ON VALUATIONS OF POLYNOMIAL RINGS
OF MANY VARIABLES

Part two

By

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I published a paper on the same theme as this in Jour. Science. Hokkaido. Univ. Vol. 21, No. 1, in 1970. The following is its continuation. So I want to use in this paper the same symbols that I used in the preceding one as much as possible.

$K$ is a field with a discrete non-archimedean valuation $V_\infty$ and $W$ is a given extension of it to a polynomial ring $K[x, y]$. Then $V_{10} = [V_\infty, V_{10}x = Wx = \mu_1]$ is a valuation of the ring $K[x]$, namely, of its quotient field $K_x$ and in the same way $V_{01} = [V_\infty, V_{01}y = Wy = \nu_1]$ is a valuation of the ring $K[y]$, consequently, of its quotient field $K_y$. And $V_{11} = [V_\infty, V_{11}x = \mu_1, V_{11}y = \nu_1]$ is a valuation of the ring $K[x, y]$ and the last simply augmented valuation $U$ which appears in the series of augmented inductive valuations binding $V_{11}$ and $W$ is decided uniquely by $W$ and the series of continuous $xy$-doubly augmented inductive valuations binding $U$ and $W$ is decided uniquely by $W$. I proved all of them in Part one.

In this paper, first in §11, I make series of continuous simply augmented inductive valuations binding $V_{11}$ and $U$. Next I expect to investigate structures of key-polynomials which produce simply augmented valuations or doubly augmented ones and structures of the residue class rings of $K[x, y]$ by these valuations. But for the sake of it, I found a fact that we can inspect these structures naturally after we can solve the next problem which I name "Principal Problem" in this paper.

Principal Problem

$V_{p_0}$ is a given discrete non-archimedean valuation of $K[x]$ and $V_{q_1}$ is a given one of $K[y]$ and both $V_{p_0}$ and $V_{q_1}$ induce the same valuation $V_\infty$ in $K$.

In this case, the problem is how we should make such valuations $W$ of $K[x, y]$ that induce $V_{p_0}$ in $K[x]$ and $V_{q_1}$ in $K[y]$.

To make such valuations is my chief study in this paper.
§ 11. Series of continuous simply augmented inductive valuations

Among these series, I presented the two simplest ones in Part one; the first one in §8,

\[ V_{11} < V_{21} < \cdots < V_{p_1} < V_{p_2} < \cdots < V_{p_q} = U \]

and the second one in §9

\[ V_{11} < V_{12} < V_{1q_1} < V_{2q_1} < \cdots < V_{p_1q_1} = U \]

In addition to these two, there are many other series of continuous simply augmented inductive valuations binding \( V_{11} \) and \( U \). A method to make one of them is as follows:

If we want to make a continuous \( x \)-simply augmented valuation of \( V_{11} \), first of all we must inspect whether such a polynomial \( f(x) \) of \( x \) for which \( Wh(x) > V_{11} f(x) \) exists. If such a polynomial \( f(x) \) does not exist, then no \( x \)-simply augmented valuation of \( V_{11} \) exists. If such polynomials \( f(x) \) exist, then out of them we select such polynomials \( g(x) \) whose degrees are minimum and next out of these \( g(x) \) we pick up such a polynomial \( \phi_2(x) \) for which \( Wh_2(x) - V_{11} \phi_2(x) \) is minimum and whose leading coefficient is 1.

Then \( \phi_2(x) \) becomes an \( x \)-key polynomial of \( V_{11} \) which is a valuation of the coefficient field \( K_y \) has the valuation \( V_{01} \). We make \( V_{21} \) as follows:

\[ V_{21} = [V_{11}, \ V_{21} \phi_2 = W\phi_2 = \mu_2]. \]

\( V_{21} \) is a continuous \( x \)-simply augmented valuation of \( V_{11} \).

In the same way as above, we continue such processes and we obtain a series of continuous \( x \)-simply augmented inductive valuations

\[ V_{11} < V_{21} < V_{31} < \cdots < V_{s1} \]

Next, if we want to make a continuous \( y \)-simply augmented valuation of \( V_{s1} \), first of all, we must inspect whether such a polynomial \( h(y) \) of \( y \) for which \( Wh(y) > V_{s1} h(y) \) exists. If such a polynomial does not exist, no \( y \)-simply augmented valuation of \( V_{s1} \) exists.

If such a polynomial exists, we make a valuation \( V_{s0} \) which \( V_{s1} \) induces in \( K[x] \). And we select such polynomials \( l(x, y) \) whose degrees with respect to \( y \) are minimum, out of such polynomials \( h(x, y) \) for which \( Wh(x, y) > V_{s1} h(x, y) \). And moreover out of these \( l(x, y) \) we pick up a polynomial \( \phi_2(x, y) \) for which \( Wh_2(x, y) - V_{s1} \phi_2(x, y) \) is minimum and whose coefficient of the term of the highest power of \( y \) is 1. Then \( \phi_2(x, y) \) is a \( y \)-key polynomial of \( V_{s1} \) which is a valuation of the coefficient field \( K_x \) has
the valuation $V_{x_0}$. Let be an expansion of an arbitrary polynomial $k(x, y)$ by $\phi_2(x, y) = \phi_2$

$$k(x, y) = \sum_{i=0}^{n} k_i(x, y) \phi_i$$

, where $\deg_y k_i(x, y) < \deg_y \phi_2$ for $i = 0, 1, 2, \ldots , n$.

Then the function $V_{x_0}$ defined as follows is a continuous $y$-simply augmented valuation of $V_{x_1}$ of the ring $K_x[y]$;

$$V_{x_0}k(x, y) = \min \left[ V_{x_1}k_i(x, y) + iW\phi_2 \right]$$

$V_{x_0}$ is denoted simply as

$$V_{x_0} = [V_{x_1}, V_{x_2}\phi_2 = W\phi_2 = v_2]$$

By repeating such processes we obtain a series of continuous $y$-simply augmented inductive valuations

$$V_{x_1} < V_{x_2} < \ldots < V_{x_t}$$

Next, if we want to make a continuous $x$-simply augmented valuation of $V_{x_t}$, we must inspect whether such a polynomial $f(x)$ of $x$ for which $Wf(x) > V_{x_t}f(x)$ exists. If such a polynomial does not exist, then no $x$-simply augmented valuation of $V_{x_t}$ exists. If such a polynomial $f(x)$ exists, we can make an $x$-simply augmented valuation of $V_{x_t}$. But for the sake of it, as I explained in §4, we must transform the valuation $V_{x_t}$ of the ring $K_x[y]$ into a valuation $V_{x_t'}$ of the ring $K_y[x]$. Namely, first we must make a valuation $V_{x_0'}$ which $V_{x_t}$ induces in $K[y]$. In this case each $y$-key polynomial which appears at each stage on the way while we make a series of continuous $y$-augmented inductive valuations from $V_{x_1}$ to $V_{x_t'}$, may be generally different from the $y$-key polynomials which produced respectively $V_{x_1}, V_{x_2}, \ldots , V_{x_t}$ and $t$ is not always equal to $t'$, as I showed in Example 10. 6. Next we make the valuation $V_{x_t}$ as the last valuation $V_{x_t'}$ of a series of continuous $x$-simply augmented inductive valuations of the polynomial ring $K_x[y]$ whose coefficient field $K_y$ has the valuation $V_{x_0'}$.

$$V_{x_t} = V_{x_t'} = [V_{x_0'}, V_{x_t'}x = WX, V_{x_t'}\phi'_2 = W\phi'_2, \ldots , V_{x_t'}\phi'_s = W\phi'_s]$$

Herein $\phi'_2, \phi'_3, \ldots , \phi'_s$ are generally polynomials of $x$ and $y$ and may generally be respectively different from the polynomials $\phi_2, \phi_3, \ldots , \phi_s$ which produce $V_{x_1}, V_{x_2}, \ldots , V_{x_t}$ respectively and $s'$ is not always equal to $s$.

Namely $V_{x_t}$ is a valuation of the polynomial ring $K_x[y]$ whose coefficient field $K_x$ has the valuation $V_{x_0}$, but only after we transform $V_{x_t}$ into a valua-
tion $V_{s',r'}$ of the polynomial ring of $K_y[x]$ whose coefficient field $K_y$ has the valuation $V_{0r'}$, we can make an $x$-augmented valuation of $V_{s',r'}$.

After we make $V_{s',r'}$, out of such polynomial $l(x, y)$ for which $WL(x, y) > V_{s',r'}l(x, y)$ we pick up such a polynomial $\phi_{r'+1}$ whose degree with respect to $x$ is minimum and whose difference between $W\phi_{r'+1} - V_{s',r'}\phi_{r'+1}$ is minimum and whose coefficient of the term of the highest power of $x$ is 1. And with $\phi_{r'+1}$ we make a continuous $x$-simply augmented valuation $V_{s'+1, r'}$ of $V_{s',r'}$.

$$V_{s'+1, r'} = [V_{s',r'}, V_{s'+1, r'}\phi_{r'+1} = W\phi_{r'+1}].$$

Thus, as last we can attain $U$, because $U$ is a discrete valuation. Therefore many series of continuous simply augmented inductive valuations binding $V_{11}$ and $U$ exist.

Now, I want to prepare for solution of Principal Problem, but for the sake of it I must use many notions which MacLane used. So I explain these notions as simply as possible.

§ 12. Homogeneous inductive valuations

MacLane introduced a notion of “homogeneous inductive valuations” as “normal forms” of valuations in $M$ §16.

We first choose in $K$ a complete set $K'$ of “representatives” with respect to $V_0$, such that each element of $K$ is equivalent in $V_0$ to one and only one representative.

A series of augmented inductive valuations of $K[x]$

$$V_1 < V_2 < \cdots < V_{i-1} < V_i < \cdots < V_k$$

is called that of homogeneous inductive valuations when they satisfy the following conditions;

let be

$$V_i = \left[ V_{i-1}, V_i\phi_i(x) = \mu_i \right] \quad \text{for } i = 1, 2, \ldots, k,$$

and

$$\phi_i = \sum_j a_j\phi_i^{m_1}\phi_2^{m_2} \cdots \phi_i^{m_i}$$

(12.1)

then

$$m_{ij} < \frac{\deg \phi_{i+1}}{\deg \phi_p}$$

for all terms in (12.1)

and $a_j$ belongs to $K'$ and

$$V_{i-1}\phi_i = V_i a_j\phi_i^{m_1}\phi_2^{m_2} \cdots \phi_i^{m_i}$$

for all terms in (12.1).

According to M. Theorem 16.3, any inductive constructed valuation from a discrete valuation $V_0$ of $K$ is equal to a homogeneous inductive valuation and two equal homogeneous valuations of $K[x]$ are identical by M. Theorem 16.4. (See §27).
Namely, if \( V_\delta \) and \( W_\tau \) are two homogeneous valuations of \( K[x] \) and for any polynomial \( f(x) \)
\[
V_\delta f(x) = W_\tau f(x)
\]
and
\[
V_\delta = [V_0, V_1 x = \mu_1, \cdots, V_\delta \phi_\delta = \mu_\delta]
\]
\[
W_\tau = [W_0, W_1 x = \nu_1, \cdots, W_\tau \phi_\tau = \nu_\tau]
\]
and
\[
1 < \deg \phi_\delta < \deg \phi_\delta \mu \cdots < \deg \phi_\delta
\]
\[
1 < \deg \phi_\tau < \deg \phi_\tau < \cdots < \deg \phi_\tau
\]
then
\[
V_0 = W_0 \quad \text{and} \quad k = t
\]
and
\[
\phi_\delta = \phi_\delta \quad (i = 1, 2, \cdots, k)
\]
\[
\nu_\tau = \nu_\tau \quad (i = 1, 2, \cdots, k).
\]

In the same way a series of continuous \( x \)-augmented homogeneous inductive valuations from \( V_\delta \) to \( W \) is decided uniquely by \( W \) according to Theorem 2.5. and every key polynomial \( \phi_k(x) \) and its value \( \mu_k = V_k \phi_k \) are decided uniquely by \( W \).

Hence I let be all valuations homogeneous in this paper. For example, in this paper I write the key polynomial \( \phi_{k+1}(x) \) as follows;
\[
\phi_{k+1} = \phi_{k+1} + \cdots + a_t(x)\phi_t + \cdots + a_0(x)
\]
(12, 2).
In (12, 2) I dropped all such terms \( a_j(x)\phi_{k+1} \) which appear in the expansion of \( \phi_{k+1} = f(x) \) by M. Theorem 9.4. and for which \( V_k a_j(x)\phi_{k+1} > V_k \phi_{k+1} \).

So, for all terms that appear in (12, 2) the followings hold;
\[
V_k \phi_{k+1} = V_k a_t(x)\phi_t = V_k \phi_{k+1} = V_k a_0(x)
\]
and
\[
\deg a_t(x) < \deg \phi_k.
\]

§ 13. Approximants

MacLane established a paper "A construction for prime ideals as absolute values of an algebraic field" in "Duke Jour. Vol. 2, No. 3." in 1936. In this paper I will often quote this paper of his. For example, D. Theorem 5.2. is Theorem 5.2. in this paper.

In it, MacLane studies to seek such series of augmented inductive valuations of \( K[x] \).
\[
V_1 < V_2 < \cdots < V_{k-1} < V_k
\]
(13. 1).
Let be
\[
V_1 = [V_0, V_1 x = \mu_1]
\]
\[
V_k = [V_{k-1}, V_k \phi_k(x) = \mu_k] \quad \text{for} \quad k = 2, 3, \cdots
\]
and for a given polynomial \( G(x) \) which is irreducible in \( K[x] \), the following
relations hold

\[ V_1 G(x) < V_2 G(x) < \cdots < V_k G(x) < V_k G(x) \]

and moreover \( V_k G(x) \) must be \( \infty \) or \( \lim_{k \to \infty} V_k G(x) \) must be \( \infty \).

\[ G(x) = g_m(x) \phi_k^m + \cdots + g_1(x) \phi_k^1 + \cdots + g_0(x) \]

where \( \deg g_i(x) < \deg \phi_k \) for \( i = 0, 1, 2, \ldots, m \).

Among the exponents \( j \) for which \( V_k G(x) = V_k \{ g_j \phi_k^j \} \), let \( \alpha \) be the largest and \( \beta \) the smallest. The difference \( \alpha - \beta \), which depends on both \( K \) and \( G(x) \), is called the project of \( V_k \) (Symbol: \( \text{Proj} \ V_k \) (D §3)). And in (13.1), if such a valuation \( V_{k-1} \) for which \( V_k G(x) < V_{k+1} G(x) \) exists, then this case is said that the "terminating" case does not occur by the \( k \)-th stage, and if \( V_k G(x) = \infty \), then the case is said that the "terminating" case occurs by the \( k \)-th stage.

A necessary and sufficient condition that the "terminating" case does not occur by the \( k \)-stage is that \( \text{Proj} \ V_k \) is positive. \( V_k \) is called a \( k \)-th approximant to \( G(x) \). (D. Definition 3.3.)

When \( G(x) \) is given, generally there are many series of approximants of \( G(x) \), and MacLane gives the two following theorems;

D. Theorem 5.2.

If the "terminating" case does not occur by the \( k \)-th stage, there is a finite number of \( k \)-th approximants, such that

\[ \sum (\text{Proj} \ V_k) (\deg \phi(V_k)) = \deg G(x), \]

where \( \phi(V) \) represents the last key of \( V \) and the sum is taken over all such augmented valuations \( V_k \).

D. Theorem 5.3.

If there is a non-finite homogeneous inductive value \( V_k \) with \( V_k G(x) = \infty \), then for \( i < k \) the value \( V_i \) from which \( V_k \) is obtained is the only \( i \)-th approximant.

If \( G(x) \) can be decomposed in \( V_{k-1} \) as in D §5 (4),

\[ G(x) \sim e(x) \phi_{k-1}(x)^{a_1} \phi_1(x)^{a_1} \cdots \phi_1(x)^{a_1} \]

in \( V_{k-1} \),

namely one homogeneous key polynomial \( \phi_1(x) \) appears in this decomposition, by giving a suitable value \( V_k \phi_1(x) \) greater than \( V_{k-1} \phi_1(x) \) to \( \phi_1(x) \), we can make as follows;

\[ V_{k-1} G(x) < V_k G(x) \neq \infty. \]
Therefore, when $V_k G(x) = \infty$, $G(x)$ itself must be a homogeneous key polynomial in $V_{k-1}$.

**Definition 13.1.**

\[
V_1 < V_2 < \cdots < V_k \quad (\alpha) \\
V_1' < V_2' < \cdots < V_k' \quad (\beta)
\]

Both $(\alpha)$ and $(\beta)$ are series of augmented inductive valuations which are extensions of $V_0$ to $K[x]$ and when we make both $(\alpha)$ and $(\beta)$ into series of continuous augmented inductive valuations by making dense the ways of the two series as in $\S$ 3, the two series of continuous augmented inductive valuations completely coincide with each other. In this case we say that $(\alpha)$ and $(\beta)$ are on the same course.

Then by D. Theorem 5.2. we obtain the following corollary.

**Corollary 13.2.**

If $G(x)$ is irreducible in $K[x]$ and $(\alpha)$ and $(\beta)$, the two following series of augmented inductive valuations which are extensions of $V_0$ to $K[x]$ satisfy the following conditions

\[
V_1 G(x) < V_2 G(x) < \cdots < V_{k-1} G(x) < V_k G(x) \\
V_1' G(x) < V_2' G(x) < \cdots < V_{k-1}' G(x) < V_k' G(x)
\]

and $V_k = [V_{k-1}, V_k G(x) = \mu_k]$, that is, $G(x)$ is the last key polynomial that produces $V_k$ and $V_k G(x) = V_k' G(x)$, then $(\alpha)$ and $(\beta)$ are on the same course and $V_k = V_k'$.

In this corollary $V_k^{\phi_k} = \mu_k$ may be finite.

**Theorem 13.3.**

If

\[
V_1 < V_2 < \cdots < V_{k-1} < V_k \quad (\gamma)
\]

is a series of augmented inductive valuations which are extensions of $V_0$ to $K[x]$ and

\[
V_k = [V_{k-1}, V_k^{\phi_k} = \mu_k],
\]

and

\[
V_1' < V_2' < \cdots < V_{i-1}' < V_i' \quad (\delta)
\]

is another series of augmented inductive valuations which are extensions of $V_0$ to $K[x]$ and

\[
V_1^{\phi_k} < V_2^{\phi_k} < \cdots < V_{i-1}^{\phi_k} < V_i^{\phi_k} = \mu_k,
\]
then \( V_k = V' \) or \( V_i \) is an augmented valuation of \( V_k \).

Proof.

\( \phi_k \) is the last key polynomial, so \( \phi_k \) is irreducible and \( V_1 \phi_k < V_2 \phi_k < \cdots < V_{k-1} \phi_k < V_k \phi_k = \mu_k \).

We can make both \((\tau)\) and \((\delta)\) into series of continuous augmented inductive valuation by making dense the ways of \((\tau)\) and \((\delta)\) as in §3, then according to Corollary 13.2. and Theorem 2.6, \( V_k \) equals to \( V'_i \) or \( V_k \) is equal to a valuation between \( V'_{i-1} \) and \( V'_i \).

\section{14. Decompositions of the last key polynomial in \( V_2 \)}

**Theorem 14.1.**

If

\[ V_1 < V_2 < \cdots < V_{p-1} < V_p \]

is a series of homogeneous augmented inductive valuations which are extensions of \( V_0 \) to \( K[x] \), then

\[ \phi_p(x) \sim \phi_i(x)^{b_i} \quad \text{in } V_{i-1} \]

for \( i = 2, 3, \ldots, P \),

where

\[ b_i = \deg \phi_p / \deg \phi_i \quad \text{for } i = 2, 3, \ldots, P. \]

Therefore especially \( \phi_p \sim \phi_2^{b_2} \) in \( V_1 \).

In order to prove this theorem, I will prove the two following lemmas before it.

**Lemma 14.2.**

In the series (14.1), if

\[ V_i = [V_{i-2}, V_{i-1} \phi_{i-1} = \mu_{i-1}, V_i \phi_i = \mu_i] \]

and \( h(x) \) is a homogeneous polynomial in \( V_{i-1} \)

namely

\[ h(x) = g_n(x) \phi_{i-1}^n + \cdots + g_k(x) \phi_{i-1}^k + \cdots + g_0(x), \]

where

\[ \deg g_k(x) < \deg \phi_{i-1} \quad \text{for all terms in} \]

and

\[ V_{i-1} h(x) = V_i \phi_{i-1} g_{i-1} \quad \text{for all terms in} \]

then

\[ h(x) \sim g_0 \phi_{i-1} \quad \text{in } V_{i-2} \]

and

\[ V_{i-1} h(x) - V_{i-2} h(x) = n [V_{i-1} \phi_{i-1} - V_{i-2} \phi_{i-1}] \].

Proof.

\[ \deg g_k < \deg \phi_{i-1} \]

so

\[ V_{i-1} g_k = V_{i-2} g_k \quad \text{for all } g_k \text{ in (14.2.)} \]

So, when \( n > k \)
Lemma 14.3.
In Lemma 14.2 if
\[ \deg h(x) < \deg \phi \]
then
\[ V_{t-1} h(x) - V_{t-2} h(x) < V_{t-1} \phi - V_{t-2} \phi. \]

Proof.
Let be
\[ \phi = \phi_{t-1} + \cdots + g_{t} \phi + \cdots + g_{0}(x) \quad (14.3). \]
\( \phi \) is homogeneous in \( V_{0} \), so
\[ V_{t-1} \phi = V_{t-1} g_{t} \phi \quad \text{for all terms in (14.3)}. \]
\[ h(x) = g_{n} \phi_{n-1} + \cdots + g_{0}(x) \]
\[ \deg h(x) < \deg \phi \]
\[ n < C. \]
So
\[ V_{t-1} \phi = V_{t-2} \phi \]
\[ = C[V_{t-1} \phi_{t-1} - V_{t-2} \phi_{t}] \quad \text{owing to Lemma 14.2}. \]
\[ > n[V_{t-1} \phi_{t-1} - V_{t-2} \phi_{t}] = V_{t-1} h(x) - V_{t-2} h(x). \]
It is evident that both lemmas 14.2 and 14.3 hold when \( i \) moves from 3 to \( p \) in the series (14.1).

Now I can prove Theorem 14.1.

Proof of Theorem 14.1.
\[ \phi_{p} = \phi_{p-1}^{p-1} + \cdots + h_{k}(x) \phi_{p-k}^{k} + \cdots + h_{0}(x) \quad (14.4) \]
Owing to Lemma 14.2.
\[ \phi_{p} \sim \phi_{p-1}^{p-1} \quad \text{in} \quad V_{p-2} \]
and
\[ V_{p-2} \phi_{p-1}^{p-1} < V_{p-2} h_{k} \phi_{k}^{k} \quad \text{for all terms in (14.4)} \quad (14.5) \]
Let be \( \phi_{p-1} = \phi_{p-2} + \cdots + h'(x) \)
then in the same way
\[
\phi_{p-1} \sim \phi_{p-2} \quad \text{in } V_{p-3}
\]
\[
V_{p-3} \phi_{p-1}^e - V_{p-3} \phi_{p-1}^e \geq (k + 1) [V_{p-2} \phi_{p-1} - V_{p-3} \phi_{p-1}] \\
\therefore \ b_{p-1} > k \quad \text{and} \quad V_{p-3} \phi_{p-1} > V_{p-3} \phi_{p-1}
\]
\[
= (V_{p-2} \phi_{p-1} - V_{p-3} \phi_{p-1}) + k(V_{p-2} \phi_{p-1} - V_{p-3} \phi_{p-1}) \\
> (V_{p-2} h_k - V_{p-3} h_k) + k(V_{p-2} \phi_{p-1} - V_{p-3} \phi_{p-1}) \\
\therefore \ \deg h_k < \deg \phi_{p-1}, \quad \text{so Lemma 14.2 holds.}
\]
\[
= V_{p-2} h_k \phi_{p-1}^k - V_{p-3} h_k \phi_{p-1}^k.
\]
Namely
\[
V_{p-3} \phi_{p-1}^e - V_{p-3} \phi_{p-1}^e > V_{p-2} h_k \phi_{p-1}^k - V_{p-3} h_k \phi_{p-1}^k \\
V_{p-3} h_k \phi_{p-1}^k > V_{p-3} \phi_{p-1}^e + (V_{p-2} h_k \phi_{p-1}^k - V_{p-3} \phi_{p-1}^e) \\
> V_{p-3} \phi_{p-1}^e \quad \text{by (14.5)}
\]
So
\[
\text{Min} \ [V_{p-3} h_k \phi_{p-1}^k] = V_{p-3} \phi_{p-1}^e.
\]
Therefore \( \phi_{p-1} \sim \phi_{p-2}^e \) \( \text{in } V_{p-3} \), \( \phi_{p-1} \sim \phi_{p-2}^e \) \( \text{in } V_{p-3} \)
then
\[
\phi_{p-1} \sim \phi_{p-2}^e \quad \text{in } V_{p-3}.
\]
\[
\deg \phi_{p} = b_{p-1} \deg \phi_{p-1} = Cb_{p-1} \deg \phi_{p-2},
\]
so
\[
Cb_{p-1} = \deg \phi_{p} / \deg \phi_{p-2} = b_{p-2}
\]
\[
\therefore \ \phi_{p-1} \sim \phi_{p-2}^e \quad \text{in } V_{p-3}. \quad \deg \phi_{p} = \deg \phi_{p-2}^e.
\]
Let be an expansion of \( \phi_{p} \) by \( \phi_{p-2} \)
\[
\phi_{p} = \phi_{p-2}^e + \cdots + l_k(x) \phi_{p-2}^k + \cdots + l_0(x)
\]
(14.6)
\[
V_{p-3}; l_k \phi_{p-2}^k > V_{p-3} \phi_{p-2}^e \quad \text{for all terms in (14.6)},
\]
because
\[
\phi_{p-2} \sim \phi_{p-2}^e \quad \text{in } V_{p-3}.
\]
Then we can prove that \( \phi_{p} \sim \phi_{p-3}^e \) \( \text{in } V_{p-4} \) in the same way as when I proved that \( \phi_{p} \sim \phi_{p-2}^e \) \( \text{in } V_{p-3} \).
Thus at last we can prove that \( \phi_{p} \sim \phi_{p}^e \) \( \text{in } V_{1} \).

§ 15. Effective degrees

There is a famous theorem called "Hensel's Lemma" in theory of valuation. In order to solve the Principal Problem in my own way, I want to make an extension of this "Hensel's Lemma" to \( V_k \) in \( K[x] \). For the sake of it I must arrange some lemmas.
\[ V_k = [V_{k-1}, \ V_k \phi_k = \mu_k]. \]

And the following is an expansion of an arbitrary polynomial \( f(x) \);
\[
f(x) = f_n(x)\phi_k^n + f_{n-1}(x)\phi_k^{n-1} + \cdots + f_1(x)\phi_k + \cdots + f_0(x) \quad (15.1)
\]
\[
\deg f_j(x) < \deg \phi_k \quad \text{for all terms in (15.1).}
\]

Then, the largest exponent \( \alpha \) for which \( V_k f(x) = V_k (f \phi_k^\alpha) \) is called the effective degree of \( f(x) \) in \( \phi_k \) and is denoted by \( D_\phi f \) or simply by \( D_f \).

If \( h(x) \sim l(x) \) in \( V_k \), then \( D_\phi h(x) = D_\phi l(x) \) and for any two polynomials \( f(x) \) and \( g(x) \)
\[
D_\phi (f \cdot g) = D_\phi f + D_\phi g.
\]

If \( D_\phi e(x) = 0 \), then the polynomial \( e(x) \) is called an equivalent unit in \( V_k \) or simply a unit.

**Lemma 15.**

Every unit is equivalent to a unit whose degree is less than that of \( \phi_k \) and a set of all equivalent-units in \( V_k \) is a group with respect to multiplication. (D § 4)

**§ 16. Canon polynomials**

\( \Gamma_{k-1} \) is a group of all values \( V_{k-1} f(x) \), where \( f(x) \in k[x] \) and \( \tau_k \) is the smallest integer such that \( \tau_k \mu_k \in \Gamma_{k-1} \). Then there exists such a polynomial \( Q(x) \) that
\[
V_k Q(x) \phi_k^\tau = 0 \quad \text{and} \quad \deg Q(x) < \deg \phi_k.
\]

In this paper, hence I write this polynomial \( Q(x) \phi_1^\tau \) as \( \Phi \). Then \( \Delta_k \), the residue class ring of \( K[x] \) by \( V_k \) is \( F_k[X] \), where \( F_k \) is an algebraic extension of \( F_0 \) which is the residue class field of \( K \) by \( V_0 \) and \( X = H_k \Phi \) is a transcendental element with respect to \( F_k \), herein \( H_k \) is a natural homomorphism by which \( \Phi \) corresponds to \( X \). (M. Theorem 12.1)

Then a set \( \{ e(x) | D_\phi e(x) = 0 \ \text{and} \ \ V_k e(x) = 0 \} \) is a subgroup of a group of all units in \( V_k \).

**Lemma 16.**

If \( \phi_k \not\sim l(x) \) in \( V_{k-1} \), then \( l(x) \) is a unit in \( V_k \).

**Proof.**

By M. Lemma 9.2. \( V_k l(x) = V_{k-1} l(x) \),
so according to M. Lemma 9.1, there exists such a polynomial \( l'(x) \) that

\[
l(x)l'(x)\sim 1 \quad \text{in} \quad V_k.
\]

Therefore

\[
0 = D_x l = D_x(ll') = D_x l + D_x l'
\]

\[
\therefore \quad D_x l(x) = 0.
\]

**Lemma 16.2.**

If \( V_k f(x) = 0 \) and \( \frac{D_x f(x)}{\tau_k} = m > 0 \), then there exists such a polynomial \( f^*(x) \) that

\[
f(x) \sim f^*(x) = a'_m(x) (\phi^m + \cdots + b_i(x) \phi^i + \cdots + b_q(x) \phi^q)
\]

in \( V_k \)

and

\[
H_k f(x) = H_k^* f(x) = a'_m (X^m + \cdots + \tilde{b}_i X^i + \cdots + \tilde{b}_q X^q)
\]

where \( a'_m = H_k a'_m(x) \neq 0 \) and \( \tilde{b}_i = H_k \tilde{b}_i(x) \neq 0 \)

and

\[
\deg a'_m(x) < \deg \phi_k
\]

\[
\deg b_i(x) < \deg \phi_k \quad \text{for all terms in (16.1)}
\]

**Proof.**

Let be

\[
f(x) = \sum_{i=0}^{n} a_i(x) \phi^i
\]

where

\[
\deg a_i(x) < \deg \phi_k \quad \text{for all terms in (16.2)}.
\]

First we drop all such terms \( a_i(x) \phi^i \) for which

\[
V_k a_i \phi^i > V_k f(x), \quad \text{from 16.2}.
\]

Then we obtain

\[
f(x) \sim a_m'(x) \phi_m' + \cdots + a_i'(x) \phi_i' + \cdots + a_q'(x) \phi_q'
\]

in \( V_k \).

\[
0 = V_k f(x) = V_k (a_i f^i) \quad \text{for all terms in (16.3)}
\]

\[
\therefore \quad \tau_k \mid i' \quad \text{so} \quad i' = i\tau_k.
\]

Namely

\[
f(x) \sim a_m'(\phi_k^m) + \cdots + a_i'(\phi_k^i) + \cdots + a_q'(\phi_k^q)
\]

in \( V_k \).

Owing to M. Lemma 9.1, there exists such a polynomial \( Q'(x) \) that
\( Q(x)Q'(x) \sim 1 \) in \( V_k \) and \( V_k Q'(x) = V_{k-1} Q'(x) \).

Then

\[
f(x) \sim a_m Q'^m(Q \phi_k)^m + \cdots + a_i Q'^i (Q \phi_k)^i + \cdots + a_q Q'^q (Q \phi_k)^q
\]

in \( V_k \), because values of all terms in (16.4) are zero.

\[
f(x) \sim a_m Q'^m \left[ \phi^m + \cdots + \frac{a_i Q'^i}{a_m Q'^m} \phi^i + \cdots + \frac{a_q Q'^q}{a_m Q'^m} \phi^q \right]
\] (16.5) in \( V_k \).

According to Lemma 15, every coefficient in (16.5) is equivalent to a unit whose degree is less than that of \( \phi_k \).

\[
f(x) \sim a_m' \phi^m + \cdots + b_i \phi^i + \cdots + b_q \phi^q
\]

in \( V_k \).

So,

\[
H_k f(x) = a_m' (X^m + \cdots + \bar{b}_i x^i + \cdots + \bar{b}_q x^q)
\]

where all coefficients \( a_m' \) and \( \bar{b}_i \) belong to \( F_k \) and none of them is zero.

**Lemma 16.3.**

If \( V_k g(x) = \alpha > 0 \), then there exists such a polynomial \( c(x) \phi_k^l g^*(x) \) that

\[
g(x) \sim c(x) \phi_k^l g^*(x)
\]

in \( V_k \)

where

\[
V_k c(x) \phi_k^l = \alpha
\]

and

\[
g^*(x) = \Phi^m + \cdots + b_i \Phi^i + \cdots + b_q \Phi^q
\]

(16.6)

and

\[
\deg c(x) < \deg \phi_k \quad \text{and} \quad \deg b_i(x) < \deg \phi_k
\]

\[
0 \neq \bar{b}_i = H_k b_i(x) \in F_k \quad \text{for all terms in (16.6).}
\]

**Proof.**

Let be an expansion of \( g(x) \) by \( \phi_k \) as follows:

\[
g(x) = g_s(x) \phi_k^s + \cdots + g_j(x) \phi_k^j + \cdots + g_0(x)
\]

(16.7).

We drop all such terms \( g_j(x) \phi_k^j \) for which

\[
V_k g(x) < V_k g_j \phi_k^j,
\]

from (16.7) and we obtain

\[
g(x) \sim g_p(x) \phi_k^p + \cdots + g_j(x) \phi_k^j + \cdots + g_1(x) \phi_k^1
\]

(16.8) in \( V_k \)

and

\[
V_k g(x) = V_k g_j \phi_k^j = \alpha
\]

for all terms in (16.8).

\[
g(x) \sim g_j' \left( \frac{g_p}{g_l} \phi_k^{p-l} + \cdots + \frac{g_j}{g_l} \phi_k^{j-l} + \cdots + 1 \right)
\]

in \( V_k \)

\[
V_k \left( \frac{g_p}{g_l} \phi_k^{p-l} + \cdots + 1 \right) = 0 \quad \therefore \quad p - l = m\tau_k .
\]

Let be

\[
p - l = m\tau_k ,
\]
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then owing to Lemma 16.2

$$\frac{g_n}{g_1} \phi_k^n + \cdots + 1 \sim a_m(x) (\Phi^n + \cdots + b_t(x) \Phi^t + \cdots + b_0(x)) \quad \text{in } V_k.$$ 

So, consequently

$$g(x) \sim g_t a_m(x) \phi_k^t (\Phi^n + \cdots + b_t(x) \Phi^t + \cdots + b_0(x)) \quad \text{in } V_k.$$ 

$$\deg g_t < \deg \phi_k \quad \text{and} \quad \deg a_m(x) < \deg \phi_k,$$

then owing to Lemma 15,

$$g_t a_m(x) \sim c(x) \quad \text{in } V_k \quad \text{and} \quad \deg c(x) < \deg \phi_k.$$ 

\[ \therefore \quad g(x) \sim c(x) \phi_k^t (\Phi^n + \cdots + b_t(x) \Phi^t + \cdots + b_0(x)) \quad \text{in } V_k \]

$$V_k c(x) \phi_k^t = V_k g(x) = \alpha.$$ 

**Corollary 16.4.** In Lemma 16.3, we can make \( l \) less than \( \tau_k \). 

**Proof.** When \( l \geq \tau_k \), \( l = h \tau_k + l' \), where 

\[ 0 \leq l' \leq \tau_k - 1. \]

\[ c(x) \phi_k^t = c(x) \phi_k^{t'} (\phi_k^{t''})^h \]

\[ \sim c(x) \left( Q'(x)^h \right)^{t'} \phi_k^{t''} \left( Q(x) \phi_k^{t''} \right)^h \sim c'(x) \phi_k^{t''} \Phi^h \quad \text{in } V_k, \]

where \( c'(x) \sim c(x) \left( Q'(x)^h \right)^{t'} \) in \( V_k \) and \( \deg c'(x) < \deg \phi_k \).

Therefore

$$g(x) \sim c'(x) \phi_k^{t''} \left( \Phi^{n-h} + \cdots + b_t(x) \Phi^{t-h} + \cdots + b_0(x) \Phi^h \right) \quad \text{in } V_k$$

where 

\[ 0 \leq l'' \leq \tau_k - 1 \]

and

\[ V_k c'(x) \phi_k^{t''} = V_k g(x) = \alpha. \]

**Definition 16.5.** A polynomial \( e(x) \) for which \( D_k e(x) = 0 \) and \( V_k e(x) = 0 \) is called a \( V_k \)-unit and \( H_k e(x) = \overline{e(x)} \) is called a \( F_k \)-unit.

**Definition 16.6.** If \( f(x) = \Phi^m + \cdots + b_t(x) \Phi^t + \cdots + b_0(x) \Phi^0 \), where all \( b_i(x) \) are \( V_k \)-units and \( m \geq 1 \), then \( f(x) \) is called a canon polynomial in \( V_k \)

and

\[ H_k f(x) = X^m + \cdots + b_t(x) X^t + \cdots + b_0(x) X \]

where all \( b_i(x) \) are \( F_k \)-units, is called a canon class of \( f(x) \).

**Definition 16.7.** \( f(x) \) is a canon polynomial
and 
\[ f(x) = (Q(x) \phi_k^i)^m + \cdots + b_k(x)(Q(x) \phi_k^i)^t + \cdots + b_0(x) \]
and 
\[ f(x) \sim Q(x)^m [\phi_k^{m_i} + \cdots + c_i(x)\phi_k^{t_i} + \cdots + c_0(x)] \], \quad \text{in } V_k, 
where 
\[ \deg c_i(x) < \deg \phi_k \]
\[ \phi_k^{m_i} + \cdots + c_i(x)\phi_k^{t_i} + \cdots + c_0(x) \] is called a key part of \( f(x) \).

Then we have the following theorem.

**Theorem 16.8.** A necessary and sufficient condition that a polynomial is a homogeneous key polynomial in \( V_k \) is that it is a key part of a canon polynomial whose canon class is irreducible in \( F_k[X] \). (M. Lemma 11.2. and M. Theorem 9.4.)

**§ 17. Prime each other in \( V_k \)**

**Theorem 17.1.** If both \( g(x) \) and \( h(x) \) are canon polynomials and there exist such polynomials \( l(x) \) and \( m(x) \) that
\[ g(x)l(x) + h(x)m(x) \sim 1 \quad \text{in } V_k, \]
and
\[ V_k l(x) \geq 0 \quad \text{and} \quad V_k m(x) \geq 0, \]
then
\[ V_k l(x) = V_k m(x) = 0 \]
and
\[ (Hg(x))(Hl(x)) + (Hh(x))(Hm(x)) = 1 \quad \text{in } F_k[X]. \]

**Theorem 17.2.** If all \( G(X), H(X), L(X) \) and \( M(X) \) are residue classes in \( F_k[X] \) and
\[ G(X) L(X) + H(X) M(X) = 1, \]
then there are such polynomials \( g(x), h(x), l(x) \) and \( m(x) \) whose values by \( V_k \) are all zero and whose residue classes in \( F_k[X] \) are respectively \( G(X), H(X), L(X) \) and \( M(X) \).

**Definition 17.3.** In Theorem 17.1 we say that \( g(x) \) and \( h(x) \) are prime each other in \( V_k \). In the same way we say that many canon polynomials are prime each other in \( V_k \), when their residue classes in \( F_k[X] \) are prime each other in \( F_k[X] \).

**Proof of Theorem 17.1 and Theorem 17.2.** Assuming that
\[ V_k l(x) > 0 \quad \text{and} \quad V_k m(x) > 0, \]
\[ 0 = V_k 1 = V_k (gl + hm) \supseteq \text{Min}[V_k gl, V_k hm] > 0. \]
This is a contradiction.
Next, assuming that \( V_k l(x) > 0 \) and \( V_k m(x) = 0, \)
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\[ V_k g l > 0 = V_k h m = V_k 1 \]

so
\[ h(x)m(x) \sim 1 \quad \text{in } V_k. \]

But \( D_0 h(x) > 1 \), so this is also a contradiction.

When we assume that \( V_k l(x) = 0 \) and \( V_k m(x) > 0 \), in the same way a contradiction takes place.

Therefore
\[ V_k l(x) = V_k m(x) = 0. \]

Then we can prove both Theorem 17.1 and 17.2 naturally, because \( H_k \) is a homomorphism which makes every polynomial \( f(x) \) of \( x \) whose value by \( V_k \) is non-negative, correspond to its residue class \( H_k f(x) \) in \( F_k[X] \).

**Definition 17.4.** If \( V_k (f(x) - l(x)) = \omega > 0 \), then \( f(x) \) and \( l(x) \) are said to be equivalent higher than \( \omega \) and are denoted as
\[ f(x) \equiv l(x) \quad \text{in } V_k. \]

§ 18. Extension of Hensel’s Lemma

**Theorem 18.1.** Both \( g_0(x) \) and \( h_0(x) \) are canon polynomials which are prime each other in \( V_k \) and \( f(x) \) is a polynomial and \( \omega \) is an arbitrary given positive number.

If
\[ f(x) \sim g_0(x) h_0(x) \quad \text{in } V_k, \]

then there are such two polynomials \( g(x) \) and \( h(x) \) that
\[ f(x) \equiv g(x) h(x) \quad \text{in } V_k, \]
where
\[ g(x) \sim g_0(x) \quad \text{and} \quad h(x) \sim h_0(x) \quad \text{in } V_k \]
and
\[ \deg g(x) = \deg g_0(x). \]

**Proof.** \( f(x) \sim g_0 h_0 \) in \( V_k \).

Let be
\[ V_k (f - g_0 h_0) = \alpha > 0, \]
then owing to Corollary 16.4
\[ f - g_0 h_0 \sim c(x) \phi_k^l f^*(\Phi) \quad \text{in } V_k \]
where\n\[ \deg c(x) < \deg \phi_k \]
and\n\[ 0 \leq l \leq \tau_k - 1 \]
and\n\[ V_k c(x) \phi_k^l = \alpha \]
and\n\[ f^*(\Phi) \]
is a canon polynomial.

Let be\n\[ f = g_0 h_0 + c(x) \phi_k^l f^* + B_1 \quad (18.1) \]
then
\[ V_k B_1 > V_k (f - g_0 h_0) = \alpha. \]
$g_0$ and $h_0$ are prime each other in $V_k$, then there are two polynomials $l(x)$ and $m(x)$ such that

$$g_0l + h_0m \sim 1 \quad \text{in} \quad V_k,$$

where

$$V_k l = V_k m = 0 \quad \text{by Theorem 17.1}.$$

$$g_0 f^* + h_0 m f^* \sim f^* \quad \text{in} \quad V_k,$$

where $f^* = f^*(\Phi)$.

In $F_k[X]$, let be $H_k(mf^*) = (H_k g(x))(H_k g_0(x)) + H_k r(x)$, namely

$$V_k f^* = \bar{g}_0 \bar{g}_0 + \bar{r},$$

where $\deg z \bar{r} = \deg z H_k(r(x)) < \deg z H_k(g_0(x)) = \deg \bar{g}_0$.

$$g_0 l^* + h_0 m f^* \sim f^* \quad \text{in} \quad V_k.$$

So in $F_k[X]$, when we write $H_k l(x)$ as $\overline{l}$,

$$\bar{f}^* = \bar{g}_0 \bar{f}^* + \bar{h}_0 \overline{m f^*}$$

$$= \bar{g}_0 \bar{f}^* + \bar{h}_0 (\bar{q} \bar{g}_0 + \bar{r}) = \bar{g}_0 (l \bar{f}^* + \bar{h}_0 \bar{q}) + \bar{h}_0 \bar{r}$$

Let be

$$l \bar{f}^* + \bar{h}_0 \bar{q} = \overline{l'},$$

then $\bar{f}^* = \bar{g}_0 l' + \bar{h}_0 \bar{r}$

so

$$f^* \sim g_0 l'(x) + h_0 r(x) \quad \text{in} \quad V_k \quad (18.2)$$

and

$$D_0 r(x) < D_0 g_0(x).$$

Let be

$$F(x) = f - \left( g_0 + c(x) h_0 r(x) \right) \left( h_0 + c(x) h'_0 l'(x) \right)$$

, then $F(x) = (f - g_0 h_0) - c(x) \phi_k^2 (g_0 l'(x) + h_0 r(x)) - (c(x) \phi'_k)^2 r l'(x).$

(By 18.1) $F(x) = B_1 - c(x) \phi^2_k (g_0 l' + h_0 r - f^*) - (c(x) \phi'_k)^2 r l'.$

By (18.2) $V_k (g_0 l' + h_0 r - f^*) > 0$

so

$$V_k c(x) \phi^2_k (g_0 l' + h_0 r - f^*) > \alpha,$$

and

$$V_k (c(x) \phi'_k)^2 r l' = 2\alpha + V_k r l' = 2\alpha > \alpha.$$

$V_k B_1 > \alpha$. Therefore $V_k F(x) > \alpha$.

When

$$\deg c(x) \phi'_k r(x) \geq \deg g_0(x),$$

$$l \leq \tau_k - 1 \quad \text{and} \quad \deg z H_k r(x) < \deg z H_k g_0(x),$$

so we can select such a polynomial $R(x)$

that

$$R(x) \sim c(x) \phi'_k r(x) \quad \text{in} \quad V_k$$

and

$$\deg R(x) < \deg g_0(x).$$

So

$$R(x) = c(x) \phi'_k r(x) + B_2,$$

where $V_k B_2 > \alpha$. 
And we make $F'(x)$ as follows:

$$F'(x) = f - \left(g_0 + R(x)\right)\left(h_0 + c(x)\phi'_l(x)\right)$$

$$= f - \left(g_0 + c(x)\phi'_kr(x) + B_2\right)\left(h_0 + c(x)\phi'_l(x)\right)$$

$$= F(x) - B_2\left(h_0 + c(x)\phi'_l(x)\right).$$

$$V_k\left(B_2\left(h_0 + c(x)\phi'_l(x)\right)\right) = V_kB_2 + V_k\left(h_0 + c(x)\phi'_l(x)\right) > \alpha.$$  

And $V_kF(x) > \alpha$, so consequently $V_kF'(x) > \alpha$.  

Then let be

$$g_0(x) + R(x) = g_1(x)$$

and

$$h_0 + c(x)\phi'_l(x) = h_1(x)$$

$$V_k\left(f - g_1(x)h_1(x)\right) > \alpha.$$  

and

$$g_1(x) \sim g_0(x) \quad \text{in } V_k \quad (18.3)$$

When $\deg g_0 > \deg c(x)\phi'_k(x)$, let be $g_1(x) = g_0(x) + c(x)\phi'_k(x)$.

In this case, also $g(x) \sim g_0(x)$ in $V_k$.

In the same way

$$h_1(x) \sim h_0(x) \quad \text{in } V_k \quad (18.4)$$

$$g_d + h_0m \sim 1 \quad \text{in } V_k,$$

so by (18.3) and (18.4)

$$g_1 + h_1m \sim 1 \quad \text{in } V_k.$$

And

$$\deg g_1(x) = \deg g_0(x).$$

Therefore both $g_1(x)$ and $h_1(x)$ satisfy all conditions given to $g_0(x)$ and $h_0(x)$ and moreover

$$V_k\left(f - g_1h_1\right) > V_k\left(f - g_0h_0\right).$$

So we can repeat the same processes as above and obtain two sequences of canon polynomials $(g_1, g_2, \ldots, g_p, \ldots)$ and $(h_1, h_2, \ldots, h_p, \ldots)$, for which

$$g_p \sim g_0 \quad \text{in } V_k \quad \text{for } p = 1, 2, \ldots.$$  

$$h_p \sim h_0 \quad \text{in } V_k \quad \text{for } p = 1, 2, \ldots.$$  

and

$$g_p + h_pm \sim 1 \quad \text{in } V_k.$$  

and

$$\deg g_p = \deg g_0 \quad \text{for } p = 1, 2, \ldots.$$  

$$V_k\left(f - g_ph_p\right)$$

increases, as $p$ increases.
$V_k$ is a discrete valuation, then at last $V_k(f - g_p h_p)$ can exceed $\omega$. So

$$f \approx g_p(x) h_p(x) \quad \text{in } V_k.$$  

On extensions of Hensel’s Lemma I will again relate in §23.

**Corollary 18.2.**

All $\phi_1(x), \phi_2(x), \ldots, \phi_t(x)$ are canon polynomials that are prime each other in $V_k$ and $f(x)$ is a polynomial and $\omega$ is an arbitrary given positive number.

If

$$f(x) \approx \phi_1 \phi_2 \cdots \phi_t \quad \text{in } V_k,$$

then there exist such $t$ polynomials $\Psi_1(x), \Psi_2(x), \ldots, \Psi_t(x)$ that

$$f(x) \approx \Psi_1 \Psi_2 \cdots \Psi_t \quad \text{in } V_k,$$

where

$$\Psi_i \approx \phi_i \quad \text{in } V_k \quad \text{for } i = 1, 2, \ldots, t$$

and

$$\deg \Psi_i = \deg \phi_i \quad \text{for } i = 1, 2, \ldots, t - 1.$$

**Proof.** $\phi_1$ and $(\phi_2 \phi_3 \cdots \phi_t)$ are prime each other in $V_k$, so owing to Theorem 18.1, there exist two polynomials $\Psi_1(x)$ and $h_2(x)$ that

$$f \approx \Psi_1 h_2 \quad \text{in } V_k,$$

where

$$\Psi_1 \approx \phi_1 \quad \text{in } V_k$$

$$h_2 \approx \phi_2 \phi_3 \cdots \phi_t \quad \text{in } V_k$$

and

$$\deg \Psi_1 = \deg \phi_1.$$

$$V_k \Psi_1 = V_k \phi_1 = 0.$$

$\phi_2$ and $(\phi_3 \phi_4 \cdots \phi_t)$ are prime each other in $V_k$, so again by Theorem 18.1, there exist two polynomials $\Psi_2(x)$ and $h_3(x)$ that

$$h_2 \approx \Psi_2 h_3 \quad \text{in } V_k \quad \text{and} \quad \Psi_2 \approx \phi_2 \quad \text{in } V_k$$

$$h_3 \approx \phi_3 \phi_4 \cdots \phi_t \quad \text{in } V_k$$

and

$$\deg \Psi_2 = \deg \phi_2.$$

$$V_k(f - \Psi_1 \Psi_2 h_3) = V_k \left[ (f - \Psi_1 h_2) + \Psi_1 (h_2 - \Psi_2 h_3) \right]$$

$$\geq \min \left[ V_k(f - \Psi_1 h_2), V_k \Psi_1 + V_k (h_2 - \Psi_2 h_3) \right]$$

$$> \omega.$$

Because $V_k \Psi_1 = 0$. 


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Therefore 
\[ f \approx \Psi_1 \Psi_2 h_3 \quad \text{in } V_k. \]

Thus, by repeating the same processes as above, we can prove this corollary.

Generally by D. §5 (4) a polynomial \( f(x) \) can be decomposed as follows:
\[ f(x) \sim e(x) \phi_1 \phi_1 \cdots \phi_t \quad \text{in } V_k. \] (18.5)

Let be 
\[ n_0 = q \tau_k + l \quad \text{and} \quad 0 \leq l \leq \tau_k - 1. \]

Then, by Theorem 16.8, we can change (18.5) as follows;
\[ e'(x) \phi_k^{\tau_k - l} f(x) \sim \phi_1 \phi_2 \cdots \phi_t \quad \text{in } V_k \]
where 
\[ \deg e'(x) < \deg \phi_k \quad \text{and} \quad 1 \leq \tau_k - l \leq \tau_k \]
and \( \phi_1, \phi_2, \ldots, \phi_t \) are canon polynomials whose key parts are respectively \( \phi_k^{q+1}, \phi_1', \ldots, \phi_t' \).

Then owing to Corollary 18.2 and Theorem 16.8
\[ e'(x) \phi_k^{\tau_k - l} f(x) \approx \Psi_1 \Psi_2 \cdots \Psi_t \quad \text{in } V_k, \] (18.6)
where
\[ \Psi_0 \sim \phi_0 \sim \phi_k^{q+1} e_0(x) \quad \text{in } V_k \]
\[ \Psi_i \sim \phi_i \sim e_i(x) \phi_i \quad \text{in } V_k \quad \text{for } i = 1, 2, \ldots, t, \]
and
\[ \deg e_i(x) < \deg \phi_k \quad \text{for } i = 0, 1, 2, \ldots, t. \]

§ 19. Application of Hensel’s lemma

When 
\[ f(x) \sim e(x) \phi_1 \phi_1 \cdots \phi_t \quad \text{in } V_k, \]
if we want to make an augmented valuation \( V_{k+1} \) of \( V_k \) for which \( V_{k+1} f(x) < V_k f(x) \), we must select one polynomial \( \phi_i \) out of \( \phi_1, \phi_2, \ldots, \phi_t \) and must increase a value of \( \phi_i \). And \( \phi_1, \phi_2, \ldots, \phi_t \) are prime each other in \( V_k \), so if
\[ V_k \phi_1 < V_{k+1} \phi_1, \quad \text{then} \]
\[ V_k \phi_i = V_{k+1} \phi_i \quad \text{for } i = 2, 3, \ldots, t. \]

Consequently, when we make an augmented valuation \( V_{k+t} \) of \( V_k \) for which \( V_k f(x) < V_{k+t} f(x) \), by (18.6)
if
\[ V_k \Psi_1 < V_{k+1} \Psi_1, \quad \text{then} \]
\[ V_k \Psi_i = V_{k+1} \Psi_i \quad \text{for } i = 2, 3, \ldots, t. \] (18.7)

Let be
\[ V_k f < V_{k+1} f < \cdots < V_t f \quad \text{and} \]
\[ V_t f - V_k f = \omega. \] (18.8)

Then
\[ V_t (e'(x) \phi_k^{\tau_k - l} f - \Psi_0 \Psi_1 \cdots \Psi_t) \]
\[ \geq V_k (e' \phi_k^{\tau_k - l} f - \Psi_0 \Psi_1 \cdots \Psi_t) \]
\[ v_k \text{ is an augmented valuation of } V_k \]
\[ > \omega + V_k(e' \phi_k^{k-t} f) \quad \text{by (18.6)} \]
\[ = \omega + V_k f + V_k e' \phi_k^{k-t} = V_s f + V_k e' \phi_k^{k-t} \quad \text{(by 18.8)} \]
\[ = V_s(e' \phi_k^{k-t} f) \]
\[ \therefore V_k e' \phi_k^{k-t} = V_s e' \phi_k^{k-t} \]

So
\[ e' \phi_i^{k-t} f \sim \psi_0 \psi_1 \ldots \psi_t \quad \text{in } V_s. \]

By (18.6)
\[ V_s(e' \phi_i^{k-t} f) = V_s(\psi_0 \psi_1 \ldots \psi_t). \]

\[ \therefore V_s(e' \phi_i^{k-t} f) - V_k(e' \phi_i^{k-t} f) = V_s(\psi_0 \ldots \psi_t) - V_k(\psi_0 \ldots \psi_t), \]

by (18.7)
\[ V_s f - V_k f = V_s \psi_1 - V_k \psi_1. \]

\[ V_s f - V_k f = \omega \quad (18.8) \]

Therefore, if we want to make a series of augmented inductive valuations
\[ V_k < V_{k+1} < \cdots < V_s \]
for which
\[ V_k f < V_{k+1} f < \cdots < V_s f = \omega + V_k f, \]
we may make such a series of augmented inductive valuations
\[ V_k < V_{k+1} < \cdots < V_s \]
for which
\[ V_k \psi_1 < V_{k+1} \psi_1 < \cdots < V_s \psi_1 = \omega + V_k \psi_1, \]
where \( \psi_1 \) is one factor in (18.6).

§ 20. Residue class ring of \( K[y] \) with respect to \( V_{0,q} \).

\( V_{0,q} \) is a valuation of \( K[y] \) and is the last valuation of a series of augmented inductive valuations which are extensions of \( V_{00} \) to \( K[y] \).

\[ V_{01} < V_{02} < \cdots < V_{0,q-1} < V_{0,q} \]
\[ V_{01} = [V_{00}, V_{01} y = \nu_1] \]
\[ V_{0i} = [V_{0,i-1}, V_{0i} \xi(y) = \nu_i] \quad \text{for } i = 2, 3, \ldots, q. \]
\[ V_{0q} = [V_{0,q-1}, V_{0q} \zeta(y) = \nu_q] \]

Then the residue class ring of \( K[y] \) by \( V_{0,q} \) is made as follows by M. Theorem 12.1.

\( \Gamma_{0,q-1} \) is a set of values of all polynomials \( f(y) \) by \( V_{0,q-1} \), and \( \sigma \) is the smallest integer such that \( \sigma \nu_0 e \in \Gamma_{0,q-1} \). \( a(y) \) is such a polynomial of \( y \) for which
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$V_{oq}a(y)\zeta_q^e = 0$.

$H_{oq}$ is the homomorphism which makes a polynomial $f(y)$ whose value by $V_{oq}$ is non-negative, correspond to its residue class $H_{oq}f(y)$ in $F_{oq} [Y]$, where $H_{oq}(a(y)\zeta_q^e) = Y$ is a transcendental element with respect to $F_{oq}$ and $F_{oq}$ is an algebraic extension of the residue class field $F_{oq}$ of $K$ by $V_{oq}$.

$V_{1q}$ is a valuation of a ring of polynomials of $x K_y [x]$ whose coefficient field $K_y$ has the valuation $V_{oq}$.

Let be

$V_{1q} = [V_{oq}, \ V_{1q}x = \mu_1],$

namely, if

$f(x, y) = \sum_{i=0}^{n} f_i(y) x^i,$

then

$V_{1q}f(x, y) = \text{Min} \{ V_{oq}f_i(y) + i\mu_1 \}.$

Now I will make the residue class ring of $K_y [x]$ by $V_{1q}$.

$\beta$ is the smallest integer such that $\beta \mu_i \in \Gamma_{oq},$

where $\Gamma_{oq}$ is a set of values of all polynomials of $y$ by $V_{oq}$. Then $b(y)\zeta_q^e$ is such a polynomial of $y$ that

$V_{1q} b(y)\zeta_q^e x^\beta = 0,$

where $\text{deg} \ b(y) < \text{deg} \ \zeta_q^e(y)$ and $0 \leq \beta < \sigma$.

$H_{1q}$ is the homomorphism that makes polynomials $f(x, y)$ of $y$ and $x$ whose values by $V_{1q}$ are non-negative, correspond to their residue classes $H_{1q}f(x, y)$. Let be $H_{1q} b(y)\zeta_q^e x^\beta = X_1$, which is a transcendental element with respect to the residue class field $F_{oq}(Y)$ of the coefficient field $K_y$ by $V_{oq}$.

Then the residue class ring of $K_y [x]$ by $V_{1q}$ is $F_1^+ [X_1]$, where $F_1^+ = F_{oq}(Y)$, owing to M. Theorem 10.2. But, here I will specially seek the residue class of $g(x, y)$ which is a polynomial of $x$ and $y$ and that of $\phi(x)$ which is a polynomial of $x$, because we shall use them, when we solve the "Principal Problem".

Let be

$V_{1q} g(x, y) = 0.$

$g(x, y) \sim \sum_i g_i(y) x^i \quad \text{in } V_{1q},$

and

$g_i(y) \sim \sum_j l_{ij}(y) \zeta_q^j \quad \text{in } V_{1q},$

where $\text{deg}_y l_{ij}(y) < \text{deg}_y \zeta_q^e(y)$.

Then

$g(x, y) \sim \sum_{i, j} l_{ij} x^i \quad \text{in } V_{1q}.$

Let be $b'(y)$ and $a'(y)$ such polynomials that

$a(y)a'(y) \sim 1$ and $b(y)b'(y) \sim 1 \quad \text{in } V_{oq}.$
and 

\[ \deg b'(y) < \deg \zeta_q(y). \]

\( \beta | i, \) because \( V_{\nu} l_{ij} \zeta_{\nu}^i x^j = 0 \quad \therefore \quad i = \beta \gamma. \)

\[ l_{ij} \zeta_{\nu}^i x^j \sim \frac{(b(y) \zeta_{\nu}^i x^j)^{\gamma}}{(a(y) \zeta_{\nu}^i)^{\delta}} \frac{(b'(y)) l_{ij}}{(a'(y))^{\delta}} \quad \text{in } V_{\nu}, \]

where \( j = \pi r - \sigma \theta. \)

\[ \therefore \quad H_{\nu}(l_{ij} \zeta_{\nu}^i x^j) = \frac{X_1^r}{Y^\delta} d_{\nu}, \]

where \( d_{\nu} = H_{\nu} \frac{(b'(y))^r l_{ij}}{(a'(y))^{\delta}} \in F_{0\nu}. \)

From (20.1) \( \pi r - \sigma \theta \geq 0, \) so \( i \geq \frac{\sigma}{\pi} \theta, \)

where \( \frac{\sigma}{\pi} \) is a constant number.

So \( H_{\nu} g(x, y) = \sum_{\nu} d_{\nu} \frac{X_1^r}{Y^\delta}, \)

where \( d_{\nu} \in F_{0\nu} \) and \( r \geq \frac{\sigma}{\pi} \theta. \)

§ 21. Residue class of \( \phi(x) \)

**Definition 21.1.** When \( f(x, y) \) is a polynomial of \( x \) and \( y \) and an expansion of \( f(x, y) \) by \( \zeta_q(y) \) and \( x \) is

\[ f(x, y) = \sum_{ij} a_{ij}(y) \zeta_{ij}^i x^j, \]

\( \deg_y a_{ij}(y) < \deg_y \zeta_q(y) \) for all terms in (21.1)

if \( V_{\nu} f(x, y) = V_{\nu} a_{ij}(y) \zeta_{ij}^i x^j \) for all terms in (21.1)

then \( f(x, y) \) is said to be homogeneous in \( V_{\nu}. \)

**Lemma 21.2.** If both \( f(x) = \sum_i f_i x^i \) and \( g(x) = \sum_i g_i x^i \) are homogeneous in \( V_{10} \) and

\[ f(x) \sim g(x) \quad \text{in } V_{10}, \]

then \( f_i \sim g_i \quad \text{in } V_{00} \) for all terms in \( f(x) \) and \( g(x). \)

**Proof.** This is evident from its definition.

**Lemma 21.3.** If both \( f(x, y) = \sum_{i,j} f_{ij}(y) \zeta_{ij}^i x^i \)

(21.2)
and $$g(x, y) = \sum_{i,j} g_{i,j}(y) \zeta_q^i x^j$$ (21.3) are homogeneous in $V_{1q}$, where 
$$\deg f_{i,j}(y) < \deg \zeta_q(y)$$ and 
$$\deg g_{i,j}(y) < \deg \zeta_q(y),$$ for all terms in (21.2) and (21.3),
then 
$$g_{i,j}(y) \sim f_{i,j}(y)$$ in $V_{1q}$ for all terms in (21.2) and (21.3).

**Proof.** Let be 
$$f(x, y) = \sum_i (\sum_j f_{i,j}(y) \zeta_q^j) x^i$$ and 
$$g(x, y) = \sum_i (\sum_j g_{i,j}(y) \zeta_q^j) x^i$$ 
$$f(x, y) \sim g(x, y)$$ in $V_{1q}$.

Then by Lemma 21.2 
$$\sum_j f_{i,j}(y) \zeta_q^j \sim \sum_j g_{i,j}(y) \zeta_q^j$$ in $V_{0q}$.

Then, owing to M. Lemma 16.2

$$f_{i,j}(y) \sim g_{i,j}(y)$$ for all terms in (21.2) and (21.3).

Now I will seek the residue class of $\phi(x)$ in the residue class ring of $K_y[x]$ by $V_{1q}$.

When $\Gamma_{00}$ is a set of values of all elements in $K$ by $V_{00}$, let be $\tau$ the smallest integer such that $\tau \mu_i \in \Gamma_{00}$ and $\lambda$ the smallest integer that $\lambda \mu_i \in \Gamma_{0,q-1}$, where $\Gamma_{0,q-1}$ is a set of all values of $f(y)$ by $V_{0,q-1}$.

$V_{1q} x^i \in \Gamma_{0,q-1}$, then there exists such a polynomial $c(y)$ that

$$V_{1,q-1}(c(y)x^i) = 0,$$

where 

$$\deg c(y) < \deg \zeta_q(y).$$

We multiply the both sides by the least common multiple of all denominators in (21.4), then

$$c(y) (a(y))^\gamma x^i \sim \sum_{i,j} d_{i,j}^\gamma(y) \frac{(b(y) \zeta_q^i x^j)^\gamma}{(c(y) \zeta_q^j)^\gamma}$$ in $V_{1q}$ (21.5).

We decrease degrees of all coefficients in (21.5) less than that of $\zeta_q(y)$. 

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\[ c'(y)x^i \sim \sum d'(y)\zeta_q^x x^{\beta r}/\zeta_q^\sigma \quad \text{in } V_{1q}. \]  
(21.6)

Then, owing to Lemma 21.3, \( c'(y)x^i \) is equivalent to only one term \( d'(y)\zeta_q^x x^{\beta r}/\zeta_q^\sigma \) in the left side of (21.6).

So
\[ c'(y)x^i \sim d'(y)\zeta_q^x x^{\beta r}/\zeta_q^\sigma \quad \text{in } V_{1q}, \]
where
\[ on = \pi \tau \quad \text{and} \quad \lambda = \beta \tau \]
and
\[ c'(y) \sim d'(y) \quad \text{in } V_{0q}. \]

\[ \therefore \quad H_{1q}(c(y)x^i) = d \frac{X_r^i}{Y_n}, \]
where \( \tau \) is a constant number and \( n = \frac{\pi}{\sigma} \tau \) is also a constant number, and \( d \in F_{0q}. \)

Now we can seek the residue class of a polynomial \( \phi(x) \) which is a polynomial of \( x \), in the residue class ring of \( K_v[x] \) by \( V_{1q} \). We select such a number \( k \) in \( K \) that
\[ V_{1q}k\phi(x) = V_{10}k\phi(x) = 0. \]

And let the homogeneous part of \( k\phi(x) \) be \( \sum_{i} a_i x^i \). (M. Lemma 16.2)
\[ V_{10}(a_i x^i) = 0 \quad \therefore \quad \lambda | i, \quad \text{so} \quad i = \lambda i'. \]
\[ a_i x^i = a_i x^{i'\prime} \sim (c(y)x^i)^{i''}(c'(y))^{i'''}a_i \quad \text{in } V_{1q}, \]
where
\[ c'(y)c(y) \sim 1 \quad \text{in } V_{1q} \quad \text{and deg } c'(y) < \text{deg } \zeta_q(y). \]
\[ a_i x^i \sim (c(y)x^i)^{i''} b_i(y) \quad \text{in } V_{1q}, \]
where
\[ b_i(y) \quad \text{is a } V_{1q}-\text{unit.} \]

So
\[ H_{1q}(a_i x^i) = b_i \left( \frac{X_r^i}{Y_n} \right)^{i''} \]
\[ H_{1q}(k\phi(x)) = \sum \bar{b}_i \left( \frac{X_r^i}{Y_n} \right)^{i''} \quad \text{where} \quad \bar{b}_i, \in F_{0q} \]  
(21.7)

and both \( \tau \) and \( n \) are constant numbers.

Therefore, if we substitute \( z \) for \( \frac{X_r^i}{Y_n} \), then
\[ H_{1q}(k\phi(x)) = \sum \bar{b}_i z^{i''}, \]  
(21.8)
where
\[ z = H_{1q}(c(y)x^i). \]
§ 22. Solution of the "Principal Problem"

I finished making preparations for solution of the Principal Problem, now I will begin the solution.

Principal Problem

$V_{0q}$ is a given valuation of $K[y]$ and $V_{10}$ is a given one of $K[x]$ and $V_{0q}$ and $V_{10}$ induce the same valuation $V_{00}$ in $K$. In the case, this problem is how we should make such valuations $W$ of $K[x,y]$ that induce $V_{0q}$ in $K[y]$ and $V_{10}$ in $K[x]$.

Solution

Let be

$$V_{0q} = [V_{00}, V_{01}y = \nu_1, \ldots, V_{0q}y(y) = \nu_q]$$

and

$$V_{p0} = [V_{00}, V_{10}x = \mu_1, \ldots, V_{p0}y(y) = \mu_p]. \quad (22.1)$$

Then this problem is that we make such extensions $W$ of the valuation $V_{0q}$ to the ring $K_y[x]$ that induce in $K[x]$ the valuation $V_{p0}$.

Let be $W_1 = [V_{0q}, W_1x = V_{10}x = \mu_1]$ an extension of $V_{0q}$ to $K_y[x]$ and the shortest series of $x$-simply augmented inductive valuations of $K_y[x]$ binding $W_1$ and $W$, as follows; (§ 3)

$$W_1 < W_2 < \cdots < W_{t-1} < W_t = W. \quad (22.2)$$

A series of augmented inductive valuations which the valuations in (22.2) induce in $K[x]$ is

$$V_{10} < V'_1 < \cdots < V'_{t-1} < V'_t.$$

When we bind $V_{10}$ and $V'_1$ with a series of continuous augmented inductive valuations and obtain the following series;

$$V_{10} < V'_2 < V''_3 < \cdots < V''_{t-1} < V'_{t}. \quad (22.3)$$

Then

$$W_1\phi_p < W_2\phi_p < \cdots < W_{t-1}\phi_p < W_t\phi_p = W\phi_p = \mu_p,$$

and

$$V_{10}\phi_p < V'_2\phi_p < \cdots < V'_{t-1}\phi_p < V'_{t}\phi_p = \mu_p.$$

and

$$V_{10}\phi_p < V'_2\phi_p < \cdots < V''_{t-1}\phi_p < V''_{t}\phi_p = \mu_p = V_{p0}\phi_p. \quad (22.4)$$

Then, from (22.1) and (22.4), owing to Theorem (13.2) we see that the two series (22.1) and (22.3) are on the same course and $V'_t = V_{p0}$.

Namely in this problem we must find such a series (22.2) that satisfies all conditions that I related above. But when we practically make a series of $x$-simply augmented inductive valuations binding $W_1$ and $W$, we need not make the shortest series, but we may make a series of $x$-simply augmented inductive valuations which appear naturally at that time.
I am afraid that you, readers of this paper, may think that my explanation here is very complex. The reason why I explained as above is as follows; if I make a series \( \alpha \) of augmented inductive valuations of \( K_y[x] \) binding \( W_1 \) and \( W \) and make a series \( \beta \) of augmented inductive valuations which \( \alpha \) induces in \( K[x] \), it may happen that \( \beta \) is not a series of augmented inductive valuations as I shall explain later in this paper.

Now we want to make such an augmented valuation \( W_2 \) of \( W_1 \) of \( K_y[x] \) that \( W_1 \phi_p < W_2 \phi_p \). For the sake of it, we must find a decomposition of \( \phi_p(x) \) in \( W_1 \) as in D. §5 (4). \( W_1 \) induces \( V_{10} \) in \( K[x] \) and according to Theorem 14.1

\[
\phi_p(x) \sim \phi_2(x)^{b_2} \quad \text{in } V_{10} \quad \text{and} \quad \deg \phi_p(x) = \deg \phi_2(x)^{b_2}.
\]

Namely

\[
\phi_p(x) \sim \phi_2(x)^{b_2} \quad \text{in } W_1,
\]

where \( \phi_2(x) \) is the key polynomial which produces

\[
V_{20} = \left[ V_{10}, V_{10} \phi_2(x) = \mu_2 \right].
\]

Therefore in this case we may find a decomposition of \( \phi_2(x)^{b_2} \) in \( W_1 \) of \( K_y[x] \).

\[
\phi_2(x) = x^{m+ \cdots} + a_\tau x^{\tau} + \cdots + a_0,
\]

where

\[
a_\tau \in K \quad \text{and} \quad V_{10} \phi_2 = W_1 \phi_2 = V_{10} a_\tau x^{\tau}
\]

for all terms in (22.4), because \( \phi_2 \) is homogeneous in \( V_{10} \), namely in \( W_1 \) and \( \tau \) is the smallest integer such that \( \tau \mu_i = V_{10} x^{\tau} \in \Gamma_{10} \) which is a set of values of all elements in \( K \). Let be \( k \) such a number in \( K \) that \( W_1(k \phi_2(x)) = 0 \). In order to find a decomposition of \( (k \phi_2(x))^{b_2} \) in \( W_1 \), we must find factorization of the residue class \( H(k \phi_2)^{b_2} \) in the residue class ring \( F_{1}^*[X_1] \) of the ring \( K_y[x] \) by \( W_1 \), where \( H \) is the natural homomorphism which makes every polynomial \( f(x, y) \) of \( x \) and \( y \) whose value by \( W_1 \) is non-negative, correspond to its residue class \( Hf(x, y) \). By (21.7) and (21.8) and from the fact that \( \phi_p \sim \phi_2^{b_2} \) in \( W_1 \),

\[
H(k^r \phi_p) = H(k \phi_2)^{b_2} = \left( \sum_i b_i \left( \frac{X_i}{Y^n} \right)^{b_2} \right) = \left( \sum_i b_i z_i^{r^{b_2}} \right), \quad \text{where } b_i \in F_{0y}.
\]

Both \( r \) and \( n \) are constants, so factorization of \( \left( \sum_i b_i \left( \frac{X_i}{Y^n} \right)^{r^{b_2}} \right) \) in \( F_{1}^*[X_1] \), where \( F_{1}^* = F_{0y}(Y) \), is the same as that of \( \left( \sum_i b_i z_i^{r} \right) \) in \( F_{0y}[z] \).

So

\[
\bar{b}(\sum_i b_i z_i^{r})^{b_2} = \bar{t}_1(z)^{a_1 b_2} \cdots \bar{t}_n(z)^{a_n b_2},
\]

(22.6)
where \( b \) is a suitable \( F_q \)-unit and every \( \overline{l_i(z)} \) is an irreducible canon class. In (22.6), \( z \) does not appear as a factor, because the last term of \( \sum \overline{h_i} z^i \), namely the constant term, is not zero.

Let be 
\[
c(y)x^i = \Phi \quad \text{and} \quad W_1 \Phi = 0
\]
then
\[
b(y)k^p \phi_p \sim b(y)(k\phi_2)^{s_2} \sim l_1(\Phi)^{s_1} \cdots l_t(\Phi)^{s_t} \quad \text{in} \quad W_1,
\]
where
\[
H\left(b(y)\right) = \overline{b} \quad \text{and} \quad \deg b(y) < \deg \zeta_q(y)
\]
and every \( l_i(\Phi) \) is an irreducible canon polynomial in \( W_1 \) whose irreducible canon class is \( \overline{l_i(z)} \) in \( F_{1*}[X] \) and
\[
z = \frac{X_1'}{Y^n}
\]
and
\[
z = \frac{X_1'}{Y^n} = H\left(c(y)x^j\right) = H(\Phi).
\]
Then according to Theorem 18.1
\[
b(y)k^p \phi_p(x) = \Psi_1 \Psi_2 \cdots \Psi_t \quad \text{in} \quad W_1,
\]
where
\[
\Psi_i \sim l_i(\Phi)^{s_i} \quad \text{in} \quad W_1 \quad \text{for} \quad i = 1, 2, \cdots, t.
\]
According to explanation in §19, herein we decide a value of \( \omega \) so that
\[
\omega = V_{\nu_0} \phi_p(x) - W_1 \phi_p(x).
\]
And we must make such a series of \( x \)-augmented inductive valuations that
\[
W_1 < W_2 < \cdots < W_t
\]
and
\[
W_1 \Psi_1 < W_1 \Psi_1 < \cdots < W_t \Psi_1 = \omega + W_1 \Psi_1.
\]
We may select any \( \Psi_i \) out of \( \Psi_1, \cdots, \Psi_t \), so herein we selected \( \Psi_1 \).
\[
\Psi_1 \sim l_1(\Phi)^{s_1} \quad \text{in} \quad W_1.
\]
Let the key part of \( l_i(\Phi) \) be \( \phi(x) = \sum l_i(y)x^{d_i} \),
then
\[
d(y) \Psi_1 \sim (\phi(x))^{s_1} \quad \text{in} \quad W_1,
\]
where \( \phi(x) \) is a homogeneous key polynomial in \( W_1 \), and
\[
\deg d(y) < \deg \zeta_q(y) \quad \text{and} \quad \deg l_i(y) < \deg \zeta_q(y).
\]
Let \( \sum h_i(y, x)(\phi(x))^i \) be an expansion of \( d(y) \Psi_1 \) by \( \phi(x) \), then \( m \geq s_ib_i \) and \( h_i(y, x) \) are polynomials of \( x \) and \( y \), because the coefficient of the term of the highest power of \( \phi \) is 1. And degrees of \( h_i(y, x) \) with respect to \( x \) are less than that of \( \phi(x) \). We write it as follows; \( \deg x h_i(y, x) < \deg x \phi(x) \).

Now we must make such an augmented valuation \( W_2 \) of \( W_1 \) that
$W_2 Y > W_1 Y$.

Here the three following cases occur.

α-case is the one when $d(y) Y = (\varphi(x))^{s,b_1}$.

β-case is the one when $W_1 h_0(y, x) < \omega + W_1 d(y) Y$.

γ-case is the one when $W_1 h_0(y, x) > \omega + W_1 d(y) Y$.

When the α-case occurs, we may decide a value of $\varphi(x)$ by $W_2$ so that

$$W_2 (\varphi(x))^{s,b_1} = \omega + W_1 Y + W_1 d(y),$$

namely

$$W_2 \varphi(x) = \frac{1}{s_1 b_2} \left( W_1 Y + \omega + W_1 d(y) \right).$$

$$d(y) Y \sim (\varphi(x))^{s,b_1} \quad \text{in } W_1$$

so

$$W_1 d(y) + W_1 Y = s_1 b_2 W_1 \varphi(x)$$

$$W_2 \varphi(x) = \frac{1}{s_1 b_2} \left( s_1 b_2 W_1 \varphi(x) + \omega \right)$$

$$W_2 \varphi(x) > W_1 \varphi(x).$$

Then as I explained in §19, where I used $f(x)$ for $\phi_p(x)$,

$$W_2 \phi_p(x) = V_{s0} \phi_p(x) = \mu_p.$$

So, when the α-case occurs, thus the problem is solved completely. When the β-case occurs, we must decide the value $W_2 \varphi(x)$ so that $d(y) Y$ can be decomposed in $W_2$ as follows;

$$d(y) Y \sim \varphi(x)^{n_1} \rho_1(x)^{n_1} \ldots \rho_r(x)^{n_r} \quad \text{in } W_2, \quad (22.7)$$

where every $\rho_r(x)$ is a homogeneous key polynomial in $W_2$ and at least such one $\rho_r(x)$ must appear in this decomposition.

For the sake of it, plot the points $P_i = (s_i b_2 - i, W_1 (h_1(y, x) \varphi(x)^i))$ in a cartesian plane, where $i$ moves from zero to $s_i b_2$ and make the Newton polygon of these points. (D §5)

Let one side of the polygon be $P_a P_{r_c}$, where $a > c$.

Namely

$$P_a = (s_i b_2 - a, W_1 (h_a \varphi^a))$$

$$P_c = (s_i b_2 - c, W_1 (h_c \varphi^c)).$$

Now we decide the value $W_2 \varphi$ so that $W_2 (h_a \varphi^a) = W_2 (h_c \varphi^c)$.

Then $W_2 h_a = W_1 h_a$ and $W_2 h_c = W_1 h_c$, so

$$W_2 \varphi(x) = \frac{1}{a - c} (W_1 h_c - W_1 h_a).$$
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The slope of $P_aP_e$ is $\frac{W_1(h_\varphi)-W_1(h_\varphi^n)}{a-c}$ which is equal to $W_2\varphi(x)-W_1\varphi(x)$.

The ordinate of $P_{a,b_1}$ is the lowest, so the slope of $P_aP_e$ is positive. Then $W_2\varphi(x)>W_1\varphi(x)$.

$\varphi(x)$ is a homogeneous key polynomial in $W_1$. So $W_2$ is an augmented valuation of $W_1$ such that $W_2\varphi(x)>W_1\varphi(x)$.

And all other points of $P_0, P_1, P_2, \ldots, P_{a,b_1}$ except $P_a$ and $P_e$ are above the straight line $P_aP_e$ or are on it. From this fact, through simple calculation of analytical geometry we find that

$$W_2d(y)\psi_1 = \min_i [W_1 h_i\varphi] = W_2 h_a\varphi^n = W_2 h_1\varphi$$

and

$$d(y)\psi_1 \sim h_1\varphi^n + \cdots + h_r\varphi$$

in $W_2$ and we obtain the decomposition (22.7). Namely $n_0 = a-c$ in (22.7).

When $m>s\varepsilon_2$, by simple calculation we find that $W_2(h_\varphi^n) > \min_i [W_2 h_i\varphi]$.

The Newton Polygon which I present here is pretty different from that which MacLane presented in D §5. But I use this polygon in this paper.

The first reason is that a number of sides of this polygon is much less than that of the one which MacLane presented in his paper. The second reason is that, when we make an augmented valuation $W_2$ of $W_1$, we elevate each point $P_i$ and we must make at least one straight line binding these two points be parallel to $x$-axis so that projection of $W_2$ of $d(y)\psi_1$ becomes positive.

On my polygon, the position of $P_{a,b_1}$ is the lowest, because $d(y)\psi_1 \sim h_a\varphi^n$ in $W_1$. And as the valuations increase from $W_1$ to $W_2$, we can inspect geometrically elevating movement of value of each term $h_i\varphi^i$, namely elevating movement of each point $P_i$. And $W_2\varphi(x)$ is always a sum of $W_1\varphi(x)$ and the slope of $P_aP_e$.

I call this Newton Polygon "Neo-Newton Polygon".

Again, from (22.7), by Theorem 18.1 we obtain the following decomposition;

$$e(y, x)\psi_1 \sim \psi_0^* \psi_{-1}^* \cdots \psi_{-r}^*$$

in $W_2$.

where

$$e_0(y, x)\psi_{-0}^* \sim \varphi(x)^{n_0}$$

in $W_2$

$$e_i(y, x)\psi_{-i}^* \sim \rho_i(x)^{n_i}$$

in $W_2$ for $i=1, 2, \ldots, r$.

and

$$\deg x e(y, x) < \deg x \varphi(x)$$

and

$$\deg x e_i(y, x) < \deg x \varphi(x)$$

for $i=0, 1, 2, \ldots, r$. 

$\varphi(x)$ and $\rho_i(x)$ may include $y$, but they are $x$-key polynomials, so purposely I wrote like this.

Now we must make such an augmented valuations $W_3$ of $W_2$ that increases value of one $\Psi_{*t}$ and only $\Psi_{*t}$ out of $\Psi_{*t}, \ldots, \Psi_{*r}$,

$$e_i(y, x)\Psi_{*1} \sim \rho_1(x)^n \quad \text{in} \ W_2.$$

So in the same way, the three cases occur.

If the $\alpha$-case occurs, then this problem can be solved as I did.

If the $\beta$-case occurs, then we repeat the same process that I did and we obtain such an augmented valuation $W_3$ of $W_2$ that

$$W_3\Psi_{*1}> W_2\Psi_{*1}.$$

Thus we get the following series of augmented inductive valuations

$$W_1 < W_2 < W_3 < \cdots,$$

where

$$W_1\Psi_1 < W_2\Psi_1 < W_3\Psi_1 < \cdots$$

and

$$W_t\phi_p(x) - W_{t-1}\phi_p(x) = W_t\Psi_1 - W_{t-1}\Psi_1$$

for $i = 2, 3, 4, \ldots$.

And $\mu_p = V_{p0}\phi_p(x)$ is a finite number and all augmented valuations that appear in this problem are discrete. So at last $W_k\phi_p$ arrives at $\mu_p$. As long as the $\beta$-cases occur, $W_k\phi_p$ can not arrive at $\mu_p$. So, at last the $r$-case or the $\alpha$-case must occur.

If the $r$-case occurs, then we can solve this problem at once as follows.

Let be

$$d(y, x)\Psi \sim \sigma(x)^r \quad \text{in} \ W_t$$

where $\sigma(x)$ is a homogeneous key polynomial in $W_t$ which is an augmented valuation of $W_1$ and $\deg_x d(y, x) < \deg_x \phi'(x)$, where $\phi'(x)$ is the key polynomial that produces $W_t$. Such value $W_{t+1}d\Psi = \Omega$ that makes $W_{t+1}\phi_p(x) = \mu_p$, can be calculated.

Let an expansion of $d\Psi$ by $\sigma(x)$ be

$$\sum_{t=0}^{m} l_t(y, x) (\sigma(x))^t$$

where

$$\deg_x l_t < \deg_x \sigma(x) \quad \text{for every term}$$

and

$$m \geq s \quad \text{and} \quad l_t(y, x) = 1.$$

The $r$-case occurs, namely $W_t l_0(y, x) \geq \Omega$.

Plot the points $P_i = (s-i, W_t l_i(y, x)\sigma^i))$ in a cartesian plane. But this time, $i$ moves from 1 to $s$. We do not plot $P_0$ and instead of $P_0$ we plot the point $Q = (s, \Omega)$. Here $\sigma = \sigma(x)$ and $l_i = l_i(x, y)$. 

Next we make the Neo-Newton Polygon of \( Q, P_1, P_2, \ldots, P_s \). Let one side of the polygon which passes through \( Q \) be \( QP_c \).

\[ P_c = (s - c, W_t(l_c \sigma^t)). \]

Then assign the value \( W_{t+1} \sigma(x) \) so that

\[ W_{t+1} \left( l_c(y, x) \left( \sigma(x) \right)^t \right) = \Omega. \]

\[ \text{deg}_x l_c(y, x) < \text{deg}_x \sigma(x), \quad \text{so} \quad W_{t+1} l_c(y, x) = W_t l_c(y, x). \]

So

\[ W_{t+1} \sigma(x) = \frac{1}{c} \left( \Omega - W_t l_c(y, x) \right). \]

Therefore

\[ W_{t+1} \sigma(x) > W_t \sigma(x). \]

\( \sigma(x) \) is a homogeneous key polynomial in \( W_t \), so \( W_{t+1} \) is an augmented valuation of \( W_t \). All other points of \( Q, P_1, \ldots, P_s \) except \( Q \) and \( P_c \) are above or on the straight line \( QP_c \) and from this fact, by simple calculation of analytical geometry, we can find that

\[ W_{t+1} d \Psi = \text{Min} \left[ W_{t+1} l_c(y, x) \left( \sigma(x) \right)^t \right] = \Omega. \]

Thus at last \( W_{t+1} \phi_p(x) \) can arrive at \( \mu_p \). And the Principal Problem is completely solved in every case. Because when \( s < t+1 \), \( W_s \phi_p < W_{t+1} \phi_p = V_{\nu_0} \phi_s(x) \). Therefore by Theorem 13.2 \( W_{t+1} \) induces the valuation \( V_{\nu_0} \) in \( K[x] \).

§ 23. Generalization of Hensel's lemma

Theorem 18.1 is pretty different from Hensel's Lemma. It is convenient for me to solve the Principal Problem with it. But in this paragraph I will try to make it approach the primitive type of Hensel's lemma as nearly as possible.

For the sake of it, we must make a complete ring of \( K[x] \) with respect to the valuation \( V_k \).

As I related in §12, MacLane introduced notion of "homogeneous" in M §16. First he made a set of representatives of elements of \( K \) and defined homogeneous polynomials and homogeneous inductive valuations and in M. Lemma 16, 2, he related that every polynomial \( f(x) \) is equivalent in \( V_k \) to one and only one homogeneous polynomial \( h(x) \). The \( h(x) \) is called the "homogeneous part" of \( f(x) \). Then an arbitrary polynomial \( f(x) \) is equivalent to
its homogeneous part \( f_1(x) \).

\[ f(x) \sim f_1(x) \quad \text{in } V_k. \]

Let be

\[ f(x) - f_1(x) = g_2(x), \]

\[ V_k g_2(x) > V_k f(x) = V_k f_1(x). \]

\( g_2(x) \) is equivalent to its homogeneous part \( f_2(x) \).

\[ f_2(x) \sim g_2(x) \quad \text{in } V_k \]

so, let be

\[ f(x) - f_1(x) - f_2(x) = g_3(x), \quad \text{then} \]

\[ V_k g_3(x) > V_k f_2(x) = V_k g_2(x) > V_k f_1(x). \]

When we repeat such processes, then \( g_{\eta+1}(x) \) becomes zero or

\[ \lim_{n \to \infty} V_k g_{\eta+1}(x) = \infty, \]

where

\[ g_{\eta+1}(x) = f(x) - f_1(x) - f_2(x) - \cdots - f_n(x), \]

because \( V_k \) is a discrete valuation.

In this case we let \( f(x) \) be equal to

\[ f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots. \]

Even if a number of terms which are included in \( f(x) \) is finite, it happens often that this sequence lasts endlessly. For example, when we select 0, 1, 2 as representatives of residue classes by mod 3 in 3-adic field, \( \frac{1}{2} \) is a sum of endless sequence as follows:

\[ \frac{1}{2} = 2 - 2 \cdot 3 + 2 \cdot 3^2 + \cdots + \left( -1 \right)^n 2 \cdot 3^n + \cdots. \]

We consider all such sequences that satisfy the following conditions;

\( h_1(x), h_2(x), \ldots, h_n(x), \cdots \)

are all homogeneous polynomials and a number of terms which are included in each \( h_n(x) \) is finite and

\[ V_k h_1(x) < V_k h_2(x) < \cdots < V_k h_n(x) < V_k h_{\eta+1}(x) < \cdots \]

Then a set \( M \) of all sums of these polynomials is a complete ring with respect to \( V_k \).

\[ M = \left\{ h_1(x) + h_2(x) + \cdots + h_n(x) + h_{\eta+1}(x) + \cdots \right\} \]

Let be

\[ h(x) = h_1(x) + \cdots + h_n(x) + \cdots. \]
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and 
\[ g(x) = g_1(x) + \cdots + g_n(x) + \cdots, \]
then a necessary and sufficient condition that \( h(x) = g(x) \) is that \( h_n(x) = g_n(x) \) for \( n=1, 2, \cdots \). And we let \( V_k h(x) \) be \( V_k h_1(x) \).

It may happen that some tails of these series consist of zero only.

**Lemma 23.1.**

If \( K[x] \) is a complete ring with respect to a homogeneous valuation \( V_k \) which a key polynomial \( \phi_k \) produces and \( V_k a(x) = 0 \) and \( \deg a(x) < \deg \phi_k \), then in \( K[x] \) there exists such a polynomial \( \alpha(x) \) that \( a(x) \alpha(x) = 1 \).

**Proof.**

\( \phi_k \) is irreducible in \( K[x] \) and \( \deg a(x) < \deg \phi_k \), so in \( K[x] \) there are such two polynomials \( \alpha'(x) \) and \( q(x) \) that

\[ a(x) \alpha'(x) + q(x) \phi_k = 1 \]

where \( \deg \alpha' < \deg \phi_k \) and \( \deg q < \deg a < \deg \phi_k \).

By M. Lemma 4.3

\[ V_k \phi_k > V_{k-1} q \phi_k \geq V_{k-1} 1 = 0 = V_k a \alpha' \].

& \therefore \quad \frac{1}{a(x)} = \alpha'(x) + \frac{1}{a(x)} q \phi_k. \quad (23.1) \]

Now in \((23.1)\) we substitute \( \alpha'(x) + \frac{1}{a(x)} q \phi_k \) for \( \frac{1}{a(x)} \) repeatedly and we obtain the following sequence

\[ \frac{1}{a(x)} = \alpha'(x) + \alpha'(x) q \phi_k + \alpha'(x) (q \phi_k)^2 + \cdots + \alpha'(x) (q \phi_k)^n \frac{1}{a(x)} \]

\[ V_k (q \phi_k) > V_k (a \alpha') = 0. \]

So we can transform the right side of \((23.2)\) into such a polynomial \( \alpha(x) \) in \( K[x] \) that \( a(x) \alpha(x) = 1 \), because \( K[x] \) is complete with respect to \( V_k \).

**Theorem 23.2.**

If \( K[x] \) is a ring in Lemma 23.1 and in \( K[x] \) there are such five polynomials \( f(x), g_0(x), h_0(x), l(x), m(x) \) that

\[ f(x) \sim g_0(x) h_0(x) \quad \text{in} \quad V_k \]

\[ g_0(x) l(x) + h_0(x) m(x) \sim 1 \quad \text{in} \quad V_k \]

and \( V_k l(x) \geq 0 = V_k g_0(x) = V_k h_0(x) \)

and \( D_{\phi_k} g_0(x) > 0 \) and \( D_{\phi_k} h_0(x) > 0 \).
then in $K[x]$ there exist such two polynomial $g(x)$ and $h(x)$ that $f(x) = g(x)h(x)$

and $g(x) \sim g_0(x)$ in $V_k$ and $h(x) \sim h_0(x)$ in $V_k$.

Proof.

$V_k g_0(x) = 0$ and $D_{\phi_k} g_0(x) > 0$

$V_k h_0(x) = 0$ and $D_{\phi_k} h_0(x) > 0$

so by Lemma 16.2 there are such two canon polynomials $g'(x)$ and $h'(x)$ in $V_k$ that

\[ g_0(x) \sim a(x)g'(x) \]
\[ h_0(x) \sim b(x)h'(x) \quad \text{in } V_k, \]

where

$V_k a(x) = V_k b(x) = 0$

and $\deg a(x) < \deg \phi_k$ and $\deg b(x) < \deg \phi_k$.

According to Lemma 23.1 let be

$a(x)\alpha(x) = b(x)\beta(x) = 1$.

$f \sim g_0h_0 \sim ag'bh'$ in $V_k$

so $a(x)\beta(x)f(x) \sim g'(x)h'(x)$ in $V_k$.

Owing to Theorem 18.1 there exist such polynomials $g_n(x)$ and $h_n(x)$ that

$a(x)\beta(x)f(x) \approx g_n(x)h_n(x)$ in $V_k$

where

$g_n(x) \sim g'(x)$ and $h_n(x) \sim h'(x)$ in $V_k$.

We repeat this process endlessly, then

\[ \lim_{n \to \infty} \omega = \infty \]

and

\[ a(x)\beta(x)f(x) = g^*(x)h^*(x) \]

where

$g^*(x) = \lim_{n \to \infty} g_n(x) \in K[x]$

$h^*(x) = \lim_{n \to \infty} h_n(x) \in K[x]$

because $K[x]$ is complete with respect to $V_k$.

$f(x) = a(x)\alpha(x)b(x)\beta(x)f(x) = a(x)g^*(x)b(x)h^*(x)$.

Now let be $g(x) = a(x)g^*(x)$ and $h(x) = b(x)h^*(x)$

then $f(x) = g(x)h(x)$. 
and
\[ g(x) = a(x)g^*(x) \sim a(x)g'(x) \sim g_0(x) \]
\[ h(x) = b(x)h^*(x) \sim b(x)h'(x) \sim h_0(x) \quad \text{in } V_x. \]

§ 24. Some examples of Principal Problem.

Example 24.1

Let \( V_{p_0} \) be an extension of \( V_{o_0} \) to \( K[x] \) and \( V_{o_p} \) an extension of \( V_{o_0} \) to \( K[y] \) and
\[ V_{p_0} = \left[ V_{o_0}, V_{10}x = \mu_1, V_{20}\phi_2(x) = \mu_2, \ldots, V_{p_0}\phi_p(x) = \mu_p \right] \]
\[ V_{o_p} = \left[ V_{o_0}, V_{o_1}y = \mu_1, V_{o_2}\phi_2(y) = \mu_2, \ldots, V_{p_0}\phi_p(y) = \mu_p \right]. \]

Namely the key polynomial \( \phi_i(y) \) which produces \( V_{o_p} \) of \( K[y] \) is equal to the polynomial which is made by substitute \( y \) for \( x \) in the key polynomial \( \phi_i(x) \) which produces \( V_{o_0} \). And their key values are equal to each other.

In this case, we can very easily make a valuation \( W \) of \( K[x, y] \) that induces \( V_{p_0} \) in \( K[x] \) and \( V_{o_p} \) in \( K[y] \).

Let \( W_1 = [V_{o_p}, W_1x = \mu_1] \) be an extension of \( V_{o_p} \) to \( K[y][x] \).

By the remainder theorem
\[ \phi_p(x) - \phi_p(y) = (x - y)Q(x, y) \quad \text{.} \quad (24.1) \]

Now we want to make such an augmented valuation \( W_2 \) of \( W_1 \) of \( K_y[x] \) that \( W_2\phi_p(x) = \mu_p \).

By M. Corollary 13.2 \( x - y \) is a key polynomial in \( W_1 \).

\[ W_1\phi_p(y) > W_1\phi_p(x), \quad \text{because} \]
\[ W_1\phi_p(y) = V_{o_p}\phi_p(y) = \mu_p = V_{o_0}\phi_p(x) > V_{10}\phi_p(x) = W_1\phi_p(x). \]

So
\[ \mu_p = W_1\phi_p(y) > W_1\phi_p(x) = W_1[(x - y)Q(x, y)]. \]
\[ W_1(x - y) = \text{Min} \left[ W_1x, W_1y \right] = \mu_1. \]

Let be
\[ \mu_p - W_1[(x - y)Q(x, y)] = \omega \quad \text{.} \quad (24.2) \]

Then we may decide so that \( W_2(x - y) = \mu_1 + \omega + 1 \), and
\[ W_2 = \left[ W_1, W_2(x - y) = \mu_1 + \omega + 1 \right]. \]

\( W_2 \) is an augmented valuation of \( W_1 \) of \( K_y[x] \) and induces \( V_{p_0} \) in \( K[x] \).

Because
\[ W_2[(x - y)Q(x, y)] = W_2(x - y) + W_2Q(x, y) \]
\[ \geq \mu_1 + \omega + 1 + W_1Q(x, y) \]
\[ = \mu_p + 1 \quad \because \text{by (24.2)} \]
\[ > \mu_p = V_{o_p}\phi_p(y) = W_2\phi_p(y). \]
So \[ W_2[(x-y)Q(x, y)] > W_2\phi_p(y) = \mu_p = W_2\phi_p(x) \] by (24.1).

If \( V_10 < V_20 < \cdots < V_{p_0} \) is the shortest series of augmented inductive valuations, when we make \( V_{p_0} \) from \( V_{o} \) in \( K[x] \); we must pass at least \( P \) stages by M. Theorem 15.3 and M. Theorem 16.3.

But in this case, we can make such an augmented valuation \( W_2 \) of \( W_1 \) of \( K_y[x] \) that induces \( V_{p_0} \) in \( K[x] \), only by one stage.

**Example 24.2.**

\( K \) is a set of all rational numbers and \( V_{o} \) is 3-adic valuation.

\( V_{10} = [V_{o}, V_{10}x = 0] \) is a valuation of \( K[x] \).

\( x^2 + 1 \) is a key polynomial in \( V_{10} \) of \( K[x] \) which was explained in Example 10.1. Then \( V_{20} = [V_{10}, V_{20}(x^2 + 1) = 1] \) is an augmented valuation of \( V_{10} \) of \( K[x] \).

\( \phi(x) = (x^2 + 1)^3 - 9(x^2 + 1) + 27 \) is a key polynomial in \( V_{20} \).

The reason is as follows.

\( F_1 \), the residue class field of \( K \) by \( V_{o} \) is \((-1, 0, 1)\).

\( V_{10}x = 0 \), so \( H_1x = x_1 \), then the residue class ring of \( K[x] \) by \( V_{10} \) is \( F_1[X_1] \), where \( H_1 \) is the natural homomorphism which makes the polynomials \( f(x) \) whose values by \( V_{10} \) are non-negative, correspond to their residue classes \( H_1f(x) \) in \( F_1[X_1] \).

\( x^2 + 1 \) is a key polynomial which produces \( V_{20} \).

\[ H_1(x^2 + 1) = x_1^2 + 1 = 0. \]

One root of this equation is \( i = \sqrt{-1} \).

\[ V_{20}\left(\frac{x^2 + 1}{3}\right) = 0 \] and \[ H_2\left(\frac{x^2 + 1}{3}\right) = x_2. \]

\[ F_2 = F_1(i). \] \( H_2 \) is the natural homomorphism of \( V_2 \).

Then the residue class ring of \( K[x] \) by \( V_{20} \) is \( F_2[X_2] \).

\[ H_2\left(\frac{(x^2 + 1)^3 - 9(x^2 + 1) + 27}{27}\right) = x_2^2 - x_2 + 1. \]

If \( x_2^2 - x_2 + 1 \) is irreducible in \( F_2[X_2] \), then \( (x^2 + 1)^3 - 9(x^2 + 1) + 27 \) is equivalent irreducible in \( V_{20} \).

If \( x_2^2 - x_2 + 1 \) is reducible in \( F_2[X_2] \), one of its factors must be an expression of \( X_2 \) of one degree. Let the factor be \( X_2 - (a + bi) \). Here both \( a \) and \( b \) are elements of \( F_1 \), because \( F_2 = F_1(i) \). We substitute \( a + bi \) for \( X_2 \) in \( X_2^3 - X_2 + 1 \), then
\[(a+bi)^3-(a+bi)+1\]
\[= a^3-b^3i-a-bi+1 \quad \therefore \quad V_{90} \text{ is 3-adic valuation.}\]
\[= a(a-1)(a+1)+1-ib(b^2+1)\]

If this expression is zero, then
\[a(a-1)(a+1)+1 = 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (24.3)\]
and
\[b(b^2+1) = 0 \]

\[F_1 = (-1, 0, 1), \quad \text{so (24.3) never happens.}\]

Then
\[(x^2+1)^3-x^2_2+1 \quad \text{is irreducible in } F_3[X] \text{ and } \]
\[(x^2+1)^3-9(x^2+1)+27 \quad \text{is equivalent irreducible in } V_{20}.\]

This expression is a key polynomial in \(V_{20}\) by M. Theorem 9.4.

\[V_{20} \varphi(x) = V_{20} \left[ (x^2+1)^3-9(x^2+1)+27 \right] = \text{Min} [3, 3, 3] = 3.\]

So \(V_{30} = [V_{20}, V_{30} \varphi(x) = 4]\) is an augmented valuation of \(V_{20}\) of \(K[x]\).

Next we make a valuation of \(K[y]\). Let be
\[V_{01} = \left[ V_{00}, V_{01} y = 0 \right] \quad \text{and} \quad V_{02} = \left[ V_{01}, V_{02}(y^2+1) = \frac{1}{2} \right].\]

Now we shall make such an extension \(W'\) of \(V_{30}\) to \(K_x[y]\) that induces \(V_{02}\) in \(K[y]\).

Let be \(W'_1 = [V_{30}, W'_1 y = 0]\) which is an extension of \(V_{30}\) to \(K_x[y]\).

\[y^2+1 \sim y^2-x^2 \quad \text{in } W'_1,\]

because
\[W'_1(y^2+1) = \text{Min} [W'_1 y^2, W'_1 1] = 0\]
\[W'_1 \left[ (y^2+1)-(y^2-x^2) \right] = W'_1(x^2+1) = V_{30}(x^2+1) = 1 > 0.\]
\[y^2+1 \sim (y-x)(y+x) \quad \text{in } W'_1.\]

So, in order to make such an augmented valuation \(W'_2\) of \(W'_1\) in \(K_x[y]\) that \(W'_2(y^2+1) > W'_1(y^2+1)\), we must make such a valuation \(W'_2\) that increases a value of \(y-x\) or that of \(y+x\).

By M. Corollary 13.2, both \(y-x\) and \(y+x\) can be key polynomials in \(W'_1\).

Let be
\[W'_2(y-x) = \frac{1}{2} > 0 = W'_1(y-x).\]

Then
\[y^2+1 = (y-x)^2 + 2x(y-x)+(x^2+1)\]
\[\quad \therefore \quad W'_2(y^2+1) = \text{Min} \left[ 1, \frac{1}{2}, 1 \right] = \frac{1}{2} = V_{02}(y^2+1).\]
Therefore $W_2$ induces $V_{02}$ by Theorem 13.2.
Next let us try to make an extension $W$ of $V_{02}$ to $K_y[x]$ that induces $V_{03}$ in $K[x]$.

\[
V_{02} = \left[ V_{02}, V_{01}, y = 0, \frac{V_{02}(y^2+1)}{2} \right].
\]

First we decide that $W_1 = [V_{02}, W_1, x=0]$.
$W_1$ is an extension of $V_{02}$ to $K_y[x]$.
Next we try to make such an augmented valuation $W_2$ of $W_1$ that

\[
W_2(x^2+1) = 1 > 0 = W_1(x^2+1).
\]

Because

\[
W_1[(x^2+1)-(x^2-y^2)] = V_{02}(y^2+1) = \frac{1}{2} > W_1(x^2+1)
\]

\[
x^2-y^2 \sim (x-y)(x+y) \quad \text{in } W_1.
\]

So, again by M. Corollary 13.2, both $x-y$ and $x+y$ can be key polynomials in $W_1$.

Let be

\[
W_2(x-y) = \frac{1}{2} > 0 = W_1(x-y).
\]

Then $W_2 = \left[ W_1, W_2(x-y) = \frac{1}{2} \right]$ is an augmented valuation of $W_1$ in $K_y[x]$.

\[
x^2+1 = (x-y)^2 + 2y(x-y) + (y^2+1)
\]

\[
W_2(x^2+1) = \text{Min} \left[ W_2(x-y)^2, W_2 2x(x-y), W_2(y^2+1) \right]
\]

\[
W_2(x^2+1) = \text{Min} \left[ 1, \frac{1}{2}, \frac{1}{2} \right] = \frac{1}{2}.
\]

\[
\frac{1}{2} = W_2(x^2+1) < 1 = V_{02}(x^2+1).
\]

But this is a natural result from the fact that

\[
W_1[(x^2+1)-(x-y)(x+y)] = \frac{1}{2}, \quad W_1(x^2+1) = 0
\]

and

\[
x^2+1 \sim x^2-y^2 \quad \text{in } W_1.
\]
In order to make such an augmented valuation $W_3$ of $W_2$ in $K_v[x]$ that $W_3(x^2+1) > W_2(x^2+1)$, there are two ways.

One way is that we make a decomposition of $x^2+1$ in $W_2$ and pick one $x$-key polynomial $\phi_3(x)$ out of factors of $x^2+1$ and make such an $x$-augmented valuation $W_3$ of $W_2$ that $W_3 \phi_3(x) > W_2 \phi_3(x)$.

The other way is that by Theorem 18.1 we find such two polynomials $g(x)$ and $h(x)$ that

$$x^2 + 1 \cong g(x) h(x) \quad \text{in } W_1,$$

where

$$g(x) \sim x - y \quad \text{in } W_1.$$

First, let us take the former way.

$$W_2(x - y)^2 = 1 = W_2(x^2 + 1).$$

So

$$x^2 + 1 \sim 2y(x - y) + y^2 + 1 \quad \text{in } W_1.$$

$$x^2 + 1 \sim 2y \left( (x - y) + \frac{y^2 + 1}{2y} \right) \quad \text{in } W_2.$$

Here we can simplify

$$x - y + \frac{y^2 + 1}{2y}$$

as follows.

Because

$$W_2(x - y) = W_2 \frac{y^2 + 1}{2y},$$

and

$$2 \sim 1 \quad \text{and} \quad \frac{1}{-y} \sim y \quad \text{in } W_2,$$

so

$$(x - y) + \frac{y^2 + 1}{2y} \sim (x - y) + y(y^2 + 1) \quad \text{in } W_2.$$

This is a decomposition of $x^2 + 1$ in $W_2$ and $(x - y) + (y^2 + y) = x + y^3$ is a homogeneous key polynomial $\phi_3(x)$ in $W_2$, because

$$W_1(x - y) = \frac{1}{2} = W_1 y(y^2 + 1) = \frac{1}{2}.$$

So we assign a value of $W_3(x + y^3)$ equal to 1 which is greater than

$$\frac{1}{2} = W_2(x + y^3) = \text{Min} \left[ W_2(x - y), W_1(y^2 + y) \right].$$
Then $W_3$ is an $x$-augmented valuation of $W_2$ and $W_3(x^2+1)=1$, because:

$$x^2 + 1 = (x + y)^2 - 2y(x + y) + (y^2 + 1) \quad (24.4)$$

$$y^2 + 1 = (y^2 + 1)^2 - 3(y^2 + 1)^2 + 3(y^2 + 1)$$

.$.$ $W_3(y^2 + 1) = \min \left[ \frac{3}{2}, 2, \frac{3}{2} \right] = V_{e_2}(y^2 + 1) = \frac{3}{2}$

$$W_3(x^2 + 1) = \min \left[ W_3(x + y)^2, W_3y^2(x + y^3), W_3(y^2 + 1) \right]$$

$$= \min \left[ 2, 1, \frac{3}{2} \right] = 1 \quad (24.5)$$

Next, let us take the latter way.

$$x^2 + 1 \sim (x - y)(x + y) \quad \text{in } W_1.$$  

Because

$$W_1 = [V_{e_2}, W_1, x = 0]$$

$$W_1(x - y) = 0 = W_1(x + y)$$

$$(x^2 + 1) - (x - y)(x + y) = y^2 + 1$$

$$W_1[(x^2 + 1) - (x - y)(x + y)] = W_1(y^2 + 1) = \frac{1}{2}.$$  

Therefore, along the proof of Theorem 18.1 we must find such two polynomials $l$ and $m$ of $x$ and $y$ that

$$W_1[(x^2 + 1) - \{(x - y) + (y^2 + 1)m\} \{(x + y) + (y^2 + 1)l\}] > \frac{1}{2}. \quad (24.6)$$

$$W_1(y^2 + 1) + W_1 \left[ 1 - (x - y)l - (x + y)m - (y^2 + 1)lm \right] > \frac{1}{2}$$

so

$$W_1 \left[ 1 - (x - y)l - (x + y)m - (y^2 + 1)lm \right] > 0$$

$$W_1(y^2 + 1)lm \geq W_1(y^2 + 1) = \frac{1}{2}.$$  

If we can find such two polynomial $l$ and $m$ that

$$1 - (x - y)l - (x + y)m = 1 + y^2$$

then

$$W_1 \left[ 1 - (x - y)l - (x + y)m \right] = W_1(1 + y^2) > 0 \quad \text{and } (24.6) \text{ holds.}$$

Assumed that

$$1 - x(l + m) + y(l - m) = 1 + y^2,$$

then

$$l + m = 0 \quad \text{and} \quad l - m = y.$$
namely
\[ l = \frac{y}{2}, \quad m = -\frac{y}{2}. \]

\[ x - y + (y^2 + 1)m = x - y - \frac{y}{2}(y^2 + 1) \]

\[ \sim x - y + y(y^2 + 1) \]

\[ x - y + (y^2 + 1)m \sim x + y^3 \quad \text{in } W_1 \]

because
\[ 1 \sim \frac{1}{-2} \quad \text{in } W_1. \]

\[ x + y + (y^2 + 1)l = x + y + (y^2 + 1)\frac{y}{2} \]

\[ \sim x + y - y(y^2 + 1) \]

\[ x + y + (y^2 + 1)l \sim x - y^3 \quad \text{in } W_1. \]

So, let be \( g(x) = x + y^3 \) and \( h(x) = x - y^3 \)
then
\[ g(x) \sim x - y \quad \text{and} \quad h(x) \sim x + y \quad \text{in } W_1 \]
and
\[ W_1[(x^2 + 1) - g(x)h(x)] = W_1(y^8 + 1) = \frac{3}{2} = V_{a_2}(y^8 + 1). \]

So,
\[ x^2 + 1 \approx g(x)h(x) \quad \text{in } W_1 \]
and
\[ g(x) \sim x - y \]
\[ h(x) \sim x + y \quad \text{in } W_1. \]

\[ W_1[(x^2 + 1) - g(x)h(x)] = \frac{3}{2} > 1, \]
so let be
\[ W^*g(x) = W^*(x + y^3) = 1, \quad \text{then} \]
\[ W^*(x^2 + 1) = 1, \quad \text{for } W^*g(x)h(x) = 1. \]

So
\[ W^* = W_3. \]

Let be
\[ \phi = (x^2 + 1)^3 - 9(x^2 + 1) + 27. \]

And we must make such an augmented valuation \( W_4 \) that \( W_4 \phi > W_3 \phi \).

But, here we meet a problem which we must solve immediately. We consider about it in \( \S \) 25.

\( \S \) 25. Greater valuation

\[ W_1 < W_2 < W_3 \]

This is a series of \( x \)-simply augmented inductive valuations which are exten-
sions of \( V_{g2} \) to the ring \( K_2[x] \).

\[
W_1 = [V_{g2}, W_1, x = 0] \\
W_2 = \left[ W_1, W_2(x-y) = \frac{1}{2} \right] \\
W_3 = \left[ W_2, W_3(x+y^2) = 1 \right]
\]

\[
W_1(x^2+1) = 0, \quad W_2(x^2+1) = \frac{1}{2}, \quad W_3(x^2+1) = 1.
\]

so

\[
W_1(x^2+1) < W_2(x^2+1) < W_3(x^2+1).
\]

Let the valuations which \( W_1, W_2, \) and \( W_3 \) induce in \( K[x] \), be respectively \( W'_1, W'_2 \) and \( W'_3 \).

\[
W'_1 = [V_{g0}, W_1, x = 0 = W'_1] \\
W'_2 = \left[ W'_1, W'_2(x^2+1) = \frac{1}{2} \right] \\
W'_3 = \left[ W'_2, W'_3(x^2+1) = 1 \right]
\]

Because \( x^2+1 \) is a key polynomial in \( W'_1 \) of \( K[x] \) which I proved in Example 10.1. But here \( W'_3(x^2+1) > W'_2(x^2+1) \).

Here I give newly some definitions.

**Definition 25.1.**

If \( U_1 \) and \( U_2 \) are valuations of \( K[x] \) and \( U_1 \neq U_2 \) and for every polynomial \( f(x) \)

\[
U_1 f(x) \leq U_2 f(x),
\]

then \( U_2 \) is called a greater valuation of \( U_1 \).

**Definition 25.2.**

If \( U_2 \) is a greater valuation of \( U_1 \), but \( U_2 \) does not change a value of the key polynomial \( \phi_1(x) \) which produces \( U_1 \),

namely

\[
U_1 \phi_1(x) = U_2 \phi_1(x),
\]

then \( U_2 \) is called an augmented valuation of \( U_1 \).

These definitions never contradict those which MacLane gave. And the definition of an augmented valuation of a valuation which I give here, coincides completely with what MacLane gave.

Here \( W'_3 \) is a greater valuation of \( W'_2 \), but it is not an augmented valuation of \( W'_2 \). In this case, if we make newly such a valuation \( W'_3^* \)
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\[ W_3^* = \left[ W_1', W_3^*(x^2 + 1) = 1 \right] \]

then \( W_3^* \) is an augmented valuation of \( W_1' \) and \( W_3^* = W_1' \). This means that \( W_3^* f(x) = W_1' f(x) \) for every polynomial \( f(x) \).

When we discuss about valuations, the most important valuation is the last valuation of a series of augmented inductive valuations.

In Example 24.2, of course, the valuation that we want to make is the last valuation \( W \) which is an extension of \( V_{02} \) to \( K_y[x] \) and induces \( V_{p0} \) in \( K[x] \).

If we make such a valuation \( \bar{W} \) of \( K_y[x] \) that
\[
\bar{W} = \left[ W_1, \bar{W}(x + y^3) = 1 = W_3(x + y^3) \right]
\]
then by M. Lemma 15.1 \( \bar{W} = W_3 \) and \( \bar{W} \) is an augmented valuation of \( W_1 \) and \( \bar{W} \) induces \( W_3^* \) in \( K[x] \).

So \( W_1 < \bar{W} = W_3 \).

Thus, from a series of augmented inductive valuations, we can drop excessive valuations not to change the last valuation of the series.

Like this, if we drop \( W_2 \) from a series,
\[
W_1 < W_2 < W_3 = \bar{W}
\]
then we obtain a new series which induces in \( K[x] \) a new series \( W' < W_3^* = W_3' \), which does not change the last valuation \( W_3' \).

So in Example 24.2, we drop \( W_2 \) and we continue our study about the new series \( W_1 < W_3 \). So hence \( W_3 = \bar{W} \).

§ 26. Continuation of solution of Example 24.2

Again we consider about \( \phi = (x^2 + 1)^3 - 9(x^2 + 1) + 27 \). From (24.4) and (24.5)
\[
x^2 + 1 = (x + y^3)^2 - 2y^2(x + y^3) + (y^6 + 1)
\]
\[
W_3(x + y^3)^2 = 2, \quad W_3(y^6 + 1) = \frac{3}{2} \quad \text{and} \quad W_3 \left[ -2y^3(x + y^3) \right] = 1.
\]

Now let be \( (x + y^3)^2 + (y^6 + 1) = \alpha \) and \( W_3 \alpha = \frac{3}{2} \) and \(-2y^3(x + y^3) = \xi \) and \( W_3 \xi = 1 \).
\[
\phi(x) = \phi = (x^2 + 1)^3 - 9(x^2 + 1) + 27
\]
\[
= (\alpha + \xi)^3 - 9(\alpha + \xi) + 27
\]
\[
= (\alpha^2 + 3\alpha^2 \xi + 3\alpha \xi^2) - 9\alpha + (\xi^2 - 9\xi + 27). \tag{26.1}
\]
\[ W_3 \alpha^2 = 4.5, \quad W_3(3\alpha \xi) = 5, \quad W_3(3\alpha \xi^2) = 4.5 \]

Therefore

\[ W_3(\alpha^2 + 3\alpha \xi + 3\alpha \xi^2) \geq 4.5 \quad (26.2) \]

\[ W(-9\alpha) = 3.5 \quad (26.3) \]

\[ W_3(\xi^3 - 9\xi + 27) = W_3 \left[ \left( -2y^3(x + y^3) \right)^3 - 9 \left( -2y^3(x + y^3) \right) + 27 \right] = \text{Min} [3, 3, 3] = 3 \quad (26.4) \]

From (26.1), (26.2), (26.3), and (26.4)

\[ W_3 \phi = W_3(\xi^3 - 9\xi + 27) = 3 \]

In this example, we must make such an augmented valuation \( W \) that \( W\phi = 4 = V_{30}\phi \).

Let be

\[ \xi^3 - 9\xi + 27 = P. \]

\[ P = -8y^9(x+y^3)^3 + 18y^3(x+y^3) + 27 \]

\[ y^2 \sim -1, \quad \frac{1}{y} \sim y \quad \text{and} \quad 2 \sim -1 \quad \text{in} \quad W_3 \]

\[ \frac{P}{3^3} \sim y \left[ \left( \frac{x+y^3}{3} \right)^3 + \frac{x+y^3}{3} - y \right] \quad \text{in} \quad W_3 \]

\[ \left( \frac{x+y^3}{3} \right)^3 + \frac{x+y^3}{3} - y = \frac{1}{3^3} \left[ (x+y^3)^3 + 9(x+y^3) - 27y \right] . \]

By M. Theorem 9.4, if we prove only that \((x+y^3)^3 + 9(x+y^3) - 27y\) is equivalent irreducible in \( W_3 \), then this expression is a homogeneous key polynomial in \( W_3 \).

For the sake of it, we may prove that \( H_3^*(\frac{P}{3^3}) \) is irreducible in the residue class ring of \( K[y] \) by \( W_3 \), where \( H_3^* \) is a natural homomorphism of \( W_3 \).

\[ V_{\partial_1} = [V_{\infty}, V_{\partial_1} y = 0] \quad \text{is a valuation of} \quad K[y]. \]

So \( H_1 K[y] = F_1[Y_1] \), where \( H_1 \) is a natural homomorphism of \( V_{\partial_1} \) and \( H_1 K[y] \) is the residue class ring of \( K[y] \) by \( V_{\partial_1} \) and \( H_1[y] = Y_1 \).

\[ V_{\partial_2} = \left[ V_{\partial_1}, V_{\partial_2} (y^2+1) = \frac{1}{2} \right] . \]

\[ H_1(y^2+1) = Y_1 + 1 = 0 \quad \therefore \quad Y_1 = i = \sqrt{-1} \]

\[ V_{\partial_2} \left( \frac{y^2+1}{3} \right) = 0 \quad \text{and} \quad H_2 \left( \frac{y^2+1}{3} \right) = Y_2 \]
Then \( H_2 K[y] = F_2 [Y_2] \), where \( F_2 = F_1(i) \)
and \( H_2 \) is a natural homomorphism of \( V_{\nu_2} \), and \( H_2 K[y] \) is the residue class ring of \( K[y] \) by \( V_{\nu_2} \).

\[ W_1 = [V_{\nu_2}, W_1 x = 0] \]

\( W_1 \) is a valuation of \( K_\nu[x] \).

\[ H_1^* x = X_1 \quad \text{and} \quad H_1^* K_\nu[x] = F_2^* [X_1] \]

where \( F_2^* = F_2 (Y_2) \) and \( H_1^* \) is a natural homomorphism of \( K_\nu[x] \) by \( W_1 \).

\[ W_3 = [W_1, W_3 (x + y^3) = 1] \]

\[ H_1^* (x + y^3) = H_1^* x + (H_1^* y)^3 \]
\[ = X_1 + Y_1^3 \quad \text{or} \quad X_1 + i^2 = X - i. \]

\[ X_1 - i = 0 \quad X_1 = i \]

\[ H_3^* \frac{x + y^3}{3} = X_3 \quad \text{then} \]

\[ H_3^* K_\nu[x] = F_3^* [X_3], \quad \text{where} \quad F_3^* = F_2 (Y_2) \]

because \( F_3(i) = F_2 \).

\[ H_3^* \left( \frac{P}{3^3} \right) = i (X_1 + X_3 - i), \quad \text{because} \quad H_3^* y = i. \]

Assumed that \( X_3^* + X_3 - i \) can be factorized in \( F_3^* [X_3] \),
one of its factors is \( X_3 - (a + bi) \), where both \( a \) and \( b \) are 0, 1, or \(-1\) in \( F_1 \),
for \( Y_2 \) and \( X_3 \) are algebraically independent with respect to \( F_1 \).

We substitute \( a + bi \) for \( X_3 \), then

\[ (a + bi)^3 + a + bi - i = 0 \]
\[ a^3 - b^3 i + a + bi - i = 0 \quad \therefore \quad H_3^* 3 = 0 \]
\[ b^3 - b + 1 = b(b + 1)(b - 1) + 1 = 0 \]

Such \( b \) does not exist in \( F_1 \). So \( b^3 - b + 1 \) can not equal zero.

And \( X_3^* + X_3 - i \) is irreducible in \( F_3^* [X_3] \).

Then \( (x + y^3)^3 + 9(x + y^3) - 27 y \) is a key polynomial in \( W_3 \).

Now we decide an augmented valuation \( W_4 \) of \( W_3 \) as

\[ W_4 [(x + y^3)^3 + 9(x + y^3) - 27 y] = 3.5, \]
where \[ 3.5 > W_3 [(x + y^3)^2 + 9(x + y^5) - 27y] = 3. \]

Then \[ W_4 P = 3.5. \]

Let be \[ (x + y^3)^2 + 9(x + y^5) - 27y = \phi_4, \]
then \[ W_4 \phi_4 = 3.5. \]

In (26.3) \[ -9\alpha = - (9x + y^5)^2 - 9(y^6 + 1) \]
\[ W_4 [-9(x + y^5)^2] = W_3 [-9(x + y^5)^2] = 4 \]
and \[ W_4 [-9(y^6 + 1)] = W_1 [-9(y^6 + 1)] = 3.5 \]
\[ \text{deg}_x [-9(x + y^5)^2] < \text{deg}_x \phi_4. \]

So \[ W_4 \phi = \text{Min} \left[ W_4 \{-9(x + y^5)^2\}, W_4 \{-9(y^6 + 1)\}, W_4 P \right] \]
\[ = \text{Min} [4, 3.5, 3.5] \quad \therefore \quad W_4 (\alpha^3 + 3\alpha^2 \xi + 3\alpha \xi^2) \geq 4.5 \]
\[ = 3.5 > W_4 \phi = 3. \]

And \[ P - 9(y^6 + 1) \sim \phi \quad \text{in} \quad W_4. \quad (26.5) \]

Therefore, if we make such an augmented valuation \( W_s \) of \( W_4 \) that \[ W_5[P - 9(y^6 + 1)] = 4, \]
then \[ W_5 \phi = 4 \quad \text{by} \quad (26.5). \]

And this example 24.2 is solved completely.

\[ P - 9(y^6 + 1) = -8y^9(x + y^5)^3 + 18y^5(x + y^5) + 27 - 9(y^6 + 1). \]
\[ -2 \sim 1, \quad y^2 \sim -1 \quad \text{and} \quad -\frac{1}{y} \sim y \quad \text{in} \quad W_4 \]
\[ P - 9(y^6 + 1) \sim y [(x + y^5)^3 + 9(x + y^5) - 27y + 9y(y^6 + 1)] \]
\[ P - 9(y^6 + 1) \sim y \left[ \phi_4 + 9y(y^6 + 1) \right] \quad \text{in} \quad W_4. \quad (26.6) \]
\[ W_4 [9y(y^6 + 1)] = 3.5 = W_4 \phi_4. \]

By M. Corollary 13.2, \( \phi_4 + 9y(y^6 + 1) \) is a key polynomial in \( W_4 \).

Then we decide an augmented valuation \( W_s \) of \( W_4 \) that \[ W_5 [\phi_4 + 9y(y^6 + 1)] = 4 > W_4 [\phi_4 + 9y(y^6 + 1)]. \]

By (26.6) \[ W_5 [P - 9(y^6 + 1)] = W_5 y [\phi_4 + 9y(y^6 + 1)] \]
\[ = 4. \]
And

\[ W_5 \phi = 4 = V_3 \phi \quad \text{by (26.5)}. \]

In (26.1)

\[ W_5 (\alpha^3 + 3\alpha^2 \xi + 3\alpha \xi^2) \geq W_3 (\alpha^3 + 3\alpha^2 \xi + 3\alpha \xi^2) \geq 4.5. \]

Therefore when we make such an augmented valuation \( W_5 \) of \( W_3 \) that \( W_5 \phi \leq 3 = W_3 \phi \), we may neglect this term. Thus, example 24.2 is solved completely.

§ 27. Homogeneous valuations

**Lemma 27.1.**

If \( W = [V, W \phi (x) = \mu] \) is an augmented valuation of a homogeneous valuation \( V \) of \( K \{x\} \) and we can insert such a valuation \( U \) between them that \( V < U < W \), then \( \phi(x) \) is not a homogeneous polynomial in \( V \).

**Proof.** Let be \( U = [V, U \phi (x) = \nu] \), and \( \phi(x) \) homogeneous in \( V \).

For every polynomial \( f(x) \) whose degree with respect to \( x \) is less than that of \( \phi(x) \)

\[
Wf(x) = Vf(x).
\]

\[
W \phi (x) = U \phi (x) > V \phi (x). \tag{27.1}
\]

So \( \deg \phi(x) = \deg \phi(x) \), for \( \deg \phi(x) \geq \deg \phi(x) \) by (27.1).

Let be \( \phi(x) = \phi(x) + r(x) \)

then \( \deg r(x) < \deg \phi(x) = \deg \phi(x) \).

\( \phi(x) = \phi(x) + r(x) \) is a key polynomial in \( U \) which the key polynomial \( \phi(x) \) produces from \( V \), so by M. Theorem 9.4. \( U r(x) = U \phi(x) > V \phi(x) \)

while \( V r(x) = U r(x) \quad \therefore \quad \deg r(x) < \deg \phi(x) \).

Namely \( V r(x) > V \phi(x) \).

So \( \phi(x) \) is not a homogeneous polynomials in \( V \). For \( \phi \) is homogeneous in \( V \).

Therefore, from this Lemma we obtain immediately the following theorem.

**Theorem 27.2.**

Every series of augmented inductive homogeneous valuations of \( K \{x\} \)

\[ V_1 < V_2 < \cdots < V_n \]

is a series of continuous augmented inductive valuations of \( K \{x\} \), where all \( V_1, V_2, \cdots, V_n \) are extensions of \( V_0 \) which is a valuation of \( K \).
Example

\[ V_2 = \left[ V_0, V_1, x = 0, V_2(x^2 + i) = 1 \right] \]

\( V_2 \) is the valuation in Example 10.1 in which I substitute \( x \) for \( y \).

Let be \( W = \left[ V_2, W \{(x^2 + 1) + 3 = 3\} = 3 \right] \).

Then we can insert \( U \) so that

\[ V_2 < U < W \]

where \( U = \left[ V_2, U \{(x^2 + 1) + 3\} = 2 \right] \).

\((x^2 + 1) + 3 + 3^2 \) is not a homogeneous polynomial in \( V_2 \), for \( 12 \sim 3 \) in \( V_0 \), so \( 12 \) is not a representative.

But let be \((x^2 + 1) + 3 = \phi\)

and \((x^2 + 1) + 3 + 3^2 = \phi\).

\( \phi = \phi + 3^2 \) and \( U\phi = U\phi = U3^2 \)

namely \( \phi \) is a homogeneous polynomial in \( U \).

So in a series of continuous inductive valuations

\[ V < U < W \]

every key polynomial is homogeneous.

In this paper, I often use the two following lemmas which are related as Remark 3 and Remark 4, but I omit their proofs, for they are very easy.

**Remark 3.** If \( a(x) \sim b(x) + c(x) \) and \( b(x) \sim b'(x) \) in \( V \)

and \( Va(x) = Vb(x) = Vc(x) \),

then \( a(x) \sim b'(x) + c(x) \) in \( V \).

**Remark 4.**

If \( \phi(x) \) is a key polynomial which produces an augmented valuation \( W \) of a valuation \( V \) of \( K[x] \) and \( a(x) \sim \phi(x) \) in \( V \) and \( \text{deg} \{a(x) - \phi(x)\} < \text{deg} \phi(x) \)

and \( V(a(x) - \phi(x)) \geq \delta + V\phi(x) \)

and \( W\phi(x) = \delta + V\phi(x) \),

then \( Wa(x) = W\phi(x) \).
On Valuations of Polynomial Rings of many Variables

I want to relate what I can not do in this paper, in Part three of the same theme which will be published in future.

References


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