SOME PROPERTIES OF KURAMOCHI BOUNDARIES OF HYPERBOLIC RIEMANN SURFACES

By

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1. Constantinescu and Cornea [1] remarked that Kuramochi boundary points share many properties enjoyed by interior points. Let F and K be mutually disjoint compact sets in a hyperbolic Riemann surface and let μ be a positive measure on K. It is known that the balayaged measure (with respect to the Green potential) of μ onto F is supported by the boundary of F. In this paper, we shall prove that a similar result is also valid for a closed set F on the Kuramochi compactification by considering Kuramochi boundary points like interior points (Theorem 1).

As an application, we shall prove that if the set of all non-minimal Kuramochi boundary points is non-empty, then it is uncountable (Theorem 2). A corresponding result for the Martin boundary was proved by Ikegami [2] and Toda [5].

2. Let R be a hyperbolic Riemann surface. We shall use the same notation as in [1], for instance, $\tilde{g}_b$, $\tilde{p}^{\mu}$, $\tilde{s}_{\tilde{F}}$ f^F, R_N^*, Δ_N etc. For a subset A of R, we denote by ∂A the relative boundary of A in $\overset{\circ}{R}$ and by $\bar{A}$ the closure of A in R_N^*. Let K_0 be a closed disk in R and $R_0^*=R_N^*-K_0$. The Kuramochi boundary Δ_N is decomposed into two mutually disjoint parts, the minimal part Δ_1 and the non-minimal part Δ_0. By a measure μ on R_0^*, we always mean a positive measure μ on R_N^* such that $\mu(K_0)=0$. For a measure μ on R_0^*, we shall denote by $S\mu$ the support of μ. If a measure μ on R_0^* satisfies $\mu(\Delta_0)=0$, then it is called canonical. It is known that if μ is a measure on R_0^*, then there exists a unique canonical measure ν such that $\tilde{p}^{\mu}=\tilde{p}^{\nu}$. For a closed set F in R and a measure μ, we denote by μ_F the canonical measure associated with $\tilde{p}^{\mu}_{\tilde{F}}$. We note that $S\mu_F \subset \bar{F}$.

The following properties are known ([1]).

(A) Let F be a non-polar[1)] closed set in R and f a Dirichlet function[2)]

1) A subset A of R is called *polar* if there exists a positive superharmonic function s on R such that $s(a)=+\infty$ at every point a of A.

2) This is called eine Dirichletsche Funktion in [1].

on R. If G is a component of $R-F$, then $f^F=f^{\partial G}$ on G.

(B) Let F be a closed set in R. If s is a Dirichlet function on R, $s=0$ on K_0 and s is a non-negative full-superharmonic function[3] on R_0, then

$$s_{\tilde{F}}=s^{K_0\cup F} \qquad \text{on } R_0-F.$$

(C) Let b be any point of $R_0\cup\Delta_1$. If F is a closed set in R such that $\bar{F}$ is a neighborhood of b in R_N^*, then $(\tilde{g}_b)_{\tilde{F}}=\tilde{g}_b$.

(D) Let μ be a measure on R_0^*. If F is a closed set in R, then

$$\left(\int\tilde{g}_b d\mu(b)\right)_{\tilde{F}}=\int(\tilde{g}_b)_{\tilde{F}}\, d\mu(b).$$

(E) Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of canonical measures on R_0^*. If μ_n converges vaguely to a measure μ as $n\to\infty$ and each $\tilde{p}^{\mu_n}$ is dominated by a fixed full-superharmonic function, then μ is canonical.

We shall prove

Theorem 1. *Let F be a closed subset of R_0 and let μ be a measure on R_0^* such that $S\mu\cap(\bar{F}\cup K_0)=\emptyset$. Then $S\mu_{F}\subset\bar{F}\cap\overline{R-F}$.*

Proof. (i) First suppose $S\mu$ is compact in R_0. Since $S\mu_F\subset\bar{F}$, it is sufficient to prove that $S\mu_F\subset\overline{R-F}$. Let b_0 be an arbitrary point of $\Delta_N-\overline{R-F}$. Let U be an open neighborhood of b_0 in R_N^* such that $\overline{U\cap R}\cap\overline{R-F}=\emptyset$ and let $G=U\cap R$. We shall prove that $\mu_F(U)=0$. Let D be a relatively compact open set in R such that $\bar{D}\cap(K_0\cup\bar{F})=\emptyset$ and ∂D consists of a finite number of analytic Jordan curves. We set $s=\tilde{p}^{\mu}{}_{\widetilde{\partial D}}$ and $f=s$ on R_0 and $=0$ on K_0. Then we see that f is a bounded continuous Dirichlet function on R. Since $(\tilde{g}_b)_{\widetilde{\partial D}}=\tilde{g}_b$ on $R_0-\bar{D}$ for $b\in D$, it follows from (D) that $s=\tilde{p}^{\mu}{}_{\widetilde{\partial D}}=\tilde{p}^{\mu}$ on $R_0-\bar{D}$. Hence $\tilde{p}^{\mu}_{\tilde{F}}=s_{\tilde{F}}$ and $\tilde{p}^{\mu}{}_{\widetilde{F-G}}=s_{\widetilde{F-G}}$. Since the measure associated with s is supported by ∂D and $\partial D\subset R-G$, by (C) and (D), we obtain that $s_{\widetilde{R-G}}=s$ on R_0. By (B) and (A), we have that $s_{\widetilde{F-G}}=f^{K_0\cup(F-G)}=f^{\partial G}$ on G. Thus $\tilde{p}^{\mu}{}_{\widetilde{F-G}}=s_{\widetilde{F-G}}=f^{\partial G}=s_{\widetilde{R-G}}=s=\tilde{p}^{\mu}$ on G. Similarly, by (A) and (B), we obtain that $\tilde{p}^{\mu}{}_{\widetilde{F-G}}=\tilde{p}^{\mu}_{\tilde{F}}$ on R_0-F. On the other hand, since $\tilde{p}^{\mu}{}_{\widetilde{F-G}}=\tilde{p}^{\mu}_{\tilde{F}}=\tilde{p}^{\mu}$ $q.p.$[4]. on $F-G$, we see that $\tilde{p}^{\mu}{}_{\widetilde{F-G}}=\tilde{p}^{\mu}_{\tilde{F}}$ $q.p.$ on R_0. Hence $\tilde{p}^{\mu}{}_{\widetilde{F-G}}=\tilde{p}^{\mu}_{\tilde{F}}$ on R_0. Therefore it follows from the uniqueness of canonical measure that $\mu_F=\mu_{F-G}$. Hence $S\mu_F\subset\overline{F-G}=\bar{F}-U$, so that $\mu_F(U)=0$.

(ii) Next suppose $S\mu$ is not necessarily compact in R_0. By the as-

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3) This is called $\overline{\text{superharmonic}}$ by Kuramochi ([3]) and "positive vollsuperharmonisch" in [1].

4) We shall say that a property holds $q.p.$ on a set E if it holds on E except for a polar set; cf. footnote 1).

sumption, we can find a closed subset V of R_0 such that $\overline{V}$ is a neighborhood of $S\mu$ in R_N^* and $\overline{V}\cap(K_0\cup\overline{F})=\emptyset$. Let $\{K_n\}_{n=1}^{\infty}$ be an increasing sequence of compact sets in R such that $\bigcup_{n=1}^{\infty}K_n=R$. We write $\mu_{V\cap K_n}=\mu_n$ and $(\mu_n)_F=\nu_n$ for simplicity. Since $\tilde{p}^{\nu_n}\leqq\tilde{p}^{\mu_n}\leqq\tilde{p}^{\mu}$ for each n, by the aid of (E), we can show that ν_n converges vaguely to a canonical measure ν. Since $S\nu_n\subset\overline{F}\cap\overline{R-F}(n=1, 2, \cdots)$ by (i), we see that $S\nu\subset\overline{F}\cap\overline{R-F}$. By Hilfssatz 16.2 in [1], we obtain that $\tilde{p}^{\nu_n}$ increases to $\tilde{p}^{\nu}$ as $n\to\infty$. By Satz 15.3, b) in [1], we see that $\tilde{p}^{\mu_n}$ increases to $\tilde{p}^{\mu}{}_{\tilde{V}}$ as $n\to\infty$. Since $\tilde{p}^{\mu}{}_{\tilde{V}}=\tilde{p}^{\mu}$ by (C) and (D), it follows that $\tilde{p}^{\mu_n}$ increases to $\tilde{p}^{\mu}$ as $n\to\infty$. Furthermore it follows from Satz 15.3, b) in [1] that $\tilde{p}^{\mu_n}_{\tilde{F}}=\tilde{p}^{\nu_n}$ increases to $\tilde{p}^{\mu}_{\tilde{F}}$ as $n\to\infty$. Hence we have that $\tilde{p}^{\mu_F}=\tilde{p}^{\nu}$. Since ν is cananical, $\mu_F=\nu$. Therefore $S\mu_F\subset\overline{F}\cap\overline{R-F}$.

3. By a discussion similar to the proof of Théorème 14 in [4], we shall prove

Lemma. *Let F be a closed subset of R_0 and let $b_0\in\Delta_N-\overline{F}$. Then there exists a measure μ such that $S\mu\subset\overline{F}\cap\overline{R-F}$ and $(\tilde{g}_{b_0})_{\tilde{F}}\leqq\tilde{p}^{\mu}\leqq\tilde{g}_{b_0}$.*

Proof. First we note the following. Let b be an arbitrary point of R_0. Let α_0 be a positive real number such that $\{z;\ \tilde{g}_b(z)\geqq\alpha_0\}$ is a compact set in R_0. For each $\alpha\geqq\alpha_0$, min $(\tilde{g}_b, \alpha)$ is a potential and the associated measure, say λ_α, is supported by $\{z;\ \tilde{g}_b(z)=\alpha\}$. Then $\tilde{p}^{\lambda_\alpha}$ can be continuously extended over R_0^*. We denote by $\tilde{p}^{\lambda_\alpha}=\min(\tilde{g}_b, \alpha)$ the continuous extension again. Let ν be a measure on R_0^*. Since $\int\tilde{p}^{\nu}d\lambda_\alpha=\int\min(\tilde{g}_b, \alpha)d\nu$, by letting $\alpha\to\infty$, we have that $\lim_{\alpha\to\infty}\int\tilde{p}^{\nu}d\lambda_\alpha=\tilde{p}^{\nu}(b)$.

Now we shall prove the lemma. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of points in R_0-F such that $b_n\to b_0$ as $n\to\infty$. We denote by μ_n the canonical measure associated with $(\tilde{g}_{b_n})_{\tilde{F}}$. By Theorem 1, we see that $S\mu_n$ is contained in $\overline{F}\cap\overline{R-F}(n=1, 2, \cdots)$. Since $\mu_n(\overline{F}\cap\overline{R-F})\leqq1(n=1, 2, \cdots)$, we can find a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ of $\{\mu_n\}_{n=1}^{\infty}$ such that μ_{n_k} converges vaguely to a measure μ supported by $\overline{F}\cap\overline{R-F}$ as $k\to\infty$. Since $\int\tilde{p}^{\lambda_\alpha}d\mu=\int\tilde{p}^{\mu}d\lambda_\alpha$ and $\int\tilde{p}^{\lambda_\alpha}d\mu_n=\int\tilde{p}^{\mu_n}d\lambda_\alpha(n=1, 2, \cdots)$, we obtain that $\lim_{k\to\infty}\int\tilde{p}^{\mu_{n_k}}d\lambda_\alpha=\int\tilde{p}^{\mu}d\lambda_\alpha$. Since $(\tilde{g}_{b_0})_{\tilde{F}}\leqq\underline{\lim}_{n\to\infty}(\tilde{g}_{b_n})_{\tilde{F}}\leqq\tilde{}_{b_0}$, we obtain that

$$\int(\tilde{g}_{b_0})_{\tilde{F}}d\lambda_\alpha\leqq\int\underline{\lim}_{k\to\infty}(\tilde{g}_{b_{n_k}})_{\tilde{F}}d\lambda_\alpha\leqq\underline{\lim}_{k\to\infty}\int(\tilde{g}_{b_{n_k}})_{\tilde{F}}d\lambda_\alpha=\underline{\lim}_{k\to\infty}\int\tilde{p}^{\mu_{n_k}}d\lambda_\alpha$$
$$=\int\tilde{p}^{\mu}d\lambda_\alpha\leqq\int\tilde{g}_{b_0}d\lambda_\alpha.$$

By letting $\alpha\to\infty$, we have that $(\tilde{g}_{b_0})_{\tilde{F}}(b)\leqq\tilde{p}^{\mu}(b)\leqq\tilde{g}_{b_0}(b)$.

Theorem 2. *If the set Δ_0 of all non-minimal Kuramochi boundary points is non-empty, then it is uncountable.*

Proof. Let b_0 be an arbitrary point of Δ_0. We set $D(r)=\{b\in R_N^*; d(b, b_0)<r\}$ and $C(r)=\{b\in R_N^*; d(b, b_0)=r\}$ for $r>0$, where d is a metric on R_N^*. Suppose there exists a sequence of positive real numbers $\{r_n\}_{n=1}^{\infty}$ such that $C(r_n)\cap\Delta_0=\emptyset$, $r_n>r_{n+1}(n=1, 2, \cdots)$ and $\lim_{n\to\infty} r_n=0$. For each n, if we apply the Lemma to $F=R-D(r_n)\cap R$ and the above b_0, then we obtain a measure μ_n supported by $\bar{F}\cap\overline{R-F}\subset C(r_n)$ such that

$$(*)\qquad (\tilde{g}_{b_0})_{\tilde{F}}\leqq\tilde{p}^{\mu_n}\leqq\tilde{g}_{b_0}.$$

Since $\mu_n(R_0^*)\leqq 1$ and $S\mu_n\subset C(r_n)\subset D(r_1)\cup C(r_1)$ for each n, we can choose a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ of $\{\mu_n\}_{n=1}^{\infty}$ such that μ_{n_k} converges vaguely to a measure μ supported by $\bigcap_{n=1}^{\infty}\left(D(r_n)\cup C(r_n)\right)=\{b_0\}$ as $k\to\infty$. Since b_0 is non-minimal and μ is not identically equal to zero by (*), μ is not canonical. On the other hand, since each μ_n is canonical, it follows from (E) that μ is canonical. This is a contradiction. Hence there exists an $r_0>0$ such that $C(r)\cap\Delta_0\neq\emptyset$ for all $r(0<r<r_0)$. If $r\neq r'$, then $C(r)\cap C(r')=\emptyset$. Thus Δ_0 is uncountable.

Corollary. *If the set Δ_0 is non-empty, then no point of Δ_0 is isolated in Δ.*

References

[1] C. CONSTANTINESCU and A. CORNEA: *Ideale Ränder Riemannscher Flächen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[2] T. IKEGAMI: *On the non-minimal Martin boundary points*, Nagoya Math. J., Vol. **29** (1967), 287-290.

[3] Z. KURAMOCHI: *Potentials on Riemann surfaces*, J. Fac. Sci. Hokkaido Univ., Vol. **16** (1962), 5-79.

[4] L. NAÏM: *Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel*, Ann. Inst. Fourier, **7** (1957), 183-281.

[5] N. TODA: *A remark on Ikegami's paper "On the non-minimal Martin boundary points"*, Proc. Japan Acad., **43** (1967), 308-309.

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