NOTE ON SEPARABILITY OF ENDOMORPHISM RINGS

By

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It is well known that the endomorphism ring $\Omega = \mathrm{End}({}_RE)$ of a finitely generated projective module E over a commutative ring R is a separable R-algebra with $R/\mathfrak{a}$ its center, where $\mathfrak{a}$ is the annihilator ideal of E in R. Then, there comes out a problem whether or not this theorem holds in case R is non commutative and E is an R-R-bimodule. Partially, an affirmative answer was given in Theorem 1 [11]. In this paper we give some sufficient conditions for Ω to be separable over R in the case where M is an R-R-bimodule. In §3 we give our main results. For example, if ${}_RM_R$ is left R-finitely generated projective and R is isomorphic to a direct summand of a finite direct sum of copies of M as R-R-module, Ω is separable over R (Theorem 6). This is a generalization of the well known result in commutative case because every finitely generated projective module E over a commutative ring R is an $R/\mathfrak{a}$-progenerator. Furthermore, we obtain that for a ring extension $\Lambda|\Gamma$ such that Λ is left Γ-progenerator, $\Omega = \mathrm{End}({}_\Gamma\Lambda)$ is separable over Λ if and only if Γ is a Γ-Γ-direct summand of Λ (Theorem 7). The first two sections are devoted to the preperations for §3. And we introduce the notions of M-semisimplisity, M-separability and centrally M-separability. These are equal to the notions of semisimple, separable and H-separable extensions respectively, in case $M = S \supset R$. In §4 we give some commutor theory for some separable extensions.

1. Separability and semisimplicity with respect to bimodules.

We assume all rings have the identities and all subrings contain the identity of the over ring. Furthermore, we assume that all modules are unitary and every module which has more than one operator rings is associative with respect to the operations. Let ${}_SM_R$ be a left S- and right R-bimodule for any rings S and R. We denote ${}^*_RM_S = {}_R\mathrm{Hom}({}_SM, {}_SS)_S$ and ${}_RM^*_S = {}_R\mathrm{Hom}(M_R, R_R)_S$. Clearly, both are R-S-bimodules by the well known methods. Throughout this paper for any left S-modules M and N, we shall denote mf insted of $f(m)$ for $m \in M$ and $f \in \mathrm{Hom}({}_SM, {}_SN)$. Hence ${}_SM$ becomes an S-Ω-bimodule ${}_SM_\Omega$ with $\Omega = [\mathrm{End}({}_SM)]^0$. Now, consider the map

$$\pi_M;\ M \otimes_R \mathrm{Hom}({}_SM, {}_SS) \to S\ (\pi_M(m \otimes f) = mf \text{ for } m \in M \text{ and } f \in {}^*_RM_S)$$

This is clearly an S–S-map. In case the map π_M splits as S–S-map, we shall say that S is M-separable over R. In case $M=S\supset R$, S is M-separable over R if and only if S is a separable extension of R in the sense of [2].

π_M S-S-splits if and only if there exist $f_i\in {}^*_R M_S$ and $m_i\in M$, $i=1,2,\cdots,n$, such that $\sum sm_i\otimes f_i=\sum m_i\otimes f_i s$ for all $s\in S$ and $\sum m_i f_i=1$. We shall call $\{m_i, f_i\}$ a system of M-separability. Hence if S is M-separable over R, M is an S-generator. Conversely, if M is a left S-generator, S is M-separable over $\Omega=[\mathrm{End}\,({}_SM)]^0$ since ${}_SM\otimes_\Omega \mathrm{Hom}\,({}_SM, {}_SS)_S\cong {}_SS_S$. If ${}_SP_R$ is right R-finitely generated projective, P gives an adjoint triple of functors (G, F, H) between ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$, the category of left R-, resp. S-modules, by $F(M)=P\otimes_R M$, $G(N)=P^*\otimes_S N$ and $H(N)=\mathrm{Hom}({}_SP, {}_SN)$. In this case the P-separability of S over R is equal to the condition (a) of the definition of strongly separable pairing of [1]. First, we give some formal properties of M-separability.

Lemma 1. *Let R, S and T be rings and ${}_SM_R$ be an S–R-bimodule such that S is M-separable over R. Then*

(1) *For any S–T-bimodule ${}_SX_T$, the map $\pi_{(M,X)}: M\otimes_R Hom\,({}_SM, {}_SX)\to X$ such that $\pi_{(M,X)}(m\otimes f)=mf$ splits as S–T-map.*

(2) *For any T–S-bimodule ${}_TY_S$, the map $\iota_{(M,Y)}: Y\to Hom\,(M_R, Y\otimes_S M_R)$ such that $\iota_{(M,Y)}(y)(m)=y\otimes m$ splits as T–S-map.*

Proof. (1). Define a map φ of X to $M\otimes_R \mathrm{Hom}\,({}_SM, {}_SX)$ to be $\varphi(x)=\sum m_i\otimes f_i\circ x^r$ where $\{m_i, f_i\}$ is a system of M-separability and x^r is the map of the right multiplication of x of S to X. Then φ is an S–T-map with $\pi\varphi=1_X$. (2). We have a T–S-map $\gamma: Y\otimes_S M\to \mathrm{Hom}(\mathrm{Hom}({}_SM, {}_SS)_S, Y_S)$ such that $\gamma(y\otimes m))(f)=y(mf)$ for $m\in M$, $y\in Y$ and $f\in {}^*M$. Let ψ be a map of $\mathrm{Hom}\,(M_R, Y\otimes_S M_R)$ to Y such that $\psi(g)=\sum\gamma(g(m_i))(f_i)$ for $g\in\mathrm{Hom}\,(M_R, Y\otimes_S M_R)$ and m_i, f_i as in (1). Then ψ is a T–S-map with $\psi\circ\iota_{(M,Y)}=1_Y$.

Proposition 1. *Let R, S and T be rings and ${}_TP_S$, ${}_SM_R$ be T–S-, S–R-bimodules, respectively. Then*

(1) *If T is P-separable over S and S is M-separable over R, T is $P\otimes_S M$-separable over R.*

(2) *If T is $P\otimes_S M$-separable over R, T is P-separable over S.*

Proof. (1) By Lemma 1, the map φ: $M\otimes_R \mathrm{Hom}\,({}_SM, {}_S\mathrm{Hom}\,({}_TP, {}_TT))\to \mathrm{Hom}\,({}_TP, {}_TT)$ with $\varphi\,(m\otimes\alpha)=m\alpha$ for $m\in M$, $\alpha\in\mathrm{Hom}\,({}_SM, {}^*_SP)$ splits as S–T-map. Then we have the following commutative diagram of T-T-maps

$$\begin{array}{ccccc}
 & & P\otimes_S \mathrm{Hom}\,({}_TP, {}_TT) & \longleftarrow & T \\
 & \swarrow & \downarrow & & \downarrow 1_T \\
P\otimes_S M\otimes_R \mathrm{Hom}\,({}_SM, {}_S\mathrm{Hom}\,({}_TP, {}_TT)) & \xrightarrow{1\otimes\varphi} & P\otimes_S \mathrm{Hom}\,({}_TP, {}_TT) & \xrightarrow{\pi_P} & T \\
\downarrow & & & \nearrow & \\
P\otimes_S M\otimes_R \mathrm{Hom}\,({}_TP\otimes_S M, {}_TT) & & \pi_{P\otimes M} & &
\end{array}$$

Thus we see $\pi_{P\otimes M}$ splits as T–T-map and T is $P\otimes_S M$-separable over R. (2). Let φ be the map defined in (1). Then we have a commutative diagram of T–T-maps

$$\begin{array}{ccccc} P\otimes_S M\otimes_R \mathrm{Hom}\,({}_TP\otimes_S M,\ {}_TT) & & \xrightarrow{\quad\pi_{P\otimes M}\quad} & & T \\ \wr\| & & & & \downarrow 1_T \\ P\otimes_S M\otimes_R \mathrm{Hom}\,({}_SM,\ {}_S\mathrm{Hom}\,({}_TP,\ {}_TT)) & \xrightarrow[1_P\otimes\varphi]{} & P\otimes_S \mathrm{Hom}\,({}_TP,\ {}_TT) & \xrightarrow[\pi_P]{} & T \end{array}$$

Then since $\pi_{P\otimes M}$ splits, π_P splits. Thus T is P-separable over S.

Remark. (1) of Proposition 1 is essentially different from Proposition 3.8 [1], because we do not assume that ${}_SM$ is finitely generated projective.

Corollary 1. *Let R, S be ring and R' a subrings of R. Then for an S–R-bimodule ${}_SM_R$, we have*

(1) *If S is M-separable over R and R is a separable extension of R', S is M-separable over R'.*

(2) *If S is M-separable over R', S is M-separable over R.*

Proof. In the situation of ${}_SM_R$, ${}_RR_{R'}$, apply Proposition 1.

Theorem 1. *Let ${}_SM_R$ be an S–R-bimodule and $\Omega=[End\,({}_SM)]^0$. Then we have*

(1) *In case M is an S-generator, if Ω is a separable extension of R, S is M-separable over R.*

(2) *In case M is S-finitely generated projective, if S is M-separable over R, Ω is a separable extension of R.*

(3) *In case M is an S-progenerator, S is M-separable over R if and only if Ω is a separable extension of R.*

Proof. (1) follows immediately from Corollary 1 (2), since S is M-separable over Ω. (2). Since M is S-finitely generated projective by assumption, we have an Ω-Ω-isomorphism η: $\mathrm{Hom}\,({}_SM,\ {}_SS)\otimes_S M\to\Omega$ with $\eta(f\otimes m)(x)=xfm$ for $f\in{}^*M$, m, $x\in M$. Then the S–S-splitting of π_M: $M\otimes_R^* M\to S$ implies the Ω–Ω-splitting of π': $\Omega\otimes_R\Omega\to\Omega$, by the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}\,({}_SM,\ {}_SS)\otimes_S M\otimes_R \mathrm{Hom}({}_SM,\ {}_SS)\otimes_S M & \xrightarrow{\quad 1_{*M}\otimes\pi\otimes 1_{*M}\quad} & \mathrm{Hom}({}_SM,\ {}_SS)\otimes_S S\otimes_S M \\ \downarrow \theta\otimes\theta & & \downarrow \theta \\ \Omega\otimes_R\Omega & \xrightarrow[\quad\pi'\quad]{} & \Omega \end{array}$$

where $\pi'(\omega\otimes\omega_1)=\omega\omega_1$. Thus Ω is separable over R. (3) follows from (1) and (2).

The next proposition is a result of R. Cunningham [1].

Proposition 2. (Proposition 3.2 [1]). *Let ${}_SM_R$ be an S–R-bimodule such that M is R-finitely generated projective, and $E=End(M_R)$. Then S is M-separable over R if and only if there exists an S–S-homomorphism h of E to S with $h(1_M)=1$, i.e., ${}_SS_S<\oplus{}_SE_S$.*

Proof. Since M_R is finitely generated projective, we have an S–S-isomorphism

$\Psi: P\otimes_R\mathrm{Hom}({}_SP, {}_SS)\to\mathrm{Hom}({}_SE, {}_SS)$ $(\Psi(p\otimes f)(\sigma)=\sigma(p)f,\ \sigma\in E,\ f\in{}^*M,\ m\in M)$

Since Ψ is an S–S-map, Ψ induces an isomorphism $[M\otimes_R\mathrm{Hom}({}_SM, {}_SS)]^S\cong[\mathrm{Hom}({}_SE, {}_SS)]^S=\mathrm{Hom}({}_SE_S, {}_SS_S)$. Therefore, there exists $\sum m_i\otimes f_i\in(M\otimes{}_R^*M)^S$ such that $\sum m_if_i=1$ if and only if there exists an $h\in\mathrm{Hom}({}_SE_S, {}_SS_S)$ with $h(1_M)=1$.

Remark. In case $M=S\supset R$, Proposition 2 induces Hilfssatz 43 [7] and Proposition 1 [8]. Note that the 'only if' part is valid if M is not R-finitely generated projective.

The next proposition is a generalization of Proposition 2 [11].

Proposition 3. *Let an S–R-bimodule ${}_SM_R$ be R-finitely generated projective, and $\{g_j, m_j\}$ be a dual basis of M_R. Then S is M-separable over R if and only if there exists an R–S-map α of ${}_RM_S^*$ to ${}_R^*M_S$ such that $\sum\alpha(g_j)m_j=1$.*

Proof. Since M_R is finitely generated projective, we have an S–S-isomorphism

$$\sigma: M\otimes_R\mathrm{Hom}({}_SM, {}_SS)\to\mathrm{Hom}({}_R\mathrm{Hom}(M_R, R_R), {}_R\mathrm{Hom}({}_SM, {}_SS))$$

with $\sigma(m\otimes f)(g)=g(m)\circ f$ for $f\in{}_R^*M_S$, $g\in{}_RM_S^*$ and $m\in M$. Consider the following commutative diagram of S–S-maps

$$\begin{array}{ccc} M\otimes_R{}^*M & \xrightarrow{\ \sigma\ } & \mathrm{Hom}({}_RM^*, {}_R{}^*M) \\ & {\scriptstyle\pi_M}\searrow \quad \swarrow{\scriptstyle\Phi} & \\ & S & \end{array}$$

with $\Phi(\beta)=\sum\beta(g_j)m_j$ for $\beta\in\mathrm{Hom}({}_RM^*, {}_R^*M)$. Then since σ induces an isomorphism $(M\otimes{}_R^*M)^S\cong\mathrm{Hom}({}_RM_S^*, {}_R^*M_S)$, π_M splits as S–S-map if and only if there exists an $\alpha\in\mathrm{Hom}({}_RM_S^*, {}_R^*M_S)$ with $\Phi(\alpha)=1$, i.e. $\sum\alpha(g_j)m_j=1$.

Let ${}_SM_R$ be an S–R-bimodule for rings S, R. We shall consider the following two S-homomorphisms for left S-module X and right S-module Y.

$\pi_{(M,X)}: M\otimes_R\mathrm{Hom}({}_SM, {}_SX)\to X$ $(\pi_{(M,X)}(m\otimes f)=mf$ for $m\in M$, $f\in\mathrm{Hom}({}_SM, {}_SX))$

$\iota_{(M,Y)}: Y\to\mathrm{Hom}(M_R, Y\otimes_SM_R)$ $(\iota_{(M,Y)}(y)(m)=y\otimes m$ for $m\in M$, $y\in Y)$

If $\pi_{(M,X)}$ S-splits for every left S-module X, we shall call that S is left M-semisimple over R, and if $\iota_{(M,Y)}$ S-splits for every right S-module Y, we shall call that S is right M-semisimple over R. In case $M=S\supset R$, they are left (resp. right) semisimple extensions in the sense of [2]. Similar to M-separability we have

Proposition 4. *Let R, S and T be rings and ${}_TP_S$, ${}_SM_R$ be T–S- and S–R-bimodules, respectively. Then,*

(1) *If T is left (resp. right) P-semisimple over S and S is left (resp. right) M-semisimple over R, T is left (resp. right) $P\otimes_S M$-semisimple over R.*

(2) *If T is left (resp. right) $P\otimes_S M$-semisimple over R, T is left (resp. right) P-semisimple over S.*

Proof. For left semisimplicities, if we take ${}_TX$ instead of T in the proof of Proposition 2, both (1) and (2) can be proved similarly. For any right T-module Y_T, $\iota_{(P\otimes M,Y)}$ is naturally equivalent to $\mathrm{Hom}(P, \iota_{(M,Y\otimes P)})\circ\iota_{(P,Y)}$. Thus both (1) and (2) are also clear for right P-semisimplicity.

Corollary 2. *Let S, R be rings and R' a subring of R. Then for an S–R-bimodule ${}_SM_R$,*

(1) *If S is left (resp. right) M-semisimple over R and R is a left (resp. right) semisimple extension of R', S is left (resp. right) M-semisimple over R'.*

(2) *If S is left (resp. right) M-semisimple over R', S is left (resp. right) M-semisimple over R.*

Proof. In the situation of ${}_SM_R$, ${}_RR_{R'}$, apply Proposition 4.

Theorem 2. *Let ${}_SM_R$ an S–R-bimodule and $\Omega=[End({}_SM)]^0$ for rings S and R. Then we have,*

(1) *If S is M-separable over R, S is left as well as right M-semisimple over R.*

(2) *In case M is an S-generator, if Ω is a left (resp. right) semisimple extension of R, S is left (resp. right) M-semisimple over R.*

(3) *In case M is S-finitely generated projective, if S is left (resp. right) M-semisimple over R, Ω is a left (resp. right) semisimple extension of R.*

Proof. (1) is clear by Lemma 1. (2) follows immediately from (1) and Corollary 2 (1). (3). Since by assumption ${}_SM$ is finitely generated projective, $\mathrm{Hom}({}_SM, {}_SS)\otimes_S N\cong\mathrm{Hom}({}_SM, {}_SN)$ ($f\otimes n\to g$ such that $mg=mfn$, for $m\in M$, $n\in N$, $f\in{}^*M$ and $g\in\mathrm{Hom}({}_SM, {}_SN)$) for any left S-module N. Then the splitting of $\pi_{(M,N)}$ implies that every left Ω-module X such that ${}_\Omega X\cong{}_\Omega\mathrm{Hom}({}_SM, {}_SN)$) for some ${}_SN$ is (Ω, R)-projective. But ${}_\Omega X\cong{}_\Omega^*M\otimes_S M\otimes_\Omega X\cong{}_\Omega\mathrm{Hom}({}_SM, {}_SM\otimes_\Omega X)$. Thus if S is left M-semisimple over R, every left Ω-module is (Ω, R)-projective, and Ω is left semisimple over R. If S is right M-semisimple over R, $\iota_{(M,Y\otimes^*M)}$ splits for every right Ω-module Y. Then $\iota_{(M,Y\otimes^*M)}\otimes 1_M$ splits, which is equivalent to $\iota_{(\Omega,Y)}$. Thus every right Ω-module is (Ω, R)-injective, and Ω is right semisimple over R.

Let M be a left S-progenerator and $\Omega=[\mathrm{End}({}_SM)]^0$. S is a semisimple ring if and only if Ω is a semisimple ring, and S is a left (resp. right) hereditary ring if and only if Ω is a left (resp. right) hereditary ring, since gl. dim S = gl. dim Ω by Morita Theorem. Furthermore, S is QF-ring if and only if Ω is a QF-ring, since S is QF if and only if every left S-injective module is projective by Theorem 5. 3 [14]. From these remarks and Theorem 2 it follows.

Proposition 5. *Let S and R be rings such that S is left M-semisimple over R for an S–R-bimodule ${}_SM_R$. Suppose M is left S-finitely generated projective. Then we have*

(1) *If R is a semisimple ring, then both S and Ω are semisimple rings.*

(2) *In case M is R-flat, if R is left hereditary, then both S and Ω are left hereditary.*

(3) *In case M is R-flat, if R is a FQ-ring, then both S and Ω are QF-rings.*

Proof. Since $\pi_{(M,S)}$ is an epimorphism, M is an S-progenerator, and Ω is a left semisimple extension of R by Theorem 2. So we need only to prove for Ω. (1) is clear by Corollary 1.7 [2]. (2). Since M is an Ω-generator and R-flat, Ω is R-flat as right R-module. Hence if R is left hereditary, Ω is left hereditary by Corollary 1.9 [2]. (3). This follows from the next proposition, since Ω is right R-flat.

Proposition 6. *Let a ring Λ be a left semisimple extension of a subring Γ. Then, if Γ is a QF-ring and Λ is right Γ-flat, Λ is a QF-ring.*

Proof. Let P be an arbitrary injective left Λ-module. Then P is Γ-injective, since Λ is right Γ-flat. Since Γ is QF, P is Γ-projective by Faith-Walker's theorem. Then, P is Λ-projective by Proposition 1.6 [2]. Thus every injective Λ-module is projective, and Λ is a QF-ring.

2. Special type of M-separability.

Following K. Hirata [4], we shall say that an R–R-bimodule is centrally projective over R if M is isomorphic to a direct summand of a finite direct sum of copies of R as two sided R-module. In this section we shall concider the case where $M\otimes_R\mathrm{Hom}({}_SM, {}_SS)$ is centrally projective over S. In case $M=S\supset R$, the above condition is equivalent to the condition that S is an H-separable extension of R in the sense of [9] and [4]. In this paper we shall call that S is centrally M-separable over R in case $M\otimes_R^* M$ is centrally projective over S.

In case M is left S-reflexive, the map θ: $\mathrm{Hom}\,({}_SM_R,\ {}_SM_R)\to\mathrm{Hom}\,({}_SM\otimes{}_R\mathrm{Hom}\,({}_SM,\ {}_SS)_S,\ {}_SS_S)$ such that $\theta(\alpha)(m\otimes f)=f(\alpha(m))$ for $\alpha\in\mathrm{End}\,({}_SM_R)$, $m\in M$ and $f\in{}^*M$, is an C-isomorphism, where C is the center of S.

By these remarks and Theorem 1.2 [3], we have

Proposition 7. *Let an S–R-module M be S-reflexive. Then S is centrally M-separable over R if and only if the map*

$$\eta:\ M\otimes_R\mathrm{Hom}\,({}_SM,\ {}_SS)\to\mathrm{Hom}\,(\mathrm{End}\,({}_SM_R)_C,\ S_C)$$

such that $\eta(m\otimes f)(\sigma)=\theta(\sigma)(m\otimes f)=\sigma(m)f$ *for* $m\in M$, $f\in{}^*M$ *and* $\sigma\in End\,({}_SM_R)$ *is an S–S-isomorphism and* $End\,({}_SM_R)$ *is a finitely generated projective C-module.*

As in the case of ring extensions, we have

Theorem 3. *For any S–R-bimodule* ${}_SM_R$ *which is a faithful and reflexive S-module, if S is centrally M-separable over R, S is M-separable over R.*

Proof. Since M is S-faithful, $\mathrm{End}\,({}_SM_R)$ is C-faithful, where C is the center of S. Then by Proposition 7, $\mathrm{End}\,({}_SM_R)$ is a C-generator and we have a commutative diagram of S–S-maps

$$\begin{array}{ccc} M\otimes_R\mathrm{Hom}\,({}_SM,\ {}_SS) & \xrightarrow{\ \ \eta\ \ } & \mathrm{Hom}\,({}_C\mathrm{Hom}\,({}_SM_R,\ {}_SM_R),\ {}_CS) \\ & {}_{\pi_M}\searrow \quad \swarrow{}_{\Psi} & \\ & S & \end{array}$$

where η is the S–S-isomorphism defined in Proposition 7 and Ψ is such that $\Psi(\alpha)=\alpha(1_M)$ for $\alpha\in\mathrm{Hom}\,({}_C\mathrm{End}\,({}_SM_R),\ {}_CS)$. Then the same method as Theorem 2.2 [3] shows that S is M-separable over R.

The next proposition is a similar result to Proposition 3.2 [1].

Proposition 8. *Let* ${}_SM_R$ *be an S–R-bimodule for rings S and R. Then*

(1) *In case* M_R *is finitely generated projective, if* $E=End\,(M_R)$ *is centrally projective over S, S is centrally M-separable over R.*

(2) *In case* ${}_SM$ *is reflexive, if S is centrally M-separable over R, E is centrally projective over S.*

Proof. Both (1) and (2) follows from the following two isomorphisms

$$\mathrm{Hom}\,(M\otimes{}_R^*M_S,\ S_S)\cong\mathrm{Hom}\,(M_R,\ \mathrm{Hom}\,({}^*M_S,\ S_S)_R)\cong\mathrm{Hom}\,(M_R,\ M_R)=E$$

and $M\otimes{}_R^*M\cong\mathrm{Hom}\,({}_SE,\ {}_SS)$.

Corollary 3. *Let* Λ *be a ring and* Γ *a subring of* Λ *such that* Λ *is right* Γ*-finitely generated projective. Then* Λ *is H-separable over* Γ *if and only if*

$E=End(\Lambda_\Gamma)$ is centrally projective over Λ.

Proof. Take $M=S=\Lambda$ and $R=\Gamma$ in Proposition 8. Note that the 'only if' part is valid without the assumpltion that Λ is right Γ-finitely generated projective.

Now, we state some properties of centrally M-separability without proof, as these are similar to the prooves of Propositions 1, 2 and Theorem 1.

Proposition 9. *Let R, S and T be rings and ${}_TP_S$, ${}_SM_R$ be any T–S-, S–R-bimodules, respectively with ${}_SM$ finitely generated projective. Then we have*

(1) *If T is centrally P-separable over R and S is centrally M-separable over R, then T is centrally $P\otimes_S M$-separable over R.*

(2) *If T is centrally $P\otimes_S M$-separable over R and S is M-separable over R, then T is centrally P-separable over S.*

Proof. Since ${}_SM$ is finitely generated projective by assumption, we see

$$P\otimes_S M\otimes_R \mathrm{Hom}({}_SM, {}_SS)\otimes_S \mathrm{Hom}({}_TP, {}_TT)\cong P\otimes_S M\otimes_R \mathrm{Hom}({}_TP\otimes_S M, {}_TT)$$

Corollary 4. *Let S, R be rings and R′ a subring of R. Then for any S–R-bimodule ${}_SM_R$,*

(1) *If S is centrally M-separable over R and R is an H-separable extension of R′, then S is M-centrally separable over R′.*

(2) *If S is centrally M-separable over R′ and R is a separable extension of R′, then S is centrally M-separable over R.*

Theorem 4. *Let ${}_SM_R$ be an S–R-bimodule and $\Omega=[End({}_SM)]^0$. Then,*

(1) *In case M is a left S-generator, if Ω is an H-separable extension of R, S is centrally M-separable over R.*

(2) *In case M is S-finitely generated projective, if S is centrally M-separable over R, Ω is an H-separable extension of R.*

(3) *In case M is an S-progenerator, Ω is an H-separable extension of R if and only if S is centrally M-separable over R.*

3. Applications to endomorphism rings.

In this section we shall apply the results of previous sections to endonorphism rings. A part of the next proposition is a result of T. Kanzaki [6].

Theorem 5. *Let R be a commutative ring, Λ an R-algebra with C its center, M a left Λ-module and $\Omega=[End({}_\Lambda M)]^0$. Then we have*

(1) *In case M is a left Λ-generator, if Ω is a separable R-algebra, Λ is a separable R-algebra.*

(2) *In case M is left Λ-finitely generated projective, if Λ is a separable*

R-algebra, Ω is a separable R-abgebra and the center of Ω is equal to $C1_M$.

Proof. (1) is a consequence of (2), since M is Ω-finitely generated projective and $\Lambda=\mathrm{End}\,(M_\Omega)$. While, (2) is a consequence of (1), since Λ has the property (PFG) and M is a Λ-generator by Theorem 6.1[12]. But we shall give the prooves independently. (1). The center of $\Omega=V_\Omega(\Omega)=\mathrm{Hom}({}_\Lambda M_\Omega, {}_\Lambda M_\Omega)$ $=V_\Lambda(\Lambda)=C$, since $\Lambda=\mathrm{Hom}\,(M_\Omega, M_\Omega)$. Hence if Ω is separable over R, Ω is central separable over C and C is separable over R. Then by Theorem 4 (1), $M\otimes_C\mathrm{Hom}\,({}_\Lambda M, {}_\Lambda\Lambda)$ is Λ-centrally projective. But since ${}_\Lambda\Lambda_C<\oplus_\Lambda(M\oplus\cdots\oplus M)_C$, $\Lambda\otimes_C\Lambda$ is isomorphic to a direct summand of a finite direct sum of copies of $M\otimes_C^* M$ as Λ-Λ-module. Thus $\Lambda\otimes_C\Lambda$ is Λ-centrally projective. This implies that Λ is separable over C. Hence Λ is separable over R. (2). This is Theorem 1 [6], but we shall give a different proof in this peper. If Λ is R-separable, C is R-separable and $\Lambda\otimes_C\Lambda$ is Λ-centrally projective. Since M is Λ-finitely generated projective by assumption, M is an Ω-generator, and we have ${}_\Lambda M_C<\oplus_\Lambda(\Lambda\oplus\cdots\oplus\Lambda)_C$ and ${}_C^*M_\Lambda<\oplus_C(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$ and $\Omega_C<\oplus$ $(M\oplus\cdots\oplus M)_C<\oplus\,(C\oplus\cdots\oplus C)_C$. Then we see ${}_\Lambda M\otimes_C^* M_\Lambda<\oplus_\Lambda(\Lambda\otimes_C\Lambda\oplus\cdots\oplus\Lambda\otimes_C\Lambda)_\Lambda$. Then $M\otimes_C^* M$ is Λ-centrally projective, and Ω is H-separable over C by Theorem 4 (2). But since Ω is C-finitely generated projective, Ω is central separable over C by Corollary 1.2 [10].

Proposition 10. *Let R be a commutative ring, Λ an R-algebra which is R-finitely generated projective, M a finitely generated projective left Λ-module and $\Omega=[End\,({}_\Lambda M)]^0$. Then if Λ is a semisimple R-algebra, Ω is a semisimple R-algebra.*

Proof. Since every ring epimorphic image of Λ is a semisimple R-algebra, we can assume M is Λ-faithful and Λ is R-faithful. Thus $\sum=$ $\mathrm{End}\,({}_R M)$ is a central separable R-algebra and Λ is a semisimple R-subalgebra of $\sum$. Then by Theorem 6.1 [12] and Theorem 3.5 [14], $\Omega(=[\mathrm{End}\,({}_R M)]^\Lambda=$ $V_\Sigma(\Lambda))$ is semisimple R-algebra.

Theorem 6. *Let M be a two sided R-module over a ring R and $\Omega=[End\,({}_R M)]^0$. Then,*

(1) *If M is left R-finitely generated projective and ${}_R R_R<\oplus_R(M\oplus\cdots\oplus M)_R$, then Ω is a separable extension of $R\cdot 1_M$.*

(2) *If ${}_R M_R<\oplus_R(R\oplus\cdots\oplus R)_R$, Ω is an H-separable extension of $R\cdot 1_M$.*

Proof. (1). ${}_R R_R<\oplus_R(M\oplus\cdots\oplus M)_R$ if and only if there exist $m_i\in M^R$ $(=\{m\in M\,|\,rm=mr$ for all r in $R\})$ and $f_i\in\mathrm{Hom}\,({}_R M_R, {}_R R_R)$, $i=1, 2, \cdots, n$, such that $\sum f_i(m_i)=1$. Then in $M\otimes_R^* M$, $r\sum m_i\otimes f_i=\sum m_i r\otimes f_i=\sum m_i\otimes rf_i$ $=\sum m_i\otimes f_i r$ for all $r\in R$. Thus R is M-separable over R, and Ω is separable over R by Theorem 1 (2). (2). This is Theorem 1 [11].

Theorem 7. *Let Λ be a ring, Γ a subring of Λ and $\Omega=[End({}_\Gamma\Lambda)]^0$. Then we have*

(1) *In case ${}_\Gamma\Lambda$ is finitely generated projective, if ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma\Lambda_\Gamma$, Ω is a separable extension of Λ.*

(2) *In case ${}_\Gamma\Lambda$ is a generator, if Ω is a separable extension of Λ, ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma\Lambda_\Gamma$.*

(3) *In case ${}_\Gamma\Lambda$ is a progenerator, Ω is a separable extension of Λ if and only if ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma\Lambda_\Gamma$.*

Proof. (1). By assumption and Theorem 1, (2), Ω is separable over Γ in this case, hence Ω is separable over Λ. (2). If ${}_\Gamma\Lambda$ is a generator and Ω is separable over Λ, Γ is Λ-separable over Λ by Theorem 1 (1). Hence the map π: $\Lambda\otimes_\Lambda\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)\to\Gamma$ splits as Γ-Γ-map. But since ${}_\Gamma\Lambda\otimes_\Lambda\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma\cong{}_\Gamma\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma$, π splits as Γ–Γ-map if and only if the map π': $\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)\to\Gamma$ such that $\pi'(f)=f(1)$ for $f\in\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ splits as Γ–Γ-map. This is the case if and only if there exists an $h\in[\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)]^\Gamma=\mathrm{Hom}({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ such that $\pi'(h)=h(1)=1$. Thus ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma\Lambda_\Gamma$ in this case. (3). This is clear by (1) and (2).

Proposition 11. *Let Λ, Γ, and Ω be as in Theorem* 6. *Then we have*

(1) *If Λ is centrally projective over Γ, Ω is an H-separable extension of Λ.*

(2) *In case ${}_\Gamma\Lambda$ is a progenerator, Ω is an H-separable extension of Λ if and only if Λ if Γ-centrally projective.*

Proof. (1). By assumption Λ is left Γ-finitely generated projective and ${}_\Gamma\Lambda\otimes_\Lambda\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma\cong{}_\Gamma\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma<\oplus_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$. Hence by Theorem 4 (2), Ω is H-separable over Λ. (2). We need only to prove the 'only if' part. Suppose Ω is H-separable over Λ. Then by Theorem 7 (2), ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma\Lambda_\Gamma$, and by Theorem 4 (1) ${}_\Gamma\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma\cong{}_\Gamma\Lambda\otimes_\Lambda\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma<\oplus_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$. Then taking the second dual of ${}_\Gamma\Lambda$, we obtain ${}_\Gamma\Lambda_\Gamma<\oplus_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$ since ${}_\Gamma\Lambda$ is finitely generated projective.

We end this section by noting

Proposition 12. *Let Λ be a ring with its center C, Γ a subring of Λ, $\Omega=End(\Lambda_\Gamma)$, and $\Delta=V_\Lambda(\Gamma)$, the centralizer of Γ in Λ. Then in case $\Lambda\otimes_\Lambda\Lambda$ is Λ-centrally projective, (i.e., Λ is H-separable over Γ), we have*

(1) *Ω is separable over Λ if and only if Δ is separable over C.*

(2) *Ω is H-separable over Λ if and only if Δ is central separable over C.*

Proof. If Λ is H-separable over Γ, Δ is C-finitely generated projective and $\Omega\cong\Lambda\otimes_C\Delta^\circ$. (1). If Δ is separable over C, $\Omega=\Lambda\otimes_C\Delta^\circ$ is separable over

Λ. Conversely, if Ω is separable over Λ, $V_\Omega(\Lambda)=[\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)]^\Lambda=\mathrm{Hom}({}_\Lambda\Lambda_\Gamma, {}_\Lambda\Lambda_\Gamma)=\Delta$ is separable over C by Theorem 2 [11]. (2). This is Theorem 1.1 [10].

4. Some commutor theory for separable extensions.

In [10] we studied a commutor theory in some H-separable extensions. Here we shall study a commutor theory in general separable extension $\Lambda|\Gamma$ such teat Λ is Γ-centrally projective. In this case Λ is H-separable over $\Gamma'=V_\Lambda(V_\Lambda(\Gamma))$ and Γ' is centrally projective and separable over Γ by Theorem 2 [11]. Hence between Λ and Γ' the commutor theory of Theorem 1.2 and Corollary 1.4 [10] holds. Now we shall study the commutors between Γ' and Γ. Let C and S be the centers of Γ and Λ, respectively. By the proof of Theorem 2 [11], we see that $V_{\Gamma'}(\Gamma)=$the center of $V_\Lambda(\Gamma)=$ the center of Γ', and $\Gamma'=\Gamma\otimes_C V_{\Gamma'}(\Gamma)$. Hence we shall consider a general ring extension $\Lambda|\Gamma$ which satisfies the following conditions (♯)

(♯) (1) Λ is a separable extension of Γ such that $V_\Lambda(\Gamma)=S$.

(2) Λ is Γ-centrally projective.

Proposition 13. *Let $\Lambda|\Gamma$ be a ring extension such that Λ is separable over Γ and Γ-centrally projective. Then for any subring B of Λ which contains Γ, if Λ is B-centrally projective, B is separable over Γ and B is a B–B-direct summand of Λ.*

Proof. Let T be the center of B. If Λ is B-centrally projective, $\Lambda=V_\Lambda(B)\otimes_T B$ and $V_\Lambda(B)$ is T-finitely generated projective. Hence ${}_BB_B<\oplus{}_B\Lambda_B$. B is also Γ-centrally projective, since Λ is so. Hence $B=V_B(\Gamma)\otimes_C\Gamma$ and $V_B(\Gamma)$ is C-finitely generated projective. Thus $\Lambda=V_\Lambda(B)\otimes_T V_B(\Gamma)\otimes_C\Gamma$ and $V_\Lambda(B)\otimes_T V_B(\Gamma)$ is C-finitely generated projective. Then by Lemma 3 [10], $V_\Lambda(\Gamma)=V_\Lambda(B)\otimes_T V_B(\Gamma)$, which is separable over C by Theorem 2 [11]. Hence $V_B(\Gamma)$ is separable over C by Corollary 2.12 [2] as ${}_TT<\oplus{}_TV_\Lambda(B)$. Then $B(=V_B(\Gamma)\otimes_C\Gamma)$ is separable over Γ.

In case $\Lambda|\Gamma$ satisfies the condition (♯), the converse of Proposition 13 holds.

Proposition 14. *Let $\Lambda|\Gamma$ be a ring extension which satisfies the condition* (♯). *Then for any intermediate subring B between Λ and Γ, the following conditions are equivalent*

(1) *Λ is centrally projective over B.*

(2) *B is a separable extension of Γ and ${}_BB_B<\oplus{}_B\Lambda_B$*

If B satisfies either of the condition (1) *or* (2), *$\Lambda|B$ satisfies the condition* (♯).

Proof. (1)⇒(2) has been proved in Proposition 13. Suppose (2). Then

$V_\Lambda(\Gamma)=S$ is a commutative separable C-algebra and C-finitely generated projective. On the other hand, ${}_BB_B<\oplus{}_B\Lambda_B$ implies that B is Γ-centrally projective, and $R=V_B(\Gamma)$ is a commutative C-separable subalgebra of S. Hence S is R-finitely generated projective. Then $\Lambda\cong S\otimes_C\Gamma<\oplus(\sum^n\oplus R)\otimes_C\Gamma\cong\sum^n\oplus B$ as B–B-module. Thus Λ is B-centrally projective, and we have proved (2)$\Rightarrow$(1). Since $S\subset V_\Lambda(B)\subset V_\Lambda(\Gamma)=S$, the last part of this proposition is evident.

Let $\Lambda|\Gamma$ be a ring extension with the condition ($\sharp$), $\mathfrak{A}$ be the class of subrings B of Λ such that B is a separable extension of Γ and ${}_BB_B<\oplus{}_B\Lambda_B$, and $\mathfrak{B}$ be the class of C-separable subalgebras of S. Consider the maps $V: B\rightsquigarrow B\cap S=V_B(\Gamma)$ for $B\in\mathfrak{A}$ and $U: R\rightsquigarrow R\cdot\Gamma\cong R\otimes_C\Gamma$ for $R\in\mathfrak{B}$. By the proof of Proposition 14, we can easily see that $V(B)\in\mathfrak{B}$ and $U(R)\in\mathfrak{A}$ for $B\in\mathfrak{A}$ and $R\in\mathfrak{B}$. Hence the above U and V are correspondences between $\mathfrak{A}$ and $\mathfrak{B}$. Furthermore, $UV(B)=V_B(\Gamma)\cdot\Gamma=B$ for $B\in\mathfrak{A}$ and $VU(R)=(R\otimes_C\Gamma)^\Gamma=R\otimes_C C=R$ for $R\in\mathfrak{B}$ by Lemma 3 [11]. Thus we have

Theorem 8. *Let $\Lambda|\Gamma$ be a ring extension with the condition ($\sharp$), and $\mathfrak{A}$ and $\mathfrak{B}$ be as above. Then there exist one to one correspondences between $\mathfrak{A}$ and $\mathfrak{B}$ such that $V: B\rightsquigarrow B\cap S$ and $U: R\rightsquigarrow R\Gamma$ for $B\in\mathfrak{A}$, $R\in\mathfrak{B}$ with $UV=1_{\mathfrak{A}}$ and $VU=1_{\mathfrak{B}}$.*

Remark. In [10] we considered the ring extension $\Lambda|\Gamma$ which satisfies the following condition (*),

(*) (1) Λ is an H-separable extension of Γ with ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma\Lambda_\Gamma$

(2) $V_\Lambda(\Gamma)=C$ and $V_\Lambda(C)=\Gamma$, where C is the center of Γ.

In this case $\Omega=\mathrm{End}({}_\Gamma\Lambda)=C\otimes_S\Lambda$ and C is a commutative S-separable algebra and is S-finitely generated projective, where S is the center of Λ. Then clearly, the center of $\Omega=C=V_\Omega(\Lambda)$, and $\Omega|\Lambda$ satisfies the condition ($\sharp$). Let $\mathfrak{A}'$ be the class of separable extensions Σ of Λ such that ${}_\Sigma\Sigma_\Sigma<\oplus{}_\Sigma\Omega_\Sigma$, $\mathfrak{B}'$ be the class of separable S-subalgebras of C, and V', U' be such that $V'(\Sigma)=\Sigma\cap C$ for $\Sigma\in\mathfrak{A}'$ and $U'(R)=R\Lambda$ for $R\in\mathfrak{B}'$. Then by Theorem 8, V' and U' provide one to one correspondences between $\mathfrak{A}'$ and $\mathfrak{B}'$ such that $U'V'=1_{\mathfrak{A}'}$, and $V'U'=1_{\mathfrak{B}'}$. Furthermore, V' and U' induce one to one correspondences between the class $\mathfrak{A}_1$ of members of $\mathfrak{A}'$ which are of the form $\mathrm{End}({}_B\Lambda)$ with B an H-separable subextension of Γ in Λ, and the class $\mathfrak{B}_1$ of members of $\mathfrak{B}'$ which are commutor subrings in Λ, by $V': \mathrm{End}({}_B\Lambda)\rightsquigarrow V_\Lambda(B)$ and $U': R\rightsquigarrow\mathrm{End}({}_B\Lambda)$ with $B=V_\Lambda(R)$. The proof is very easy if we use Proposition 3.1 [4] and Theorem 1.3 [10]. Hence we shall omit it.

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Added in proof. In the subsequent paper: On centralizers in separable extensions II, to appear in Osaka J. Math., the author will show that for $\mathfrak{A}_1$, $\mathfrak{A}'$, $\mathfrak{B}_1$ and $\mathfrak{B}'$ in the remark in § 4, the equalities $\mathfrak{A}_1=\mathfrak{A}'$ and $\mathfrak{B}_1=\mathfrak{B}'$ hold.