ON GALOIS EXTENSIONS AND CROSSED PRODUCTS

Dedicated to Professor K. Asano for his 60th birthday

By

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Introduction.

In this paper, we study Frobenius extensions (in the sense of Kasch [14]), in particular, Galois extensions ([1] [5] [6]), and crossed products ([13]). In Section 1, we give a relation between invertible submodules and ring automorphisms (Th. 1. 3). In Section 2, we prove a fundamental relation between generalized crossed products and Galois extensions (Th. 2. 12). This is our main theorem. In Section 3, we define an equivalence relation concerning the ring homomorphisms, and prove that "Galois extension" and "crossed product" are completely determined by an equivalence class (Th. 3. 4 and Th. 3. 5). In Section 4, we prove the splitting property of crossed products, which includes Nobusawa [22]. In Section 5, we see that a particular generalized crossed product is a symmetric extension (which is defined in § 2). Further we derive directly Williamson and Silver's results [28], [25] concerning hereditary orders, in our situation. Finally we correct the errors in our previous paper "Galois extensions and crossed products, J. Fac. Sci. Hokkaido Univ., Ser. I, 20 (1968)".

Throughout this paper, all rings have identities, and modules are unital. A subring of a ring contains the identity. A ring homomorphism means a ring homomorphism such that the image of 1 is 1.

§ 1. Invertible submodules and automorphisms.

Let ${}_AM$, ${}_AN$ be left A-modules. By N^n we denote the direct sum of n copies of ${}_AN$. If ${}_AM$ is isomorphic to a direct summand of ${}_AN^n$ for some n, then we write ${}_AM|{}_AN$. If ${}_AM|{}_AN$ and ${}_AN|{}_AM$, then we write ${}_AM\sim{}_AN$ (*similar*). By $\mathrm{Hom}_r({}_AM, {}_AN)$ (resp. $\mathrm{Hom}_l({}_AM, {}_AN)$) we denote the module of A-homomorphisms from ${}_AM$ to ${}_AN$ acting on the right (resp. left) side. We denote $\mathrm{Hom}_r({}_AM, {}_AM)$ (resp. $\mathrm{Hom}_l({}_AM, {}_AM)$) by $\mathrm{End}_r({}_AM)$ (resp. $\mathrm{End}_l({}_AM)$). Let f be a mapping from a set S to a set T. For any s in S, the image of s by f is written as $f(s)={}^fs=(s)f=s^f$, and f is written as

$f=(s\to f(s))$ $(s\in S)$, etc. To be easily seen, ${}_AM|{}_AN$ if and only if there are A-homomorphisms $f_1, \cdots, f_n$ in $\mathrm{Hom}_r({}_AM, {}_AN)$ and $g_1, \cdots, g_n$ in $\mathrm{Hom}_r({}_AN, {}_AM)$ such that $\sum_i f_i g_i = id_M$, or equivalently, $\mathrm{Hom}_r({}_AM, {}_AN)\cdot \mathrm{Hom}_r({}_AN, {}_AM)=\mathrm{End}_r({}_AM)$.

Then the following are well known:

S. 1. If ${}_AM|{}_AA$ (i.e. ${}_AM$ is finitely generated and projective) then $A^*_{A^*}|M_{A^*}$, where $A^*=\mathrm{End}_r({}_AM)$.

S. 2. If ${}_AA|{}_AN$ (i.e. ${}_AN$ is a generator of the category of left A-modules) then $N_{A^+}|A^+_{A^+}$ and $\mathrm{End}_l(N_{A^+})\simeq A$, where $A^+=\mathrm{End}_r({}_AN)$.

To be easily seen, the following holds. (Cf. Hirata [9], [10].)

S. 3. If ${}_AM|{}_AN$ then $\mathrm{Hom}_r({}_AM, {}_AN)_{A^+}|A^+_{A^+}$, ${}_{A^+}\mathrm{Hom}({}_AN, {}_AM)|_{A^+}A^+$, $\mathrm{End}_l(\mathrm{Hom}_r({}_AM, {}_AN)_{A^+})=A^*$, and $\mathrm{End}_r({}_{A^+}\mathrm{Hom}_r({}_AN, {}_AM))=A^*$.

The first half is a direct consequence of standard properties of Hom and $\oplus$. To see the latter half, let $f_1, \cdots, f_n$ and $g_1, \cdots, g_n$ be A-homomorphisms in $\mathrm{Hom}_r({}_AM, {}_AN)$ and $\mathrm{Hom}_r({}_AN, {}_AM)$ respectively such that $\sum_i f_i g_i = 1_M$. And, let φ be in $\mathrm{End}_l(\mathrm{Hom}_r({}_AM, {}_AN)_{A^+})$. Then, for any f in $\mathrm{Hom}_r({}_AM, {}_AN)$, $\varphi(f)=\varphi(\sum_i f_i g_i f)=\sum_i \varphi(f_i)g_i f$, and $\varphi(f_i)g_i$ is in A^*. Further, $\mathrm{Hom}_r({}_AM, {}_AN)\cdot\mathrm{Hom}_r({}_AN, {}_AM)=A^*$ implies that ${}_{A^*}\mathrm{Hom}({}_AM, {}_AN)$ is faithful. Similarly we know that $A^*=\mathrm{End}_r({}_{A^+}\mathrm{Hom}({}_AN, {}_AM))$.

Let ${}_AM_{A'}$ be a left A, right A'-module. If ${}_AM\sim{}_AA$ and $\mathrm{End}_r({}_AM)\simeq A'$, then we call ${}_AM_{A'}$ a *Morita module.* Note that this definition is right-left symmetric. By a ring extension A/B, we mean a ring homomorphism from B to A such that the image of 1 is 1. Then every A-module may be considered as a B-module by f. Let $f: B\to A$ and $f^*: A^*\to B^*$ be ring homomorphisms, and let M be a left A, right B^*-module. If both ${}_AM_{A^*}$ and ${}_BM_{B^*}$ are Morita modules, we say that ${}_{A/B}M_{B^*/A^*}$ is a *Morita module.* Note that, in this case, f and f^* are injections, ${}_BA\sim{}_BB$, and $B^*_{A^*}\sim A^*_{A^*}$.

Let A/B be a ring extension. By $V_A(B)$ we denote $\{x\in A | bx=xb$ for all $b\in B\}$, and call it the *centralizer* of B in A/B.

Let A be a ring, and R a subring of A. Let X_R be an R-submodule of A. If there is an R-submodule ${}_RY$ of A such that $1\in XY$ and $YX\subseteq R$, then X_R is said to be *right invertible* in A (cf. Maranda [15]). (Symmetrically we define left invertible submodules.) Evidently $S=XY$ is a subring of A. X and Y are a left S-submodule and a right S-submodule of A, respectively. Further, $S=\{x\in A | xX\subseteq X\}=\{x\in A | Yx\subseteq Y\}$, $Y=\{x\in A | xX\subseteq R\}$, and $X=\{x\in A | Yx\subseteq R\}$. Thus we call ${}_RY$ the *right inverse* of X_R in A. In this case we say briefly that ${}_SX_R$ is right invertible in A. Since $S=XY\ni 1$, 1 is written as $1=\sum_i v_i w_i (v_i\in X, w_i\in Y)$. Then $x=\sum_i v_i(w_i x)$ for all x in X. As $w_i x\in R$, this means that $X_R|R_R$ (i.e. X_R is finitely generated and projective,) and evidently $l_A(X)=0$, where $l_A(X)=\{x\in A|xX=0\}$. The canonical homomorphism $X\otimes_R A\to XA=A$ is an isomorphism with the inverse mapping $(x\to\sum_i v_i\otimes w_i x)$ $(x\in A)$. Since X_R is projective, we have $XW=X\otimes_R W$ for any R-submodule W of A, in particular, $XY=X\otimes_R Y$. Let f be any R-homomorphism from X_R to A_R. Then $A=X\otimes_R A\xrightarrow{f\otimes 1}A\otimes_R A\to A$ is a right A-homomorphism, where $A\otimes A\to A$ is the contraction mapping. Therefore, for some b in A, $bx=f(x)$ for all x in X. From this fact, it follows that $S\simeq\mathrm{End}_l(X_R)$ and $S\simeq\mathrm{End}_r({}_RY)$ as rings, canonically.

Next we assume that $X_R|R_R$, $l_A(X)=0$, and every R-homomorphism from X_R to R_R is given by a left multiplication by an element of A. Then, as is easily seen, X_R is a right invertible submodule with the right inverse $Y=\{x\in A|xX\subseteq R\}$. Thus we obtain the following

Proposition 1.1. *Let A be a ring, R a subring of A, and X an R-submodule of A. Then the following are equivalent:*

(i) *There is an R-submodule ${}_RY$ of A such that $1\in XY$ and $YX\subseteq R$ (i.e. X_R is right invertible in A).*

(ii) *$X_R|R_R$ and $A=XA=X\otimes_R A$.*

(iii) *$X_R|R_R$, $l_A(X)=0$, and every R-homomorphism from X_R to R_R is given by a left multiplication by an element of A.*

In this case, if we set $S=XY$, then X is a left S-submodule, and Y is a right S-submodule. $S=\{x\in A|xX\subseteq X\}\simeq\mathrm{End}_l(X_R)$, and $S=\{x\in A|Yx\subseteq Y\}\simeq\mathrm{End}_r({}_RY)$ as rings, canonically. Every R-homomorphism from X_R into A_R is given by a left multiplication by an element of A. $X=\{x\in A|Yx\subseteq R\}$, and $Y=\{y\in A|yX\subseteq R\}$.

Let R, S be subrings of A, and ${}_SX_R$ a submodule of A. If there is a submodule ${}_RY_S$ of A such that $XY=S$ and $YX=R$, we call ${}_SX_R$ an *invertible submodule* of A. Evidently this is equivalent to that ${}_SX_R$ is right and left invertible in A. Y is denoted by ${}_SX_R^{-1}$, and is called the *inverse* of ${}_SX_R$.

Next we consider a special class of ring homomorphisms. Let S, A be rings, and let σ, τ be ring homomorphisms from S to A. Let ${}_AAu_\sigma$ be a free A-module which is isomorphic to ${}_AA$ by the mapping $a\to au_\sigma$ $(a\in A)$. If we define $u_\sigma s=\sigma(s)u_\sigma$ $(s\in S)$ then we obtain a left A, right S-module. A submodule ${}_\sigma J_\tau$ of A is defined by ${}_\sigma J_\tau=\{x\in A|\sigma(s)a=a\cdot\tau(s)$ for all s in $S\}$. This is a left $\mathrm{V}_A(\sigma(S))$, right $\mathrm{V}_A(\tau(S))$-submodule of A, where $\mathrm{V}_A(\sigma(S))$ means the centralizer of $\sigma(S)$ in A.

The proof of the following lemma may be omitted.

Lemma 1.2. ${}_\sigma J_\tau\simeq\mathrm{Hom}_r({}_AAu_{\sigma S}, {}_AAu_{\tau S})$ *by the correspondence* $a\mapsto(xu_\sigma\to$

xau_τ) ($x \in A$). *In particular,* $\mathrm{V}_A(\sigma(S)) = {}_\sigma J_\sigma \simeq \mathrm{End}_r({}_A A u_{\sigma S})$ *as rings.* ${}_\sigma J_\tau$ *is a left* $\mathrm{V}_A(\sigma(S))$, *right* $\mathrm{V}_A(\tau(S))$*-module, and* ${}_\sigma J_\tau \cdot {}_\tau J_\sigma$ *is an ideal of* $\mathrm{V}_A(\sigma(S))$.

Corollary. ${}_A A u_{\sigma S} | {}_A A u_{\tau S}$ *if and only if* ${}_\sigma J_\tau \cdot {}_\tau J_\sigma = \mathrm{V}_A(\sigma(S))$. *In this case,* ${}_{V_A(\sigma(S))\sigma} J_{\tau V_A(\tau(S))}$ *is right invertible in* A, *and* ${}_\tau J_\sigma$ *is its right inverse.*

Now, let A be a ring, and R, S subrings of A. Let ${}_S X_R$ be a right invertible submodule of A, and ${}_R Y$ its right inverse. Then, $XY = X \otimes_R Y = S \ni 1 = \sum_i u_i \otimes_i v_i$ for some $u_i \in X$, $v_i \in Y$. For any a in $\mathrm{V}_A(R)$, the mapping $x \otimes y \mapsto xay$ from $X \otimes_R Y$ to A is well defined. Since $s = \sum_i su_i \otimes v_i = \sum_i u_i \otimes v_i s$ for all s in S, we have $\sum_i su_i a v_i = \sum_i u_i a v_i s$ for all a in $\mathrm{V}_A(R)$. Put $\sigma_X(a) = \sum_i u_i a v_i$. Then, as is easily seen, σ_X is a ring homomorphism from $\mathrm{V}_A(R)$ to $\mathrm{V}_A(S)$. In the sequel, for any subring R of A, we denote $\mathrm{V}_A(R)$ by R^*. Since $XY \ni 1$, we have $l_l(X) = 0$, so that the ring homomorphism σ_X is characterized as a mapping from R^* to S^* such that $\sigma_X(a)u = ua$ for any $a \in R^*$ and any $u \in X$. Similarly, $v \cdot \sigma_X(a) = av$ for any $a \in R^*$ and any $v \in Y$. Therefore $X \subseteq {}_{\sigma_X} J_1$, and $Y \subseteq {}_1 J_{\sigma_X}$, and hence ${}_{\sigma_X} J_1$ is right invertible in A as a left $\sigma_X(R^*)$, right R^{**}-module. As $R^{***} = R^*$, the ring homomorphism induced by ${}_{\sigma_X} J_1$ is the same with σ_X. By Cor. to Lemma 1. 2, we know that ${}_A A u_{\sigma_X R^*} | {}_A A_{R^*}$. Assume that $R^{**} = R$. Then, since ${}_1 J_{\sigma_X} \cdot {}_{\sigma_X} J_1 \subseteq R^{**} = R$, we have $Y \supseteq ({}_1 J_{\sigma_X} \cdot {}_{\sigma_X} J_1) Y \supseteq {}_1 J_{\sigma_X} \cdot XY \supseteq {}_1 J_{\sigma_X}$, and so ${}_R Y = {}_R {}_1 J_{\sigma_X}$. Similarly, $X_R = {}_{\sigma_X} J_{1_R}$, and hence $S = \mathrm{End}_l(X_R) = \mathrm{End}_l({}_{\sigma_X} J_{1_R}) = (\sigma_X(R^*))^*$. (In general, it is easily seen that $XR^{**}Y \subseteq (\sigma_X(R^*))^*$ and $Y \cdot (\sigma_X(R^*))^* \subseteq R^{**}$. From these we have $XR^{**}Y = (\sigma_X(R^*))^*$, and so ${}_{(\sigma_X(R^*))^*} XR^{**}_{R^{**}}$ is right invertible. Hence the argument in the case $R^{**} = R$ implies $XR^{**} = {}_{\sigma_X} J_1$, because $\sigma_X = \sigma_{XR^{**}}$. Thus we have also $R^{**}Y = {}_1 J_{\sigma_X}$.) Conversely, we take a subring R of a ring A and a ring homomorphism σ from R to A such that ${}_A A u_{\sigma_R} | {}_A A_R$. Then, by Lemma 1. 2, ${}_{(\sigma(R))^*\sigma} J_{1_{R^*}}$ is right invertible in A, where $R^* = \mathrm{V}_A(R)$, and the ring homomorphism induced by ${}_\sigma J_{1_{R^*}}$ is evidently an extension of σ to R^{**}.

Let ${}_S X_R$ be right invertible in A, and ${}_R Y_S$ its right inverse, and let $X'_{S'}$ be right inversible, and ${}_{S'} Y'$ its right inverse. Assume that $S' \subseteq S$. Then $X'X \cdot YY' = X'SY' \ni 1$, and $YY' \cdot X'X \subseteq YS'X \subseteq YX \subseteq R$. Hence ${}_{X'SY'} X'X_R$ is right invertible, and YY' is its right inverse. Therefore, if $S' = S$, then ${}_{X'Y'} X'X_R$ is right invertible. Hence, if ${}_T X'_S$ and ${}_S X_R$ are invertible, then so is ${}_T X'X_R$, and $(X'X)^{-1} = X^{-1}X'^{-1}$. To be easily seen, $\sigma_{X'X} = \sigma_{X'}\sigma_X$, and $\sigma_R = id_{R^*}$. Therefore, if ${}_S X_R$ is invertible then σ_X: $R^* \simeq S^*$ (isomorphism), and $(\sigma_X)^{-1} = \sigma_{X^{-1}}$. Since ${}_A A u_{\sigma_{X^{-1}} S^*} | {}_A A_{S^*}$ implies ${}_A A_{R^*} \simeq {}_A A u_{\sigma_{X^{-1}}} \otimes_{S^*} S^* u_{\sigma_X R^*} | {}_A A u_{\sigma_X R^*}$, we have ${}_A A u_{\sigma_X R^*} \sim {}_A A_{R^*}$.

Conversely we take a ring isomorphism σ: $R \simeq S$ between subrings of A such that ${}_A A u_{\sigma_R} \sim {}_A A_R$. Then ${}_{S^*\sigma} J_{1R^*}$ is invertible (Lemma 1. 2). Let τ: $S \simeq T$ be also a ring isomorphism between subrings of A such that ${}_A A u_{\tau S} \sim$

${}_A A_S$. Then ${}_\tau J_1 \cdot {}_\sigma J_1 \subseteq {}_{\tau\sigma} J_1$, because ${}_{\tau\sigma} J_\sigma = {}_\tau J_1$. Since ${}_{\tau\sigma} J_1 \cdot {}_1 J_\sigma \subseteq {}_{\tau\sigma} J_\sigma = {}_\tau J_1$, we obtain ${}_{\tau\sigma} J_1 \subseteq {}_\tau J_1 \cdot {}_\sigma J_1$. Hence ${}_\tau J_1 \cdot {}_\sigma J_1 = {}_{\tau\sigma} J_1$. ${}_\sigma J_1$ induces ρ: $R^{**} \simeq S^{**}$, and ρ is an extension of $\sigma : R \simeq S$. Therefore $R = R^{**}$ and $S = S^{**}$ are equivalent.

From the above argument, the set of all invertible submodules of A is a groupoid. This is denoted by $\mathfrak{G}(A)$. The set of identities is the set of subrings of A. By $\mathfrak{G}^*(A)$ we denote the groupoid of all ring isomorphisms σ: $R \simeq S$ between subrings of A such that ${}_A A u_{\sigma_R} \sim {}_A A_R$. The set of identities is the set of identity mappings of subrings of A.

Summarizing the above we state the following

Theorem 1.3. *Let A be a ring. Then the mapping ${}_S X_R \mapsto \sigma_X$ is a groupoid homomorphism from $\mathfrak{G}(A)$ to $\mathfrak{G}^*(A)$, where σ_X is the ring isomorphism from R^* to S^* such that $\sigma_X(a)u = ua$ for any a in R^* and any u in X. The mapping σ: $R \simeq S \mapsto {}_\sigma J_1$ is a groupoid homomorphism from $\mathfrak{G}^*(A)$ to $\mathfrak{G}(A)$, where ${}_\sigma J_1 = \{u \in A \mid \sigma(a)u = ua$ for all a in $R\}$. For an invertible ${}_S X_R$, $X = {}_{\sigma_X} J_1$, $S^{**} = S$, and $R^{**} = R$ are equivalent conditions. For σ: $R \simeq S$ in $\mathfrak{G}^*(A)$, $\sigma = \sigma_{{}_\sigma J_1}$, $S^{**} = S$, and $R^{**} = R$ are equivalent conditions. Therefore, if we put $\mathfrak{G}_C(A) = \{{}_S X_R \in \mathfrak{G}(A) \mid R^{**} = R\}$ and $\mathfrak{G}_C^*(A) = \{\sigma \in \mathfrak{G}^*(A) \mid \sigma$: $R \to A$, $R^{**} = R\}$ then, by the above homomorphism, $\mathfrak{G}_C(A) \simeq \mathfrak{G}_C^*(A)$ as groupoids.*

Let A be a ring, and R a subring of A. By $\mathfrak{G}(A/R)$ we denote the group of all R-R-invertible submodules of A. By $\mathfrak{G}^*(A/R)$ we denote the group of automorphisms σ of R such that ${}_A A u_{\sigma_R} \sim {}_A A_R$. By $\mathrm{Aut}(A/R)$ we denote the group of R-automorphisms of A acting on the left side. If R^{**} is equal to R, we call R a *commutor subring* of A, where R^* is the centralizer of R in A.

Corollary. *If R is a commutor subring of A, then $\mathfrak{G}(A/R) \simeq \mathfrak{G}^*(A/R^*)$ as groups. Particularly, $\mathfrak{G}(A/C) \simeq \mathfrak{G}^*(A/A)$, where C is the center of A. If B is a subring of A, then the image of $\mathfrak{G}(B^*/C)$ under the above isomorphism is the group of B-automorphisms σ of A such that ${}_A A u_{\sigma_A} \sim {}_A A_A$.*

Theorem 1.4. *Let $A \supseteq R \supseteq B$ be rings, and R a commutor subring of A. Assume that ${}_A A \otimes_B R_R \sim {}_A A_R$. Then $\mathfrak{G}(B^*/R^*) \simeq Aut$ (R/B) as groups.*

Proof. It suffices to prove that ${}_A A u_{\sigma_R} \sim {}_A A_R$ for all σ in Aut (R/B). Since ${}_A A \otimes_B R_R \sim {}_A A_R$, we have ${}_A A_R \sim {}_A A \otimes_B R_R \simeq {}_A A \otimes_B R u_{\sigma_R} \simeq {}_A A \otimes_B R \otimes_R R u_{\sigma_R}$ $\sim {}_A A \otimes_R R u_{\sigma_R} \simeq {}_A A u_{\sigma R}$. Thus ${}_A A_R \sim {}_A A u_{\sigma_R}$.

Corollary. *If A is a central separable C-algebra then $\mathfrak{G}(A/C) \simeq Aut(A/C)$ as groups.*

Let M be a module (i.e. additive abelian group). Put $A = \mathrm{End}_l(M)$ (the ring of all endomorphisms of M), and let S, T be commutor subrings of

A. Let σ, τ be ring isomorphisms from T^* to S^*, where $T^*=\mathrm{End}_l\,({}_TM)$. ${}_SM$ may be considered as a left T^*-module by σ. We denote this by ${}_{T^*}(\sigma, M)$. Then, for a in A, a is a left T^*-homomorphism from ${}_{T^*}(\sigma, M)$ to ${}_{T^*}(\tau, M)$ if and only if $a\cdot\sigma(t^*)m=\tau(t^*)am$ for any $t^*\in T^*$ and any $m\in M$ (i. e. $a\in{}_\tau J_\sigma$). Therefore ${}_{T^*}(\sigma, M)\sim{}_{T^*}M$ if and only if ${}_\sigma J_1\cdot{}_1J_\sigma=S$ and ${}_1J_\sigma\cdot{}_\sigma J_1=T$, or equivalently, ${}_AAu_{\sigma_{T^*}}\sim{}_AA_{T^*}$. However, if ${}_SM\sim{}_SS^*$ and ${}_{T^*}M\sim{}_{T^*}T^*$ then ${}_{T^*}(\sigma, M)\sim{}_{T^*}(\sigma, S^*)\simeq{}_{T^*}T^*\sim{}_{T^*}M$, and so ${}_{T^*}(\sigma, M)\sim{}_{T^*}M$. Thus we have proved the following

Theorem 1.5. *Let M be a module, and put $A=\mathrm{End}\,(M)$. Let $S\supseteq T$ be commutor subrings of A. Then $\mathfrak{G}(S/T)\simeq\{\sigma\in Aut(T^*/S^*)|{}_{T^*}(\sigma, M)\sim{}_{T^*}M\}$ as groups. In particular, if ${}_{T^*}M\sim{}_{T^*}T^*$, then $\mathfrak{G}(S/T)\simeq Aut(T^*/S^*)$ as groups.*

We conclude this section with the following

Proposition 1.6. *Let $A\supseteq S$ be rings, and τ a ring homomorphism from S to A. Then the following conditions are equivalent:*

(i) ${}_AA_S|{}_AAu_{\tau S}$.

(ii) *τ is a momomorphism, and* ${}_AAu_{\tau^{-1}\ \tau(S)}|{}_AA_{\tau(S)}$.

Proof. (ii) $\Rightarrow$(i) From ${}_AAu_{\tau^{-1}\tau(S)}|{}_AA_{\tau(S)}$, we obtain ${}_AA_S\simeq{}_AAu_{\tau^{-1}}\otimes_{\tau(S)}\tau(S)u_{\tau S}$ $|{}_AA\otimes_{\tau(S)}\tau(S)u_{\tau S}\simeq{}_AAu_{\tau S}$. (i) $\Rightarrow$(ii) If $\tau(s)=0$ for s in S, then ${}_AA_S|{}_AAu_{\tau S}$ means that $As=0$. Hence $s=0$. To be easily seen, ${}_1J_\tau={}_{\tau^{-1}}J_1$ and ${}_\tau J_1={}_1J_{\tau^{-1}}$. Therefore, $1\in{}_1J_\tau\cdot{}_\tau J_1={}_{\tau^{-1}}J_1\cdot{}_1J_{\tau^{-1}}$, and hence ${}_AAu_{\tau^{-1}\ \tau(S)}|{}_AA_{\tau(S)}$.

§ 2. Centralizer of Frobenius extensions.

Let σ be a ring homomorphism from S to R. Then R may be considered as a left S, right S-module by σ. R/S is called a Frobenius extension, if $R_S|S_S$ and $R\simeq\mathrm{Hom}\,(R_S, S_S)$ as left S, right R-modules (cf. Kasch [14]). If R/S is Frobenius, then there are $h:\ {}_SR_S\to{}_SS_S$, and r_i, $l_i\in R (i=1, \cdots, n)$ such that $x=\sum_i r_i\cdot h(l_ix)=\sum_i h(xr_i)l_i$ for all x in R, and conversely (cf. Onodera [23]). In this case, h is called a Frobenius homomorphism, and is characterized as the image of 1 under an S-R-isomorphism $R\simeq\mathrm{Hom}\,(R_S, S_S)$.

The proof of the following lemma is evident.

Lemma 2.1. *Let R/S be a ring extension, and R' a ring. Let M be a left R', right R-module.*

(1) $\mathrm{End}_l\,({}_{R'}M_S)\simeq\mathrm{Hom}_l\,({}_{R'}M\otimes_SR_R, {}_{R'}M_R)$ *by the mapping* $f\mapsto(m\otimes r\to f(m)r)$ $(m\in M,\ r\in R)$.

(2) $\mathrm{End}_l\,({}_{R'}M_S)\simeq\mathrm{Hom}_l\,({}_{R'}M_R, {}_{R'}\mathrm{Hom}_l\,(R_S,M_S)_R)$ *by the mapping* $f\mapsto(m\to(r\to f(mr)))$. *The inverse of this mapping is the mapping* $g\mapsto(m\to g(m)(1))$ $(m\in M)$.

The proof of the following is easily checked.

Lemma 2. 2. *Let R/S be a Frobenius extension with (h, r_i, l_i) (i.e. $x=\sum_i h(xr_i)\,l_i=\sum_i r_i\cdot h(l_ix)$ for all x in R), and let R' be a ring. Let M be a left R', right S-module. Then $\mathrm{Hom}_l(R_S, M_S)\simeq M\otimes_S R$ as left R', right R-modules, by the mapping $f\mapsto\sum_i f(r_i)\otimes l_i$. The inverse of this mapping is the mapping $m\otimes r\mapsto(x\to m\cdot h(rx))$ $(m\in M,\ r,\ x\in R)$.*

Corollary. *Under the same notations and assumptions, we have an isomorphism $\mathrm{End}_l\,({}_{R'}M_S)\simeq\mathrm{Hom}\,({}_{R'}M_R,\ {}_{R'}M\otimes_S R_R)$, $f\mapsto(m\to\sum_i f(mr_i)\otimes l_i)$.*

Proof. This follows from Lemma 2. 1 (2) and Lemma 2. 2.

Proposition 2. 3. *Let R/S be a Frobenius extension with (h, r_i, l_i), and let M be a left R', right R-module.*

(1) *${}_{R'}M\otimes_S R_R|{}_{R'}M_R$ if and only if there are f_j, g_j $(j=1, \cdots, t)$ in $\mathrm{End}_r\,({}_{R'}M_S)$ such that $\sum_j (m^{f_j}\cdot x)^{g_j}=m\cdot h(x)$ for any $m\in M$ and any $x\in R$.*

(2) *${}_{R'}M_R|{}_{R'}M\otimes_S R_R$ if and only if there is a left R', right S-endomorphism k of M such that $\sum_i (mr_i)^k\cdot l_i=m$ for all m in M.*

Proof. By Lemma 2. 1 and Cor. to Lemma 2. 2, ${}_{R'}M\otimes_S R_R|{}_{R'}M_R$ is equivalent to that there are f_j, g_j in $\mathrm{End}_r({}_{R'}M_S)$ such that $\sum_{i,j} (m^{f_j}\cdot r_i)^{g_j}\otimes l_i=m\otimes 1$ in $M\otimes_S R$ for all m in M. By making use of the inverse of the isomorphism in Lemma 2. 2, the above equality is equivalent to that $\sum_j (m^{f_j}\cdot x)^{g_j}=m\cdot h(x)$ for any $m\in M$ and any $x\in R$. Similarly we can prove (2).

Proposition 2. 4. *Let both A/R and R/S be ring extensions, and assume that R/S is a Frobenius extension with (h, r_i, l_i).*

(1) *${}_A A\otimes_S R_R|{}_A A_R$ (cf. Hirata [10]) if and only if there are f_j, g_j in $V_A(S)$ such that $\sum_j f_j\cdot\tilde{x}\cdot g_j=\widetilde{h(x)}$ for all x in R, where $\tilde{x}$ is the image of x in A. (Note that the latter is right-left symmetric.)*

(2) *${}_A A_R|{}_A A\otimes_S R_R$ if and only if there is an element k in $V_A(S)$ such that $\sum_i \tilde{r}_i k\tilde{l}_i=1$.*

Proof. Put $M=R'=A$ in Prop. 2. 3.

Remark 1. In the above Corollary, let $\alpha: S\to R$ and $\beta: R\to A$, and assume that ${}_A A\otimes_S R_R|{}_A A_R$. Then, if $\beta(x)=0$ for x in R, then $\beta\alpha h(x)=0$, because $0=\sum_j f_j\cdot\beta(x)g_j=\beta\alpha h(x)$. $\tilde{h}$ is defined by $\tilde{h}(\beta(x))=\beta\alpha h(x)$ $(x\in R)$. Then, to be easily seen, $\beta(R)/\beta\alpha(S)$ is a Frobenius extension with $(\tilde{h}, \beta(r_i)$, $\beta(l_i))$, and ${}_A A\otimes_{\beta\alpha(S)}\beta(R)_{\beta(R)}|{}_A A_{\beta(R)}$. Further, if ${}_A A_R|{}_A A\otimes_S R_R$ then ${}_A A_{\beta(R)}|$ ${}_A A\otimes_{\beta\alpha(S)}\beta(R)_{\beta(R)}$ holds.

Remark 2. In Prop. 2. 3, we put $R'=$the ring of integers and $A=\mathrm{End}_r(M)$. And assume that both R and S are subrings of A. Then, by Prop. 2. 3 and Prop. 2. 4, $M\otimes_S R_R|M_R$ if and only if ${}_A A\otimes_S R_R|{}_A A_R$. Similarly, $M_R|M\otimes_S R_R$ if and only if ${}_A A_R|{}_A A\otimes_S R_R$.

Remark 3. Let R/S be a ring extension, and M a right R-module. Assume that $R_R|M_R$ and $M_S|S_S$. Then $M\otimes_S R_R|S\otimes_S R_R\simeq R_R|M_R$. Thus $M\otimes_S R_R|M_R$ (cf. [18]).

Lemma 2.5. *Let $\bar{R}\supseteq\bar{S}$ be a Frobenius extension with (h, r_i, l_i), and let $R\supseteq S$ be subrings of $\bar{R}$ such that $\bar{S}\supseteq S$. Assume that $h(R)\subseteq S$ and that $r_i, l_i\in R$ for all i. Then R/S is a Frobenius extension, and $\bar{R}=R\otimes_S\bar{S}=\bar{S}\otimes_S R$. Therefore, if $S=\bar{S}$, then $R=\bar{R}$. If $R=\bar{R}$ and ${}_SS|{}_SR$, then $S=\bar{S}$.*

Proof. The canonical homomorphism $R\otimes_S\bar{S}\to\bar{R}$, $r\otimes\bar{s}\mapsto r\bar{s}$ is an isomorphism with the inverse mapping $\bar{r}\mapsto\sum_i r_i\otimes h(l_i\bar{r})$. The remainder will be easily seen.

Let A be a ring. In the sequel, for any subring R of A, R^* means the centralizer of R in A. Let $R\supseteq S$ be subrings of A such that R/S is a Frobenius extension with (h, r_i, l_i), and assume that ${}_AA\otimes_S R_R|{}_AA_R$. Then, by Prop. 2.4, there are a_j, b_j in S^* such that $\sum_j a_j x b_j=h(x)$ for all x in R. Now, $R\otimes_S R\simeq \mathrm{Hom}_l(R_S, R_S)$ as R-R-modules, by the correspondence $x\otimes y\to xhy$ (cf. Onodera [23]). Since the image of $\sum_i r_i\otimes l_i$ is 1, we know that $\sum_i xr_i\otimes l_i=\sum_i r_i\otimes l_i x$ for all x in R. Therefore, for any s^* in S^*, $\sum_i xr_i s^* l_i=\sum_i r_i s^* l_i x$. Hence $\sum_i r_i s^* l_i\in R^*$ for any s^* in S^*. Put $\sum_i r_i s^* l_i=h^*(s^*)$. Then h^* is an R^*-R^*-homomorphism from S^* to R^*. Further, $\sum_j h^*(s^*a_j)b_j=\sum_{i,j} r_i s^* a_j l_i b_j=\sum_i r_i s^*\cdot h(l_i)=\sum_i r_i\cdot h(l_i)s^*=s^*$. Similarly, we obtain $\sum_j a_j\cdot h^*(b_j s^*)=s^*$. Hence S^*/R^* is a Frobenius extension with (h^*, a_j, b_j) such that ${}_AA\otimes_{R^*}S^*{}_{S^*}|{}_AA_{S^*}$. Thus we have proved the main part of the following theorem.

Theorem 2.6. *Let A be a ring, and $R\supseteq S$ subrings of A such that R/S is a Frobenius extension with (h, r_i, l_i), and assume that ${}_AA\otimes_S R_R|{}_AA_R$. Let a_j, b_j be elements in S^* such that $\sum_j a_j x b_j=h(x)$ for all x in R, where S^* is the centralizer of S in A. Then S^*/R^* is a Frobenius extension with (h^*, a_j, b_j) such that ${}_AA\otimes_{R^*}S^*_{S^*}|{}_AA_{S^*}$, where $h^*(s^*)=\sum_i r_i s^* l_i (s^*\in S^*)$. In this case the following hold.*

(1) *${}_AA_R|{}_AA\otimes_S R_R$ is equivalent to ${}_{R^*}R^*|{}_{R^*}S^*$.*

(2) *If ${}_RR_R|{}_RR\otimes_S R_R$ (i.e. R/S is separable) then ${}_{R^*}R^*_{R^*}|{}_{R^*}S^*_{R^*}$.*

(3) *If ${}_SS_S|{}_SR_S$ then ${}_{S^*}S^*_{S^*}|{}_{S^*}S^*\otimes_{R^*}S^*_{S^*}$.*

(4) *If ${}_RR\otimes_S R_R|{}_RR_R$ (i.e. R/S is H-separable) then ${}_{R^*}S^*_{R^*}|{}_{R^*}R^*_{R^*}$.*

(5) *If ${}_SR_S|{}_SS_S$ then ${}_{S^*}S^*\otimes_{R^*}S^*_{S^*}|{}_{S^*}S^*_{S^*}$.*

(6) *If ${}_SS|{}_SR$ then ${}_AA_{S^*}|{}_AA\otimes_{R^*}S^*_{S^*}$.*

Proof. It remains to prove (1), $\cdots$, (6). (1) ${}_AA_R|{}_AA\otimes_S R_R$ is equivalent to that $h^*(s^*)=1$ for some s^* in S^*. And the latter is equivalent to that ${}_{R^*}R^*$ is a direct summand of ${}_{R^*}S^*$. (2) ${}_RR_R|{}_RR\otimes_S R_R$ is equivalent to that

$\sum_i r_i c l_i = 1$ for some c in $R \cap S^*$. Then the mapping $s^* \mapsto h^*(s^*c)$ from S^* to R^* is an R^*-R^*-homomorphism such that $h^*(r^*c) = r^*$ for all r^* in R^*. (3) ${}_SS_S|{}_SR_S$ is equivalent to that $(\sum_j a_j c b_j =)\, h(c) = 1$ for some c in $R \cap S^*$. Then ${}_{S^*}S^*_{S^*}|{}_{S^*}S^* \otimes_{R^*} S^*_{S^*}$, because $c \in R \cap S^* \subseteq R^{**} \cap S^*$. (4) ${}_RR \otimes_S R_R|{}_RR_R$ is equivalent to that a_j, b_j can be taken in $R \cap S^*$. Then, as $R \cap S^* \subseteq R^{**} \cap S^*$, we know ${}_{R^*}S^*_{R^*}|{}_{R^*}R^*_{R^*}$, because $s^* = \sum_j h^*(s^* a_j) b_j$ for all s^* in S^*. (5) ${}_SR_S|{}_SS_S$ is equivalent to that r_i, l_i can be taken in $R \cap S^*$, because $\mathrm{Hom}_l({}_SR_S, {}_SS_S) = h\cdot(R \cap S^*)$ and $\mathrm{Hom}_l({}_SS_S, {}_SR_S) \simeq R \cap S^*$ canonically. Therefore ${}_S S^* \otimes_{R^*} S^*_{S^*}|{}_{S^*}S^*_{S^*}$, because $R \cap S^* \subseteq R^{**} \cap S^*$. (6) If ${}_SS|{}_SR$ then $h(a) = 1$ for some $a \in R$. Then, since $h(a) = \sum_j a_j a b_j$, we know ${}_AA_{S^*}|{}_AA \otimes_{R^*} S^*_{S^*}$ (Prop. 2. 4 (2)).

Corollary. *Let A be a ring, C its center, and R a commutor subring of A containing C. Put R^* = the centralizer of R in A. Then the following are equivalent*:

(i) *R/C is a Frobenius extension such that ${}_AA \otimes_C R_R|{}_AA_R$.*

(ii) *A/R^* is a Frobenius extension such that ${}_AA \otimes_{R^*} A_A|{}_AA_A$.*

Remark. Let $A \supseteq T \supseteq R \supseteq S$ be rings such that ${}_AA \otimes_S R_R|{}_AA_R$ and ${}_AA \otimes_R T_T|{}_AT_T$. Then ${}_AA \otimes_S T_T \simeq {}_AA \otimes_S R \otimes_R T_T|{}_AA \otimes_R T_T|{}_AA_T$.

Let R/S be a Frobenius extension with (h, r_i, l_i). Then, φ: $R \simeq \mathrm{Hom}_l\,(R_S, S_S)$, $x \mapsto hx$, as left S, right R-modules. For f in $\mathrm{Hom}_l\,(R_S, S_S)$ and a in $\mathrm{V}_R(S)$ (the centralizer of S in R), we define $(af)(x) = f(xa)\ (x \in R)$. Then $\mathrm{Hom}\,(R_S, S_S)$ is a left $\mathrm{V}_R(S)$-module. In this sense, φ is a left $S, \mathrm{V}_R(S)$, right R-isomorphism if and only if $h(ax) = h(xa)$ for any a in $\mathrm{V}_R(S)$ and any x in R. Noting these facts, the proof of the following is virtually the same as that in Onodera [23].

Proposition 2. 7. *For a ring extension R/S, the following are equivalent*:

(i) *$R_S|S_S$, and $R \simeq \mathrm{Hom}\,(R_S, S_S)$ as left $S, \mathrm{V}_R(S)$, right R-modules.*

(ii) *There are h: ${}_SR_S \to {}_SS_S$, r_i, $l_i \in R$ $(i = 1, \cdots, n)$ such that $h(xa) = h(ax)$ and $x = \sum_i h(xr_i) l_i = \sum_i r_i \cdot h(l_i x)$ for any x in R and any a in $\mathrm{V}_R(S)$.*

If a ring extension R/S satisfies (i) we call R/S a symmetric extension. Note that a Frobenius extension R/S such that $\mathrm{V}_R(S)$ = the center of R is a symmetric extension.

Theorem 2. 8. *In Th.* 2.6, *if $S = S^{**}$, and R/S is a symmetric extension, then S^*/R^* is also a symmetric extension.*

Proof. In the proof of Th. 2. 6, we may assume that $\sum_j a_j x a b_j = \sum_j a_j a x b_j$ for any x in R and any a in $\mathrm{V}_R(S)$. Then, as is easily seen, $\sum_i r_i h a l_i = \sum_i r_i a h l_i = (x \to xa)\ (x \in R)$ for any a in $\mathrm{V}_R(S)$. Hence $\sum_i r_i \otimes a l_i = \sum_i r_i a \otimes l_i\ (\in R \otimes_S R)$ for any a in $\mathrm{V}_R(S)$, because $R \otimes_S R \simeq \mathrm{Hom}_l\,(R_S, R_S)$, by the mapping $x\otimes y\mapsto xhy$. Consequently, for any s^* in S^*, $h^*(s^*a)=\sum_i r_is^*al_i=\sum_i r_ias^*l_i=h^*(as^*)$. Since $V_{S^*}(R^*)=V_R(S)$, S^*/R^* is a symmetric extension.

Lemma 2.9. *Let R/S be a Frobenius extension, and M a right R-module such that $M\otimes_S R_R|M_R$. Put $A=\mathrm{End}_r(M)$, and let $\tilde{R}$ and $\tilde{S}$ be the images of R and S in A, respectively. Then $\tilde{R}/\tilde{S}$ is a Frobenius extension, and $M\otimes_{\tilde{S}}\tilde{R}_R\simeq M\otimes_S R_R$ canonically. Therefore $M\otimes_{\tilde{S}}\tilde{R}_{\tilde{R}}|M_{\tilde{R}}$.*

Proof. If $Mr=0$ for r in R, then $M\otimes r=0$, and so $M\cdot h(r)=0$. Then, as in Remark 1 to Prop. 2.4, we can prove that $\tilde{R}/\tilde{S}$ is a Frobenius extension. By $\tilde{x}$, we denote the image of x in A. Then the mapping $(m, \tilde{x})\mapsto m\otimes x$ from $M\times\tilde{R}$ to $M\otimes_S R$ is well defined. Because, if $\tilde{x}=\tilde{y}$, then $\widetilde{x-y}=0$, and so $M\otimes(x-y)=0$, so that $m\otimes x=m\otimes y$ in $M\otimes_S R$ for any m in M. Then the remainder is obvious.

Theorem 2.10. *Let R/S be Frobenius, and M a right R-module such that $M\otimes_S R_R|M_R$. Put $\mathrm{End}_r(M_R)=R^*$ and $\mathrm{End}_r(M_S)=S^*$. Then S^*/R^* is a Frobenius extension such that $M\otimes_{R^*}S^*_{S^*}|M_{S^*}$. If both S and R are subrings of $\mathrm{End}_r(M)$, $\mathrm{End}_r(M_{S^*})=S$, and R/S is a symmetric extension, then S^*/R^* is also a symmetric extension.*

Proof. Put $\mathrm{End}_r(M)=A$. Then, by Lemma 2.9, we may assume that both S and R are subrings of A. Then, by Remark 2 to Prop. 2.4 and Th. 2.6, $M\otimes_S R_R|M_R \Rightarrow {}_AA\otimes_S R_R|{}_AA_R \Rightarrow {}_AA\otimes_{R^*}S^*_{S^*}|{}_AA_{S^*}\Rightarrow M\otimes_{R^*}S^*_{S^*}|M_{S^*}$. The part concerning symmetric extensions follows at once from Th. 2.8.

Next we consider crossed products. Let G be a group, and let $\Omega\supseteq A$ be rings, and C the center of A. Let U be a group homomorphism from G to $\mathfrak{G}(\Omega/A)$. If $\Omega=\sum_{\sigma\in G}\oplus U_\sigma$ (direct sum) then we call $(\Omega/A, U)$ a *generalized crossed product* of A with G. (Note that our definition is more general than that in Kanzaki [13].) Then each U_σ induces a ring automorphism of C. We denote this by σ, too. For any subgroup H of G, we set $C^H=\{c\in C|\tau(c)=c$ for all τ in $H\}$. Assume that $(G:H)<\infty$, and let $G=\sigma_1H\cup\cdots\cup\sigma_rH$ be the coset decomposition of G. Then the mapping $c\mapsto t_{G:H}(c)=\sum_i\sigma_i(c)$ from C^H to C^G is a C^G-C^G-homomorphism, and is independent of the choice of $\sigma_1, \cdots, \sigma_r$.

Theorem 2.11. *Let $\Omega/A=\sum_{\sigma\in G}\oplus U_\sigma$ be a generalized crossed product of A with G.*

(1) *Let H be a subgroup of G such that $(G:H)<\infty$, and put $\Omega_H=\sum_{\tau\in H}\oplus U_\tau$. Then Ω/Ω_H is a Frobenius extension. Ω/Ω_H is separable if and only if $t_{G:H}(c)=1$ for some c in C^H.*

(2) *Let N be a normal subgroup, and put $\Omega_N=\sum_{\rho\in N}\oplus U_\rho$. Then Ω/Ω_N*

is a generalized crossed product of Ω_N with G/N.

Proof. (1) Let $G=\sigma_1 H\cup\cdots\cup\sigma_r H$ be the coset decomposition of G. For any $x=\sum_{\sigma\in G}x_\sigma(x_\sigma\in U_\sigma)$ in Ω, we define $h(x)=\sum_{\tau\in H}x_\tau$. Then h is evidently an Ω_H-Ω_H-homomorphism from Ω to Ω_H. For any σ_j, 1 is written as $1=\sum_i x_{j,i}y_{j,i}(x_{j,i}\in U_{\sigma_j},\ y_{j,i}\in U_{\sigma_j^{-1}})$. Then $\sum_j\sum_i h(xx_{j,i})y_{j,i}=\sum_j\sum_i\sum_{\sigma\in G}h(x_\sigma x_{j,i})y_{j,i}=\sum_j\sum_i\sum_{\tau\in H}x_{\tau\sigma_j^{-1}}x_{j,i}y_{j,i}=\sum_j\sum_{\tau\in H}x_{\tau\sigma_j^{-1}}=x$. Similarly, $\sum_j\sum_i x_{j,i}\cdot h(y_{j,i}x)=x$ for all x in Ω. Thus Ω/Ω_H is a Frobenius extension with $(h,\ x_{j,i},\ y_{j,i})$. Therefore Ω/Ω_H is separable if and only if $\sum_{j,i}x_{j,i}xy_{j,i}=1$ for some x in $V_\Omega(\Omega_H)$. In this case, let $x=\sum_{\sigma\in G}x_\sigma(x_\sigma\in U_\sigma)$. Then $x\in V_\Omega(\Omega_H)$ implies $x_1\in C^H$, and $t_{G:H}(x_1)=1$ is easily seen. Thus we have proved (1). (2) Let $G=\cup_{\lambda\in\Lambda}\sigma_\lambda N$ be the coset decomposition of G with respect to a normal subgroup N. For any subset S of G, we set $\Omega_s=\sum_{\sigma\in S}U_\sigma$. Then $\Omega=\sum_\lambda\oplus\Omega_{\sigma_\lambda N}$, and $\Omega_{\sigma N}\Omega_{\tau N}=\Omega_{\sigma\tau N}$ for any σ, τ in G. Thus we have proved (2).

Corollary. *Let* $\Omega/A=\sum_{\sigma\in G}\oplus U_\sigma$ *be a generalized crossed product of* A *with a finite group* G. *Then* Ω/A *is a Frobenius extension.* Ω/A *is separable if and only if* $\sum_{\sigma\in G}\sigma(c)=1$ *for some* c *in* C.

Let A be a ring, and $R\supseteq S$ subrings of A with ${}_AA\otimes R_R|{}_AA_R$. For any subring T of A, we denote the centralizer of T in A by T^*. Then, by Th. 2. 6, if R/S is Frobenius, then so is S^*/R^*, and ${}_AA\otimes_{R^*}S^*_{S^*}|{}_AA_{S^*}$. Now, let $R/S=\sum_{\sigma\in G}\oplus U_\sigma$ be a generalized crossed product of A with a finite group G. Then each U_σ induces an automorphism of S^*. This defines a group homomorphism from G to $\mathrm{Aut}(S^*/R^*)$, and we obtain a G-ring S^*. Since the projection to S is a Frobenius homomorphism of R/S, by assumption, there are a_j, b_j in S^* such that $\sum_j a_j(\sum_{\sigma\in G}z_\sigma)b_j=z_1$ for any $z_\sigma\in U_\sigma$ $(\sigma\in G)$. Then, $\sum_j a_j\cdot\sigma(b_j)=\delta_{1,\sigma}$ for any σ in G. Since $S^{*G}=V_A(R)=R^*$, this means that S^*/R^* is a G-Galois extension. By Th. 2. 6 (1), ${}_{R^*}S^*$ is a generator if and only if ${}_AA_R|{}_AA\otimes_SR_R$.

Next we assume that R/S is a G-Galois extension. Then, since ${}_RR_R|{}_RR\otimes_SR_R$, we have ${}_AA_R|{}_AA\otimes_SR_R$, so that ${}_AA_R\sim{}_AA\otimes_SR_R$. Hence, by the proof of Th. 1. 4, ${}_AAu_{\sigma_R}\sim{}_AA_R$ for all σ in G. Thus ${}_\sigma J_1\in\mathfrak{G}(S^*/R^*)$. Put ${}_\sigma J_1=U_\sigma$. Then $R^*=U_1\subseteq\sum_{\sigma\in G}U_\sigma\subseteq S^*$, and $U_\sigma U_\tau=U_{\sigma\tau}$. By hypothesis, there are a_j, b_j in R such that $\sum_j a_j\cdot\sigma(b_j)=\delta_{1,\sigma}$ $(\sigma\in G)$. Let $z=\sum_{\sigma\in G}z_\sigma(z_\sigma\in U_\sigma)$ be any element of $\sum_{\sigma\in G}U_\sigma$. Then $\sum_j\tau(a_j)(\sum_{\sigma\in G}z_\sigma)b_j=\sum_j\sum_{\sigma\in G}\tau(a_j)\sigma(b_j)z_\sigma=z_\tau$. Hence $\sum_{\sigma\in G}U_\sigma=\sum_{\sigma\in G}\oplus U_\sigma$. Put $\sum_{\sigma\in G}U_\sigma=S_0^*$. Then, for any x in R, we have $x=\sum_j t_G(xa_j)b_j=\sum_j a_j\cdot t_G(b_jx)$, so that R/S is a Frobenius extension with $(t_G,\ a_j,\ b_j)$. For any σ in G, 1 is written as $1=\sum_i x_{\sigma,i}y_{\sigma,i}(x_{\sigma,i}\in U_\sigma,\ y_{\sigma,i}\in U_{\sigma^{-1}})$. Then, for any x in R, $t_G(x)=\sum_{\sigma,i}x_{\sigma,i}xy_{\sigma,i}$. Let h be an R^*-R^*-homomorphism from S^* to R^* defined by $h(s^*)=\sum_j a_js^*b_j$. Then both

S^*/R^* and S_0^*/R^* are Frobenius extensions with $(h, x_{\sigma,i}, y_{\sigma,i})$, so that $S^*=S_0^*$ (Lemma 2.5). Hence $S^*/R^*=\sum_{\sigma\in G}\oplus U_\sigma$ is a generalized crossed product. Noting Th. 1.4 we have the following

Theorem 2.12. *Let A be a ring, and $R\supseteq S$ subrings of A such that ${}_A A\otimes_S R_R|{}_A A_R$, and let G be a finite group. We set $S^*=\mathrm{V}_A(S)$ and $R^*=\mathrm{V}_A(R)$.*

(1) If R/S is a generalized crossed product of S with G, then S^*/R^* is a G-Galois extension. In this case, ${}_{R^*}S^*$ is a generator if and only if ${}_A A_R|{}_A A\otimes_S R_R$.

(2) If R/S is a G-Galois extension, then S^*/R^* is a generalized crossed product with G. Further, if ${}_S R$ is a generator then ${}_A A_{S^*}|{}_A A\otimes_{R^*}S^*_{S^*}$.

Therefore, if $S^{**}=S$ and $R^{**}=R$ then that R/S is G-Galois is equivalent to that S^*/R^* is a generalized crossed product. In this case, let $\{U\}$ be the set of all monomorphisms U: $G\to\mathfrak{G}(S^*/R^*)$ such that $S^*=\sum_{\sigma\in G}\oplus U_\sigma$, and $\{u\}$ the set of all monomorphisms u: $G\to Aut(R/S)$ such that R/S is G-Galois. Then there are a group isomorphism $\mathfrak{G}(S^*/R^*)\simeq Aut(R/S)$, $X\to\sigma_X$, and a bijection $U\to u$ from $\{U\}$ to $\{u\}$ such that the diagram

$$\begin{tikzcd} & \mathfrak{G}(S^*/R^*) \arrow[dd] \\ G \arrow[ur, "U"] \arrow[dr, "u"'] & \\ & Aut(R/S) \end{tikzcd}$$

is commutative.

Corollary 1. *Let A be a ring, and C its center, and let G be a finite group. Let R be a commutor subring of A such that $R\supseteq C$. Then the following are equivalent:*

(i) R/C is a Frobenius extension such that ${}_A A\otimes_C R_R|{}_A A_R$.

(ii) A/R^* is a Frobenius extension such that ${}_A A\otimes_{R^*}A_A|{}_A A_A$.

In this case the following hold:

(1) R/C is G-Galois if and only if A/R^* is a generalized crossed product of R^* with G.

(2) R/C is a generalized crossed product of C with G if and only if A/R^* is G-Galois.

Corollary 2. (*cf.* [6][8][12]) *Let A be a ring, and C its center. Let G be a finite group. Then the following are equivalent:*

(i) A/C is a G-Galois extension.

(ii) A/C is separable, and is a generalized crossed product of C with G.

(iii) A/C is a generalized crossed product with G, and a Frobenius homomorphism of A/C is given by a member of $A\otimes_C A^{0p}$.

(iv) A/C is a generalized crossed product with G, and $(G:1)1$ is a

unit of C.

Proof. (i) $\Rightarrow$(ii) In this case, as is well known, A/C is separable, and A/C is a generalized crossed product of C with G (Th. 2.12). The implication (ii) $\Rightarrow$(i) follows at once from Th. 2.12. By Cor. to Th. 2.11, (ii)$\Leftrightarrow$(iv). (ii)$\Rightarrow$(iii) is obvious, and (iii)$\Rightarrow$(i) follows from Th. 2.12 and Prop. 2.4.

Corollary 3. *Let A be a ring, and C its center, and let G be a finite group. Let B be a commutative subring of A such that $V_A(B)=B$. Then the following conditions are equivalent:*

(i) *A/B is a finite G-Galois extension, and ${}_AA\otimes_B A_A|{}_AA_A$.*

(ii) *B/C is a genealized crossed product with G, and ${}_AA\otimes_C B_B|{}_AA_B$.*

Corollary 4. *With the same notations and assumptions as in Cor. 3, the following conditions are equivalent:*

(i) *A/B is a generalized crossed product of B with G, and ${}_AA\otimes_B A_A|{}_AA_A$.*

(ii) *B/C is a finite G-Galois extension, and ${}_AA\otimes_C B_B|{}_AA_B$.*

(iii) *A/C is separable, and B/C is a finite G-Galois extension.*

(iv) *A/C is separable, A/B is a generalized crossed product of B with G, and ${}_AA_B|{}_AA\otimes_C B_B$.*

Proof. (i)$\Leftrightarrow$(ii) follows at once from Cor. 1. (ii)$\Rightarrow$(iii) Since ${}_CC|{}_CB$, we have ${}_AA_A|{}_AA\otimes_B A_A$ by Th. 2.6 (3). Therefore both A/B and B/C are separable, and hence A/C is separable. (iii)$\Rightarrow$(ii) Since B/C is separable, $A_C|C_C$ implies that $A_B|B_B$, and so $B_B|A_B$, because B is commutative. Then ${}_AA\otimes_C B_B|{}_AA\otimes_C A_B|{}_AA_B$. Hence ${}_AA\otimes_C B_B|{}_AA_B$. ((i)$\Leftrightarrow$(ii)$\Leftrightarrow$) (iii) $\Rightarrow$(iv) As ${}_BB_B|{}_BB\otimes_C B_B$, we obtain ${}_AA_B|{}_AA\otimes_C B_B$. (iv)$\Rightarrow$(i) Since ${}_AA_B|{}_AA\otimes_C B_B$, we have ${}_AA\otimes_B A_A|{}_AA\otimes_C B\otimes_B A_A\simeq{}_AA\otimes_C A_A|{}_AA_A$. Hence ${}_AA\otimes_B A_A|{}_AA_A$. This completes the proof.

Theorem 2.13. *Let $R\supseteq S$ be rings such that R/S is a Frobenius extension, and let M be a non-zero right R-module such that $M\otimes_S R_R|M_R$. Let G be a finite group. Put $S^*=\mathrm{End}_r(M_S)$ and $R^*=\mathrm{End}_r(M_R)$.*

(1) *If R/S is a G-Galois extension, then S^*/R^* is a generalized crossed product of R^* with G.*

(2) *If R/S is a generalized crossed product of S with G, then S^*/R^* is a G-Galois extension.*

Proof. We set $\mathrm{End}_r(M)=A$ (the ring of all endomorphisms of M), and let φ be the ring homomorphism from R to A such that $u^{\varphi(r)}=ur$ for any $u\in M$ and any $r\in R$. Then, by Lemma 2.9, $M\otimes_{\varphi(S)}\varphi(R)_R\simeq M\otimes_S R_R$ canonically. Put $\mathfrak{A}=\mathrm{Ker}\,\varphi$. (1) If a is in A, then $Ma=0$, and so $M\otimes a=0$, because $M\otimes_S R_R|M_R$. Then, for any σ in G, $M\otimes\sigma(a)=0$, so that

$M\cdot\sigma(a)=0$. Hence $\sigma(\mathfrak{A})\subseteq\mathfrak{A}$ for all $\sigma\in G$. Therefore $\varphi(R)/\varphi(R)^G$ is a G-Galois extension, canonically (cf. [16; Th. 5. 6]). Now there are a_j, b_j in R such that $\sum_j a_j\cdot\sigma(b_j)=\delta_{1,\sigma}$ $(\sigma\in G)$. Let r be an element of R with $\varphi(r)\in\varphi(R)^G$. Then $m\cdot\sigma(r)=mr$ for any m in M, and so $m\otimes r=m\otimes\sum_\sigma\sum_j\sigma(ra_j)\,b_j=\sum_j m\sum_\sigma\sigma(ra_j)\otimes b_j=\sum_j mr\cdot\sum_\sigma\sigma(a_j)\otimes b_j=\sum_j mr\otimes\sum_\sigma\sigma(a_j)\,b_j=mr\otimes 1$ in $M\otimes_S R$. From this fact we can see that $M\otimes_{\varphi(R)^G}\varphi(R)_R\simeq M\otimes_{\varphi(S)}\varphi(R)_R$ canonically. Therefore, if we set $\mathrm{End}_r(M_{\varphi(R)^G})=S_0^*$, then $S_0^*\subseteq S^*$, and R^*/S_0^* is a generalized crossed product of S_0^* with G. It remains to prove that $S_0^*=S^*$. Let f be in S^*. Then $f(m)\otimes r=f(mr)\otimes 1$, and so $f(m)\,\varphi(r)=f(m)r=f(mr)$ $=f(m\cdot\varphi(r))$ for any $m\in M$ and any $\varphi(r)\in\varphi(R)^G$. Thus $S_0^*=S^*$, as required. (2) Let $R/S=\sum_{\sigma\in G}\oplus U_\sigma$ be a generalized crossed product of S with G. Let $x=\sum_\sigma x_\sigma\,(x_\sigma\in U_\sigma)$ be in $\mathfrak{A}$. Then $Mx=0$ implies $M\otimes x=0$ in $M\otimes_S R$. Since $M\otimes_S R=\sum_{\sigma\in G}\oplus(M\otimes_S U_\sigma)$, we have $M\otimes x_\sigma=0$ for any σ in G. Hence $Mx_\sigma=0$ for all σ in G. Thus $\mathfrak{A}=\sum_{\sigma\in G}\oplus(\mathfrak{A}\cap U_\sigma)$. As is well known, each $\mathfrak{A}\cap U_\sigma$ is written as $\mathfrak{a}_\sigma U_\sigma$ with an ideal $\mathfrak{a}_\sigma$ of S. Then, to be easily seen, $\mathfrak{a}_\sigma=A\cap S$ $(=\mathfrak{a}_1)$ for all σ in G. Thus $\mathfrak{A}=(\mathfrak{A}\cap S)R$. Similarly $\mathfrak{A}=R(\mathfrak{A}\cap S)$. Then we can easily check that $\varphi(R)/\varphi(S)=\sum_{\sigma\in G}\varphi(U_\sigma)$ is a generalized crossed product of $\varphi(S)$ with G. By Lemma 2. 9, $M\otimes_{\varphi(S)}\varphi(R)_{\varphi(R)}|M_{\varphi(R)}$, and hence S^*/R^* is a G-Galois extension (Th. 2. 12).

§ 3. Equivalence relation concerning the ring homomorphisms.

Lemma 3. 1. *Let both $f\colon B\to A$ and $f'\colon B'\to A'$ be ring homomorphisms, and let both ${}_BN_{B'}$, and ${}_AM_{A'}$ be Morita modules. Let φ be a left B, right B'-homomorphism from N to M such that $A\otimes_B N\simeq M$ as left A, right B'-modules, by the corresponcence $a\otimes n\mapsto a\cdot\varphi(n)$. Then the following hold.*

(1) *$N\otimes_{B'}A'\simeq M$ as left B, right A'-modules, by the correspondence $n\otimes a'\mapsto\varphi(n)a'$.*

(2) *$\mathrm{Hom}_r({}_BN, {}_BB)\otimes_B A\simeq\mathrm{Hom}_r({}_AM, {}_AA)$ as left B', right A-modules, by the correspondence $f\otimes a\mapsto(x\cdot\varphi(n)\to x\cdot n^f\cdot a)$ $(f\in\mathrm{Hom}_r({}_BN, {}_BB)$, a, $x\in A$, $n\in N)$. Similarly $A'\otimes_{B'}\mathrm{Hom}_l(N_{B'}, B'_{B'})\simeq\mathrm{Hom}_l(M_{A'}, A'_{A'})$ as left A', right B-modules.*

(3) *The following diagram is commutative:*

$$\begin{array}{ccc}\mathrm{Hom}_r({}_BN, {}_BB) & \xrightarrow{\alpha} & \mathrm{Hom}_l(N_{B'}, B'_{B'}) \\ \varphi^*\big\downarrow & & \varphi^+\big\downarrow \\ \mathrm{Hom}_r({}_AM, A) & \xrightarrow{\beta} & \mathrm{Hom}_l(M_{A'}, A'_{A'})\end{array}$$

where α, β are canonical isomorphisms (cf. [18] or [21]), $\varphi^*(g)=(a\cdot\varphi(n)\to a\cdot n^g)$ $(g\in\mathrm{Hom}_r({}_BN, {}_BB)$, $a\in A$,

$n\in N$), *and* $\varphi^{+}(g')=(\varphi(n)a'\to g'(n)a')$ $(g'\in \mathrm{Hom}_l(N_{B'}, B'_{B'})$, $n\in N$, $a'\in A')$.

(4) Ker $\varphi=(\mathrm{Ker}\ f)\cdot N=N\cdot \mathrm{Ker}\ f'$. *Therefore* Ker $f=0$, Ker $f'=0$, *and* Ker $\varphi=0$ *are equivalent conditions. In this case, by identification*, $B=\{a\in A\,|\,aN\subseteq N\}$.

Proof. (1) There hold $N\otimes_{B'}A'\simeq N\otimes_{B'}\mathrm{Hom}({}_AM, {}_AM)\simeq N\otimes_{B'}\mathrm{Hom}({}_AA\otimes_B N, {}_AM)\simeq N\otimes_{B'}\mathrm{Hom}_r({}_BN, {}_B\mathrm{Hom}_r({}_AA, {}_AM))\simeq N\otimes_{B'}\mathrm{Hom}_r({}_BN, {}_BM)\simeq N\otimes_{B'}\mathrm{Hom}_r({}_BN, {}_BB)\otimes_B M\simeq M$ as left B, right A'-modules. To follow the above sequence of isomorphisms, we take $f_i: {}_BN\to {}_BB$ and $n_i\in N$ such that $\sum_i n^{f_i}\cdot n_i=n$ for all $n\in N$. Then, for any $n_0\otimes a'$ in $N\otimes_{B'}A'$, $n_0\otimes a'\mapsto n_0\otimes(m\to ma')$ $(m\in M)\mapsto n_0\otimes(a\otimes n\to a\cdot\varphi(n)a')$ $(a\in A,\ n\in N)\mapsto n_0\otimes(n\to(a\to a\cdot\varphi(n)a'))\mapsto n_0\otimes(n\to \varphi(n)a')\mapsto \sum_i n_0\otimes f_i\otimes\varphi(n_i)a'\mapsto\sum_i(n_0^{f_i}\cdot n_i)a'=\varphi(n_0)a'$. (2) $\mathrm{Hom}_r({}_BN, {}_BB)\otimes_B A\simeq \mathrm{Hom}_r({}_BN, {}_BB)\otimes_B\mathrm{Hom}_r({}_BB, {}_BA)\simeq\mathrm{Hom}_r({}_BN, {}_BA)\simeq\mathrm{Hom}_r({}_BN, {}_B\mathrm{Hom}_r({}_AA, {}_AA))\simeq\mathrm{Hom}_r({}_AA\otimes_B N, {}_AA)\simeq\mathrm{Hom}_r({}_AM, {}_AA)$ as left B', right A-module. If we follow the above sequence of isomorphisms we obtain the required one. (3) is easily checked. In fact, for any g in $\mathrm{Hom}_r({}_BN, {}_BB)$, $\alpha(g)=(n\to(n_1\to(n_1)^g\cdot n))$ $(n, n\in N)$. (4) Since $\varphi(\mathrm{Ker}\ f)N)=(\mathrm{Ker}\ f)\varphi(N)=0$, we have $(\mathrm{Ker}\ f)N\subseteq\mathrm{Ker}\ \varphi$. On the other hand, Ker φ is written as Ker $\varphi=\mathfrak{a}N$ with an ideal $\mathfrak{a}$ of B. Then $0=\varphi(\mathfrak{a}N)=\mathfrak{a}\cdot\varphi(N)$ means $0=\mathfrak{a}\cdot\varphi(N)A'=\mathfrak{a}M$. Thus $\mathfrak{a}\subseteq\mathrm{Ker}\ f$. To prove the latter half, let $aN\subseteq N$ for a in A. Then there is an element b in B such that $an=bn$ for all n in N. Then $0=(a-b)NA'=(a-b)M$, and so $a=b$.

Let both A/B and A'/B' be ring extensions. If there are Morita modules ${}_AM_{A'}$ and ${}_BN_{B'}$ such that $A\otimes_B N\simeq M$ as left A, right B'-modules, we write $A/B\sim A'/B'$. In this case, let $\sigma: A\otimes N\simeq M$ as left A, right B'-modules and let φ be the B-B'-homomorphism from N to M defined by $\varphi(n)=(1\otimes n)$ $(n\in N)$. Then $\sigma(a\otimes n)=a\cdot\sigma(1\otimes n)=a\cdot\varphi(n)$ $(a\in A,\ n\in N)$, and so $N\otimes_{B'}A'\simeq M$ as left B, right A'-modules, $n\otimes a'\mapsto\varphi(n)a'=\sigma(1\otimes n)a'$, by Lemma 3. 1.

Proposition 3. 2. $\sim$ *is an equivalence relation.*

Proof. By Lemma 3. 1 (2), $\sim$ is symmetric, because both ${}_{B'}\mathrm{Hom}_r({}_BN, {}_BB)_B$ and ${}_{A'}\mathrm{Hom}({}_AM, {}_AA)$ are Morita modules. Therefore, it remains to prove that $\sim$ is transitive. Let $A/B\sim A'/B'$ and $A'/B'\sim A''/B''$ by $({}_BN_{B'}, {}_AM_{A'})$ and $({}_{B'}N'_{B''}, {}_{A'}M'_{A''})$, respectively. Then, $A\otimes_B N\otimes_{B'}N'\simeq M\otimes_{B'}N'\simeq M\otimes_{A'}A'\otimes_{B'}N'\simeq M\otimes_{A'}M'$ as left A, right B'-modules. Thus $A/B\sim A''/B''$ by $({}_BN\otimes_{B'}N_{B''}, {}_AM\otimes_{A'}M'_{A''})$.

Remark 1. Let both A and A' be rings. Then, $id_A\sim id_{A'}$ if and only if there is a Morita module ${}_AM_{A'}$.

Remark 2. Let $f: B\to A$ and $f': B'\to A'$ be ring homomorphisms, and let $\sigma: B'\to B$ and $\tau: A'\to A$ be ring isomorphisms rendering the diagram

$$\begin{array}{ccc} B' & \xrightarrow{f'} & A' \\ \sigma\big\downarrow & & \big\downarrow\tau \\ B & \xrightarrow{f} & A \end{array}$$ commutative. Then ${}_A A\otimes_B Bu_{\sigma_{B'}}\simeq {}_A Au_{\tau B'}$, $a\otimes bu_\sigma\mapsto a\cdot f(b)u_\tau$ $(a\in A,\ b\in B)$. Thus $f\sim f'$.

Let $A\supseteq B$ be rings. By $\mathfrak{R}(A/B)$, we denote the lattice of all B-B-submodules of A. Then $\mathfrak{R}(A/B)$ is a semi-group with respect to multiplication. And B is the identity, and $\{0\}$ is the zero element of a semi-group $\mathfrak{R}(A/B)$.

Proposition 3.3. *Let $A\supseteq B$ and $A'\supseteq B'$ be rings, and let ${}_A M_{A'}\supseteq {}_B N_{B'}$ be Morita modules such that $M=A\otimes_B N=N\otimes_{B'}A'$. Then the following hold.*

(1) *There is a lattice and a semi-group isomorphism $\mathfrak{R}(A/B)\simeq\mathfrak{R}(A'/B')$, $X\mapsto X'$ such that $XN=NX'$. This induces a group isomorphism $\mathfrak{G}(A/B)\simeq\mathfrak{G}(A'/B')$.*

(2) *Under the above isomorphism, the set of all intermediate rings of A/B and the set of all intermediate rings of A'/B' correspond to each other. If $T\leftrightarrow T'$ then $A/T\sim A'/T'$ and $T/B\sim T'/B'$.*

(3) *Under the above isomorphism, the set of all B-T-submodules of A and the set of all B'-T'-submodules of A' correspond to each other. Let ${}_B Y_T\leftrightarrow {}_{B'}Y'_{T'}$. Then Y_T is right invertible in A if and only if so is $Y'_{T'}$ in A'.*

(4) *There is a ring isomorphism $\mathrm{V}_A(B)\simeq\mathrm{V}_{A'}(B')$, and this induces $\mathrm{V}_A(A)\simeq\mathrm{V}_{A'}(A')$ and $\mathrm{V}_B(B)\simeq\mathrm{V}_{B'}(B')$.*

(5) *Under the correspondence in (2), $B\cdot\mathrm{V}_A(B)\leftrightarrow B'\cdot\mathrm{V}_{A'}(B')$, and $\mathrm{V}_A(\mathrm{V}_A(B))\leftrightarrow\mathrm{V}_{A'}(\mathrm{V}_{A'}(B'))$.*

Proof. (1)Since ${}_B N_{B'}$ is a Morita module, the mapping $X_B\to X\otimes_B N_{B'}$ is a bijection from the set of all right B-submodules of A to the set of all right B'-submodules of M, and this induces a bijection from the set of all B-B-submodules of A to the setof all B-B'-submodules of M. The remainder is obvious. (2) For a B-B-submodule T of A, T is an intermediate ring of A/B if and only if $T\supseteq B$ and $T\cdot T=T$. Thus the first half is evident. To prove the latter half, it suffices to prove that ${}_T TN_{T'}=NT'$ is a Morita module. Evidently ${}_T TN\sim{}_T T$. Put $T''=\mathrm{End}_r({}_T TN)$. Then $T'\subseteq T''$. However, $TN=N\otimes_{B'}T''$ by Lemma 3.1(1), and so $N\otimes_{B'}(T''/T')=0$. Thus $T''/T'=0$, that is, $T''=T'$. (3) Let $X\leftrightarrow X'$ under the correspondence in (1). If $XT=X$ then $NX'=XN=XTN=XNT'=NX'T'$. Hence $X'=X'T'$. This prove the first half. The remainder is easily seen. (4) Let c be in $\mathrm{V}_A(B)$. Then the mapping $a\otimes n\to ac\otimes n$ is a left A, right B'-endomorphism of M, and hence there is a unique $c'\in\mathrm{V}_{A'}(B')$ such that $ac\otimes n=$

$(a\otimes n)c'$ for any $a\in A$ and any $n\in N$. Then the mapping $c\mapsto c'$ is evidently a ring isomorphism from $\mathrm{V}_A(B)$ to $\mathrm{V}_{A'}(B')$. Evidently this induces a ring isomorphism $\mathrm{V}_B(B)\simeq\mathrm{V}_{B'}(B')$ (cf. Lemma 3. 1 (4)). If c is in $\mathrm{V}_A(A)$, then $can=acn=anc'$ for any $a\in A$ and any $n\in N$, so that $cm=mc'$ for any m in M. Thus, as is well known, $\mathrm{V}_A(A)\simeq\mathrm{V}_{A'}(A')$. (5) By making use of (4), $\mathrm{V}_A(B)BN=\mathrm{V}_A(B)N=N\mathrm{V}_{A'}(B')=NB'\mathrm{V}_{A'}(A')$. Thus $\mathrm{V}_A(B)B\leftrightarrow\mathrm{V}_{A'}(B')B'$. Let $\mathrm{V}_A(\mathrm{V}_A(B))\leftrightarrow T'$, and let t' be in T'. For any n in N, $nt'\in\mathrm{V}_A(\mathrm{V}_A(B))N$, and hence nt' is written as $nt'=\sum_i a_in_i$ $(a_i\in\mathrm{V}_A(\mathrm{V}_A(B))N,\ n_i\in N)$. Then, for any c' in $\mathrm{V}_{A'}(B')$, $(nt')c'=\sum_i a_in_ic'=\sum_i a_icn_i=c\sum_i a_in_i=c(nt')=nc't'$, where $c\to c'$ under the isomorphism in (4). Thus $0=AN(t'c'-c't')=M(t'c'-c't')$. Hence $t'c'=c't'$ for all c' in $\mathrm{V}_{A'}(B')$, that is, $t'\in\mathrm{V}_{A'}(\mathrm{V}_{A'}(B'))$. This completes the proof.

From Prop. 3. 3, we obtain at once the following

Theorem 3. 4. *Let $A\supseteq B$ and $A'\supseteq B'$ be rings such that $A/B\sim A'/B'$, and let G be a group. Let $\{U\}$ be the set of all monomorphisms from G to $\mathfrak{G}(A/B)$ such that $A=\sum_{\sigma\in G}\oplus U_\sigma$, and $\{U'\}$ the set of all monomorphisms from G to $\mathfrak{G}(A'/B')$ such that $A'=\sum_{\sigma\in G}\oplus U'_\sigma$. Then there are a group isomorphism $\mathfrak{G}(A/B)\simeq\mathfrak{G}(A'/B')$ and a bijection $U\mapsto U'$ from $\{U\}$ to $\{U'\}$ such that the diagram*

$$\begin{tikzcd} & \mathfrak{G}(A/B) \arrow[dd] \\ G \arrow[ur, "U"] \arrow[dr, "U'"'] & \\ & \mathfrak{G}(A'/B') \end{tikzcd}$$

is commutative.

Let $A\supseteq B$ and $A'\supseteq B'$ be rings such that $A/B\sim A'/B'$, and G a finite group. Let ${}_AM_{A'}\supseteq{}_BN_{B'}$ be Morita modules such that $M=A\otimes_BN=N\otimes_{B'}A'$.

Assume that there is a group monomorphism $u:\ G\to\mathrm{Aut}(A/B)$ such that A/B is a G-Galois extension, and set $\varDelta=\mathrm{End}_l(A_B)$. Then $\mathrm{End}_r({}_\varDelta A)\simeq B$, canonically. Since $M=A\otimes_BN$, M may be considered as a left $\varDelta$-module. If $\delta M=0$ for δ in $\varDelta$, then $0=\delta(A)NA'=\delta(A)M=0$, and so $\delta=0$. Therefore ${}_\varDelta M$ is faithful. If f is in $\mathrm{End}_r({}_\varDelta M)$, then there is an element a' in A' such that $m^f=ma'$ for any m in M. Now, $N\simeq\mathrm{Hom}_r({}_\varDelta A,{}_\varDelta A\otimes_BN)$, $n\mapsto(a\to a\otimes n)$ $(n\in N,\ a\in A)$, because ${}_BN|{}_BB$, and $\mathrm{End}_r({}_\varDelta A)=B$. Therefore

$\mathrm{Hom}_r({}_\varDelta A\otimes_BN,{}_\varDelta A\otimes_BN)\simeq\mathrm{Hom}_r({}_BN,{}_B\mathrm{Hom}_r({}_\varDelta A,{}_\varDelta A\otimes_BN))\simeq\mathrm{Hom}_r({}_BN,{}_BN)\simeq B'$. Hence $\mathrm{End}_r({}_\varDelta M)\simeq B'$, canonically. $M_{B'}=A\otimes_BN_{B'}|B\otimes_BN_{B'}\simeq N_{B'}|B'_{B'}$, and so $M\otimes_{B'}A'_{A'}|B'\otimes_{B'}A'_{A'}\simeq A'_{A'}|M_{A'}$. And $A\otimes_BA_A|B\otimes_BA_A\simeq A_A$. Applying Th. 1. 5 to both ${}_{\varDelta/A}M_{A'/B'}$ and ${}_{\varDelta/A}A_{A/B}$, we obtain group isomorphisms $\mathfrak{G}(\varDelta/A)\simeq\mathrm{Aut}(A'/B')$ and $\mathfrak{G}(\varDelta/A)\simeq\mathrm{Aut}(A/B)$. Hence $\mathrm{Ant}(A/B)\simeq\mathrm{Aut}(A'/B')$. Then Th. 2. 12 and Th. 2. 13 yields directly the following

Theorem 3.5. *Let $A \supseteq B$ and $A' \supseteq B'$ be rings such that $A/B \sim A'/B'$, and let G be a finite group. Assume that A/B is a G-Galois extension. Then A'/B' is also a G-Galois extension. Further there are a group isomorphism Aut $(A/B) \simeq Aut(A'/B')$ and a bijection $u \mapsto u'$ from the set of all monimorphisms $u: G \to Aut(A/B)$ such that A/B is G-Galois to the set of all monomorphisms $u': G \to Aut(A'/B')$ such that A'/B' is G-Galois rendering the diagram* $\quad Aut(A/B)$ *commutative.*

$$\begin{tikzcd}
 & Aut(A/B) \arrow[d] \\
G \arrow[ur, "u"] \arrow[dr, "u'"'] & \\
 & Aut(A'/B')
\end{tikzcd}$$

Remark. In fact the group isomorphism $\mathrm{Aut}(A/B) \simeq \mathrm{Aut}(A'/B')$, $\sigma \to \sigma'$ is the following: $\sum_i \sigma(x_i) n_i = \sum_j u_j \cdot \sigma'(x'_j)$ for any $\sum_i x_i n_i = \sum_j u_j x'_j$ ($x_i \in A$, $x'_j \in A'$, n_i, $u_j \in N$) in M, where both ${}_B N_{B'} \subseteq {}_A M_{A'}$ are Morita modules such that $M = A \otimes_B N = N \otimes_{B'} A'$.

The following lemma is easily checked.

Lemma 3.6. *Let A/T be a ring extension, and let both ${}_A M$ and ${}_A N$ be left A-modules such that ${}_A M | {}_A N$, and L a left T-module. Put* $\mathrm{End}_r({}_A M) = A^*$, $\mathrm{End}_r({}_A N) = A^+$, *and* $\mathrm{End}_r({}_T L) = T'$. *Let $f_i: {}_A M \to {}_A N$ and $g_i: {}_A N \to {}_A M$ ($i = 1, \cdots, n$) be A-homomorphisms such that $\sum_i m^{f_i g_i} = m$ for all $m \in M$.*

(1) $\mathrm{Hom}_r({}_T L, {}_T N) \otimes_{A^+} \mathrm{Hom}_r({}_A N, {}_A M) \simeq \mathrm{Hom}_r({}_T L, {}_T M)$, $k \otimes g \mapsto kg$, *as left T', right A^*-modules. The inverse of this isomorphism is the mapping $h \mapsto \sum_i h f_i \otimes g_i$.*

(2) $\mathrm{Hom}_r({}_T L, {}_T M) \simeq \mathrm{Hom}_l(\mathrm{Hom}_r({}_A M, {}_A N)_{A^+}, \mathrm{Hom}_r({}_T L, {}_T N)_{A^+})$, $h \mapsto (f \to hf)$ $(f \in \mathrm{Hom}_r({}_A M, {}_A N))$, *as left T', right A^*-modules. The inverse of his isomorphism is the mapping $\phi \mapsto \sum_i {}^{\phi} f_i \cdot g_i$.*

(3) $\mathrm{Hom}_r({}_A M, {}_A N) \otimes_{A^+} \mathrm{Hom}_r({}_T N, {}_T L) \simeq \mathrm{Hom}_r({}_T M, {}_T L)$, $f \otimes k \mapsto fk$, *as left A^*, right T'-modules. The inverse of this isomorphism is the mapping $h \mapsto \sum_i f_i \otimes g_i h$.*

(4) $\mathrm{Hom}_r({}_T M, {}_T L) \simeq \mathrm{Hom}_r({}_{A^+}\mathrm{Hom}_r({}_A N, {}_A M), {}_{A^+}\mathrm{Hom}_r({}_T N, {}_T L))$, $h \mapsto (g \to gh)$, *as left A^*, right T'-modules. The inverse of this isomorphism is the mapping $\varphi \mapsto \sum_i f_i \cdot g_i^{\varphi}$.*

By Lemma 3.6 (1) and (3), we obtain an isomorphism:
$\mathrm{Hom}_r({}_A M, {}_A N) \otimes_{A^+} \mathrm{Hom}_r({}_T N, {}_T L) \otimes_{T'} \mathrm{Hom}_r({}_T L, {}_T N) \otimes_{A^+} \mathrm{Hom}_r({}_A N, {}_A M) \to \mathrm{Hom}_r({}_T M, {}_T L) \otimes_{T'} \mathrm{Hom}_r({}_T L, {}_T M)$, $f \otimes k \otimes k' \otimes g \mapsto fk \otimes k'g$. Putting ${}_T L = {}_T N$, we obtain the following

Corollary 1. *Under the same notations and assumptions as in Lemma 3.6,* $\mathrm{Hom}_r({}_A M, {}_A N) \otimes_{A^+} T^+ \otimes_{A^+} \mathrm{Hom}_r({}_A N, {}_A M) \simeq T^*$, $f \otimes t^+ \otimes g \mapsto f \cdot t^+ \cdot g$, *where*

$T^+ = \mathrm{End}_r\,({}_T N)$, *and* $\mathrm{End}_r\,({}_T M) = T^*$.

Corollary 2. *Under the same notations and assumptions as in Cor.* 1, $T^+ \otimes_{A^+} \mathrm{Hom}_r\,({}_A N,\ {}_A M) \simeq \mathrm{Hom}_r\,({}_T N,\ {}_T M)$, $t^+ \otimes g \mapsto t^+ \cdot g$, *as left* T^+, *right* A^*-*modules.* $\mathrm{Hom}_r\,({}_A M,\ {}_A N) \otimes_{A^+} T^+ \simeq \mathrm{Hom}_r\,({}_T M,\ {}_T N)$, $f \otimes t^+ \mapsto f \cdot t^+$, *as left* A^*, *right* T^+-*modules.*

Proof. Put ${}_T L = {}_T N$ in Lemma 3. 6.

Proposition 3. 7. *For two ring extensions* A/B *and* A'/B', *the following are equivalent*:

(1) $A/B \sim A'/B'$, *and* $A_B \sim B_B$.

(2) *For some* B^*/A^*, *there are Morita modules* ${}_{B^*/A^*}M_{A/B}$ *and* ${}_{B^*/A^*}M'_{A'/B'}$.

Proof. (1) $\Rightarrow$(2) In this case, $B \to A$ is a monomorphism. Put $\mathrm{End}_l\,(A_B) = \Delta$. Then, by hypothesis, ${}_{\Delta/A}A_{A/B}$ is a Morita modules. Let ${}_B N_{B'}$ be a Morita module such that $\mathrm{End}_r\,({}_A A \otimes_B N)/B' = A'/B'$. Then ${}_{\Delta/A}A \otimes_B N_{A'/B'}$ is a Morita module. (2) $\Rightarrow$(1) By Cor. 2 to Lemma 3. 6, $A \otimes_B \mathrm{Hom}\,({}_{B^*}M,\ {}_{B^*}M')$ $\simeq \mathrm{Hom}_r\,({}_{A^*}M,\ {}_{A^*}M')$, as left A, right B'-modules. As is well known, both ${}_B\mathrm{Hom}_r\,({}_{B^*}M,\ {}_{B^*}M')_{B'}$ and ${}_A\mathrm{Hom}\,({}_{A^*}M,\ {}_{A^*}M')_{A'}$ are Morita modules. Hence $A/B \sim A'/B'$. As $A_B \sim M_B \sim B_B$, we have $A_B \sim B_B$.

Let both A/B and B^*/A^* be ring extensions. If there is a Morita module ${}_{B^*/A^*}M_{A/B}$, then we write $B^*/A^* \square A/B$. Note that $A_B \sim B_B$, in this case.

Proposition 3. 8. *If* $A'/B' \sim A/B \square B^+/A^+$ *then* $A'/B' \square B^+/A^+$.

Proof. Let both ${}_{A'}M_A$ and ${}_{B'}N_B$ be Morita modules such that ${}_{B'}N \otimes_B A_A \simeq {}_{B'}M_A$, and ${}_{A/B}W_{B^+/A^+}$ a Morita module. Then ${}_{A'}M \otimes_A W_{A^+}$ is a Morita module and ${}_{B'}M \otimes_A W_{A^+} \simeq {}_{B'}N \otimes_B A \otimes_A W_{A^+} \simeq {}_{B'}N \otimes_B W_{A^+}$. Therefore $\mathrm{End}\,({}_{B'}M \otimes_A W)/A^+ \simeq \mathrm{End}_r\,({}_B W)/A^+ = B^+/A^+$. In this sense, ${}_{A'/B'}M \otimes_A W_{B^+/A^+}$ is a Morita module, because ${}_{B'}M \otimes_A W \simeq {}_{B'}N \otimes_B W \sim {}_{B'}N \otimes_B B \simeq {}_{B'}N \sim {}_{B'}B'$.

Proposition 3. 9. *Let* R/S *be a Frobenius extension such that* ${}_S S | {}_S R$. *Then, for any ring extension* S'/R', $R/S \square S'/R'$ *and* $S'/R' \square R/S$ *are equivalent.*

Proof. Since ${}_R\mathrm{Hom}_r\,({}_S R,\ {}_S S)_S \simeq {}_R R_S$, it follows that $\mathrm{End}_r\,({}_R\mathrm{Hom}_r\,({}_S R,\ {}_S S))/S \simeq R/S$. We set $\Delta = \mathrm{End}_r\,({}_S R)$ and $R_1 = \mathrm{End}_r\,({}_R\mathrm{Hom}_r\,({}_S R,\ {}_S S))$. Then ${}_{\Delta/R}\mathrm{Hom}_r\,({}_S R,\ {}_S S)_{R_1/S}$ is a Morita module. Consequently $\Delta/R \square R_1/S \simeq R/S$. Now, assume that $S'/R' \square R/S$. Then $S'/R' \sim \Delta/R$ by Prop. 3. 7, and so $R/S \square \Delta/R \sim S'/R'$. By Prop. 3. 8, we obtain $R/S \square S'/R'$, as desired.

Let ${}_B N | {}_B B$, and $\mathrm{End}_r\,({}_B N) = B'$. Let φ be a ring homomorphism from B to A. Then ${}_A A \otimes_B N | {}_A A$. Putting $\mathrm{End}_r\,({}_A A \otimes_B N) = A'$, we obtain a ring homomorphism φ' : $B' \to A'$, where $(a \otimes n) \cdot \varphi'(b') = a \otimes (nb')$ $(a \in A,\ n \in N,\ b' \in B')$.

As $A'=\mathrm{End}_r({}_AA\otimes_BN)\simeq\mathrm{Hom}_r({}_BN, {}_B\mathrm{Hom}({}_AA, {}_AA\otimes_BN))\simeq\mathrm{Hom}_r({}_BN, {}_BA\otimes_BN)$ $\simeq\mathrm{Hom}_r({}_BN, {}_BB)\otimes_BA\otimes_BN$, we know $\mathrm{Hom}_r({}_BN, {}_BB)\otimes_BA\otimes_BN$ is isomorphic to A' as a B'-B'-module, by the correspondence $f\otimes a\otimes n\mapsto(x\otimes u\mapsto x^f\cdot a\otimes n)$ ($f\in\mathrm{Hom}_r({}_BN, {}_BB)$, a, $x\in A$, n, $u\in N$). Consequently $\mathrm{Hom}_r({}_BN, {}_BB)\otimes_BA\otimes_BN$ is a ring by the multiplication $(f\otimes a\otimes n)(f'\otimes a'\otimes n')=f\otimes a\cdot n^{f'}\cdot a'\otimes n'$, and we have a commutative diagram of ring homomorphisms: $B'=$ $\mathrm{Hom}_r({}_BN, {}_BB)\otimes_B\mathrm{Hom}({}_BB, {}_BN)\xrightarrow{\alpha}\mathrm{Hom}_r({}_BN, {}_BB)\otimes_BA\otimes_BN$, where $\alpha(f\otimes g)=$

$$\begin{array}{ccc} \varphi' & & \\ & \searrow \quad \swarrow & \\ & A' & \end{array}$$

$f\otimes 1\otimes 1^g$. Let T/B be another ring extension, and put $\mathrm{End}_r({}_TT\otimes_BN)=T'$. Then there is a homomorphism $k\mapsto k'$ from $\mathrm{Hom}_l({}_BA_B, {}_BT_B)$ to $\mathrm{Hom}_l({}_{B'}A'_{B'}, {}_{B'}T'_{B'})$ such that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_r({}_BN, {}_BB)\otimes_BA\otimes_BN & \to & {}_{B'}A'_{B'} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_r({}_BN, {}_BB)\otimes T\otimes_BN & \to & {}_{B'}T'_{B'} \end{array}$$

is commutative. If k is a monomorphism (resp. an epimorphism), then so is k'. To be easily seen, if k is a ring homomorphism from A/B to T/B then k' is also a ring homomorphism from A'/B' to T'/B'. Therefore, if k is a ring isomorphism, then so is k'. Evidently, if $A/B=T/B$, then the correspondence $k\mapsto k'$ is a ring homomorphism from $\mathrm{End}_l({}_BA_B)$ to $\mathrm{End}_l({}_{B'}A'_{B'})$. As $A'\simeq\mathrm{Hom}_r({}_BN, {}_BB)\otimes_BA\otimes_BN$ as B'-B'-modules, we have a B-B'-isomorphism $N\otimes_{B'}A'\simeq N\otimes_{B'}\mathrm{Hom}_r({}_BN, {}_BB)\otimes_BA\otimes_BN$. Let π be the canonical homomorphism $n\otimes f\mapsto n^f$ from $N\otimes_{B'}\mathrm{Hom}_r({}_BN, {}_BB)$ to B. Then we have a commutative diagram:

$$\begin{array}{ccccc} N\otimes_{B'}A' & \to & N\otimes_{B'}\mathrm{Hom}({}_BN, {}_BB)\otimes_BA\otimes_BN & \xrightarrow{\pi\otimes 1} & A\otimes_BN \\ \downarrow{\scriptstyle 1\otimes k'} & & \downarrow{\scriptstyle 1\otimes 1\otimes k\otimes 1} & & \downarrow{\scriptstyle k\otimes 1} \\ N\otimes_{B'}T' & \to & N\otimes_{B'}\mathrm{Hom}({}_BN, {}_BB)\otimes_BT\otimes_BN & \xrightarrow{\pi\otimes 1} & T\otimes_BN \end{array}$$

Therefore, if ${}_BN_{B'}$ is a Morita module, there is a bijection $k\mapsto k'$ from the set of all B-ring homomorphisms from A to T to the set of all B'-ring homomorphisms from A' to T' such that the diagram

$$\begin{array}{ccc} N\otimes_{B'}A' & \xrightarrow{\beta} & A\otimes_BN \\ \downarrow{\scriptstyle 1\otimes k'} & & \downarrow{\scriptstyle k\otimes 1} \\ N\otimes_{B'}T' & \longrightarrow & T\otimes_BN \end{array}$$

is commutative, where $\beta(n\otimes a')=(1\otimes n)a'$.

Proposition 3.10. *Let A/B be a Frobenius extension with a Frobenius homomorphism h (which acts on the right side), and let ${}_BN_{B'}$ be a Morita module. Put $\mathrm{Hom}_r({}_BN, {}_BB)\otimes_BA\otimes_BN=A'$. Then A'/B' is a Frobenius extension with a Frobenius homomorphism $1\otimes h\otimes 1$.*

Proof. That ${}_{B'}A'|_{B'}B'$ will be easily seen. Since ${}_AA_B \simeq {}_A\mathrm{Hom}_r({}_BA, {}_BB)_B$, $a \mapsto ah$ $(a \in A)$, we have a sequence of isomorphisms: $A' = \mathrm{Hom}_r({}_BN, {}_BB) \otimes_B A \otimes_B N \simeq \mathrm{Hom}_r({}_BN, {}_BB) \otimes_B \mathrm{Hom}_r({}_BA, {}_BB) \otimes_B N \simeq \mathrm{Hom}_r({}_BN, {}_BB) \otimes_B \mathrm{Hom}_r({}_BA, {}_BN) \simeq \mathrm{Hom}_r({}_BN, {}_B\mathrm{Hom}_r({}_BA, {}_BN)) \simeq \mathrm{Hom}_r({}_BA \otimes_B N, {}_BN)$. If we follow the above sequence of isomorphisms, we obtain an isomorphism $A' \simeq \mathrm{Hom}_r({}_BA \otimes_B N, {}_BN)$, $f \otimes a \otimes n \mapsto (x \otimes u \to (x \cdot u^f \cdot a)^h \cdot n)$ $(f \in \mathrm{Hom}_r({}_BA, {}_BN),\ a, x \in A,\ n, u \in N)$. Further, since ${}_{B'}\mathrm{Hom}_r({}_BN, {}_BB)_B$ is a Morita module, $\mathrm{Hom}_r({}_BA \otimes {}_BN, {}_BN) \simeq \mathrm{Hom}_r({}_{B'}\mathrm{Hom}_r({}_BN, {}_BB) \otimes {}_BA \otimes {}_BN, {}_{B'}B')$ canonically. Thus $A' \simeq \mathrm{Hom}_r({}_{B'}A', {}_{B'}B'), f \otimes a \otimes n \mapsto (f \otimes a \otimes n)(1 \otimes h \otimes 1)$, as desired.

§ 4. Splitting property of crossed products.

Let $\Omega/A = \sum_{\sigma \in G} \oplus U_\sigma$ be a generalized crossed product of A with a finite group G, and C the center of A. Put $\mathrm{End}_r({}_A\Omega) = A^*$. Then A^*/Ω is a finite G-Galois extension (Th. 2. 13), where Ω is identified with all right multiplications by elements of Ω. Putting $\Delta = \mathrm{End}_l(A^*_\Omega)$, Δ/A^* is a trivial crossed product. Then, by Prop. 3. 7, $\Omega/A \sim \Delta/A^*$. Let p_σ be the projection from Ω to U_σ. Then $p_\sigma \in A^*$, $\sum_\sigma p_\sigma = 1$, and $p p_\tau = \delta_{\sigma,\tau} p_\sigma$ for all σ, τ in G. If we put $T = \sum_{\sigma \in G} Ap_\sigma$, then T is a subring of A^* containing A (=the set of all right multiplications by elements of A). Evidently T is isomorphic to the ring of $(G:1)$ copies of A, and it is easily seen that $T = \{x \in A^* \ (U_\sigma)x \subseteq U_\sigma$ for all $\sigma \in G\}$. Let $(A=)\, U_\sigma U_{\sigma^{-1}} \ni 1 = \sum_i u_{\sigma,i} v_{\sigma,i} (u_{\sigma,i} \in U_\sigma,\ v_{\sigma,i} \in U_{\sigma^{-1}})$. Then, for any $z = \sum_\rho z_\rho (z_\rho \in U_\rho)$ in Ω, we have $z^{\sigma(p_\tau)} = \sum_i u_{\sigma,i}(v_{\sigma,i}z)^{p_\tau} = \sum_\rho \sum_i u_{\sigma,i}(v_{\sigma,i}z_\rho)^{p_\tau} = \sum_i u_{\sigma,i} v_{\sigma,i} z_{\sigma\tau} = z_{\sigma\tau} = (z)^{p_{\sigma\tau}}$. Thus $\sigma(p_\tau) = p_{\sigma\tau}$, and hence the G-Galois extension A^*/Ω induces a trivial G-Galois extension T/A. Therefore $A^* = \Omega \otimes_A T = T \otimes_A \Omega$ and $\Omega \cap T = A$ (cf. [16; Th. 5. 1]).

$$\begin{tikzcd}[column sep=small, row sep=small] & A^* \arrow[d, dash] \\ \Omega \arrow[ur, dash] \arrow[d, dash] & T \\ A \arrow[ur, dash] & \end{tikzcd}$$

For any σ, τ in G, $(U_\tau) U_\sigma T U_{\sigma^{-1}} \subseteq U_\tau$ holds, and so $U_\sigma T U_{\sigma^{-1}} \subseteq T$ for all σ in G. Thus $U_\sigma T = TU_\sigma$ for all σ in G. Hence $A^*/T = \sum_{\sigma \in G} \oplus (T \otimes_A U_\sigma)$ is a generalized crossed product of T with G.

In the sequel, we fix a group homomorphism from G to $\mathrm{Aut}(A)$ (which may be trivial), and assume that there are invertible elements u_σ of Ω such that $U_\sigma = Au_\sigma$, $u_\sigma x = \sigma(x) u_\sigma (x \in A)$. Put $A_1 = \{\sum_\sigma \sigma^{-1}(x) p_\sigma | x \in A\}$. Then, for any τ in G, $\tau(\sum_\sigma \sigma^{-1}(x) p_\sigma) = \sum_\sigma \sigma^{-1}(x) p_{\tau\sigma} = \sum_\rho \rho^{-1}\tau(x) p_\rho$. Therefore the correspondence $x \mapsto \sum_\sigma \sigma^{-1}(x) p_\sigma$ $(x \in A)$ is a ring isomorphism and G-admissible. Put $A^G = B$. Then, as is easily seen $A \cap A_1 = B$. Therefore $A = B$ and $A = A_1$ are equivalent. Let $u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau}$. Then each $a_{\sigma,\tau}$ is an invertible element of C. As $a_{\sigma,\tau} a_{\sigma\tau,\rho} = {}^\sigma a_{\tau,\rho} a_{\sigma,\tau\rho}$, we have $a_{\tau,\rho} = \sum_\sigma ({}^\sigma a_{\tau,\rho})_l p_\sigma = (\sum_\sigma (a_{\sigma,\tau})_l p_\sigma)$

$(\sum_{\omega}(a_{\omega\tau,\rho})_l p_{\omega})(\sum_{\nu}(a_{\nu,\tau\rho}^{-1})_l p_{\nu}$, where x_l means the left multiplication by an element x of C. Put $f_{\tau}=\sum_{\sigma}(a_{\sigma,\tau})_l p_{\sigma}$. Then each f_{τ} is an invertible element of the center of T. Since $u_{\tau}p_{\mu}u_{\tau}^{-1}=p_{\mu\tau^{-1}}$, we have $u_{\tau}f_{\rho}u_{\tau}^{-1}=\sum_{\sigma}(a_{\sigma,\rho})_l p_{\sigma\tau^{-1}}=\sum_{\omega}(a_{\omega\tau,\rho})_l p_{\omega}$. Hence $a_{\tau,\rho}=f_{\tau}\cdot {}^{\tau}f_{\rho}f_{\tau\rho}^{-1}$, where ${}^{\tau}f_{\rho}=u_{\tau}f_{\rho}u_{\tau}^{-1}$. Finally, for any $\sum_{\sigma}x_{\sigma}p_{\sigma}$ in T, $u_{\tau}(\sum_{\sigma}x_{\sigma}p_{\sigma})u_{\tau}^{-1}=\sum_{\sigma}\tau(x_{\sigma})p_{\sigma\tau^{-1}}$. Therefore it is easily seen that $T^{G}=A_1$. Further, $\sum_{\tau}u_{\tau}p_1u_{\tau}^{-1}=\sum_{\tau}p_{\tau^{-1}}=1$, $\sum_{\sigma}u_{\tau}p_{\sigma}u_{\tau}^{-1}\cdot p_{\sigma}=\sum_{\sigma}p_{\sigma\tau^{-1}}p_{\sigma}=\delta_{1,\tau}$. Thus T/A_1 is a finite G-Galois extension, and $\mathrm{End}(T_{A_1})/T\simeq A^*/T$. This is the splittig property of $\Omega/A=\sum_{\sigma\in G}\oplus Au_{\sigma}$. If A/B is G-Galois, then $T=A\otimes_B A_1=A_1\otimes_B A$ (cf. [16; Th. 5. 1]). Conversely, if $AA_1=T$, then there are x_i, y_i in A such that $\sum_i x_i\sum_{\sigma}\sigma^{-1}(y_i)p_{\sigma}=p_1$. Then $\sum_i x_i\cdot\sigma(y_i)=\delta_{1,\sigma}$. Hence A/B is G-Galois. Similarly, if $A_1A=T$ then A/B is G-Galois. (Cf. Nobusawa [22])

$$\begin{tikzcd}[column sep=small, row sep=small]
 & A^* \arrow[dd, dash] \\
\Omega \arrow[ur, dash] \arrow[dd, dash] & \\
 & T \arrow[dd, dash] \\
A \arrow[ur, dash] \arrow[dd, dash] & \\
 & A_1 \\
B \arrow[ur, dash] &
\end{tikzcd}$$

§ 5.

Theorem 5. 1. *Let G be a finite group, and let $A\supseteq B$ be rings such that $\mathrm{V}_A(B)=\mathrm{V}_A(A)$. We give a group homomorphism from G to $Aut(A/B)$ (which may be trivial), and let $\Delta/A=\sum_{\sigma\in G}\oplus Au_{\sigma}$ be a trivial crossed product of A with G. Then the following are equivalent:*

(i) *$\sum_{\sigma}\sigma(c)=1$ for some c in the center of A.*

(ii) *Δ/A is a separable extension.*

(iii) *If Y is a left Δ, right B-submodule of Δ such that ${}_AY_B|{}_AA_B$, then ${}_{\Delta}Y_B|{}_{\Delta}\Delta_B$.*

(iv) ${}_{\Delta}A_B|{}_{\Delta}\Delta_B$.

Proof. (i) $\Rightarrow$(ii) follows from Cor. to Th. 2. 11. (ii) $\Rightarrow$(iii) As ${}_{\Delta}\Delta_{\Delta}|{}_{\Delta}\Delta\otimes_A\Delta_{\Delta}$, we have ${}_{\Delta}Y_B|{}_{\Delta}\Delta\otimes_A Y_B|{}_{\Delta}\Delta\otimes_A A_B\simeq{}_{\Delta}\Delta_B$. (iii) $\Rightarrow$(iv) Put $\sum_{\sigma}u_{\sigma}=u$. Then $u_{\sigma}u=u$ for all σ in G, and hence Au is a left Δ, right B-submodule of Δ which is isomorphic to A as a left Δ, right B-module. Therefore, by assumption, ${}_{\Delta}A_B|{}_{\Delta}\Delta_B$. (iv) $\Rightarrow$(i) To be easily seen, $\mathrm{V}_A(B)\simeq\mathrm{Hom}_r({}_{\Delta}A_B, {}_{\Delta}\Delta_B)$, $c\mapsto(x\to xuc)$ $(x\in A)$, and $\mathrm{V}_A(B)\simeq\mathrm{Hom}_r({}_{\Delta}\Delta_B, {}_{\Delta}A_B)$, $c\mapsto(\delta\to\delta(c))$ $(\delta\in\Delta)$. Then, ${}_{\Delta}A_B|{}_{\Delta}\Delta_B$ implies that there are $c, \cdots, c_n$; $d, \cdots, d_n$ in $\mathrm{V}_A(B)$ such that $\sum_{\sigma}\sigma(\sum_i c_i d_i)=1$.

Remark. In general, if A/B is G-Galois and $\sum_{\sigma}\sigma(d)=1$ for some d in $\mathrm{V}_A(B)$, then ${}_BB_B|{}_BA_B$, and hence Δ/A is separable (Th. 2. 6 (3)).

Corollary. *Under the same notations and assumptions, the following are equivalent:*

(i) Δ/A *is separable, and if* ${}_AX_B$ *is a submodule of* ${}_AA_B^n$ *for some* n *then* ${}_AX_B|{}_AA_B$.

(ii) *If* ${}_\Delta Y_B$ *is a submodule of* ${}_\Delta\Delta_B^n$ *for some* n, *then* ${}_\Delta Y_B|{}_\Delta\Delta_B$.

Proof. (i) $\Rightarrow$(ii) follows from Th. 5. 1 and the fact that ${}_A\Delta_B\simeq{}_AA_B^g$, where $g=(G:1)$. (ii) $\Rightarrow$(i) It suffices to prove the latter half. Since Δ_A is projective, ${}_\Delta\Delta\otimes_AX_B$ is a submodule of ${}_\Delta\Delta\otimes_AA_B^n$ $(\simeq{}_\Delta\Delta_B^n)$. Then, by assumption, ${}_\Delta\Delta\otimes_AX_B|{}_\Delta\Delta_B$. Since ${}_AA_A|{}_A\Delta_A$, we have ${}_AX_B|{}_A\Delta\otimes_AX_B$. Hence we obtain ${}_AX_B|{}_AA_A$, because ${}_A\Delta_B\simeq{}_AA^g{}_B$.

Lemma 5. 2. *Let* $A\supseteq B$ *be rings, and let* σ *be an automorphism of* A/B *such that* ${}_AAu_{\sigma A}\sim{}_AA_A$. *Then* $\sum_{i=0,\pm1,\pm2,\cdots,\sigma^i}J_1$ *is a commutative subring of* $\mathrm{V}_A(B)$.

Proof. From ${}_AAu_{\sigma A}\sim{}_AA_A$, it follows ${}_AA_A\sim{}_AAu_{\sigma^{-1}A}$, so that ${}_AAu_{\sigma^i A}\sim{}_AA_A$ for all integer i. Thus, by Th. 1. 3, ${}_{\sigma^i}J_1\in\mathfrak{G}(\mathrm{V}_A(B)/C)$ for all integer i, where C is the center of A. For $j\geqslant1$, ${}_{\sigma^j}J_1=({}_\sigma J_1)^j$, ${}_{\sigma^{-j}}J_1=({}_{\sigma^{-1}}J_1)^j$, and ${}_{\sigma^{-1}}J_1={}_1J_\sigma=({}_\sigma J_1)^{-1}$. Therefore it suffices to prove that $xy=yx$ for any x, y in ${}_{\sigma^{\pm1}}J_1$. First, assume that ${}_\sigma J_1=Cu$ for some u. Then u is invertible, and $({}_\sigma J_1)^{-1}=Cu^{-1}$. Therefore R is commutative. Next we return to general case. For any maximal ideal $\mathfrak{p}$ of C, let $\varphi_\mathfrak{p}:\mathrm{V}_A(B)\to\mathrm{V}_A(B)\otimes_CC_\mathfrak{p}$ be the localization. Then ${}_{\sigma^{\pm1}}J\otimes_CC_\mathfrak{p}\in\mathfrak{G}(\mathrm{V}_A(B)\otimes_CC_\mathfrak{p}/C_\mathfrak{p})$, and $({}_\sigma J_1\otimes C_\mathfrak{p})^{-1}={}_{\sigma^{-1}}J_1\otimes C_\mathfrak{p}$. Since $C_\mathfrak{p}$ is a local ring, ${}_\sigma J_1\otimes C_{\mathfrak{p}C_\mathfrak{p}}$ is a free module of rank 1. Therefore, for any x, y in ${}_{\sigma^{\pm1}}J_1$, we have $xy\otimes1=yx\otimes1$ in $\mathrm{V}_A(B)\otimes_CC_\mathfrak{p}$. Hence $xy-yx\in$ Ker $\varphi_\mathfrak{p}$ for all $\mathfrak{p}$. Then, as $\cap_\mathfrak{p}$ Ker $\varphi_\mathfrak{p}=0$, we have $xy=yx$. Hence R is commutative. This completes the proof.

Theorem 5. 3. *Let* $\Omega/A=\sum_{\sigma\in G}\oplus U_\sigma$ *be a generalized crossed product of* A *with a finite group* G. *Assume that* ${}_AU_{\sigma A}\sim{}_AA_A$ *for all* $\sigma\in G$. *Then* Ω/A *is a symmetric extension.*

Proof. Put $A^*=\mathrm{End}_r({}_A\Omega)$. Then ${}_{\Omega/A}\Omega_{A^*/\Omega}$ is a Morita module, and A^*/Ω is a finite G-Galois extension such that ${}_{A^*}A^*\otimes_\Omega A^*_{A^*}\sim{}_{A^*}A_{A^*}$ (Th. 2. 6). Put $\Delta/A^*=\mathrm{End}_l(A^*_\Omega)$. Then $\Delta=\sum_{\sigma\in G}\oplus A^*u_\sigma$, and ${}_{A^*}A^*u_{\sigma A^*}\sim{}_{A^*}A^*_{A^*}$ (Th. 2. 6). Therefore, by Th. 2. 10, we may assume that $\Omega/A=\mathrm{End}_l(A_B)$ for some finite G-Galois extension A/B such that ${}_AA\otimes_BA_A\sim{}_AA_A$. Then $\Omega/A=\sum_{\sigma\in G}\oplus Au_\sigma$ is a trivial crossed product of A with G. Let h be the projection from Ω to A. Then Ω/A is a Frobenius extension with a Frobenius homomorphism h. Therefore it suffices to prove that $h(\delta\omega)=h(\omega\delta)$ for any $\omega\in\Omega$ and any $\delta\in\mathrm{V}_\Omega(A)$ (Prop. 2. 7). Evidently $\mathrm{V}_\Omega(A)=\sum_\sigma\oplus_{\sigma^{-1}}J_1u_\sigma$. Let x be in ${}_{\sigma^{-1}}J_1$. Then $xu_\sigma\cdot yu_{\sigma^{-1}}=x\cdot\sigma(y)=yx$ and $yu_{\sigma^{-1}}\cdot xu_\sigma=y\cdot\sigma^{-1}(x)$ for any y in A. On the other hand, Lemma 5. 2 yields $\sigma^{-1}(x)=x$. Hence Ω/A is a symmetric extension.

Corollary. *Let A/B be G-Galois, and assume that ${}_A A\otimes_B A_A \sim {}_A A_A$. Then A/B is a symmetric extension.*

Proof. Put $\Delta/A=\mathrm{End}_l(A_B)$. Then $\Delta/A=\sum_{\sigma\in G}\oplus Au_\sigma$ is a trivial crossed product such that ${}_A Au_{\sigma A}\sim {}_A A_A$ for all $\sigma\in G$ (Th. 2. 6). Therefore Δ/A is symmetric, and hence A/B is a symmetric extension (Th. 2. 10), because ${}_\Delta\Delta|_\Delta A$.

Finally we note the following

Proposition 5. 4. *Let $\Omega/A=\sum_{\sigma\in G}\oplus U_\sigma$ be a generalized crossed product of A with a group G. If there is a ring homomorphism $\Omega/A\to A/A$, then Ω/A is a group ring (and conversely).*

Proof. Let φ: $\Omega/A\to A/A$. Then each $\varphi(U_\sigma)$ is an ideal, and $A=\varphi(U_\sigma U_{\sigma^{-1}})=\varphi(U_\sigma)\varphi(U_{\sigma^{-1}})$. Hence $\varphi(U)=A$. If $\varphi(x)=0$ for some x in U_σ, then $\varphi(xU_{\sigma^{-1}})=\varphi(x)\varphi(U_{\sigma^{-1}})=0$. On the other hand, as $x\cdot U_{\sigma^{-1}}\subseteq A$, we have $\varphi(x\cdot U_{\sigma^{-1}})=xU_{\sigma^{-1}}$, and so $x=0$. Thus $\varphi|U_\sigma$: $U_\sigma\simeq A$ as A-A-modules. Let $\varphi(u_\sigma)=1$ $(u_\sigma\in U_\sigma)$. Then, to be easily seen, u_σ is an invertible elements of $V_\Omega(A)$ such that $u_\sigma A=U_\sigma$. Each $u_\sigma u_\tau$ is in $U_\sigma U_\tau=U_{\sigma\tau}$, and $\varphi(u_\sigma u_\tau)=\varphi(u_\sigma)\varphi(u_\tau)=1=\varphi(u_{\sigma\tau})$. Hence $u_\sigma u_\tau=u_{\sigma\tau}$. Therefore Ω/A is a group ring of A with G.

§ 6. Correction to: Galois extensions and crossed products.

In the previous paper "Galois extensions and crossed products, J. Fac. Sci. Hokkaido Univ., Ser. I, 20 (1968)", the fact that $a_{\sigma,\tau}$ is in the center was false. Consequently I must eliminate the parts concerning the above error.

From 2 line from the bottom of p. 122 to 3 line from the top of p. 123. From 4 line from the bottom of p. 125 to 6 line from the bottom of p. 126.

Further I replace the proof of Prop. 2. 3 by the proof of Th. 2. 11 in the present paper.

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