On perturbation of closed operators in a Banach space

Didicated to Professor Yoshie Katsurada on her 60th birthday

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Introduction.

This paper deals with the perturbation problems of closed linear operators A and B in a Banach space. We give in §1 an elementary criterion in order that $A+B$ be again closed (See Theorem 1.2). We apply this result to perturbation problems of two accretive operators A and B and obtain a criterion in order that $-(A+B)$ generate a strongly continuous semi-group of contraction operators (See Theorem 2.7). However, we should note that the essence of our criterion was discovered by Trotter. In §3 we develop a Hilbert space theory and obtain several sufficient conditions, covering Nelson's condition (See Theorems 3.7 and 3.10).

§1. A criterion for closedness of $A+B$.

Consider two linear operators A and B in a Banach space X. We define a third operator $A+B$ by

(1.1) $$(A+B)x = Ax + Bx \quad \text{for} \quad x \in \mathbf{D}(A+B) = \mathbf{D}(A) \cap \mathbf{D}(B).$$ [1]

We exclude from our consideration the trivial case $\mathbf{D}(A) \cap \mathbf{D}(B) = 0$.

Now we pose the following

PROBLEM 1.1. Assume that A and B be closed. When is $A+B$ also closed?

The following result is our partial answer to this problem.

THEOREM 1.2. *Assume that the resolvent set of A, $\mathbf{P}(A)$, be non-empty and that there be a $\lambda \in \mathbf{C}$ such that $\lambda + B$ is of closed range and invertible. Let $-\mu \in \mathbf{P}(A)$. Then the following two conditions are equivalent*:

(1.2) $$A+B \text{ is closed and } -\mu \in \mathbf{P}(A+B);$$

(1.3) $$-1 \in \mathbf{P}\left(B(\mu+A)^{-1}\right).$$

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1) $\mathbf{D}(T)$ stands for the definition domain of a linear operator T in X. We shall denote by $\mathbf{R}(T)$ the range of T, and by $\mathbf{N}(T)$ the null-space of T.

Here $B(\mu+A)^{-1}$ *is a closed operator with* $\mathbf{D}(B(\mu+A)^{-1})=\{x\in X;\ (\mu+A)^{-1}x \in \mathbf{D}(B)\}$.

REMARK 1.3. This theorem is quite formal, but it enables us to reduce the Problem 1.1 to a '*better*' behaving operator $B(\mu+A)^{-1}$. In later sections, we apply it to perturbation problems of (say) accretive operators.

In the rest of this section, we are going to prove Theorem 1.2. For later convenience, the proof is devided into several propositions. First, we verify that $B(\mu+A)^{-1}$ is closed so that the statement (1.3) has a meaning. Namely, we prove

PROPOSITION 1.4. *Under the assumptions of Theorem* 1.2, $B(\mu+A)^{-1}$ *is a closed operator.*

PROOF. Let $x_n\in\mathbf{D}(B(\mu+A)^{-1})$ be such that $x_n\to x$ and $B(\mu+A)^{-1}x_n\to y$ for some x and y in X. Let λ be as in Theorem 1.2. Then, since $(\mu+A)^{-1}$ is bounded, $(\lambda+B)(\mu+A)^{-1}x_n\to\lambda(\mu+A)^{-1}x+y$. Since $\mathbf{R}(\lambda+B)$[1] is closed, there is a $z\in\mathbf{D}(B)$ such that $\lambda(\mu+A)^{-1}x+y=(\lambda+B)z$. On the other hand, $\lambda+B$ being invertible, $(\lambda+B)^{-1}$: $R(\lambda+B)\to X$ is bounded by the closed graph theorem. It follows that $(\mu+A)^{-1}x_n\to z$, whence $(\mu+A)^{-1}x=z$. In other words, $x\in\mathbf{D}(B(\mu+A)^{-1})$ and $(\lambda+B)(\mu+A)^{-1}x=(\lambda+B)z=\lambda(\mu+A)^{-1}x+y$. Thus, $B(\mu+A)^{-1}x=y$, and the proposition is proved. Q.E.D.

PROPOSITION 1.5. $\mathbf{D}(B(\mu+A)^{-1})=(\mu+A)(\mathbf{D}(A)\cap\mathbf{D}(B))$, $-\mu\in\mathbf{P}(A)$.

PROOF. Let $x\in\mathbf{D}(B(\mu+A)^{-1})$. Then $(\mu+A)^{-1}x\in\mathbf{D}(A)\cap\mathbf{D}(B)$. That is, $x\in(\mu+A)(\mathbf{D}(A)\cap\mathbf{D}(B))$. Conversely, let $x\in(\mu+A)(\mathbf{D}(A)\cap\mathbf{D}(B))$. Then $(\mu+A)^{-1}x\in\mathbf{D}(B)\cap\mathbf{D}(A)$. That is, $x\in\mathbf{D}(B(\mu+A)^{-1})$. Q.E.D.

PROPOSITION 1.6. $\mathbf{R}(\mu+A+B)=\mathbf{R}(I+B(\mu+A)^{-1})$, $-\mu\in\mathbf{P}(A)$.

PROOF. Let $z\in\mathbf{R}(\mu+A+B)$. Then there is a $x\in\mathbf{D}(A)\cap\mathbf{D}(B)$ such that $z=\mu x+Ax+Bx$. Let $y=\mu x+Ax$. Then $y\in\mathbf{D}(B(\mu+A)^{-1})$ and $z=y+B(\mu+A)^{-1}y$. This shows $\mathbf{R}(\mu+A+B)\subset\mathbf{R}(I+B(\mu+A)^{-1})$. Conversely, let $z\in\mathbf{R}(I+B(\mu+A)^{-1})$. Then there is a $y\in\mathbf{D}(B(\mu+A)^{-1})$ such that $z=y+B(\mu+A)^{-1}y$. Let $x=(\mu+A)^{-1}y$. Then $x\in\mathbf{D}(A)\cap\mathbf{D}(B)$ and $z=\mu x+Ax+Bx\in\mathbf{R}(\mu+A+B)$. This shows $\mathbf{R}(I+B(\mu+A)^{-1})\subset\mathbf{R}(\mu+A+B)$. Q.E.D.

PROPOSITION 1.7. *Let* $-\mu\in\mathbf{P}(A)$. $\mu+A+B$ *is invertible if and only if* $I+B(\mu+A)^{-1}$ *is invertible.*

PROOF. Assume that $\mu+A+B$ be invertible. Let $y\in\mathbf{D}(B(\mu+A)^{-1})$ be such that $y+B(\mu+A)^{-1}y=0$. Set $x=(\mu+A)^{-1}y$. Then $x\in\mathbf{D}(A)\cap\mathbf{D}(B)$ and $\mu x+Ax+Bx=0$. It follows that $x=0$ and so $y=0$. Now suppose that $I+B(\mu+A)^{-1}$ be invertible. Let $x\in\mathbf{D}(A)\cap\mathbf{D}(B)$ be such that $\mu x+Ax+Bx=0$. Set $y=(\mu+A)x$. Then $y\in\mathbf{D}(B(\mu+A)^{-1})$ and $y+B(\mu+A)^{-1}y=0$. Thus, $y=0$ and so $x=0$. Q.E.D.

REMARK 1.8. In Propositions 1.5–1.7, our arguments relied only on algebraic properties. In other words, we did not employ in an essential way the topological requirements on A and B such as closedness of their graphs.

PROPOSITION 1.9. *If* $-1 \in \mathbf{P}(B(\mu+A)^{-1})$, then $A+B$ is closed.

PROOF. Let $x_n \in \mathbf{D}(A) \cap \mathbf{D}(B)$ be such that $x_n \to x$ and $z_n = \mu x_n + A x_n + B x_n \to z$ for some x and z in X. Let $y_n = (\mu+A)x_n$. Then $y_n \in \mathbf{D}(B(\mu+A)^{-1})$ and $z_n = \{I+B(\mu+A)^{-1}\}(\mu+A)x_n = \{I+B(\mu+A)^{-1}\}y_n$. Since $\{I+B(\mu+A)^{-1}\}^{-1}$ is bounded, $y_n \to y = \{I+B(\mu+A)^{-1}\}^{-1}z$. We see thus $y \in \mathbf{D}(B(\mu+A)^{-1})$. We also have that $x_n \to x$ and $y_n = (\mu+A)x_n \to y$. Since A is closed, it follows that $x \in \mathbf{D}(A)$ and $y = (\mu+A)x$. Hence, $x = (\mu+A)^{-1}y \in \mathbf{D}(A) \cap \mathbf{D}(B)$, and $(\mu+A+B)x = \{I+B(\mu+A)^{-1}\}y = \{I+B(\mu+A)^{-1}\}\{I+B(\mu+A)^{-1}\}^{-1}z = z$. Thus, closedness of $A+B$ is proved. Q.E.D.

Now we proceed to a

PROOF of Theorem 1.2.

(i) (1.2) *implies* (1.3). $\mu+A+B$ is invertible and $\mathbf{R}(\mu+A+B) = X$. Thus, by Propositions 1.6 and 1.7, $I+B(\mu+A)^{-1}$ is invertible and $\mathbf{R}(I+B(\mu+A)^{-1}) = X$. Then, by Proposition 1.4, $I+B(\mu+A)^{-1}$ is closed and so is $\{I+B(\mu+A)^{-1}\}^{-1}$. Hence, by the closed graph theorem, $\{I+B(\mu+A)^{-1}\}^{-1}$ is bounded and $-1 \in \mathbf{P}(B(\mu+A)^{-1})$.

(ii) (1.3) *implies* (1.2). $I+B(\mu+A)^{-1}$ is invertible and $\mathbf{R}(I+B(\mu+A)^{-1}) = X$. Then, by Propositions 1.6 and 1.7, $\mu+A+B$ is invertible and $\mathbf{R}(\mu+A+B) = X$. Then, by Proposition 1.9, $\mu+A+B$ is closed, and so is $(\mu+A+B)^{-1}$. Using again the closed graph theorem, $(\mu+A+B)^{-1}$ is bounded and $-\mu \in \mathbf{P}(A+B)$. Thus the theorem is proved. Q.E.D.

REMARK 1.10. In the statements of Theorem 1.2, the statement

(1.2) is equivalent to the following:

(1.2′) $\mu+A+B$ is invertible and $\mathbf{R}(\mu+A+B) = X$.

In fact, (1.2′) implies (1.3) and so (1.2).

REMARK 1.11. The requirement that X be a Banach space is not essential as can easily be seen from our proof of Theorem 1.2. For any topological linear space where the closed graph theorem is valid, Theorem 1.2 would hold good.

§ 2. Applications to perturbation problems of accretive operators.

We begin by introducing a semi-inner product in a Banach space X in order to define *accretivity* (or *dissipativity*) of operators in X. Then we state the well-known characterization of the infinitesimal generators of strongly continuous semi-groups of contraction operators in terms of thus defined accretivity of operators. Using these terminologies and Theorem 1.2, we give a criterion for perturbations of accretive operators. We then apply our criterion to particular cases to re-discover or to obtain more practical sufficient condition.

PROPOSITION 2.1. (Lumer [4]). *There is a mapping* $[\ ,\ ]$:

(2.1) $$X\times X\ni(x, y)\longrightarrow[x, y]\in\mathbf{C}$$

with the following properties:

(2.2) $$[x+y, z]=[x, z]+[y, z] \quad \textit{for any } x, y, z\in X;$$

(2.3) $$[\lambda x, y]=\lambda[x, y] \quad \textit{for any } \lambda\in\mathbf{C} \textit{ and } x, y\in X;$$

(2.4) $$|[x, y]|\leqq\|x\|\|y\| \quad \textit{for any } x, y\in X$$

and

(2.5) $$[x, x]=\|x\|^{2}$$

$[x, y]$ *is called a semi-inner product of* x *and* y (*compatible with the norm of* X).

PROOF. See [4]. [5] or [10]. Q.E.D.

REMARK 2.2. If X is a Hilbert space, there is a unique semi-inner product compatible with the norm of X. This unique semi-inner product is nothing but the inner product of X.

In the rest of this section we only consider a fixed semi-inner product compatible with the norm of X unless the contrary is explicitly stated.

DEFINITION 2.3. (Lumer-Phillips [5]). A linear operator T in X is said to be accretive if

(2.6) $$\mathrm{Re}[Tx, x]\geqq 0 \quad \text{for all } x\in\mathbf{D}(T).$$

T is called dissipative if $-T$ is accretive.

PROPOSITION 2.4. (Lumer-Phillips [5]). *If* T *is a densely defined accretive operator, then* T *is closable. The closure* $T^{\sim}$ *of* T *is accretive in a semi-inner product* (*possibly different from the original but compatible with the norm of* X).

PROOF. See Lumer-Phillips [5]. p. 693–694. Q.E.D.

Note that for a closed accretive operator T, $\lambda+T$ is invertible and of closed range for any $\mathrm{Re}\,\lambda>0$.

The following result is well-known.

PROPOSITION 2.5. (Lumer-Phillips [5]). *Let* T *be a densely defined* *closed operator in X. Then the following three conditions are equivalent*:

(2. 7) $-T$ *generates a strongly continuous semi-group of contraction operators in* X;

(2. 8) T *is accretive and* $\mu \in \mathbf{P}(-T)$ *for some* μ, $\mathrm{Re}\,\mu>0$;

(2. 9) T *is accretive and* $\mu \in \mathbf{P}(-T)$ *for any* μ, $\mathrm{Re}\,\mu>0$.

Now we state our problem of this section.

PROBLEM 2. 6. Assume that A and B be two closed operator in X with $\mathbf{D}(A) \cap \mathbf{D}(B)$ dense in X. Assume furthermore that $-A$ generate a strongly continuous semi-group of contraction operators in X. Then the question is: When does $-(A+B)$ generate a strongly continuous semi-group of contraction operators in X?

The following is our answer to this problem, by means of Theorem 1. 2.

THEOREM 2. 7. *Let* $-A$ *be the infinitesimal generator of a strongly continuous semi-group of contraction operators in* X. *Let* B *be a closed operator in* X *such that* $\mathbf{D}(A) \cap \mathbf{D}(B)$ *is dense in* X *and that* $A+B$ *is accretive. Then the following three conditions are mutually equivalent*:

(2. 10) $A+B$ *is closed and* $-(A+B)$ *generates a strongly continuous semi-group of contraction operators in* X;

(2. 11) $$-1 \in \mathbf{P}\left(B(\mu+A)^{-1}\right) \quad \textit{for some} \quad \mu, \mathrm{Re}\,\mu>0;$$

(2. 12) $$-1 \in \mathbf{P}\left(B(\mu+A)^{-1}\right) \quad \textit{for any} \quad \mu, \mathrm{Re}\,\mu>0.$$

REMARK 2. 8. Close results to this are known (e. g., Trotter [9]. See our Introduction). However, it seems that a result in this form has never been stated explicitly. Nevertheless, an exact statement like this may permit us a new scope.

PROOF of Theorem 2. 7. Proposition 2. 5 shows the equivalence of (2. 10) with each of the following two conditions:

(2. 13) $$-\mu \in \mathbf{P}(A+B) \quad \text{for some} \quad \mu, \mathrm{Re}\,\mu>0;$$

(2. 14) $$-\mu \in \mathbf{P}(A+B) \quad \text{for any} \quad \mu, \mathrm{Re}\,\mu>0.$$

By Theorem 1. 2, (2. 13) is equivalent to (2. 11) and (2. 14) to (2. 12). Q. E. D.

We state some supplements to our Theorems 2. 7 and 1. 2. This is a slight generalization of Trotter's lemma [9]. (cf. Kato [2]).

COROLLARY 2. 9. *Let* A *and* B *be two closed operators in* X *such that* $\mathbf{D}(A) \cap \mathbf{D}(B)$ *is dense in* X. *Assume that* $-A$ *generate a strongly continuous semi-group of contraction operators in* X *and that* $A+B$ *be accre-*

tive. Then the following two conditions are equivalent.

(2.15) *The closure $-(A+B)^{\sim}$ of $-(A+B)$ generates a strongly continuous semi-group of contraction operators in X;*

(2.16) $$-1\in \mathbf{P}\left(B(\mu+A)^{-1}\right)\cup \Sigma_C\left(B(\mu+A)^{-1}\right) \qquad \textit{for some } \mu, \mathrm{Re}\,\mu>0 .$$

Here $\Sigma_C(B(\mu+A)^{-1})$ denotesthe set $\{\lambda\in \mathbf{C}; \lambda-B(\mu+A)^{-1}$ has a densely defined inverse, closed but not bounded$\}$.

PROOF. We only need to discuss the case when $\mathbf{R}(\mu+A+B)$ is dense in X but not identical with X. This happens if and only if $-1\in \Sigma_C(B(\mu+A)^{-1})$ by virtue of Propositions 1.6 and 1.7. Q.E.D.

In the rest of this section, we confine ourselves to the case $\mathbf{D}(A)\subset\mathbf{D}(B)$. First we prepare some terminologies.

DEFINITION 2.10. (Kato [2]). Let A and B be two linear operators with $\mathbf{D}(A)\subset\mathbf{D}(B)$. B is said to be relatively bounded with respect to A (in short, A-bounded) if there are positive constants a and b such that

(2.17) $$\|Bx\|\leqq a\|x\|+b\|Ax\| \qquad \text{for all } x\in\mathbf{D}(A).$$

If X is a Hilbert space, (2.17) is equivalent to:

(2.18) $$\|Bx\|^2\leqq a_1^2\|x\|^2+b_1^2\|Ax\|^2 \qquad \text{for all } x\in\mathbf{D}(A)$$

with some positive constants a_1 and b_1.

DEFINITION 2.11. (Kato [2]). The lower bound b_0 of all possible b in (2.17) is called the A-bound of B. If X is a Hilbert space, b_0 coincides with the lower bound of all possible b_1 in (2.18).

If both A and B are closed, then $\mathbf{D}(A)\subset\mathbf{D}(B)$ is equevalent to that B is A-bounded. This can be seen by the closed graph theorem. Furthermore, if $\mathbf{P}(A)\neq\emptyset$, then $B(\mu+A)^{-1}$, $-\mu\in\mathbf{P}(A)$, is a bounded operator. Then the relation between the A-bound of B and the norm of $B(\mu+A)^{-1}$ is given by the following

PROPOSITION 2.12. *Assume that A and B be closed with $\mathbf{D}(A)\subset\mathbf{D}(B)$. Let b_0 be the A-bound of B. If $-A$ generates a strongly continuous semi-group of contraction operators in X, then we have*

(2.19) $$b_0\leqq \liminf \|B(\lambda+A)^{-1}\|\leqq \limsup \|B(\lambda+A)^{-1}\|\leqq 2b_0 .$$

In particular, if X is a Hilbert space, then

(2.20) $$b_0=\lim \|B(\lambda+A)^{-1}\| .$$

In (2.19) and(2.20) the limits are taken for $\lambda\to\infty$.

The proof is straightforward and so omitted.

Using this proposition, we prove the following slight generalizations of Nelson's and Yosida's results as corollaries to Theorem 2.7.

COROLLARY 2.13. (Nelson [6]). *Assume that A and B be two closed operators in X with $\mathbf{D}(A)\subset\mathbf{D}(B)$ such that $A+B$ is accretive. If $-A$ generates a strongly continuous semi-group of contraction operators in X and if the A-bound b_0 of B is less than 1/2, then $A+B$ is closed and $-(A+B)$ generates a strongly continuous semi-group of contraction operators. If X is a Hilbert space, the conclusion is true when $b_0<1$.*

PROOF. Since $b_0<1/2$, there is a $\lambda>0$ such that $\|B(\lambda+A)^{-1}\|<1$. This is true when $b_0<1$ if X is a Hilbert space. Thus $-1\in\mathbf{P}(B(\lambda+A)^{-1})$ and the corollary follows from Theorem 2.7. Q.E.D.

COROLLARY 2.14. (Yosida [11]). *Assume that A and B be two closed operators in X with $\mathbf{D}(A)\subset\mathbf{D}(B)$ such that $A+B$ is accretive. If $-A$ generates a strongly continuous semi-group of contraction operators and if $\mathbf{D}(A^\alpha)\subset\mathbf{D}(B)$ for some α, $0<\alpha<1$, then $A+B$ is closed and $-(A+B)$ generates a strongly continuous semi-group of contraction operators. In the above, A^α denotes the fractional power of A (See for example Yosida* [10] *or Komatsu* [3]).

PROOF. For any $\lambda>0$, $\|B(\lambda+A)^{-1}\|=\|B(1+A)^{-\alpha}(1+A)^{\alpha}(\lambda+A)^{-1}\|$. By the assumption, $\|B(1+A)^{-\alpha}\|<\infty$. On the other hand, $\|(1+A)(\lambda+A)^{-1}\|\leqq \|\{(1+A)(\lambda+A)^{-1}\}^{\alpha}\|\,\|(\lambda+A)^{\alpha-1}\|\leqq M\lambda^{\alpha-1}$ since $(1+A)(\lambda+A)^{-1}$ is uniformly bounded for $\lambda>1$. Hence, $\lim\|B(\lambda+A)^{-1}\|=0$ $(\lambda\to\infty)$ and the corollary reduces to Corollary 2.13. Q.E.D.

For the sake of completeness, we state Gustafson's result.

COROLLARY 2.15. (Gustafson [1]). *Assume that A and B be two closed accretive operators in X with $\mathbf{D}(A)\subset\mathbf{D}(B)$. If $-A$ generates a strongly continuous semi-group of contraction operators in X and if the A-bound b_0 of B is less than* 1, *then $A+B$ is closed and $-(A+B)$ generates a strongly continuous semi-group of contraction operators.*

PROOF. See Gustafson [1] p. 337. Q.E.D.

Now we introduce another definition.

DEFINITION 2.16. (Kato [2]). Let A and B be two linear operators in X with $\mathbf{D}(A)\subset\mathbf{D}(B)$. B is said to be relatively compact with respect to A (in short, A-compact) if for any sequence $\{x_n\}$ in $\mathbf{D}(A)$ with both $\{\|x_n\|\}$ and $\{\|Ax_n\|\}$ bounded, $\{Bx_n\}$ contains a convergent subsequence.

A-compactness of B implies A-boundedness of B. In particular, if both

A and B be closed and $\mathbf{P}(A)\neq\emptyset$, then A-compactness of B is equivalent to compactness of $B(\mu+A)^{-1}$ for $-\mu\in\mathbf{P}(A)$.

The following is another corollary to Theorem 2.7.

COROLLARY 2.17. *Assume that A and B be two closed operators in X with $\mathbf{D}(A\subset\mathbf{D}(B)$ such that $A+B$ is accretive. If $-A$ generates a strongly continuous semi-group of contraction operators and if B is A-compact, then $A+B$ is closed and $-(A+B)$ generates a strongly continuous semi-group of contraction operators. Furthermore, $A+B$ has compact resolvents if and only if A has compact resolvents.*

PROOF. Since $B(\mu+A)^{-1}$, $\operatorname{Re}\mu>0$, is compact, it suffices to show that $I+B(\mu+A)^{-1}$ is invertible. By Proposition 1.7, we only need to check that $\mu+A+B$, $\operatorname{Re}\mu>0$, is invertible. Let $x\in\mathbf{D}(A)$ be such that $\mu x+Ax+Bx=0$. Then, by accretivity of $A+B$, $0=[\mu x+Ax+Bx, x]\geqq\operatorname{Re}\mu\|x\|^2$.
It follows that $x=0$, and hence, $\mu+A+B$ is invertible. Now note $\mu+A+B$ $=(I+B(\mu+A)^{-1})(\mu+A)$. Since both sides are boundedly invertible, we have $(\mu+A+B)^{-1}=(I+B(\mu+A)^{-1})^{-1}(\mu+A)^{-1}$ and
$(\mu+A)^{-1}=(\mu+A+B)^{-1}(I+B(\mu+A)^{-1})$. If $\dim X<\infty$ the last statement is trivial. If $\dim X=\infty$, then $(I+B(\mu+A)^{-1})^{-1}$ is not compact. Hence the corollary is proved. Q.E.D.

REMARK 2.18. Our proof of Corollary 2.13 is essentially the same as Nelson's original. This is an example that Theorem 2.7 has implicitly been employed. Our proof of Corollary 2.14 is a little different from Yosida's original proof. Corollary 2.17 follows also from Kato's stability theorem. (Kato [2]).

§3. Further discussions in the case when X is a Hilbert space.

If X is a Hilbert space, things are much simplified. For example any semi-inner product compatible with the norm of X is the inner product $\langle,\rangle$ of X. Thus the notion of accretivity becomes easier to handle. In this section, we intend to transcribe the results of the previous section, using the numerical range of an operator.

DEFINITION 3.1. Let T be a linear operator in a Hilbert space X. The set

$$\mathbf{W}(T)=\{\langle Tx, x\rangle;\ x\in\mathbf{D}(T),\ \|x\|=1\} \tag{3.1}$$

is called the numerical range of T.

We enumerate some properties of the numerical range (See, e.g., Kato [2]).

PROPOSITION 3. 2. (Hausdorff-Stone. See Stone [8]) $\mathbf{W}(T)$ *is a convex set in the complex plane.*

Thus if we denoteby $\Gamma=\Gamma(T)$ the closure of $\mathbf{W}(T)$, then $\mathbf{C}-\Gamma$ has at most two components. In the following proposition, $\mathrm{nul}\, S=\dim \mathbf{N}(S)$, and $\mathrm{def}\, S=\dim (X/\mathbf{R}(S))$, for an operator S. nul S is called the nullity of S, and def S the deficiency of S.

PROPOSITION 3. 4. (Kato [2]). *Let T be a closed operator. Then, for $\mu\notin\Gamma(T)$, $T-\mu$ has closed range,* $\mathrm{nul}\,(T-\mu)=0$ *and* $\mathrm{def}\,(T-\mu)=$ *const. in each component of* $\mathbf{C}-\Gamma(T)$. *If* $\mathrm{def}\,(T-\mu)=0$, *then* $\mu\in\mathbf{P}(T)$ *and* $\|(\mu-T)^{-1}\|\leqq 1/\mathrm{dist}\,(\mu,\Gamma)$. *In particular, if T is bounded, then* $\mathbf{C}-\Gamma(T)\subset\mathbf{P}(T)$.

As we have mentioned at the beginning of this section the notion of accretivity in a Hilbert space is easier to handle. The following statements give a summary to this point.

DIFINITION 3. 4. A linear operator T is said to be accretive if

$$\mathrm{Re}\,\langle Tx, x\rangle\geqq 0 \quad \text{for} \quad x\in\mathbf{D}(T)\,. \tag{3.2}$$

T is said to be m-accretive if T is closed, accretive and $\mathbf{P}(-T)$ contains all μ, $\mathrm{Re}\,\mu>0$.

PROPOSITION 3. 5. (Kato [2]). *An m-accretive operator is maximal in the sense that T does not permit any proper accretive extension of T. T is necessarily densely defined. In particular, T is m-accretive if and only if $-T$ generates a strongly continuous semi-group of contraction operators in X.*

In the rest of this section we confine ourselves to the case $\mathbf{D}(A)\subset\mathbf{D}(B)$. Before stating our first result, we need to make a supplementrary remark to Corollary 2. 9 in the case when X is a Hilbert space.

PROPOSITION 3. 6. *Let A and B be two closed operators with* $\mathbf{D}(A)\subset\mathbf{D}(B)$. *Assume that A be m-accretive and $A+B$ be accretive. Then the following three conditions are mutually equivalent.*

(3. 3) *The closure* $(A+B)^{\sim}$ *of* $A+B$ *is not m-accretive*;

(3. 4) -1 *is an eigenvalue of* $(B(\mu+A)^{-1})^{*}$ *for some* μ, $\mathrm{Re}\,\mu>0$;

(3. 5) -1 *is an eigenvalue of* $(B(\mu+A)^{-1})^{*}$ *for any* μ, $\mathrm{Re}\,\mu>0$.

PROOF. (3. 3) is equivalent to that $\mathbf{R}(\mu+A+B)$ is non-dense in X for some μ, $\mathrm{Re}\,\mu>0$. And if $\mathbf{R}(\mu+A+B)$ is non-dense in X for some μ, $\mathrm{Re}\,\mu>0$, then $\mathbf{R}(\mu+A+B)$ is not dense in X for any μ, $\mathrm{Re}\,\mu>0$. On the other hand, if $\mathbf{R}(\mu+A+B)$ is non-dense in X, then so is $\mathbf{R}(I+B(\mu+A)^{-1})$, by virtue of Proposition 1. 6. -1 is an eigenvalue of $(B(\mu+A)^{-1})^{*}$ if and only if $\mathbf{R}(I+B(\mu+A)^{-1})$ is not dense in X. Thus the proposition is proved. Q.E.D.

We have a sufficient condition in order that the closure $(A+B)^{\sim}$ be m-accretive. Namely,

THEOREM 3.7. *Let A and B be two closed operators with $\mathbf{D}(A)\subset\mathbf{D}(B)$. Assume that A be m-accretive and $A+B$ be accretive. If, for some μ, $\mathrm{Re}\,\mu>0$, -1 is not an interior point of $\mathbf{W}(B(\mu+A)^{-1})$, then $(A+B)^{\sim}$ is m-accretive. In particular, if -1 is not in the closure of $\mathbf{W}(B(\mu+A)^{-1})$, then $A+B$ is m-accretive.*

PROOF. We are going to prove that if $(A+B)^{\sim}$ is not m-accretive, then for any μ, $\mathrm{Re}\,\mu>0$, -1 is an interior point of $\mathbf{W}(B(\mu+A)^{-1})$. Since $(A+B)^{\sim}$ is not m-accretive, -1 an eigenvalue of $(B(\lambda+A)^{-1})^{*}$ for any λ, $\mathrm{Re}\,\lambda>0$. Hence there is an eigenvector $y(\lambda)\in X$ such that $\|y(\lambda)\|=1$ and

$$(B(\lambda+A)^{-1})^{*}y(\lambda)+y(\lambda)=0\,. \tag{3.6}$$

On the other hand, by the resolvent equation,

$$B(\lambda+A)^{-1}-B(\mu+A)^{-1}=(\mu-\lambda)B(\lambda+A)^{-1}(\mu+A)^{-1}$$

for any λ, μ, $\mathrm{Re}\,\lambda>0$, $\mathrm{Re}\,\mu>0$. Taking the adjoints, we have

$$(B(\lambda+A)^{-1})^{*}-(B(\mu+A)^{-1})^{*}=(\bar{\mu}-\bar{\lambda})(\bar{\mu}+A^{*})^{-1}(B(\lambda+A)^{-1})^{*}\,. \tag{3.7}$$

Here note that A^{*} is also m-accretive. Now applying (3.7) to $y(\lambda)$, and using (3.6), we have

$$y(\lambda)+(B(\mu+A)^{-1})^{*}y(\lambda)=(\bar{\mu}-\bar{\lambda})(\bar{\mu}+A^{*})^{-1}y(\lambda)\,. \tag{3.8}$$

Take the inner product of (3.8) with $y(\lambda)$. Then

$$1+\langle B(\mu+A)^{-1}y(\lambda),\ y(\lambda)\rangle=(\mu-\lambda)\langle(\mu+A)^{-1}y(\lambda),\ y(\lambda)\rangle\,. \tag{3.9}$$

Now choose

$$\lambda=\mu+\nu,\ \nu=re^{i\theta},\ 0<r<|\mu|,\ -\pi\leqq\theta\leqq\pi\,. \tag{3.10}$$

Then $e^{-i\theta}(1+\langle B(\mu+A)^{-1}y(\lambda),\ y(\lambda)\rangle)=-r\langle(\mu+A)^{-1}y(\lambda),\ y(\lambda)\rangle$. It follows that, for any θ, $-\pi\leqq\theta\leqq\pi$, there is a λ such that

$$\mathrm{Re}\,e^{-i\theta}\big(1+\langle B(\mu+A)^{-1}y(\lambda),\ y(\lambda)\rangle\big)<0\,. \tag{3.11}$$

In fact, set $z=(\mu+A)^{-1}y(\lambda)$. Then $z\neq 0$, and

$$\begin{aligned}\mathrm{Re}\langle(\mu+A)^{-1}y(\lambda),\ y(\lambda)\rangle&=\mathrm{Re}\langle z,\ (\mu+A)z\rangle\\&=(\mathrm{Re}\,\mu)\|z\|^{2}+\mathrm{Re}\,\langle z,\ Az\rangle\\&\geqq(\mathrm{Re}\,\mu)\|z\|^{2}>0\,.\end{aligned}$$

Since $\mathbf{W}(B(\mu+A)^{-1})$ is convex, and $-1\in\mathbf{W}(B(\mu+A)^{-1})$ (take $\lambda=\mu$ in (3.9)), (3.11) means that -1 is an interior point of $\mathbf{W}(B(\mu+A)^{-1})$. In fact, this is due to the following Lemma 3.8. Thus the theorem is proved. Q.E.D.

LEMMA 3.8. *Let K be a convex set in the complex plane. Assume that $0\in K$. If, for each θ, $-\pi\leqq\theta\leqq\pi$, there is a $w\in K$ such that* $\mathrm{Re}\{e^{i\theta}w\}<0$, *then* 0 *is an interior point of K.*

PROOF. If 0 were not an interior point, then there would be a $\zeta\in\mathbf{C}$ such that $K\subset\{z\in\mathbf{C}\}$; $\mathrm{Re}\langle z,\zeta\rangle\geqq 0\}$. Q.E.D.

REMARK 3.9. The above theorem shows in particular that if we have

$$\|Bx\|^2\leqq a\|x\|^2+\|Ax\|^2,\quad a>0,\quad x\in\mathbf{D}(A), \tag{3.10}$$

then $(A+B)^{\sim}$ is m-accretive. (see Okazawa [7]). In fact in this case, -1 cannot be an interior point of $\mathbf{W}(B(\mu+A)^{-1})$, $\mu^2=a$.

Our next result is the following. The criterion given below contains the Kato-Nelson criterion, as can be easily seen.

THEOREM 3.10. *Let A be m-accretive and B closed accretive with $\mathbf{D}(A)\subset\mathbf{D}(B)$. Then we always have, for some real a and a non-negative b,*

$$\mathrm{Re}\langle Ax, Bx\rangle\geqq a\|x\|^2-b\|Ax\|^2. \tag{3.11}$$

If the estimate (3.11) *holds with $0\leqq b<1$, then $A+B$ is m-accretive. If* (3.11) *holds with $b=1$, then $(A+B)^{\sim}$ is m-accretive.*

PROOF. We devide the proof into two parts, $0<b<1$, and $b=1$.

FIRST CASE: $0\leqq b<1$. Since B is accretive, (3.11) implies

$$\mathrm{Re}\langle(\lambda+A)x, Bx\rangle\geqq a\|x\|^2-b\|Ax\|^2\qquad\text{for }\lambda>0.$$

Thus

$$\begin{aligned}\mathrm{Re}\langle B(\lambda+A)^{-1}y, y\rangle&\geqq a\|(\lambda+A)^{-1}y\|^2-b\|A(\lambda+A)^{-1}y\|^2\\&\geqq-(|a|/\lambda^2+b\|A(\lambda+A)^{-1}\|^2)\|y\|^2.\end{aligned}$$

Since $b\|A(\lambda+A)^{-1}\|^2<1$, we have, for sufficiently large λ, $\mathrm{Re}\langle B(\lambda+A)^{-1}y, y\rangle>-\|y\|^2$. Thus -1 is not in the closure of $\mathbf{W}(B(\lambda+A)^{-1})$. Hence $A+B$ is m-accretive.

SECOND CASE: $b=1$. We see by the previous case that $A+B/2$ is m-accretive. Then, by (3.11), taking $A+B/2$ and $B/2$ instead of A and B in (2.18), respectively, we see that (2.18) holds with $b_1=1$. It follows from Remark 3.9 that $(A+B)^{\sim}=(A+B/2+B/2)^{\sim}$ is m-accretive. Q.E.D.

COROLLARY 3.11. *Let A and B be as in Theorem* 3.10. *Assume that, for some θ, $-\pi/2<\theta<\pi/2$, we have*

(3. 12) $\quad \mathrm{Re}\, e^{i\theta}\langle Ax, Bx\rangle \geqq a\|x\|^2 - b\|Ax\|^2$, *where* $a \in \mathbf{R}$ *and* $b>0$.

If (3. 16) *holds with* $0 \leqq b < \cos^2\theta$, *then* $A+B$ *is m-accretive. If* (3. 16) *holds with* $a=0$ *and* $b=\cos^2\theta$, *then* $(A+B)^{\sim}$ *is m-accretive.*

PROOF. This can be shown as in the first case of the proof of Theorem 3. 10. Q.E.D.

REMARK 3. 12. It seems that Re $\langle Ax, Bx\rangle$ was first considered by Okazawa [7]. He treated the case $a=0$ and $b=0$ in (3. 11). The requirement $a=0$ and $b=0$ seems to be too particular. However, Mr. Yoshio Konishi has called to our attention that our Theorem 3. 10 can be proved by means of Okazawa's result. We note furthermore that in a condition of the form stated in Theorem 3. 10, we can get rid of the assumption $\mathbf{D}(A) \subset \mathbf{D}(B)$ if we assume, for instance, that $\mathbf{R}(\zeta\mu + \zeta A + B) = X$ for some ζ, Re $\zeta < 0$, and for some μ, Re μ sufficiently large >0.

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