ON VALUATIONS OF POLYNOMIAL RINGS OF MANY VARIABLES

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Author(s)
Inoue, Hiroshi

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Part one

By

Hiroshi INOUE

In his paper “A construction for absolute values in polynomial rings”, “Trans. Amer. Math. Soc. Vol. 40 (1936)”, MacLane developed a valuation theory for a polynomial ring of one variable determining every possible valuation in the ring whose coefficient field has a discrete valuation. In the present paper the writer wants to extend MacLane’s theory to polynomial rings of two variables and also to those of many variables.

We shall refer the above MacLane’s paper to as M. For example, M. Theorem 8.1 means Theorem 8.1 in the MacLane’s paper. While the present paper is concerned with polynomial rings of many variables, we shall start for completeness with considerations on valuations of a polynomial ring of one variable.

§ 1. Valuations of \( K(x) \)

Let \( K \) be a field. Suppose \( K \) has a discrete non-archimedean valuation \( V_0 \), namely, \( V_0 \) is a real valued function of \( K \) satisfying the following conditions;
If \( a \in K \) and \( a \neq 0 \), then \( V_0 a \neq \pm \infty \) and \( V_0 0 = \infty \),
\( V_0 ab = V_0 a + V_0 b \),
\( V_0 (a + b) \geq \min \{ V_0 a, V_0 b \} \).

These are called the three conditions of a valuation.

In particular, if \( V_0 a > V_0 b \), then \( V_0 (a + b) \) must equal \( V_0 b \). We shall sometimes call this inequality the triangle law.

All real numbers which are the values taken by \( V_0 \) in \( K \) form an isolated point set \( \Gamma_0 \) and is a cyclic additive group. Therefore in this paper we may assume that \( \Gamma_0 \) is a set of all rational integers.

If a non-zero integral multiple of a real number \( \mu \) belongs to \( \Gamma_0 \), then \( \mu \) is said to be commensurable with \( \Gamma_0 \), and if otherwise \( \mu \) is called incommensurable.

Let \( R \) be an integral domain and \( W \) a valuation of \( R \). Let \( a \) and \( b \) be
two elements of $R$. We shall say that $a$ and $b$ are equivalent in $W$ (notation: $a \sim b$) if $W(a-b) > Wa = Wb$, while $b$ is equivalence divisible by $a$ (notation: $a \mid b$ in $W$) if there exists an element $c$ in $R$ such that $ac \sim b$ in $W$. Generally when a valuation $U_1$ of $K[x]$ is given, a polynomial $\phi_2(x)$ in $K[x]$ is called an $x$-key polynomial over $U_1$, if it satisfies the following three conditions:

1. if $\phi_2(x)|f(x)g(x)$ in $U_1$, then $\phi_2(x)|f(x)$ or $\phi_2(x)|g(x)$ in $U_1$,
2. if $\phi_2(x)|h(x)$ in $U_1$ and $h(x) \neq 0$, then $\deg_x \phi_2(x) \leq \deg_x h(x)$, where $\deg_x h(x)$ means the degree of $h(x)$ with respect to $x$,
3. the leading coefficient of $\phi_2(x)$ is 1. (M. Definition 4.1)

Let $\phi_2(x)$ be an $x$-key polynomial over $U_1$ and $\mu_2$ a real number greater than $U_1 \phi_2(x)$. Then the function $U_2$ defined as follows is a valuation of $K[x]$;

$$U_2 h(x) = \min \left[ U_1 h_i(x) + i\mu_2 \right]$$

where $h(x)$ is any polynomial in $K[x]$ and $h(x) = \sum_i h_i(x) \phi_2(x)^i$, $\deg_x h_i(x) < \deg_x \phi_2(x)$ for $i = 0, 1, 2, \cdots$, is the expansion of $h(x)$ with respect to $\phi_2(x)$. (M. Theorem 4.2)

**Definition 1.1.** The above defined function $U_2$ is called an augmented valuation of $U_1$ in narrow sense and is denoted as $U_2 = [U_1, \ U_2 \phi_2 = \mu_2]$ and this relation is denoted as $U_1 < U_2$.

Similarly, if $\phi_3(x)$ is an $x$-key polynomial over $U_2$, then we define an augmented valuation $U_3$ of $U_2$, using $\phi_3(x)$. Continuing in this way, we obtain valuations of $K[x]$ as follows;

$$U_1 < U_2 < \cdots < U_i < U_{i+1} < \cdots$$

(s).

The $x$-key polynomials $\phi_i(x)$ corresponding to this series must satisfy the following two conditions besides the above mentioned three conditions (1), (2) and (3);

4. $\deg_x \phi_i(x) \leq \deg_x \phi_{i-1}(x)$,
5. $\phi_i(x)$ and $\phi_{i-1}(x)$ are not equivalent in $U_i$,
both for $i = 1, 2, \cdots$. (M. Definition 6.1)

**Definition 1.2.** We call the above series (s) a series of $x$-augmented inductive valuations and when $b > a$, $U_b$ in the series is called an $x$-augmented valuation of $U_a$ in broad sense and $U_a$ an $x$-descended valuation of $U_b$.

An $x$-augmented valuation of a valuation $V$ of $K[x]$ in narrow sense
is an $x$-augmented valuation of $V$ in broad sense. So, hence in this paper whenever we say that $W$ is an $x$-augmented valuation of $V$ or write as \("W>V\)”, it means that $W$ is an $x$-augmented valuation of $V$ in broad sense. Therefore, when $W>V$, $W$ and $V$ induce the same valuation in $K$ and $W: x= U: x$.

Now, let $\mu_1$ be a real number. Then the function $V_{10}$ defined as follows is an extended valuation of $V_{00}$ to $K[x]$;

\[
V_{10}f(x) = \min \left\{ V_{00}f(x) + i\mu_1 \right\},
\]

where $f(x) = \sum f_i x^i$, $f_i \in K$, is any polynomial in $K[x]$. (M. Theorem 3.1).

If a polynomial $\phi_2(x)$ is an $x$-key polynomial over $V_{10}$, we define an $x$-augmented valuation $V_{20}$ of $V_{10}$ using $\phi_2(x)$ in the same way as we defined $U_2$ from $U_1$. Thus we obtain a series of $x$-augmented inductive valuations;

\[
V_{10} < V_{20} < V_{30} < \cdots < V_{\nu_0} < V_{\nu_1,0} < \cdots.
\]

(M. Definition 6.1)

In this case for every non-zero polynomial $g(x)$

\[
V_{\nu_{i+1},0}g(x) \geq V_{\nu_0}g(x) \quad \text{for} \quad i=1, 2, 3, \cdots.
\]

Therefore $V_{\nu_{i-1},0}$ is called an augmented valuation of $V_{\nu_0}$.

If the above series lasts infinitively and if for every non-zero polynomial $f(x)$ the sequence $V_{10}f(x)$, $V_{20}f(x)$, \cdots does not tend to $\infty$, then the limit valuation of this sequence is denoted by $V_{\nu_0}f(x)$. $V_{\nu_0}$ is also a valuation of $K[x]$.

Every discrete extended valuation of $V_{00}$ to $K[x]$ can be represented either as an $x$-augmented valuation or as a limit valuation. (M. Theorem 8.1)

All values $V_{\nu_0}f(x)$ form a cyclic additive group $\Gamma_{\nu_0}$ for $k=0, 1, 2, \cdots$ and $\mu_{k+1} = V_{\nu_{k+1},0}\phi_{k+1}(x)$ is always commensurable with $\Gamma_{\nu_0}$ in this paper, so there is a smallest positive integer $\tau_{k+1}$ such that $\tau_{k+1}\mu_{k+1}$ is in $\Gamma_{\nu_0}$.

Let $f(x) = \sum_{i=0}^{n} f_i(x)(\phi_k(x))^i$ be the expansion of $f(x)$ with respect to $\phi_k(x)$ which is the $x$-key polynomial producing the valuation $V_{\nu_0}$. Then the polynomial $f(x)$ can be an $x$-key polynomial with which we can produce an augmented valuation $V_{\nu_{k+1},0}$ of $V_{\nu_0}$, if and only if the three following conditions hold:

1. $f_0(x) = 1$,
2. $V_{\nu_0}f(x) = V_{\nu_0}f_0(x) = V_{\nu_0}(\phi_k(x))^n = n\mu_k$,
3. $f(x)$ is equivalence-irreducible in $V_{\nu_0}$, so owing to (2) $n=0 \pmod{\tau_k}$.

(M. Theorem 9.4)

Let $S_k$ be a ring which consists of all polynomials $f(x)$ with $V_{\nu_0}f(x) \geq 0$.
and $A_k$ the residue class ring of $V_{x_0}$ in $K[x]$. Namely two polynomials $f(x)$ and $g(x)$ for which $V_{x_0}f(x) = V_{x_0}g(x) = 0$ belong to the same residue class of $A_k$, if and only if

$$V_{x_0}(f(x) - g(x)) > V_{x_0}f(x) = 0.$$ 

And the natural homomorphism $H_k$ of $S_k$ on the residue class ring $A_k$ has the following properties:

$$H_k(f(x) + h(x)) = H_kf(x) + H_kh(x)$$

$$H_k(f(x)h(x)) = (H_kf(x))(H_kh(x))$$

and

$$H_kl(x) = 0 \text{ if and only if } V_{x_0}l(x) > 0.$$ 

A commensurable inductive valuation $V_{x_0}$ of $K[x]$ may be conveniently symbolized thus:

$$V_{x_0} = [V_{x_0}, V_{x_0}x = \mu_1, V_{x_0}x^2 = \mu_2, \ldots, V_{x_0}x^n = \mu_t]$$

or more briefly $V_{x_0} = [V_{k-1,0}, V_{x_0}x = \mu_1].$

Let $F_0$ be a residue class field of the original valuation $V_0$ of $K$. Then about a residue class ring $A_k$ we have the following theorem (M. Theorem 12.1):

There is a sequence of fields $F_1 = F_0, F_2, F_3, \ldots, F_t$, such that each field is an algebraic extension of its preceding field and such that for any $t = 1, 2, \ldots, k$, the $V_{x_0}$-residue class ring $A_{x_0}$ of $K[x]$ is isomorphic to the ring $F_t[y]$ of polynomials in a variable $y$ with coefficients in $F_t$.

§ 2. Continuous augmented valuations

Definition 2.1. $Q, V, U$ and $W$ are elements in an ordered set $L$. If $Q < V < W$ and $Q < U < W$ implies always $U < V$ or $U = V$, $L$ is said to be partially linearly ordered.

We shall show that a set of all descended valuations of a valuation of $K[x]$ is partially linearly ordered, while a set of all descended valuations of a valuation of $K[x, y]$ is not always partially linearly ordered.

Definition 2.2. An $x$-augmented valuation $W$ of a valuation $V$ of $K[x]$ is called a continuous $x$-augmented valuation of $V$ if there is no valuation $T$ of $K[x]$ such that $V < T < W$.

Definition 2.3. Let $V_{x_0} < V_{x_1} < \cdots < V_{x_p-1,0} < V_{x_0} < \cdots < V_{x_0}$ be a series of $x$-augmented inductive valuations of $K[x]$ and suppose, for every $k$ between $p$ and $t-1$, $V_{x_0+k+1,0}$ is a continuous $x$-augmented valuation of $V_{x_0}$. Then the
series is said to be continuous from $V_{\psi^0}$ to $V_\psi$, $t-p$ is called the $x$-distance between $V_{\psi^0}$ and $V_\psi$ and $V_{\psi^0}\prec V_{\psi^{1,0}}\prec\cdots\prec V_\psi$ is called a series of continuous $x$-augmented inductive valuations of $K[x]$.

**Theorem 2.4.** Let $W$ be a discrete valuation of $K[x]$ which induces a valuation $V_\psi$ in $K$ and let $Wx = \mu_1$. Let $V_{\psi^0}$ be the valuation of $K[x]$ defined as follows;

$$V_{\psi^0}f(x) = \text{Min}\left[V_\psi f_\xi + i\mu_1\right],$$

where $f(x) = \sum f_\xi x^\xi$, $f_\xi \in K$, is in $K[x]$.

Then $W$ can be represented either as the last member of a series of continuous $x$-augmented inductive valuations starting from $V_{\psi^0}$ or as a limit valuation of it.

**Proof.** Suppose that $V_{\psi^0} < W$. Consider then the set $P$ of all those polynomials $f(x)$ for which $V_{\psi^0}f(x) < Wf(x)$. Out of $P$ pick up all those polynomials whose degrees are minimum and whose leading coefficients are 1.

Furthermore out of these polynomials choose a polynomial $\phi_2(x)$ so that $W\phi_2(x) - V_{\psi^0}\phi_2(x)$ becomes minimum. Since both $W$ and $V_{\psi^0}$ are discrete valuations of $K[x]$, it is evident that such a polynomial $\phi_2(x)$ does exist. Then $\phi_2(x)$ is an $x$-key polynomial over $V_{\psi^0}$ of $K[x]$. Let $h(x)$ be a polynomial, and let the expansion in $\phi_2(x)$ of $h(x)$ be

$$h(x) = \sum_{\xi=0}^{n} h_\xi(x)\phi_\xi$$

(deg $h_\xi(x) < \text{deg} \phi_2$, for $i = 0, 1, 2, \ldots, n$)

and

$$\mu_2 = V_{\psi} \phi_2 = W\phi_2 > V_{\psi^0} \phi_2$$

then a function $V_{\psi^0}$ defined by

$$V_{\psi^0} h(x) = \text{Min}_i \left[ V_{\psi} h_\xi(x) + i\mu_2 \right]$$

is a continuous $x$-augmented valuation of $V_{\psi^0}$ of $K[x]$.

It was proved in M. Theorem 8.1 that $V_{\psi^0}$ is an $x$-augmented valuation of $V_{\psi^0}$. Suppose that there were a valuation $T$ such that $V_{\psi^0} < T < V_{\psi^0} \leq W$. $T$ appears in a series of $x$-augmented inductive valuations which starts from $V_{\psi^0}$ and ends at $W$. If $t(x)$ is the $x$-key polynomial over $V_{\psi^0}$ of $K[x]$ which produces $T$, it follows according to M. Theorem 6.4

$$Tt(x) = Wt(x) > V_{\psi^0}t(x).$$

Then because of the quality of $\phi_2(x)$, we have

$$\text{deg} \phi_2(x) \leq \text{deg} t(x).$$

Sometimes we denote $\text{deg} t(x)$ briefly as $\text{deg} t(x)$. Since $V_{\psi^0}$ is an $x$-augmented inductive valuation of $T$, we have
By M. Lemma 6.3

\[ V_{10} \phi_2(x) = V_{10} t(x) < W t(x) = T t(x) < V_{20} \phi_2(x) = W \phi_2(x). \]

Therefore \[ W t(x) - V_{10} t(x) < W \phi_2(x) - V_{10} \phi_2(x). \]

This contradicts the above conditions imposed on \( \phi_2(x) \), and thus \( V_{20} \) is a continuous \( x \)-augmented valuation of \( V_{10} \).

In case \( V_{20} < W \), we can derive \( V_{30} \) from \( V_{20} \) in the same way as above, so that \( V_{30} \) is a continuous \( x \)-augmented valuation of \( V_{20} \) of \( K[x] \). By continuing this method we can establish our theorem in the same way as in M. Theorem 8.1.

**Theorem 2.5.** The series of continuous \( x \)-augmented inductive valuations from \( V_{10} \) to \( W \) or their limit valuation in Theorem 2.4 is uniquely determined by \( W \) and hence the \( x \)-distance between \( V_{10} \) and \( W \) is also uniquely determined.

**Proof.** Consider two series of continuous \( x \)-augmented inductive valuations from \( V_{10} \) to \( W \);

\[ V_{10} < V_{20} < \cdots < V_{a-1,0} < V_{a0} < \cdots < V_{p0} = W \]
\[ V_{10} < V'_{20} < \cdots < V'_{a-1,0} < V'_{a0} < \cdots < V'_{q0} = W, \]

and assume that \( V_{20} = V'_{20}, V_{30} = V'_{30}, \ldots, V_{a-1,0} = V'_{a-1,0}. \) We shall then show that \( V_{a0} = V'_{a0}. \)

Both \( V_{a0} \) and \( V'_{a0} \) are continuous \( x \)-augmented valuations of \( V_{a-1,0} \) of \( K[x] \), so let \( \phi_a(x) \) and \( \phi'_a(x) \) be respectively \( x \)-key polynomials over \( V_{a-1,0} \) that produce \( V_{a0} \) and \( V'_{a0} \). Then

\[ \text{deg} \phi_a = \text{deg} \phi'_a > \text{deg} r(x) \]

where \( r(x) = \phi_a(x) - \phi'_a(x) \), and moreover

\[ W \phi_a = V_{a} \phi_a > V_{a-1,0} \phi_a \]
\[ W \phi'_a = V'_{a} \phi'_a > V'_{a-1,0} \phi'_a. \]

This implies

\[ V_{a0} r(x) = V_{a-1,0} r(x) = W r(x) \geq \text{Min} [W \phi_a, W \phi'_a] \]
\[ > \text{Min} [V_{a-1,0} \phi_a, V_{a-1,0} \phi'_a], \]

and so

\[ V_{a-1,0} r(x) > V_{a-1,0} \phi_a = V_{a-1,0} \phi'_a. \]

As both \( V_{a0} \) and \( V'_{a0} \) are continuous \( x \)-augmented valuations of \( V_{a-1,0} \)
Hence we have
\[ W\phi_a = W\phi'_a. \]
Since \( r(x) = \phi_a - \phi'_a \), it follows by the triangle law \( Wr(x) \geq W\phi_a = W\phi'_a \).
Namely \( V_{n-1,0} r(x) \geq V_{n0} \phi_a = V_{n0} \phi'_a \).
\[
\phi'_a = \phi_a - r(x)
\]
\[
V_{n0} \phi'_a = \min \left[ V_{n0} \phi_a, \, V_{n-1,0} r(x) \right] = V_{n0} \phi_a = W\phi_a = W\phi'_a = V_{n0} \phi'_a.
\]
Let the expansion in \( \phi'_a \) of an arbitrary polynomial \( h(x) \) be \( \sum h_k(x) \phi'_a \). Then
\[
V_{n0} h(x) = \min \left[ V_{n-1,0} h_k(x) + i V_{n0} \phi'_a \right]
\]
\[
= \min \left[ V_{n-1,0} h_k(x) + i V_{n0} \phi'_a \right]
\]
\[
= \min \left[ V_{n0} h_k(x) \phi'_a \right]
\]
\[
\leq V_{n0} \left[ \sum h_k(x) \phi'_a \right]
\]
\[
= V_{n0} h(x).
\]
In the same way,
\[
V_{n0} h(x) \geq V_{n0} h(x).
\]
Thus
\[
V_{n0} h(x) = V_{n0} h(x).
\]

**Theorem 2.6.** If \( V_{10} < V_{20} < \cdots < V_{n0} = W \) is a series of continuous \( x \)-augmented inductive valuations and \( U \) is a descended valuation of \( W \), then \( U \) coincides with one of \( V_{10}, \, V_{20}, \cdots, \, V_{n0} \) and \( x \)-distance between \( U \) and \( W \) is uniquely determined by \( U \) and \( W \).

**Proof.** According to Theorem 2.4 one can build a series of continuous \( x \)-augmented inductive valuations
\[
V_{10} < V_{20} < \cdots < V_{n0} = W
\]
which goes through \( U \); say \( U = V'_{a0} \) \((0 < a \leq q)\). By Theorem 2.5 this is the same as
\[
V_{10} < V_{20} < \cdots < V_{n0} = W.
\]
Hence \( U \) must coincide with one of \( V_{10}, \cdots, V_{n0} \) and \( x \)-distance between \( U \) and \( W \) is uniquely determined by \( U \) and \( W \).

**Corollary 2.7.** If \( W \) is an augmented valuation of \( U \) of \( K[x] \), then we can build a series of continuous \( x \)-augmented inductive valuations from \( U \) to \( W \).

This corollary is self-evident from Theorem 2.6.
Theorem 2.8. A set of all x-descended valuations of a valuation of $K[x]$ is partially linearly ordered.

Proof. Assume that $Q$, $U$ and $W$ are valuations of $K[x]$ and $Q < W$ and $U < W$. Then we can build a series of continuous $x$-augmented inductive valuations from $V_{10}$ to $W$, according to Theorem 2.4. And owing to Theorem 2.6, $Q$ and $U$ must appear in the series, therefore $Q < U$ or $Q > U$ or $Q = U$.

§ 3. The shortest series of $x$-augmented inductive valuations

Mac Lane gave the following lemma (M. Lemma 15.1); if an inductive valuation

$$V_{k} = [V_{k-2}, V_{k}, \phi_{k-1} = \mu_{k-1}, V_{k} \phi_{k} = \mu_{k}]$$

has two key polynomials $\phi_{k-1}$ and $\phi_{k}$ of the same degree, then

$$W = [V_{k-2}, W \phi_{k} = \mu_{k}]$$

is an inductive valuation equal to $V_{k}$.

Herein $W \phi_{k} = \mu_{k} = V_{k} \phi_{k}$, so instead of $W = [V_{k-2}, W \phi_{k} = \mu_{k}]$, we may write this conclusion as $V_{k} = [V_{k-2}, V_{k} \phi_{k} = \mu_{k}]$.

According to this lemma, we can shorten every series of continuous $x$-augmented inductive valuations.

Let

$$V_{10} < V_{20} < \cdots < V_{10} = W$$

be a series of continuous $x$-augmented inductive valuations and $\phi_{k}$ an $x$-key polynomial which produces $V_{k}$, for $k = 1, 2, \ldots, p$:

$$V_{10} \phi_{1} = \mu_{1}, \quad V_{20} \phi_{2} = \mu_{2}, \quad \ldots, \quad V_{p0} \phi_{p} = \mu_{p}.$$

Suppose

$$1 = \deg \phi_{1} = \deg \phi_{2} = \cdots = \deg \phi_{a-1} = \cdots = \deg \phi_{b} < \deg \phi_{b+1} = \cdots = \deg \phi_{c} < \deg \phi_{c+1} = \cdots = \deg \phi_{p}.$$

Then according to M. Theorem 15.2

$$W = V_{p0} = [V_{00}, V_{10} \phi_{1} = \mu_{1}, V_{20} \phi_{2} = \mu_{2}, \ldots, V_{p0} \phi_{p} = \mu_{p}].$$

Definition 3.1. Let $W = [V_{00}, V_{10} \phi = \mu_{1}, V_{20} \phi_{2} = \mu_{2}, \ldots, V_{p0} \phi_{p} = \mu_{p}]$. If $\deg \phi_{3} < \deg \phi_{3} < \cdots < \deg \phi_{p}$, the series $V_{10} < V_{20} < \cdots < V_{p}$ is called the shortest series of $x$-augmented inductive valuations from $V_{12}$ to $W$. 
When a series of continuous augmented inductive valuations of a valuation \( W \) is given, construction of the shortest series of augmented inductive valuations of \( W \) is very easy, as we have shown above.

Conversely when the shortest series of augmented inductive valuations of a valuation \( W \) is given, we consider the method to construct a series of continuous augmented inductive valuations of \( W \).

Let the shortest series of augmented inductive valuations of \( W \) be given as follows;

\[
V_1 < V_2 < \cdots < V_a < V_{a+1} < \cdots < V_p = W.
\]

Now we want to construct a series of continuous augmented valuations between \( V_a \) and \( V_{a+1} \) as follows;

\[
V_a < U_1 < U_2 < \cdots < U_b < V_{a+1} \ldots \ldots \ldots (1).
\]

Let \( \phi_1, \phi_2, \ldots, \phi_b, \phi_{a+1} \) be respectively the \( x \)-key polynomials which produce \( V_a, U_1, U_2, \ldots, U_b, V_{a+1} \). Then according to M. Lemma 15.1, the following relations hold.

\[
\deg \phi_a = \deg \phi_1 = \deg \phi_2 = \cdots = \deg \phi_b = \deg \phi_{a+1}.
\]

Put \( V_a \phi_a = \mu_a, U_1 \phi_1 = \mu_1, U_2 \phi_2 = \mu_2, \ldots, U_b \phi_b = \mu_b \) and \( V_{a+1} \phi_{a+1} = \mu_{a+1} (\equiv V_a \phi_{a+1}) \).

If \( \Gamma_\epsilon \) is a set of all values that \( V_a \) takes in \( K[x] \), then \( \Gamma_a \) contains \( \mu_1, \mu_2, \ldots, \mu_b \) and \( V_{a+1} \phi_{a+1} \). For, \( V_{a+1} \phi_{a+1} > U_\epsilon \phi_\epsilon = \mu_\epsilon \) according to M. Lemma 6.3 and \( \phi_\epsilon \) is a key polynomial which produces \( U_\epsilon \), for \( i = 1, 2, \ldots, b \), so \( V_{a+1} \phi_\epsilon = U_\epsilon \phi_\epsilon \) for each \( i \).

Let \( \phi_{a+1} = \phi_\epsilon + C_\epsilon(x) \). Then \( \deg C_\epsilon(x) < \deg \phi_\epsilon = \deg \phi_{a+1} \) so by the definition of \( V_{a+1} \)

\[
V_{a+1}C_\epsilon(x) = V_aC_\epsilon(x).
\]

Therefore, owing to the triangle law

\[
V_{a+1} \phi_{a+1} > V_{a+1} \phi_\epsilon = \mu_\epsilon = V_a C_\epsilon(x),
\]

so each \( \mu_\epsilon = U_\epsilon \phi_\epsilon \) belongs to \( \Gamma_a \).

According to M. Lemma 6.3

\[
U_\epsilon \phi_{a+1} = U_\epsilon \phi_\epsilon = \mu_\epsilon.
\]

Therefore a necessary and sufficient condition for that a series

\[
V_a < U_1 < U_2 < \cdots < U_b < V_{a+1}
\]

be a series of continuous augmented inductive valuations is that the series

\[
V_a \phi_{a+1} > \mu_1 > \mu_2 > \cdots > \mu_b > V_{a+1} \phi_{a+1}
\]
On Valuations of Polynomial Rings of many Variables

is most dense in \( I_a \) except \( V_{a+1} \).
Namely we pick up all those numbers in \( I_a \) that are between \( V_a \phi_a + 1 \) and \( V_{a+1} \phi_{a+1} \).
Such numbers exist, for \( I_a \) is a set of isolated points.
Let
\[ \phi_{a+1} = (\phi_a^n)^n + \cdots + 1(x) \]
be an expansion of \( \phi_{a+1} \) with respect to \( \phi_a \), where \( V_a \phi_a \) belongs to \( I_{a-1} \).
Then according to M. Lemma 9.3, there exists a polynomial \( G_i(x) \) such that
\[ V_a G_i(x) = \rho_i \]
and
\[ \deg G_i(x) < \deg \phi_{a+1} \]
where
\[ i = 1, 2, \ldots, b \]
Put
\[ \psi_1 = \phi_{a+1} - \sum_{i=1}^b G_i(x) \]
Then \( \psi_1 \) is a key polynomial over \( V_a \),
because
\[ V_a (\sum G_i) = \min_i [\rho_i] = \rho_1 > V_a \phi_{a+1} \]
Therefore
\[ \psi_1 \sim \phi_{a+1} \] in \( V_a \).
\( \phi_{a+1} \) is a key polynomial which produces \( V_{a+1} \) from \( V_a \), so \( \psi_1 \) can be a key polynomial over \( V_a \), for \( \deg \psi_1 = \deg \phi_{a+1} \).
On the other hand, \( \psi_1 \) produces \( U_1 \), then we may define \( U_1 \) as follows.
\[ U_1 \psi_1 = V_{a+1} \psi_1 = V_{a+1} (\phi_{a+1} - \sum G_i) = \rho_1 \]
because
\[ V_{a+1} \phi_{a+1} = \mu_{a+1} > \rho_1 \]
If we put
\[ \phi_2 = \psi_1 + G_1(x) \]
then we can prove in the same way that \( \phi_2 \) is a key polynomial over \( U_1 \) and we may define \( U_2 \) as \( U_1 \psi_2 = \rho_2 \). Next we put \( \phi_3 = \phi_2 + G_2(x) \), and so on. Thus we can show that the series
\[ V_a < U_1 < U_2 < \cdots < U_b < V_{a+1} \]
is a series of continuous augmented inductive valuations between \( V_a \) and \( V_{a+1} \).

§ 4. Valuations of \( K(x, y) \)

Now we consider valuations of polynomial rings of two variables. A valuation of a ring \( R \) can be extended uniquely to its quotient field. (M. Thorem 2.1)
We assume that a valuation \( W \) of a field \( K[x, y] \) is given, where \( x \) and \( y \) are algebraically independent with respect to \( K \). Let \( W \) induce a valuation
$V_0$ in $K$ and a valuation $V_{p_0}$ in $K[x]$.

Since $V_{p_0}$ is an extended valuation of $V_0$ to $K[x]$, we can build by Theorem 2.4 a series of continuous $x$-augmented inductive valuations of $K[x]$ such that

$$V_{p_0} = [V_0, V_1 x = \mu_1, V_{20} \phi_2 = \mu_2, \ldots, V_{q0} \phi_q = \mu_q].$$

If we denote $K(x) = K_x$, then $W$ is an extended valuation of $V_{p_0}$ to a ring $K_x[y]$ whose coefficient field $K_x$ has a valuation $V_{p_0}$. Therefore again by Theorem 2.4, we can build a series of continuous $y$-augmented inductive valuations of $K_x[y]$ such that

$$V_{p_0} = [V_{p_0}, V_{p_1} y = \nu_1, V_{p_2} \phi_2 = \nu_2, \ldots, V_{p_q} \phi_q = \nu_q],$$

where $\phi_{j+1}$ is a $y$-key polynomial over $V_{p_j}$ of $K_x[y]$ and

$$V_{p_{j+1}} \phi_{j+1} = \nu_{j+1} > V_{p_j} \phi_{j+1} \quad \text{for} \quad j = 1, 2, \ldots, q-1.$$

$V_{p_0}$ is denoted briefly as $V_{p_0} = [V_{p-1}, V_{p_0} \psi = \nu_q]$. $V_{p_0}$ is a valuation of $K(x, y)$, so it is equal to $W$.

Suppose $W$ induces another valuation $V_{q_0}$ in $K[y]$. Then we can build in the same way a series of continuous $y$-augmented inductive valuations of $K[y]$ such that

$$V_{q_0} = [V_0, V_{q_0} y = \nu_1, V_{q_0} \zeta_2 = \rho_2, \ldots, V_{q_0} \zeta_s = \rho_s].$$

Since $V_{q_0}$ is a valuation of a quotient field $K(y) = K_y$, we can obtain in the same way a series of continuous $x$-augmented inductive valuations of $K_y[x]$ such that

$$V_{s_0} = [V_{q_0}, V_1 x = \mu_1, V_{20} \xi_2 = \sigma_2, \ldots, V_{s_0} \xi_s = \sigma_s],$$

where $\xi_{i+1}$ is an $x$-key polynomial over $V_{s_i}$ in $K_y[x]$ and

$$V_{s_{i+1}} \xi_{i+1} = \sigma_{i+1} > V_{s_i} \xi_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, s-1.$$  

$V_{s_0}$ is denoted briefly as $V_{s_0} = [V_{s-1}, V_{s_0} \xi = \sigma_s]$ and we can also conclude that $V_{s_0}$ is equal to $W$ in the same way.

Thus a valuation $W$ of $K[x, y]$ has two different expressions $V_{p_0} = [V_{p-1}, V_{p_0} \psi(y) = \nu_q]$ and $V_{s_0} = [V_{s-1}, V_{s_0} \xi(x) = \sigma_s]$. Namely $V_{p_0}$ is an expression of $W$ in the case when $W$ is regarded as the last member of a series of continuous $y$-augmented inductive valuations of the ring $K_x[y]$ whose coefficient field $K_x$ has the valuation $V_{p_0}$ which $W$ induces in $K_x$, while $V_{s_0}$ is an expression of $W$ in the case when $W$ is regarded as the last member of a series of continuous $x$-augmented inductive valuations of the ring $K_y[x]$ whose coefficient field $K_y$ has the valuation $V_{q_0}$ which $W$ induces in $K_y$. 


§ 5. **y-augmented valuations**

Now we want to build a \( y \)-augmented valuation of a valuation \( W \) of \( K[x, y] \) which was given in § 4. In this case we must naturally use \( V_{pq} \) as \( W \). First we define a \( y \)-key polynomial \( \phi_{q+1}(y) \) over \( V_{pq} \) in the ring \( K_s[y] \) in the same way when we defined an \( x \)-key polynomial \( \phi_2(x) \) over \( U_1 \) in the ring \( K[x] \) in § 1. But this time \( \phi_{q+1}(y) \) is a polynomial of \( y \) with coefficients in \( K_x \).

Let any polynomial \( h(y) \) of \( y \) with coefficients in \( K_x \) have the following expansion with respect to \( \phi_{q+1}(y) \):

\[
h(y) = \sum_i h_i(y)(\phi_{q+1}(y))^i
\]

where \( h_i(y) \) is a polynomial of \( y \) in \( K_x[y] \) and

\[
\deg_y h_i(y) < \deg_y \phi_{q+1}(y) \quad \text{for} \quad i = 0, 1, 2, \ldots.
\]

Then the function \( V_{p,q+1} \) defined as follows is called a \( y \)-augmented valuation of \( V_{pq} \):

\[
V_{p,q+1} h(y) = \text{Min}_i \left[ V_{pq} h_i(y) + i\nu_{q+1} \right],
\]

where \( \nu_{q+1} = V_{p,q+1} \phi_{q+1}(y) \) is an arbitrary real number greater than \( V_{pq} \phi_{q+1}(y) \).

\( V_{p,q+1} \) is denoted briefly as \( V_{p,q+1} = [V_{pq}, V_{p,q+1} \phi_{q+1}(y) = \nu_{q+1}] \) and this relation is denoted as \( V_{pq} < V_{p,q+1} \).

\( V_{p,q+1} \) is a valuation of \( K_s[y] \), so it induces the same valuation in the subring \( K[x, y] \) of \( K_s[y] \). Then by M. Theorem 5.1, for every polynomial \( f(x, y) \) in \( K[x, y] \)

\[
V_{p,q+1} f(x, y) \geq V_{p,q} f(x, y),
\]

and owing to the definition of the \( y \)-augmented valuation \( V_{p,q+1} \) for every polynomial \( g(x) \) in \( K[x] \)

\[
V_{p,q+1} g(x) = V_{pq} g(x).
\]

Consequently for a polynomial \( 1(x, y) \) in \( K[x, y] \) if

\[
V_{p,q+1} 1(x, y) > V_{pq} 1(x, y)
\]

then \( y \) actually appears in the polynomial \( 1(x, y) \).

Next we define a \( y \)-key polynomial \( \phi_{q+2}(y) \) over \( V_{p,q+1} \) in \( K_s[y] \) and with it we define a \( y \)-augmented valuation \( V_{p,q+2} \) of \( V_{p,q+1} \). Thus we obtain a series of \( y \)-augmented inductive valuations in \( K_s[y] \) as follows;

\[
V_{pq} < V_{p,q+1} < V_{p,q+2} < \cdots.
\]
In this series, when \( b > a \), \( V_{pa} \) is called a \( y \)-augmented valuation of \( V_{pa} \) and \( V_{pa} \) a \( y \)-descended valuation of \( V_{pa} \) and this relation is denoted as \( V_{pa} < V_{pb} \). Therefore when \( V_{pa} < V_{pb} \), \( V_{pa} \) and \( V_{pb} \) induce the same valuation \( V_{pa} \) in \( K_x \) and \( V_{pb} = V_{pa} y \).

And completely in the same way when we established Definition 2.2. and Definition 2.3., we define a continuous \( y \)-augmented valuation in \( K[x, y] \) and the \( y \)-distance between two valuations in \( K[x, y] \).

**Definition 5.1.** If \( W_2 \) is a \( y \)-augmented valuation of \( W_1 \) in \( K[x, y] \) and there exists a polynomial \( f(y) \) in \( K[y] \) such that \( W_2 f(y) > W_1 f(y) \), then \( W_2 \) is called a \( y \)-simply augmented valuation of \( W_1 \) and this relation is denoted as 
\[
W_1 y < W_2.
\]

In the following series of \( y \)-simply augmented inductive valuations of \( K[x, y] \)
\[
W_a y < W_{a+1} y < \ldots < W_b y < \ldots,
\]
\( W_b \) is called a \( y \)-simply augmented valuation of \( W_c \) and \( W_a \) a \( y \)-simply descended valuation of \( W_b \).

**Definition 5.2.** If \( W_2 \) is a \( y \)-augmented valuation of \( W_1 \) in \( K[x, y] \) but is not a \( y \)-simply augmented valuation of \( W_1 \), then \( W_2 \) is called an \( x \)-\( y \)-doubly augmented valuation of \( W_1 \) and \( W_1 \) an \( x \)-\( y \)-doubly descended valuation of \( W_2 \) and is denoted as 
\[
W_1 xy < W_2.
\]

Namely when \( W_2 \) is an \( x \)-\( y \)-doubly augmented valuation of \( W_1 \), if \( W_2 f(x, y) > W_1 f(x, y) \), then both \( x \) and \( y \) actually appear in the polynomial \( f(x, y) \).

§ 6. \( x \)-augmented valuations

Next we want to build an \( x \)-augmented valuation of the valuation \( W \) given in § 4. In this case naturally we must use \( V_{sr} \) as \( W \). Completely in the same way when we built a \( y \)-augmented valuation of \( V_{pq} \) of the ring \( K_x [y] \), we define an \( x \)-key polynomial \( \xi_{s+1}(x) \) with coefficients in \( K_y \) over \( V_{sr} \) and with \( \xi_{s+1}(x) \) we define an \( x \)-augmented valuation \( V_{x:1,r} \) of \( V_{sr} \) in the ring \( K_y[x] \) whose coefficient field \( K_y \) has the valuation \( V_{sr} \). And furthermore we define a series of \( x \)-augmented inductive valuations as follows:
\[
V_{sr} x < V_{s+1,r} x < \ldots < V_{ar} x < \ldots < V_{br} x < \ldots.
\]
We call $V_{b_r}$ in this series an $x$-augmented valuation of $V_{sr}$ and also in the same way as in § 5 we define a continuous $x$-augmented valuation of $V_{sr}$ and the $x$-distance between two valuations.

$V_{s+1,r}$ is an $x$-augmented valuation of $V_{sr}$ in $K[x, y]$, so again by M. Theorem 5.1, for every polynomial $f(x, y)$

$$V_{s+1,r}f(x, y) \leq V_{sr}f(x, y).$$

And owing to the definition of an $x$-augmented valuation for every polynomial $g(y)$ in $K[y]$

$$V_{s+1,r}g(y) = V_{sr}g(y),$$

consequently if

$$V_{s+1,r}1(x, y) > V_{sr}1(x, y)$$

then $x$ actually appears in the polynomial $1(x, y)$.

**Definition 6.1.** If $W_2$ is an $x$-augmented valuation of a valuation $W_1$ in $K[x, y]$ and there exists a polynomial $f(x)$ in $K[x]$ such that $W_2f(x) > W_1f(x)$, then $W_2$ is called an $x$-simply augmented valuation of $W_1$ and is denoted as $W_1 <^x W_2$.

**Definition 6.2.** When $U$ is an $x$-augmented valuation of $V$ or a $y$-augmented valuation of $V$, $U$ is called an augmented valuation of $V$ and is denoted as $V < U$.

**Definition 6.3.** When $U$ is an $x$-simply augmented valuation of $V$ in $K[x, y]$ or a $y$-simply augmented valuation of $V$, $U$ is called a simply augmented valuation of $V$ and is denoted as

$$V <^s U.$$

**Definition 6.4.** In the following series of augmented inductive valuations in $K[x, y]$

$$U_1 < U_2 < \cdots < U_a < \cdots < U_b < \cdots$$

$U_b$ is called an augmented valuation of $U_a$ and $U_a$ an descended valuation of $U_b$.

**Definition 6.5.** In the following series of simply augmented valuations in $K[x, y]$

$$U_1 <^s U_2 <^s \cdots <^s U_a <^s \cdots <^s U_b <^s \cdots$$

$U_b$ is called a simply augmented valuation of $U_a$ and $U_a$ a simply descended valuation of $U_b$.

When $U$ and $V$ are two valuations of a ring $R$, if for every element $r$
in \( R \) \( Ur = Vr \), then we say that \( U \) equals \( V \) and denote as \( U = V \). We say that the valuation \( W \) given in \( \S 4 \) has two expressions

\[ V_{pq} = [V_{p,q-1}, V_{pq}\varphi(y) = \nu_q] \quad \text{and} \quad V_{sr} = [V_{s-1,r}, V_{sr}\xi_r(x) = \sigma_s]. \]

**Theorem 6.6.** No \( y \)-simply augmented valuation of a valuation \( W \) in \( K[x, y] \) equals an \( x \)-augmented valuation of \( W \).

**Proof.** This is evident according to their definitions.

Similarly no \( x \)-simply augmented valuation of a valuation \( W \) in \( K[x, y] \) equals an \( y \)-augmented valuation of \( W \).

**Theorem 6.7.** If \( W_2 \) is an \( x \)-augmented valuation of a valuation \( W_1 \) in \( K[x, y] \) but not an \( x \)-simply augmented valuation of \( W_1 \), then \( W_2 \) is an \( xy \)-doubly augmented valuation of \( W_1 \) in \( K[x, y] \).

**Proof.** \( W_2 \) is an \( x \)-augmented valuation of \( W_1 \) but not an \( x \)-simply augmented valuation of \( W_1 \), so for every polynomial \( f(x) \) in \( K[x] \) \( W_2 f(x) = W_1 f(x) \). Namely \( W_2 \) and \( W_1 \) induce the same valuation \( V_{p0} \) in the field \( K_x \) and \( W_2 y = W_1 y \). So regarding \( W_2 \) as a \( y \)-augmented valuation of \( W_1 \) in the ring \( K_x[y] \), we can build the following series of continuous \( y \)-augmented inductive valuations by Corollary 2.7.;

\[ W_1 = V_{p0} < V_{p,q-1} < \cdots < V_{pq} = W_2. \]

But \( W_2 \) is an \( x \)-augmented valuation of \( W_1 \) in \( K[x, y] \), so for every polynomial \( g(y) \) in \( K[y] \) \( W_2 g(y) = W_1 g(y) \). Therefore no valuation in this series is a \( y \)-simply augmented valuation of its preceding valuation. So \( W_2 \) is an \( xy \)-doubly augmented valuation of \( W_1 \).

**§ 7. Doubly augmented valuations**

**Theorem 7.1.** Let \( U \) be an \( x \)-augmented valuation of a valuation \( Q \) of \( K[x, y] \) and \( W \) also an \( x \)-augmented valuation of \( U \).

Then, for a polynomial \( f(x, y) \),

\[ Uf(x, y) = Q(x, y) \quad \text{implies} \quad Wf(x, y) = Uf(x, y). \]

**Proof.** Since \( U \) is an \( x \)-augmented valuation of \( Q \) in \( K[x, y] \), \( Ua(y) = Qa(y) \) for every polynomial of \( y \).

Also since \( W \) is an \( x \)-augmented one of \( U \),

\[ Wa(y) = Ua(y) = Qa(y). \]

Therefore \( W, U \) and \( Q \) induce the same valuation \( V_{or} \) in the ring \( K[y] \), and
On Valuations of Polynomial Rings of many Variables

\[ Wx = Ux = Qx. \]

Owing to Corollary 2.7 we can build a series of continuous \(x\)-augmented inductive valuations of a ring \(K_y[x]\) from \(Q\) to \(W\) through \(U\):

\[
\begin{align*}
Q &= V_1^x < V_2^x < \cdots < V_{p+1}^x < U^x < V_{q+1}^y < \cdots < V_m^y = W,
V_{pr}^x f(x, y) &= Qf(x, y) = Uf(x, y) = V_{pr}^y f(x, y).
\end{align*}
\]

Hence it follows from M. Theorem 6.5 that

\[
V_{pr}^x f(x, y) = V_{pr}^y f(x, y) = Wf(x, y).
\]

**Theorem 7.2.** If \(U\) is an \(xy\)-doubly augmented valuation of a valuation \(Q\) of \(K[x, y]\) and \(W\) is an augmented valuation of \(U\), then \(W\), \(U\) and \(Q\) induce the same valuations both in \(K[x]\) and in \(K[y]\) and \(W\) is also an \(xy\)-doubly augmented valuation of \(U\).

**Proof.** Since \(U\) is an \(x\)-augmented valuation of \(Q\) and also a \(y\)-augmented one of \(Q\), \(U\) and \(Q\) induce the same valuation \(V_{or}\) in \(K[y]\) and also the same valuation \(V_{p0}\) in \(K[x]\). So, for every polynomial \(a(x)\) of \(x\) and \(b(y)\) of \(y\), we have

\[
Ua(x) = Qa(x) \quad \text{and} \quad Ub(y) = Qb(y).
\]

Assume that \(W\) is an \(x\)-simply augmented valuation of \(U\). Then there exists a polynomial \(l(x)\) of \(x\) such that

\[
Wl(x) > Ul(x).
\]

But \(W\) is an \(x\)-simple augmented valuation of \(U\), so \(W\) is an \(x\)-augmented valuation of \(U\). Hence

\[
Wl(x) > Ul(x) = Ql(x).
\]

This is however impossible owing to Theorem 7.1. Thus \(W\) is not an \(x\)-simply augmented valuation of \(U\). It is evident that Theorem 7.1 holds when we interchange the letter \("x"\) with the letter \("y"\).

So \(W\) is not a \(y\)-simply augmented valuation of \(U\), even if \(W\) is a \(y\)-augmented valuation of \(U\). But \(W\) is an augmented valuation of \(U\), hence \(W\) must be an \(xy\)-doubly augmented valuation of \(U\).

**Theorem 7.3.** The set of all \(xy\)-doubly descended valuations of a valuation of \(K[x, y]\) is partially linearly ordered.

**Proof.** Let \(V\), \(U\), and \(U'\) be \(xy\)-doubly descended valuations of \(W\) and let \(V < U < V\) and \(V < U' < V\).

Then according to Theorem 7.2 they all induce the same valuation \(V_{or}\) in the ring \(K[y]\) and besides
\[ Wx = Ux = U'x = Vx. \]

Therefore they may be considered \( x \)-descended valuations of \( W \) in \( K_y[x] \) whose coefficient field \( K_y \) has a valuation \( V_{\alpha} \). Hence owing to Theorem 2.8 they are partially linearly ordered.

**Theorem 7.4.** If \( U \) is an \( xy \)-doubly augmented valuation of a valuation \( Q \) of \( K[x, y] \) and \( W \) is an augmented valuation of \( U \), then there exists only one series of continuous \( xy \)-doubly augmented inductive valuations of \( K[x, y] \) from \( Q \) to \( W \) through \( U \) and the \( x \)-distance between \( U \) and \( W \) is equal to the \( y \)-distance between \( U \) and \( W \).

**Proof.** Since \( W \), \( U \) and \( Q \) induce the same valuation \( V_{\alpha} \) in \( K_y \) and \( Wx = Ux = Qx \), we can build according to Corollary 2.7 a series of continuous \( x \)-augmented inductive valuations of \( K_y[x] \) from \( Q \) to \( W \) through \( U \):

\[
(1) \quad Q = V_1^x < V_2^x < \ldots < V_i^x = W.
\]

Since \( W \) and \( U \) are \( xy \)-doubly augmented valuations of \( Q \), \( W \), \( U \) and \( Q \) induce the same valuation \( V_{\alpha} \) in the ring \( K[x] \) and \( W \) and \( U \) may be considered as \( y \)-augmented valuations of \( Q \) of \( K_x[y] \) whose coefficient field \( K_x \) has the valuation \( V_{\alpha} \) because \( Wy = Uy = Qy \).

Therefore again by Corollary 2.7 we can build a series of continuous \( y \)-augmented valuations from \( Q \) to \( W \) through \( U \):

\[
(2) \quad Q = V_1'^y < V_2'^y < \ldots < V_i'^y = W.
\]

\( W \) is an \( xy \)-doubly augmented valuation of \( Q \), so \( V_1', V_2', \ldots, V_i' \) are all \( xy \)-doubly augmented valuations of \( Q \) in \( K[x, y] \). But an \( xy \)-doubly augmented valuation of \( Q \) is a \( y \)-augmented valuation of \( Q \). Therefore \( V_1', \ldots, V_i' \) may be considered as a series of \( y \)-augmented inductive valuations of \( Q \) in \( K[x, y] \).

Hence owing to Theorem 2.6 all \( V_1', V_2', \ldots, V_q' \) must appear in the series (2). Similarly all \( V_1''', V_2''', \ldots, V_q''' \) must appear in the series (1).

Therefore both the series (1) and (2) coincide. Since \( U \) appears in the series, the \( x \)-distance between \( U \) and \( W \) is equal to the \( y \)-distance between \( U \) and \( W \).

We shall later show that a set of all simply descended valuations of a valuation of \( K[x, y] \) is not always partially linearly ordered.

**§ 8. Series of augmented inductive valuations in \( K[x, y] \)**

The following theorem can be proved in the same way as M. Theorem 3.1.
Theorem 8.1. If $R$ is an integral domain and has a valuation $V_\infty$, then for any polynomial $f(x) = \sum f_i x^i$ in $R[x]$, the function $V_{10}$ defined as follows is a valuation of the ring $R[x]$;

$$V_{10}(x) = \text{Min} \left[ V_\infty f_i + i \mu_1 \right]$$

where $\mu_1 = V_{10} x$ is an arbitrary real number. The valuation is denoted by

$$V_{10} = [V_\infty, V_{10} x = \mu_1].$$

When a valuation $W$ of $K[x, y]$ is given, in Theorem 8.1 we substitute $W x$ for $V_{10} x = \mu_1$, and we obtain

$$V_{10} = [V_\infty, W x = \mu_1].$$

Generally a polynomial $f(x, y)$ of $x$ and $y$ can be expressed as $\sum f_i(x) y^i$, where $f_i(x)$ are polynomials of $x$. Then the following theorem is an immediate consequence of Theorem 8.1.

Theorem 8.2. If $W$ is a valuation of $K[x, y]$ and $V_{10} = [V_\infty, W x = \mu_1]$ is a valuation defined above, then for any polynomial $f(x, y) = \sum f_i(x) y^i$, where $f_i(x)$ are polynomials of $x$, the function $V_{11}$ defined as follows is a valuation of $K[x, y]$;

$$V_{11}(x, y) = \text{Min} \left[ V_{10} f_i(x) + i \nu_1 \right]$$

where $\nu_1 = V_{11} y = W y$.

Now we want to build a series of augmented inductive valuations of $K[x, y]$ from $V_{11}$ to $W$.

Theorem 8.3. If a valuation $W$ of $K[x, y]$ induces in $K[x]$ a series of continuous $x$-augmented inductive valuations

$$V_{10} < V_{20} < \ldots < V_{p0},$$

then for any polynomial $f(x, y) = \sum f_i(x) y^i$, the functions $V_t$, defined as follows are $x$-simply augmented valuations of $V_{11}$ in $K[x, y]$, where $t$ ranges between 1 and $p$;

$$V_{11}(x, y) = \text{Min} \left[ V_{10} f_i(x) + i \nu_1 \right]$$

where $\nu_1 = W y$, and moreover

(1) $V_{11} < V_{21} < \ldots < V_{p1}$.
gives a series of continuous $x$-simply augmented inductive valuation of $K[x, y]$.

**Proof.** Owing to Theorem 8.1 it is evident that each $V_{t_{1}}$ is an $x$-simply augmented valuation of $V_{11}$ for $t=1, 2, \ldots, p$. Assume that the series (1) is not continuous $x$-simply augmented inductive valuations. Then there exists a valuation $U$ of $K[x, y]$ such that

$$V_{11} < V_{21} < \cdots < V_{n_{1}} < U < V_{n_{1}+1} < \cdots < V_{p_{1}}.$$  

So for suitable polynomials $g(x)$ and $h(x)$

$$V_{n_{1}} g(x) < U g(x) \quad \text{and} \quad U h(x) < V_{n_{1}+1} h(x).$$

In $K[x]$ $V_{n_{1}}$, $U$ and $V_{n_{1}+1}$ induce respectively valuations $V_{n_{1}0}$, $U'$ and $V_{n_{1}+10}$ and $V_{n_{1}} g(x) = V_{n_{10}} g(x)$, so

$$V_{n_{0}} g(x) < U' g(x) \quad \text{and} \quad U' h(x) < V_{n_{1}+10} h(x).$$

Namely $V_{n_{0}} < U' < V_{n_{1}+10}$.

But this is impossible because $V_{n_{1}+10}$ is a continuous $x$-augmented valuation of $V_{n_{0}}$ in $K[x]$. Thus (1) is a series of continuous $x$-simply augmented inductive valuations.

$W$ and $V_{p_{1}}$ induce in $K[x]$ the same valuation $V_{p_{0}}$, so owing to Theorem 2.4 we can build a series of continuous $y$-augmented inductive valuations of $K_{x}[y]$ from $V_{p_{1}}$ to $W$;

$$V_{p_{1}} < V_{p_{1}2} < \cdots < V_{p_{1}q} < \cdots < V_{p_{1}} = W.$$  

In this series, according to Theorem 7.2, there must exist a valuation $V_{pq}$ such that the subseries between $V_{p_{1}}$ and $V_{pq}$ is a series of continuous $y$-simply augmented inductive valuations and the subseries between $V_{pq}$ and $W$ is a series of continuous $xy$-doubly augmented inductive valuations.

**§ 9. Last simply augmented valuation**

When $W$ is a valuation, we define a valuation $V_{11}$ of $K[x, y]$ as in §8. But by interchanging $x$ with $y$ in construction of the series, we obtain another series;

$$V_{11} \rightarrow V_{12} < \cdots < V_{1q} < V_{2q} < \cdots < V_{pq} < V_{n_{1}+1} < \cdots < V_{sp} = W.$$  

Namely, the beginning part of this series between $V_{11}$ and $V_{1q}$ is a series of continuous $y$-simply augmented inductive valuations of $K[x, y]$, the middle part of the series between $V_{1q}$ and $V_{pq}$ is a series of continuous $x$-simply
augmented inductive valuations of \( K[x, y] \) and the last part of the series between \( V_{pq} \) and \( W \) is a series of continuous \( xy \)-doubly augmented inductive valuations.

Thus we can build the two different series of augmented inductive valuations from \( V_{11} \) to \( W \). We shall however show in Part 2 that we can build many other series of continuous augmented inductive valuations which connect \( V_{11} \) and \( W \).

**Theorem 9.1.** Let \( W \) be a valuation of \( K[x, y] \) and \( V_{11} \) a valuation of \( K[x, y] \) defined in \( \S 8 \). Then in every series of continuous augmented inductive valuations which begins at \( V_{11} \) and comes to end at \( W \), there appears a valuation \( U \) such that the beginning part of the series between \( V_{11} \) and \( U \) is a series of continuous simply augmented inductive valuations of \( K[x, y] \) and the last part of the series between \( U \) and \( W \) is a series of continuous \( xy \)-doubly augmented inductive valuations of \( K[x, y] \); and such a valuation \( U \) is determined uniquely by \( W \).

**Definition 9.2.** The valuation \( U \) in Theorem 9.1 is called the last simply augmented valuation of \( W \).

**Proof of Theorem 9.1.** We prove first the uniqueness.

Let the following series be as in our Theorem:

\[
V_{11}^{s} < \cdots < U^{s} < W^{s}.
\]

By connecting the series (1) in Theorem 8.3 with the other series (2) in \( \S 8 \), we obtain the following series:

\[
V_{11}^{s} < V_{21}^{s} < \cdots < V_{p_{1}}^{s} < V_{p_{2}}^{s} < \cdots < V_{p_{q}}^{s} < V_{p_{q+1}}^{s} < \cdots < V_{p_{l}}^{s} = W.
\]

Namely, \( W \) is an \( xy \)-doubly augmented valuation of \( V_{pq} \), so \( W \) and \( V_{pq} \) induce the same valuation \( V_{p_{0}} \) in the ring \( K[x] \) and \( W \) is also an \( xy \)-doubly augmented valuation of \( U \). Hence \( W, U \) and \( V_{pq} \) induce the same valuation \( V_{p_{0}} \) in the field \( K_{x} \). Besides

\[
W_{y} = U_{y} = V_{pq}y.
\]

Therefore according to Theorem 2.6, \( U \) must coincide with one of valuations in the series (4) between \( V_{pq} \) and \( W \). Assume that \( U \neq V_{pq} \). Then the part of the series (4) between \( V_{pq} \) and \( U \) is a series of continuous simply augmented inductive valuations of \( K[x, y] \) due to its construction, while the part of the series (4) between \( V_{pq} \) and \( U \) is simultaneously a series of continuous \( xy \)-doubly augmented inductive valuations of \( K[x, y] \) according to the construction of the other series. This is, however, impossible, so that we have
$U = V_{pq}$.

Therefore the last simply augmented valuation of $W$ is determined uniquely by $W$ and it is independent of choice of the series.

According to Theorem 7.4 there is only one series of continuous $xy$-doubly augmented inductive valuations which connects $W$ with its last simply augmented valuation $V_{pq}$.

The proof of the existence of a series of continuous simply augmented inductive valuations binding $V_{11}$ and $V_{pq}$ and the method to construct the series is not so simple. We shall show them in Part 2 of this paper, together with other topics, such as key-polynomials of simply augmented valuations and those of $xy$-doubly augmented valuations, residue class rings of a valuation of $K[x, y]$ and valuations of $K[x, y, z]$.

§ 10. Some examples of valuations of $K[x, y]$

Let $K$ be the field of all rational numbers with the 3-adic valuation. Namely, for any rational number $k$ expressed as $k = 3^n \frac{b}{c}$, where $b, c$ and $n$ are integers with $(b, c) = (c, 3) = (3, b) = 1$, $V_{00}$ is defined by

$$V_{00} k = n.$$  

The residue class field $F_0$ of $K$ with respect to $V_{00}$ is a Galois field which consists of only three elements 1, 0 and $-1$.

A valuation $V_{10}$ of $K[x]$ is defined as follows, for any polynomial $f(x) = \sum f_i x^i$, $f_i \in K$:

$$V_{10} f(x) = \min_i \{V_{00} f_i\}.$$  

$V_{10}$ is denoted as $V_{10} = [V_{00}, V_{10} x = 0]$.

Exemple 10.1. Let $K[x]$ have the above defined valuation $V_{10}$. Then a valuation $V_{11}$ of $K[x, y]$ is defined as follows, for any polynomial $f(x, y) = \sum f_i(x) y^i = \sum a_{ij} x^i y^j$, $a_{ij} \in K$:

$$V_{11} f(x, y) = \min_i \{V_{10} f_i(x)\} = \min_{i, j} \{V_{00} a_{ij}\}.$$  

$V_{11}$ is denoted as $V_{11} = [V_{10}, V_{11} y = 0]$.

In the ring $K[x][y]$ with the valuation $V_{11}$, $y^2 + 1$ can be a $y$-key polynomial over $V_{11}$.

Since $V_{10} x = 0$, the residue class ring $A_{10}$ of $K[x]$ is $F_0[X]$, where $X = H_{10} x$ is the residue class of $x$; $X$ can be a variable with respect to $F_0$. (M. Theorem 10.2). So a residue class field $A_{11}$ of a field $K_x$ is $F_0(X) = _0 F_x$. Since $V_{11} y = 0$, the residue class ring $A_{11}$ of $K_x[y]$ is a ring $F_x[Y]$, where $Y = H_{11} y$. 


is the residue class of \( y \), and \( Y \) and \( X \) are variables algebraically independent with respect to \( F_0 \).

The class \( H_{11}(y^2+1)=\langle H_{11}y \rangle^2 + H_{11}1 = Y^2 + 1 \). \( Y^2 + 1 \) is an irreducible polynomial in \( F_0[Y] \) and also is irreducible in \( \bar{F}_0[Y] \). Therefore \( y^2+1 \) is equivalence irreducible over \( V_{11} \) in \( K_\ast[y] \) due to M. Lemma 11.2.

Besides, \( y^2+1 \) satisfies all the conditions for a \( y \)-key polynomial over \( V_{11} \) (M. Theorem 9.4). So \( y^2+1 \) is a \( y \)-key polynomial over \( V_{11} \) in \( K_\ast[y] \).

Any polynomial \( f(x, y) \) can be expressed as

\[
f(x, y) = \sum_i f_i(x, y)(y^2+1)^i
\]

where the degree of each \( f_i(x, y) \) with respect to \( y \) is less than that of \( y^2+1 \) for \( i = 0, 1, 2, \ldots \). We describe it briefly as \( \deg_y f_i(x, y) < \deg_y (y^2 + 1) \), \( i = 0, 1, 2, \ldots \). Then the function defined as follows is an augmented valuation of \( V_{11} \) in the ring \( K\times[y] \):

\[
V_{12}f(x, y) = \min \left[ V_{11}f_i(x, y) + i \right],
\]

where \( V_{12}(y^2+1) = 1 > V_{11}(y^2+1) \).

\( V_{12} \) is a \( y \)-simply augmented valuation of \( V_{11} \). \( V_{12} \) induces in the ring \( K[y] \) the following valuation \( V_{02} \);

\[
V_{02} = \left[ V_{00}, \ V_{01}y = 0, \ V_{02}(y^2+1) = 1 \right].
\]

Then it is evident that for any polynomial \( f(x, y) = \sum_i f_i(x) y^i \) in \( K[x, y] \), the function \( V_{12} \) defined as follows is also a valuation of \( K_{y}[x] \) whose coefficient field \( K_{y} \) has the valuation \( V_{02} \) and \( V_{12} \) is equal to \( V_{12} \) in \( K[x, y] \);

\[
V_{12}f(x, y) = \min \left[ V_{02}f_i(y) \right]
\]

where \( V_{12}x = 0 \).

**Example 10.2.** Let \( K[x, y] \) have the same valuation \( V_{11} = [V_{10}, V_{11}y = 0] \) defined as in Example 10.1. Then \( y-x \) is a \( y \)-key polynomial over \( V_{11} \) in the ring \( K_\ast[y] \), because

\[
V_{11}y = 0 = V_{11}x \quad \text{(M. Corollary 13.2)}.
\]

Therefore for any polynomial \( f(x, y) = \sum_i f_i(x)(y-x)^i \) the function \( W_{12} \) defined as follows is a \( y \)-augmented valuation of \( V_{11} \) in \( K[x, y] \);

\[
W_{12}f(x, y) = \min \left[ W_{10}f_i(x) + i \right],
\]

where

\[
W_{12}(y-x) = 1 > V_{11}(y-x).
\]

But in this case \( W_{12} \) is an \( xy \)-doubly augmented valuation of \( V_{11} \) of
$K[x, y]$. The reason is as follows. By the remainder theorem, we have for any polynomial $g(y)$

$$g(y) = (y - x)l(x, y) + g(x).$$

According to the definition of $V_{11}$, $V_{11}g(x) = V_{11}g(y)$ and therefore $W_{12}g(y) = V_{11}g(y)$ by M. Lemma 4.3, so $W_{12}$ is not a $y$-simply augmented valuation of $V_{11}$ and consequently $W_{12}$ is an $xy$-doubly augmented valuation of $V_{11}$.

**Example 10.3.** Let $K[x, y]$ have the same valuation $V_{11}$ as in Example 10.1. Then $y^2+1$ is a $y$-key polynomial over $V_{11}$ in $K_x[y]$. So it can be seen in the same way that $x^2+1$ is an $x$-key polynomial over $V_{11}$ in $K_x[x]$, where $K_y$ has the valuation $V_{01} = [V_{00}, V_{01}y=0]$ and $V_{11} = [V_{01}, V_{11}x=0]$.

But $K[x, y]$ having the valuation $V_{12} = V_{12}$ defined as in Example 10.1, $x^2+1$ can not be an $x$-key polynomial over $V_{12}$ in $K_y[x]$. For, since

$$V_{12}((x^2+1)-(x^2-y^2)) = V_{12}(y^2+1) = 1 > 0 = V_{12}(x^2+1),$$

it follows

$$x^2+1 \sim x^2-y^2 \sim (x+y)(x-y) \text{ in } V_{12}.$$ 

Hence $x^2+1$ is an $x$-key polynomial in $V_{12}$ in $K_y[x]$, then $x+y$ or $x-y$ must be equivalence-divisible by $x^2+1$ in $V_{12}$ in $K_y[x]$. But this is impossible because

$$\text{deg}_x(x^2+1) > \text{deg}_x(x+y) = \text{deg}_x(x-y).$$

**Example 10.4.** Let $V_{20} = [V_{00}, V_{10}x=0, V_{20}(x^2+1) = 1]$ be a valuation of $K[x]$ and let $K[x, y]$ have the valuation $V_{11} = [V_{10}, V_{11}y=0]$.

Then $y-x$ is a $y$-key polynomial over $V_{11}$ in $K_x[y]$ and $W_{12} = [V_{11}, W_{12}(y-x) = 1]$ is an $xy$-doubly augmented valuation of $V_{11}$ of $K_x[y]$. But $K_x[y]$ having the valuation $V_{21} = [V_{20}, V_{21}y=0]$, $y-x$ can be a $y$-key polynomial over $V_{21}$ in $K_x[y]$, while the valuation $V_{22} = [V_{21}, V_{22}(y-x) = 1]$ of $K_x[y]$ whose coefficient field has the valuation $V_{20}$ is not an $xy$-doubly augmented valuation of $V_{21}$, but is a $y$-simply augmented valuation of $V_{21}$.

Any polynomial $f(x)$ can be expressed as $\sum f_i(x)(x^2+1)^i$, where

$$\text{deg}_x f_i(x) < \text{deg}_x(x^2+1) = 2.$$ 

Then $V_{20}$ is defined as follows;

$$V_{20}f(x) = \text{Min}[V_{10}f_i(x)+i],$$

where

$$\text{deg}_x f_i(x) < \text{deg}_x(x^2+1) = 2.$$
where \( V_{20}(x^2 + 1) = 1 > V_{20}(x^2 + 1) \).

And for any polynomial \( f(x, y) = \sum_{i} f_i(x)y^i \), \( V_{21} \) is defined as

\[
V_{21}f(x, y) = \min_i \left[ V_{20}f_i(x) \right]
\]

where \( V_{21}y = 0 \).

So \( V_{21} \) can be considered as a valuation of \( K_z[y] \). Since \( V_{21}y = 0 = V_{21}x \), again by M. Corollary 13. 2 \( y - x \) can be a \( y \)-key polynomial over \( V_{21} \) in \( K_z[y] \). Therefore for any polynomial \( f(x, y) = \sum_{i} f_i(x)(y - x)^i \) the function \( V_{22} \) defined as follows is a \( y \)-augmented valuation of \( V_{21} \) in \( K_z[y] \).

\[
V_{22}f(x, y) = \min_i \left[ V_{20}f_i(x) + i \right]
\]

where \( V_{22}(y - x) = 1 > V_{21}(y - x) \).

But \( V_{22} \) is a \( y \)-simply augmented valuation of \( V_{21} \), because

\[
y^2 + 1 = (y - x)^2 + 2x(y - x) + (x^2 + 1)
\]

\[
V_{22}(y^2 + 1) = \min \left[ V_{22}(y - x)^2, V_{22}2x(y - x), V_{22}(x^2 + 1) \right]
\]

\[
= \min[2, 1, 1] = 1 > 0 = V_{21}(y^2 + 1).
\]

Thus it is shown that the \( y \)-key polynomial over \( V \) which produces \( U \) is not always a polynomial of \( y \), even if \( U \) is a \( y \)-simply augmented valuation of \( V \) in \( K_z[y] \).

**Example 10.5.** Next we show that \( V_{22} \) defined as in Example 10. 4 induces in \( K[y] \) the valuation \( V_{02} = \left[ V_0, V_0y = 0, V_0(y^2 + 1) = 1 \right] \) defined as in Example 10. 1.

\[
V_{22}y = V_{22}[y - x + x] = \min \left[ V_{22}(y - x), V_{22}x \right] = 0 = V_{01}y
\]

\[
y^2 + 1 = (y - x)^2 + 2x(y - x) + x^2 + 1.
\]

Therefore

\[
V_{22}(y^2 + 1) = \min[2, 1, 1] = 1 = V_{02}(y^2 + 1)
\]

and

\[
y^2 + 1 \sim 2x(y - x) + x^2 + 1 \quad \text{in} \ V_{22}.
\]

Now assume that in \( K[y] \) \( V_{22} \) induces a valuation greater than \( V_{02} \), i.e. there exists a polynomial \( f(y) \) such that

\[
V_{22}f(y) > V_{02}f(y).
\]

Among such polynomials \( f(y) \), we take a polynomial \( \phi(y) \) whose degree is
minimum and whose leading coefficient is 1. Then \( \phi(y) \) is a \( y \)-key polynomial over \( V_{02} \) and

\[
V_{22}\phi(y) > V_{02}\phi(y).
\]

(1) Let \( \phi(y) = (y^2 + 1)^m + (ay + b)(y^2 + 1)^{m-1} + \cdots + (cy + d) \).

Then owing to M. Theorem 9.4,

\[
V_{02}\phi(y) = V_{02}(y^2 + 1)^m = m = V_{02}(cy + d) \leq V_{02}(ay + b)(y^2 + 1)^{m-1}.
\]

Namely

\[
m \leq V_{02}(ay + b) + m - 1
\]

\[
1 \leq \min[V_{02}ay, V_{00}b].
\]

Consequently

\[
(2) \quad 1 \leq V_{02}a.
\]

Substituting \( 2x(y-x) + x^2 + 1 \) for \( y^2 + 1 \) in (1), we have

\[
h(x, y) = \left[ 2x(y-x) + x^2 + 1 \right]^m + (ay + b)\left[ 2x(y-x) + x^2 + 1 \right]^{m-1} + \cdots + [cy + d]
\]

and

\[
\phi(y) \sim h(x, y) \quad \text{in } V_{22} \quad (4).
\]

Then the coefficient of \( (y-x)^m \) of \( h(x, y) \) is

\[
(2x)^m + a(2x)^{m-1} = (2x)^{m-1}[2x + a].
\]

\[
V_{22}\left[ (2x)^{m-1}(2x + a) \right] = V_{20}\left[ (2x)^{m-1}(2x + a) \right] = \min[V_{20}2x, V_{00}a] = 0
\]

because of (2).

\[
V_{22}h(x, y) \leq V_{22}(2x)^{m-1}(2x + a)(y-x)^m = m.
\]

But by (4)

\[
V_{22}h(x, y) = V_{22}\phi(y) > V_{02}\phi(y) = m.
\]

This is a contradiction, therefore \( V_{22} \) induces \( V_{02} \) in \( K[y] \).

**Example 10.6.** Let \( V_{11} = [V_{00}, V_{10}x = 0, V_{11}y = 0] \) be the valuation of \( K[x, y] \) defined as in Example 10.1 and \( V_{21} = [V_{11}, V_{21}(x^2 + 1) = 1] \) be a valuation of \( K_y[x] \). Then \( V_{21} \) is an \( x \)-simply augmented valuation of \( V_{11} \) and is equal to another valuation \( V_{21} = [V_{20}, V_{21}y = 0] \) of \( K_x[y] \) whose coefficient field \( K_x \) has the valuation \( V_{20} = [V_{00}, V_{10}x = 0, V_{20}(x^2 + 1) = 1] \). Moreover \( V_{22} = [V_{21}, V_{22}(y-x) = 1] \) defined as in Example 10.4 is a \( y \)-simply augmented valuation of \( V_{21} \). Namely

\[
V_{11} \xrightarrow{\text{x-simply aug.}} V_{21} = V_{21} \xrightarrow{\text{y-simply aug.}} V_{22}.
\]

Next by interchanging \( x \) and \( y \), it follows that \( V_{11} = [V_{11}, V_{12}(y^2 + 1) = 1] \) is
a $y$-simply augmented valuation of $V_{11}$ of $K_x[y]$ and it is equal to another valuation $V_{12} = [V_{02}, V_{12}, x=0]$ of $K_y[x]$ whose coefficient field $K_y$ has the valuation $V_{02} = [V_{00}, V_{01} y=0, V_{02} (y^2 + 1) = 1]$ and moreover $\overline{V_{22}} = [V_{11}, V_{22}, (x-y) = 1]$ is an $x$-simply augmented valuation of $V_{12}$ of $K_y[x]$. Namely

$$V_{11} \quad \text{y-simply aug.} \quad \overline{V_{12}} = V_{12} \quad \text{x-simply aug.} \quad \overline{V_{22}}.$$  

It is evident that $V_{21} \not= V_{12}$ and $V_{21} \not> V_{12}$ and $V_{21} \not< V_{12}$ because

$$V_{12}(x^2 + 1) = \text{Min} [V_{11}, x^2, V_{00} 1] = 0 < 1 = V_{21}(x^2 + 1).$$

$$V_{21}(y^2 + 1) = \text{Min} [V_{11}, y^2, V_{00} 1] = 0 < 1 = V_{12}(y^2 + 1).$$

Therefore next proving that $V_{22} = \overline{V_{22}}$, it follows that this example is that of the case when a set of simply descended valuations of a valuation of $K[x, y]$ is not partially linearly ordered.

We proved that $V_{22}$ induces in $K[y]$ $V_{02} = [V_{00}, V_{01} y=0, V_{02} (y^2 + 1) = 1]$ in Example 10.5. And $\overline{V_{22}}$ also induces $V_{02}$ in $K[y]$ owing to its definition.

$$V_{22}(x-y) = V_{22}(-1)(y-x) = V_{22}(y-x) = \overline{V_{22}}(x-y) = 1.$$  

Let be $f(x, y) = \sum f_i(y)(x-y)^i$ any polynomial of $K[x, y]$, then

$$\overline{V_{22}} f(x, y) = \overline{V_{22}} \left( \sum f_i(y)(x-y)^i \right) = \text{Min} \left[ V_{22} f_i(y) + i \overline{V_{22}}(x-y) \right]$$

$$= \text{Min} \left[ V_{22} f_i(y) + i \overline{V_{22}}(x-y) \right] = \text{Min} \left[ V_{22} f_i(y)(x-y)^i \right]$$

$$\leq V_{22} \left( \sum f_i(y)(x-y)^i \right) = V_{22} f(x, y).$$

Again by interchanging $x$ and $y$, in the same way it follows that $\overline{V_{22}} f(x, y) \geq V_{22} f(x, y)$.

So, consequently $V_{22} = \overline{V_{22}}$.

**Example 10.7.** Let $K[x, y]$ have the valuation $V_{22}$ defined as in Example 10.4, then $y-x - \frac{9}{x^2 + 1}$ is a $y$-key polynomial over $V_{22}$ in $K_x[y]$, because $V_{22}(y-x) = 1 = V_{22} \left( -\frac{9}{x^2 + 1} \right)$. Therefore it follows again from M. Corollary 13.2 that $y-x - \frac{9}{x^2 + 1}$ is a $y$-key polynomial over $V_{22}$ in $K_x[y]$.

Any polynomial $f(x, y)$ can be expressed as an expansion with respect to $\left( y-x - \frac{9}{x^2 + 1} \right)$;

$$f(x, y) = \sum_i a_i(x) \left( y-x - \frac{9}{x^2 + 1} \right)^i.$$


Then the function $W_{23}$ defined as follows is a $y$-augmented valuation of $V_{22}$;

$$W_{23}f(x, y) = \min_i \left[ V_{22} \frac{a_i(x)}{b_i(x)} + 2i \right],$$

where $W_{23} \left( y - x - \frac{9}{x^2 + 1} \right) = 2 > 1 = V_{22} \left( y - x - \frac{9}{x^2 + 1} \right)$. Moreover $W_{23}$ is an $xy$-doubly augmented valuation of $V_{22}$. The reason is as follows. Again by the remainder theorem any polynomial $f(y)$ can be expressed as follows;

$$f(y) = \left( y - x - \frac{9}{x^2 + 1} \right) \frac{1(x, y)}{b(x)} + f\left( x + \frac{9}{x^2 + 1} \right)$$

Let $f(y) = \sum_i f_i y^i$, $f_i \in K$. Then

$$f\left( x + \frac{9}{x^2 + 1} \right) = \sum_i f_i \left( x + \frac{9}{x^2 + 1} \right)^i$$

$$V_{22}f_i \left( x + \frac{9}{x^2 + 1} \right)^i = \min_i \left[ V_{22}f_i + iV_{22} \left( x + \frac{9}{x^2 + 1} \right) \right] = \min_i [V_{00}f_i],$$

because $V_{22}x = 0 < 1 = V_{22} \frac{9}{x^2 + 1}$ and so

$$V_{22} \left( x + \frac{9}{x^2 + 1} \right) = 0.$$

On the other hand,

$$V_{22}f(y) = \min_i [V_{00}f_i + iV_{01}y] = \min_i [V_{00}f_i]$$

$$= V_{22}f \left( x + \frac{9}{x^2 + 1} \right).$$

Hence again by M. Lemma 4.3 it follows

$$W_{23}f(y) = V_{22}f(y).$$

Thus $W_{23}$ is an $xy$-doubly augmented valuation of $V_{22}$. Furthermore,

$$V_{11} \prec V_{21} \prec V_{22} \prec W_{23},$$

where

$$V_{11} \ x\text{-simply aug.}, \ V_{21} \ y\text{-simply aug.}, \ V_{22} \ xy\text{-doubly aug.}, \ W_{23}.$$

So $V_{22}$ is the last simply augmented valuation of $W_{23}$.

**Remark 1.**

Let $V_{10}$ and $V_{20} = [V_{30}, V_{20}(x^2 + 1) = 1]$ be the two valuations of $K[x]$.
defined in Example 10. 4.

\[ V_{10}(x^2 + 1 - (x^2 - 2)) = V_{00}3 = 1 > 0 = V_{10}(x^2 + 1) \]

so \( x^2 + 1 \sim x^2 - 2 \) in \( V_{10} \) and \( \deg_2(x^2 + 1) = \deg_2(x^2 - 2) \).

Then \( x^2 - 2 \) can be an \( x \)-key polynomial over \( V_{10} \) and the function \( V_{20} = [V_{10}, V_{20}(x-2) = 1] \) is also an \( x \)-augmented valuation of \( V_{10} \) and \( V_{20} \) owing to M. Theorem 15. 3.

\[ V_{20}^*(x^2 - 2) = 1 = V_{20}^*3. \]

Then \( x^2 + 1 = (x^2 - 2) + 3 \) can be an \( x \)-key polynomial over \( V_{20}^* \) by M. Corollary 13. 2, and the function \( V_{30}^* \) defined as follows in an \( x \)-augmented valuation of \( V_{20}^* \);

\[ V_{30}^* = [V_{20}^*, V_{30}^*(x^2 + 1) = 2], \]

because \( V_{30}^*(x^2 + 1) > V_{26}^*(x^2 + 1) \).

But \( V_{30}^* \) cannot be an \( x \)-augmented valuation of \( V_{20} \). For, even if we build forcibly a function \( V_{30} \) as follows;

\[ V_{30} = [V_{20}, V_{30}(x^2 + 1) = 2] = [V_{00}, V_{10}.x = 0, V_{20}(x^2 + 1) = 1, V_{30}(x + 1) = 2], \]

it follows that the polynomial \( x^2 + 1 \) is used in \( V_{30} \) twice as an \( x \)-key polynomial. But this breaks the condition (5) for \( x \)-key polynomials in §1.

Namely \( V_{30}^* \) is an \( x \)-augmented valuation of \( V_{20}^* \) and \( V_{20} = V_{20}^* \), but in spite of it, \( V_{30}^* \) is not an \( x \)-augmented valuation of \( V_{20} \). In this case we adopt another method of definition. When valuations \( T_1, T_2, \cdots \) of \( K[x, y] \) are equal to each other, we consider that \( T_1, T_2, \cdots \) induce in \( K[x, y] \) the same valuation \( W \) which is defined abstractly as a function satisfying only the three conditions of a valuation in §1 and we regard each \( T_i \) as an expression of \( W \), namely, a concrete representation of \( W \) and if a valuation \( U \) is an augmented valuation of one \( T_i \), then we call \( U \) an augmented valuation of \( W \). In this paper we adopt this method of definition like \( W \) at the end of §4. Following this method of definition, the above valuation \( V_{20}^* \) is an \( x \)-augmented valuation of the valuation whose expressions are \( V_{20}^* \) and \( V_{20} \).

**Remark 2.**

Let \( U \) and \( V \) be two valuations of \( K[x] \) and \( U \not\equiv V \). If for every polynomial \( f(x) \) in \( K[x] \) \( Uf(x) \geq Vf(x) \), then \( U \) is called to be greater than \( V \).

Owing to M. Theorem 5. 1, an \( x \)-augmented valuation \( W \) of a valuation \( Q \) in \( K[x] \) is greater than \( Q \). But even if \( U \) is a valuation greater than a valuation \( V \) in \( K[x] \), \( U \) is not always an \( x \)-augmented valuation of \( V \).
The following is an example of this case.

Let be \( V_{10} \) and \( V_{20} = [V_{10}, V_{10}(x^2 + 1) = 1] \) the two valuations of \( K[x] \) defined in Example 10.4. \( x^2 + 1 \) is an \( x \)-key polynomial over \( V_{10} \) and \( V_{10}(x^2 + 1) = 0 \). Then the function \( V^* = [V_{10}, V^*(x^2 + 1) = \frac{1}{2}] \) is also an \( x \)-augmented valuation of \( V_{10} \), and according to their definitions it is evident that \( V_{20} \) is greater than \( V^* \).

But \( V_{20} \) cannot be an \( x \)-augmented valuation of \( V^* \). Because, if \( V_{20} \) is an \( x \)-augmented valuation of \( V^* \), then a set \( P_2 \) of all values that \( V_{20} \) takes in \( K[x] \) must include a set \( P^*_2 \) of all values that \( V^* \) takes in \( K[x] \). \( P_2 \) is a set of all integers, so \( P^*_2 \) does not include \( \frac{1}{2} = V^*(x^2 + 1) \). All valuations that are equal to each other have the same set of values. Then \( V_{20} \) cannot be an \( x \)-augmented valuation of \( V^* \), even if we adopt the method of definition in Remark 1.

In this paper when we write as \( "Ug(x) > Vf(x)" \), it means that a value of the polynomial \( g(x) \) by a valuation \( U \) is greater than a value of \( f(x) \) by \( V \). But when we write as \( "W > Q" \), it means that \( W \) is an augmented valuation of \( Q \) and it does not mean that the valuation \( W \) is greater than the valuation \( Q \).

References
