NORMALIZABLE OPERATORS

By

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1. Introduction. This is a study of a bounded linear operator T in a Hilbert space $\mathfrak{H}$ which becomes a bounded normal (resp. symmetric) operator in the Hilbert space $\mathfrak{H}_A$ associated with a norm $\|x\|_A = \|A^{\frac{1}{2}}x\|$ where A is a bounded strict positive operator. Such an operator is called weak normalizable (resp. symmetrizable) by A. Strong normalizability is distinguished by the additional condition that the adjoint of T in $\mathfrak{H}_A$ can be lifted to a bounded operator in $\mathfrak{H}$. Strong normalizable operators are nothing but normalizable operators in the sense of Zaanen [10] while an operator weak normalizable by a boundedly invertible positive operator is a scalar type spectral operator in the sense of Dunford [3]. A normal operator will be shown to be weak normalizable by a strict positive operator only when they commute with each other.

Results in § 3, which are generalization of results of Zaanen [10, 11], Reid [9], Lax [6] and Dieudonné [2], will show that the spectrum of an operator T, weak normalizable by a strict positive operator A, is only diminished and, in case of compact T, remains unchanged under transfer to $\mathfrak{H}_A$. Under suitable conditions, estimates for norm of resolvents of weak normalizable operators can be obtained on the basis of corresponding estimates for normal oprtators. These are discussed in § 4.

General intrinsic characterization of weak (or strong) normalizability is not known, which makes it difficult to decide whether weak (or strong) normalizability is retained under restriction to invariant subspaces. The only general result in this direction is that restriction of a unitary operator to any invariant subspace is strong normalizable. However situations are much favorable to compact operators. For instance, it will be shown in § 5 that a compact operator is weak normalizable if and only if the set of eigenvectors of its adjoint is total. Symmetrizability of a compact operator T can be determined by the behaviour of the exponential group $\{\exp(i\lambda T^*): -\infty<\lambda<\infty\}$.

2. Normalizability. In what follows, an *operator* means a bounded linear operator in a Hilbert space $\mathfrak{H}$ unless otherwise expressed. Given an operator T the symbols $\mathfrak{N}(T)$, $\mathfrak{R}(T)$, $\sigma(T)$ and $\rho(T)$ will denote its kernel, range, spectrum and resolvent set respectively. The orthogonal complement of $\mathfrak{N}(T)$ coincides always with the closure of $\mathfrak{R}(T^*)$. T is called *injective* in case $\mathfrak{N}(T)=\{0\}$. When T is compact, every non-zero spectre λ is an isolated eigenvalue such that $\mathfrak{R}(T-\lambda)$ is closed and $\mathfrak{N}(T-\lambda)$ is finite dimensional with $\dim(\mathfrak{N}(T-\lambda))=\dim(\mathfrak{N}(T^*-\bar{\lambda})$. Further the following three conditions are mutually equivalent: $\mathfrak{N}(T-\lambda)=\mathfrak{N}((T-\lambda)^2)$, $\mathfrak{R}(T-\lambda)\cap\mathfrak{N}(T-\lambda)=\{0\}$ and $\mathfrak{N}(T-\lambda)+\mathfrak{R}(T-\lambda)=\mathfrak{H}$. Details are found in [4], [11].

An operator A is called *strict positive* if it is positive and injective, and $A^{\frac{1}{2}}$ denotes its positive square root. A strict positive operator A introduces a new scalar product: $(x, y)_A=(Ax, y)$. The associated Hilbert space, that is, the completion of $\mathfrak{H}$ with respect to this scalar product, will be denoted by $\mathfrak{H}_A$. When there is no confusion, $\mathfrak{H}$ is identified, as a linear space, with the dense subspace of $\mathfrak{H}_A$. An operator T in $\mathfrak{H}$ can be considered as a linear operator in $\mathfrak{H}_A$ with domain $\mathfrak{H}$. If it is bounded in $\mathfrak{H}_A$, its unique bounded extension to whole $\mathfrak{H}_A$ will be denoted by the same letter T. In this case the symbols $\mathfrak{N}(T)_A$, $\mathfrak{R}(T)_A$, $\sigma(T)_A$ and $\rho(T)_A$ denote respectively the kernel, range, spectrum and resolvent set of T (more precisely, of its extension) in $\mathfrak{H}_A$.

Lemma 1. *The following conditions for an operator T are equivalent:*

(1) *There are a bounded injective operator B from $\mathfrak{H}$ to a dense range in a Hilbert space $\mathfrak{K}$ and a normal* (*resp. symmetric*) *operator N in $\mathfrak{K}$ with $BT=NB$.*

(2) *There are a strict positive operator A and a normal* (*resp. symmetric*) *operator M with $A^{\frac{1}{2}}T=MA^{\frac{1}{2}}$.*

(3) *T is normal* (*resp. symmetric*) *in $\mathfrak{H}_A$ for some strict positive operator A.*

Proof. Equivalence of (1) and (2) follows from the observation that an injective operator B with dense range has the form $B=UA^{\frac{1}{2}}$ where U is a unitary operator from $\mathfrak{H}$ to $\mathfrak{K}$ and $A=B^*B$ is strict positive. Equivalence of (2) and (3) is a consequence of the fact that $A^{\frac{1}{2}}$ defines an isometric operator from the dense subspace $\mathfrak{H}$ of $\mathfrak{H}_A$ to the Hilbert space $\mathfrak{H}$ and is extended to a unitary operator from $\mathfrak{H}_A$ to $\mathfrak{H}$.

An operator T is called *weak normalizable* (resp. *symmetrizable*) by strict positive A if it is normal (resp. symmetric) in $\mathfrak{H}_A$. T is called *strong normalizable* by A if it is normal in $\mathfrak{H}_A$ and its adjoint in $\mathfrak{H}_A$ is also the resultant of an operator in $\mathfrak{H}$.

When T is bounded in $\mathfrak{H}_A$ its adjoint T' in $\mathfrak{H}_A$ is determined by the relation: $T^*Ax=AT'x$ (in $\mathfrak{H}_A$) for every $x\in\mathfrak{H}$. Therefore T is strong normalizable by A if and only if it is weak normalizable and there is an operator S with $T^*A=AS$. Such S is called a *pseudo-adjoint* of T though it may depend on the choice of A. A weak symmetrizable operator is strong normalizable. For this reason the word "weak" will be omitted in case of symmetrizability.

The following lemma is useful to prove boundedness of a densely defined linear operator. Indeed, this was used, in its essence, by many authors in connection with symmetrizable operators.

Lemma 2. *Let T be a linear operator such that there is a dense subspace $\mathfrak{D}$ on which every iterate T^n is defined. If T has the following properties: for every $x \in \mathfrak{D}$ $\|Tx\|^2 \leq \|x\| \cdot \|T^2 x\|$ and $\overline{\lim\limits_{n\to\infty}} \|T^n x\|^{\frac{1}{n}} \leq \gamma$ for some γ, then it is bounded with $\|T\| \leq \gamma$.*

Proof. Induction based on the first property shows that for every $x \in \mathfrak{D}$ $\|Tx\|^n \leq \|x\|^{n-1} \|T^n x\|$. Now the assertion results immediately from the second property.

If T is a *hyponormal* operator, that is, $T^*T \geq TT^*$, the inequality: $\|Tx\|^2 \leq \|x\| \cdot \|T^2 x\|$ is always satisfied. Another remark is that if T is an operator in $\mathfrak{H}$, for any strict positive A

$$\|T^n x\|_A \leq \|A^{\frac{1}{2}}\| \cdot \|T^n x\| \leq \|A^{\frac{1}{2}}\| \cdot \|T\|^n \cdot \|x\| .$$

Theorem 1. *An operator T is weak normalizable by strict positive A if and only if for every $x \in \mathfrak{H}$*

$$\sup_{y \in \mathfrak{H}} \frac{|(ATy, x)|^2}{(Ay, y)} = (ATx, Tx) .$$

Proof. Suppose that T is normal in $\mathfrak{H}_A$. Then it follows, with T', the adjoint of T in $\mathfrak{H}_A$, that

$$\sup_{y \in \mathfrak{H}} \frac{|(ATy, x)|^2}{(Ay, y)} = \|T'x\|_A^2 = \|Tx\|_A^2 = (ATx, Tx)$$

because $\mathfrak{H}$ is considered a dense subspace of $\mathfrak{H}_A$. On a similar basis it suffices for the converse assertion to prove the boundedness of T in $\mathfrak{H}_A$. Since for every $x \in \mathfrak{H}$ and $n \geq 1$

$$\frac{(Tx, Tx)_A^2}{(x, x)_A} \leqslant (T^2 x, T^2 x)_A \text{ and } \|T^n x\|_A \leq \|A^{\frac{1}{2}}\| \cdot \|T\|^n \cdot \|x\|$$

the boundedness results from Lemma 2.

The following corollary is essentially due to Zaanen [10].

Corollary 1.1. *An operator T is strong normalizable by strict positive A if and only if there is an operator S such that $AT = S^*A$ and $TS = ST$.*

Proof. Suppose that such S exists. Then

$$\sup_{y\in\mathfrak{H}} \frac{|(ATy, x)|^2}{(Ay, y)} = \sup_{y\in\mathfrak{H}} \frac{|(Ay, Sx)|^2}{(Ay, y)}$$
$$= \|A^{\frac{1}{2}}Sx\|^2 = (ASx, Sx) = (ATSx, x) = (ATx, Tx),$$

hence T is (weak) normalizable by A. Obviously S becomes the adjoint of T in $\mathfrak{H}_A$, hence T is strong normalizable by A. The converse is also obvious from the above consideration.

Corollary 1.2. *An operator T is symmetrizable by strict positive A if and only if $AT=T^*A$.*

Theorem 2. *If a normal operator T is weak normalizable by strict positive A, then T commutes with A.*

Proof. By Lemma 1 there is a normal operator M with $A^{\frac{1}{2}}T=MA^{\frac{1}{2}}$. Then it follows from the commutativity theorem for normal operators [8, p. 9] that $A^{\frac{1}{2}}T^*=M^*A^{\frac{1}{2}}$. Since $A(T+T^*)=A^{\frac{1}{2}}(M+M^*)A^{\frac{1}{2}}$. The symmetric operator $T+T^*$ commutes with A. Similarly $T-T^*$ commutes with A.

Corollary 2.1. *If an operator T is weak normalizable by strict positive A as well as by its square A^2, then T is normal.*

Proof. It follows from the hypothesis and Lemma 1 that, considered in $\mathfrak{H}_A$, the normal operator T is weak normalizable by the strict positive operator A. Then by Theorem 2 T commutes with A, and consequently with $A^{\frac{1}{2}}$. The normality results now from Lemma 1.

Corollary 2.2. *If an operator T and its adjoint T^* are weak normalizable by one and the same strict positive operator A, then T is normal.*

Proof. Let T' be the adjoint of T in $\mathfrak{H}_A$. Then for bounded operators A, T^* and T' in $\mathfrak{H}_A$ the relation $T^*A=AT'$ holds. Since both T^* and T' are normal and A is strict positive, this means that the normal operator T' is weak normalizable by A^2. Then by Theorem 2 T^* commutes with A^2 hence with $A^{\frac{1}{2}}$. Now the normality results from Lemma 1.

3. Spectrum. A non-trivial property of a normal operator is that its spectrum contains at least two distinct points unless it is a scalar. The same property is shared with normalizable operators.

Theorem 3. *If T is weak normalizable by strict positive A then $\sigma(T)_A\subseteq\sigma(T)$.*

Proof. Take $\lambda\in\rho(T)$. As an operator in $\mathfrak{H}_A$ each $(T-\lambda)^{-n}$ has the common dense domain $\mathfrak{H}$. Since $T-\lambda$ is normal in $\mathfrak{H}_A$ by assumption, for every $x\in\mathfrak{H}$

$$\|(T-\lambda)^{-1}x\|_A^2 = \|(T-\lambda)(T-\lambda)^{-2}x\|_A^2 \leq \|(T-\lambda)^{-2}x\|_A \cdot \|x\|_A$$

and further for every n

$$\|(T-\lambda)^{-n}x\|_A \leqslant \|A^{\frac{1}{2}}\| \cdot \|(T-\lambda)^{-1}\|^n \cdot \|x\| .$$

Therefore $(T-\lambda)^{-1}$ is bounded in $\mathfrak{H}_A$ by Lemma 2. This shows that $\rho(T) \subseteq \rho(T)_A$, or equivalently $\sigma(T)_A \subseteq \sigma(T)$.

Corollary 3.1. *A compact operator T is weak normalizable by strict positive A if and only if for every x, $y \in \mathfrak{H}$*

$$|(ATy, x)|^2 \leq (Ay, y)(ATx, Tx) .$$

Proof. If T is weak normalizable by A, the inequality in question follows immediately from Theorem 1. Conversely it follows from this inequality that for every $x \in \mathfrak{H}$

$$\|Tx\|_A^2 \leqslant \|x\|_A \cdot \|T^2x\|_A$$

and further

$$\|T^n x\|_A \leqslant \|A^{\frac{1}{2}}\| \cdot \|T\|^n \cdot \|x\| \qquad (n=1, 2, \cdots).$$

Therefore T is bounded in $\mathfrak{H}_A$ by Lemma 2. Now the inequality

$$\sup_{y \in \mathfrak{H}} \frac{|(Ty, x)_A|^2}{\|y\|_A^2} \leqslant \|Tx\|_A^2 \qquad (x \in \mathfrak{H})$$

means that T is *hyponormal* in $\mathfrak{H}_A$. Since $T-\lambda$ is also hyponormal for every λ, the proof of Theorem 3 remains valid without change, leading to $\sigma(T)_A \subseteq \sigma(T)$. It is known [4, p. 106] that a hyponormal operator with 0 as the only limiting point of its spectrum is normal.

Lemma 3. *Let an operator T be weak normalizable by strict positive A. Then for every λ $\mathfrak{N}(T-\lambda) \cap \overline{\mathfrak{R}(T-\lambda)} = \{0\}$, where $\overline{(\ .\ )}$ denotes the closure, $\mathfrak{N}(T-\lambda) = \mathfrak{N}((T-\lambda)^2)$ and $A\mathfrak{N}(T-\lambda) \subseteq \mathfrak{N}(T^*-\bar{\lambda})$. If $\mathfrak{N}(T-\lambda)$ is finite dimensional and $\mathfrak{N}(T-\lambda)+\mathfrak{R}(T-\lambda)$ is dense, then $\mathfrak{N}(T-\lambda)$ coincides with $\mathfrak{N}(T-\lambda)_A$.*

Proof. It suffices to treat the case $\lambda=0$. Since T is normal in $\mathfrak{H}_A$, $\mathfrak{N}(T)_A$ is orthogonal to $\overline{\mathfrak{R}(T)_A}$ in $\mathfrak{H}_A$, consequently $\mathfrak{N}(T) \cap \overline{\mathfrak{R}(T)} = \{0\}$ because of $\overline{\mathfrak{R}(T)} \subseteq \overline{\mathfrak{R}(T)_A}$. Also the relation $\mathfrak{N}(T)_A = \mathfrak{N}(T^2)_A$ for normal T implies $\mathfrak{N}(T) = \mathfrak{N}(T^2)$. Let T' be the adjoint of T in $\mathfrak{H}_A$. Since $\mathfrak{N}(T)_A = \mathfrak{N}(T')_A$ because of normality, $A\mathfrak{N}(T) \subseteq \mathfrak{N}(T^*)$ follows immediately from $T^*A = AT'$. The last assertion results from the observation that if $\mathfrak{N}(T)$ is finite dimensional $\mathfrak{N}(T) + \overline{\mathfrak{R}(T)_A}$ is closed and must coincide with the closure of $\mathfrak{N}(T) + \mathfrak{R}(T)$ in $\mathfrak{H}_A$.

The following theorem is due essentially to Zaanen [10] and Lax [6].

Theorem 4. *If a compact operator T is weak normalizable by strict positive A, it is compact in $\mathfrak{H}_A$, $\sigma(T)=\sigma(T)_A$ and $\mathfrak{N}(T-\lambda)=\mathfrak{N}(T-\lambda)_A$ for every non-zero λ.*

Proof. Since every non-zero spectre of T is an eigenvalue, $\sigma(T)_A=\sigma(T)$ follows immediately from Theorem 3. Now to prove the compactness of normal T in $\mathfrak{H}_A$, it suffices to show that $\mathfrak{N}(T-\lambda)_A=\mathfrak{N}(T-\lambda)$ for every non-zero λ because $\mathfrak{N}(T-\lambda)$ is finite dimensional and every non-zero spectre is an isolated eigenvalue. Since $\mathfrak{N}(T-\lambda)=\mathfrak{N}((T-\lambda)^2)$ by Lemma 3, it follows by compactness that $\mathfrak{N}(T-\lambda)+\mathfrak{R}(T-\lambda)=\mathfrak{H}$, and finally $\mathfrak{N}(T-\lambda)_A=\mathfrak{N}(T-\lambda)$ again by Lemma 3.

4. Resolvent. One of the important properties of a normal operator is that the norm of its resolvent at λ is majorated by, and really equal to, the inverse of the distance from λ to the spectrum. There is no analogy for normalizable operators. However if T is a scalar type spectral operator such that $A^{\frac{1}{2}}T=MA^{\frac{1}{2}}$ for some boundedly invertible positive operator A and a normal operator M, then the norm of its resolvent admits the estimate: $\|(T-\lambda)^{-1}\|\leqslant\frac{\|A\|^{\frac{1}{2}}\cdot\|A^{-1}\|^{\frac{1}{2}}}{d(\lambda)}$ where $d(\lambda)$ is the distance from λ to the spectrum. The following theorems are generalizations of this result.

Theorem 5. *Let A_1 and A_2 be strict positive with $\alpha A_1\geq A_2\geq\alpha^{-1}A_1$ for some $\alpha>0$. If an operator T and its adjoint T^* are weak normalizable by A_1 and A_2 respectively, then $\sigma(T)=\sigma(T)_{A_1}$ and for every $\lambda\in\rho(T)$*

$$\|(T-\lambda)^{-1}\|\leq\frac{\sqrt{\alpha}}{d(\lambda)}$$

where $d(\lambda)$ denotes the distance from λ to $\sigma(T)$.

Proof. In view of mutual majoration between A_1 and A_2 there is a boundedly invertible operator C with $CA_1^{\frac{1}{2}}=A_2^{\frac{1}{2}}$. Since C^* gives an isometrie from the dense set $\mathfrak{H}$ of $\mathfrak{H}_{A_2}$ onto the dense set $\mathfrak{H}$ of $\mathfrak{H}_{A_1}$, it is extended to a unitary operator from $\mathfrak{H}_{A_2}$ to $\mathfrak{H}_{A_1}$. Then, considered as operators in $\mathfrak{H}_{A_1}$, $T\cdot CA_1=CA_1\cdot C^*LC^{*-1}$ where L is the adjoint of T^* in $\mathfrak{H}_{A_2}$. Since both T and C^*LC^{*-1} are normal and obviously CA is injective with dense range in $\mathfrak{H}_{A_1}$, the commutativity theorem for normal operators [8, p. 9] shows that T is unitary equivalent to C^*LC^{*-1} in $\mathfrak{H}_{A_1}$. Therefore $\sigma(T)_{A_1}=\sigma(L)=\sigma(T^*)_{A_2}^*$ where $(\ .\ )^*$ denotes complex conjugation. Take $\lambda\in\rho(T)_{A_1}$. Since T is normal in $\mathfrak{H}_{A_1}$ and $\sigma(T)_{A_1}\subseteq\sigma(T)$ by Theorem 3, $(T^*-\bar{\lambda})A_1(T-\lambda)\geq d(\lambda)^2A_1$. On the other hand, since $\rho(T)_{A_1}^*=\rho(T^*)_{A_2}$ as proved above, the normality of T^* in $\mathfrak{H}_{A_2}$ implies $(T-\lambda)A_2(T^*-\bar{\lambda})\geq d(\lambda)^2A_2$. It follows then from the inequality $\alpha A_1\geq A_2\geq\alpha^{-1}A_1$ that

$$\left[A_2^{\frac{1}{2}}(T^*-\bar{\lambda})(T-\lambda)A_2^{\frac{1}{2}}\right]^2 \geq \frac{d(\lambda)^2}{\alpha} A_2^{\frac{1}{2}}(T^*-\bar{\lambda})A_1(T-\lambda)A_2^{\frac{1}{2}} \geq \left[\frac{d(\lambda)^2}{\alpha}A_2\right]^2.$$

Further in view of the Loewner-Heinz theorem [5]

$$A_2^{\frac{1}{2}}(T^*-\bar{\lambda})(T-\lambda)A_2^{\frac{1}{2}} \geq \frac{d(\lambda)^2}{\alpha}A_2$$

and finally

$$(T^*-\bar{\lambda})(T-\lambda) \geq \frac{d(\lambda)^2}{\alpha}$$

Similar arguments give

$$(T-\lambda)(T^*-\bar{\lambda}) \geq \frac{d(\lambda)^2}{\alpha}$$

Thus $T-\lambda$ has bounded inverse with $\|(T-\lambda)^{-1}\| \leqslant \frac{\sqrt{\alpha}}{d(\lambda)}$.

In Theorem 5 the single inequality $\alpha A_1 \geq A_2$ is not sufficient to guarantee $\sigma(T)_{A_1} = \sigma(T)$ even when A_1 and A_2 are compact and commutative. This is shown by the following examle. Let $\{e_n\}$ be the canonical orthonormal basis of l^2 and $\mathfrak{H}_n$ the subspace spanned by $e_1, \cdots, e_n$. Consider the operators S_n and R_n in $\mathfrak{H}_n$, defined respectively by matrices $(\delta_{j-1,k})_{j,k=1}^n$ and $(\delta_{j,k}/n+j)_{j,k=1}^n$ with respect to the basis $e_1, \cdots, e_n$. The operator $T_n = S_n + R_n$ is symmetrizable by some strict positive B_n because its spectrum consists of simple eigenvalues $1/n+j$ $(j=1, 2, \cdots, n)$. Observe the operators T, A_1 and A_2 in the direct sum $\oplus \sum_{n=1}^{\infty} \mathfrak{H}_n$ defined respectively by $T = \oplus \sum_{n=1}^{\infty} T_n$, $A_1 = \oplus \sum_{n=1}^{\infty} \frac{B_n}{n\|B_n\|}$ and $A_2 = \oplus \sum_{n=1}^{\infty} \frac{\rho_n B_n^{-1}}{\|B_n^{-1}\|}$ with $0 < \rho_n < \frac{1}{n\|B_n\| \cdot \|B_n^{-1}\|}$. A_1 and A_2 are compact strict positive such that $A_1 \geqslant A_2$ and both $A_1 T$ and $A_2 T^*$ are symmetric. However the spectrum $\sigma(T)$ covers the whole unit disc. In fact, since every λ with $|\lambda| < 1$ is an eigenvalue of the adjoint shift in l^2, which assigns e_n to e_{n+1} and 0 to e_1, for any $\varepsilon > 0$ and sufficiently large n there is $x_n \in \mathfrak{H}_n$ such that $\|x_n\| = 1$ and $\|S_n^* x_n - \lambda x_n\| < \varepsilon$. Therefore λ is an approximate eigenvalue of T^*.

Theorem 6. *If an operator T is weak normalizable by strict positive A and if T is bounded as an operator from $\mathfrak{H}_A$ to $\mathfrak{H}$, then $\sigma(T) = \sigma(T)_A$ and for every non-zero $\lambda \in \rho(T)$*

$$\|(T-\lambda)^{-1}\| \leq \frac{d(\lambda) + \gamma\|A\|^{\frac{1}{2}}}{|\lambda| \cdot d(\lambda)}$$

where γ is the norm of T as an operator from $\mathfrak{H}_A$ to $\mathfrak{H}$ and $d(\lambda)$ is the distance from λ to $\sigma(T)$.

Proof. In view of the continuity of T as an operator from $\mathfrak{H}_A$ to $\mathfrak{H}$ there is an operator D with $T=DA^{\frac{1}{2}}$. Since $A^{\frac{1}{2}}T=(A^{\frac{1}{2}}D)A^{\frac{1}{2}}$, $\sigma(T)_A=\sigma(A^{\frac{1}{2}}D)$. If A has bounded inverse, $\sigma(A^{\frac{1}{2}}D)=\sigma(DA^{\frac{1}{2}})$ is obvious. Even if A has no bounded inverse, $\sigma(T)=\sigma(DA^{\frac{1}{2}})$ coincides with $\sigma(A^{\frac{1}{2}}D)$ with possible exception of $\lambda=0$ [4, p. 39]. However under present circumstance $0\in\sigma(DA^{\frac{1}{2}})$ implies $0\in\sigma(A^{\frac{1}{2}}D)$ because $\mathfrak{R}(A^{\frac{1}{2}})$ does not cover $\mathfrak{H}$. That $0\in\sigma(A^{\frac{1}{2}}D)=\sigma(T)_A$ implies $0\in\sigma(T)$ follows from Theorem 3. Thus the relation $\sigma(T)=\sigma(T)_A$ is proved. Since T is normal in $\mathfrak{H}_A$, for non-zero $\lambda\in\rho(T)=\rho(T)_A$

$$|\lambda|\cdot\|x\|\leq\|(T-\lambda)x\|+\|D\|\cdot\|A^{\frac{1}{2}}x\|\leq\|(T-\lambda)x\|+\frac{\|D\|\cdot\|A^{\frac{1}{2}}(T-\lambda)x\|}{d(\lambda)}$$

The assertion results from the relation $\gamma=\|D\|$.

5. Restriction. Restriction of a normal operator to a closed invariant subspace is not necessarily normal but *subnormal* [4, p. 100]. Then in view of Lemma 1 whether weak normalizability is retained under restriction to invariant subspaces is equivalent to whether every subnormal operator is weak normalizable. Corresponding equivalence concerning strong normalizability is not known. However unitary or compact operators are manageable in this respect.

Theorem 7. *Restriction of a unitary operator to any closed invariant subspace is strong normalizable, or equivalently, every isometric operator is strong normalizable.*

Proof. Given an isometric operator V, there is the maximum among all reducing subspaces on which V is unitary [4, p. 74]. Therefore it can be assumed for the proof that V admits no non-trivial reducing subspace. In this case, V is unitary equivalent to a unilateral shift U of some multiplicity [4, p. 74]. This means that U is the tensor product of the usual shift S in l^2 and the identity operator in a Hilbert space $\mathfrak{K}$. Let A be the tensor product of the operator with Hilbert matrix $\left(\frac{1}{j+k}\right)_{j,k=1}^{\infty}$ in l^2 and the identity operator in $\mathfrak{K}$. Since the Hilbert matrix is known to be bounded and strict positive [8, p. 135], A is also strict positive. Now it is easy to see that AU is symmetric. Therefore V is symmetrizable.

An operator T is called *quasi-normal* if T commutes with T^*T. Every isometric operator is quasi-normal. It is known [4, p. 69] that given a quasi-normal operator T there is a reducing subspace such that T is normal on it and the restriction to its orthogonal complement has the form $UB=BU$ where U is an isometric operator with no non-trivial reducing subspace and B is a positive operator.

Corollary 7.1. *Every quasi-normal operator is strong normalizable.*

Proof. As in the proof of Theorem 7 a quasi-normal operator T can be assumed to have the form $UB=BU$ where B is a positive operator and U is the tensor product of the usual shift in l^2 and the identity operator in a Hilbert space $\mathfrak{K}$. Then B becomes a tensor product of the identity operator in l^2 and an operator in $\mathfrak{K}$ [4, p. 73]. Then the operator A in the proof of Theorem 7 commutes with B. Therefore T is symmetrizable by A.

Normalizability for compact operators is purely spectral property. This is seen in the following and subsequent theorems.

Theorem 8. *A compact operator T is weak normalizable if and only if the set of eigenvectors of T^* is total.*

Proof. Suppose that T is weak normalizable by strict positive A. Since $\mathfrak{N}(T-\lambda)$ and $\mathfrak{N}(T^*-\bar{\lambda})$ have same finite dimension for non-zero λ, it follows from Lemma 3 that $A\mathfrak{N}(T-\lambda)=\mathfrak{N}(T^*-\bar{\lambda})$. If a vector x is orthogonal to all eigenvectors of T^*, it is orthogonal to all $\mathfrak{N}(T-\lambda)$ with non-zero λ in $\mathfrak{H}_A$. Since T is compact normal by Theorem 4, this implies $Tx=0$. On the other hand, since Ax belongs to $\mathfrak{N}(T^*)$ by Lemma 3, $(Ax, x)=0$ consequently $x=0$ in view of strict positivity. This means that the set of eigenvectors of T^* and $\mathfrak{N}_0=\overline{A\mathfrak{N}(T)}$, which is a closed subspace of $\mathfrak{N}(T^*)$, together span $\mathfrak{H}$. To prove the converse, let $\{\lambda_n\}$ be the set of non-zero eigenvalues of T. Since $\mathfrak{N}(T-\lambda_n)=\mathfrak{N}((T-\lambda_n)^2)$ by Lemma 3, there is a projection E_n such that $TE_n=E_nT$, $\mathfrak{R}(E_n)=\mathfrak{N}(T-\lambda_n)$ and $\mathfrak{N}(E_n)=\mathfrak{R}(T-\lambda_n)$. Let A be defined by

$$A=P+\sum_{n=1}^{\infty}\alpha_n E_n^* E_n$$

where $\alpha_n=\dfrac{1}{2^n\|E_n\|^2}$ and P is the orthogonal projection to $\mathfrak{N}(T^*)$. Since $E_n x=0$ means that x is orthogonal to $\mathfrak{N}(T^*-\bar{\lambda}_n)$, A becomes strict positive by the hypothesis. Then it follows from $PT=0$ that

$$\left|(ATy, x)\right|^2=\left|\sum_{n=1}^{\infty}\alpha_n\lambda_n(E_n y, E_n x)\right|^2$$
$$\leq\left\{\sum_{n=1}^{\infty}\alpha_n\|E_n y\|^2\right\}\left\{\sum_{n=0}^{\infty}\alpha_n\lambda_n^2\|E_n x\|^2\right\}\leq(Ay, y)\cdot(ATx, Tx).$$

Therefore T is weak normalizable by A in view of Corollary 3.1.

A part of the following Corollary is proved by Dieudonné [2].

Corollary 8.1. *A compact operator is symmetrizable if and only if its spectrum is real and the set of eigenvectors of its adjoint is total.*

Proof. This follows immediately from Theorems 8 and 4.

Another characterization of symmetrizable operators will be given in Theorem 11.

Corollary 8.2. *Restriction of a compact, weak normalizable operator to its closed invariant subspace is weak normalizable.*

Proof. This results from Theorem 8 and the fact that projection of every eigenvector of the adjoint to the invariant subspace is an eigenvector of the adjoint of the restriction.

Now a characterization of strong normalizability is near at hand.

Theorem 9. *A compact operator T is strong normalizable if and only if there is a closed subspace $\mathfrak{N}_0$ of the kernel of T^* such that $\mathfrak{N}_0$ and the set of eigenvectors of T^*, belonging to non-zero eigenvalues, span $\mathfrak{H}$ and if there is an operator S, vanishing on $\mathfrak{N}_0$, such that $S^*x=\lambda x$ whenever $T^*x=\bar{\lambda}x$ and $\lambda\neq 0$.*

Proof. Suppose thet T is strong normalizable by strict positive A with pseudo-adjoint S. Since $A\mathfrak{N}(T-\lambda)\subseteq\mathfrak{N}(S^*-\lambda)$, S and $\mathfrak{N}_0=\overline{A\mathfrak{N}(T)}$ meet the requirement of the assertion as in the proof of Theorem 8. To prove the converse, let A be defined as in the proof of Theorem 8 with P replaced by the orthogonal projection to $\mathfrak{N}_0$. Then $AT=S^*A$ and $T^*S^*=S^*T^*$ result immediately from the hypothesis.

Corollary 9.1. *If a compact injective operator is strong normalizable, its pseudo-adjoint is uniquely determined.*

Proof. As in the proof of Theorem 8 $\mathfrak{N}(T)=\{0\}$ implies that the set of eigenvectors of T^* belonging to non-zero eigenvalues is total. Now the assertion follows from the proof of Theorem 9.

On the basis of these corollaries it is easy to construct a compact operator which is weak but not strong normalizable. Let C be the operator with matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & \exp\left(\frac{2\pi i}{3}\right) & 0 \\ 0 & 0 & \exp\left(\frac{-2\pi i}{3}\right) \end{pmatrix}$ in a three dimensional space. Since C has simple eigenvalues 1, $\exp(2\pi i/3)$ and $\exp(-2\pi i/3)$, it is (strong) normalizable with pseudo-adjoint C^{-1}. $\|C\|<\|C^{-1}\|$ is easy to see. Let $D=C/\|C\|$ and consider operators $D_n=D\otimes\cdots\otimes D$ (n-tuple tensor product) in a 3^n dimensional space $\mathfrak{H}_n$. The operators $T=\oplus\sum_{n=1}^{\infty}\gamma^n D_n$ with $\|C\|/\|C^{-1}\|<\gamma<1$ in the direct sum $\oplus\sum_{n=1}^{\infty}\mathfrak{H}_n$ is compact and the set of eigenvectors of T^* is total. Therefore T is weak normalizable by Theorem 8. If T is strong normalizable, its pseudo-adjoint S must coincide with $\gamma^n \|C\|^{-2n} D_n^{-1}$ on $\mathfrak{H}_n$, leading to a contradiction:

$$\|S\| \geq (\gamma \|C^{-1}\|/\|C\|)^n \longrightarrow \infty \text{ as } n \longrightarrow \infty .$$

Suppose that T is strong normalizable by strict positive A with pseudo-adjoint S and that $\mathfrak{M}$ is a closed invariant subspace of T. If $\mathfrak{M}$ is invariant under S, the restriction of T to $\mathfrak{M}$ is also strong normalizable with the restriction of S to $\mathfrak{M}$ as its pseudo-adjoint. The invariance of $\mathfrak{M}$ under S is guaranteed if $\mathfrak{M}$ is spanned by eigenvectors of T or, in case of compact T, if the orthogonal complement of $\mathfrak{M}$ is spanned by eigenvectors of T^* belonging to non-zero eigenvalues. The following theorem shows that under compactness assumption the invariance of $\mathfrak{M}$ under S is also necessary.

Theorem 10. *Let T be a compact, strong normalizable, injective operator. Then its restriction to a closed invariant subspace $\mathfrak{M}$ is strongly normalizable (if and) only if $\mathfrak{M}$ is invariant under the uniquely determined pseudo-adjoint S.*

Proof. Suppose that the restriction T_1 of T to $\mathfrak{M}$ is strong normalizable with pseudo-adjoint S_1. Since $T^*x = \bar{\lambda}x$ with $\lambda \neq 0$ implies $T_1^*Px = \bar{\lambda}Px$ where P is the orthogonal projection to $\mathfrak{M}$, by Theorem 9 $S^*x = \lambda x$ and $S_1^*Px = \lambda Px$. Since the set of eigenvectors of T^* belonging to non-zero eigenvalues is total, this leads to $PS^* = S_1^*P$, hence to $PS^*(1-P) = 0$.

The following result corresponds to Lumer's characterization [7] of a scalar type spectral operator with real spectrum.

Theorem 11. *A compact injective operator T is symmetrizable if and only if the set of vectors x with $\sup_{-\infty<\lambda<\infty} \|\exp(i\lambda T^*)x\| < \infty$ is dense.*

Proof. Suppose that T is symmetrizable. If $T^*x = \mu x$, μ is real by Corollary 8.1 and

$$\sup_{-\infty<\lambda<\infty} \left\| \exp(i\lambda T^*)x \right\| = \sup_{-\infty<\lambda<\infty} \left| \exp(i\lambda\mu) \right| \cdot \|x\| \leq \|x\| .$$

The set of linear combinations of such x is dense by Corollary 8.1. To prove the converse, let

$$\theta(x) = \sup_{-\infty<\lambda<\infty} \left\| \exp(i\lambda T^*)x \right\| .$$

Suppose that the set $\mathfrak{M}$ of x with $\theta(x) < \infty$ is dense. First of all every non-zero eigenvalue μ of T is real. In fact, if $Ty = \mu y \neq 0$,

$$\left| \exp(i\lambda\mu) \right| \cdot \left| (y, x) \right| = \left| (y, \exp(-i\lambda T^*)x) \right| \leq \|y\| \cdot \theta(x) .$$

Since $\mathfrak{M}$ is dense, μ must be real. Next $\mathfrak{N}(T-\mu)\cap\mathfrak{R}(T-\mu)=\{0\}$ or equivalently $\mathfrak{N}(T-\mu)=\mathfrak{N}((T-\mu)^2)$. In fact, if $(T-\mu)^2z=0$,

$$\exp(i\lambda T)\,x=\exp(i\lambda\mu)\left\{z+i\lambda(T-\mu)\,z\right\}.$$

Arguments similar to the above show that $(T-\mu)z$ is orthogonal to $\mathfrak{M}$. Now let $\mathfrak{N}=\bigcap_{\mu}\mathfrak{R}(T-\mu)$ where μ runs over the set of non-zero eigenvalues. Then the restriction T_1 of T to $\mathfrak{N}$ can have no non-zero eigenvalue, that is, T_1 is quasinilpotent. For $y\in\mathfrak{N}$ and $x\in\mathfrak{M}$ the complex function $\varphi(\zeta)=(\exp(i\zeta T)\,y, x)$ is an entire function of exponential type 0 because of $\lim_{n\to\infty}\|T_1^n\|^{\frac{1}{n}}=0$. Since $|\varphi(\zeta)|$ is bounded by $\|y\|\cdot\theta(x)$ on the real axis, in view of the Bernstein theorem [1, p. 83] $\varphi(\zeta)$ is a constant function. Since this holds for every $y\in\mathfrak{N}$ and $x\in\mathfrak{M}$, T_1 must vanish identically. Since T is injective, $\mathfrak{N}=\{0\}$, that is, the set of eigenvectors of T^* is total. The assertion results from Corollary 8.1.

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(Received December 12, 1970)