ON A CERTAIN PROPERTY OF A CLOSED HYPERSURFACE IN A RIEMANN SPACE

By

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Introduction. Recently the properties of closed hypersurfaces in a Riemann space have been investigated by Y. Katsurada [1], [2], K. Yano [3], T. Otsuki [4] and others. First Y. Katsurada proved the following theorem with respect to the \( \nu \)-th mean curvature \( H_\nu \) = constant (\( \nu = 1, 2, \ldots, m-1 \)):

**Theorem 0.1.** Let \( \mathbb{R}^{n-1} \) be a Riemann space with constant Riemann curvature, \( V^n \) a closed orientable hypersurface in \( \mathbb{R}^{n-1} \). If there exists a one-parameter group \( G \) of conformal transformations of \( \mathbb{R}^{n-1} \) such that the scalar product \( n^i \xi_i \) of the normal vector \( n \) of \( V^n \) and the generating vector \( \xi \) of \( G \) does not change the sign on \( V^n \) and is not identically zero, and if the principal curvatures \( k_1, k_2, \ldots, k_m \) on \( V^n \) are positive and \( H_\nu \) is constant for any \( \nu (1 \leq \nu \leq m-1) \), then every point of \( V^n \) is umbilic. ([1], p. 291)

The purpose of the present paper, especially in the only case of \( \nu = 2 \), is to investigate an analogous property in a more general Riemann space admitting a one-parameter group of conformal transformations. In §1 the integral formula of Minkowski type given by Y. Katsurada [2] which holds in arbitrary Riemann space is explained. In §2, we give an integral formula for a closed orientable hypersurface with certain special second fundamental tensor in a Riemann space admitting a one-parameter group of conformal transformations. And in §3 a property of a closed orientable hypersurface with such the second fundamental tensor is discussed.

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§ 1. The generalized Minkowski formula (I). We consider a Riemann space \( \mathbb{R}^{n-1}(m \geq 3) \) of class \( C^\nu(\nu \geq 3) \) which admits a one-parameter continuous group \( G \) of transformations generated by an infinitesimal transformation

\[
\bar{x}^i = x^i + \xi^i(x) \partial \tau
\]

(1.1)

where \( x^i \) are local coordinates in \( \mathbb{R}^{n-1} \) and \( \xi^i \) are the components of a con-

1 Numbers in brackets refer to the references at the end of paper.
travariant vector \( \xi \). We suppose that the paths of these transformations cover \( R^{m+1} \) simply and that \( \xi \) is everywhere continuous and \( \neq 0 \). If \( \xi \) is a conformal Killing vector, then the group \( G \) is called conformal.

We now consider a closed orientable hypersurface \( V^m \) of class \( C^2 \) imbedded in \( R^{m+1} \), locally given by

\[
x^t = x^t(u^a);
\]

here and henceforth, Latin indices run from 1 to \( m+1 \) and Greek indices from 1 to \( m \).

To the vector \( \xi \) introduced above, there is a covariant vector \( \xi \) of \( V^m \) with the components

\[
\xi_a = B^b_a \xi_b,
\]

where \( B^b_a \) mean \( \partial x^t/\partial u^a \) and \( \xi_b \) the covariant components of \( \xi \); we shall compute its covariant derivatives along \( V^m \) : by virtue of the fact that the covariant derivatives of \( B^b_a \) are

\[
\frac{\delta}{\delta u^a} (B^b_a) = b_{a\beta} n^\beta
\]

where \( b_{a\beta} \) is the second fundamental tensor and \( n^\beta \) the unit normal vector of \( V^m \), we find

\[
(1.2) \quad \xi_{a\beta} = b_{a\beta} n^\gamma \xi_{\gamma} + B^b_a B^j_b \xi_{\gamma j}
\]

(the symbol ";" always means the covariant derivative). Multiplying (1.2) by the contravariant metric tensor \( g^{a\beta} \) of \( V^m \) and summing for \( \alpha \) and \( \beta \), we get

\[
(1.3) \quad g^{a\beta} \xi_{a\beta} = m H_1 n^\gamma \xi_{\gamma} + \frac{1}{2} g^{a\beta} B^b_a B^j_b \xi_{\gamma j} g_{ij},
\]

where \( H_1 \) is the first mean curvature \( \frac{1}{m} g^{a\beta} b_{a\beta} \) of \( V^m \) and \( \xi_{\gamma j} \) is the Lie derivative of the fundamental tensor \( g_{ij} \) of \( R^{m+1} \) with respect to the infinitesimal transformation (1.1). If we put

\[
(1.4) \quad \xi_{\gamma j} = B^b_a B^j_b \xi_{\gamma j},
\]

then (1.3) is rewritten as follows:

\[
\frac{1}{m} \xi_{a\beta} = \frac{1}{m} g^{a\beta} \xi_{a\beta} = H_1 n^\gamma \xi_{\gamma} + \frac{1}{2m} g^{a\beta} \xi_{a\beta},
\]

\( dA \) being the area element of \( V^m \), we have
On a Certain Property of a Closed Hypersurface in a Riemann Space.

\[ \int_{\Sigma_m} \xi_i^a dA = 0, \quad (5), \text{p. 31} \]

because \( V^m \) is closed and orientable. Thus we obtain the integral formula

\[ \int_{\Sigma_m} \cdots \int H_i n^i \xi_i dA + \frac{1}{2m} \int_{\Sigma_m} g^{ij} \mathcal{L}_j g_{ij} dA = 0. \]

Let the group \( G \) be conformal, that is, \( \xi \) satisfy the equation

\[ \mathcal{L}_j g_{ij} = \xi_i;j + \xi_{j;i} = 2\Phi g_{ij}, \]

then (I) becomes

\[ (I)_c \int_{\Sigma_m} \cdots \int H_i n^i \xi_i dA + \int_{\Sigma_m} \Phi dA = 0 \quad ([2], \text{p. 167}). \]

§ 2. An integral formula for a closed orientable hypersurface with certain special second fundamental tensor. We first take the covariant vector \( \partial_i \) of \( V^m \), whose components are given by

\[ \partial_i = (b^a_i) B^i_a \xi_i \]

and calculate its covariant derivatives along \( V^m \), we have

\[ \partial_{i;\theta} = (b^a_{i;\theta}) B^i_a \xi_i + \Phi \xi_{i;\theta} + (b^a_i) \Phi B^i_a \xi_i. \]

Remembering the following Gauss' formulas for a hypersurface

\[ B^{ij}_{\theta} = b_{ij} n^i, \]

we find that

\[ \partial_{i;\theta} = (b^a_{i;\theta}) B^i_a \xi_i + (b^a_i) \Phi \xi_{i;\theta} + (b^a_i) \Phi B^i_a \xi_i. \]

Multiplying by \( g^{ij} \) and summing for \( i \) and \( \theta \), we obtain

\[ (2.1) \quad g^{ij} \partial_{i;\theta} = (b^a_{i;\theta}) g^{ij} B^i_a \xi_i + (b^a_i) \Phi \xi_{i;\theta} + (b^a_i) \Phi g^{ij} B^i_a \xi_i. \]

Moreover we consider the three covariant vectors \( \partial_i, \partial_j, \partial_k \) of \( V^m \) whose components are given by

\[ \partial_i = b^a_i B^i_a \xi_i, \]

\[ \partial_j = (b^a_j) B^i_a \xi_i, \]

\[ \partial_k = (b^a_k) B^i_a \xi_i, \]

respectively. And by making use of the similar calculation mentioned above, we obtain
Accordingly putting \( \tilde{u}_{ij} = \frac{1}{2} \delta_{ij} + \gamma_{ij} \), from (2.1), (2.2), (2.3) and (2.4), we get

\[
(2.5) \quad g^{\alpha \beta} \tilde{u}_{ij} = \left( b^2 \right)_{i j} g^{\alpha \beta} B^i_j \xi_i - \frac{1}{2} \left( \left( b^2 \right)_{i j} - b^2 b^2 \right) \xi_i + \frac{1}{6} \left( b^2 \right)_{i j} - b^3 b^4 \right) \xi_i + \frac{1}{6} \left( b^2 \right)_{i j} - b^3 b^4 \right) \xi_i - \frac{1}{6} \left( b^2 \right)_{i j} - b^3 b^4 \right) \xi_i.
\]

On the other hand, if we denote by \( k_1, k_2, \ldots, k_m \) the principal curvatures of \( V_m \), then the second mean curvature \( H_2 \) and the third mean curvature \( H_3 \) of \( V_m \) are respectively given by

\[
(2.6) \quad \left( \frac{m}{2} \right) H_2 = \sum_{a<i} k_a k_i = \frac{1}{2} \left( b^2 \right)_{i j} - b^2 b^2
\]

and

\[
(2.7) \quad \left( \frac{m}{3} \right) H_3 = \sum_{a<i<j} k_a k_i k_j = \frac{1}{3} \left( \left( b^2 \right)_{i j} + 2b^2 b^2 b^2 - 3b^2 b^2 \right) \xi_i + \frac{1}{6} \left( b^2 \right)_{i j} - b^3 b^4 \right) \xi_i - \frac{1}{6} \left( b^2 \right)_{i j} - b^3 b^4 \right) \xi_i.
\]

where \( b_{ij} \) means the second fundamental tensor of \( V_m \), \( b^2 = b_{ii} g^{ij} \) and \( b^3 = b_{ij} g^{ij} \). Therefore (2.5) may be written as follows:

\[
(2.8) \quad g^{\alpha \beta} \tilde{u}_{ij} = m(m-1) H_{ij} g^{\alpha \beta} B^i_j \xi_i - 2C_{i j} g^{\alpha \beta} B^i_j \xi_i
\]

where \( C_{ij} \) means \( b^2 b_{ij} - b^2 b_{ij} \) and \( C_{ij} = C_{ij} g^{\alpha \beta} \). In (2.8), from the definition of \( C_{ij} \), we see easily that the \( C_{ij} g^{\alpha \beta} B^i_j \) of the final term of the right-hand member are symmetric in \( i \) and \( j \). Consequently we obtain

\[
(2.9) \quad g^{\alpha \beta} \tilde{u}_{ij} = m(m-1) H_{ij} g^{\alpha \beta} B^i_j \xi_i - 2C_{ij} g^{\alpha \beta} B^i_j \xi_i
\]

by virtue of (1.4).
We now assume that the group $G$ of transformations is conformal, that is, $\mathcal{L}_g g = 2\Phi g$, then the last two terms of the right-hand member of (2.9) become in the form

$$\frac{1}{2} m(m-1) H g^{\alpha \beta} \mathcal{L}_{g_{\beta}} + C^\alpha_{\beta \gamma} g^{\beta} \mathcal{L}_{g_{\gamma}} = m(m-1)(m-2) \Phi H,$$

on making use of (1.4) and $C_{\beta \gamma} g^{\beta} = m(m-1) H$. In this case (2.9) can be written as follows:

$$g^{\alpha \beta} \tilde{g}_{\gamma \delta} = m(m-1) H_{2 \mu \nu} g^{\alpha \beta} B_{\mu \nu} - 2 C_{\beta \gamma} g^{\beta} B_{\gamma}^i \tilde{z}_i$$

$$+ m(m-1)(m-2)(H n^i \tilde{z}_i + \Phi H),$$

Moreover if $V^m$ is a closed orientable hypersurface with the second fundamental tensor $b_{ij}$ such that $C_{ij}$ are covariantly constant, both the first term and the second of the right-hand member of (2.10) vanish, because of $C_{\alpha \beta} g^{\alpha \beta} = m(m-1) H$; it follows that (2.10) becomes

$$\frac{1}{m(m-1)(m-2)} \tilde{g}_{\alpha \beta} = \frac{1}{m(m-1)(m-2)} g^{\alpha \beta} \tilde{g}_{\gamma \delta}$$

$$= H n^i \tilde{z}_i + \Phi H,$$

by virtue of dividing the two sides of (2.10) by $m(m-1)(m-2)$. Since $V^m$ is closed and orientable, we have

$$\int_{V^m} \cdots \int \tilde{g}_{\alpha \beta} n^i \xi_i dA = 0.$$

Thus we finally reach the integral formula

$$\int_{V^m} \cdots \int H n^i \tilde{z}_i dA + \int_{V^m} \cdots \int \Phi H dA = 0.$$

§ 3. A property of a closed orientable hypersurface with the special second fundamental tensor in a Riemann space. In this section we shall prove the following two theorems.

**Theorem 3.1.** Let $R^{m-1}$ be a Riemann space and $V^m$ a closed orientable hypersurface in $R^{m-1}$. If there exists a one-parameter group $G$ of conformal transformations of $R^{m-1}$ such that the scalar product $n^i \xi_i$ of the normal vector $n$ of $V^m$ and the generating vector $\xi$ of $G$ does not change the sign (and $\neq 0$) on $V^m$, and if the principal curvatures $k_1, k_2, \ldots, k_m$ on $V^m$ are positive and $C_{ij}$ is covariantly constant, then every point of $V^m$ is umbilic.

**Proof.** From $C_{\alpha \beta} g^{\alpha \beta} = m(m-1) H$, $H$ is constant under our conditions.
Multiplying by $H_2$ the formula (I) in § 1, we obtain
\[
\int_{\mathcal{M}} H_1 H_2 n^t \xi_t \, dA + \int_{\mathcal{M}} \Phi H_2 \, dA = 0,
\]
and subtracting from this the formula (II) in § 2, we find
\[
(3.1) \quad \int_{\mathcal{M}} (H_1 H_2 - H_3) n^t \xi_t \, dA = 0.
\]
On the other hand, from
\[
(3.2) \quad H_1 H_2 - H_3 = \frac{2}{m^3(m-1)} \sum_{\alpha} \sum_{\beta < \ell} k_\alpha k_\beta k_\ell - \frac{3!}{m^2(m-1)(m-2)} \sum_{\alpha < \beta < \ell} k_\alpha k_\beta k_\ell,
\]
we see that
\[
(3.3) \quad H_1 H_2 - H_3 \geq 0 \quad ([1], p. 292).
\]
From (3.1), (3.3) and the fact that $n^t \xi_t$ has a fixed sign we conclude that
\[
H_1 H_2 - H_3 = 0,
\]
from which, because of (3.2), $k_1 = k_2 = \cdots = k_m$ at each point of $\mathcal{V}^n$. This means that every point of $\mathcal{V}^n$ is umbilic.

We next assume that the Riemann space $R^{m-1}$ under consideration is a space of constant curvature $K$: $R_{ijkl} = K (g_{ik} g_{jl} - g_{il} g_{jk})$ where $R_{ijkl}$ is the curvature tensor of $R^{m-1}$. As well-known, a hypersurface in a Riemann space has the following property
\[
(3.4) \quad R_{\alpha j \beta \delta} = b_{\alpha \beta} b_{j \delta} - b_{\alpha \delta} b_{j \beta} + R_{ijkl} B_i^\alpha B_j^\beta B_k^\gamma B_\delta^\delta,
\]
where $R_{\alpha j \beta \delta}$ is the curvature tensor of $\mathcal{V}^m$. Because of a space of constant curvature, (3.4) becomes
\[
R_{\alpha j \beta \delta} = b_{\alpha \beta} b_{j \delta} - b_{\alpha \delta} b_{j \beta} + K (g_{\alpha \beta} g_{j \delta} - g_{\alpha \delta} g_{j \beta}).
\]
Multiplying by $g^{\alpha \delta}$ and summing for $\alpha$ and $\delta$, we have
\[
R_{j \beta} = g^{\alpha \delta} R_{\alpha j \beta \delta} = b_{\alpha j} b_{\beta \delta} - b_{\alpha \delta} b_{j \beta} - (m-1) K g_{j \beta},
\]
from which, by covariant differentiation along $V^m$,
\[
R_{\beta j \gamma} = (b_{\beta j} b_{\gamma \delta} - b_{\beta \delta} b_{j \gamma}) g_{j \delta}, \quad \text{that is,} \quad R_{\beta j \gamma} = C_{\beta j \gamma}.
\]
Thus, we have
Corollary. Let $R^{m-1}$ be a space of constant curvature and $V^m$ a closed orientable Ricci symmetric hypersurface in $R^{m-1}$. We suppose that $R^{m-1}$ admits a one-parameter group $G$ of conformal transformations such that the scalar product $n^i \xi_i$ of the normal vector $n$ of $V^m$ and the generating vector $\xi$ of $G$ does not change the sign and is not identically zero on $V^m$, and the principal curvatures $k_1, k_2, \cdots, k_m$ on $V^m$ are positive, then every point of $V^m$ is umbilic.

Finally we consider the hypersurface with $H_2=$constant and $C_{\alpha \beta} g^{\alpha \beta}=0$. Taking account of $C_{\alpha \beta} g^{\alpha \beta}=m(m-1)H_2$, it is evident that these conditions are weaker than $C_{\alpha \beta}=0$, and also we get (II)$_e$ from (2.10). Thus we have

Theorem 3.2. Let $R^{m-1}$ be a Riemann space and $V^m$ a closed orientable hypersurface in $R^{m-1}$. If there exists a one-parameter group $G$ of conformal transformations of $R^{m-1}$ such that the scalar product $n^i \xi_i$ does not change the sign and is not identically zero on $V^m$, and if the principal curvatures $k_1, k_2, \cdots, k_m$ on $V^m$ are positive, $H_2=$constant and $C_{\alpha \beta} g^{\alpha \beta}=0$, then every point of $V^m$ is umbilic.

Remark. Now we leave us the following question: "which is more general, Theorem 0.1. given by Y. Katsurada or Theorem 3.2. in this paper?"

References


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