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ON MIXED PROBLEMS FOR REGULARLY HYPERBOLIC SYSTEMS

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1. **Introduction.** Let \( \Omega \) be a domain with a bounded boundary \( \Gamma \) of \( \mathbb{R}^n \). We consider the mixed problems for the hyperbolic system of equations

\[
\begin{align*}
(1) \quad & \sum_{j=1}^{k} a_{ij}(x, \frac{\partial}{\partial t}, D_x) u_j = f_i, \quad i = 1, \ldots, k, \quad (t, x) \in (0, T) \times \Omega, \\
& (u_j(0, x), \frac{\partial u_j}{\partial t}(0, x), \ldots, \frac{\partial^{m_j-1} u_j}{\partial t^{m_j-1}}(0, x)) \in D\left(a^{m_j}_{j, 0}\right) \times D\left(a^{m_j-1}_{j, 0}\right) \\
& \times \cdots \times D\left(a^{1}_{j, 0}\right), \quad j = 1, \ldots, k,
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
(D) \quad & u_j|_{\Gamma} = a u_j|_{\Gamma} = \cdots = a^{m_j-1}_{j, 0} u_j|_{\Gamma} = 0, \quad j = 1, \ldots, k \\
\text{or} \quad \quad (N) \quad & \left( \frac{\partial}{\partial n} + \rho(x) \right) u_j|_{\Gamma} = \left( \frac{\partial}{\partial n} + \rho(x) \right) a u_j|_{\Gamma} = \cdots \\
& = \left( \frac{\partial}{\partial n} + \rho(x) \right) a^{m_j-1}_{j, 0} u_j|_{\Gamma} = 0, \quad j = 1, \ldots, k,
\end{align*}
\]

respectively, where \( a_{ij}(x, \frac{\partial}{\partial t}, D_x) = \delta_{ij} \frac{\partial^{m_j}}{\partial t^{m_j}} + a_{ij}^1(x, D) \frac{\partial^{m_j-1}}{\partial t^{m_j-1}} + \cdots + a_{ij}^m(x, D) \)

\((\text{lower order terms})\), \( a_{ij}^s(x, D) = \sum_{|\alpha| = k} a_{ij}^s(x) D^\alpha \), \( D = (D_n, \ldots, D_n) \), \( D_s = -1 - \frac{\partial}{\partial x_i} \),

\( \delta_{ij} \) is a Kronecker delta, \( m_j \) is even for all \( j \), and let all of the roots \( \tau_s(x, \xi) \) \((s = 1, \ldots, m_1 + \cdots + m_s)\) with respect to \( \tau \) of the characteristic equation

\[
\det \left(a_{ij}^s(x, \tau, \xi)\right) = 0 \quad \text{be pure imaginary and distinct mutually, not zero uniformly for} \quad x \in \Omega \quad \text{and} \quad |\xi| = 1,
\]

where \( a_{ij}^s(x, \tau, \xi) \) is the principal part of \( a_{ij}(x, \tau, \xi) \). Furthermore let \( a(x, D) \) be a uniformly elliptic operator such that \( a(x, D) = -\sum_{i,j} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + (\text{lower order terms}) \), \( a_{ij}(x) = a_{ji}(x) \) are real-valued functions and \( \sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \delta|\xi|^2 \) \((\delta > 0)\) for \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \). Moreover \( D(a) = H^2(\Omega) \cap H^1_0(\Omega) \) or \( D(a) = \left\{ u \in H^2(\Omega) ; \left( \frac{\partial}{\partial n} + \rho(x) \right) u|_{\Gamma} = 0 \right\} \) according as we con-
sider Dirichlet type boundary conditions \((D)\) or Neumann ones \((N)\) respectively, 
\[
\frac{\partial}{\partial n}
\]
is the conormal derivative of \(a(x, D)\) at the boundary \(\Gamma'\) and \(\rho(x) \in C^\infty(\Gamma')\).

Furthermore we assume that after applying \((\text{any})\) coordinate transformation
\((U \cap \Omega, U \cap \Gamma) \rightarrow (\mathbb{R}^n = \{ y \in \mathbb{R}^n; y_n > 0 \}, \mathbb{R}^{n-1} = \{ y \in \mathbb{R}^n; y_n = 0 \})\) such that on
the boundary the conormal direction of \(a(x, D)\) is changed into the normal
direction, the coefficients of the principal part of \((1)\) containing odd power of
\[
\frac{\partial}{\partial y_n}
\]
are zero on the boundary \(\mathbb{R}^{n-1}\).

In the present paper we will extend the theory of Calderón-Zygmund's
singular integral operator to that with boundary conditions and as one of its
application we will prove that the above mixed problems are solved by means
of \(\text{the method of semi group analogous to the cases of second order.\) In
\(\S\,2.1\) we construct the theory of the singular integral operators with boundary
conditions on \(\mathbb{R}^n_+\) and next we extend it to one on the domain \(\Omega\) with
bounded boundary in \(\S\,2.2\). Here we avoid to use the principle of reflection
as we can as possible, because it will be expected that the theory of our
singular integral operators is extended to that with respect to elliptic operators
of higher order. Finally we show the Energy inequality and the existence
of resolvent in \(\S\,3\).

Concerning the mixed problems it seems to us that J. Leray asked what
ones for hyperbolic systems of equations of higher order of Petrowski-Leray-
Gårding were well-posed ([11], the third part, Introduction). Later extending
Cauchy-Kowalewski theorem to mixed problem G. F. D. Duff [7] treated the
mixed problems for the case of single equation and \(a = A\) in the quarter space
(see also S. Mizohata [13] and S. Miyatake [12]). With the progress of re-
searches of boundary value problems for elliptic equations, applying them to
the mixed problems for hyperbolic equations has been considered. In particular
S. Agmon [1] showed that the certain mixed problems where the coefficients
of the differential operators are constant and the domain is quadrant were
well-posed. On the other hand the mixed problems for systems of first order
have been developed by Friedrichs-Lax-Phillips, Agranovič and so on. However,
we seems to us to be confronted with a large difficulty in the process to reduce
the operators of higher order to the system of first order theoretically.
Furthermore concerning the difficulty of the mixed problems A. A. Dezin [6]
pointed out from the general viewpoint of partial differential equations (see
also [9]).

For the simplicity of the description we assume that the coefficients of
the operators are sufficiently smooth and bounded with their derivatives in
and the boundary $I'$ is also sufficiently smooth. Furthermore we will be concerned with only single equations (see Remarks in §3). In this case we write (1) as

$$(1') \quad Lu = a(x, \frac{\partial}{\partial t}, D_x)u = \left( \frac{\partial^{2m}}{\partial t^{2m}} + a_1(x, D)\frac{\partial^{2m-1}}{\partial t^{2m-1}} + \cdots + a_{2m}(x, D) \right) u + \text{(lower order terms)} u = f, \quad (t, x) \in (0, T) \times \Omega.$$ 

This article is the extension and details of our previous papers [3] and [18]. Finally we note that the necessity for the condition about coefficients of equations mentioned above is obtained by [19].

§ 2. Singular integral operators with boundary conditions

2.1. Singular integral operators with boundary conditions defined on $\mathbb{R}^n$.

Definition 1. Let $\mathbb{S}^n_0$ be the set of $\sigma(x, \xi)$ such that $\sigma(x, \xi) \in C^4_{x, \xi} \left( \mathbb{R}^n_+ \times (\mathbb{R}^n_+ - \{0\}) \right)$ (this means that $\sigma(x, \xi)$ is defined on $\mathbb{R}^n_+ \times (\mathbb{R}^n_+ - \{0\})$ and all derivatives of $\sigma$ with respect to $x$ and $\xi$ such that the order with respect to $x$ is not higher than four are continuous), $\sigma(x, \lambda \xi) = \sigma(x, \xi)$ for $\lambda > 0$ and for every integer $s (\geq 0)$ we have the following estimate.

$$(2) \quad \sum_{|\alpha| \leq 4} \sup_{x \in \mathbb{R}^n_+} \left| \left( \frac{\partial}{\partial x} \right)^s \left( \frac{\partial}{\partial \xi} \right)^s \sigma(x, \xi) \right| = M_s(\sigma) < \infty.$$ 

Let $\Sigma = \{ \xi \in \mathbb{R}^n_+ ; |\xi| = 1 \}$ and let $\{Y_{l,m}(\xi)\}$ be a complete orthonormal system in $L^2(\Sigma)$ such that every $Y_{l,m}(\xi)$ be a real polynomial of spherical harmonic of homogeneous order $l$ and $\int_{\Sigma} Y_{l,m}(\xi) d\Sigma = 0$.

Then for $\sigma(x, \xi) \in \mathbb{S}^n_0$, we can expand it such that

$$(3) \quad \sigma(x, \xi) = \sum_{l,m} \sigma_{l,m}(x) Y_{l,m}(\xi) \quad \text{in} \quad L^2(\Sigma),$$ 

where $\sigma_{l,m}(x) = \int_{\Sigma} \sigma(x, \xi) Y_{l,m}(\xi) d\Sigma$.

Moreover, as well known we obtain the following estimates.

$$\begin{cases} \sup_{|\alpha| \leq 4} \left| \left( \frac{\partial}{\partial \xi} \right)^s Y_{l,m}(\xi) \right| \leq c(n, k) l^{\frac{1}{2}(n-2)+|\nu|} , \\
the number of $m$ is of order $l^{n-2}$ as $l \to \infty$ , \\
\sum_{|\alpha| \leq 4} \sup_{x \in \mathbb{R}^n_+} \left| \left( \frac{\partial}{\partial x} \right)^s \sigma_{l,m}(x) \right| \leq c(n, k) M_{2k}(\sigma) l^{-2k-\frac{n}{2}} , \quad \text{where} \quad k \text{is an arbitrary non negative integer}.
\end{cases}$$

Definition 2. Let $\mathcal{A}$ be the algebra generated by $\sigma(x, \xi)$ and $f(x)$.
with the following properties: \( \sigma(x, \xi) \in \mathfrak{B}^s(\mathbb{R}^n) \) and \( f(x, 0) = 0 \), here \( x, \xi \) will denote \( x = (x_1, \ldots, x_n), \xi = (\xi_1, \ldots, \xi_n) \). For \( \alpha(x, \xi) \in \mathfrak{A} \) consider the singular integral operator \( \alpha(x, D) \) as follows:

\[
\alpha(x, D)u = F'\alpha(x, \xi)F\tilde{u} \mid_{\xi_n > 0} \quad \text{for } u \in L^2(\mathbb{R}^n),
\]

where

\[
(Fv)(\xi) = \int_{\mathbb{R}^n} e^{-ix'\xi}v(x)dx,
\]

\[
(F'v)(\xi) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix'\xi}v(x)dx \quad \text{for } v \in L^2(\mathbb{R}^n), \quad x' = \sum_{j=1}^n x_j \xi_j
\]

\( i = \sqrt{-1} \), and \( \tilde{u}(x) \) is the extended function of \( u(x) \) to \( \mathbb{R}^n \) by defining it to be the odd function or the even one with respect to \( x_n \).

Furthermore for \( \alpha(x, \xi) \in \mathfrak{A} \), let \( M_\alpha = \min(M_\alpha + \sum_{j=1}^n M_s(\sigma_j(x, x', \xi_n) f_j(x) \xi_n)) \), where \( \sigma(x, \xi) = \sigma(x, \xi) + \sum_{j=1}^n \sigma_j(x, \xi) f_j(x) \xi_n \in \mathfrak{A} \). Then \( M_\alpha(\alpha + \alpha_2) \leq M_\alpha(\alpha_1) + M_\alpha(\alpha_2) \) and \( M_\alpha(\alpha_1) \leq M_\alpha(\alpha_1)M_\alpha(\alpha_2) \).

**Definition 3.** Let \( \mathcal{A}_+ \) and \( \mathcal{A} = -\mathcal{A} \), whose definition domains are \( \{u \in H^1(\mathbb{R}^n); \frac{\partial u}{\partial x_n}(x', 0) = 0\} \), \( H^2(\mathbb{R}^n) \cap H_0^1(\mathbb{R}^n) \) and \( H^2(\mathbb{R}^n) \), respectively. Then they are selfadjoint operators on \( L^2(\mathbb{R}^n) \), \( L^2(\mathbb{R}^n) \) and \( L^2(\mathbb{R}^n) \), respectively. Let \( A_1 = (\mathcal{A}_+)\frac{1}{2}, A = (\mathcal{A})\frac{1}{2} \) and \( A = (\mathcal{A})\frac{1}{2} \) then we see that \( D(A_1) = D(A_1) = H^1(\mathbb{R}^n) \), \( D(A_1) = H^1(\mathbb{R}^n) \) and \( D(A) = H^1(\mathbb{R}^n) \).

**Lemma 1.** Let \( u(x) \in D(A_+) \) and \( \alpha(x, \xi) \in \mathfrak{A} \). Then \( \alpha(x, D)u, \alpha(x, D)^*u \in D(A) \) where in Definition 2 we assume that \( \tilde{u}(x) \) is the odd extension or even one of \( u(x) \) respectively according as the case where we consider \( A_+ \) or \( A_- \).

In what follows \( \tilde{u}(x) \) will be defined as above.

**Proof.** We will show only the assertion for \( D(A_+) \), because the proof for \( D(A_-) \) is more obvious than that for \( D(A_+) \).

Let

\[
F^{-1}u(\xi) = \int_{\mathbb{R}^n} e^{-ix'\xi} \sin(x_n \xi_n) u(x', x_n) dx, \quad \text{for } \xi_n > 0
\]

\[
F^{-1}u(\xi) = 4(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix'\xi} \sin(x_n \xi_n) u(x', x_n) dx \quad \text{for } u \in L^2(\mathbb{R}^n).
\]

Then we find that \( u \in D(A_-) \) if and only if \( F^{-1}u(\xi) \) and \( |\xi| F^{-1}u(\xi) \in L^2(\mathbb{R}^n) \). For if \( u(x) \in D(A_-) \) then from \( F^{-1}u(\xi) = \frac{i}{2} F\tilde{u}(\xi) \) \( \xi_n > 0 \) it follows that \( F^{-1}u(\xi) \) and \( |\xi| F^{-1}u(\xi) \in L^2(\mathbb{R}^n) \). Conversely let \( F^{-1}u(\xi) \) and \( |\xi| F^{-1}u(\xi) \in L^2(\mathbb{R}^n) \). Then \( F\tilde{u}(\xi) \) and \( |\xi| F\tilde{u}(\xi) \in L^2(\mathbb{R}^n) \), from which \( \tilde{u}(x) \in H^1(\mathbb{R}^n) \) and \( u(x) \in H^1(\mathbb{R}^n) \). Next,
to show \( u(x) \in H_0^1(\mathbb{R}^n) \) let us set partial Fourier transformation \( F^- \) such that for \( u(x', x_n) \in L^2(\mathbb{R}^n) \)

\[
(F^- u)(\xi', x_n) = \int_{-\infty}^{\infty} e^{ix'n} u(x', x_n) dx'.
\]

Then we see that

\[
(F^- u)(\xi', x_n) = \frac{\pi}{2} \int_0^\infty \sin (x_n \xi_n) (F^- u)(\xi', \xi_n) d\xi_n.
\]

Furthermore from the hypothesis it follows

\[
\int_0^\infty |(F^- u)(\xi', \xi_n)| d\xi_n 
\]

\[
\leq \left( \int_0^\infty (1 + \xi_n^2) (F^- u)(\xi', \xi_n)^2 d\xi_n \right)^{\frac{1}{2}} \left( \int_0^\infty (1 + \xi_n^2)^{-1} d\xi_n \right)^{\frac{1}{2}} < \infty,
\]

and

\[
|F^- u(\xi', x_n)|^2 
\]

\[
\leq c \int_0^\infty (1 + \xi_n^2) (F^- u)(\xi', \xi_n)^2 d\xi_n \int_0^\infty (1 + \xi_n^2)^{-1} d\xi_n
\]

for \( x_n > 0 \) and a.e. \( \xi' \), from which it implies by virtue of Lebesgue's theorem

\[
(F^- u)(\xi', 0) = \frac{\pi}{2} \lim_{x_n \to 0} \int_0^\infty \sin (x_n \xi_n) (F^- u)(\xi', \xi_n) d\xi_n
\]

\[
= \frac{\pi}{2} \int_0^\infty \lim_{x_n \to 0} \sin (x_n \xi_n) (F^- u)(\xi', \xi_n) d\xi_n = 0 \quad \text{for a.e. } \xi',
\]

hence

\[
\lim_{x_n \to 0} \int_{-\infty}^{\infty} |(F^- u)(\xi', x_n)|^2 d\xi' = \int_{-\infty}^{\infty} \lim_{x_n \to 0} |(F^- u)(\xi', x_n)|^2 d\xi' 
\]

\[
= \int_{\mathbb{R}^n} |(F^- u)(\xi', 0)|^2 d\xi' = 0.
\]

On the other hand we see that

\[
\int_{-\infty}^{\infty} |u(x', x_n)|^2 dx' = (2\pi)^{-n+1} \int_{-\infty}^{\infty} |(F^- u)(\xi', x_n)|^2 d\xi' \quad \text{for } x_n > 0.
\]

From (5) and (6) it follows

\[
\lim_{x_n \to 0} \int_{-\infty}^{\infty} |u(x', x_n)|^2 dx' = 0,
\]

that is, \( u(x) \in H_0^1(\mathbb{R}^n) \).

Now we will prove \( \alpha(x, D)u \in D(A_-) \) for \( u \in D(A_-) \) using the equivalent
relation stated above. Let \( \alpha(x, \xi) = f(x) \frac{\xi_n}{|\xi|} \) and \( f(x', 0) = 0 \). Then

\[
\alpha(x, D) u = F' \alpha(x, \xi) \mathcal{F} \hat{u} \bigg|_{x_n > 0} = f(x) F' \frac{\xi_n}{|\xi|} \mathcal{F} \hat{u} \bigg|_{x_n > 0}
\]

From \( f(x', 0) = 0 \) it implies \( \alpha(x, D) u \in H_0^1(\mathbb{R}_+^n) \).

Furthermore let \( \alpha(x, \xi) \in \mathcal{S}'_0 \) and \( \alpha(x, \xi', \xi_n) = \alpha(x, \xi', -\xi_n) \). Then by virtue of (3), (4) and regarding that \( \alpha(x, \xi) \) is an even function with respect to \( \xi_n \) we see that

\[
\alpha(x, \xi) = \sum_{l,m} \alpha_{l,m}(x) Y_{l,m}^+(\xi)
\]

where \( Y_{l,m}^+(\xi) = \frac{1}{2} \left( Y_{l,m}(\xi', \xi_n) + Y_{l,m}(\xi', -\xi_n) \right) \) and that it converges uniformly on \( (x, \xi) \in \mathbb{R}_+^n \times (\mathbb{R}^n - \{0\}) \).

Then we see by the characterization mentioned above that every \( \alpha_{l,m}(x) Y_{l,m}^+(D) u \in D(\Lambda) \), because

\[
F' \alpha_{l,m}(x) Y_{l,m}^+(\xi) \mathcal{F} \hat{u} \bigg|_{x_n > 0} = \alpha_{l,m}(x) F' Y_{l,m}^+(\xi) \mathcal{F} \hat{u},
\]

\( Y_{l,m}^+(\xi) \mathcal{F} \hat{u} = Y_{l,m}^-(\xi) \mathcal{F} \hat{u} \in L^2(\mathbb{R}^n_+) \). Hence we have only to show that \( \sum_{l,m} \alpha_{l,m}(x) Y_{l,m}^+(D) u \) converges in \( H^1(\mathbb{R}^n_+) \). But it is easily seen from the fact that this expression converges in \( L^2(\mathbb{R}^n_+) \) by virtue of (4). Furthermore from (4) and by the fact that \( Y_{l,m}^+(D) \Lambda u = \Lambda Y_{l,m}^+(D) u \) for \( u \in D(\Lambda) \) we see that above expression converges in \( H^1(\mathbb{R}^n_+) \).

We also find that \( \alpha(x, D)^* u \in D(\Lambda) \) for \( u \in D(\Lambda) \). To prove this let \( \alpha(x, \xi) = f(x) \frac{\xi_n}{|\xi|} \), \( f(x', 0) = 0 \). Then for every \( v \in C^\infty_c(\mathbb{R}^n_+) \)

\[
(\alpha(x, D)^* u, \Lambda v)_{x_n > 0} = \left( u, \alpha(x, D) \Lambda v \right)_{x_n > 0} = \left( u, f(x) \frac{1}{\sqrt{-1}} \frac{\partial v}{\partial x_n} \right)_{x_n > 0}
\]

from which we find that

\[
| (\alpha(x, D)^* u, \Lambda v)_{x_n > 0} | \leq C \|u\|_{L^2(\mathbb{R}^n_+)} \|v\|_{x_n > 0},
\]

where \( \|u\|_{x_n > 0} = \int_{\mathbb{R}^n_+} |u|^2 dx \) and \( (u, v)_{x_n > 0} = \int_{\mathbb{R}^n_+} u \bar{v} dx \) for \( u, v \in L^2(\mathbb{R}^n_+) \).

This implies \( \alpha(x, D)^* u \in D(\Lambda^+) = D(\Lambda) \).

Furthermore let \( \alpha(x, \xi) \in \mathcal{S}'_0 \) and \( \alpha(x, \xi', \xi_n) = \alpha(x, \xi', -\xi_n) \). Then we find that \( \alpha(x, D)^* u = \sum_{l,m} Y_{l,m}^+(D) \alpha_{l,m}(x) u \). Moreover every term on the right is in \( D(\Lambda) \) and similarly to the case stated above this expression converges in
This implies $\alpha(x, D)^* u \in D(\Lambda_-)$. q.e.d.

**Theorem 1.** Let $\alpha(x, \xi)$ and $\beta(x, \xi) \in \mathfrak{A}$. Then for $u(x) \in D(\Lambda_-)$ we obtain the following estimates:

\[
(8) \quad \left\| (\alpha(x, D)\Lambda_- - \Lambda_- \alpha(x, D)) u \right\|_{x_n \to 0} \leq CM_2(\alpha) \| u \|_{x_n \to 0},
\]
\[
(9) \quad \left\| (\alpha(x, D)^* \Lambda_- - \Lambda_- \alpha(x, D)^*) u \right\|_{x_n \to 0} \leq CM_2(\alpha) \| u \|_{x_n \to 0},
\]
\[
(10) \quad \left\| (\alpha(x, D)^* - \alpha^*(x, D)) \Lambda_- u \right\|_{x_n \to 0} \leq CM_2\left(\left[\frac{3}{2}n\right]_1\right) (\alpha) \| u \|_{x_n \to 0},
\]
\[
(11) \quad \left\| (\alpha(x, D) \beta(x, D) - (\alpha \beta)(x, D)) \Lambda_- u \right\|_{x_n \to 0} \leq CM_2\left(\left[\frac{3}{2}n\right]_1\right) (\alpha) \| u \|_{x_n \to 0},
\]

where $\alpha^*(x, D)$, $(\alpha \beta)(x, D)$ are the operators with symbol $\alpha(x, \xi)$, $\alpha(x, \xi) \beta(x, \xi)$, respectively, $\left[ \right]$ denotes Gauss symbol and the constant $C$ depends only on dimension $n$.

In the following $C$ will be used to denote various constants depending only on $n$.

To prove Theorem 1 the following Lemmas 2–11 will be needed. Hereafter the proofs of Theorem 1 and Lemmas 2–10 except Lemma 11 will be carried out only the case of $\Lambda_+$. In the case of $\Lambda_-$ the proofs are obtained by regarding $u(x)$ in the proofs mentioned below as the odd function.

**Notation.** For $a(x) \in \mathfrak{B}^k(\mathbb{R}_n^+)$, $k$: non negative integer, let $|a|_k$ be the norm of $a(x)$ such that

\[
|a|_k = \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}_n^+} \left| \frac{\partial^\alpha}{\partial x^\alpha} a(x) \right|,
\]

and for $a(x) \in \mathfrak{B}^{k, \delta}(\mathbb{R}_n^+)$, $k$: non negative integer and $\delta$ is a constant such that $0 < \delta < 1$, $|a|_{k, \delta}$ be the norm of $a(x)$ such that

\[
|a|_{k, \delta} = |a|_k + \text{Hölder constant of order } \delta.
\]

**Lemma 2.** Let $u(x) \in D(\Lambda_+)$. Then $\Lambda_+ u = \Lambda \tilde{u} |_{x_n \to 0}$.

**Proof.** Similarly to the case of $D(\Lambda_-)$ in Lemma 1, let

\[
F^+ u(\xi) = \int_{\mathbb{R}_n^+} e^{-ix'\xi} \cos (x_n \xi_n) u(x', x_n) dx,
\]
\[
F^{++} u(\xi) = 4(2\pi)^{-n} \int_{\mathbb{R}_n^+} e^{ix'\xi} \cos (x_n \xi_n) u(x', x_n) dx \quad \text{for } u(x) \in L^2(\mathbb{R}_n^+).
\]
Then
\[(F^+u)(\xi) = \frac{1}{2} (F\tilde{u})(\xi) |_{\xi_n > 0}\]
and
\[(F^{++}u)(\xi) = 2(F'\tilde{u})(\xi) |_{\xi_n > 0}.\]
From our definitions it follows that
\[F^+(A\nu u)(\xi) = |\xi| (F^+u)(\xi) \quad (\xi_n > 0),\]
and that for \(v(x) \in D(A)\)
\[F(Av)(\xi) = |\xi| (Fv)(\xi).\]
Thus it implies
\[A\nu u = F^{++}F^+A\nu u = F^{++}(|\xi| (F^+u)(\xi)) = 2F'(|\xi| (F^+\nu u)(\xi)) |_{\xi_n > 0}\]
\[= F'(|\xi| F\tilde{\nu}(\xi)) |_{\xi_n > 0} = A\tilde{\nu} |_{\xi_n > 0}.\]

**Lemma 3.** Let \(a(x) \in \mathcal{B}^4(\mathbb{R}^n)\). For \(u(x) \in D(A)\)
\[\left| \left( a(x) A_\nu A \right) u \right|_{\xi_n > 0} \leq C |a|_4 \|u\|_{\xi_n > 0}.\]

**Proof.** Let us decompose \(a(x)\) as follows:
\[a(x) = a_1(x') + a_2(x')x_n + a_3(x', x_n)x_n^2 = b_1(x) + b_2(x),\]
here
\[b_1(x) = a_1(x') + a_3(x', x_n)x_n^2, \quad b_2(x) = a_2(x')x_n.\]
Furthermore choose \(\varphi(x) \in \mathcal{B}(\mathbb{R}^n)\) such that \(0 \leq \varphi(x) \leq 1, \varphi(x) = 1\) for \(0 \leq x_n \leq 1\) and \(\varphi(x) = 0\) for \(x_n \geq 2\).
Then
\[a(x) = \varphi(x) b_1(x) + \varphi(x) b_2(x) + (1 - \varphi(x)) a(x)\]
\[= a_4(x) + a_5(x) + a_6(x).\]
From our decomposition we see that
\[\tilde{a}_4(x) \in \mathcal{B}(\mathbb{R}^n),\]
\[\tilde{a}_5(x) = \begin{cases} a_2(x') |x_n| & \text{for } |x_n| \leq 1, \\ 0 & \text{for } |x_n| \geq 2 \end{cases}, \quad a_5(x) \in \mathcal{B}(\mathbb{R}^n),\]
and
\[\tilde{a}_6(x) \in \mathcal{B}(\mathbb{R}^n).\]
Now by virtue of Lemma 2 it implies
\[
(a(x)A_+ - A_+ a(x)) u \\
= \left( \tilde{a}(x)A_+ - A_+ \tilde{a}(x) \right) \tilde{u} \bigg|_{x_n > 0} \\
= (\tilde{a}_4 A - A \tilde{a}_4) \tilde{u} \bigg|_{x_n > 0} + (\tilde{a}_5 A - A \tilde{a}_5) \tilde{u} \bigg|_{x_n > 0} + (\tilde{a}_6 A - A \tilde{a}_6) \tilde{u} \bigg|_{x_n > 0}. 
\]

Because of $\tilde{a}_4$ and $\tilde{a}_6 \in C^\infty_c(R^n)$, the first and the last terms are bounded by $C \| \tilde{u} \|_{x_n > 0} = C \| u \|_{x_n > 0}$ from the theory of the ordinary singular integrals. Hence to complete the proof of Lemma 3 it suffices to prove the following

**Lemma 4.** Let $a(x) \in C^\infty_c(R^n)$ such that

\[
a(x) = \begin{cases} 
  a_1(x') x_n & \text{for } 0 \leq x_n < 1, \\
  0 & \text{for } x_n \geq 2.
\end{cases}
\]

Then for $u(x) \in D(A_+)$

\[
\left\| (a(x)A_+ - A_+ a(x)) u \right\|_{x_n > 0} \leq C |a|_{1, \infty} \| u \|_{x_n > 0}.
\]

**Proof.** Since $C^\infty_0(R^n)$ (this means the set of all functions $u(x)$ such that $u$ is $C^\infty$ function with compact support contained in $R^n$) is dense in $D(A_+)$ it suffices to prove this lemma for $u \in C^\infty_0(R^n)$.

Here we remark that hereafter we may prove following Lemmas 5–11 for $u(x) \in C^\infty_0(R^n)$.

Let $R_j$ be the operator from $L^2(R^n)$ into itself such that

\[
(FR_j v)(\xi) = \frac{1}{\sqrt{-1}} \sum_{j} \xi_j (Fv)(\xi) \quad \text{for } v(x) \in L^2(R^n).
\]

From this it follows that

\[
A \tilde{u} = \sum_{j=1}^n \frac{\partial}{\partial x_j} R_j \tilde{u} = \sum_{j=1}^n R_j \frac{\partial}{\partial x_j} \tilde{u},
\]

and

\[
\left( \tilde{a}(x) A - A \tilde{a}(x) \right) \tilde{u} = \sum_{j=1}^n \left( \tilde{a}(x) R_j - R_j \tilde{a}(x) \right) \frac{\partial}{\partial x_j} \tilde{u} - \sum_{j=1}^n R_j \frac{\partial \tilde{a}}{\partial x_j} \tilde{u}.
\]

Where $\frac{\partial \tilde{a}}{\partial x_j}$ is not continuous but bounded, so each of the second term on the right is bounded in $L^2$-sense. Therefore we may investigate each of the first term on the right.

To this end we set

\[
v_1(x) = \int_{|x-y| > 1} \left( \tilde{a}(x) - \tilde{a}(y) \right) R_j(x-y) \frac{\partial}{\partial y_j} \tilde{u}(y) dy.
\]
\[= \int_{|z-y|<\epsilon} \left( \bar{a}(x) - \bar{a}(y) \right) R_j(x-y) \bar{u}(y) \cos \gamma_j dS, \]
\[+ \int_{|z-y|>\epsilon} \frac{\partial \bar{a}}{\partial y_j} (y) R_j(x-y) \bar{u}(y) dy \]
\[+ \int_{|z-y|>\epsilon} \left( \bar{a}(x) - \bar{a}(y) \right) \frac{\partial}{\partial x_j} R_j(x-y) \bar{u}(y) dy \]
\[\equiv v^1(x) + v^2(x) + v^3(x), \]
where \( S_\epsilon = \{ x \in \mathbb{R}^n ; |x| = \epsilon \}. \)

Then we may show the following estimates:
\[
\lim_{\epsilon \to 0} \|v^i(x)\|_{\varepsilon_n > 0} \leq C |a|_{1+i} \|u\|_{\varepsilon_n > 0} \quad (i = 1, 2, 3).
\]

We estimate \( v^1(x) \).
\[
v^1(x) = \int_{|z-y|<\epsilon} \frac{\bar{a}(x) - \bar{a}(y)}{|x-y|} \cdot |x-y| R_j(x-y) \bar{u}(y) \cos \gamma_j dS.
\]
Hence
\[
|v^1(x)| \leq |\bar{a}(x)|_1 C \int_{s_\epsilon} \epsilon \cdot \frac{1}{\varepsilon^n} |\bar{u}(x + \varepsilon \omega)| \varepsilon^{n-1} dS_1.
\]
\[
= C |a(x)|_1 \int_{s_\epsilon} |\bar{u}(x + \varepsilon \omega)| dS_1.
\]
The right hand tends to \( C |a(x)|_1 \omega_n |\bar{a}(x)| \) as \( \varepsilon \to 0 \), where \( \omega_n = \int_{s_\epsilon} dS_1 \).

Therefore
\[
\lim_{\epsilon \to 0} \|v^1(x)\|_{\varepsilon_n > 0} \leq C |a(x)|_1 \|u(x)\|_{\varepsilon_n > 0}.
\]

Next \( v^2(x) \) tends to \( v.p. R_j(x) \ast \left( \frac{\partial \bar{a}}{\partial x_j} \bar{u} \right)(x) \) as \( \varepsilon \to 0 \), hence
\[
\lim_{\epsilon \to 0} \|v^2(x)\|_{\varepsilon_n > 0} \leq |\bar{a}(x)|_1 \|\bar{u}(x)\|_{\varepsilon_n > 0} = |a(x)|_1 \|u(x)\|_{\varepsilon_n > 0}.
\]

Finally we estimate \( v^3(x) \). We decompose it as follows:
\[
v^3(x) = \int_{\varepsilon < |z-y| < 1} dy + \int_{|z-y| > 1} dy
\]
\[
= v^4(x) + v^5(x).
\]
From \( |v^5(x)| \leq 2|\bar{a}(x)|_0 \left| \left( \frac{\partial}{\partial x_j} R_j(x) \right) \right| |x| \bar{u}(x)| \) and \( \frac{\partial}{\partial x_j} R_j(x) = O(|x|^{-n-1}) \), it follows
Furthermore we estimate $v_i^j(x)$. We see that $\bar{a}(x) = a(x)$ on $x_n > 1$ and $\bar{a}(y) = a(y)$ on $x_n \leq 1$. Hence by virtue of the theory of ordinary singular integrals it follows

$$\lim_{\epsilon \to 0} \frac{1}{x_n} \sum_{j \in \mathbb{Z}^n} |v_j^i(x)|^2 dx \leq C |a|_{1+\delta} \|u\|_{x_n > 0}.$$ 

Moreover we estimate $v_i^j(x)$ where $0 < x_n < 1$. By $\bar{a}(x)$ set the odd extension of $a(x)$ to $\mathbb{R}^n$ with respect to $x_n$. Then

$$v_i^j(x) = \int_{x_n < y_n < 1} (\bar{a}(x) - \bar{a}(y)) \left( \frac{\partial}{\partial x_j} R_j \right)(x-y) \bar{u}(y) dy + \int_{x_n < y_n < 1} (\bar{a}(y) - \bar{a}(y)) \left( \frac{\partial}{\partial x_j} R_j \right)(x-y) \bar{u}(y) dy.$$

As $\bar{a}(x) = \bar{a}(x)$ on $x_n > 0$ and $\bar{a}(x) \in \mathcal{B}^{1+\delta}(\mathbb{R}^n)$, $v_i^j(x)$ is estimated similarly to the case where $x_n \geq 1$ of $v_i^j(x)$. Hence

$$\lim_{\epsilon \to 0} \frac{1}{x_n} \sum_{j \in \mathbb{Z}^n} |v_j^i(x)|^2 dx \leq C |a|_{1+\delta} \|u\|_{x_n > 0}.$$ 

To finish the proof we may estimate only $v_i^j(x)$. As $\bar{a}(y) = \bar{a}(y)$ on $y_n > 0$,

$$v_i^j(x) = 2 \int_{x_n < y_n < 1} a_1(y') y_n \left( \frac{\partial}{\partial x_j} R_j \right)(x-y) \bar{u}(y') dy'$$

$$= 2 \int_{x_n < y_n < 1} a_1(y') y_n \left( \frac{\partial}{\partial x_j} R_j \right)(x' - y', x_n + y_n) \bar{u}(y', -y_n) dy'.$$

Hence

$$|v_i^j(x)| \leq |a|_1 C \int_{\mathbb{R}^m} y_n \left( |x' - y'| + x_n + y_n \right)^{-n-1} |\bar{u}(y', -y_n)| dy'$$

$$\leq C |a|_1 \int_{\mathbb{R}^m} \left( |x' - y'| + x_n + y_n \right)^{-n} |\bar{u}(y', -y_n)| dy'.$$

Using Hilbert's inequality, we see that

$$\|v_i^j(x)\|_{x_n > 0} \leq C |a|_1 \|u(x)\|_{x_n > 0}.$$

From the above statements, by passing to the limit, we find that
Lemma 5. Let \( a(x) \in S^1(\mathbb{R}^n_+) \), \( A(\xi) \in \mathcal{E}_+^\infty \) and \( A(\xi', \xi_n) = A(\xi', -\xi_n) \). Then for \( u(x) \in D(A) \)

\[
\left\| \left( a(x) A(D) A_+ - A(D) A_+ a(x) \right) u \right\|_{x_n > 0} \leq C |a|_{1+2} \|u\|_{x_n > 0}.
\]

Proof. We define \( \tilde{A}(D) \) as follows:

\[
\tilde{A}(D)v = F^* A(\xi) Fv \quad \text{for } v \in L^2(\mathbb{R}^n).
\]

Then \( A(D) u = \tilde{A}(D) \tilde{u} |_{x_n > 0} \) for \( u \in L^2(\mathbb{R}^n_+) \). From this and Lemma 2 it follows

\[
\left\| \left( a(x) A(D) A_+ - A(D) A_+ a(x) \right) u \right\|_{x_n > 0} = \left\| \sum_{j=1}^n \left( \tilde{a}(x) \tilde{A}(D) R_j - \tilde{A}(D) R_j \tilde{a}(x) \right) \frac{\partial \tilde{u}}{\partial x_j} - \sum_{j=1}^n \tilde{A}(D) R_j \frac{\partial \tilde{a}}{\partial x_j} \tilde{u} \right\|_{x_n > 0} \leq \sum_{j=1}^n \left\| \frac{\partial \tilde{u}}{\partial x_j} \right\|_{x_n > 0} + \sum_{j=1}^n \left\| \frac{\partial \tilde{a}}{\partial x_j} \right\|_{x_n > 0}.
\]

Second term on the right is estimated as follows:

\[
\left\| \sum_{j=1}^n \tilde{A}(D) R_j \frac{\partial \tilde{a}}{\partial x_j} \tilde{u} \right\|_{x_n > 0} \leq C |a|_1 M_0(A) \|u\|_{x_n > 0}.
\]

Hence we shall estimate the first term on the right. For \( v \in L^2(\mathbb{R}^n) \),

\[
F' \left( \tilde{A}(D) R_j v \right) = \frac{1}{\sqrt{-1}} A(\xi) \frac{\xi_j}{|\xi|} Fv \quad \text{and } A(\xi) \frac{\xi_j}{|\xi|} \text{ is of positive homogeneous degree zero.}
\]

From this, provided \( \int_{S^1} A(\xi) \frac{\xi_j}{|\xi|} d\Sigma = c_1 = 0 \), dividing \( \frac{1}{\sqrt{-1}} A(\xi) \frac{\xi_j}{|\xi|} \) into \( A_1(\xi) + \frac{c_1}{\omega_n} \), it follows that there exists a function \( f(x) \in \mathcal{E}_+^\infty(\mathbb{R}^n - \{0\}) \) of homogeneous degree \((-n)\) such that

\[
F' \left( A_1(\xi)' \right)(x) = v, p, f(x), \quad \int_{S^1} f(x) dS_1 = 0 \quad \text{and}
\]

\[
\sum_{|a| < n+1+k} \sup_{|\xi| \geq 1} \left| D^a f(x) \right| \leq C \sum_{|a| < n+1+k} \sup_{|\xi| \geq 1} \left| D^a A_1(\xi) \right| \leq C \sum_{|a| < n+1+k} \sup_{|\xi| \geq 1} \left| D^a A(\xi) \right|.
\]

Replacing \( R_j(x) \) in Lemma 4 by \( f(x) \) obtained above and making use of \( k = 1 \) in the above expression, we can prove the following estimate similarly to that used in the proof of Lemma 4

\[
\left\| \left( \tilde{a}(x) \tilde{A}(D) R_j - \tilde{A}(D) R_j \tilde{a}(x) \right) \frac{\partial \tilde{u}}{\partial x_j} \right\|_{x_n > 0} \leq C |a|_1 M_{n+2}(A) \|u\|_{x_n > 0}.
\]
Therefore
\[ \left\| (a(x)A(D)A_+ - A(D)A_+ a(x))u \right\|_{\mathbb{R}_n^*} \leq C |a|_4 M_{n+2}(A) \|u\|_{\mathbb{R}_n^*}. \]

The following Corollary is obtained by the same method used above.

**Corollary.** Let \( a(x) \in \mathcal{B}^{1+\delta}(R_n^*) \) and \( A(\xi) \in \mathcal{E}_{\xi}^n \). Then for \( u \in D(A) \)
\[ \left\| (\tilde{a}(x) \tilde{A}(D)A - A \tilde{A}(D) \tilde{a}(x)) u \right\| \leq C |a|_{1+\delta} M_{n+2}(A) \|u\|. \]

Now, for \( \tilde{\eta}(\xi) = \frac{\xi_n}{|\xi|} \) define the operators \( \tilde{\eta}(D) \) and \( \tilde{\eta}(D) \) as follows:
\[ \tilde{\eta}_1(D) \tilde{u} = F^* \tilde{\eta}(\xi) F \tilde{u}, \quad \tilde{\eta}(D) u = \tilde{\eta}_1(D) \tilde{u} \quad \text{for} \quad u \in L^2(R_n^*). \]

Furthermore for convenience of calculations set \( \tilde{\eta}^2(D) \) as follows:
\[ \tilde{\eta}^2(D) u = F^* \left( \tilde{\eta}(\xi) \right)^2 F \tilde{u} \quad \text{for} \quad u \in L^2(R_n^*). \]

**Lemma 6.** Let \( a(x) \in \mathcal{B}^{4}(R_n^*) \). Then for \( u \in D(A_+) \)
\[ \left\| (\tilde{\eta}(D) a(x) - a(x) \tilde{\eta}(D)) A_+ u \right\|_{\mathbb{R}_n^*} \leq C |a|_4 \|u\|_{\mathbb{R}_n^*}. \]

**Proof.** From \( \tilde{\eta}(D) A_+ u = \frac{1}{\sqrt{-1}} \frac{\partial \tilde{u}}{\partial x_n} \) and Lemma 3 it follows
\[ \left\| (\tilde{\eta}(D) a(x) - a(x) \tilde{\eta}(D)) A_+ u \right\|_{\mathbb{R}_n^*} \leq C |a|_4 \|u\|_{\mathbb{R}_n^*} + \left\| \frac{\partial}{\partial x_n} a(x) u - a(x) \tilde{\eta}(D) A_+ u \right\|_{\mathbb{R}_n^*} \leq C |a|_4 \|u\|_{\mathbb{R}_n^*} + C |a|_4 \|u\|_{\mathbb{R}_n^*}. \]

**Lemma 7.** Let \( f(x) \in \mathcal{B}^{3}(R_n^*), \) \( f(x',0) = 0, \) \( A(\xi) \in \mathcal{E}_{\xi}^n \) and \( A(\xi', \xi_n) = A(\xi', -\xi_n). \) Then for \( u(x) \in D(A_+) \)
\[ \left\| (f(x) \tilde{\eta}(D) A(D) - A(D) f(x) \tilde{\eta}(D)) A_+ u \right\|_{\mathbb{R}_n^*} \leq C \|f\|_{1+\delta} M_{n+2}(A) \|u\|_{\mathbb{R}_n^*}. \]

**Proof.** We see that
\[ \tilde{\eta}(D) A(D) A_+ u = F^* \left( \tilde{\eta}(\xi) A(\xi) \xi F \tilde{u} \right)_{\mathbb{R}_n^*} = F^* \left( A(\xi) \xi_n F \tilde{u} \right)_{\mathbb{R}_n^*} = F^* \left( A(\xi) F \left( \frac{1}{\sqrt{-1}} \frac{\partial \tilde{u}}{\partial x_n} \right) \right)_{\mathbb{R}_n^*} = \frac{1}{\sqrt{-1}} \tilde{A}(D) \left( \frac{\partial \tilde{u}}{\partial x_n} \right)_{\mathbb{R}_n^*} \]
and \( A(D) f(x) \tilde{\eta}(D) A_+ u = \frac{1}{\sqrt{-1}} \tilde{A}(D) \left( f(x) \frac{\partial \tilde{u}}{\partial x_n} \right)_{\mathbb{R}_n^*} \) for \( \frac{\partial \tilde{u}}{\partial x_n} \) is the odd func-
tion, from which and the theory of ordinary singular integrals it implies

\[
\left\| \left( f(x) \eta(D) A(D) - A(D) f(x) \eta(D) \right) A_u \right\|_{x_n > 0} = \left\| \frac{1}{\sqrt{-1}} \left( \tilde{f}(x) \tilde{A}(D) \frac{\partial \tilde{u}}{\partial x_n} - \tilde{A}(D) \tilde{f}(x) \frac{\partial \tilde{u}}{\partial x_n} \right) \right\|_{x_n > 0} \\
\leq C \left\| \tilde{f}(x) \right\|_{1+\delta} M_{n+2}(A) \left\| \tilde{u} \right\|_{x_n > 0} \leq C \left\| f(x) \right\|_{1+\delta} M_{n+2}(A) \left\| u \right\|_{x_n > 0}.
\]

Lemma 8. Let \( f(x) \) be as in Lemma 7. Then for \( u(x) \in D(A_u) \)

\[
\left\| \left( f(x) \eta(D) A_u A, f(x) \eta(D) \right) u \right\|_{x_n > 0} \leq C \left\| f \right\|_{1+\delta} \left\| u \right\|_{x_n > 0}.
\]

Proof. From that \( \eta(D) \tilde{u} \) is the odd function it follows

\[
\left\| \left( f(x) \eta(D) A_u - A_u f(x) \eta(D) \right) u \right\|_{x_n > 0} = \left\| \tilde{f}(x) \eta_1(D) A \tilde{u} - A \tilde{f}(x) \eta_1(D) \tilde{u} \right\|_{x_n > 0} \\
\leq \left\| \tilde{f}(x) A - A \tilde{f}(x) \right\|_{x_n > 0} \leq C \left\| f \right\|_{1+\delta} \left\| u \right\|_{x_n > 0}.
\]

Lemma 9. Let \( f(x) \) be as in Lemma 7. Then for \( u(x) \in D(A_u) \)

\[
\left\| \left( f(x) \eta(D) \right)^* f(x) \eta(D) A_u u \right\|_{x_n > 0} \leq C \left\| f \right\|_{1+\delta} \left\| u \right\|_{x_n > 0}.
\]

Proof. Let \( v(x) \in C_0^\infty(R^n) \). Then

\[
\left( \left( f(x) \eta(D) \right)^* f(x) \eta(D) A_u u, v \right)_{x_n > 0} = \left( f(x) \eta(D) A_u u, v \right)_{x_n > 0} - \left( f(x) \eta(D) A_u u, v \right)_{x_n > 0} \\
= \left( f(x) \eta(D) A_u u, v \right)_{x_n > 0} - \left( f(x) \eta(D) A_u u, v \right)_{x_n > 0} \\
+ \left( f(x) \eta(D) A_u u, v \right)_{x_n > 0}.
\]

From Lemma 8 the absolute value of the first term on the right is bounded by \( C \left\| f \right\|_{1+\delta} \left\| u \right\|_{x_n > 0} \left\| v \right\|_{x_n > 0} \). The rest terms on the right are calculated as follows:

\[
\left( u, f(x) \eta(D) A_u u \right)_{x_n > 0} - \left( f(x) \eta(D) A_u u, v \right)_{x_n > 0} \\
= \left( u, f(x) \frac{1}{\sqrt{-1}} \frac{\partial \tilde{u}}{\partial x_n} \right)_{x_n > 0} - \left( f(x) \frac{1}{\sqrt{-1}} \frac{\partial \tilde{u}}{\partial x_n}, v \right)_{x_n > 0}.
\]

From this it follows
Hence
\[
\left(\left( f(x) \eta(D) \right)^* - f(x) \eta(D) \right) A_+ u, v \mathfrak{z}_{n>0} \leq C |f|_{1+\delta} \|u\|_{\mathfrak{z}_{n>0}} \|v\|_{\mathfrak{z}_{n>0}}.
\]

Because that \( C_0^\infty (R^m) \) is dense in \( L^2(R^m) \), it follows
\[
\left(\left( f(x) \eta(D) \right)^* - f(x) \eta(D) \right) A_+ u \mathfrak{z}_{n>0} \leq C |f|_{1+\delta} \|u\|_{\mathfrak{z}_{n>0}}.
\]

**Lemma 10.** Let \( f(x) \) be as in Lemma 7. Then for \( u(x) \in D(A_+) \)
\[
\left(\eta(D)f(x) \eta(D) - f(x) \eta(D) \right) A_+ u \mathfrak{z}_{n>0} \leq C |f|_{1+\delta} \|u\|_{\mathfrak{z}_{n>0}}.
\]

**Proof.** We see that
\[
\eta(D)f(x) \eta(D) A_+ u = F' \eta(\xi) F \left( \frac{1}{\sqrt{1 - \xi_n}} \frac{\partial \tilde{u}}{\partial x_n} \right)_{\mathfrak{z}_{n>0}}
\]
and \( \eta(D) A_+ u = F' \eta(\xi)\left|\xi\right| F\tilde{u} \mathfrak{z}_{n>0} = F' \eta(\xi) F \left( \frac{1}{\sqrt{1 - \xi_n}} \frac{\partial \tilde{u}}{\partial x_n} \right)_{\mathfrak{z}_{n>0}}, \) from which it follows
\[
\left(\eta(D)f(x) \eta(D) - f(x) \eta(D) \right) A_+ u \mathfrak{z}_{n>0} \leq C \left| \frac{\partial \tilde{u}}{\partial x_n} \right|_{\mathfrak{z}_{n>0}} \leq C |f|_{1+\delta} \|u\|_{\mathfrak{z}_{n>0}}.
\]

q. e. d.

Now decompose \( Y_{l,m}(\xi) \) as follows:
\[
Y_{l,m}(\xi) = Y_{l,m}(\xi', \xi_n)
\]
\[
= \frac{1}{2} \left( Y_{l,m}(\xi', \xi_n) + Y_{l,m}(\xi', -\xi_n) \right) + \frac{1}{2} \left( Y_{l,m}(\xi', \xi_n) - Y_{l,m}(\xi', -\xi_n) \right)
\]
\[
= Y_{l,m}^+(\xi', \xi_n) + Y_{l,m}^-(\xi', \xi_n).
\]

Then we find that every \( Y_{l,m}^+(\xi', \xi_n) \left( Y_{l,m}^-(\xi', \xi_n) \right) \) becomes an even (odd) function with respect to \( \xi_n \) and \( \int_{\Sigma} Y_{l,m}^-(\xi) d\Sigma = 0 = \int_{\Sigma} Y_{l,m}^+(\xi) d\Sigma. \)

**Lemma 11.** We have the following estimate for every \( l, m \) and \( \alpha. \)
\[
\sup_{|\xi| = 1} \left| \frac{\partial}{\partial \xi} \right|^\alpha Y_{l,m}^\pm(\xi', \xi_n) \leq \sup_{|\xi| = 1} \left| \frac{\partial}{\partial \xi} \right|^\alpha Y_{l,m}(\xi', \xi_n).
\]
Proof. We see that
\[ \sup_{|\xi|=1} \left| \left( \frac{\partial}{\partial \xi} \right)^n Y_{l,m}(\xi', -\xi_n) \right| = \sup_{|\xi|=1} \left| \left( \frac{\partial}{\partial \xi} \right)^n Y_{l,m}(\xi', -\xi_n) \right| \]
for every \( l, m \) and \( \alpha \). From which it implies the desired inequality. q. e. d.

We shall prove Theorem 1 using the preceding Lemmas.

Proof of Theorem 1. Similarly to the proofs of Lemmas 3–10, we may calculate this theorem for \( u(x) \in C^\infty_{\text{loc}}(\mathbb{R}^n) \) and \( v(x) \in C^\infty_{\text{loc}}(\mathbb{R}^n) \) if it needs. Without loss of generality, it suffices to prove for \( \alpha(x, \xi) = a(x, \xi)f(x) \frac{\xi}{|\xi|} \) and \( \beta(x, \xi) = b(x, \xi)g(x) \frac{\xi}{|\xi|} \). By definition 2 \( a(x, \xi) \) is the even function with respect to \( \xi_n \). Therefore, as stated in the proof of Lemma 1 we see that \( a(x, \xi) = \sum \alpha_{l,m}(x) Y_{l,m}(\xi) = \sum \alpha_{l,m}(x) Y_{l,m}(\xi) \). We furthermore remark that \( A_+ Y_{l,m}^+(D) u = Y_{l,m}^+(D) A_+ u \) for \( u \in D(A_+) \).

Proof of (8). We see that
\[
\alpha(x, D) u = F'f(x) \eta(\xi) a(x, \xi) F u \bigg|_{x_n>0} \\
= F'f(x) \eta(\xi) \sum_{l,m} a_{l,m}(x) Y_{l,m}^+(\xi) F u \bigg|_{x_n>0} \\
= \sum_{l,m} a_{l,m}(x) f(x) \eta(D) Y_{l,m}^+(D) u.
\]
From this
\[
\alpha(x, D) A_+ - A_+ \alpha(x, D) \\
= \sum_{l,m} a_{l,m}(x) f(x) \eta(D) Y_{l,m}^+(D) A_+ - A_+ \sum_{l,m} a_{l,m}(x) f(x) \eta(D) Y_{l,m}^+(D) \\
= \sum_{l,m} a_{l,m}(x) \left( f(x) \eta(D) A_+ - A_+ f(x) \eta(D) \right) Y_{l,m}^+(D) \\
+ \sum_{l,m} \left( a_{l,m}(x) A_+ - A_+ a_{l,m}(x) \right) f(x) \eta(D) Y_{l,m}^+(D) .
\]

Hence applying (2), (4) and Lemma 11 to all the terms, Lemma 8 to the first term and Lemma 3 to the second term on the right we find that
\[
\left\| (\alpha(x, D) A_+ - A_+ \alpha(x, D)) u \right\|_{x_n>0} \\
\leq CM_{2n}(\alpha) \sum \left| f \right|_{l^{2n-\frac{n}{2}}} \left| f \right|_{l^{2n-\frac{n}{2}}} \left| f \right|_{l^{2(\frac{n}{2} - 1)}} \|u\|_{x_n>0} \\
+ CM_{2n}(\alpha) \sum \left| f \right|_{l^{2n-\frac{n}{2}}} \left| f \right|_{l^{2n-\frac{n}{2}}} \left| f \right|_{l^{2(\frac{n}{2} - 1)}} \|u\|_{x_n>0} \leq CM_{2n}(\alpha) \|u\|_{x_n>0} .
\]

Proof of (9). Using (8), we see that
\[ \left| \left( (\alpha(x, D)^* A_+ - A_+ \alpha(x, D)^*) \right) u, v \right|_{x_n > 0} \]
\[ = \left| \left( u, \left( A_+ \alpha(x, D) - \alpha(x, D) A_+ \right) v \right) \right|_{x_n > 0} \leq CM_{2n}(\alpha) \| u \|_{x_n > 0} \| v \|_{x_n > 0}. \]

Therefore
\[ \left\| (\alpha(x, D)^* A_+ - A_+ \alpha(x, D)^*) u \right\|_{x_n > 0} \leq CM_{2n}(\alpha) \| u \|_{x_n > 0}. \]

**Proof of (10).** We see that
\[ (\alpha(x, D)^* - \alpha^*(x, D)) A_s \]
\[ = \sum_{l,m} \left( Y_{l,m}^*(D) \left( f(x) \eta(D) \right)^* a_{l,m}(x) - a_{l,m}(x) f(x) \eta(D) Y_{l,m}^*(D) \right) A_+ \]
\[ = \sum_{l,m} \left( Y_{l,m}^*(D) \left( f(x) \eta(D) \right)^* \left( a_{l,m}(x) A_+ - A_+ a_{l,m}(x) \right) \right) \]
\[ + \sum_{l,m} Y_{l,m}^*(D) \left( \left( f(x) \eta(D) \right)^* - f(x) \eta(D) \right) A_+ a_{l,m}(x) \]
\[ + \sum_{l,m} f(x) \eta(D) \left( Y_{l,m}^*(D) A_+ a_{l,m}(x) - a_{l,m}(x) Y_{l,m}^*(D) A_+ \right) \]
\[ + \sum_{l,m} f(x) \left( \eta(D) a_{l,m}(x) - a_{l,m}(x) \eta(D) \right) \right) A_+ Y_{l,m}^*(D). \]

Hence applying (2), (4) and Lemma 11 to all the terms, Lemma 3 to the first term, Lemma 9 to the second term, Lemma 7 to the third term, Lemma 5 to the fourth term and Lemma 6 to the last term, we find that
\[ \left\| (\alpha(x, D)^* - \alpha^*(x, D)) A_s u \right\|_{x_n > 0} \]
\[ \leq C \sum_{l} l^{1/2(n-2)} |f|_0 l^{-2n + n \over 2} l^{n-2} M_{2n}(\alpha) \| u \|_{x_n > 0} \]
\[ + C \sum_{l} l^{1/2(n-2)} |f|_{1+} l^{-2n + n \over 2} l^{n-2} M_{2n}(\alpha) \| u \|_{x_n > 0} \]
\[ + C \sum_{l} |f|_{1+} l^{1/2(n-2) + n + 2} l^{-2[2/3]n + 1} + n \over 2} l^{n-2} M_{2[2/3]n + 1}(\alpha) \| u \|_{x_n > 0} \]
\[ + C |f|_0 \sum_{l} l^{-2[2/3]n + 1} + n \over 2} l^{n-2} l^{1/2(n-2) + n + 2} M_{2[2/3]n + 1}(\alpha) \| u \|_{x_n > 0} \]
\[ + C \sum_{l} l^{-2n + n \over 2} l^{n-2} l^{1/2(n-2)} M_{2n}(\alpha) \| u \|_{x_n > 0} \]
\[ \leq CM_{2[2/3]n + 1}(\alpha) \| u \|_{x_n > 0}. \]

**Proof of (11).** We see that
\[ (\alpha \circ \beta)(x, D)u = F'f(x)\eta(\xi)a(x, \xi)g(x)\eta(\xi)b(x, \xi)F\tilde{u}|_{\xi_n > 0} \]

\[ = f(x)g(x)F'\left(\eta(\xi)\right)^2 a(x, \xi)b(x, \xi)F\tilde{u}|_{\xi_n > 0} \]

\[ = f(x)g(x)F'\left(\eta(\xi)\right)^2 \sum_{l,m} a_{l,m}(x)Y_{l,m}(\xi) \sum_{p,q} b_{p,q}(x) Y_{p,q}(\xi)F\tilde{u}|_{\xi_n > 0} \]

\[ = \sum_{l,m} \sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)g(x)\eta_2(D)Y_{l,m}^+(D)Y_{p,q}^+(D)u. \]

From this

\[ (\alpha(x, D)\beta(x, D) - (\alpha \circ \beta)(x, D)) \]

\[ = \sum_{l,m} a_{l,m}(x)f(x)\eta(D)Y_{l,m}^+(D)\sum_{p,q} b_{p,q}(x)g(x)\eta(D)Y_{p,q}^+(D)A_+ \]

\[ - \sum_{l,m} \sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)g(x)\eta_2(D)Y_{l,m}^+(D)Y_{p,q}^+(D)A_+ \]

\[ + \sum_{l,m} \sum_{p,q} a_{l,m}(x)f(x)\eta(D)Y_{l,m}^+(D)\left(b_{p,q}(x)A_+ - A_+b_{p,q}(x)\right)g(x)\eta(D)Y_{p,q}^+(D) \]

\[ + \sum_{l,m} \sum_{p,q} a_{l,m}(x)f(x)\eta(D)\left(Y_{l,m}^+(D)A_+ b_{p,q}(x) - b_{p,q}(x)Y_{l,m}^+(D)A_+ \right) \]

\[ \times g(x)\eta(D)Y_{p,q}^+(D) \]

\[ + \sum_{l,m} \sum_{p,q} a_{l,m}(x)f(x)\left(\eta(D)b_{p,q}(x) - b_{p,q}(x)\eta(D)\right)A_+Y_{l,m}^+(D)g(x)\eta(D)Y_{p,q}^+(D) \]

\[ + \sum_{l,m} \sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)\eta(D)Y_{l,m}^+(D)\left(A_+g(x)\eta(D) - g(x)\eta(D)A_+ \right)Y_{p,q}^+(D) \]

\[ + \sum_{l,m} \sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)\eta(D)\left(Y_{l,m}^+(D)g(x)\eta(D) - g(x)\eta(D)Y_{l,m}^+(D) \right) \]

\[ \times A_+Y_{p,q}^+(D) \]

\[ + \sum_{l,m} \sum_{p,q} a_{l,m}(x)b_{p,q}(x)f(x)\left(\eta(D)g(x)\eta(D) - g(x)\eta_2(D)\right)A_+Y_{l,m}^+(D)Y_{p,q}^+(D). \]

Hence applying (2) (4) and Lemma 11 to all the terms, Lemma 8 to the first term, Lemma 3 to the second term, Lemma 5 to the third term, Lemma 6 to the fourth term, Lemma 8 to the fifth term, Lemma 7 to the sixth term and Lemma 10 to the last term, similarly to the proof of (10) we find that

\[ \left\| (\alpha(x, D)\beta(x, D) - (\alpha \circ \beta)(x, D))A_+u \right\|_{\xi_n > 0} \leq CM z([\alpha_1^{-1}])(\alpha)M_{2n}(\beta)\|u\|_{\xi_n > 0}. \]

q.e.d.

Furthermore we consider the singular integral operators with boundary conditions on \( R^n_+ \) for more extended symbol.

**Definition 4.** Let \( \mathcal{A} \) be the set \( \alpha(x, \xi) \) such that
\[ \alpha(x, \xi) = \sum_{\ell=1}^{\infty} \alpha_{\ell}(x, \xi), \alpha_{\ell}(x, \xi) \in \mathfrak{U} \text{ and for every integer } s(\geq 0), \text{ there exists } M_s(\alpha)(< \infty) \text{ such that} \]
\[ \sum_{\ell=1}^{\infty} M_s(\alpha_{\ell}) \leq M_s(\alpha). \]

For \( \alpha(x, \xi) \in \mathfrak{U} \) consider the singular integral operator \( \alpha(x, D) \) as follows:
\[ \alpha(x, D)u = \sum_{\ell=1}^{\infty} \alpha_{\ell}(x, D)u \text{ in } L^2(\mathbb{R}^n) \text{ for } u \in L^2(\mathbb{R}^n). \]

Here we remark that \( \alpha(x, \xi) \in \mathcal{E}_{\infty} \) is in \( \mathfrak{U} \) if and only if \( \alpha(x', 0, \xi) \) is an even function with respect to \( \xi_n \).

**Theorem 2.** For the symbols \( \alpha(x, \xi) \) and \( \beta(x, \xi) \in \mathfrak{U} \), \( \alpha(x, D)u, \alpha(x, D)^* u \in D(A_{\pm}) \) for \( u \in D(A_{\pm}) \) and the statements of Theorem 1 are also valid.

**Proof.** The first statement is shown by virtue of Lemma 1, (12) and Theorem 1. The last statement is shown by virtue of Theorem 1 and (12) regarding \( \alpha_{\ell}(x, \xi) \) and \( \beta_{\ell}(x, \xi) \in \mathfrak{U} \). Hence we prove only (11).

\[ \| (\alpha(x, D)\beta(x, D) - (\alpha \circ \beta)(x, D)) A_{\pm} u \|_{x_n>0} \]
\[ \leq \| \left( \sum_{\ell=1}^{\infty} \alpha_{\ell}(x, D)\beta_{\ell}(x, D) - \sum_{\ell, j} (\alpha_{\ell} \circ \beta_{j})(x, D) \right) A_{\pm} u \|_{x_n>0} \]
\[ \leq \sum_{\ell, j} \| \left( \alpha_{\ell}(x, D)\beta_{j}(x, D) - (\alpha_{\ell} \circ \beta_{j})(x, D) \right) A_{\pm} u \|_{x_n>0} \]
\[ \leq C \sum_{\ell, j} M_{\mathcal{E}_{\infty}}(\alpha_{\ell}) M_{2n}(\beta_{j}) \| u \|_{x_n>0} \]
\[ \leq C \sum_{\ell} M_{\mathcal{E}_{\infty}}(\alpha_{\ell}) \sum_{j} M_{2n}(\beta_{j}) \| u \|_{x_n>0} \]
\[ \leq C M_{\mathcal{E}_{\infty}}(\alpha) M_{2n}(\beta) \| u \|_{x_n>0} . \quad \text{q.e.d.} \]

**Lemma 12.** (Kohn and Nirenberg see [8])

Let \( \sigma(x, \xi) \in \mathfrak{U} \) and \( \sigma(x, \xi) \geq d > 0 \) for \( x \in \mathbb{R}^n \) and \( |\xi| = 1 \). Then for sufficiently large \( k > 0 \)
\[ \| (\sigma(x, D)A_{\pm} + k) u \|_{x_n>0}^2 \geq (d - \varepsilon)^2 \| A_{\pm} u \|_{x_n>0}^2 + (k - \rho)^2 \| u \|_{x_n>0}^2 \]
for \( u \in D(A_{\pm}) \), where \( \varepsilon \) is an arbitrary small positive number, \( k > \rho \) and \( \rho \) depends on \( n, \sigma \) and \( \varepsilon \).

**Proof.** We see that
\[ \| (\sigma(x, D)A_{\pm} + k) u \|_{x_n>0}^2 \]
\[ = \| \sigma A_{\pm} u \|_{x_n>0}^2 + k \left( \langle \sigma A_{\pm} u, u \rangle_{x_n>0} + \langle u, \sigma A_{\pm} u \rangle_{x_n>0} \right) + k^2 \| u \|_{x_n>0}^2. \]
We at first estimate the first term on the right. Regarding that \( \sigma(x, D) = \sigma^*(x, D) \) since \( \sigma(x, \xi) \) is real,

\[
(\sigma A_x u, \sigma A_x u)_{x_n > 0} = (\sigma A_x u, (\sigma^* - \sigma^*) A_x u)_{x_n > 0} + (\sigma \circ \sigma A_x u, A_x u)_{x_n > 0}.
\]

The first two terms are bounded by \( \varepsilon_1 \| A_x u \|^2_{x_n > 0} + c(\varepsilon_1) \| u \|^2_{x_n > 0} \) by virtue of Theorem 2. Let us estimate the last term on the right. From the assumption we can choose \( \sigma_1(x, \xi) \in \mathfrak{F} \) such that \( \sigma(x, \xi) = (d - \varepsilon')^2 + (\sigma_1(x, \xi))^2, \ d > \varepsilon' > 0, \ \sigma_1(x, \xi) \geq \varepsilon' > 0 \) for \( x \in \mathbb{R}^n, \ |\xi| = 1 \).

Then

\[
(\sigma \circ \sigma A_x u, A_x u)_{x_n > 0}
= (d - \varepsilon')^2 \| A_x u \|^2_{x_n > 0} + (\sigma_1 \circ \sigma_1 A_x u, \sigma_1 A_x u)_{x_n > 0}
= (d - \varepsilon')^2 \| A_x u \|^2_{x_n > 0} + (\sigma_1^2 - \sigma_1^* \sigma_1 - \sigma_1 \sigma_1^*) A_x u, A_x u)_{x_n > 0}
+ (\sigma_1 A_x u, (\sigma_1^* - \sigma_1^*) A_x u)_{x_n > 0} + (\sigma_1 A_x u, \sigma_1 A_x u)_{x_n > 0}.
\]

The second and the third terms on the right are also bounded by \( \varepsilon_1 \| A_x u \|^2_{x_n > 0} + c(\varepsilon_1) \| u \|^2_{x_n > 0} \) by virtue of Theorem 2. Therefore

\[
\| \sigma A_x u \|^2_{x_n > 0} \geq (d - \varepsilon')^2 \| A_x u \|^2_{x_n > 0} - 2 \varepsilon_1 \| A_x u \|^2_{x_n > 0} - c(\varepsilon_1) \| u \|^2_{x_n > 0}.
\]

Next we estimate \( (\sigma A_x u, u)_{x_n > 0} + (u, \sigma A_x u)_{x_n > 0} \). From the assumption \( \sigma(x, \xi) \geq d > 0, \) we see that \( (\sigma(x, \xi))^{\frac{1}{2}} \in \mathfrak{F} \).

From this

\[
(\sigma A_x u, u)_{x_n > 0} = (\sigma^{\frac{1}{2}} \circ \sigma^{\frac{1}{2}} A_x u, u)_{x_n > 0}
= (\sigma^{\frac{1}{2}} A_x^{\frac{1}{2}} u, \sigma^{\frac{1}{2}} u)_{x_n > 0} + (\sigma^{\frac{1}{2}} - \sigma^{\frac{1}{2}}^*) A_x^{\frac{1}{2}} u, \sigma^{\frac{1}{2}} u)_{x_n > 0}
+ (\sigma^{\frac{1}{2}} (\sigma^{\frac{1}{2}} A_x - A_x^{\frac{1}{2}} u), u)_{x_n > 0} + ((\sigma^{\frac{1}{2}} \circ \sigma^{\frac{1}{2}} - \sigma^{\frac{1}{2}} \circ \sigma^{\frac{1}{2}}) A_x u, u)_{x_n > 0}.
\]

By virtue of Theorem 2 the last three terms are bounded by \( C \| u \|^2_{x_n > 0} \). From \( (\sigma A_x u, \sigma^{\frac{1}{2}} u)_{x_n > 0} \) it follows that

\[
(13)
(\sigma A_x u, u)_{x_n > 0} + (u, \sigma A_x u)_{x_n > 0} \geq - C \| u \|^2_{x_n > 0}.
\]

Therefore

\[
\| (\sigma A_x + k) u \|^2_{x_n > 0} \geq (d - \varepsilon')^2 \| A_x u \|^2_{x_n > 0} - 2 \varepsilon_1 \| A_x u \|^2_{x_n > 0} - c(\varepsilon_1) \| u \|^2_{x_n > 0}
- Ck \| u \|^2_{x_n > 0} + k^2 \| u \|^2_{x_n > 0}.
\]

By setting \( k \) sufficiently large and \( \varepsilon_1 \) sufficiently small \( (> 0) \), we obtain the
desired estimate. \[\text{q.e.d.}\]

2.2. Singular integral operators with boundary conditions defined on \(\Omega\).

We at first describe the coordinate transformations, from which we reduce above ones to singular integral operators with boundary conditions on \(\mathbb{R}^n\).

**Lemma 13.** There exist a finite covering \(\{U_a\}\) of \(\Omega\) and diffeomorphisms \(T_a\) from \(U_a\) into \(\mathbb{R}^n\) such that the following 1), 2) and 3) are satisfied:

1) If \(U_a \cap \Gamma \neq \emptyset\), then \(T_a(U_a \cap \Omega) \subset \mathbb{R}^n\), \(T_a(U_a \cap \Gamma) \subset \mathbb{R}^{n-1}\) and \(T_a(U_a \cap \Gamma)\) contains origin.

2) Let \(a_0(\xi, D)\) be the principal part of \(a(x, D)\) and let \(b_\alpha(\eta, \xi) = \sum_{i,j} b_{ij}^{(a)}(\eta) \xi_i \xi_j |\xi|^{-2} = a_0(T_a^{-1}(\eta), (dT_a^*)^i(\xi)) |\xi|^{-2}\) for \(\eta \in T_a(U_a)\) and \(\xi \in \mathbb{R}^n\setminus\{0\}\), then \(b_\alpha^{(a)}(\eta) = 0\) for \(n = 0\) and for \(n \neq j\), that is, the conormal direction of \(a(x, D)\) at \(\Gamma\) transfers to the normal direction of the surface \(\mathbb{R}^{n-1}\), here \(dT_a\) is the differential from the tangent space \(T_x\) to the tangent space \(T_{T_a(x)}\) and \(dT_a^*\) is the dual differential of \(dT_a\) from \(T_{T_a(x)}^*\) to \(T_x^*\).

3) If \(U_a \cap U_b \cap \Gamma \neq \emptyset\) and \(y \in T_a(U_a \cap U_b)\), then the \(n\)-th component of \(T_a^{-1}(y)\) is equal to \(y_n\).

**Proof.** We can choose a function \(\varphi(x)\) such that \(\varphi(x) \in C^\infty(\mathbb{R}^n)\), \(\text{grad } \varphi(x) \neq 0\) on \(U(\Gamma')\) and \(\varphi(x) = 0\) if and only if \(x \in \Gamma\). Furthermore for \(x_0 \in \Gamma\) there exists a diffeomorphism \(S'\) from some neighborhood of origin \(U'(0) \subset \mathbb{R}^{n-1}\) into \(\Gamma\) such that \(S'(0) = x_0\).

Now, for any \(y' \in U'(0)\) we construct a curve \(S'(y', y_n)\) transversal to \(\Gamma\) through \(S'(y') \in \Gamma\). To this end it suffices to solve the following ordinary differential equations:

\[
\frac{dS_i(y', y_n)}{dy_n} = \sum_{j=1}^{n} a_{ij}^{(a)}(S(y', y_n)) \frac{\partial \varphi}{\partial x_j}(S(y', y_n)) \quad i = 1, \ldots, n,
\]

\[
S(y', 0) = S'(y') \quad \text{for } y' \in U'(0).
\]

Suppose that \(S(y', y_n)\) is not transversal at \(S'(y')\), that is, there exists \(\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1}\) such that

\[
\sum_{j=1}^{n} a_{ij}^{(a)}(x) \frac{\partial \varphi}{\partial x_j}(x) = \sum_{k=1}^{n-1} \lambda_k \frac{\partial S_i'}{\partial y_k}(x) \quad i = 1, \ldots, n,
\]

where \(x = S'(y') = (S_1'(y'), \ldots, S_n'(y'))\).

From \(\varphi(S'(y')) = 0\) it follows that
\[ \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} \frac{\partial S_i'}{\partial y_k} = 0 \quad \text{for} \quad k=1, \ldots, n-1. \]

Hence
\[ \sum_{i} \frac{\partial \varphi}{\partial x_i} \sum_{j} a_{i,j}(x) \frac{\partial \varphi}{\partial x_j} = \sum_{i} \frac{\partial \varphi}{\partial x_i} \sum_{k} \lambda_k \frac{\partial S_i'}{\partial y_k} = \sum_{k} \lambda_k \sum_{i} \frac{\partial \varphi}{\partial x_i} \frac{\partial S_i'}{\partial y_k} = 0, \]
which is a contradiction for \( \text{grad} \varphi \equiv 0 \) and \( a(x, \xi) \) is elliptic. Therefore \( S(y', y_n) \) is transversal at \( S'(y') \).

Consequently, if we set \( U(0) = U'(0) \times \{ y_n ; |y_n| < \varepsilon \} \) for sufficiently small \( \varepsilon (>0) \), then we see that \( S \) indicate a diffeomorphism from \( U(0) \) into \( R^n \).

Denote \( U = S(U(0)) \) and the inverse transformation of \( S \) by \( T \). Then we find that a finite covering \( \{ U_i \} \) and diffeomorphisms \( T_i \) are obtained as \( I \) is compact. Consequently, from the construction \( 1) \) is shown and \( 3) \) is found from the facts that \( a_{i,j}(x) \) and \( \varphi(x) \) are given in global and from the uniqueness of solution of the equation \((14)\). Now we show \( 2) \).

From \( b_*(y, \xi) = a_0 \left( T_*^{-1}(y), (dT_*')(\xi) \right) \)
\[ = \sum_{i,j} \left( \sum_{k,l} a_{k,l}(S_*(y)) \frac{\partial T_{*k}}{\partial x_k} (S_*(y)) \frac{\partial T_{*l}}{\partial x_l} (S_*(y)) \xi_i \xi_j \right) \quad \text{for} \quad |\xi| = 1 \]
it follows that
\[ (15) \quad b_{n,j}^*(y) = \sum_{k,l} a_{k,l}(S_*(y)) \frac{\partial T_{*k}}{\partial x_k} (S_*(y)) \frac{\partial T_{*l}}{\partial x_l} (S_*(y)), \]
where
\[ T_*(x) = \left( T_{*1}(x), \ldots, T_{*n}(x) \right). \]
Furthermore from \( \varphi(S_*(y', 0)) = 0 \) we find that there exists a function \( \phi(y) \in C^\infty(R^n) \) such that \( \phi(y', 0) \equiv 0 \), and \( \varphi(S_*(y', y_n)) = y_n \phi(y') \), that is, \( \varphi(x) = T_{*n}(x) \phi(T_*(x)) \).

From this and \((14)\) it follows that
\[ \frac{\partial S_{*k}}{\partial y_n} (y', 0) = \sum_{j} a_{i,j}(S_*(y', 0)) \frac{\partial (T_{*n}(x))^{\phi(T_*(x))}}{\partial x_j} (S_*(y', 0)) \]
\[ = \sum_{j} \tilde{a}_{i,j}(S_*(y', 0)) \frac{\partial T_{*n}}{\partial x_j} (S_*(y', 0)) \phi(y', 0). \]

By \( (\tilde{a}_{i,j}) \) set the inverse matrix of \( (a_{i,j}^0) \). Then from \((16)\)
\[ \frac{\partial T_{*n}}{\partial x_j} (S_*(y', 0)) = \sum_{i} \tilde{a}_{i,k}(S_*(y', 0)) \frac{\partial S_{*k}}{\partial y_n} (y', 0) \cdot (\phi(y', 0))^{-1}. \]
Substituting this into \((15)\) setting \( y_n = 0 \), it follows that
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\[ b_{u*}^J(y', 0) = \sum_{k,l} \delta_{k,l}(S_u(y', 0)) \sum_{i} \alpha_{k,l}(S_u(y', 0)) \frac{\partial S_d}{\partial y_n}(y', 0) \]
\[ \times \frac{\partial T_{aJ}}{\partial x_i}(S_u(y', 0)) \cdot (\phi(y', 0))^{-1} \]
\[ = \sum_{k,l} \alpha_{k,l}(S_u(y', 0)) \frac{\partial T_{aJ}}{\partial y_n}(S_u(y', 0)) \cdot (\phi(y', 0))^{-1} \]
\[ = \sum_{i} \frac{\partial S_d}{\partial y_n}(y', 0) \frac{\partial T_{aJ}}{\partial x_i}(S_u(y', 0)) \cdot (\phi(y', 0))^{-1} \]
\[ = \delta_{y_n} \cdot (\phi(y', 0))^{-1}. \]

q.e.d.

We now take a partition of unity \( \{ \varphi_i \} \) with respect to \( \Omega \) such that \( \sum_{i=1}^{\hat{y}} \varphi_i(x) = 1 \) on \( \Omega \), \( \varphi_i, \mu \in D(a) \) for \( u \in D(a) \) and \( \{ \text{Supp} \varphi_i \} \) is a star-finite refinement of \( \{ U_u \} \) which is defined in Lemma 13 and also if \( \text{Supp} \varphi_j \cap \Gamma = \phi \), then \( (\text{Supp} \varphi_j)^o \cap \Gamma \neq \phi \), here \( G^0 \) is an interior of \( G \).

**Definition 5.** Let \( \mathbb{H}_u \) be the set of \( \sigma(x, \xi) \) such that \( \sigma(x, \xi) \in \mathbb{Z}_1(\Omega) \times (\mathbb{R}^n - \{0\}) \) and for \( \text{Supp} \varphi_j \cap \Gamma = \phi \) and \( \text{Supp} \varphi_i \cap \text{Supp} \varphi_j = \phi \)
\[ (T_{a*} \sigma \varphi_i) \cdot \tilde{\sigma} \left( T_{a*}^{-1}(y), (dT_{a*})(\xi) \right) \cdot (T_{a*} \varphi_j) \in \mathbb{H}_u. \]

In the above by \( \mathbb{Z}_1(\Omega) \times (\mathbb{R}^n - \{0\}) \) we mean the set being replaced \( \mathbb{R}^n \) by \( \Omega \) in Definition 1, \( U_u \supset \text{Supp} \varphi_i \cup \text{Supp} \varphi_j \), \( (T_{a*} \sigma u)(x) = u(T_{a*}^{-1}(x)) \) and \( \tilde{\sigma}(T_{a*}^{-1}(y), (dT_{a*})(\xi)) = (\phi(y)) \sigma \left( T_{a*}^{-1}(y), (dT_{a*})(\xi) \right) + (1 - \phi(y)) \sigma \left( T_{a*}^{-1}(0), (dT_{a*})(\xi) \right) \) for \( \phi(y) \in C_0^\infty(T_u(U_u)) \) such that \( \phi(y) = 1 \) for \( y \in T_u(\text{Supp} \varphi_i \cup \text{Supp} \varphi_j) \).

**Definition 6.** For \( \sigma(x, \xi) \in \mathbb{H}_u \) consider the singular integral operator \( \sigma(x, D) \) as follows: for \( u \in L^2(\Omega) \)
\[ \sigma(x, D)u = \sum_{ij} \sigma_{ij}(x, D)u. \]

Where if \( \text{Supp} \varphi_j \cap \Gamma = \phi \), we set
\[ \sigma_{ij}(x, D)u = \varphi_i(x) \tilde{\sigma}(x, D) \varphi_j(x)u, \]
here \( \tilde{\sigma}(x, \xi) = \phi_1(x) \sigma(x, \xi) + (1 - \phi_1(x)) \sigma(x_0, \xi) \) for \( \phi_1(x) \in C_0^\infty(\mathbb{R}^n) \)
such that \( \phi_1(x) = 1 \) for \( x \in \text{Supp} \varphi_j \), \( \text{Supp} \varphi_j \subset \Omega \), \( x_0 \in (\text{Supp} \varphi_j)^o \) and \( \tilde{\sigma}(x, D) \) is the ordinary singular integral operator. Furthermore if \( \text{Supp} \varphi_j \cap \Gamma = \phi \) and \( \text{Supp} \varphi_i \cap \text{Supp} \varphi_j = \phi \), we set
\[ \sigma_{ij}(x, D)u = T_{a*} \left( (T_{a*} \varphi_i) \cdot \tilde{\sigma} \left( T_{a*}^{-1}(y), (dT_{a*})(D_x) \right) (T_{a*} \varphi_j)u \right), \]
here \( U_a \supset \text{Supp } \varphi_i \cup \text{Supp } \varphi_j \), \( \bar{a} \) is defined by Definition 5 (see Definition 4) and \( (T_a \ast w)(x) = w(T_a(x)) \). Finally if \( \text{Supp } \varphi_j \cap \Gamma = \phi \) and \( \text{Supp } \varphi_i \cap \text{Supp } \varphi_j = \phi \), we set
\[
\sigma_{ij}(x, D)u = 0.
\]

**Definition 7.** Let \( \bar{\Lambda}_\pm \) be as follows: for \( u \in D(a^{\frac{1}{2}}) \)
\[
\bar{\Lambda}_\pm u = \sum_{i,j} \bar{\Lambda}_\pm_{ij} u.
\]
Where if \( \text{Supp } \varphi_j \cap \Gamma = \phi \), we define
\[
\bar{\Lambda}_\pm_{ij} u = \varphi_i(x) \sqrt{a_0}(x, D)A\varphi_j(x)u,
\]
where \( \sqrt{a_0}(x, \xi) = \varphi_1(x) \sqrt{a_0}(x, \frac{\xi}{|\xi|}) + \left(1 - \varphi_1(x)\right) \sqrt{a_0}(x_0 - \frac{\xi}{|\xi|}) \), \( \varphi_1(x) \) is as in Definition 6 and \( \sqrt{a_0}(x, D) \) is the ordinary singular integral operator. Furthermore if \( \text{Supp } \varphi_j \cap \Gamma = \phi \) and \( \text{Supp } \varphi_i \cap \text{Supp } \varphi_j = \phi \), we define
\[
\bar{\Lambda}_\pm_{ij} u = T_a \ast \left((T_a \ast \varphi_i) \sqrt{b_a}(y, D)A\varphi_j(x)u\right),
\]
here \( U_a \supset \text{Supp } \varphi_i \cup \text{Supp } \varphi_j \), \( \sqrt{b_a}(y, D) \) is defined by Lemma 13 and Definition 4 like as Definition 6. Finally if \( \text{Supp } \varphi_j \cap \Gamma = \phi \) and \( \text{Supp } \varphi_i \cap \text{Supp } \varphi_j = \phi \), we define
\[
\bar{\Lambda}_\pm_{ij} u = 0.
\]

Then we shall show that the definitions of \( \sigma(x, D) \) and \( \bar{\Lambda}_\pm \) are invariant with respect to the choice of \( U_a \) satisfying Lemma 13. To this end we may prove the following Lemmas 14 and 15, using theorem 7 in [15] (page 247).

**Lemma 14.** For \( \sigma(x, \xi) \in \mathbb{H} \)
\[
||\sigma(x, D)\Lambda_{\pm} u - \text{ordinary singular integral operator } \sigma(x, D)Au||_{x_n > 0} \leq C||u||_{x_n > 0}
\]
for \( u \in D(a^{\frac{1}{2}}) \) with \( \text{Supp } u \subset \{x \in \mathbb{R}^n \mid x_n \geq d > 0\} \).

**Lemma 15.** Let \( T \) be a diffeomorphism from \( (\mathbb{R}^n_+, \mathbb{R}^n_{<+}) \) to \( (\mathbb{R}^n_-, \mathbb{R}^n_{<+}) \) and satisfy 3) of Lemma 13. Then for \( \sigma(x, \xi) \in \mathbb{H} \), \( \varphi_1(x), \varphi_2(x) \in C^\infty(\mathbb{R}^n_+) \)
\[
||T_\ast \left(\varphi_2 \sigma(x, D)\Lambda_{\pm} \varphi_1 u\right) - (T_\ast \sigma_2)(\tau(y, D)\Lambda_{\pm} T_\ast \sigma_1 u)||_{y_n > 0} \leq C||u||_{x_n > 0}
\]
for \( u \in D(a^{\frac{1}{2}}) \), here \( \tau(y, \eta) = \frac{\tau_0(T^{-1}(y), (dT^\ast)(\eta))}{|\eta|} \in \mathbb{H} \) and \( \tau_0(x, \xi) = \sigma(x, \xi)|\xi| \) (see the remark next to Definition 4).

**Proof of Lemma 14.** We see that
\[
\sigma(x, D)\Lambda_{\pm} u = F' \sigma_1(x, \xi)|\xi| F\bar{n}|_{x_n > 0}, \text{ where } \sigma_1(x, \xi) \text{ is the extension of } \sigma(x, \xi)
\]
such that \( \sigma_1(x, \xi) \in \mathcal{E}_c^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \).

Set

\[
\begin{cases}
\bar{u}_1(x) = \tilde{\bar{u}}(x) & \text{for } x_n \geq d, \\
0 & \text{for } x_n < d,
\end{cases}
\]

and

\[
\bar{u}_2(x) = \bar{u}(x) - \bar{u}_1(x),
\]

then

\[
F' \sigma(x, \xi)\|\xi\|F\bar{u} = F' \sigma(x, \xi)\|\xi\|F\bar{u}_1 + F' \sigma(x, \xi)\|\xi\|F\bar{u}_2.
\]

The first term on the right is the ordinary singular integral operator \( \sigma(x, D)\bar{u}_1 \), and the symbol of the second term on the right may be considered as zero by virtue of \( \text{Supp } \bar{u}_2 \subset \{ x \in \mathbb{R}^n ; x_n \leq -d \} \), from which and the theory of the ordinary singular integrals we obtain the desired estimate. \ q. e. d.

**Proof of Lemma 15.** When \( \sigma(x, \xi) \in \mathcal{E}_c \) and \( \sigma(x, \xi', \xi_n) = \sigma(x, \xi, -\xi_n) \), we see that

\[
\varphi_2(x) \sigma(x, D) \varphi_1 u = F'(\varphi_2(x) \tilde{\sigma}(x, D) F(\varphi_1 u))|_{\xi_n > 0},
\]

where \( \tilde{\sigma}(x, \xi) \) is an arbitrary extension of \( \sigma(x, \xi) \) such that \( \tilde{\sigma}(x, \xi) \in C_0^\infty(\mathbb{R}^n \times (\mathbb{R}^n - \{0\})) \) and \( \tilde{\sigma}(x, \xi', \xi_n) = \tilde{\sigma}(x, \xi, -\xi_n) \). Furthermore \( \tilde{\sigma}(x, D) \) is the ordinary singular integral operator.

From this and 3) of Lemma 13 it follows that

\[
\left\| T_*(\varphi_2 \cdot \tilde{\sigma}(x, D) \varphi_1 u) - (T_*(\sigma_2))(\text{ordinary singular integral operator}) \right\|_{L^1(\mathbb{R}^n)} \leq \mathcal{C} \| \tilde{\varphi} \|_{L^1(\mathbb{R}^n)}
\]

by virtue of the theory of Seeley.

Hence

\[
\left\| T_*(\varphi_2 \cdot \sigma(x, D) \varphi_1 u) - (T_*(\sigma_2))(\tau(y, D_y) A\varphi_1 u)) \right\|_{L^1(\mathbb{R}^n)} \leq \mathcal{C} \| \tilde{\varphi} \|_{L^1(\mathbb{R}^n)}
\]

for \( u \in D(a^1) \).

For \( f(x) \frac{\xi_n}{|\xi|} \in \mathcal{E}_c \), we see that the proposition holds by virtue of 3) of Lemma 13.

For \( \sigma(x, \xi) = \sigma_1(x, \xi)f(x) \frac{\xi_n}{|\xi|} \in \mathcal{E}_c \), we set \( f(x) \eta(D) A\varphi_1 u = \varphi \). Then, as above, we see that

\[
\left\| T_*(\varphi_2 \cdot \sigma_1(x, D) \varphi_1 \varphi) - (T_*(\sigma_2))\sigma_1(T^{-1}(y), (dT^*)(D_y))(T_*(\sigma_2))\varphi \right\|_{L^1(\mathbb{R}^n)} \leq \mathcal{C} \| \tilde{\varphi} \|_{L^1(\mathbb{R}^n)}
\]

for \( \varphi \in L^2(\mathbb{R}^n) \).
By virtue of $v = f(x) \frac{1}{\sqrt{-1}} \frac{\partial \tilde{u}}{\partial x_n} \mid_{x_n > 0} \in L^2(R^n)$ and from Theorem 2, we obtain

$$\left\| T_{\ast} \circ (\varphi_2 \cdot \sigma(x, D) A_\pm \varphi_1 u) - (T_{\ast} \circ \varphi_2) \cdot (\tau(y, D) A_\pm (T_{\ast} \circ \varphi_1 u)) \right\|_{y_n > 0} \leq C(\|\tilde{u}\|_{-1, L^2(R^n)}$$

$$+ \|\tilde{u}\|_{L^2(R^n)}) = C(\left\| f(x) \frac{\partial \tilde{u}}{\partial x_n} \right\|_{-1, L^2(R^n)} + \|\tilde{u}\|_{L^2(R^n)}) \leq C\|\tilde{u}\|_{L^2(R^n)} \leq C\|u\|_{x_n > 0}.$$

q.e.d.

Furthermore we have the following Theorem and Lemmas similarly to Theorems and Lemmas obtained in §2.1.

**Lemma 16.** For sufficiently large $\lambda > 0$

$$\left\| (\tilde{A}_\pm + \lambda) u \right\|_{L^2(\Omega)}^2 \geq (\delta - \varepsilon)\|u\|_{L^2(\Omega)}^2 + (\lambda - \rho)^2\|u\|_{L^2(\Omega)}^2$$

for $u \in D(a^{1/2})$, $\varepsilon$ is an arbitrary small positive number, $\rho$ depends on $\varepsilon$ and $n$, and $\delta$ is a constant determined by means of the ellipticity of $a(x, D)$.

In the following $\lambda$ is assumed to be a sufficiently large constant satisfying the above inequality and be fixed.

**Proof.** Let $u \in D(a)$. From the ellipticity of $a(x, D)$ we see that

$$(17) \quad (a_0(x, D) u, u)_{L^2(\Omega)} + k\|u\|_{L^2(\Omega)}^2 \geq \delta\|u\|_{L^2(\Omega)}^2$$

for some constant $k(>0)$.

To obtain the desired inequality we at first show that

$$(18) \quad (a_0(x, D) u, u)_{L^2(\Omega)} - \|\tilde{A}_\pm u\|^2_{L^2(\Omega)} \leq \varepsilon\|u\|_{L^2(\Omega)}^2 + c(\varepsilon)\|u\|_{L^2(\Omega)}.$$ 

Using the partition of unity described next to Lemma 13 $(a_0(x, D) u, u)_{L^2(\Omega)}$ goes over into

$$(19) \quad \sum_{J \in I} (a_0(x, D) \varphi_J u, \varphi_I u)_{L^2(\Omega)} + \sum_J \left( [\varphi_J, a_0] u, u \right)_{L^2(\Omega)}.$$

The second term is bounded by $\varepsilon\|u\|_{L^2(\Omega)}^2 + c(\varepsilon)\|u\|_{L^2(\Omega)}$ by means of the interpolation. We will estimate the first term.

We may suppose $\text{Supp} \varphi_J \cap \text{Supp} \varphi_I = \emptyset$. If $\text{Supp} \varphi_J \cap \Gamma = \emptyset$ and $\text{Supp} \varphi_I \cap \Gamma = \emptyset$, then, since $\varphi, u \in D(a)$,

$$(20) \quad (a_0(x, D) \varphi_J u, \varphi_I u)_{L^2(\Omega)} = (\sqrt{a_0} \circ \sqrt{a_0} A\varphi_J u, \varphi_I u)_{L^2(\Omega)}$$

$$= (\sqrt{a_0} \cdot \sqrt{a_0} A\varphi_J u, \sqrt{a_0} \cdot \sqrt{a_0} A\varphi_I u)_{L^2(\Omega)} + (\sqrt{a_0} \ast - \sqrt{a_0} \ast) A\varphi_J u, A\varphi_I u)_{L^2(\Omega)}$$

$$+ (\sqrt{a_0} \circ \sqrt{a_0} - \sqrt{a_0} \ast \sqrt{a_0}) A\varphi_J u, A\varphi_I u)_{L^2(\Omega)}$$

$$+ (\sqrt{a_0} \circ \sqrt{a_0} - \sqrt{a_0} \ast \sqrt{a_0}) A\varphi_J u, A\varphi_I u)_{L^2(\Omega)}.$$
We see that the last three terms are bounded by $\varepsilon \|u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2$ by means of the theory of the ordinary singular integrals and the interpolation.

Furthermore if $\text{Supp} \; \varphi_j \cap \Gamma \equiv \phi$ or $\text{Supp} \; \varphi_i \cap \Gamma \equiv \phi$, then

$$(a_0(x, D)\varphi_j u, \varphi_i u)_{L^2(\Omega)} = (Jb_\alpha(y, D)A_\pm T_\alpha \circ \varphi_j u, T_\alpha \circ \varphi_i u)_{y_n > 0}$$

by using the transformation $T_\alpha$ such that $\text{Supp} \; \varphi_j \cup \text{Supp} \; \varphi_i \subset U_\alpha$, here $J$ is a Jacobian ($>0$).

By virtue of the uniform ellipticity of $b_\alpha$ and Theorem 2 it is equal to

$$(21) \quad (J\sqrt{b_\alpha}(y, D)A_\pm T_\alpha \circ \varphi_j u, \sqrt{b_\alpha}(y, D)A_\pm T_\alpha \circ \varphi_i u)_{y_n > 0}$$

and the last three terms are bounded by $\varepsilon \|A_\pm T_\alpha \circ \varphi_j u\|_{y_n > 0}^2 + \|T_\alpha \circ \varphi_i u\|_{y_n > 0}^2$.

On the other hand for $u \in D(\alpha^2)$

$$\|A_\pm u\|_{L^2(\Omega)} = \sum_{i,j,k,l} (\bar{A}_{\pm ik} u, \bar{A}_{\pm jl} u)_{L^2(\Omega)}.$$
Adding up this expression with respect to $i$ and $k$, it is equal to

$$(J \sqrt{b_n}(y, D) \Lambda_{\pm} T_{ao} \circ \varphi_m, (T_{ao} \circ \varphi_k) \sqrt{b_n}(y, D) \Lambda_{\pm} T_{ao} \circ \varphi_i u)_{y_n>0}.$$  

By virtue of Lemma 14 and theorem 7 in [15] (page 247) we can also treat the calculation in the case where $\text{Supp} \varphi_i \cap \Gamma = \emptyset$ or $\text{Supp} \varphi_i \cap \Gamma = \emptyset$ as above.

Hence from (19) — (23) we obtain (18).

We next show that for some $k_1 > 0$

$$(\tilde{A}_{\pm} u, u)_{L^2(\Omega)} + (u, \tilde{A}_{\pm} u)_{L^2(\Omega)} \geq -k_1 ||u||^2_{L^2(\Omega)}.$$  

To prove this let $\tilde{\psi}_i(x) = \left(\varphi_i(x)\right)^{1/2}$. Then we see that $\{\tilde{\psi}_i(x)\}$ is also another partition of unity with respect to $\tilde{\Omega}$ such that $\sum(\tilde{\psi}_i(x))^2 = 1$, $\tilde{\psi}_i u \in D(a \tilde{h})$ for $u \in D(a \tilde{h})$, $\psi_i u$ and $\varphi_i u \in D(a)$ for $u \in D(a)$ and $\{\text{Supp} \tilde{\psi}_i\}$ is a star-finite refinement of $\{U_n\}$. Then

$$(\tilde{A}_{\pm} u, u)_{L^2(\Omega)} = \sum_k (\Lambda_{\pm} \psi_i u, \psi_i u)_{L^2(\Omega)} + \sum_k \left(\psi_i \Lambda_{\pm} - \Lambda_{\pm} \psi_i\right) u, \psi_i u)_{L^2(\Omega)}.$$  

We estimate the first term on the right. If $\text{Supp} \psi_i \cap \Gamma = \emptyset$, from Definition 7 we see that

$$(\Lambda_{\pm} \psi_i u, \psi_i u)_{L^2(\Omega)} = \sum_{i,j} (\Lambda_{\pm} \psi_i u, \psi_i u)_{L^2(\Omega)}$$

$$= \sum_{i,j} \left(J \sqrt{b_n}(y, D) \Lambda_{\pm} (T_{ao} \circ \varphi_j u), \psi_i u\right)_{L^2(\Omega)}$$

$$= \sum_{i,j} \left(J \sqrt{b_n}(y, D) \Lambda_{\pm} (T_{ao} \circ \varphi_i u), \psi_i u\right)_{y_n>0}$$

$$= \left(J \sqrt{b_n}(y, D) \Lambda_{\pm} (T_{ao} \circ \varphi_i u), \psi_i u\right)_{y_n>0}.$$  

By virtue of (13) it follows from this

$$(\Lambda_{\pm} \psi_i u, \psi_i u)_{L^2(\Omega)} + (\psi_i u, \Lambda_{\pm} \psi_i u)_{L^2(\Omega)} \geq -k_1 ||T_{ao} \circ \varphi_i u||^2_{y_n>0}.$$  

We can estimate above expression when $\text{Supp} \psi_i \cap \Gamma = \emptyset$ similarly to the case when $\text{Supp} \psi_i \cap \Gamma = \emptyset$.

Furthermore $\tilde{A}_{\pm} \psi_i - \psi_i \Lambda_{\pm}$ can be extended to be a bounded operator from $L^2(\Omega)$ into itself, which will be proved at the following Theorem 3. Hence we obtained (24).

Therefore from (17), (18) and (24) we have...
\[
\| (\tilde{A}_\pm + \lambda) u \|_{L^2(\Omega)}^2 \\
= \| \tilde{A}_\pm u \|_{L^2(\Omega)}^2 + \lambda \left( (\tilde{A}_\pm u, u)_{L^2(\Omega)} + (u, \tilde{A}_\pm u)_{L^2(\Omega)} \right) + \lambda^2 \| u \|_{L^2(\Omega)}^2 \\
= (a_0(x, D) u, u)_{L^2(\Omega)} + \left( \| \tilde{A}_\pm u \|_{L^2(\Omega)}^2 - (a_0(x, D) u, u)_{L^2(\Omega)} \right) \\
+ \lambda \left( (\tilde{A}_\pm u, u)_{L^2(\Omega)} + (u, \tilde{A}_\pm u)_{L^2(\Omega)} \right) + \lambda^2 \| u \|_{L^2(\Omega)}^2 \\
\geq \delta \| u \|_{L^2(\Omega)}^2 - k \| u \|_{L^2(\Omega)}^2 - \varepsilon \| u \|_{L^2(\Omega)}^2 - c(\varepsilon) \| u \|_{L^2(\Omega)}^2 - \lambda k_1 \| u \|_{L^2(\Omega)}^2 + \lambda^2 \| u \|_{L^2(\Omega)}^2 \\
= (\delta - \varepsilon) \| u \|_{L^2(\Omega)}^2 + (\lambda - k - c(\varepsilon) - \lambda k_1) \| u \|_{L^2(\Omega)}^2.
\]

We take \( \varepsilon \) sufficiently small such that \( \delta > \varepsilon (\varepsilon > 0) \), next take \( \lambda, \rho \) sufficiently large such that \( \rho^2 - k - c(\varepsilon) - \lambda k_1 > 0 \) and \( \lambda > \rho \). Then

\[
\| (\tilde{A}_\pm + \lambda) u \|_{L^2(\Omega)}^2 \geq (\delta - \varepsilon) \| u \|_{L^2(\Omega)}^2 + (\lambda - \rho)^2 \| u \|_{L^2(\Omega)}^2.
\]

For \( u \in D(\tilde{A}_\pm) \) we obtain the estimate by passing to the limit. q.e.d.

From Lemma 16 we may define \( D(\tilde{A}_\pm) \) as the definition domain of the closed operator \( \tilde{A}_\pm \) and denote it by \( D(\tilde{A}_\pm) \).

**Lemma 17.** Let \( \alpha(x, \xi) \in \mathbb{H}_o \) and \( u \in D(\tilde{A}_\pm) \). Then \( \alpha(x, D) u, \alpha(x, D)^* u \in D(\tilde{A}_\pm) \).

**Proof.** This is shown by virtue of Theorem 2, Definition 5 and Lemma 13. q.e.d.

From Lemmas 14, 15, 17 and Theorem 2 it implies the following theorem by the same consideration used by Seeley (see theorem 2 in [15] (page 262)).

**Theorem 3.** For \( \alpha(x, \xi), \beta(x, \xi) \in \mathbb{H}_o \) the operators
\[
\alpha(x, D) \tilde{A}_\pm - \tilde{A}_\pm \alpha(x, D), \quad \alpha(x, D)^* \tilde{A}_\pm - \tilde{A}_\pm \alpha(x, D)^*,
\]
\[
(\alpha(x, D)^* - \alpha^*(x, D)) \tilde{A}_\pm, \quad \left( (\alpha \circ \beta)(x, D) - \alpha(x, D) \beta(x, D) \right) \tilde{A}_\pm
\]
are extended to operators of \( B(L^2(\Omega), L^2(\Omega)) \), where the symbols of \( \alpha^*(x, D) \) and \( (\alpha \circ \beta)(x, D) \) are \( \overline{\alpha(x, \xi)} \) and \( \alpha(x, \xi) \beta(x, \xi) \), respectively.

Furthermore we obtain the following lemmas needed for the next section.

**Lemma 18.** Let \( u \in D(\tilde{A}_\pm) \). Then \( u \in D(\tilde{A}_\pm^*) \) and
\[
\| (\tilde{A}_\pm - \tilde{A}_\pm^*) u \|_{L^2(\Omega)} \leq C \| u \|_{L^2(\Omega)}.
\]

**Proof.** It is seen from Definition 7, Lemmas 14, 15, 17 and Theorem 2 that the symbols of \( \tilde{A}_\pm \) and \( \tilde{A}_\pm^* \) are the same. It implies this lemma.

**Lemma 19.** \( \tilde{A}_\pm + \lambda \) has a bounded inverse from \( L^2(\Omega) \) into itself such that
\[
\| (\tilde{A}_\pm + \lambda)^{-1} u \|_{L^2(\Omega)} \leq (\lambda - \rho)^{-1} \| u \|_{L^2(\Omega)}.
\]
Proof. We take a partition of unity \( \{ \Phi_h \} \) with respect to \( \bar{\Omega} \) such that \( \sum_{n} \Phi_h^2 = 1 \) on \( \bar{\Omega} \), \( \Phi_h u, \Phi_h^2 u \in D(a) \) for \( u \in D(a) \). Furthermore we assume that if \( \text{Supp } \Phi_h \cap \Gamma \neq \emptyset \) then \( (\text{Supp } \Phi_h)^0 \cap \Gamma \neq \emptyset \). \( \{ \text{Supp } \Phi_h \} \) is a star-finite refinement of \( \{ U_n \} \). Let us choose \( x_h \in (\text{Supp } \Phi_h)^0 \) for every \( \Phi_h \) and in particular \( x_h \in (\text{Supp } \Phi_h)^0 \) if \( (\text{Supp } \Phi_h) \cap \Gamma \neq \emptyset \). Then we may assume that if \( U_n \supset \text{Supp } \Phi_h \), for any \( y \in R^n \), \( \sup |\sigma_a(y, \xi) - \sigma_a(y, \xi_1)| < \varepsilon_i \) for sufficiently small positive number \( \varepsilon_i \), which is determined later. Where \( y_h = T_a(x_h), \sigma_a(y, D) = \sqrt{\tilde{D}}_a(y, D) \) if \( \text{Supp } \Phi_h \cap \Gamma \neq \emptyset \) and setting \( x = y, \sigma_a(y, D) = \sqrt{\tilde{D}}_a(x, D) \) if \( \text{Supp } \Phi_h \cap \Gamma = \emptyset \).

Furthermore we take another partition of unity \( \{ \varphi_j \} \) with respect to \( \bar{\Omega} \) described next to Lemma 13. Here we assume that for every \( \{ \varphi_j \} \), there exists \( \text{Supp } \varphi_j \subset \text{Supp } \Phi_h \). \( \text{Supp } \varphi_j \subset \text{Supp } \Phi_h \) if \( \text{Supp } \varphi_j \cap \Gamma = \emptyset \), then \( U_n \supset \text{Supp } \Phi_h \), hereafter we write one of such \( U_n \) by \( U_n \).

Then we can write \( A \lambda _{\pm} + \lambda \) as

\[
(A \lambda _{\pm} + \lambda )u = \sum _{j} T^* \Phi_h (\sigma_a(y, D)A_{\pm} + \lambda ) (T^* \Phi_h u) \quad \text{for } u \in D(A_{\pm}) ,
\]

here \( T_a = I \) if \( \text{Supp } \varphi_j \cap \Gamma = \emptyset \), \( I \) is an identity operator and \( A_{\pm} = A_{\pm} \).

As \( y_h \) is in \( R^{n-1} \) when \( (\text{Supp } \Phi_h)^0 \cap \Gamma \neq \emptyset \), \( \sigma_h(y, \xi) \) is the even function with respect to \( \xi_n \) from the construction of \( \sigma_a(y, \xi) \). Hence \( \sigma_h(y, D)A_{\pm} + \lambda \) has a bounded inverse from \( L^2(R^n) \) onto \( D(A_{\pm}) \).

Then set

\[
R \cdot = \sum _{j} T^* (T^* \Phi_h (\sigma_a(y, D)A_{\pm} + \lambda )^{-1} T^* \Phi_h u)
\]

and calculate \( (A \lambda _{\pm} + \lambda )R \).

\[
(A \lambda _{\pm} + \lambda )Ru = \sum _{j} T^* (T^* \Phi_h (\sigma_a(y, D)A_{\pm} + \lambda )^{-1} T^* \Phi_h u)\times T^* \Phi_h \sigma_a(y, D)A_{\pm} + \lambda )^{-1} T^* \Phi_h u)
\]

for \( u \in L^2(\Omega) \).

We may assume that \( \text{Supp } \varphi_i \cap \text{Supp } \varphi_j \neq \emptyset \), \( \text{Supp } \varphi_i \cap \text{Supp } \Phi_h \neq \emptyset \). Then from the above convention we can choose \( \alpha, \beta \) the same index. Hence

\[
(A \lambda _{\pm} + \lambda )Ru = \sum _{j} \sum _{h} T^* (T^* \Phi_h (\sigma_a(y, D)A_{\pm} + \lambda )T^* \Phi_h \sigma_a(y, D)A_{\pm} + \lambda )^{-1} T^* \Phi_h u) + Bu
\]

\[
= \sum _{j} T^* (T^* \Phi_h (\sigma_a(y, D)A_{\pm} + \lambda )^{-1} T^* \Phi_h u) + Bu ,
\]

where \( \sigma_a(y, D)A_{\pm} + \lambda \) is the bounded inverse from \( L^2(R^n) \) onto \( D(A_{\pm}) \).

Moreover we assume that \( \text{Supp } \varphi_i \cap \text{Supp } \varphi_j \neq \emptyset \), \( \text{Supp } \varphi_i \cap \text{Supp } \Phi_h \neq \emptyset \). Then from the above convention we can choose \( \alpha, \beta \) the same index. Hence

\[
(A \lambda _{\pm} + \lambda )Ru = \sum _{j} \sum _{h} T^* (T^* \Phi_h (\sigma_a(y, D)A_{\pm} + \lambda )^{-1} T^* \Phi_h u) + Bu
\]

\[
= \sum _{j} T^* (T^* \Phi_h (\sigma_a(y, D)A_{\pm} + \lambda )^{-1} T^* \Phi_h u) + Bu ,
\]

where \( \sigma_a(y, D)A_{\pm} + \lambda \) is the bounded inverse from \( L^2(R^n) \) onto \( D(A_{\pm}) \).
here $\|Bu\|_{L^2(\Omega)} \leq \frac{C}{\lambda - \rho} \|u\|_{L^2(\Omega)}$ because of Lemmas 12, 14 and 15. Furthermore we find that

$$
\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)(T_{h^*} \circ \Phi_h)\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)^{-1}(T_{h^*} \circ \Phi_h u)
$$

$$
= \left[\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)(T_{h^*} \circ \Phi_h)\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)^{-1}(T_{h^*} \circ \Phi_h u)
\right] + (T_{h^*} \circ \Phi_h)\left(\sigma_h(y, D) - \sigma_h(y, D)\right)A_{\pm h}^{-1}(T_{h^*} \circ \Phi_h u)
$$

$$
= \left[\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)(T_{h^*} \circ \Phi_h)\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)^{-1}(T_{h^*} \circ \Phi_h u)
\right] + (T_{h^*} \circ \Phi_h)\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)(T_{h^*} \circ \Phi_h u).
$$

From the choosing of $\Phi_h$, $\|\left(T_{h^*} \circ \Phi_h\right)(\sigma_h(y, D) - \sigma_h(y, D))A_{\pm h} u\|_{y_n > 0} \leq (\epsilon_1 + \epsilon) \times \|A_{\pm h} u\|_{y_n > 0} + c(\epsilon)\|u\|_{y_n > 0}$ for $u \in D(A_{\pm})$ similarly to Lemma 12. Moreover from Lemma 12 we find that

$$
\|\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)^{-1} u\|_{y_n > 0} \leq (\lambda - \rho)^{-1}\|u\|_{y_n > 0},
$$

$$
\|A_{\pm h}^{-1}(\sigma_h(y, D)A_{\pm h} + \lambda)^{-1} u\|_{y_n > 0} \leq (\delta^2 - \epsilon')^{-1}\|u\|_{y_n > 0}
$$

for $u \in L^2(\mathbb{R}^n_+)$, here $\delta$ is a constant determined by means of the ellipticity of $a(x, D)$.

We remark here that in the case where $\text{Supp} \Phi_h \cap \Gamma = \phi$ we use the theory of the ordinary singular integrals.

From this and Theorem 2 we find that

$$
\left\| \sum_{\ell,h} T_{h^*} \circ (T_{h^*} \circ \varphi_h)\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)(T_{h^*} \circ \Phi_h)\left(\sigma_h(y, D)A_{\pm h} + \lambda\right)^{-1}(T_{h^*} \circ \Phi_h u)\right\|_{L^2(\Omega)}
$$

$$
+ \|Bu - u\|_{L^2(\Omega)}
$$

$$
\leq \frac{C'}{\lambda - \rho} \|u\|_{L^2(\Omega)} + \frac{\epsilon_1 + \epsilon}{\delta^2 - \epsilon'} \|u\|_{L^2(\Omega)} + \frac{c(\epsilon)}{\lambda - \rho} \|u\|_{L^2(\Omega)} + \frac{\epsilon_1}{\lambda - \rho} \|u\|_{L^2(\Omega)}
$$

$$
= \left(\frac{C'}{\lambda - \rho} + \frac{\epsilon_1 + \epsilon}{\delta^2 - \epsilon'} + \frac{c(\epsilon)}{\lambda - \rho} + \frac{\epsilon_1}{\lambda - \rho}\right)\|u\|_{L^2(\Omega)}.
$$

We choose $\epsilon_1$ so small that $\frac{\epsilon_1}{\delta^2 - \epsilon'} < 1$ and also $\epsilon$ small such that $\frac{\epsilon_1 + \epsilon}{\delta^2 - \epsilon'} < 1$, and then $\lambda$ so large that $\frac{C'}{\lambda - \rho} + \frac{\epsilon_1 + \epsilon}{\delta^2 - \epsilon'} + \frac{c(\epsilon)}{\lambda - \rho} + \frac{\epsilon_1}{\lambda - \rho} < 1$. Then $(I - (\lambda_{\pm} + \lambda)R)^{-1} = ((\lambda_{\pm} + \lambda)R)^{-1}$ exists and is bounded from $L^2(\Omega)$ into itself.

From $(\lambda_{\pm} + \lambda)R((\lambda_{\pm} + \lambda)R)^{-1} = I$, we find that

$$
(\lambda_{\pm} + \lambda)^{-1} = R((\lambda_{\pm} + \lambda)R)^{-1}.
$$
Moreover from Lemma 16 we find that
\[
\|(\bar{A}_\pm + \lambda)^{-1} u\|_{L^2(\Omega)} \leq (\lambda - \rho)^{-1} \|u\|_{L^2(\Omega)}. \quad \text{q.e.d.}
\]

**Lemma 20.** We find that \(u \in D(a)\) if and only if \(u \in D(\bar{A}_\pm \bar{A}_\pm^*)\).

**Proof.** Let \(u \in D(a)\). We set
\[
a_i(x, D) = -\sum_{ij} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + k_1,
\]
where \(k_1\) is a positive constant, and \(a_i\) is assumed a positive definite self-adjoint operator with the domain \(D(a_i) = D(a)\).

Hence using the same consideration in Lemma 16 for \(\varphi \in D(a^A)\)
\[
(a_i(x, D)u, \varphi)_{L^2(\Omega)} = (\bar{A}_\pm u, \bar{A}_\pm \varphi)_{L^2(\Omega)} + B_1(u, \varphi)_{L^2(\Omega)},
\]
where
\[
|B_1(u, \varphi)_{L^2(\Omega)}| \leq C_1 \|u\|_{1, L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}.
\]
Then
\[
|(\bar{A}_\pm u, \bar{A}_\pm \varphi)_{L^2(\Omega)}| \leq \left(\|a_i u\|_{L^2(\Omega)} + C_1 \|u\|_{1, L^2(\Omega)} \right) \|\varphi\|_{L^2(\Omega)}.
\]
Consequently, from the fact that \(D(a^A)\) is dense in \(L^2(\Omega)\), we find that \(\bar{A}_\pm u \in D(\bar{A}^*_\pm)\), that is, \(u \in D(\bar{A}^*_\pm \bar{A}_\pm)\).

The fact that if \(u \in D(\bar{A}^*_\pm \bar{A}_\pm)\) then \(u \in D(a)\) is shown by virtue of the similar method stated above.

**Lemma 21.** \(D(\bar{A}_\pm) = D(\bar{A}^*_\pm)\) and \(D(a) = D(\bar{A}^*_\pm)\).

**Proof.** From Lemma 19, we find that
\[
(\bar{A}^*_\pm + \lambda) u = (I + (\bar{A}^*_\pm - \bar{A}_\pm)(\bar{A}_\pm + \lambda)^{-1})(\bar{A}_\pm + \lambda) u
\]
for \(u \in D(\bar{A}_\pm)\).

Moreover from Lemma 18 and 19 we find that
\[
\left\|(\bar{A}^*_\pm - \bar{A}_\pm)(\bar{A}_\pm + \lambda)^{-1}\right\|_{L^2(\Omega) \to L^2(\Omega)} \leq \frac{C}{\lambda - \rho} < 1
\]
for large \(\lambda\).

From this it follows that \((I + (\bar{A}^*_\pm - \bar{A}_\pm)(\bar{A}_\pm + \lambda)^{-1})(\bar{A}_\pm + \lambda)\) has a bounded inverse from \(L^2(\Omega)\) onto \(D(\bar{A}_\pm)\), from which it implies this lemma because \(D(\bar{A}_\pm) \subset D(\bar{A}^*_\pm)\) and Lemma 19. q.e.d.

Hereafter we set \(\tilde{a}(x, D) = a(x, D) + \lambda, \bar{A}^*_{\pm, \lambda} = \bar{A}_\pm + \lambda\).

**Corollary.** We find that for \(u \in D(a)\)
\[
\left\|(a(x, D) - \bar{A}^*_{\pm, \lambda}) u\right\|_{L^2(\Omega)} \leq C \|\bar{A}^*_{\pm, \lambda} u\|_{L^2(\Omega)}.
\]

**Proof.** From (25), Lemma 16 and 18 it implies this corollary.
Lemma 22. Let \( \sigma(x, \xi) \in \mathbb{R}_d \) with \( \sigma(x, \xi) \geq d > 0 \) for \( x \in \bar{\Omega} \) and \( |\xi| = 1 \). Then for sufficiently large \( k(>0) \)

\[
\left\| (\sigma(x, D) \Lambda_\pm + k) u \right\|_{L^p(Q)} \geq (d - \varepsilon)^2 \| \Lambda_\pm u \|_{L^p(Q)} + (k - \rho)^2 \| u \|_{L^p(Q)}
\]

for \( u \in D(\Lambda_\pm) \), here \( \varepsilon \) is an arbitrary small positive number, \( k > \lambda \), \( k > \rho \) and \( \rho \) depends on \( n \), \( \sigma \) and \( \varepsilon \).

Proof. We can prove this lemma like as Lemma 12 by virtue of Lemma 16 and Theorem 3.

Lemma 23. Let \( \sigma(x, \xi) \) be as in Lemma 22. Then for sufficiently large \( k > 0 \) \( \sigma(x, D) \Lambda_\pm + k \) has a bounded inverse from \( L^2(\Omega) \) into itself such that

\[
\left\| (\sigma(x, D) \Lambda_\pm + k)^{-1} u \right\|_{L^2(\Omega)} \leq (k - \rho)^{-1} \| u \|_{L^2(\Omega)}
\]

for \( u \in L^2(\Omega) \).

Proof. \( \sigma(x, \xi) \sqrt{a_0(x, \xi)} > d \sqrt{\delta} > 0 \) for \( x \in \bar{\Omega} \) and \( |\xi| = 1 \), which implies this lemma from Lemmas 14, 15, 22 and Theorem 3 similarly to Lemma 19.

q. e. d.

3. Reductions to the theory of semi group

We may consider \( \rho(x) = 0 \) on \( \Gamma \) in the Neumann case. For, we set

\[
\psi(x) = \varphi(x) \frac{\| \text{grad} \varphi(x) \|}{\sum_{i,j} a_{ij}^{(0)}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j}} \quad \rho(x) \quad \text{on} \quad U(\Omega),
\]

where \( \bar{\rho}(x) \) is an extended function of \( \rho(x) \) defined on \( U(\Omega) \) such that \( \bar{\rho}(x) \in C^\infty(U(\Omega)) \) and \( \bar{\rho}(x) = 0 \) on \( \Omega_\varepsilon = \{ x \in \Omega ; \text{dist} (\Gamma, x) \geq \varepsilon \} \), \( \varepsilon \) is an arbitrary small but a fixed number. We furthermore set

\[
u(x) = e^{-\psi(x)} \psi(x).
\]

Then we see that

\[
\left( \frac{\partial}{\partial n} + \rho(x) \right) u = \frac{\partial}{\partial n} v \quad \text{and} \quad \left( \frac{\partial}{\partial n} + \rho(x) \right) a^l u = \frac{\partial}{\partial n} a^l v,
\]

where \( a^l = e^{\psi(x)} \sigma_{1} \varphi(x) \), \( l = 1, 2, \ldots \).

Here we remark that the principal part of \( a^l(x, D) \) is the same as one of \( a(x, D) \). Hereafter we assume that \( \rho(x) = 0 \) and set \( \bar{a}(x, D) = a(x, D) + \lambda \) for a sufficiently large, but fixed constant \( \lambda \).

We state the following lemma to reduce the equation (1') to the system.

Lemma 24. If \( a_k(x, D) \) satisfies the assumption stated in §1, then we find that
\[ a_{2k}(x, D)u = \alpha_{2k}(x, D)\tilde{a}^k u + \beta_{2k} u, \]
\[ a_{2k+1}(x, D)u = \alpha_{2k+1}(x, D)\tilde{\lambda}_{2k+1} \tilde{a}^k u + \beta_{2k+1} u \]

for \( u \in D(a^k) \) or \( u \in D(a^{k-\frac{1}{2}}) \) \((k \geq 0)\) respectively, where \( \alpha_k(x, \xi) = \frac{a_k(x, \xi)}{(a_0(x, \xi))^k} \in \mathcal{M}_0 \), \( \beta_k \) is a linear operator such that \( \beta_{2k}\tilde{a}^{-(k-1)}\tilde{\lambda}_{2k+1}^{-1} \in B(L^2(\Omega), L^2(\Omega)) \).

**Proof.** From Lemma 15 the statement that \( \alpha_k(x, \xi) = \frac{a_k(x, \xi)}{(a_0(x, \xi))^k} \in \mathcal{M}_0 \) is shown from corollary in [15] (page 244) and the first two expressions in this lemma. Here we remark that \( \left( a_0 \left( x, \frac{\xi}{|\xi|} \right) \right)^{-\frac{1}{2}} \in \mathcal{M}_0 \).

Hence we will show the first two expressions.

We first consider for \( a_{2k}(x, D) \). Let \( u \in D(a^k) \). Using the partition of unity described next to Lemma 13, we have

\[ a_{2k}(x, D)u = \sum_j a_{2k}(x, D)\varphi_j u + \sum_j [\varphi_j, a_{2k}] u \]

and that \( \sum_j [\varphi_j, a_{2k}] \) is a differential operator of order at most \((2k-1)\) and \( \sum_j [\varphi_j, a_{2k}]\tilde{a}^{-(k-1)}\tilde{\lambda}_{2k+1}^{-1} \in B(L^2(\Omega), L^2(\Omega)) \).

If \( \text{Supp} \varphi_j \cap \Gamma = \emptyset \), then \( \varphi_j u \in D(a^k) \) and

\[ a_{2k}(x, D)\varphi_j u = \alpha'_{2k}(x, D)a^k \varphi_j u + \beta'_{2k} u = a'_{2k}(x, D)\varphi_j \tilde{a}^k u + \beta'_{2k} u, \]

where \( \alpha'_{2k}(x, \xi) = \frac{a_{2k}(x, \xi)}{(a_0(x, \xi))^k} \varphi_j(x) \in \mathcal{M}_0 \) and \( \beta'_{2k}\tilde{a}^{-(k-1)}\tilde{\lambda}_{2k+1}^{-1} \in B(L^2(\Omega), L^2(\Omega)) \).

Let \( \text{Supp} \varphi_j \cap \Gamma \neq \emptyset \). Using coordinate transformation \( T_y \) such that \( U_{a'} \supset \text{Supp} \varphi_j \) we replace \( \frac{\partial^2}{\partial y_n^2} \) in the principal part of \( a_{2k}(T_{a'}^{-1}(y), (dT_{a'}^*)(D_y)) \) by

\[ \frac{\partial^2}{\partial y_n^2} = b_{nn}^{-1} \left( a(T_{a'}^{-1}(y), (dT_{a'}^*)(D_y)) - \sum_{i,j} b_{ij} \frac{\partial^2}{\partial y_i \partial y_j} - \sum_{\bar{r}} b_{\bar{r}} \frac{\partial^2}{\partial y_\bar{r}_1 \partial y_\bar{r}_2} \right) + \text{(lower order terms)} \]

Then from the assumption with respect to \( a_{2k} \) stated in \( \S 1 \),

\[ a_{2k} \left( T_{a'}^{-1}(y), (dT_{a'}^*)(D_y) \right) T_{a'}^{-1} \varphi_j u \]

\[ = \sum_{|\eta|=n} \sum_{\eta \neq \emptyset} f(x, y) X_{i_1} \cdots X_{i_l} Y_{i_1} \cdots Y_{i_l} \left( a \left( T_{a'}^{-1}(y), (dT_{a'}^*)(D_y) \right) \right)^{\eta} T_{a'}^{-1} \varphi_j u + \text{(lower order terms)} u, \]

where \( X_{i_j} = D_i D_j, \ Y_t = y_{n+1}(y) D_i D_n \) \((i, j, l \neq n)\), \( \eta = (\eta_1, \ldots, \eta_p) \), \( \bar{r} = (\bar{r}_1, \ldots, \bar{r}_q) \), \( \| \eta \| = \eta_1 + \cdots + \eta_p \) and \( |\bar{r}| = r_1 + \cdots + r_q \).
From this we see that
\[ a_{2k}(x, D)\psi \mu = \sum A_{\gamma \theta}(x, D) \left( a(x, D) \right)^{2k} \psi \mu + (\text{lower order terms}) u \]
using inverse coordinate transformation \( T^{-1}_\alpha \), where \( A_{\gamma \theta}(x, D) = T^{-1}_\alpha \left( f_{\gamma \theta}(x) X_{i_j}^{\gamma \theta} \cdots X_{p}^{\gamma \theta} Y_{i_q}^{\gamma \theta} \cdots Y_{q}^{\gamma \theta} \right) \). Furthermore using \( \psi \mu \in D(a) \) for \( u \in D(a) \), \( a \psi \mu = \psi \mu + (\text{lower order terms}) u \).

Hence
\[ a_{2k}(x, D)\psi \mu = \sum A_{\gamma \theta}(x, D) \left( a(x, D) \right)^{2k-1} \psi \mu + (\text{lower order terms}) u. \]
Repeating this method for \( a_{2k}(x, D)\psi \mu \), we find that
\[ a_{2k}(x, D)\psi \mu = \sum A_{\gamma \theta}(x, D) \psi a^k u + (\text{lower order terms}) u \]
for \( u \in D(a^k) \).

Using coordinate transformation \( T^{-1}_\alpha \) again, this goes over into
\[ a_{2k} \left( T^{-1}(y), (dT^*_a)(Dy) \right) T^{-1}_\alpha \psi \mu = \sum f_{\gamma \theta}(y) X_{i_j}^{\gamma \theta} \cdots X_{p}^{\gamma \theta} Y_{i_q}^{\gamma \theta} \cdots Y_{q}^{\gamma \theta} T^{-1}_\alpha \psi \mu(a^k u) + (\text{lower order terms}) u \]
Next we consider for \( a_{2k+1}(x, D) \). Let \( u \in D(a^{k+\frac{1}{2}}) \). Similarly to the case for \( a_{2k}(x, D) \) we can show that after using the partition of unity
\[ \sum_j \left[ \psi \mu, a_{2k+1} \right] \tilde{a}^{-k} \in B \left( L^2(\Omega), L^2(\Omega) \right), \]
\[ a_{2k+1}(x, D)\psi \mu = a'_{2k+1}(x, D) \psi \mu a^k u + \beta''_{2k+1} u \]
if \( \text{Supp} \psi \mu \cap \Gamma = \phi \), where
\[ a'_{2k+1}(x, \xi) = \frac{a_{2k+1}(x, \xi)}{a(x, \xi)} \psi \mu(x) \in \mathbb{H}_a \] and \( \beta''_{2k+1} \tilde{a}^{-k} \in B \left( L^2(\Omega), (L^2(\Omega)) \right) \), and if \( \text{Supp} \psi \mu \cap \Gamma = \phi \),
\[ a_{2k+1} \left( T^{-1}(y), (dT^*_a)(Dy) \right) T^{-1}_\alpha \psi \mu = \sum f_{\gamma \theta}^1(y, \frac{\partial}{\partial y}) X_{i_j}^{\gamma \theta} \cdots X_{p}^{\gamma \theta} Y_{i_q}^{\gamma \theta} \cdots Y_{q}^{\gamma \theta} T^{-1}_\alpha \psi \mu(a^k u) \]
\[ + (\text{lower order terms}) u \],
where \( f_{\gamma \theta}^1(y, \frac{\partial}{\partial y}) \) is a differential operator of order one. Furthermore from the assumption with respect to the coefficients stated in §1, the coefficients of \( \frac{\partial}{\partial y} \) are zero at \( y_n = 0 \). Hence we may prove only the following

**Lemma 25.** Let \( X_{i_j} \) and \( Y_{i_q} \) be as described before. Then every term
\[ \sum_{|i| + |\ell| = k} X_{i_j}^{\gamma \theta} \cdots X_{p}^{\gamma \theta} Y_{i_q}^{\gamma \theta} \cdots Y_{q}^{\gamma \theta} T^{-1}_\alpha \psi \mu \] (Supp \( \psi \mu \subset U_a \)) is represented as \( \alpha(x, D)a^k u + \beta \mu \) for \( u \in D(a^k) \) considering it in \( \Omega \), where \( \alpha(x, \xi) \in \mathbb{H}_a \) and \( \beta \tilde{a}^{-k-1} \tilde{\alpha}_a^{-1} \in \).
B(L^2(Q), L^2(Q)).

Proof. We prove this lemma by the methods of induction with respect to \( k \).

At first we consider the case \( k=1 \). Let \( u \in D(\alpha) \). Then \( u \in D(X_{ij}) \) and \( u \in D(Y_t) \) in \( R^n \), where \( D(X_{ij}) = D(Y_t) = H^2(\mathbb{R}^n) \), \( X_{ij} \) and \( Y_t \) are being considered as only differential operators.

We see that

\[ \xi \xi = \alpha_{ij}(y, \xi)a_0 \left( y, \frac{\xi}{|\xi|} \right) |\xi|^2, \quad y_n \xi_n \xi = \alpha_{ij}(y, \xi)a_0 \left( y, \frac{\xi}{|\xi|} \right) |\xi|^2, \text{ and } \alpha_{ij}(y, \xi), \alpha_i(y, \xi) \in \mathbb{R}. \]

Hence from Theorem 2, Lemma 21 and definition of \( \eta(\mathcal{D}) \) we find that for \( u \in D(\alpha) = D(\alpha_0) \)

\[ T^*_{\alpha_0} \left( X_{ij}T^*_{\alpha_0} \varphi_i u \right) = T^*_{\alpha_0} \left( \left( \alpha_{ij}(y, \xi)a_0 \left( y, \frac{\xi}{|\xi|} \right) \right) \xi^2 \right) + \beta_{ij} \varphi_i u, \]

where \( \beta_{ij}, a \in B(L^2(Q), L^2(Q)) \).

Next we will show that our assertion holds for the case \( k \) supposing that it holds for the case \( (k-1) \). To this end it suffices to prove this for the one of \( k \)-times product of \( X_{ij} \) and \( Y_t \). For example we consider \( X_{k-1}Y \equiv X_{i_{j_1}} \cdots X_{i_{j_{k-1}}}Y_{i_k} \). Furthermore we set \( X^k = X_{i_{j_1}}X^{k-2}Y \). By virtue of the case \( k=1 \) we find that

\[ T^*_{\alpha_0} \left( X_{i_{j_1}}X^{k-2}Y \right) T^*_{\alpha_0} \varphi_i u \]

\[ = \alpha_{i_{j_1}}(x, D) aT^*_{\alpha_0} \left( X^{k-2}Y \right) \varphi_i u + \beta_{i_{j_1}} u, \]

where \( \beta_{i_{j_1}} \in B(L^2(Q), L^2(Q)) \).

Furthermore

\[ aT^*_{\alpha_0} \left( X^{k-2}Y \right) \varphi_i u = T^*_{\alpha_0} \left( X^{k-2}Y \right) \varphi_i u + \left[ a, T^*_{\alpha_0} \left( X^{k-2}Y \right) \varphi_i \right] u. \]

From \( u \in D(\alpha^k) \), we see that \( au \in D(\alpha^{k-1}) \). Hence by methods of induction

\[ T^*_{\alpha_0} \left( X^{k-2}Y \right) \varphi_i u = \alpha_{k-1}(x, D) \varphi_i a^{k-1}u + \beta_{k-1} u \]

\[ = \alpha_{k-1}(x, D) \varphi_i a^{k-1}u + \beta_{k-1} u. \]

Since \( [a, T^*_{\alpha_0}(X^{k-2}Y)\varphi_i] \) is a differential operator of order at most \((2m-1)\),
we can apply it to \( u \in D(a^k) \), and \([a, T^*_a \circ (X^{k-2} Y) \varphi_1] \tilde{A}^{-1}_{x, t} \in B(L^2(\Omega), L^2(\Omega)) \), which complete the proof of this lemma.

By virtue of Lemma 24, the equation (1') is rewritten as

\[
Lu = \frac{\partial^m}{\partial t^m} + \alpha_1(x, D) \tilde{A}_{x, t} + \alpha_2(x, D) \frac{\partial^{m-1}}{\partial t^{m-1}} \tilde{A}_{x, t} + \alpha_{2m-1}(x, D) \frac{\partial}{\partial t} + \alpha_{2m}(x, D) \tilde{A}^{m-1} u
\]

for \( (u, \frac{\partial u}{\partial t}, \cdots, \frac{\partial^{m-1} u}{\partial t^{m-1}}) \in D(a^m) \times D(a^{m-1}) \times \cdots \times D(a^1) \), where \( \beta_j \) is a linear operator such that \( \beta_{2\ell} \tilde{a}^{-\ell-1} \tilde{A}_{x, t}^{-1}, \beta_{2\ell-1} \tilde{a}^{-(\ell-1)} \in B \left( L^2(\Omega), L^2(\Omega) \right) \).

Hence setting

\[
A = \begin{pmatrix}
0 & 1 \\
0 & \ddots & \ddots & \ddots \\
-\alpha_{2m-1} \tilde{A}_{x, t}^{m-1} & \cdots & -\alpha_1 \tilde{A}_{x, t}^{m-1} & 0 & 1 \\
0 & \ddots & \ddots & \ddots \\
-\beta_{2m} & \cdots & -\beta_2 & -\beta_1
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
0 \\
0 & \ddots & \ddots & \ddots \\
-\beta_{2m} & \cdots & -\beta_2 & -\beta_1
\end{pmatrix},
\]

\[
U = \left( u, \frac{\partial u}{\partial t}, \cdots, \frac{\partial^{m-1} u}{\partial t^{m-1}} \right), \quad F = \ell(0, \cdots, 0, f),
\]

we can rewrite (26) as

\[
(t^2 \frac{d}{dt} + A + Q)U + F = (A + Q)U + F,
\]

\( E : \) unit matrix.

From now on we consider only Neumann problem as we can consider Dirichlet problem similar to Neumann one.

Here \( D(a) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial n} \bigg|_\Gamma = 0 \right\} \).

Then the basic space is \( \mathcal{E}(\tilde{A}) = D(a^{m-1}) \times D(a^{m-2}) \times \cdots \times D(a^1) \times L^2(\Omega) \), and the domain of \( A \) is \( D(a^m) \times D(a^{m-1}) \times \cdots \times D(a^1) \). Furthermore we define
Similarly to Leray's method, for the roots \( \tau_j(x, \xi) \) \((j=1, \cdots, 2m)\) of the equation \( \tau^{2m} + \frac{a_1(x, \xi)}{a_0(x, \xi)} \tau^{2m-1} + \cdots + \frac{a_{2m}(x, \xi)}{(a_0(x, \xi))^m} = 0 \), we set \( \mu_j(x, \xi) \). From the assumption that \( \tau_j(x, \xi) \) are pure imaginary, we see that \( \mu_j(x, \xi) \) are real. Now we set

\[
R(x, \xi) = \begin{pmatrix} 1 & \cdots & 1 \\
\mu_1(x, \xi) & \cdots & \mu_{2m}(x, \xi) \\
\cdots & \cdots & \cdots \\
(\mu_1(x, \xi)^{2m-1} & \cdots & (\mu_{2m}(x, \xi)^{2m-1}) 
\end{pmatrix}, \quad S = R R',
\]

and \( B(x, \xi) = S^{-1} \text{det } S \), whose components are polynomials of the coefficients of the above equation.

Here we give an equivalent norm in \( \mathcal{E}(\mathcal{A}_a) \) with the aid of \( B(x, \xi) \), which will be needed to get the a priori estimate of \((27)\). To this end we state the following Lemmas.

**Lemma 26.** (Leray)

Each component of \( B(x, \xi) \) belongs to \( \overline{\mathcal{U}}_a \) (From now on we write it as \( B(x, \xi) \in \overline{\mathcal{U}}_a \)).

**Lemma 27.** Let \( G(x, \xi) \) be a \((2m, 2m)\) matrix such that \( G(x, \xi) \in \overline{\mathcal{U}}_a \) and \( G(x, \xi) \gg \delta, \text{E} > 0 \). Then \( (G(x, \xi))^{\frac{1}{2}} \in \overline{\mathcal{U}}_a \).

**Proof.** It is seen from

\[
(G(x, \xi))^{\frac{1}{2}} = \frac{1}{2\pi i} \int_{r} \lambda^{\frac{1}{2}}(\lambda \text{E} - G)^{-1} d\lambda,
\]

where \( r \) is a sufficiently large circle whose radius is larger than the absolute value of all eigenvalues with respect to \( G(x, \xi) \) and \( M_s(G) \) (see \((2)\)).

**Lemma 28.** Let \( G(x, \xi) \) be as in Lemma 27. Then for an arbitrary small \( \varepsilon(>0) \) there exists a constant \( c(\varepsilon) \) such that

\[
\text{Re} \left( G(x, D) \mathcal{A}_a U, \mathcal{A}_a U' \right)_{L^2(\Omega)} \geq (\delta_1 - \varepsilon) \| \mathcal{A}_a U \|_{L^2(\Omega)}^2 - c(\varepsilon) \| U \|_{L^2(\Omega)}^2
\]

for all \( U \in (D(\mathcal{A}_a))^{2m} \), here \( (U, V)_{L^2(\Omega)} = \sum_{i=1}^{2m} (\mu_i, v_i)_{L^2(\Omega)} \) for \( U, V \in (L^2(\Omega))^{2m} \).

**Proof.** From Lemma 27 we see that \( (G(x, \xi))^{\frac{1}{2}} \in \overline{\mathcal{U}}_a \). And \( G^{\frac{1}{2}}(x, D) = G^{\frac{1}{2}}(x, D) \), from which we can prove this lemma by virtue of Theorem 3 and interpolation (see the proof of Lemma 12).
Now we will give an equivalent norm in $\mathcal{E}(\tilde{a}_+).$ We set

$$B_1 = E(i)^{-1} \frac{B + B^*}{2} E(i) + \frac{\beta}{2} (\tilde{A}_+^{-1} + A_+^{-1})$$

and set

$$\|U\|_2^2 = (B_1 E(\tilde{a}_+) U, E(\tilde{a}_+) U)_{L^2(\Omega)}$$

for $U \in \mathcal{E}(\tilde{a}_+),$ where

$$E(\tilde{a}_+) = \begin{pmatrix} \tilde{a}_{m-1} & \cdots & \tilde{a} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m-1} & \cdots & \tilde{a} \\ 1 & \cdots & 1 \end{pmatrix}, \\ E(i) = \begin{pmatrix} i^{2m-1} \\ \vdots \\ i^{2m-1} \\ 1 \end{pmatrix},$$

$i = \sqrt{-1}, \beta: \text{a sufficiently large constant}.$

**Lemma 29.** $C_1 \|E(\tilde{a}_+) U\|^2_{L^2(\Omega)} \leq \|U\|^2 \leq C_2 \|E(\tilde{a}_+) U\|^2_{L^2(\Omega)}$ for $U \in \mathcal{E}(\tilde{a}_+),$ here $C_1$ and $C_2$ are independent of $U.$

**Proof.** $S$ is real symmetric and $S \gg \delta_S E > 0,$ from which it follows that $R(x, \xi)$ is also real symmetric and $\mathcal{R} \gg \delta_R E > 0.$

Then by Lemma 28 for $U \in (D(\tilde{a}_+))^m$ and for sufficiently small $\epsilon(>0)$ we obtain

$$(B_1 \tilde{A}_+ \cdot U, \tilde{A}_+ \cdot U)_{L^2(\Omega)} = \Re \left( E(i) \cdot B \cdot E(i) \cdot \tilde{A}_+ \cdot U, \tilde{A}_+ \cdot U \right)_{L^2(\Omega)} + \beta \Re (\tilde{A}_+^{-1} \tilde{A}_+ \cdot U, \tilde{A}_+ \cdot U)_{L^2(\Omega)}$$

$$\geq \left( \tilde{a}_m - \epsilon \right) \|\tilde{A}_+ \cdot U\|^2_{L^2(\Omega)} - \epsilon \|U\|^2_{L^2(\Omega)} + \beta \Re (\tilde{A}_+ \cdot U, U)_{L^2(\Omega)}.$$

Moreover from (24) it follows that

$$\Re (\tilde{A}_+ \cdot U, U)_{L^2(\Omega)} \geq C_1 \|U\|^2_{L^2(\Omega)} \quad \text{where} \quad C_1 > 0.$$

Therefore for sufficiently large $\beta$

$$(B_1 \tilde{A}_+ \cdot U, \tilde{A}_+ \cdot U)_{L^2(\Omega)} \geq (\tilde{a}_m - \epsilon) \|\tilde{A}_+ \cdot U\|^2_{L^2(\Omega)}.$$

Next for $V \in \mathcal{E}(\tilde{a}_+),$ we see that $\tilde{A}_+^{-1} E(\tilde{a}_+) V \in (D(\tilde{a}_+))^m,$ from which it follows that

$$(B_1 E(\tilde{a}_+) V, E(\tilde{a}_+) V)_{L^2(\Omega)} \geq (\tilde{a}_m - \epsilon) \|E(\tilde{a}_+) V\|^2_{L^2(\Omega)}.$$
Moreover it is easily seen that
\[ \left( B_1, E(\tilde{A}_+) V, E(\tilde{A}_+) V \right)_{L^1(\Omega)} \leq C_2 \left\| E(\tilde{A}_+) V \right\|_{L^1(\Omega)}^2, \quad \text{q.e.d.} \]

**Lemma 30.** $B_1$ is a bounded selfadjoint operator from $(L^2(\Omega))^{2m}$ onto itself and also has a bounded inverse.

**Proof.** It is seen from the expression of $B_1$ and Lemma 29.

**Lemma 31.** (a priori estimate)

For $U \in D(A)$ there exists a constant $\tau_0(>0)$ such that
\[ \left\| \left( \tau E_0 (A + Q_0) U \right) \right\| \geq (\tau - \tau_0) \left\| U \right\| \quad \text{for} \quad \tau > \tau_0. \]

**Proof.** We see that
\[
\left( (A + Q) U, U \right) = \left( B_1 E(\tilde{A}_+) (A + Q) U, E(\tilde{A}_+) U \right)_{L^1(\Omega)},
\]
where
\[
E(\tilde{A}_+) A = \begin{pmatrix}
0 & 1 & & \\
0 & & & 1 \\
-\alpha_{2m} & -\alpha_{2m-1} & \cdots & -\alpha_2 & -\alpha_1 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & 0 & K\tilde{A}_+ & & \\
0 & K\tilde{A}_+ & & \\
-\alpha_{2m}K\tilde{A}_+ & -\alpha_{2m-2}K\tilde{A}_+ & & \\
\end{pmatrix} + \begin{pmatrix}
0 & 0 & & \\
0 & 0 & \cdots & \\
0 & 0 & \cdots & \\
\end{pmatrix}
\]

\[
\equiv P\tilde{A}_+ E(\tilde{A}_+), \quad K = \tilde{a} - \tilde{A}_+. \]

From Corollary stated above Lemma 22 $Q_1$ is a bounded operator from $(L^2(\Omega))^{2m}$ into itself.

Hence we find that
\[ ((A + Q) U, U) = \left( B_1 P\tilde{A}_+ E(\tilde{A}_+) U, E(\tilde{A}_+) U \right)_{L^1(\Omega)} + \left( B_1 Q_1 E(\tilde{A}_+) U, E(\tilde{A}_+) U \right)_{L^1(\Omega)} + \left( B_1 Q_1 E(\tilde{A}_+) U, E(\tilde{A}_+) U \right)_{L^1(\Omega)}, \]

\[ \text{(28)} \]
where \( E(\Lambda_+) \Omega = Q_2 E(\Lambda_+) \) and \( Q_2 \in B(\left( L^2(\Omega) \right)^{2m}, \left( L^2(\Omega) \right)^{2m}) \).

To estimate the first term on the right of (28) we set \( P = iE(i)^{-1}P_1 E(i) \). Then we find that the symbol of the principal part of \( B_1P \) is \( iE(i)^{-1}B(x, \xi) P_1(x, \xi) E(i) \) which is anti symmetric, that is, \( \text{Re}(iE(i)^{-1}B(x, \xi) \times P_1(x, \xi) E(i)) = 0 \) (see [11]), hence
\[
\text{Re}(B_1P \Lambda_+ E(\Lambda_+) U, E(\Lambda_+) U)_{L^2(\Omega)} \leq C \| E(\Lambda_+) U \|_{L^2(\Omega)}^2 \leq C \| U \|^2.
\]

The other terms on the right of (28) are also bounded by \( C \| U \|^2 \). Therefore
\[
(29) \quad \left| \text{Re}\left( (A + Q) U, U \right) \right| \leq C \| U \|^2.
\]

It follows from this that for \( U \in D(A) \) there exists a constant \( \tau_0(>0) \) such that
\[
\left\| \left( \tau E - (A + Q) \right) U \right\| \geq (\tau - \tau_0) \| U \| \quad \text{for} \quad \tau > \tau_0.
\]

**Theorem 4.** (Existence of resolvent)

For sufficiently large constant \( \tau(\tau_0) \), \( \tau E - (A + Q) \) has a bounded inverse \( R(\tau, A + Q) \) from \( \mathcal{E}(\Lambda_+) \) into itself such that
\[
\left\| R(\tau, A + Q) \right\| \leq \frac{1}{\tau - \tau_0}.
\]

**Proof.** From Lemma 31, we may prove that for any \( F \in (L^2(\Omega))^{2m} \) there exists a solution \( U \in (D(\Lambda_+))^{2m} \) of the equation
\[
\left( \tau E - P \Lambda_+ \right) \left( Q_1 + Q_2 \right) U = F.
\]

This can be shown by the same consideration in Lemma 19 indicating the operator \( R \). To this end, let \( x_0 \) be a fixed point in \( \Gamma \). Then we may show that \( (\tau E - P(T_\sigma^{-1}y_0, \langle dT_\sigma^* \rangle(D_\sigma)(\sigma_\sigma(y_0, D)A_+ + \lambda) U \) has a similar estimate in Lemma 16, that is, for a sufficiently large \( \tau \)
\[
\left\| \left( \tau E - P \right) \left( T_\sigma^{-1}y_0, \langle dT_\sigma^* \rangle(D_\sigma)(\sigma_\sigma(y_0, D)A_+ + \lambda) U \right|_{y_0 \geq 0}
\geq C(\left\| U \right\|_{y_0 \geq 0} + (\tau - \rho) \| U \|_{y_0 \geq 0})
\]
\[
\text{for} \quad U \in (D(A_+))^{2m}, \quad \text{and also has a bounded inverse, here} \quad x_0 \in U_+, \quad y_0 = T_\sigma(x_0).
\]

Set
Then $D(y_0, \xi) \in \mathbb{N}$, $N(y_0, \xi) \mathbf{P}(y_0, \xi) = iD(y_0, \xi) N(y_0, \xi)$ and from Theorem 2, $N(\tau E - \mathbf{P}(\sigma_0 A_+ + \lambda) - (Q_1 + Q_2)) N^{-1} = (\tau E - iD(\sigma_0 A_+ + \lambda) - Q_3$, where $Q_3$ is a bounded operator and $D(y_0, D)(\sigma_0(y_0, D)A_+ + \lambda)$ is a selfadjoint operator on $(L^2(\mathbb{R}^n))^{2m}$ with its domain $(D(A_+))^{2m}$. Thus our assertion is proved. q.e.d.

By virtue of Lemma 31 and Theorem 4 we will solve the equation (27) applying the theory of semi group.

For $U(0) = u(0, x), \cdots, \frac{\partial^{2m-1} u}{\partial \tau^{2m-1}}(0, x) \in D(A)$ and $F(t) \in C^0([0, T]; \mathcal{E}(A_+))$, $(A + Q) F(t) \in C^0([0, T]; \mathcal{E}(A_+))$, there exists a unique solution $U(t)$ of the equation (27) and

$$\left\| U(t) \right\| \leq \exp(\tau_0 t) \left( \left\| U(0) \right\| + \int_0^t \left\| f(s) \right\|_{L^2(\Omega)} ds \right),$$

that is,

$$\left\| u(t, x) \right\|_{L^2(\mathbb{R}^n)} + \left\| \frac{\partial u(t, x)}{\partial \tau} \right\|_{L^2(\mathbb{R}^n)} + \cdots + \left\| \frac{\partial^{2m-1} u}{\partial \tau^{2m-1}}(t, x) \right\|_{L^2(\Omega)} \leq C \exp(\tau_0 t) \left( \left\| u(0, x) \right\|_{L^2(\mathbb{R}^n)} + \left\| \frac{\partial u}{\partial t}(0, x) \right\|_{L^2(\mathbb{R}^n)} + \cdots + \left\| \frac{\partial^{2m-1} u}{\partial \tau^{2m-1}}(0, x) \right\|_{L^2(\Omega)} + \int_0^t \left\| f(s) \right\|_{L^2(\Omega)} ds \right).$$

Moreover for $f(t) \in C^1([0, T]; L^2(\Omega))$, we choose $f_j(t)$ such that $f_j(t) \in C^0([0, T]; D(A_+)) \cap C^1([0, T]; L^2(\Omega))$ and passing to the limit we find the existence of solution $u(t)$. Furthermore using the equation itself we find that
\begin{equation}
\left\| u(t, x) \right\|_{D(a^{m})} + \left\| \frac{\partial u}{\partial t}(t, x) \right\|_{D(a^{m-\frac{1}{2}})} + \cdots + \left\| \frac{\partial^{m} u}{\partial t^{m}}(t, x) \right\|_{L^{2}(\Omega)} \\
\leq C \exp \left( \tau_{\delta} t \right) \left( \left\| u(0, x) \right\|_{D(a^{m})} + \left\| \frac{\partial u}{\partial t}(0, x) \right\|_{D(a^{m-\frac{1}{2}})} + \cdots + \left\| \frac{\partial^{m} u}{\partial t^{m}}(0, x) \right\|_{L^{2}(\Omega)} \right) \end{equation}

\textbf{Remark 1.} As seen directly in our proofs of \S 2.1, in the statements of Theorems 1, 2, \E_{t}^{\infty} can be replaced by \E_{t, \delta}^{\infty}(0<\delta<1) which is the set of \sigma(x, \xi) such that the first order derivatives \frac{\partial \sigma}{\partial x_{i}}(x, \xi) exist and uniformly Hölder continuous of order \delta. Furthermore we can drop Hölder continuity \delta by the similar treatment as one in [4].

\textbf{Remark 2.} For the system mentioned in Introduction we can also apply the same treatment as one in \S 3. Because setting \frac{\partial^{i} u_{i}}{\partial t^{i}} = u_{i, i, 1} we may consider \left( u_{11}, u_{12}, \ldots, u_{1, m_{1}}, u_{21}, \ldots, u_{2, m_{2}}, \ldots, u_{k, 1}, \ldots, u_{k, m_{k}} \right) and the corresponding system. Then using the partition of unity with respect to \tilde{Q} \times \Sigma we can construct \R(x, \xi) locally and the norm same as one treating in Cauchy problem (see [14]).

\textbf{Remark 3.} By the method of reflection we can reduce our mixed problems where \Q=\R^{n} and \sigma(x, D)=\Delta into Cauchy problems over the whole space \R^{n}. Here we must remark that the coefficients of principal part of reduced equation have Lipshitz continuity. Therefore for the case of single equation we can apply the theory of Leray and Gårding. Furthermore for the case of system we can extend the theory of Calderón-Zygmund's singular integral operator to one for symbols which are introduced by the reduced equations (see Remark 1, Corollary of Lemma 5 and [4]). Thus in this case we can obtain the results in \S 3.

\textbf{Remark 4.} In the case for single equation it is easily seen that by coordinate transformation in Lemma 13 \sigma(x, D) is transformed to \sigma(y, D) with \sigma_{i}(y, 0)=0 \begin{equation} (i=1, \ldots, n-1) \end{equation} therefore our boundary conditions (D) and (N) are reduced to \left( \frac{\partial^{i}}{\partial t^{i}} + (\text{lower order terms}) \right) u = 0 on \Gamma, i=2j (j=0, 1, \ldots, m-1), i=2j+1 (j=0, 1, \ldots, m-1), respectively. Therefore by the transformation such that \begin{equation} t' = t + \varepsilon \sum_{i=1}^{n-1} y_{i}^{2} \end{equation} the form of the boundary conditions are invariant, we can prove that local uniqueness theorem is valid, that is, the speed of propagation of waves is finite. Moreover by the same consideration this fact is also valid for the case of systems.
As mentioned Remark 2–4, we have various methods for solving our problems. To apply these ones we must treat the perturbation with respect to boundary operators by using the theory of elliptic operators for boundary value problems and gather local solutions. As long as we are concerned with the mixed problems mentioned in Introduction, we can avoid such trouble-someness by considering our singular integral operators. Furthermore by our method our system can be reduced into the first order system written in our singular integral operators, which are as invariant for certain change of variables as formulation of our problems. Moreover we obtain (29) directly by our method.

References


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