ; we shall compute its covariant derivatives along V^m: by virtue of the fact that the covariant derivatives of B^i_α are

$$\frac{\delta}{\partial u^\beta}(B^i_\alpha) = b_{\alpha\beta} n^i$$

where $b_{\alpha\beta}$ is the second fundamental tensor and n^i the unit normal vector of V^m, we find

$$\bar{\xi}_{\alpha;\beta} = b_{\alpha\beta} n^i \xi_i + B^i_\alpha B^j_\beta \xi_{i;j} \tag{1.2}$$

(the symbol ";" always means the covariant derivative). Multiplying (1.2) by the contravariant metric tensor $g^{\alpha\beta}$ of V^m and summing for α and β, we get

$$g^{\alpha\beta}\bar{\xi}_{\alpha;\beta} = mH_1 n^i \xi_i + \frac{1}{2} g^{\alpha\beta} B^i_\alpha B^j_\beta \underset{\xi}{\mathfrak{L}}\, g_{ij}, \tag{1.3}$$

where H_1 is the first mean curvature $\frac{1}{m} g^{\alpha\beta} b_{\alpha\beta}$ of V^m and $\underset{\xi}{\mathfrak{L}}\, g_{ij}$ is the Lie derivative of the fundamental tensor g_{ij} of R^{m+1} with respect to the infinitesimal transformation (1.1). If we put

$$\underset{\xi}{\mathfrak{L}}\, g_{\alpha\beta} = B^i_\alpha B^j_\beta \underset{\xi}{\mathfrak{L}}\, g_{ij}, \tag{1.4}$$

then (1.3) is rewritten as follows:

$$\frac{1}{m}\bar{\xi}^\alpha_{;\alpha} \equiv \frac{1}{m} g^{\alpha\beta}\bar{\xi}_{\alpha;\beta} = H_1 n^i \xi_i + \frac{1}{2m} g^{\alpha\beta} \underset{\xi}{\mathfrak{L}}\, g_{\alpha\beta}.$$

dA being the area element of V^m, we have

$$\int_{V^m}\cdots\int \xi^{\alpha}_{;\alpha}\, dA = 0\,, \qquad ([5], \text{ p. } 31)$$

because V^m is closed and orientable. Thus we obtain the integral formula

$$\text{(I)} \qquad \int_{V^m}\cdots\int H_1 n^i \xi_i\, dA + \frac{1}{2m}\int_{V^m}\cdots\int g^{\alpha\beta} \underset{\xi}{\mathfrak{L}}\, g_{\alpha\beta}\, dA = 0\,.$$

Let the group G be conformal, that is, ξ satisfy the equation

$$\underset{\xi}{\mathfrak{L}}\, g_{ij} = \xi_{i;j} + \xi_{j;i} = 2\Phi g_{ij}\,,$$

then (I) becomes

$$\text{(I)}_c \qquad \int_{V^m}\cdots\int H_1 n^i \xi_i\, dA + \int_{V^m}\cdots\int \Phi\, dA = 0 \qquad ([2], \text{ p. } 167)\,.$$

§ 2. An integral formula for a closed orientable hypersurface with certain special second fundamental tensor. We first take the covariant vector ${}_1\eta$ of V^m, whose components are given by

$${}_1\eta_\gamma = (b^\alpha_\alpha)^2 B^i_\gamma \xi_i$$

and calculate its covariant derivatives along V^m, we have

$${}_1\eta_{\gamma;\theta} = (b^\alpha_\alpha)^2_{;\theta} B^i_\gamma \xi_i + (b^\alpha_\alpha)^2 B^i_{\gamma;\theta}\xi_i + (b^\alpha_\alpha)^2 B^i_\gamma B^j_\theta \xi_{i;j}\,.$$

Remembering the following Gauss' formulas for a hypersurface

$$B^i_{\beta;\gamma} = b_{\beta\gamma} n^i\,,$$

we find that

$${}_1\eta_{\gamma;\theta} = (b^\alpha_\alpha)^2_{;\theta} B^i_\gamma \xi_i + (b^\alpha_\alpha)^2 b_{\gamma\theta} n^i \xi_i + (b^\alpha_\alpha)^2 B^i_\gamma B^j_\theta \xi_{i;j}\,.$$

Multiplying by $g^{\gamma\theta}$ and summing for γ and θ, we obtain

$$\text{(2. 1)} \qquad g^{\gamma\theta}\,{}_1\eta_{\gamma;\theta} = (b^\alpha_\alpha)^2_{;\theta} g^{\gamma\theta} B^i_\gamma \xi_i + (b^\alpha_\alpha)^3 n^i \xi_i + (b^\alpha_\alpha)^2 g^{\gamma\theta} B^i_\gamma B^j_\theta \xi_{i;j}\,.$$

Moreover we consider the three covariant vectors ${}_2\eta$, ${}_3\eta$, ${}_4\eta$ of V^m whose components are given by

$${}_2\eta_\alpha = b^\beta_\alpha b^\gamma_\beta B^i_\gamma \xi_i\,,$$

$${}_3\eta_\gamma = (b^\beta_\alpha b^\alpha_\beta)\, B^i_\gamma \xi_i\,,$$

$${}_4\eta_\beta = (b^\alpha_\alpha)\, b^\gamma_\beta B^i_\gamma \xi_i$$

respectively. And by making use of the similar calculation mentioned above, we obtain

$$(2.2)\qquad g^{\alpha\theta}{}_2\eta_{\alpha;\theta}=(b^\beta_\alpha b^\gamma_\beta)_{;\theta}g^{\alpha\theta}B^i_\gamma\xi_i+b^\beta_\alpha b^\gamma_\beta b^\alpha_\gamma n^i\xi_i+b^\beta_\alpha b^\gamma_\beta g^{\alpha\theta}B^i_\gamma B^j_\theta\xi_{i;j}\,,$$

$$(2.3)\qquad g^{\gamma\theta}{}_3\eta_{\gamma;\theta}=(b^\beta_\alpha b^\alpha_\beta)_{;\theta}g^{\gamma\theta}B^i_\gamma\xi_i+b^\gamma_\gamma(b^\beta_\alpha b^\alpha_\beta)n^i\xi_i+(b^\beta_\alpha b^\alpha_\beta)g^{\gamma\theta}B^i_\gamma B^j_\theta\xi_{i;j}$$

and

$$(2.4)\qquad g^{\beta\theta}{}_4\eta_{\beta;\theta}=(b^\alpha_\alpha b^\gamma_\beta)_{;\theta}g^{\beta\theta}B^i_\gamma\xi_i+b^\alpha_\alpha(b^\gamma_\beta b^\beta_\gamma)n^i\xi_i+b^\alpha_\alpha b^\gamma_\beta g^{\beta\theta}B^i_\gamma B^j_\theta\xi_{i;j}\,.$$

Accordingly putting $\bar{\eta}_{\gamma;\theta}={}_1\eta_{\gamma;\theta}+2{}_2\eta_{\gamma;\theta}-{}_3\eta_{\gamma;\theta}-2{}_4\eta_{\gamma;\theta}$, from (2. 1), (2. 2), (2. 3) and (2. 4), we get

$$(2.5)\qquad \begin{aligned} g^{\gamma\theta}\bar{\eta}_{\gamma;\theta}=&\left\{(b^\alpha_\alpha)^2-b^\alpha_\beta b^\beta_\alpha\right\}_{;\theta}g^{\gamma\theta}B^i_\gamma\xi_i-2\left\{b^\alpha_\alpha b^\gamma_\beta-b^\alpha_\beta b^\gamma_\alpha\right\}_{;\theta}g^{\beta\theta}B^i_\gamma\xi_i\\ &+\left\{(b^\alpha_\alpha)^3+2b^\beta_\alpha b^\gamma_\beta b^\alpha_\gamma-3b^\alpha_\alpha(b^\gamma_\beta b^\beta_\gamma)\right\}n^i\xi_i+\left\{(b^\alpha_\alpha)^2-b^\alpha_\beta b^\beta_\alpha\right\}g^{\gamma\theta}B^i_\gamma B^j_\theta\xi_{i;j}\\ &-2\left\{b^\alpha_\alpha b^\gamma_\beta-b^\alpha_\beta b^\gamma_\alpha\right\}g^{\beta\theta}B^i_\gamma B^j_\theta\xi_{i;j}\,.\end{aligned}$$

On the other hand, if we denote by $k_1, k_2, \cdots, k_m$ the principal curvatures of V^m, then the second mean curvature H_2 and the third mean curvature H_3 of V^m are respectively given by

$$(2.6)\qquad \binom{m}{2}H_2=\sum_{\alpha<\beta}k_\alpha k_\beta=\frac{1}{2}\left\{(b^\alpha_\alpha)^2-b^\alpha_\beta b^\beta_\alpha\right\}$$

and

$$(2.7)\qquad \binom{m}{3}H_3=\sum_{\alpha<\beta<\gamma}k_\alpha k_\beta k_\gamma=\frac{1}{3!}\left\{(b^\alpha_\alpha)^3+2b^\beta_\alpha b^\gamma_\beta b^\alpha_\gamma-3b^\alpha_\alpha(b^\gamma_\beta b^\beta_\gamma)\right\},$$

where $b_{\alpha\beta}$ means the second fundamental tensor of V^m, $b^\beta_\alpha=b_{\alpha\gamma}g^{\gamma\beta}$ and $b^\alpha_\alpha=b_{\alpha\beta}g^{\beta\alpha}$. Therefore (2. 5) may be written as follows:

$$(2.8)\qquad \begin{aligned} g^{\gamma\theta}\bar{\eta}_{\gamma;\theta}=&\,m(m-1)H_{2;\theta}g^{\gamma\theta}B^i_\gamma\xi_i-2C^\gamma_{\beta;\theta}g^{\beta\theta}B^i_\gamma\xi_i\\ &+m(m-1)(m-2)H_3n^i\xi_i+m(m-1)H_2g^{\gamma\theta}B^i_\gamma B^j_\theta\xi_{i;j}\\ &-2C^\gamma_\beta g^{\beta\theta}B^i_\gamma B^j_\theta\xi_{i;j}\,,\end{aligned}$$

where $C_{\beta\gamma}$ means $b^\alpha_\alpha b_{\beta\gamma}-b^\alpha_\beta b_{\alpha\gamma}$ and $C^\gamma_\beta=C_{\beta\alpha}g^{\alpha\gamma}$. In (2. 8), from the definition of $C_{\beta\gamma}$, we see easely that the $C^\gamma_\beta g^{\beta\theta}B^i_\gamma B^j_\theta$ of the final term of the right-hand member are symmetric in i and j. Consequently we obtain

$$(2.9)\qquad \begin{aligned} g^{\gamma\theta}\bar{\eta}_{\gamma;\theta}=&\,m(m-1)H_{2;\theta}g^{\gamma\theta}B^i_\gamma\xi_i-2C^\gamma_{\beta;\theta}g^{\beta\theta}B^i_\gamma\xi_i\\ &+m(m-1)(m-2)H_3n^i\xi_i+\frac{1}{2}m(m-1)H_2g^{\gamma\theta}\underset{\xi}{\pounds}g_{\gamma\theta}\\ &-C^\gamma_\beta g^{\beta\theta}\underset{\xi}{\pounds}g_{\gamma\theta}\,,\end{aligned}$$

by virtue of (1. 4).

We now assume that the group G of transformations is conformal, that is, $\underset{\xi}{\pounds} g_{ij}=2\Phi g_{ij}$, then the last two terms of the right-hand member of (2. 9) become in the form

$$\frac{1}{2} m(m-1) H_2 g^{\gamma\theta} \underset{\xi}{\pounds} g_{\gamma\theta} - C^{\gamma}_{\beta} g^{\beta\theta} \underset{\xi}{\pounds} g_{\gamma\theta} = m(m-1)(m-2)\Phi H_2 ,$$

on making use of (1. 4) and $C_{\beta\gamma} g^{\beta\gamma} = m(m-1) H_2$. In this case (2. 9) can be written as follows:

$$\text{(2. 10)} \qquad \begin{aligned} g^{\gamma\theta} \bar{\eta}_{\gamma;\theta} &= m(m-1) H_{2;\theta} g^{\gamma\theta} B^i_{\gamma} \xi_i - 2C^{\gamma}_{\beta;\theta} g^{\beta\theta} B^i_{\gamma} \xi_i \\ &\quad + m(m-1)(m-2)(H_3 n^i \xi_i + \Phi H_2) . \end{aligned}$$

Moreover if V^m is a closed orientable hypersurface with the second fundamental tensor $b_{\alpha\beta}$ such that $C_{\alpha\beta}$ are covariantly constant, both the first term and the second of the right-hand member of (2. 10) vanish, because of $C_{\alpha\beta} g^{\alpha\beta} = m(m-1) H_2$; it follows that (2. 10) becomes

$$\text{(2. 11)} \qquad \begin{aligned} \frac{1}{m(m-1)(m-2)} \bar{\eta}^{\alpha}_{;\alpha} &\equiv \frac{1}{m(m-1)(m-2)} g^{\gamma\theta} \bar{\eta}_{\gamma;\theta} \\ &= H_3 n^i \xi_i + \Phi H_2 , \end{aligned}$$

by virtue of dividing the two sides of (2. 10) by $m(m-1)(m-2)$. Since V^m is closed and orientable, we have

$$\int_{V^m} \cdots \int \bar{\eta}^{\alpha}_{;\alpha} dA = 0 .$$

Thus we finally reach the integral formula

$$\text{(II)}_c \qquad \int_{V^m} \cdots \int H_3 n^i \xi_i dA + \int_{V^m} \cdots \int \Phi H_2 dA = 0 .$$

§ 3. A property of a closed orientable hypersurface with the special second fundamental tensor in a Riemann space. In this section we shall prove the following two theorems.

Theorem 3. 1. *Let R^{m+1} be a Riemann space and V^m a closed orientable hypersurface in R^{m+1}. If there exists a one-parameter group G of conformal transformations of R^{m+1} such that the scalar product $n^i\xi_i$ of the normal vector n of V^m and the generating vector ξ of G does not change the sign (and $\neq 0$) on V^m, and if the principal curvatures $k_1, k_2, \cdots, k_m$ on V^m are positive and $C_{\alpha\beta}$ is covariantly constant, then every point of V^m is umbilic.*

Proof. From $C_{\alpha\beta} g^{\alpha\beta} = m(m-1) H_2$, H_2 is constant under our conditions.

Multiplying by H_2 the formula $(\mathrm{I})_c$ in § 1, we obtain

$$\int\cdots\int_{V^m} H_1 H_2 n^i \xi_i \, dA + \int\cdots\int_{V^m} \Phi H_2 dA = 0\,,$$

and subtracting from this the formula $(\mathrm{II})_c$ in § 2, we find

$$\int\cdots\int_{V^m} (H_1 H_2 - H_3)\, n^i \xi_i dA = 0\,. \tag{3. 1}$$

On the other hand, from

$$H_1 H_2 - H_3 = \frac{2}{m^2(m-1)} \sum_{\alpha} k_\alpha \sum_{\beta<\gamma} k_\beta k_\gamma - \frac{3!}{m(m-1)(m-2)} \sum_{\alpha<\beta<\gamma} k_\alpha k_\beta k_\gamma \tag{3. 2}$$

$$= \frac{3!}{m^2(m-1)(m-2)} \sum_{\alpha<\beta<\gamma} k_\alpha (k_\beta - k_\gamma)^2\,,$$

we see that

$$H_1 H_2 - H_3 \geqq 0 \qquad ([1],\ \text{p. } 292). \tag{3. 3}$$

From (3. 1), (3. 3) and the fact that $n^i \xi_i$ has a fixed sign we conclude that

$$H_1 H_2 - H_3 = 0\,,$$

from which, because of (3. 2), $k_1 = k_2 = \cdots = k_m$ at each point of V^m. This means that every point of V^m is umbilic.

We next assume that the Riemann space R^{m+1} under consideration is a space of constant curvature K: $R_{ijkl} = K(g_{ik} g_{jl} - g_{il} g_{jk})$ where R_{ijkl} is the curvature tensor of R^{m+1}. As well-known, a hypersurface in a Riemann space has the following property

$$R_{\alpha\beta\gamma\delta} = b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta} + R_{ijkl} B^i_\alpha B^j_\beta B^k_\gamma B^l_\delta\,, \tag{3. 4}$$

where $R_{\alpha\beta\gamma\delta}$ is the curvature tensor of V^m. Because of a space of constant curvature, (3. 4) becomes

$$R_{\alpha\beta\gamma\delta} = b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta} + K(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})\,.$$

Multiplying by $g^{\alpha\delta}$ and summing for α and δ, we have

$$R_{\beta\gamma} \equiv g^{\alpha\delta} R_{\alpha\beta\gamma\delta} = b^\alpha_\alpha b_{\beta\gamma} - b^\alpha_\beta b_{\alpha\gamma} - (m-1) K g_{\beta\gamma}\,,$$

from which, by covariant differentiation along V^m,

$$R_{\beta\gamma;\theta} = (b^\alpha_\alpha b_{\beta\gamma} - b^\alpha_\beta b_{\alpha\gamma})_{;\theta}\,, \quad \text{that is,} \quad R_{\beta\gamma;\theta} = C_{\beta\gamma;\theta}\,.$$

Thus, we have

Corollary. *Let R^{m+1} be a space of constant curvature and V^m a closed orientable Ricci symmetric hypersurface in R^{m+1}. We suppose that R^{m+1} admits a one-parameter group G of conformal transformations such that the scalar product $n^i\xi_i$ of the normal vector n of V^m and the generating vector ξ of G does not change the sign and is not identically zero on V^m, and the principal curvatures $k_1, k_2, \cdots, k_m$ on V^m are positive, then every point of V^m is umbilic.*

Finally we consider the hypersurface with $H_2=$constant and $C_{\alpha\theta;\beta}g^{\alpha\beta}=0$. Taking account of $C_{\alpha\beta}g^{\alpha\beta}=m(m-1)H_2$, it is evident that these conditions are weaker than $C_{\alpha\theta;\beta}=0$, and also we get $(\mathrm{II})_c$ from (2. 10). Thus we have

Theorem 3. 2. *Let R^{m+1} be a Riemann space and V^m a closed orientable hypersurface in R^{m+1}. If there exists a one-parameter group G of conformal transformations of R^{m+1} such that the scalar product $n^i\xi_i$ does not change the sign and is not identically zero on V^m, and if the principal curvatures $k_1, k_2, \cdots, k_m$ on V^m are positive, $H_2=$constant and $C_{\alpha\theta;\beta}g^{\alpha\beta}=0$, then every point of V^m is umbilic.*

Remark. Now we leave us the following question: "which is more general, Theorem 0. 1. given by Y. Katsurada or Theorem 3. 2. in this paper?"

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