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<th>Title</th>
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<tbody>
<tr>
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SOME PROPERTIES OF KURAMOCHI BOUNDARIES
OF HYPERBOLIC RIEMANN SURFACES

By

Hiroshi TANAKA

1. Constantinescu and Cornea [1] remarked that Kuramochi boundary points share many properties enjoyed by interior points. Let $F$ and $K$ be mutually disjoint compact sets in a hyperbolic Riemann surface and let $\mu$ be a positive measure on $K$. It is known that the balayaged measure (with respect to the Green potential) of $\mu$ onto $F$ is supported by the boundary of $F$. In this paper, we shall prove that a similar result is also valid for a closed set $F$ on the Kuramochi compactification by considering Kuramochi boundary points like interior points (Theorem 1).

As an application, we shall prove that if the set of all non-minimal Kuramochi boundary points is non-empty, then it is uncountable (Theorem 2). A corresponding result for the Martin boundary was proved by Ikegami [2] and Toda [5].

2. Let $R$ be a hyperbolic Riemann surface. We shall use the same notation as in [1], for instance, $\mathcal{A}_\phi$, $\mathcal{P}_\mu$, $\mathcal{F}_\mu$, $R^*_\mu$, $\mathcal{A}_\mu$ etc. For a subset $A$ of $R$, we denote by $\partial A$ the relative boundary of $A$ in $R$ and by $\bar{A}$ the closure of $A$ in $R^*_\mu$. Let $K_0$ be a closed disk in $R$ and $R_0^* = R^*_\mu - K_0$. The Kuramochi boundary $\mathcal{A}_\mu$ is decomposed into two mutually disjoint parts, the minimal part $\mathcal{A}_\mu$ and the non-minimal part $\mathcal{A}_0$. By a measure $\mu$ on $R^*_\mu$, we always mean a positive measure $\mu$ on $R^*_\mu$ such that $\mu(K_0) = 0$. For a measure $\mu$ on $R^*_\mu$, we shall denote by $S\mu$ the support of $\mu$. If a measure $\mu$ on $R^*_\mu$ satisfies $\mu(\mathcal{A}_0) = 0$, then it is called canonical. It is known that if $\mu$ is a measure on $R^*_\mu$, then there exists a unique canonical measure $\nu$ such that $\mathcal{P}_\nu = \mathcal{P}_\mu$. For a closed set $F$ in $R$ and a measure $\mu$, we denote by $\mu_F$ the canonical measure associated with $\mu_F$. We note that $S\mu_F \subseteq F$.

The following properties are known ([1]).

(A) Let $F$ be a non-polar\(^1\) closed set in $R$ and $f$ a Dirichlet function\(^2\)

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\(^1\) A subset $A$ of $R$ is called polar if there exists a positive superharmonic function $s$ on $R$ such that $s(a) = +\infty$ at every point $a$ of $A$.

\(^2\) This is called eine Dirichletsche Funktion in [1].
on $R$. If $G$ is a component of $R-F$, then $f^F=f^{aG}$ on $G$.

(B) Let $F$ be a closed set in $R$. If $s$ is a Dirichlet function on $R$, $s=0$ on $K_0$ and $s$ is a non-negative full-superharmonic function\(^3\) on $R_0$, then

$$s_F = s_{R \cup F}$$
on $R_0-F$.

(C) Let $b$ be any point of $R_0 \cup A$. If $F$ is a closed set in $R$ such that $\bar{F}$ is a neighborhood of $b$ in $R^*_y$, then $(\bar{\sigma}_b)_F = \bar{\sigma}_b$.

(D) Let $\mu$ be a measure on $R_0^*$. If $F$ is a closed set in $R$, then

$$\left( \int \bar{\sigma}_b d\mu(b) \right)_F = \int \bar{\sigma}_b d\mu(b).$$

(E) Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of canonical measures on $R_0^*$. If $\mu_n$ converges vaguely to a measure $\mu$ as $n \to \infty$ and each $\bar{\mu}_n$ is dominated by a fixed full-superharmonic function, then $\mu$ is canonical.

We shall prove

**Theorem 1.** Let $F$ be a closed subset of $R_0$ and let $\mu$ be a measure on $R_0^*$ such that $S\mu \cap (\bar{F} \cup K_0) = \emptyset$. Then $S\mu_F \subset R \cap R-F$.

**Proof.** (i) First suppose $S\mu$ is compact in $R_0$. Since $S\mu_F \subset \bar{F}$, it is sufficient to prove that $S\mu_F \subset R-F$. Let $b_0$ be an arbitrary point of $A_0-R-F$. Let $U$ be an open neighborhood of $b_0$ in $R^*_y$ such that $U \cap \bar{R} \cap R-F = \emptyset$ and let $G = U \cap R$. We shall prove that $\mu_s(U) = 0$. Let $D$ be a relatively compact open set in $R$ such that $D \cap (K_0 \cup \bar{F}) = \emptyset$ and $\partial D$ consists of a finite number of analytic Jordan curves. We set $s = \bar{\mu}_s$ and $f = s$ on $R_0$ and $0$ on $K_0$. Then we see that $f$ is a bounded continuous Dirichlet function on $R$. Since $(\bar{\sigma}_b)_F = \bar{\sigma}_b$ on $R_0-D$ for $b \in D$, it follows from (D) that $s = \bar{\mu}_s = \bar{\mu}_s$ on $R_0-D$. Hence $\bar{\mu}_s = \bar{s}_s$ and $\bar{\mu}^{\bar{s}} = s_{\bar{s}}$. Since the measure associated with $s$ is supported by $\partial D$ and $\partial D \subset R-G$, by (C) and (D), we obtain that $s_{\bar{s}} = s$ on $R_0$. By (B) and (A), we have that $s_{\bar{s}} = f_{\bar{s}} \cap (F-G) = f^{aG}$ on $G$. Thus $\bar{\mu}_s = s_{\bar{s}} = f^{aG} = \tilde{\mu}_s$ on $G$. Similarly, by (A) and (B), we obtain that $\bar{\mu}_s = \tilde{\mu}_s$ on $R_0$. On the other hand, since $\bar{\mu}_s = \bar{\mu}_s = \tilde{\mu}_s = \tilde{\mu}_s$, on $F-G$, we see that $\bar{\mu}_s = \tilde{\mu}_s$ on $R_0$. Therefore it follows from the uniqueness of canonical measure that $\mu_{\bar{s}} = \mu_{\tilde{\mu}}$. Hence $S\mu_{\bar{s}} \subset F-G = F-U$, so that $\mu_{\bar{s}}(U) = 0$.

(ii) Next suppose $S\mu$ is not necessarily compact in $R_0$. By the as-

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\(^3\) This is called _superharmonic_ by Kuramochi ([3]) and "positive vollsuperharmonisch" in [1].

\(^4\) We shall say that a property holds _q.p._ on a set $E$ if it holds on $E$ except for a polar set; cf. footnote 1).
Some Properties of Kuranouchi Boundaries of Hyperbolic Riemann Surfaces

Some Properties of Kuranouchi Boundaries of Hyperbolic Riemann Surfaces

By a discussion similar to the proof of Théorème 14 in [4], we shall prove

**Lemma.** Let $F$ be a closed subset of $R_0$ and let $b_0 \in A_F \subset R$. Then there exists a measure $\mu$ such that $S\mu \subset \overline{F} \cap R - F$ and $(\bar{\sigma}_{b_0})_{\overline{F}} \leq \bar{\sigma}^* \leq \sigma_{b_0}$.

**Proof:** First we note the following. Let $b$ be an arbitrary point of $R_0$. Let $\sigma_0$ be a positive real number such that $\{z; \bar{\sigma}_b(z) \leq \sigma_0 \}$ is a compact set in $R$. For each $\sigma_0 \geq 1$, we define $\bar{\sigma}_b(\sigma) = \sigma$, say, $\mu_0$ is a potential and the associated measure, say $\nu$, is supported by $\{z; \bar{\sigma}_b(\zeta) = \sigma \}$. Then $\bar{\sigma}_b$ can be continuously extended over $R_0^*$. We denote by $\bar{\sigma}_b = \min(\bar{\sigma}_b, \sigma)$ the continuous extension again. Let $\nu$ be a measure on $R_0^*$. Since $\int \bar{\sigma}_b^* d\nu = \int \min(\bar{\sigma}_b, \sigma) d\nu$, by letting $\sigma \to \infty$, we have that $\lim_{\sigma \to \infty} \int \bar{\sigma}_b^* d\nu = \bar{\sigma}_b(b)$.

Now we shall prove the lemma. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of points in $R_0 - F$ such that $b_n \to b_0$ as $n \to \infty$. We denote by $\mu_n$ the canonical measure associated with $(\bar{\sigma}_{b_n})_{\overline{F}}$. By Theorem 1, we see that $S\mu_n$ is contained in $\overline{F} \cap R - F$ as $n \to \infty$. Since $\mu_n(\overline{F} \cap R - F) \leq 1 (n=1, 2, \ldots)$, we can find a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ of $\{\mu_n\}_{n=1}^{\infty}$ such that $\mu_{n_k}$ converges vaguely to a measure $\mu$ supported by $\overline{F} \cap R - F$ as $k \to \infty$. Since $\int \bar{\sigma}_b^* d\mu = \int \bar{\sigma}_b d\lambda_b$ and $\int \bar{\sigma}_b^* d\mu_n = \int \bar{\sigma}_b^* d\lambda_b (n=1, 2, \ldots)$, we obtain that $\lim_{n \to \infty} \int \bar{\sigma}_b^* d\lambda_b = \int \bar{\sigma}_b^* d\lambda_b$. Since $(\bar{\sigma}_{b_0})_{\overline{F}} \leq \lim_{n \to \infty} (\bar{\sigma}_{b_n})_{\overline{F}} \leq \sigma_{b_n}$, we obtain that

$$\int (\bar{\sigma}_{b_n})_{\overline{F}} d\lambda_b \leq \lim_{k \to \infty} (\bar{\sigma}_{b_n})_{\overline{F}} d\lambda_b \leq \lim_{k \to \infty} (\bar{\sigma}_{b_n})_{\overline{F}} d\lambda_b = \lim_{k \to \infty} \int (\bar{\sigma}_{b_n})_{\overline{F}} d\lambda_b = \int (\bar{\sigma}_{b_n})_{\overline{F}} d\lambda_b \leq \int (\bar{\sigma}_{b_n})_{\overline{F}} d\lambda_b.$$
Theorem 2. If the set $\Delta_0$ of all non-minimal Kuramochi boundary points is non-empty, then it is uncountable.

Proof. Let $b_0$ be an arbitrary point of $\Delta_0$. We set $D(r)=\{b \in \mathbb{R}^*_+; \, d(b, b_0)<r\}$ and $C(r)=\{b \in \mathbb{R}^*_+; \, d(b, b_0)=r\}$ for $r>0$, where $d$ is a metric on $\mathbb{R}^*_+$. Suppose there exists a sequence of positive real numbers $\{r_n\}_{n=1}^{\infty}$ such that $C(r_n) \cap \Delta_0 = \emptyset$, $r_n > r_{n+1}$ ($n=1, 2, \ldots$) and $\lim_{n \to \infty} r_n = 0$. For each $n$, if we apply the Lemma to $F=R-D(r_n) \cap R$ and the above $b_0$, then we obtain a measure $\mu_n$ supported by $F \cap R-\overline{F} \subset C(r_n)$ such that

\[
(\ast) \quad (\widehat{\mu}_n)_F \preceq \overline{\mu}_n \preceq \widehat{\mu}_n.
\]

Since $\mu_n(R^*_+ \leq 1$ and $S\mu_n \subset C(r_n) \subset D(r_1) \cup C(r_1)$ for each $n$, we can choose a subsequence $\{\mu_n\}_{n=1}^{\infty}$ of $\{\mu_n\}_{n=1}^{\infty}$ such that $\mu_{n_k}$ converges vaguely to a measure $\mu$ supported by $\bigcap_{n=1}^{\infty} (D(r_n) \cup C(r_n))=\{b_0\}$ as $k \to \infty$. Since $b_0$ is non-minimal and $\mu$ is not identically equal to zero by (\ast), $\mu$ is not canonical. On the other hand, since each $\mu_n$ is canonical, it follows from (E) that $\mu$ is canonical. This is a contradiction. Hence there exists an $r_0>0$ such that $C(r) \cap \Delta_0 = \emptyset$ for all $r(0<r<r_0)$. If $r \neq r'$, then $C(r) \cap C(r') = \emptyset$. Thus $\Delta_0$ is uncountable.

Corollary. If the set $\Delta_0$ is non-empty, then no point of $\Delta_0$ is isolated in $\Delta$.

References


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