A remark on the Steenrod representation of \( B(\mathbb{Z}_p \times \mathbb{Z}_p) \)

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§1. Introduction

For a topological space \( X, z \in H_n(X; \mathbb{Z}) \) is Steenrod representable if there exists a closed oriented smooth \( n \)-manifold \( M \) and a continuous map \( f: M \to X \) such that \( f_*(\sigma) = z \), where \( \sigma \) is a fundamental homology class of \( M \). In [4], Thom showed that for a finite polyhedron \( X \) any \( z \in H_n(X; \mathbb{Z}) \) is representable if \( n \leq 6 \), but if \( n \geq 7 \) not everything is representable. He exhibited a class in \( H_7(L'(3) \times L'(3); \mathbb{Z}) \) which was not, where \( L'(3) \) is 7-dimesional lens space mod 3. Moreover Burdick [1] extended to \( B(\mathbb{Z}_p \times \mathbb{Z}_p) \), classifying space of \( \mathbb{Z}_p \times \mathbb{Z}_p \), and computed all representable elements. He determined \( E^\infty \) terms of bordism spectral sequence of \( B(\mathbb{Z}_3 \times \mathbb{Z}_3) \) and used necessary condition of representability of Thom [4].

In this note we show the case \( p = 2 \) and any odd prime \( p \). Latter case we use the same methods as Burdick's.

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§2. Homology groups of \( B(\mathbb{Z}_p \times \mathbb{Z}_p) \)

Let \( X = B(\mathbb{Z}_p \times \mathbb{Z}_p) \), \( Y = B(\mathbb{Z}_p) \).

Case (a): \( p = 2 \).

Let \( RP^n \) be the \( n \) dimensional real projective space, \( RP^\infty \) be the direct limit of it. Then we can consider \( Y = RP^\infty \), and so \( X = Y \times Y \). The cell structure of \( RP^n \) and its boundary operations are given as follows:

\[
RP^n = e_0 \cup e_1 \cup \cdots \cup e_n,
\]
(1.1) \[ \partial e_{2i} = 2e_{2i-1}, \quad \partial e_{2i+1} = 0, \]
where \( e_i \) is the \( i \) dimensional cell. \( Y \) is a CW complex with one cell \( e_i \) in each dimension. We will use the same symbol \( e_i \) for the homology class containing \( e_i \).

Let \( C_\ast(X) \) and \( C_\ast(Y) \) be the chain complexes as CW complex \( X \) and \( Y \) respectively. \( C_\ast(X) \cong C_\ast(Y) \otimes C_\ast(Y) \) by cross product, thus \( C_n(X) = H_n(X^n; X^{n-1}; Z) \otimes H_j(Y^j, Y^{j-1}; Z) \), where \( X^n \) and \( Y^n \) are \( n \)-skeleton of \( X \) and \( Y \) respectively. Therefore \( C_n(X) \) is generated by \( e_i \) for \( i = 0, 1, \ldots, n \) and \( \partial_n : C_n(X) \to C_{n-1}(X) \) is given as follows:

\[
(1.2) \quad \begin{align*}
\partial_n(e_{2i-1} \otimes e_{2j-1}) &= 0, \\
\partial_n(e_{2i} \otimes e_{2j-1}) &= 2e_{2i-1} \otimes e_{2j-1}, \\
\partial_n(e_{2i-1} \otimes e_{2j}) &= -2e_{2i-1} \otimes e_{2j-1}, \\
\partial_n(e_{2i} \otimes e_{2j}) &= 2e_{2i-1} \otimes e_{2j} + e_{2i} \otimes e_{2j-1}.
\end{align*}
\]

Then we have

(1.3) \( H_{2n}(X; Z) \) is generated by \( e_{2i-1} \otimes e_{2n-2i+1} \) for \( i = 1, \ldots, n \) and \( H_{2n-1}(X; Z) \) is generated by \( e_{2i-1} \otimes e_{2n-2i} + e_{2i} \otimes e_{2n-2i-1} \) for \( i = 0, 1, \ldots, n \) and every elements are order 2. \( H_n(X; Z) \cong \mathbb{Z} \) generated by \( e_0 \otimes e_0 \).

Case (b): \( p \) is the odd prime.

Let \( S^{2n+1} \) be the unit \((2n+1)\)-sphere. A point of \( S^{2n+1} \) is represented by a \((n+1)\)-tuple of complex numbers \((z_0, z_1, \ldots, z_n)\) with \( \sum_{i=0}^{n} |z_i|^2 = 1 \). Let \( T \) be the rotation of \( S^{2n+1} \) defined by \( T(z_0, z_1, \ldots, z_n) = (\lambda z_0, \lambda z_1, \ldots, \lambda z_n) \), where \( \lambda = \exp(2\pi i/p) \). \( T \) generates a fixed point free topological transformation group of \( S^{2n+1} \) of order \( p \), so we will say it \( Z_p \) action on \( S^{2n+1} \). Then the lens space mod \( p \) is defined to be the orbit space \( L^{2n+1}(p) = S^{2n+1}/Z_p \). This is the closed orientable \((2n+1)\) smooth manifold. For \( m < n \) consider \( S^{2m+1} \) as contained in \( S^{2n+1} \) with \((z_0, \ldots, z_m) = (z_0, \ldots, z_m, 0, 0, \ldots) \). Then \( L^1(p) \subset L^2(p) \subset \ldots \). Let \( L^\infty(p) \) be the direct limit of this sequence, then we can consider \( Y = L^\infty(p) \), and so \( X = Y \times Y \). The cell structure of \( L^{2n+1}(p) \), and its boundary relations are given as follows:

\[
L^{2n+1}(p) = e_0 \cup e_1 \cup \cdots \cup e_{2n+1},
\]

(1.4) \[ \partial e_{2i} = pe_{2i-1}, \quad \partial e_{2i+1} = 0. \]

\( Y \) is a CW complex with one cell \( e_i \) in each dimension and the \((2n+1)\)-skeleton is \( L^{2n+1}(p) \). \( C_n(X) \) is generated by \( e_i \otimes e_{n-i} \) \((i = 0, 1, \ldots, n)\) and \( \partial_n : C_n(X) \to C_{n-1}(X) \) is given as follows:
(1. 5) \[ \partial_n(e_{2t-1} \otimes e_{2j-1}) = 0, \]
\[ \partial_n(e_{2t} \otimes e_{2j-1}) = p e_{2t-1} \otimes e_{2j-1}, \]
\[ \partial_n(e_{2t-1} \otimes e_{2j}) = -p e_{2t-1} \otimes e_{2j-1}, \]
\[ \partial_n(e_{2t} \otimes e_{2j}) = p e_{2t-1} \otimes e_{2j} + p e_{2t} \otimes e_{2j-1}. \]

Then we have

\[ (1. 6) \quad H_{2n}(X; Z) \text{ is generated by } e_{2t-1} \otimes e_{2n-2t+1} \text{ (} i = 1, \ldots, n), \quad H_{2n-1}(X; Z) \text{ is generated by } e_{2t-1} \otimes e_{2n-2t+1} \text{ (} i = 0, 1, \ldots, n) \] and every elements are order $p$. $H_0(X; Z) \cong \mathbb{Z}$ generated by $e_0 \otimes e_0$.

§ 3. Theorems

Let $\Omega_n(X, A)$ be $n$-dimensional oriented bordism group of $(X, A)$. There is a natural homomorphism $\mu : \Omega_n(X, A) \to H_n(X, A; \mathbb{Z})$. Given $[B^n, f] \in \Omega_n(X, A)$, let $\sigma_n \in H_n(B^n, \partial B^n; \mathbb{Z})$ denote the fundamental homology class of $B^n$. Then $\mu$ is defined $\mu([B^n, f]) = f_\# (\sigma_n) \in H_n(X, A; \mathbb{Z})$. The image of $\mu$ is the subgroup of integral homology classes representable in the sense of Steenrod. $\mu$ has following properties which are proved by Conner-Floyd.

**Theorem 2.** (Conner-Floyd) ([2], (7. 2))

The edge homomorphism $\Omega_n(X, A) = J_{n, 0} \to E_{n, 0} \to E_{n, 0}^2 = H_n(X, A; \mathbb{Z})$ of the bordism spectral sequence coincides with the homomorphism $\mu : \Omega_n(X, A) \to H_n(X, A; \mathbb{Z})$.

**Theorem 3.** (Conner-Floyd) ([2], (15. 1))

If $(X, A)$ is a CW pair then the bordism spectral sequence is trivial if and only if $\mu : \Omega_n(X, A) \to H_n(X, A; \mathbb{Z})$ is an epimorphism for all $n \geq 0$.

**Theorem 4.** (Conner-Floyd) ([2], (15. 2))

If $(X, A)$ is a CW pair such that each $H_n(X, A; \mathbb{Z})$ is finitely generated and has no odd torsion, then the bordism spectral sequence is trivial.

Next theorem is useful to obtain the manifold with $\mathbb{Z}_p$ action.

**Theorem 5.** (Conner-Floyd) ([2], (46. 1))

Consider the generating set $\alpha_{2k-1}; k = 1, 2, \ldots$ for $\Omega_*(\mathbb{Z}_p)$, $p$ an odd prime, where $\alpha_{2k-1} = [T, S^{2k-1}]$. Then there exist closed oriented manifolds $M^{4k}$, $k = 1, 2, \ldots$, such that for each $k$, $p \alpha_{2k-1} + [M^4] \alpha_{2k-5} + [M^8] \alpha_{2k-9} + \cdots = 0$ in $\Omega_*(\mathbb{Z}_p)$.

§ 4. Proof of Theorem 1.

Case (a): $p = 2$.

This case follows immediately from Theorems 3 and 4. Because each
$H_n(B(Z_4 \times Z_2); Z)$ is finitely generated and has no odd torsion from (1, 3).

REMARK. $e_0 \otimes e_0$, $e_{2i-1} \otimes e_0$, $e_0 \otimes e_{2j-1}$ and $e_{2i-1} \otimes e_{2j-1}$ are explicitly represented by $RP^0 \times RP^0$, $RP^{2i-1} \times RP^0$, $RP^0 \times RP^{2j-1}$ and $RP^{2i-1} \times RP^{2j-1}$ respectively. $e_{2i-1} \otimes e_{2n-2i} + e_{2i} \otimes e_{2n-1-2i}$ is represented by $H_{2i, 2n-2i}$ which is the subset in $RP^{2i} \times RP^{2n-2i}$ defined by the equation

$$x_0 y_0 + x_1 y_1 + \cdots + x_m y_m = 0,$$

where $m = \min(2i, 2n-2i)$, and $(x_0, \cdots, x_{2i})$ and $(y_0, \cdots, y_{2n-2i})$ are the standard homogeneous coordinates in $RP^{2i}$ and $RP^{2n-2i}$ respectively. It is a smooth submanifold of codimension 1, and orientable because its first Stiefel-Whitney class $w_1 = 0$. Consider the intersection of $H_{2i, 2n-2i}$ and 1 cycles of $RP^{2i} \times RP^{2n-2i}$ we can see that $i_\ast: H_{2n-1}(H_{2i, 2n-2i}; Z) \rightarrow H_{2n-1}(RP^{2i} \times RP^{2n-2i}; Z)$ is non-trivial, that is onto.

Case (b): $p$ an odd prime.

By Theorem 5 there exists compact orientable $2n$ dimensional manifold $V^{2n}$ with $\partial V^{2n} = \partial S^{2n-1} \cup M^4 \times S^{2n-5} \cup M^8 \times S^{2n-9} \cup \cdots$ and an action of $Z_p$ restricted to $M^{4k} \times S^{2n-4k-1}$ is $id \times T$. We can chose following classifying maps from the property of classifying space:

$$f_{2n}: V^{2n}/Z_p \rightarrow Y = B(Z_p)$$

such that $f_{2n}(V^{2n}/Z_p) \subseteq Y^{2n}$, $f_{2n}(M^{4k} \times S^{2n-4k-1}/Z_p) \subseteq Y^{2n-4k-1}$

and $f_{2n\ast}(\sigma_{2n}) = e_{2n}$, where $\sigma_{2n}$ is fundamental homology class of $V^{2n}/Z_p$.

Let $f_{0}: V^{0}/Z_p \rightarrow Y^0$ and let $f_{2n-1}: S^{2n-1}/Z_p \rightarrow Y^{2n-1}$ be inclusion, then $f_{2n-1\ast}(\sigma'_{2n-1}) = e_{2n-1}$, where $\sigma'_{2n-1}$ is fundamental class of $S^{2n-1}/Z_p$.

Next let $G = Z_p \times Z_p$ and choose classifying maps

$$g_j: S^{2j-1} \times S^{2n-2j+1}/G \rightarrow X^{2n},$$

$$h_j: V^{2j} \times S^{2n-2j-1}/G \rightarrow X^{2n-1},$$

$$k_j: S^{2j-1} \times V^{2n-2j}/G \rightarrow X^{2n-1},$$

$$l_j: V^{2j} \times V^{2n-2j}/G \rightarrow X^{2n}$$

such that $h_j(M^{4k} \times S^{2j-4k-1} \times S^{2n-2j-1}/G) \subseteq X^{2n-4k-2}$,

$k_j(M^{4k} \times S^{2j-1} \times S^{2n-2j-4k-1}/G) \subseteq X^{2n-4k-2}$,

and $l_j\left(\{M^{4k} \times V^{2j} \times S^{2n-2j-4k-1}/G\} \cup \{M^{4k} \times S^{2j-4k-1} \times V^{2n-2j}/G\}\right) \subseteq X^{2n-4k-1}$.

Then each fundamental class is mapped onto $e_{2j-1} \otimes e_{2n-2j-1}$, $e_{2j} \otimes e_{2n-2j-1}$, $e_{2j-1} \otimes e_{2n-2j}$ and $e_{2j} \otimes e_{2n-2j}$ by $g_{j\ast}$, $h_{j\ast}$, $k_{j\ast}$ and $l_{j\ast}$ respectively.
A remark on the Steenrod representation of $B(Z_p \times Z_p)$

Let
\[
\alpha_j = [g_j, S^{2j-1} \times S^{2n-2j+1}/G], \quad j = 1, \ldots, n,
\]
\[
\delta_j = [l_j, V^{2j} \times V^{2n-2j}/G], \quad j = 0, \ldots, n,
\]
\[
\beta_j^{2n-1} = [h_j, V^{2j} \times S^{2n-2j-1}/G], \quad j = 0, \ldots, n-1,
\]
\[
\gamma_j^{n-1} = [k_j, S^{2j} \times V^{2n-2j}/G], \quad j = 1, \ldots, n.
\]

Then $\alpha_j$ and $\delta_j$ generate $\Omega_*(X^{2n}, X^{2n-1})$ freely over $\Omega_*$, and $\beta_j^{2n-1}$ and $\gamma_j^{n-1}$ generate $\Omega_*(X^{2n-1}, X^{2n-2})$ freely over $\Omega_*$, because $\mu : \Omega_*(X^r, X^{r-1}) \rightarrow H_*(X^r, X^{r-1}; \Omega_*)$ is an $\Omega_*$ isomorphism.

**Lemma.**

$C^2$-term of bordism spectral sequence of $X = B(Z_p \times Z_p)$ is generated over $\Omega_*$ by $\delta_0$, $\alpha_j$, $\beta_j^{2n-1}$, $\gamma_j^{n-1}$, and $\langle \beta_j^{2n-1} + \gamma_j^{n-1} \rangle$, $(n = 1, 2, \ldots; i = 1, \ldots, n; j = 1, \ldots, n-1)$ and $B^2$-term is generated over $\Omega_*$ by $p\alpha_j$, $p\beta_j^{2n-1}$, $p\gamma_j^{n-1}$ and $p(\beta_j^{2n-1} + \gamma_j^{n-1})$, $(n = 1, 2, \ldots; i = 1, \ldots, n; j = 1, \ldots, n-1)$.

**Proof.**

\[
C_n^2 = \text{Ker}(\partial : \Omega_*(X^r, X^{r-1}) \rightarrow \Omega_*(X^{r-1}, X^{r-2}))
\]
\[
= \mu^{-1}( \text{Ker}\ \partial : H_*(X^r, X^{r-1}; \Omega_*) \rightarrow H_*(X^{r-1}, X^{r-2}; \Omega_*))
\]
and
\[
\partial \mu(\delta_j) = p e_{2j-1} \otimes e_{2n-2j} + p e_{2j} \otimes e_{2n-2j-1},
\]
\[
\partial \mu(\alpha_j) = 0,
\]
and
\[
\partial \mu(\gamma_j^{n-1}) = -p e_{2j-1} \otimes e_{2n-2j-1} \quad \text{therefore $C^2$-term follows.}
\]
\[
B_2 = \text{Im}(\partial : \Omega_*(X^{r-1}, X^r) \rightarrow \Omega_*(X^r, X^{r-1}))
\]
\[
= \mu^{-1}( \text{Im}(\partial : H_*(X^{r-1}, X^r; \Omega_*) \rightarrow H_*(X^r, X^{r-1}; \Omega_*))
\]
and
\[
\mu^{-1}(e_{2j} \otimes e_{2n-2j}) = p \alpha_j, \quad \mu^{-1}(e_{2j} \otimes e_{2n-2j+2}) = -p \alpha_j, \quad \mu^{-1}(e_{2j} \otimes e_{2n-2j-2}) = p \gamma_j^{n-1},
\]
and
\[
\mu^{-1}(e_0 \otimes e_n) = p \alpha_1^n, \quad \mu^{-1}(e_0 \otimes e_0) = p \gamma_n^{n-1} \quad \text{and} \quad \mu^{-1}(e_{2j-1} \otimes e_{2n+1-2j}) = 0,
\]
so we have $B^2$-term.

Next theorem essentially is the same as the case $p=3$ proved by Burdick [1].

**Theorem 6.** The bordism spectral sequence of $X = B(Z_p \times Z_p)$ is as follows:

\[
E^2 \cong \cdots \cong E^5, \quad E^6 \cong \cdots \cong E^\infty
\]

$E^\infty$ is generated by $\delta_0$, $\alpha_j$, $\beta_j^{2n-1}$, $\gamma_j^{n-1}$ $(n = 1, 2, \ldots; i = 1, 2, \ldots, n)$, \{$(\beta_j^{2n-1} + \gamma_j^{n-1})$ \}
and \{$(\beta_j^{2n-1} + \gamma_j^{n-1} + \cdots) \}$ with relations

$[M^q][\alpha_i - \alpha_{i-2}] = 0$, and every element except $\delta_0$ has order $p$.

**Proof.**

Every elements of $H_*(X; Z)$ have order $p$ (odd prime). $\Omega_n$ is free group
if \( n \equiv 0 \pmod{4} \) and 2-torsion groups if \( n \not\equiv 0 \pmod{4} \).

\[
E^2_{m,n} = H_m(X; \Omega_n) = H_m(X; \mathbb{Z}) \otimes \Omega_n + H_{m-1}(X; \mathbb{Z}) \ast \Omega_n.
\]

Therefore we have \( d^2 = d^3 = d^4 = 0 \), so \( E^2 \simeq \cdots \simeq E^5 \). Now recall the definition of \( d^r_{m,n} : \)

\[
\begin{align*}
\Omega_{m+n}(X^{m-1}, X^{m-r}) & \xrightarrow{i_*} \Omega_{m+n}(X^m, X^{m-r}) \\
& \quad \downarrow \partial_2
\end{align*}
\]

\[
\begin{align*}
\Omega_{m+n}(X^m, X^{m-r}) & \xrightarrow{j_*} \Omega_{m+n}(X^m, X^{m-1}) \\
& \quad \downarrow \partial_1
\end{align*}
\]

where \( i, j, i', j' \) are inclusion maps and \( \partial_1, \partial_2, \partial_3 \) are boundary homomorphisms of triple, then there exist homomorphism \( \psi \) such that \( \psi = \partial_1 \cdot j_* = i'_* \cdot \partial_2 \), every triangles are commutative.

Let \( C_{r,n} = \text{Im} j_* \), \( C_{r+1,n} = \text{Im} j'_* \), \( B_{m-r,n+r-1} = \text{Im} \partial_2 \) and \( B_{m-r,n+r-1} = \text{Im} \partial_3 \). Then the definition of \( d^r_{m,n} \) is composition of \( d^r_{m,n} : E^r_{m,n} = C^r_{m,n}/B^r_{m,n} \rightarrow C^r_{m,n}/C^r_{m+1,n} \xrightarrow{\partial_1} \)

\[
\text{Im} \psi \xrightarrow{i_*^{-1}} B^r_{m-r,n+r-1}/B^r_{m-r,n+r-1} \rightarrow C^r_{m-r,n+r-1}/B^r_{m-r,n+r-1} = E^r_{m-r,n+r-1}.
\]

Here let \( r = 5 \) then \( d^5(\partial_0) = d^5(\alpha_2^n) = d^5(\beta_2^n) = d^5(\gamma_2^n) = 0 \). Because \( \partial_0, \alpha_2^n, \beta_2^n, \gamma_2^n \) are represented by closed manifolds \( \partial_1 \) will kill them.

For \( j = 1, \ldots, n - 1 \) let \( N^n_{2j-1} \) be the manifold obtained from \( V^{2j} \times S^{2n-2j-1} \cup S^{2j-1} \times V^{2n-2j} \) by joining \( pS^{2j-1} \times S^{2n-2j-1} \) in \( \partial(V^{2j} \times S^{2n-2j-1}) \) to \( -pS^{2j-1} \times S^{2n-2j-1} \) in \( \partial(S^{2j-1} \times V^{2n-2j}) \).

Then \( \partial N^n_{2j-1} = M^4 \times S^{2j-5} \times S^{2n-2j-1} \cup M^4 \times S^{2j-1} \times S^{2n-2j-5} \)

\[
\cup M^8 \times S^{2j-9} \times S^{2n-2j-1} \cup M^8 \times S^{2j-1} \times S^{2n-2j-9} \cup \cdots
\]

There is an induced action of \( G = \mathbb{Z}_p \times \mathbb{Z}_p \) on \( N^n_{2j-1} \). Choose classifying maps \( \phi_j : N^n_{2j-1}/G \rightarrow X^{2n-1} \) such that \( \phi_j(\partial(N^n_{2j-1}/G)) \subseteq X^{2n-6} \) and such that

\[
\begin{align*}
N^n_{2j-1}/G & \xrightarrow{\phi_j} X^{2n-1} \\
& \quad \downarrow h_j \cup k_j
\end{align*}
\]

\[
(V^{2j} \times S^{2n-2j-1}/G) \cup (S^{2j-1} \times V^{2n-2j}/G)
\]

commutes up to homotopy.

Then \( \phi_j(\sigma) = e_{2j} \otimes e_{2n-2j-1} + e_{2j-1} \otimes e_{2n-2j} \), where \( \sigma \) is a fundamental class of
A remark on the Steenrod representation of $B(Z_p \times Z_p)$

$N^{2n-1}_j/G$. Thus $[\phi_j, N^{2n-1}_j/G] = \beta^{2n-1}_j + \gamma^{2n-1}_j$ in $\Omega_* (X^{2n-1}, X^{2n-2})$.

By the definition of $d^5$, $d^5(\beta^{2n-1}_j + \gamma^{2n-1}_j) = [M'] [\alpha^{2n-6}_j - \alpha^{2n-6}_3]$. Therefore $\text{Ker } d^5$ is generated by $\alpha^{2n}_0, \alpha^{2n}_1, \beta^{2n-1}_n, \gamma^{2n-1}_n$ and $(\beta^{2n-1}_1 + \gamma^{2n-1}_1), (\beta^{2n-1}_3 + \gamma^{2n-1}_3) + \ldots$ and $(\beta^{2n-1}_4 + \gamma^{2n-1}_4 + \ldots)$.

Let $K^{-1}$ be the identification manifold obtained from $V^2 \times S^{2n-3} \cup S^1 \times V^{2n-2} \cup V^6 \times S^{2n-7} \cup S^5 \times V^{2n-6} \cup V^{10} \times S^{2n-11} \cup \ldots$

by identifying pair-wise of boundary components of this manifold.

Then $K^{-1}$ is an orientable closed $(2n-1)$-manifold with induced natural action of $G = Z_p \times Z_p$.

Let $\Psi_1 : K^{-1}/G \rightarrow X^{2n-1}$ be a classifying map, then $[\Psi_1, K^{-1}/G] = (\beta^{2n-1}_1 + \gamma^{2n-1}_1) + (\beta^{2n-1}_3 + \gamma^{2n-1}_3) + \ldots$ in $\Omega_*(X^{2n-1}, X^{2n-2})$. Likewise construct $K^{-1}$ from $V^4 \times S^{2n-5} \cup S^3 \times V^{2n-4} \cup V^8 \times S^{2n-9} \cup S^7 \times V^{2n-8} \cup \ldots$

and $\Psi_2 : K^{-1}/G \rightarrow X^{2n-1}$ with $[\Psi_2, K^{-1}/G] = (\beta^{2n-1}_1 + \gamma^{2n-1}_1) + (\beta^{2n-1}_4 + \gamma^{2n-1}_4) + \ldots$

Therefore every generator of $E^6$ can be represented by a closed manifolds, so $d^8 = d^7 = \ldots = 0$ and hence $E^6 \cong \cdots \cong E^{\infty}$.

**Proof of Theorem 1.**

The classes listed in Theorem 6 really belong to $E_*, 0$. Therefore from Theorems 2 and 6 $e_0 \otimes e_0$, $e_0 \otimes e_{2j-1}$, $e_{2j-1} \otimes e_0$ and $e_{2j-1} \otimes e_{2j-1}$ are represented by $V^0 \times V^0/G$, $V^0 \times S^{2j-1}/G$, $S^{2n-1} \times V^0/G$ and $V^{2n-1} \times V^{2j-1}/G$ respectively.

$$(e_0 \otimes e_{2j-1} + e_0 \otimes e_{2j}) + (e_0 \otimes e_{2j-1} + e_{2j} \otimes e_0) + \ldots$$

and

$$(e_0 \otimes e_{2j-2} + e_0 \otimes e_{2j-3}) + (e_0 \otimes e_{2j-7} + e_1 \otimes e_{2j-6}) + \ldots$$

are represented by $K^{-1}_{2j+1}/G$ and $K^{-1}_{2j+1}/G$ respectively.

The proof of Theorem 1 is completed.

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**References**


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