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Doctoral Thesis

A Study on Noise-Based Global Asymptotic Stabilization and Optimization Method

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Abstract

Noise degrades the performance of systems in most cases. However, noise can be used to improve the performance compared to the case of the absence of noise. This thesis studies the noise-based methods for the global asymptotic stabilization and the optimization problem in control theory. For the asymptotic stabilization problem, this study establishes the method for designing feedback controllers using Wiener processes. For the optimization problem, this study proposes an extremum seeking method that guarantees the convergence of estimation variables to optimum values.

Although the global asymptotic stabilization problem is the one of the fundamental problems in the literature of control theory, there exist systems that cannot be stabilized by any smooth time-invariant feedback controllers. This study employs a method using stochastic feedback controllers to stabilize such systems. When the stochastic feedback controllers are used to stabilize deterministic nonlinear systems, the closed-loop systems are often modeled as Stratonovich stochastic differential equations. In the stabilization method using a stochastic feedback controller, the constructive method for designing controllers for general nonlinear affine systems has not been established when closed-loop systems are given by Stratonovich stochastic differential equations. This thesis proposes a constructive design method based on stochastic control Lyapunov functions.

For the optimization problem, this study considers a stochastic extremum seeking method. In extremum seeking methods, dither signals are added to given systems to approximate the gradient of objective functions, and the optimum is estimated by updating the estimation variable based on the approximated gradient. In previous extremum seeking methods, although the estimation variables approach the optimum sufficiently, the estimation variables do not converge to the optimum. This thesis shows a stochastic extremum seeking method that can guarantee the convergence of the estimation variables to the optimum by introducing the updating mechanism of the estimation variables based on the stochastic Lyapunov stability theory.

Chapter 1 states the backgrounds and the objectives of this thesis, and Chapter 2 introduces the mathematical preliminaries, which includes the fundamentals of stochastic process, manifolds.

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Chapter 3 shows the noise-based stabilization method and the method for designing stochastic feedback controllers. This chapter first shows the problem setting of the global asymptotic stabilization and the design of the controller. Then, we define a stochastic control Lyapunov function for the design of stochastic controllers. The design method is shown based on the stochastic control Lyapunov function. Further, this chapter gives the proof that the designed controllers by the proposed method globally asymptotically stabilize given systems. Moreover, the numerical examples show the global asymptotic stabilization of a nonholonomic system and non-Euclidean systems. In addition, since the designed controller can be seen as an extension of the Sontag-type controller, the designed controllers satisfy inverse optimality. By the inverse optimality, the controllers have a stability margin.

Chapter 4 considers homogeneous stochastic systems and discusses their stability, which can be applied to improve the convergence of the stabilization by the noise-based stabilization. This chapter first explains the homogeneity, and then gives the definition of homogeneous stochastic systems as an extension of homogeneous deterministic systems. Then, the author shows the relation between the homogeneity and the convergence speed of stable homogeneous stochastic systems. Further, a homogeneous feedback controller is shown to preserve the homogeneity of systems and to guarantee the convergence speed of the closed-loop systems. Finally, this chapter also shows the redesign method of the controllers designed by the method described in Chapter 3 to improve the convergence speed in the stabilization of driftless systems.

Chapter 5 shows a stochastic extremum seeking method that can guarantee the convergence of estimation variables to an optimum value. After showing the objective of the stochastic extremum seeking method and the problem setting of the optimization problem, the proposed method is shown, which uses the Wiener process to approximate the gradients of objective functions. The proposed method uses a high-pass filter with a state-dependent parameter obtained from the stochastic Lyapunov stability analysis. Also, this chapter gives the proof of the convergence of the estimation variables by the stochastic Lyapunov theory.

Chapter 6 states the conclusion of this thesis.

Keywords: Stochastic systems, Global asymptotic stabilization, optimization, Noise-based stabilization

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Chapter 1. Introduction

Generally speaking, noise is regarded as disturbance in the most cases. On the other hand, unexpectedly, noise can cause beneficial effects on systems. We call methods using such beneficial effects of noise as noise-based methods in this thesis. The one of such methods in engineering is a method using stochastic resonance [49]. Roughly speaking, stochastic resonance is a phenomenon where a nonlinear system shows better performance by the existence of stochastic noise. The stochastic resonance, especially in the area of signal processing, has attracted attention because small analogue noise is digitally encoded with noise better than the case of the absence of noise. From the perspective of the effect of randomness, we can consider algorithms using noise, such as the simulated annealing and the genetic algorithm, as the noise-based method.

In this thesis, we study noised-based methods for the global asymptotic stabilization and the optimization problem. In the stabilization problem, there exist nonlinear dynamical systems such that when we stabilize an equilibrium by using a state feedback controller, there might exist equilibria that are obstructions for stabilizing the desired equilibrium. To solve this, we deal with noise-based stabilization, which introduces noise into a continuous feedback controller [54, 55]. The noise can have effects for removing the equilibria and the feedback controller can asymptotically stabilize the desired equilibrium. Also, we study an optimization method called extremum seeking method [2, 53, 67], which has received attention in the literature of control engineering. The extremum seeking method estimates the optimal solution by approximating the gradient of an objective function by using dither signals. In recent years, extremum seeking methods using stochastic signal for the dither signal have been studied for the case of many optimization parameters, which are called as stochastic extremum seeking methods [44, 40]. In this thesis, we consider a stochastic extremum seeking method that guarantees the convergence of optimization parameters. In the following, we state backgrounds, motivations, and outlines of both the global asymptotic stabilization method and the optimization method.

1.1 Global Asymptotic Stabilization Method: Noise-Based Stabilization

Many dynamical systems are modeled by differential equations. When a dynamical system has inputs, the subject is to steer the state of the dynamical system from a given value to a desired value. In the control theory, we consider how we give inputs to steer the state of dynamical systems depending on purposes. There are many purposes of control, such as stabilization, servo control, optimal control and many other controls.

In such problems, the asymptotic stabilization problem is the one of the most fundamental problems in control theory. The asymptotic stabilization is to give an input to steer given initial state to an equilibrium of a system. When we consider the stabilization for any initial state, the problem is called global asymptotic stabilization. For linear systems, the stabilization has been extensively studied, and the stabilization method has been established by many researchers. Linear systems are stabilized by continuous state feedback laws, and the feedback controller is designed by well-established methods such as the pole assignment. Further, asymptotic stability in linear systems also means global asymptotic stability.

For nonlinear systems, there exist systems that cannot be stabilized by any continuous state feedback laws. The fact is shown by Brockett [13] and Sontag [64]. Brockett [13] shows a necessary condition for asymptotic stabilization of general nonlinear systems. The systems not satisfying the condition include nonholonomic systems. The well-known examples of such nonholonomic systems are the models of vehicles and unicycles and many other models [12]. Also, Sontag [64] shows that the systems whose state space is noncontractible cannot be asymptotically stabilized by any continuous feedback laws. Systems whose state includes angular displacement can be modeled as noncontractible systems.

To stabilize the systems that cannot be stabilized by any continuous time-invariant feedback laws, such as nonholonomic systems and non-Euclidean systems, discontinuous feedback laws and time-varying feedback laws have been studied [16, 17, 57, 51]. In the literature of discontinuous feedback laws, the stability analysis is conducted by the theory of discontinuous dynamical systems [16, 17]. Although the discontinuous feedback methods are powerful for the stabilization, they sometimes need the effort in the analysis or synthesis. Also, time-varying feedback laws are studied in [57, 51]. The time-varying feedback methods do not have a constructive design method of stabilizing feedback laws, except for the design method of time-varying controllers for driftless systems [57], to the best knowledge of the author.

In recent years, Nishimura et al. have proposed global asymptotic stabilization method using continuous feedback laws with Wiener processes [54, 55]. This method can be applied to the stabilization of general deterministic nonlinear affine systems, including nonholonomic systems and non-Euclidean systems. Roughly speaking, the noise has effect to remove equilibria, except for a desired equilibrium, which might appear when continuous time-invariant feedback control laws are used. The method using Wiener processes can be considered as an extension of the time-varying feedback control methods. In the time-varying feedback control methods, a periodic function is employed as a time-varying component. On the other hand, Gaussian white noise is employed as a time-varying component, and the closed-loop system is modeled by a stochastic differential equation. Since this method uses noise, we call this method a noise-based method.

For this method, the constructive design method of controllers has been proposed when closed-loop systems are model as Itô stochastic differential equations [54]. Some studies have shown that deterministic nonlinear dynamical systems driven by white noise are modeled as Stratonovich stochastic differential equations in general [33, 3]. Then, in [55], a design method of a control law for nonholonomic systems is shown in the case of Stratonovich stochastic differential equations. For general nonlinear affine systems, a constructive design method has not been proposed for the case where the closed-loop systems are modeled as Stratonovich stochastic differential equations.

This thesis studies a constructive design method for systems given by Stratonovich stochastic differential equation based on the results in [54, 55]. The design method of noise-based controller is based on the notion of a stochastic control Lyapunov function, which is a common policy of the stabilization of nonlinear systems in the literature of nonlinear control theory. This thesis proposes a definition of stochastic control Lyapunov functions fitted with the noise-based stabilization method. Based on the stochastic control Lyapunov function, a constructive method is given to design a stabilizing controller, which can be seen as an extension of Sontag-type controller [63]. A main result shows that if there exists a stochastic control Lyapunov function for a given system, a designed controller globally asymptotically stabilizes the origin of the system with probability one. The proposed design method can be used to stabilize nonholonomic systems and non-Euclidean systems, and numerical examples of the stabilization of such systems are shown.

Besides the design method, this thesis shows the robustness of the designed controllers, and considers stochastic Lyapunov stability theory on manifold for non-Euclidean systems. This thesis shows a stability margin of the designed controllers based on their inverse optimality. In addition, this thesis considers stochastic Lyapunov stability theory on manifold because, to the author's best knowledge, there does not exist the study on the global asymptotic stability in probability for systems on a manifold.

Moreover, this thesis considers the method for the improvement of the convergence speed of the noise-based method for driftless systems. In the noise-based stabilization method, the state of a closed-loop system sometimes shows the slow convergence. To avoid such problem, this thesis shows the redesign method of the noise-based controller for driftless systems. The redesign method uses the homogeneity as seen in the redesign method of time-varying feedback laws [50]. To develop the method, we need to consider the stochastic homogeneous systems and their stability. Thus, this thesis also shows the results of the stochastic homogeneous systems.

The noise-based stabilization methods have been studied before Nishimura et al. [54, 55]. First, the stabilization of linear systems by using noise was studied by Khasminskii [37]. Arnold [5, 4] showed a condition of linear systems that can be stabilized by noise. The stabilization of nonlinear systems was first studied by Mao [45]. In [45], the sufficient condition of nonlinear systems for noise-based stabilization was shown. After the study of Mao, Appleby et al. [1] generalized Mao's results by relaxing the restrictions on control systems. The noise-based method of Nishimura et al. and that in this thesis can be seen as a generalization of these noise-based stabilization methods.

1.2 Optimization Method: Stochastic Extremum Seeking

In engineering, the one of the most common issues is designing or operating systems to obtain the best performance under a given performance measure. The problem is called optimization problem, which we often encounter in the field of engineering. The performance measure is called an objective function in optimization. When a parameter and an objective function are continuous, the most common strategy to solve a optimization problem is to use a gradient of the objective function. However, in real problems, to obtain an exact model needs considerable effort. Thus, we often cannot obtain the gradient of the objective function. To address the optimization without exact information of models, many non-model based optimization methods have been studied, such as hill climbing method, and genetic algorithm.

Extremum seeking methods have been much studied in the literature of control engineering [2, 53, 67, 8, 58]. These methods solve the optimization problem in continuous-time systems. In addition, the methods can be applied to the real-time optimization of dynamical systems. Extremum seeking methods are able to solve optimization problems by using only the values of inputs and outputs. Thus, extremumm seeking is categorized as a non-model based algorithm. To be more precise, in order to solve an optimization problem, the method approximates the value of the gradient of an objective function by adding a dither signal to a given system.

In recent years, stochastic extremum seeking methods are studied by some researchers [44, 40, 65]. Although previous extremum seeking methods use periodic signals for dither signal, the estimation of the gradients might be difficult due to the interaction between the periodic signals when the number of parameters is large. Stochastic noise is introduced into the dither signal to avoid such problems. In addition, one of the features of stochastic extremum seeking is the possibility of the implementation by using noise existing in a given system. This feature can lead to simplifying the device implementing optimization.

This thesis considers an extension of stochastic extremum seeking methods such that the estimation variable of optimal solution converges to the optimum. In previous methods using periodic signal and stochastic noise, the residual of the estimation variable to the optimal value remains after sufficient time has elapsed due to the persisting effect of dither signal. Previous methods improve the convergence of the estimation variable by introducing a high-pass filter into the mechanism updating the estimation variable. However, the strict convergence of the estimation variable is not guaranteed even if high-pass filters are introduced in the existing literature. To address this problem, we introduce a high-pass filter with a state-dependent parameter determined from the stochastic Lyapunov stability analysis. It guarantees the convergence of the estimation parameters.

In addition, if an objective function is smooth and unimodal, the estimation variable convergences with probability one for any initial value. The proposed method can be applied for static systems, and the application to dynamical systems is included in future works.

1.3 Outline

This thesis is organized as follows. Following this section, Section 2 introduces the mathematical preliminaries, including the fundamentals of stochastic systems, the Lyapunov stability theory, and manifolds.

Chapter 3 shows the noise-based stabilization and the method for designing stochastic feedback controllers. We use the noise-based stabilization method to stabilize mainly nonholonomic systems and non-Euclidean systems. Since we aim to stabilize non-Euclidean systems, we need the stability analysis of stochastic systems on manifolds. Thus, first, this chapter discusses the stochastic systems on manifolds and the stabilization of the systems. Further, the stochastic Lyapunov stability theory on the manifolds is shown. Then, this chapter defines stochastic control Lyapunov function, and shows the design method of the stochastic controllers. This chapter gives the proof of the convergence of the closed-loop system with the designed controller. Numerical examples are shown for the stabilization of a nonholonomic system and non-Euclidean systems. Further, the inverse optimality and the stability margin of the designed controllers are shown.

Chapter 4 discusses stochastic homogeneous systems and their stability. The definition of homogeneous stochastic systems is introduced, and the relation between the homogeneity and the convergence speed of stable homogeneous systems are shown. Further, a homogeneous feedback controller is shown to stabilize homogeneous systems and to guarantee the homogeneity. The results on the stability is used in the improvement of the slow convergence in the noise-based stabilization. This chapter shows the redesign method of the noise-based controller for driftless systems.

Chapter 5 shows a stochastic extremum seeking method that guarantees the convergence of the estimation variables to the optimum. This chapter first shows the problems settings of the optimization problem. Then, we show tree scheme of the proposed stochastic extremum seeking method. We also give the proof of the convergence of the estimation variable to the optimum when we use the one of the proposed schemes.

Chapter 6 gives the conclusion of this thesis.

Chapter 2. Mathematical Preliminaries

2.1 General Notation and Definition

This section introduces the mathematical notation used in this thesis.

The symbol $\mathbb{R}$ denotes a Euclidean space. Thus, the symbol $\mathbb{R}^n$ denotes an n-dimensional Euclidean space. The superscript T denotes the transpose of a vector or matrix. The expression $\|\cdot\|$ denotes a 2-norm. Lastly, $L_F V$ denotes the Lie derivative of a function $V : M \to \mathbb{R}$ along a vector field F on M.

2.2 Stochastic Systems

This section introduces fundamentals of stochastic systems, such as stochastic process and stochastic differential equation. In this thesis, we consider the noise-based methods for a global asymptotic stabilization method, and an optimization method. In these methods, variables are under the effect of stochastic noise. Therefore, we focus on stochastic systems to analyze these methods. This section introduces the fundamentals of stochastic systems along the line of [37, 3, 39, 56].

First, a probability space is introduced as the basis of the analysis of stochastic systems.

Definition 2.1 (σ-algebra)
A family of subsets of a set Ω, $\mathcal{F}$, is called a σ-algebra if

1. $\Omega \in \mathcal{F}$,
2. $A \in \mathcal{F} \;\Rightarrow\; \Omega \setminus A \in \mathcal{F}$,
3. $A_1, A_2, \cdots \in \mathcal{F} \;\Rightarrow\; \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Definition 2.2 (probability space)
For a set Ω and a σ-algebra $\mathcal{F}$ on Ω, a function $\mathrm{P} : \mathcal{F} \to [0, 1]$ is said to be a probability if the conditions

1. $A_i \cap A_j = \emptyset \;\Rightarrow\; \mathrm{P}\left(\cap_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{P}(A_i)$ for $\forall\, i, j$,
2. $P(\emptyset) = 0$,
3. $P(\Omega) = 1$,

are satisfied. A tuple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Definition 2.3 (expectation)
For a probability space $(\Omega, \mathcal{F}, P)$, the expectation of a random variable $X(\omega)$ is defined to be an integral

$$\mathrm{E}\{X\} = \int_\Omega X(\omega)P(d\omega).$$

Then, we introduce the conditional expectation and conditional probability. Let us to define a indicator function.

Definition 2.4 (indicator function)
For a set A, a function $I_A : \Omega \to \{0, 1\}$ defined by

$$I_A(\omega) = \begin{cases} 1 & \text{for } \omega \in A, \\ 0 & \text{for } \omega \notin A, \end{cases} \tag{2.1}$$

is said to be a indicator function.

Definition 2.5 (conditional expectation and conditional probability)
For a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, consider a sub-$\sigma$-algebra $\mathcal{G}$. Let X be a $\mathcal{F}$-measurable. A $\mathcal{G}$-measurable random variable Y is said to be a conditional expectation of C on $\mathcal{G}$ if a equation

$$\int_C Y dP = \int_C X dP \text{ for all } C \in \mathcal{G}$$

are satisfied. The conditional expectation is denoted as

$$Y = E\{X|\mathcal{G}\}.$$

Furthermore, a conditional probability of A on $\mathcal{G}$ is defined to be

$$\mathrm{P}(A|\mathcal{G}) = \mathrm{E}\{I_A|\mathcal{G}\}.$$

Next, we consider a stochastic process, which is a set of random variables parameterized by a time variable. The definition is given in the following.

Definition 2.6 (stochastic process)
Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let I be a interval $(I \subset \mathbb{R})$. The function $X : I \times \Omega \to \mathbb{R}^d$ is said to be a stochastic process.

For a fixed ω, the stochastic process is a function of time, and $X(\cdot, \omega)$ is called a sample path. For each t, $X(t, \cdot)$ is a random variable.

Definition 2.7 (filtration)
For a set Ω and a σ-algebra $\mathcal{F}$ on Ω, a sequence of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ is said to be a filtration if for $s < t$, $\mathcal{F}_s \subset \mathcal{F}_t$.

Then, martingale is introduced, which is a basis of stochastic Lyapunov stability theory.

Definition 2.8 (martingale)
Let $X(t,\omega)$ be a stochastic process, and let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Assume that for any $t \in \mathbb{R}$, $X(t,\omega)$ is measurable on $\mathcal{F}_t$ and that $\mathrm{E}\{X(t,\omega)\}$ has finite-value. A tuple $(X(t,\omega),\mathcal{F}_t)$ is said to be a martingale if for any $s < t$, $\mathrm{E}\{X(t)|\mathcal{F}_s\} = X(s)$ holds almost surely. Further, $(X(t,\omega),\mathcal{F}_t)$ is said to be a supermartingale if for any $s < t$,

$$\mathrm{E}\{X(t)|\mathcal{F}_s\} \leq X(s) \text{ a.s.}$$

is satisfied. On the other hand, when $\mathrm{E}\{X(t)|\mathcal{F}_s\} \geq X(s)$ holds almost surely for any $s < t$, $(X(t,\omega),\mathcal{F}_t)$ is said to be a submartingale.

The well-known results on a supermartingle are introduced without proof.

Theorem 2.1 ([20])
Assume that $(X(t,\omega),\mathcal{F}_t, t \geq 0)$ is continuous a.s. supermartingale. Then, for any $\epsilon > 0$ and $p \geq 1$,

$$\mathrm{P}\left\{\sup_{t_0 \leq t \leq T} |X(t,\omega)| > \epsilon\right\} \leq \frac{\mathrm{E}\{|X(T,\omega)|^p\}}{\epsilon^p}$$

holds.

Theorem 2.2 ([20])
Assume that $(X(t,\omega),\mathcal{F}_t, t \geq 0)$ is supermartingale and that $X(t,\omega)$ has positive value. Then, there exists a limit $X_\infty = \lim_{t\to\infty} X(t,\omega)$ almost surely, and X_∞ is finite. Further,

$$\mathrm{E}\{X_\infty\} = \lim_{t\to\infty} \mathrm{E}\{X(t,\omega)\}$$

holds.

These results are the basis for the stochastic Lyapunov stability theory.

Then, the Wiener process is introduced.

Definition 2.9 (Wiener process)
A stochastic process $W(t,\omega)$ is said to be a Wiener process if it satisfies that

1. $W(t,\omega)$ has independent increments,
2. the increment $W(t) - W(s)$ has normal distribution, and

$$\mathrm{E}\{W(t) - W(s)\} = 0 \quad \mathrm{E}\{(W(t) - W(s))^2\} = \sigma^2|t-s|$$

   hold, where $\sigma > 0$.
3. $\mathrm{P}\{W(0) = 0\} = 1$

The Wiener process is martingale, and differentiable nowhere. Roughly speaking, for an increment of the Wiener process dW, and a white noise $\xi(t)$, we have $dW \approx \xi dt$.

When we consider systems driven by white noise, we use stochastic differential equations to model such systems. Stochastic differential equations have two definitions, Itô stochastic differential equations, and Stratonovich stochastic differential equations. In this thesis, the both two definitions are used, and the definitions are given as below.

To discuss stochastic differential equations, we present the definitions of stochastic integrals.

Definition 2.10 (Itô stochastic integral)
Let $W(t)$ be a standard Wiener process defined on a probabitlity space $(\Omega, \mathcal{F}, P)$. Consider a function $f : \mathfrak{B} \times \mathcal{F} \to \mathbb{R}$. Assume that $f(t, \omega)$ is $\mathfrak{B} \times \mathcal{F}$-measurable,

$$\mathrm{E}\left[\int_{t_0}^{t} f(t,\omega)^2 dt\right] < \infty$$

holds, and for almost all $\omega \in \Omega$, $t \mapsto f(t,\omega)$ is continuous. An Itô stochastic integral is defined as

$$\int_{t_0}^{T} f(t,\omega)dW(t) = \underset{\max_i(t_{i+1}-t_i)\to 0}{\mathrm{l.i.m.}} \sum_{i=0}^{N-1} f(t_i,\omega)[W(t_{i+1}) - W(t_i)],$$

where t_i are partitions of an interval $[t_0, T]$, and l.i.m. means limit in the mean.

Definition 2.11 (Stratonovich stochastic integral)
Let $W(t)$ be a standard Wiener process defined on a probabitlity space $(\Omega, \mathcal{F}, P)$. Consider a function $f : \mathfrak{B} \times \mathcal{F} \to \mathbb{R}$. Assume that $f(t, \omega)$ is $\mathfrak{B} \times \mathcal{F}$-measurable,

$$\mathrm{E}\left[\int_{t_0}^{t} f(t,\omega)^2 dt\right] < \infty$$

holds, and for almost all $\omega \in \Omega$, $t \mapsto f(t,\omega)$ is continuous. A Stratonovich stochastic integral is defined as

$$\int_{t_0}^{T} f(t,\omega) \circ dW(t) = \underset{\max_i(t_{i+1}-t_i)\to 0}{\mathrm{l.i.m.}} \sum_{i=0}^{N-1} f(\frac{t_{i+1}-t_i}{2},\omega)[W(t_{i+1}) - W(t_i)],$$

where t_i are partitions of an interval $[t_0, T]$.

Then, we present the definitions of stochastic differential equations. Denote the increments of a stochastic process $X(t)$ as $dX(t)$.

Definition 2.12
Consider vector fields $F : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and a m-dimensional standard Wiener process $W(t)$. A stochastic process is said to be a solution of an Itô stochastic differential equation

$$dX(t) = F(X(t))dt + G(X(t))dW(t),$$

if the stochastic process $X(t)$ satisfies

$$X(t) = X(t_0) + \int_{t_0}^{t} F(X(s))ds + \int_{t_0}^{t} G(X(s))dW(s).$$

Definition 2.13
Consider vector fields $F : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and a m-dimensional standard Wiener process $W(t)$. A stochastic process is said to be a solution of a Stratonovich stochastic differential equation

$$dX(t) = F(X(t))dt + G(X(t)) \circ dW(t),$$

if the stochastic process $X(t)$ satisfies

$$X(t) = X(t_0) + \int_{t_0}^{t} F(X(s))ds + \int_{t_0}^{t} G(X(s)) \circ dW(s).$$

A Stratonovich stochastic differential equation

$$dX(t) = F(X(t))dt + G(X(t)) \circ dW(t)$$

is converted into an equivalent Itô stochastic differential equation as

$$\begin{aligned} dX(t) &= F(X(t))dt + G(X(t)) \circ dW(t) \\ &= \left\{ F(X(t)) + \frac{1}{2}\frac{\partial G}{\partial x}(X(t))G(X(t)) \right\} dt + G(X(t))dW(t), \end{aligned}$$

(see [66]).

Theorem 2.3 ([56])
Let $X(t)$ be a solution of a stochastic differential equation given by

$$dX(t) = F(X(t))dt + G(X(t))dW(t).$$

Consider a function $W : \mathbb{R}^n \to \mathbb{R}$, and assume that $W(x)$ is twice continuously differentiable. Then, $Z(X(t))$ is a solution of a stochastic differential equation given by

$$\begin{aligned} dZ(t) = L_F Z(X(t))dt &+ \frac{1}{2}\mathrm{tr}\left\{ G(X(t))^T \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial x}\right)(X(t))G(X(t)) \right\} dt \\ &+ \frac{\partial Z}{\partial x}(X(t))G(X(t))dW(t). \end{aligned} \tag{2.2}$$

The infinitesimal generator is used in the stochastic Lyapunov stability theory.

Definition 2.14
Let $X(t)$ be n-dimensional stochastic process, and consider a function $Z : \mathbb{R}^n \to \mathbb{R}$. A infinitesimal generator of $X(t)$ is given by

$$\mathcal{L}Z(x) = \lim_{\delta\to 0} \frac{E\{Z(X(t+\delta))|X(t) = x\} - Z(x)}{\delta}.$$

Theorem 2.4 ([56])
Consider a stochastic process $X(t)$ which is a solution of a stochastic differential equation

$$dX(t) = F(X(t))dt + G(X(t))dW(t).$$

For a twice continuously differentiable function $Z : \mathbb{R}^n \to \mathbb{R}$, an infinitesimal generator of $X(t)$ is given by

$$\mathcal{L}Z(x) = L_F Z(x) + \frac{1}{2}\mathrm{tr}\left\{G(x)^T \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial x}\right)(x)G(x)\right\}. \tag{2.3}$$

By an infinitesimal generator $\mathcal{L}(\cdot)$, the equation (2.2) is rewritten as

$$dZ(t) = \mathcal{L}Z(X(t))dt + \frac{\partial Z}{\partial x}(X(t))G(X(t))dW(t).$$

We also introduce the following theorem, which is known as the Dynkin's formula.

Theorem 2.5 ([56])
Suppose that x_t is a right continuous strong Markov process and τ is a random time with $E\{\tau | X(0) = x\} < \infty$. Let $Z : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function $\mathcal{L}Z(x) = V(x)$. Then

$$\begin{aligned} E\{Z(X(\tau)) \mid X(0) = x\} - W(x) &= E\left\{\int_0^\tau V(X(s))ds \mid X(0) = x\right\} \\ &= E\left\{\int_0^\tau \mathcal{L}Z(X(s))ds \mid X(0) = x\right\} \end{aligned}$$

Finally, we shows the following relation between the solution of a stochastic differential equation and a transition probability density.

Theorem 2.6 ([56])
Consider an Itô stochastic differential equation

$$dX(t) = F(X(t))dt + G(X(t))dW(t), \tag{2.4}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and W is a m-dimensional standard Wiener process. The probability density of the solution of (2.4), $p(t, x)$ follows a partial differential equation

$$\frac{\partial p}{\partial t}(x,t) = -\sum_i \frac{\partial}{\partial x_i}\left(F_i(x)p(x,t)\right) + \frac{1}{2}\sum_{i,j,k} \frac{\partial^2}{\partial x_i x_j}\left(G_{ik}(x)G_{jk}(x)p(x,t)\right). \tag{2.5}$$

The partial differential equation is called Fokker-Planck equation.

2.3 Lyapunov stability theory

This section introduces stochastic Lyapunov stability theory. The Lyapunov theory is used for the stability analysis of dynamical systems without knowing any analytic solutions. Lyapunov studied the case of dynamical systems given by deterministic differential equations [41]. For stochastic systems, Khasminkii [37] and Kushner [39] contributed to the stochastic Lyapunov stability theory (See also [46, 19]). Since this thesis deals with the noise-based method, this section introduces the stochastic Lyapunov stability theory along the line of Khasminskii [37].

Before introducing the stochastic Lyapunov stability theory, we show the Lyapunov stability theory for deterministic dynamical systems (See [36, 64]).

First, Consider a deterministic dynamical system

$$\dot{x}(t) = F(x(t)), \quad x(0) = x_0 \tag{2.6}$$

where $x \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ and $F(0) = 0$.

Then, the stability of the equilibrium point $x = 0$ of (2.6) is defined as follows.

Definition 2.15 (stability)
The equilibrium point $x = 0$ of (2.6) is said to be stable if, for $\forall\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that if $\|x(0)\| < \delta$ then

$$\|x(t)\| < \epsilon, \quad \forall t \geq 0.$$

Definition 2.16 (asymptotic stability)
The equilibrium point $x = 0$ of (2.6) is said to be asymptotically stable if it is stable and δ can be chosen such that if $\|x(0)\| < \delta$ then

$$\lim_{t\to\infty} \|x(t)\| = 0. \tag{2.7}$$

Further, if the equation (2.7) holds for any $x(0) \in \mathbb{R}^n$, the equilibrium is said to be globally asymptotically stable.

To discuss Lyapunov theory, some technical terminology are introduced.

Definition 2.17 (positive definiteness)
A function $W : U \subset \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite when $W(0) = 0$ and $W(x) > 0$ for $\forall x \neq 0$.

Definition 2.18 (properness)
A function $W : U \subset \mathbb{R}^n \to \mathbb{R}$ is said to be proper if, for each small $c > 0$, the level sets $\{x \in U | W(x) \leq c\}$ are compact.

The Lyapunov theory of deterministic systems are shown as bellow.

Definition 2.19
A function $V : U \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a local Lyapunov function for (2.6) if V is differentiable, positive definite on U, proper, and it satisfies

$$L_F V(x) < 0 \text{ for } x \in U \setminus 0.$$

Definition 2.20
A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a global Lyapunov function for (2.6) if V is differentiable, positive definite on $\mathbb{R}^n$, proper, and if satisfies

$$L_F V(x) < 0 \text{ for } x \in \mathbb{R}^n \setminus 0.$$

Theorem 2.7 ([36])
If there exists a local (respectively, global) Lyapunov function for (2.6), then the equilibrium $x = 0$ of (2.6) is locally (respectively, globally) asymptotically stable.

In the following, the stochastic Lyapunov theory are introduced. Let us consider a stochastic system given by

$$dx(t) = F(x(t))dt + \Sigma(x(t))dw, \quad x(0) = x_0, \tag{2.8}$$

where $x \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\Sigma : \mathbb{R}^{m\times n} \to \mathbb{R}^n$ satisfying $F(0) = 0$ and $\Sigma(0) = 0$, and w is a m-dimensional standard Wiener process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

First, stability in probability is defined as follows.

Definition 2.21 (stability in probability)
The origin $x = 0$ of the system (2.8) is said to be stable in probability if for any $\epsilon > 0$,

$$\lim_{x_0\to 0} \mathrm{P}\left(\sup_{t\geq 0} \|x(t)\| > \epsilon\right) = 0 \tag{2.9}$$

holds.

Next, the asymptotic stability in probability and global asymptotic stability in probability are introduced.

Definition 2.22 (asymptotic stability in probability)
The origin of (2.8) is said to be asymptotically stable in probability if the equilibrium $x = 0$ is stable in probability, and

$$\lim_{x_0\to 0} \mathrm{P}\left(\lim_{t\to\infty} \|x(t)\| = 0\right) = 1. \tag{2.10}$$

Definition 2.23 (global asymptotic stability in probability)
The origin of (2.8) is said to be globally asymptotically stable in probability if the equilibrium $x = 0$ is stable in probability, and

$$\mathrm{P}\left(\lim_{t\to\infty} \|x(t)\| = 0\right) = 1, \tag{2.11}$$

holds for any $x_0 \in \mathbb{R}^n$.

Definition 2.24 (local stochastic Lyapunov function)
A function $W : U \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a stochastic Lyapunov function of the system (2.8) if $W(x)$ is C^2, positive definite, and proper, and $\mathcal{L}W(x)$ is negative definite.

Definition 2.25 (global stochastic Lyapunov function)
A function $W : \mathbb{R}^n \to \mathbb{R}$ is said to be a global stochastic Lyapunov function of the system (2.8) if $W(x)$ is C^2, positive definite, and proper, and $\mathcal{L}W(x)$ is negative definite.

In the stochastic Lyapunov stability theory, the main results are stated as follows.

Theorem 2.8 ([37])
If a local stochastic Lyapunov function of the system (2.8) exists, the origin is asymptotically stable in probability.

Theorem 2.9 ([37])
If a global stochastic Lyapunov function of the system (2.8) exists, the origin is globally asymptotically stable in probability.

The proof of Theorem 2.9 is given in Section 3.2, which includes the case of systems on manifolds.

2.4 Manifolds

This section introduces the notion of manifolds to address the stabilization problem of systems on manifolds in this thesis (See Chapter 3). Related notions, such as a topological space, and a Hausdorff space, are also introduced along the line of [48, 47].

First, the definition of a topological space is given as follows.

Definition 2.26 (topological space)
Let S be a non-empty set. Let $\mathfrak{O}$ be a collection of subsets of S. The tuple $(S, \mathfrak{O})$ is said to be a topological space if following conditions.

1. $S \in \mathfrak{O}$, and $\emptyset \in \mathfrak{O}$.
2. For $O_1,\ O_2 \in \mathfrak{O}$, $O_1 \cap O_2 \in \mathfrak{O}$.
3. For any finite collection of sets $(O_\lambda)_{\lambda\in\Lambda}$, $\cup_{\lambda\in\Lambda} O_\lambda \in \mathfrak{O}$.

Definition 2.27 (neighborhood)
A set in $\mathfrak{O}$ is said to be an open set. For $x \in S$, a subset $V \subset S$ is said to be a neighborhood of x if there exists an open set U such that

$$x \in U,\ U \subset V.$$

We denote the collection of all neighborhoods of x as $N(x)$. The collection $N(x)$ is called as the neighborhood system.

Next, the definition of a Hausdorff space is shown.

Definition 2.28 (Hausdorff space)
A topological space $(S, \mathfrak{O})$ is said to be a Hausdorff space if for any $x, y \in S$, there exist neighborhoods of x and y, $U \in N(x)$ and $V \in N(y)$, such that

$$U \cap V = \emptyset,$$

where $N(x)$ and $N(y)$ denote the neighborhood systems of x and y.

Under the definition of a Hausdorff space, the definition of a manifold is given as follows.

Definition 2.29 (differentiable manifold)
The topological space M is said to be a n-dimensional differentiable manifold if M is a Hausdorff space, and there exist a collection of open neighborhoods and a collection of homeomorphisms $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{R}^n$ such that

- $\cup_i U_i = M$
- For any open subsets U_i, U_j satisfying $U_i \cap U_j \neq \emptyset$, the following map is smooth map.
$$\varphi_i \circ \left(\varphi_j^{-1}|\varphi_j(U_i \cap U_j)\right) : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$
.

We call (U_i, φ_i) as a local coordinate. The set $\{(U_i, \varphi_i)\}$ is said to be a local coordinate system.

Chapter 3. Noise-Based Stabilization Method

3.1 Introduction

This chapter presents a noise-based stabilization method and a method for designing a stabilizing controller with noise [28]. The method can be used for stabilizing nonholonomic systems and non-Euclidean systems. Previous studies showed that these systems cannot be stabilized by any continuous feedback laws. The results are stated as follows. Consider a system given by

$$\dot{x} = f(x) + g(x)u, \tag{3.1}$$

where $x \in X$, $u \in U$, $f : X \to TX$, $g : X \to TX$.

Proposition 3.1 (Sontag [64])
If the state space X is not contractible, no continuously differentiable feedback laws exist such that the origin of the system (3.1) becomes globally asymptotically stable.

Proposition 3.2 (Brockett [13])
Let a mapping $\gamma : X \times U \to \mathbb{R}^n$ be defined by

$$\gamma(x, u) = f(x) + g(x)u.$$

A necessary condition for the existence of a continuously differentiable feedback law that makes the origin of (3.1) asymptotically stable is that the mapping γ should be onto an open set containing 0.

These results imply that peculiar feedback laws are needed for the stabilization of such systems. For such systems, discontinuous feedback laws [6, 42, 52, 43, 16] and time-varying feedback laws [57, 69, 50] have been studied. Although the discontinuous feedback methods are effective for the stabilization of such systems (See [17, 16]), the methods might need effort in the analysis or the synthesis in individual problems. Moreover, the time-varying feedback methods do not have any constructive design methods of stabilizing feedback laws, except for the design method for driftless nonlinear systems in [57].

Noise-based stabilization for such systems has been developed by Nishimura et al. [54, 55]. The method can be basically considered as an extension of a time-varying feedback law, where a time-varying component is replaced with noise. In [54], the noise-based stabilization method, and the constructive design method of feedback laws have been proposed in the case that the closed-loop systems are given by Itô stochastic differential equations. In [55], they showed the stabilization of a nonholonomic system by using a noise-based feedback controller when the closed-loop systems are given by Stratonovich stochastic differential equations.

This chapter considers the stabilization of the nonlinear affine system by using a noise-based controller when the closed-loop system is modeled as a Stratonovich stochastic differential equation. Some studies have been discussed which stochastic differential equation should be used when deterministic systems are driven by white noise [33, 3, 56]. In [33, 3], it is stated that Stratonovich stochastic differential equations are more appropriate for deterministic systems driven by white noise. Further, a representative result by Wong and Zakai [70] is stated as follows. Consider an ordinary differential equation that approximates a stochastic differential equation, such as

$$\dot{x}(t) = f(x(t)) + \sigma(x(t))\gamma(t),$$

where $\gamma(t)$ is white noise. Roughly speaking, we may say that $\gamma(t) = \dot{w}(t)$. Let $w^{(n)}(t)$ be an approximation of a one-dimensional Wiener process $w(t)$ defined by

$$w^{(n)}(t) := w^{(n)}(t_i) + \frac{w^{(n)}(t_{i+1}) - w^{(n)}(t_i)}{t_{i+1}^{(n)} - t_i^{(n)}}(t - t_i^{(n)})$$
$$t_i \le t \le t_{i+1} \quad i = 1, 2, \dots, n-1.$$

Under these settings, let us denote a solution of an ordinary differential equation

$$dx^{(n)}(t) = f(x^{(n)}(t))dt + \sigma(x^{(n)}(t))dw^{(n)}(t) \tag{3.2}$$

by $x^{(n)}(t)$. The Wong–Zakai theorem is stated as follows.

Theorem 3.1 (Wong and Zakai [70])
Consider the equation (3.2), and assume the following conditions:

- $\sigma'(x)\frac{\partial \sigma(x)}{\partial x}$ is continuous in x.
- There exists $K > 0$ such that
  - $|f(x) - f(x_0)| \le K\,|x - x_0|$,
  - $|\sigma(x) - \sigma(x_0)| \le K\,|x - x_0|$,
  - $|\sigma'(x)\sigma(x) - \sigma'(x_0)\sigma(x_0)| \le K\,|x - x_0|$.

Then, consider the limit $n \to \infty$ in order that $\max_i \left(t_{i+1}^{(n)} - t_i^{(n)}\right) \to 0$. The sequence of solutions $x^{(n)}(t)$ of (3.2) converges to $x(t)$ that is the solution of an Itô-type stochastic differential equation

$$dx(t) = f(x(t))dt + \sigma(x(t))dw(t) + \frac{1}{2}\sigma'(x(t))\sigma(x(t))dt, \tag{3.3}$$

in the mean.

The system is equivalent to the system given by the Stratonovich stochastic differential equation

$$dx(t) = f(x(t))dt + \sigma(x(t)) \circ dw(t). \tag{3.4}$$

This result shows that an approximated ordinary differential equation by (3.2) gives the system (3.4) as its limit. Thus, in the remainder of this chapter, we deal with systems given by Stratonovich stochastic differential equations.

This chapter presents a constructive design method for designing a stabilizing controller when closed-loop systems are modeled by Stratonovich stochastic differential equations based on the results in [55]. The design method is based on the notion of a stochastic control Lyapunov function, and a stabilizing controller is designed by using the Sontag's formula [63]. In this thesis, we propose a definition of stochastic control Lyapunov function for designing noise-based feedback laws. A proposed stochastic control Lyapunov function is a smooth strict Lyapunov function for a closed-loop system in the sense of stochastic systems. In general, it is difficult or impossible to obtain a smooth strict control Lyapunov function for nonholonomic systems and non-Euclidean systems. As in the stabilization methods using Sontag's formula, the method shows that if the stochastic control Lyapunov function satisfies a small control property, a designed feedback laws can globally asymptotically stabilize a given system with probability one. The proposed method can stabilize nonholonomic systems and non-Euclidean systems, and the numerical examples shows the effectiveness of the method.

Some kind of robustness can be guaranteed when there exists a strict Lyapunov function for a given system. In addition to the design method of the controller, this chapter also shows the robustness of the proposed controller based on its inverse optimality. Inverse optimality of a controller means that the controller minimizes some cost functionals. Using the inverse optimality, Sepulchre et al. [61] showed that the stability margin of the Sontag-type controller for deterministic nonlinear control systems. For stochastic systems, Deng et al. showed the inverse optimality of the Sontag-type controller [18]. In the following, the inverse optimality and stability margin of the proposed controller are shown. This chapter considers the margin in the diffusion coefficient as well as in the feedback term. The diffusion coefficient appears in the expected value of the time derivative of the Lyapunov function with a quadratic form, which is different from that in [61, 18]. Since most time-varying controllers with a periodic signal use weak Lyapunov functions, the guarantee of robustness is an advantage of the proposed method over those methods.

3.2 Stability of Stochastic Systems on a Manifold

Since a controller with white noise transforms a given deterministic system into a stochastic system, we introduce a stochastic differential equation on a manifold [32] and its stability issues.

Let a triple $(\Omega, \mathcal{F}, \mathrm{P})$ denote a probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-field, and P is a probability measure. Let M be a connected σ-compact n-dimensional C^∞-manifold.

Consider a Stratonovich stochastic differential equation

$$dZ(t) = A_0(Z(t))dt + A_1(Z(t)) \circ dw, \tag{3.5}$$

where $Z(t) \in M$, A_0, A_1 are smooth vector fields on M, and w is a one-dimensional standard Wiener process. We assume that (4.1) has a unique equilibrium, and we denote it by $Z = 0$, i.e., $A_0(0) = 0$ and $A_1(0) = 0$. We express local coordinate representations of vector fields in local coordinates $z = (z^1, z^2, \dots, z^n)$ as

$$A_i = \sum_{j=1}^{n} a_i^j(z) \frac{\partial}{\partial z^j}, \quad i = 0, 1.$$

Their vector expressions are denoted by

$$a_i(z) = \left(a_i^1(z), \dots, a_i^n(z)\right)^T, \quad i = 0, 1.$$

A solution $Z(t)$ of (4.1) with $Z(0) = Z_0$ is given by patching local solutions in local coordinates. Let (U, φ) be a chart, where U is an open subset of M containing the initial state Z_0, and φ is a homeomorphism from U to an open subset of a Euclidean space with local coordinates. By a Stratonovich stochastic differential equation

$$dz(t) = a_0(z(t))dt + a_1(z(t)) \circ dw \tag{3.6}$$

under the local coordinates $z = (z^1, z^2, \dots, z^n)$, we obtain a unique solution $z(t)$ of (3.6) in the local coordinates, which specifies a local solution on M as $Z(t) = \varphi^{-1}(z(t))$ for $0 \le t < \tau_U = \inf\{t; Z(t) \notin U\}$. Then, we choose another chart $(\widetilde{U}, \widetilde{\varphi})$, where $\widetilde{U}$ includes the point $\lim_{t\uparrow\tau_U} \varphi^{-1}(z(t))$. We can construct a unique solution $\widetilde{z}(t)$ in $\widetilde{U}$, which defines a local solution of $Z(t)$ for $\tau_U \le t < \tau_{\widetilde{U}} = \inf\{t; t > \tau_U \text{ and } Z(t) \notin \widetilde{U}\}$. We repeat the above procedure to patch local solutions together into a global solution (with respect to the state space) [32].

Given a twice-differentiable function $W : M \to \mathbb{R}$, the infinitesimal generator $\mathcal{L}$ of the solution of (4.1) for $W(Z)$ is given by

$$\mathcal{L}W(Z) = L_{A_0}W(Z) + \frac{1}{2}L_{A_1}L_{A_1}W(Z).$$

The above equation does not seem to be in agreement with the definition of that for an Itô stochastic differential equation, but we can show that both definitions are essentially equivalent by using Itô-Stratonovich drift conversion. Then, we define the stability of the equilibrium of (4.1). Although the stability of stochastic systems has been studied by Khasminskii [37], Kushner [39], Mao [46], and Deng et al. [19] for Euclidean-space cases, we define a stability suitable for stochastic differential equations on manifolds by adapting the results of [39] to our case. Let Z_0 be an initial value of the solution of $Z(t)$ at $t = 0$.

Definition 3.1 (stability in probability)
The equilibrium of the system (4.1) is said to be stable in probability if for any open set $Q \subset M$ containing the equilibrium $Z = 0$ and any $\rho > 0$, there exists an open set $S \subset M$ such that

$$\mathrm{P}\{Z(t) \in Q \quad \text{for all } 0 \le t < \infty \mid Z(0) = Z_0\} \ge 1 - \rho \quad \forall Z_0 \in S.$$

Definition 3.2 (global asymptotic stability in probability)
The equilibrium of the system (4.1) is said to be globally asymptotically stable in probability if it is stable in probability and for any $Z_0 \in M$,

$$\mathrm{P}\left\{\lim_{t\to\infty} Z(t) = 0 \mid Z(0) = Z_0\right\} = 1.$$

The Lyapunov stability theory for stochastic systems is discussed below.

Definition 3.3 (global Lyapunov function)
A function $V : M \to \mathbb{R}$ of the system (4.1) is said to be a global Lyapunov function if $V(Z)$ is twice continuously differentiable, positive definite, and proper, and there exists a continuous positive definite function $W : M \to \mathbb{R}$ such that

$$\mathcal{L}V(Z) = -W(Z), \quad \forall Z \in M.$$

For global asymptotic stability in probability, we can obtain the following stochastic Lyapunov theorem.

Theorem 3.2
If a global Lyapunov function of the system (4.1) exists, then the equilibrium of (4.1) is globally asymptotically stable in probability.

Proof
We essentially follow the proofs presented by Kushner [39] and Khasminskii [37]. We modify the proof such that we do not use the norm of the variable Z and we make some modifications. The notation $t_1 \cap t_2$ is used to denote $\min(t_1, t_2)$ in the following.

Consider two level sets of the Lyapunov function

$$Q_r = \{Z \in M \mid V(Z) < r\}, \quad Q_m = \{Z \in M \mid V(Z) < m\},$$

for $m > r > 0$. Assume that $Z(0) \in Q_m$. Let τ_m be the first exit time from Q_m of the solution $Z(t)$ of (4.1). By the condition of the theorem,

$$\mathcal{L}V(Z) = -W(Z) \le 0$$

holds. By Dynkin's formula,

$$\begin{aligned} E\{V(Z(t\cap\tau_m)) \mid Z(0) = Z_0\} - V(Z_0) &= E\left\{\int_0^{t\cap\tau_m} \mathcal{L}V(Z(\tau))d\tau \mid Z(0) = Z_0\right\} \\ &= -E\left\{\int_0^{t\cap\tau_m} W(Z(\tau))d\tau \mid Z(0) = Z_0\right\} \\ &\le 0 \end{aligned}$$

is satisfied, and therefore,

$$E\left\{V(Z(t\cap\tau_m)) \mid Z(0)=Z_0\right\} \leq V(Z_0)$$

holds. Thus, it is concluded that $V(Z(t\cap\tau_m))$ is a supermartingale. By the supermartingale inequality,

$$\mathrm{P}\left\{\sup_{0\leq t<\infty} V(Z(t\cap\tau_m)) \geq \lambda \mid Z(0)=Z_0\right\} \leq \frac{V(Z_0)}{\lambda}$$

holds, where $0<\lambda\leq m$. By considering the case of $\lambda=m$, we obtain

$$P\left\{Z(t\cap\tau_m)\in Q_m, \text{for all } t<\infty \mid Z(0)=Z_0\right\} > 1-\frac{r}{m},$$

for the initial state in Q_r. This shows the stability in probability of the equilibrium $Z=0$.

Let B_m be a set given by

$$B_m=\left\{\omega\in\Omega \mid Z(t)\in Q_m \text{ for all } t<\infty\right\}.$$

We can conclude that for almost all $\omega\in B_m$, there exists a stochastic variable $c(\omega)$ such that $0\leq c(\omega)\leq m$ and $\lim_{t\to\infty} V(Z(t))=c(\omega)$ according to the supermartingale convergence theorem.

We will prove that $c(\omega)=0$ by showing $\liminf_{t\to\infty} V(Z(t\cap\tau_m))=0$ with probability one relative to B_m. Consider

$$Q_\epsilon=\left\{Z\in M \mid V(Z)<\epsilon\right\},$$

where $\epsilon>0$ such that $\epsilon<m$. Because of the continuity and positive definiteness of $W(Z)$, there exists $b>0$ such that

$$\mathcal{L}V(Z)=-W(Z)\leq -b<0, \quad \forall Z\in Q_m\setminus Q_\epsilon.$$

Let $I_{\xi_0}(\tau,\omega,Q_\epsilon)$ be the indicator function that takes one if $Z(\tau,\omega)\in Q_m\setminus Q_\epsilon$ and zero otherwise. Define

$$T_{\xi_0}(t,Q_\epsilon)=\int_{t\cap\tau_m}^{\tau_m} I_{\xi_0}(\tau,\omega,Q_\epsilon)d\tau,$$

which represents the total time that $Z(\tau,\omega)\in Q_m\setminus Q_\epsilon$ from t to τ_m $(\leq\infty)$. Note that $T_{\xi_0}(t,Q_\epsilon)=0$ if $\tau_m\leq t$. Then, we obtain

$$\begin{aligned}
&E\left\{V(Z(t\cap\tau_m)) \mid Z(0)=Z_0\right\} - E\left\{V(Z(\tau_m)) \mid Z(0)=Z_0\right\} \\
&= -E\left\{\int_{t\cap\tau_m}^{\tau_m} \mathcal{L}V(Z(\tau))d\tau \mid Z(0)=Z_0\right\} \\
&\geq bE\left\{\int_{t\cap\tau_m}^{\tau_m} I(\tau,\omega,Q_\epsilon)d\tau \mid Z(0)=Z_0\right\} \\
&= bE\left\{T_{\xi_0}(t,Q_\epsilon) \mid Z(0)=Z_0\right\},
\end{aligned}$$

and

$$E\left\{T_{\xi_0}(t, Q_\epsilon) \mid Z(0) = Z_0\right\} \leq \frac{E\left\{V(Z(t \cap \tau_m)) \mid Z(0) = Z_0\right\}}{b} \leq \frac{V(Z(0))}{b} < \infty,$$

from the positive definiteness of $V(Z)$. This implies that $T_{\xi_0}(t, Q_\epsilon) < \infty$ with probability one. Therefore,

$$\lim_{t\to\infty} T_{\xi_0}(t, Q_\epsilon) = 0 \tag{3.7}$$

holds. Since (3.7) holds for any $\epsilon > 0$ such that $\epsilon < m$,

$$\liminf_{t\to\infty} V(Z(t)) = 0 \tag{3.8}$$

holds with probability one relative to B_m. We can conclude that

$$\lim_{t\to\infty} V(Z(t)) = 0,$$

since there exists $\lim_{t\to\infty} V(Z(t)) = c(\omega)$. Since we can choose any value of $m > 0$, we obtain

$$P\left\{\lim_{t\to\infty} V(Z(t)) = 0 \mid Z(0) = Z_0\right\} = 1 \quad \text{for any } Z_0 \in M.$$

Recalling the positive definiteness of $V(Z)$,

$$P\left\{\lim_{t\to\infty} Z(t) = 0 \mid Z(0) = Z_0\right\} = 1 \quad \text{for any } Z_0 \in M$$

holds. This completes the proof.

3.3 Problem Statement

This section presents the problem setting of the global asymptotic stabilization of deterministic nonlinear affine systems using noise-based controllers. The noise-based controller consists of an ordinary term and a noise term.

Let M be a smooth n-dimensional manifold and consider a deterministic control system given by

$$\dot{X}(t) = F(X(t)) + \sum_{i}^{m} u_i G_i(X(t)), \tag{3.9}$$

where $X(t) \in M$ is the state, $u_i \in \mathbb{R}$, $i = 1, \ldots, m$ are the inputs, and F and G_i, $i = 1, \ldots, m$, are smooth vector fields on M with $F(0) = 0$. This chapter considers the global asymptotic stabilization of the equilibrium of the deterministic system (3.3). Nonholonomic systems and non-Euclidean systems can be represented by (3.3) According to Proposition 3.1 and 3.2, such systems cannot not be stabilized by any smooth time-invariant feedback laws. Thus, we use an input with noise to stabilize such systems, which is given by

$$u_i dt = v_i dt + B_i(X(t)) \circ dw, \quad i = 1, \ldots, m, \tag{3.10}$$

where $v_i \in \mathbb{R}$ are the new inputs, $B_i : X \to \mathbb{R}$ are the diffusion coefficient functions satisfying $B_i(0) = 0$ for all $i = 1, \ldots, m$, and w is a one-dimensional standard Wiener process. The system with the input (3.10) is given by a Stratonovich stochastic differential equation

$$dX(t) = F(X(t))dt + \sum_{i=1}^{m} v_i G_i(X(t))dt + \sum_{i=1}^{m} B_i(X(t))G_i(X(t)) \circ dw. \quad (3.11)$$

To stabilize the system, we consider a design method that gives the functions v_i and $B_i(X)$ $(i = 1, \ldots, m)$ stabilizing the equilibrium of the system (3.11).

Given a chart (U, φ), in a local coordinate $x = (x_1, \ldots, x_n)$, the vector fields are expressed as

$$F(\varphi^{-1}(x)) = \sum_{j=1}^{n} f^j(x) \frac{\partial}{\partial x^j}, \quad G_i(\varphi^{-1}(x)) = \sum_{j=1}^{n} g^j(x) \frac{\partial}{\partial x^j}. \quad (3.12)$$

Further, using notations,

$$f(x) = \begin{pmatrix} f^1(x) \\ \vdots \\ f^n(x) \end{pmatrix},$$

$$g(x) = (g_1(x), \ldots, g_m(x)), \quad g_i(x) = \begin{pmatrix} g_i^1(x) \\ \vdots \\ g_i^n(x) \end{pmatrix},$$

$$v = \begin{pmatrix} v_1, \\ \vdots, \\ v_m \end{pmatrix}, \quad B(x) = \begin{pmatrix} B_1(\varphi^{-1}(x)) \\ \vdots \\ B_m(\varphi^{-1}(x)) \end{pmatrix},$$

the system (3.11) is expressed as

$$dx(t) = f(x(t))dt + g(x(t))vdt + g(x(t))B(x(t)) \circ dw. \quad (3.13)$$

In the following, for the simplicity, we use the expression (3.13).

Remark 3.1
We may be able to use an Itô stochastic differential equation to express the system with noise inputs [54]. In the case, the system is given by an Itô stochastic differential equation

$$dx(t) = f(x(t))dt + g(x)vdt + g(x)B(x)dw. \quad (3.14)$$

The equivalent system of (3.13) by Itô formulation given by

$$dx(t) = f(x(t))dt + g(x)vdt + \frac{1}{2} \left\{ \frac{\partial}{\partial x} (g(x)B(x))\, g(x)B(x) \right\} dt + g(x)B(x)dw, \quad (3.15)$$

which is different from (3.14).

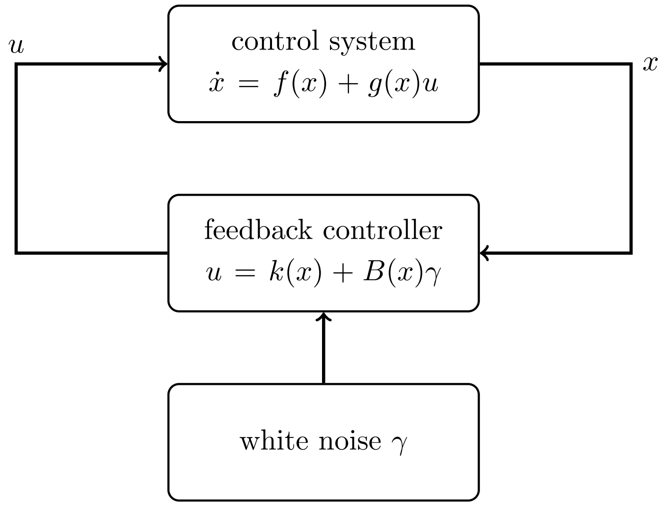

Fig. 3.1: Noise-based stabilization

3.4 Main Results: Design Method of Stabilizing Controller

This section presents a method for designing a stabilizing noise-based controller. The design method is based on the notion of a control Lyapunov function, which is a common notion in the literature of the stabilization of nonlinear systems [7]. After designing a control Lyapunov function, the method can obtain a stabilizing controller by using Sontag's formula [63]. The obtained controller can be seen as an extension of the Sontag controller [63].

3.4.1 Stochastic Control Lyapunov Function

This subsection provides the notion and definition of stochastic control Lyapunov function. Let $V : M \to \mathbb{R}$ be a twice continuously differentiable Lyapunov function candidate for the closed loop system (3.11), and let $\mathcal{L}$ denote the generator of (3.11). Then, we obtain

$$\begin{aligned}\mathcal{L}V(x) &= L_f V(x) + L_g V(x)v + L_{(g\cdot B)} L_{(g\cdot B)} V(x) \\ &= L_f V(x) + L_g V(x)v + \frac{1}{2} L_g V(x) \frac{\partial B}{\partial x}(x) g(x) B(x) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{m} L_{g_i} L_{g_j} V(x) B_i(x) B_j(x).\end{aligned}$$

Considering a symmetric-matrix-valued function $H(x)$ whose elements are given by

$$H_{ij}(x) = \frac{1}{2}\left(L_{g_i} L_{g_j} V(x) + L_{g_j} L_{g_i} V(x)\right), \tag{3.16}$$

we have

$$\mathcal{L}V(x) = L_f V(x) + L_g V(x)v + \frac{1}{2} L_g V(x) \frac{\partial B}{\partial x}(x) g(x) B(x) + \frac{1}{2} B(x)^T H(x) B(x).$$

Further, for v, we use a minor feedback [55],

$$v = v' - \frac{1}{2}\frac{\partial B}{\partial x}(x)g(x)B(x) \tag{3.17}$$

where v' is a new input, given by $v' = (v'_1, \dots, v'_m)$, and $v'_i \in \mathbb{R}$, $i = 1, \dots, m$. Then, we obtain

$$\mathcal{L}V(x) = L_f V(x) + L_g V(x) v' + \frac{1}{2} B(x)^T H(x) B(x). \tag{3.18}$$

We find from (3.18) that $\mathcal{L}V(x)$ may be negative even when $L_f V(x) \geq 0$ and $L_g V(x) = 0$ if the last term on the right-hand side of (3.18) is negative. In (3.18), the last term $1/2B(x)^T H(x) B(x)$ is a quadratic form with respect to $B(x)$. Thus, if the matrix $H(x)$ has a negative eigenvalue and we choose $B(x)$ properly. we can make $\mathcal{L}V(x)$ negative. A discussion similar to the above was done in [55], and it is generalized and clarified here. A contribution here is the establishment of the expression of (3.18). Based on the above discussion, a stochastic control Lyapunov function is defined as follows.

Definition 3.4 (stochastic control Lyapunov function)
A function $V : M \to \mathbb{R}$ is said to be a stochastic control Lyapunov function of (3.11) if $V(x)$ is twice continuously differentiable except at the origin, positive definite, and proper, and $V(x)$ satisfies that

$$L_f V(x) < 0$$
$$\text{for } x \in \{x \in M \mid x \neq 0,\ L_g V(x) = 0,\ \text{and } \lambda_i(x) \geq 0, \forall\ (i = 1, \dots, m)\},$$

where $\lambda_i(x)$ is the i-th eigenvalue of the matrix $H(x)$ given by (3.16).

Remark 3.2
Although a stochastic control Lyapunov function is used to stabilize nonlinear stochastic systems [21], Definition 3.4 is different from that of [21]. The stochastic control Lyapunov function here is defined to be suitable for our problem setting, that is, to design the diffusion coefficient in addition to the ordinary inputs.

3.4.2 Design of Controller with Noise

A design method is presented here for a stabilizing controller of a noise-based stabilization. The existence of a stochastic control Lyapunov function implies that a given system can be stabilized by a noise-based feedback controller. Then, the problem to be solved is the design of a stabilizing controller. This subsection addresses the design problem.

As seen from the choice of the input v_i in (3.17), the proposed method requires the differentiability of $B(x)$. To ensure the differentiability, we make the following assumptions.

Assumption 3.1
All the eigenvalues of the matrix $H(x)$ are distinctive on M.

Assumption 3.2
The eigenvalues $\lambda_i(x)$, $i = 1, 2, \ldots, m$ of the matrix $H(x)$ are locally bounded.

Lemma 3.1
If Assumption 3.1 holds, then the eigenvalues and eigenvectors are continuously differentiable.

Appendix A gives the proof of Lemma 3.1.

As a first step of the design procedure, we first put the input v as the minor feedback controllers (3.17). Then, let $V(x)$ be a stochastic control Lyapunov function of (3.11). We put the diffusion coefficient in a form

$$B(x) = \alpha(x)P(x)\Xi(x)\begin{pmatrix}\sqrt{K_1(x)}\\ \vdots \\ \sqrt{K_m(x)}\end{pmatrix}, \tag{3.19}$$

where the matrix-valued function $P(x)$ satisfies

$$H(x) = P(x)\begin{pmatrix}\lambda_i(x) & & \\ & \ddots & \\ & & \lambda_m(x)\end{pmatrix}$$

and

$$P(x)P(x)^T = I.$$

Since the matrix-valued function $H(x)$ is symmetric, the columns of $P(x)$ are unit eigenvectors of $H(x)$. According to Lemma 3.1, the functions $\lambda_i(x)$ and the components of $P(x)$ are differentiable. We set the function $\Xi : M \to \mathbb{R}^{m\times n}$ as

$$\Xi(x) = \sqrt{2}\begin{pmatrix}\xi(\lambda_1(x)) & & \\ & \ddots & \\ & & \xi(\lambda_m(x))\end{pmatrix}$$

where $\xi : \mathbb{R} \to \mathbb{R}$. The functions $\alpha(x)$ and $\xi(\lambda)$ are functions properly chosen by the designer of the controller. The function $\alpha(x)$ should satisfy a positive definiteness and a differentiability of $\alpha^2(x)$ except at the origin. The function $\xi : \mathbb{R} \to \mathbb{R}$ should satisfy $\xi(\lambda) = 0$ for $\lambda \geq 0$, $\xi(\lambda)$ for $\lambda < 0$, and a differentiability of $\sqrt{-\lambda}\xi^2(\lambda)$. We will determine the feedback $v' = \beta(x) = (\beta_1(x), \ldots, \beta_m(x))^T$ and the function $K(x) = (K_1(x), \ldots, K_m(x))^T$ later. By using (3.19), (3.18) can be expressed as

$$\mathcal{L}V(x) = L_f V(x) + G(x)\mu(x) \tag{3.20}$$

where

$$G(x) = (L_g V(x), G_B(x)), \tag{3.21}$$

$$\mu(x) = \begin{pmatrix} \beta(x) \\ K(x) \end{pmatrix}, \tag{3.22}$$

with

$$G_B(x) = \alpha^2(x) \left(\lambda_1(x)^2, \ldots, \lambda_m(x)^2\right).$$

The stability of the system (3.11) is guaranteed if the function $\mu(x)$ is designed properly so that $\mathcal{L}V(x)$ is negative definite. One of such function $\mu(x)$ is obtained by using Sontag's formula [63] as

$$\mu(x) = \begin{pmatrix} \beta(x) \\ K(x) \end{pmatrix} = \begin{pmatrix} \beta_s(x) \\ K_s(x) \end{pmatrix} = -k(x)G(x) = -k(x) \begin{pmatrix} L_g V(x)^T \\ G_B(x)^T \end{pmatrix}, \tag{3.23}$$

where the gain function $k(x)$ is given by

$$k(x) = \begin{cases} \dfrac{L_f V(x) + \sqrt{(L_f V(x))^2 + (G(x)G(x)^T)^2}}{G(x)G(x)^T} & (G(x) \neq 0) \\ 0 & (G(x) = 0). \end{cases} \tag{3.24}$$

The function $k(x)$ is differentiable because of the differentiability of $G(x)G(x)^T$ and the nature of the Sontag-type controller. The elements of $K_s(x)$ are nonnegative because $k(x)$ is nonnegative and the elements of $G_B(x)$ are nonpositive.

The small control property is introduced to discuss the continuity of the feedbacks (3.23) with (3.24).

Definition 3.5 (small control property)
A stochastic control Lyapunov function of (3.11) is said to satisfy the small control property if there exist $\beta_c(x)$ and $K_c(x)$ such that $\mathcal{L}V(x) < 0$ for $x \neq 0$ with $\beta(x) = \beta_c(x)$, $K(x) = K_c(x)$, and $\beta_c(x) \to 0$, $K_c(x) \to 0$ as $x \to 0$.

Then, the main results of the proposed noise-based stabilization are stated as follows.

Theorem 3.3
Let $V : M \to \mathbb{R}$ be a stochastic control Lyapunov function of (3.11), and assume that the matrix-valued function $H(x)$ satisfies Assumption 3.1. By using the controller (3.10) with (3.17), (3.19), and (3.23), the equilibrium of the system (3.11) is globally asymptotically stable in probability. Furthermore, suppose that Assumption 3.2 holds. If $V(X)$ satisfies the small control property, then the functions of the controller defined by (3.23) also satisfy $\beta(x) \to 0$, $K(x) \to 0$ as $x \to 0$.

Proof

When the controller is given by (3.19) and (3.23), the infinitesimal generator of the closed-loop system (3.11) for $V(x)$ is obtained as

$$\begin{aligned}\mathcal{L}V(x) &= L_fV(x) - k(x)G(x)G(x)^T \\ &= -\sqrt{(L_fV(x))^2 + \left(G(x)G(x)^T\right)^2} < 0,\end{aligned}$$

for $x \neq 0$. Thus, according to Theorem 3.2, the origin of the closed-loop system is globally asymptotically stable in probability.

Next, we prove that $\beta(x) \to 0$ and $K(x) \to 0$ as $t \to 0$ under the small control property of the stochastic control Lyapunov function $V(x)$. According to the nature of the Sontag-type controller and the assumption of $V(x)$, it is obvious that the function $k(x)$ is locally bounded, except at the origin. Thus, the local-boundedness of $k(x)$ in the neighborhood of the origin is not guaranteed in general. When the stochastic control Lyapunov function $V(x)$ satisfies the small control property, there exist functions $v = \beta_c(x)$ and $K(x) = K_c(x)$ such that $\beta_c(x)$ and $K_c(x)$ make the function $\mathcal{L}V(x)$ negative definite, and

$$\beta_c(x) \to 0, \quad K_c(x) \to 0 \quad \text{as } x \to 0.$$

Then, we show that the functions $v' = \beta(x) = -k(x)(L_gV(x))^T$ and $K(x) = -k(x)G_B(x)^T$ in (3.23) satisfies

$$\beta(x) \to 0, \quad K(x) \to 0 \quad \text{as } x \to 0.$$

When $L_fV(x) \leq 0$,

$$|\beta(x)| \leq |L_gV(x)| \tag{3.25}$$

$$|K(x)| \leq |G_B(x)| \tag{3.26}$$

holds, so we can see that $\beta(x) \to 0$, $K(x) \to 0$ as $x \to 0$ in this case. When $L_fV(x) \geq 0$,

$$|L_fV(x)| \leq |G(x)| \cdot \left|\begin{pmatrix}\beta_c(x) \\ K_c(x)\end{pmatrix}\right|. \tag{3.27}$$

should be satisfied due to (3.20) and $\mathcal{L}V(x) < 0$ for $x \neq 0$. By substituting (3.27) into (3.23) and (3.24), we obtain

$$\left|\begin{pmatrix}\beta(x) \\ K(x)\end{pmatrix}\right| \leq \left|\begin{pmatrix}\beta_c(x) \\ K_c(x)\end{pmatrix}\right| + \sqrt{\left|\begin{pmatrix}\beta_c(x) \\ K_c(x)\end{pmatrix}\right|^2 + G(x)G(x)^2} \tag{3.28}$$

from the fact $L_fV(x) \geq 0$. Since the right-hand side of (3.28) converges to 0 as $x \to 0$, we have

$$\beta(x) \to 0, \quad K(x) \to 0 \quad \text{as } x \to 0.$$

Thus, the proof is complete.

Remark 3.3
Since the minor feedback in (3.17) includes the partial derivative $B(x)$ and $\sqrt{k(x)}$ is not differentiable at $x = 0$, one might expect that the minor feedback is not well-defined. However, the minor feedback is well-defined because $B(x) = 0$ at $x = 0$. This is confirmed by

$$\frac{\partial B}{\partial x} g(x)B(x) = k(x)\frac{\partial \Xi'(x)}{\partial x} g(x)\Xi'(x) - \sqrt{k(x)}\Xi'(x)\frac{\partial k(x)}{\partial x} g(x)\Xi'(x).$$

Remark 3.4
The small control property is defined as the property of a stochastic control Lyapunov function for simplicity. However, strictly speaking, the small control property depends on the choice of the functions $\xi(\lambda)$ and $\alpha(x)$ as well as the stochastic control Lyapunov function.

The mechanism of the noise-based stabilization is described as follows. We often encounter the absence of a smooth control Lyapunov function in the stabilization of nonholonomic systems and non-Euclidean systems by time-invariant feedback. In such cases, we may have a weak smooth Lyapunov function V, i.e., there exists a set where $\dot{V} = 0$ for all inputs, except the origin. When we find a weak Lyapunov function and obtain a feedback law such as the Jurdjevic-Quinn like feedback, such set becomes an invariant manifold in many cases. In the noise-based stabilization, noise is injected into a system to cause the state to escape from such set (Fig. 3.2). The diffusion coefficient function $B(x)$ of the proposed controller is designed to have nonzero values on the set. The proposed method uses the eigenvalues of matrix $H(x)$, and $H(x)$ coincides with

$$H(x) = g(x)^T \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)^T (x)g(x).$$

when $L_g V(x) = 0$. This shows that the stochastic control Lyapunov function is given to be partially concave along a direction in span $(g_1(x), \ldots, g_m(x))$ on the set. In the design method, since the matrix $H(x)$ has some negative eigenvalues on the set where $V(x)$ is partially concave, $B(x)$ has nonzero values on such set. Thus, as shown in Fig. 3.2, the noise affect the evolution of the state, and the state escape from the sets. We can expect that the expected value of the function $V(x)$ will decrease due to the partial concavity of $V(x)$ along the direction of the effect of noise. Since the diffusion coefficient $B(x)$ is given by such mechanisms, $\mathcal{L}V(x)$ can be negative definite by using the designed controller. Then, according to Theorem 3.2, the origin becomes globally asymptotically stable in probability.

Remark 3.5
Discontinuous feedback methods have been studied for nonholonomic systems [6, 42] and non-Euclidean systems [43, 52]. Moreover, time-varying feedback methods have been developed in [57, 69, 50]. The noise-based stabilization method can be seen as a variation of time-varying feedback methods, where the time-varying signal, such as

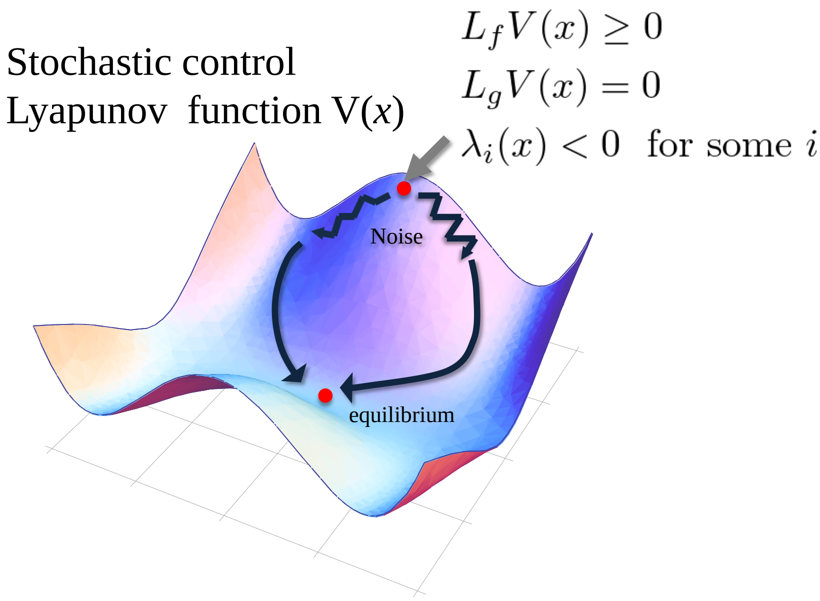

Fig. 3.2: Conceptual illustration of the stabilization mechanism

periodic signal, in the feedback is replaced with the white noise. Thus, the functions $v' = \beta(x)$ and $B(x)$ can be continuous with respect to the state variable.

Remark 3.6
As shown in Definition 3.4, the control Lyapunov function can be a smooth strict control Lyapunov function in the sense of stochastic systems. This fact shows that out method is superior to discontinuous feedback methods and time-varying feedback methods. The discontinuous feedback methods use non-smooth control Lyapunov functions. We might encounter the difficulty in the analysis of the feedback controller due to the non-smoothness. Moreover, the time-varying feedback methods often use weak control Lyapunov functions. On the other hand, the control Lyapunov function defined by Definition 3.4 can be a smooth strict control Lyapunov function. This difference is related to whether the controllers guarantee a kind of robustness, which is discussed in Section 3.5.

3.4.3 Numerical Examples

This subsection presents numerical examples of stabilization of non-Euclidean systems and a nonholonomic system by noise-based stabilization.

Non-Euclidean System

Consider a system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= a_1 \sin(x_1) - a_2 x_2 + u, \end{aligned} \tag{3.29}$$

where $(x_1, x_2) \in \mathbb{S} \times \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the input, and a_1, a_2 are constant parameters where $a_1 = 10.0, a_2 = 0.1$. The state x_1 is a point on a circle $\mathbb{S}$, which means that we equate $x_1 = 0$ with $x_1 = 2i\pi$ $(i = \dots, -1, 0, 1, \dots)$. The state space $\mathbb{S} \times \mathbb{R}$ is noncontractible. Thus, according to Proposition 3.1, no continuous time-invariant feedback law stabilize this system. Thus, we stabilize the system (3.29) by the noise-based stabilization method. A stochastic control Lyapunov function of (3.29) is designed as

$$V(x) = \log\left(2 - \cos(x_1) + \frac{1}{2}(x_2 + \sin(x_1))^2 + \frac{1}{4}(x_2^2 - 1)^2(1 - \cos(x_1))\right). \tag{3.30}$$

Figure 3.3 shows the shape of the stochastic control Lyapunov function (3.30). The system (3.29) has $L_F V(x) = 0$ and $L_G V(x) = 0$ at $(x_1, x_2) = (\pi, 0)$, where $F = x_2 \frac{\partial}{\partial x_1} + (a_1 \sin(x_1) - a_2 x_2) \frac{\partial}{\partial x_2}$ and $G = \frac{\partial}{\partial x_2}$. However, since $H(x) = -1$ at $(x_1, x_2) = (\pi, 0)$, the system (3.29) can be stabilized by the proposed method. Thus, we obtain a stabilizing controller by using stochastic control Lyapunov function and by following the proposed method. In the design of the controller, we use the following functions,

$$\begin{aligned} \xi(\lambda)^2 &= \begin{cases} \tanh(-\lambda^3), & \text{if } \lambda < 0 \\ 0, & \text{if } \lambda \geq 0 \end{cases} \\ \alpha(x)^2 &= c(x_1^2 + x_2^2), \end{aligned} \tag{3.31}$$

where $c = 0.5$.

Figure 3.4 shows the trajectory of the state variable on the state space $\mathbb{S} \times \mathbb{R}$ embedded in $\mathbb{R}^3$ with initial value $(x_1, x_2) = (\pi, 0)$. The figure shows that the state variable converges to the equilibrium. Figure 3.5 shows the values of the input v and the diffusion coefficient $B(x)$. The results indicates that the values of v and $B(x)$ tend to zero as the state variable converges to the equilibrium. The diffusion coefficient $B(x)$ has nonzero value on the set $\{x \mid H(x) < 0\}$ only when the state escape from the neighborhood of critical points other than the equilibrium. Thus, the diffusion coefficient is zero in most cases.

Nonholonomic System

Then, the proposed designed method is applied to stabilize a nonholnomic system called Brockett integrator. The system is given by

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= -2x_1 u_2 + 2x_2 u_1. \end{aligned} \tag{3.32}$$

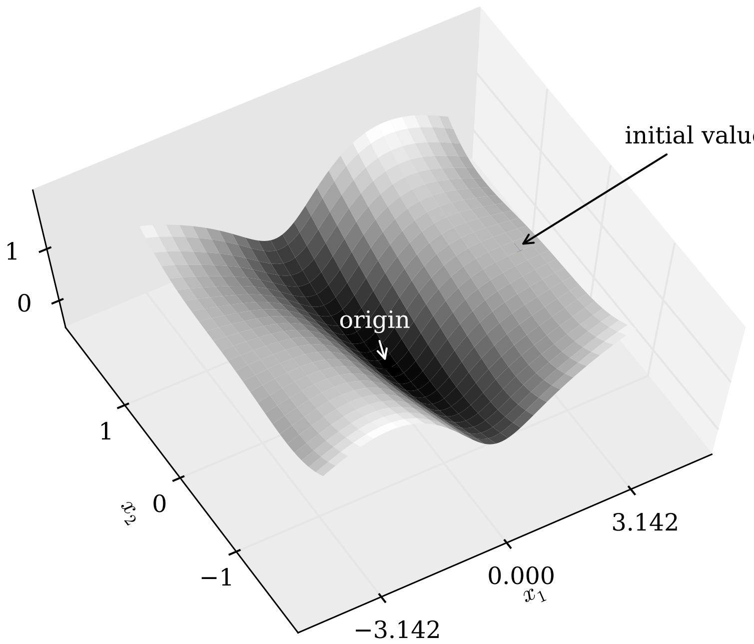

Fig. 3.3: Shape of the stochastic control Lyapunov function (3.30)

According to Proposition 3.2, any smooth time-invariant controller cannot stabilize the Brockett integrator. Indeed, let us set $x = (x_1, x_2, x_3)$, $u = (u_1, u_2)$, and consider the map given by

$$\gamma(x,u) = \begin{pmatrix} u_1 \\ u_2 \\ -2x_1u_2 + 2x_2u_1 \end{pmatrix}.$$

Let $\epsilon \neq 0$, then, the point given by

$$\gamma(x,u) = \begin{pmatrix} 0 \\ 0 \\ \epsilon \end{pmatrix}$$

is not contained in the image of γ. Thus, the Brockett integrator does not satisfy the necessary condition in Proposition 3.2. However, the proposed method can stabilize the Brockett integrator. A stochastic control Lyapunov function is designed as

$$V(x) = 2\left|x_1^2 + 2x_2^2\right|^{1+x_3^2} - \frac{1}{2}(x_1^2 + 2x_2^2)(1 + x_3^2) + 10x_3^2. \tag{3.33}$$

When $x_1 = x_2 = 0$ and $x_3 \neq 0$, $L_gV(x) = 0$ holds. The value of the matrix $H(x)$ becomes

$$H(x) = \begin{pmatrix} -2(1+x_3^2) & 0 \\ 0 & -4(1+x_3^2) \end{pmatrix}$$

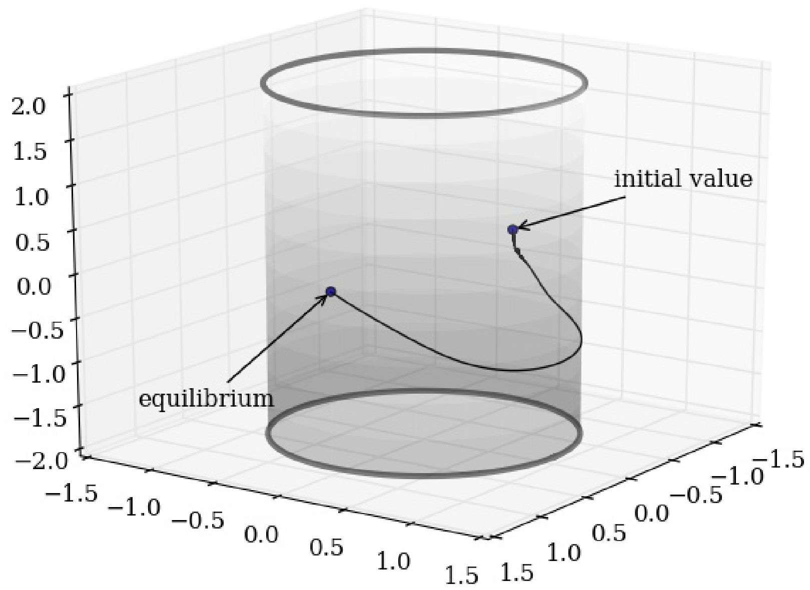

Fig. 3.4: Trajectory of state variables of (3.29) with proposed controller on the state space $\mathbb{S} \times \mathbb{R}$ embed in $\mathbb{R}^3$

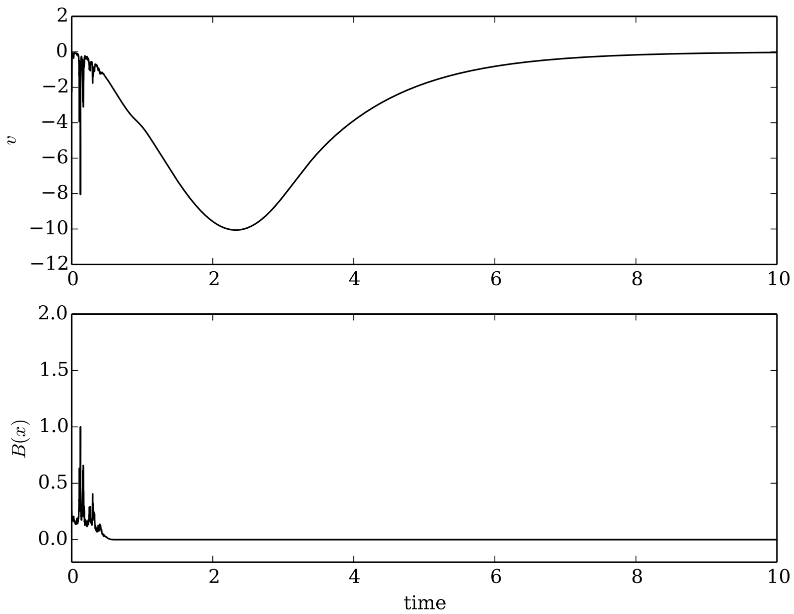

Fig. 3.5: Time responses of the ordinary input term and the diffusion coefficient

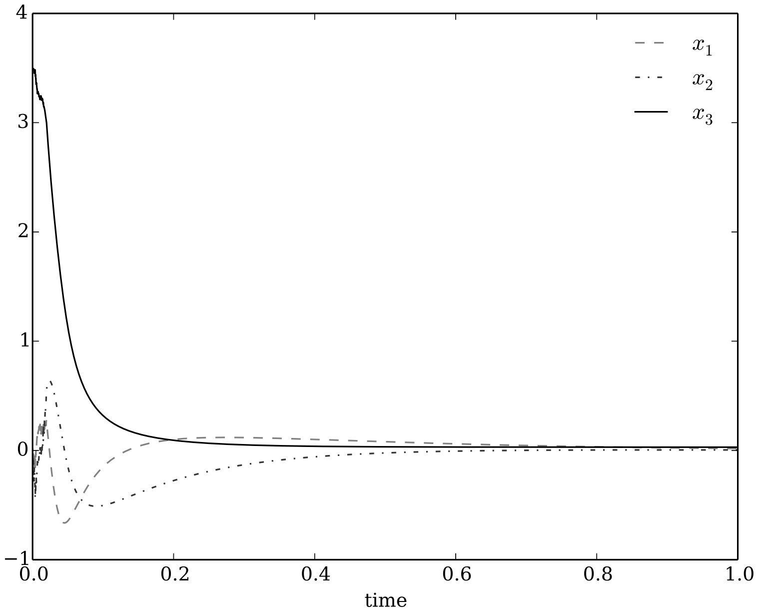

Fig. 3.6: Time responses of state variables of (3.32) with the proposed controller

when $x_1 = x_2 = 0$ and $x_3 \neq 0$. Thus, we can design a stabilizing controller by the design procedure. In the design of the controller, the function $\xi(\lambda)$ is given by

$$\xi(\lambda)^2 = \begin{cases} 0.1\lambda^2, & \text{if } \lambda < 0 \\ 0, & \text{if } \lambda \geq 0 \end{cases}$$

and the function $\alpha(x)$ is given by (3.31) with $c = 0.01$. Figure 3.6 shows time responses of the state variable with the initial value $(x_1, x_2, x_3) = (0, 0, 3.5)$. The result shows that the sample path of the state variable converges to the origin.

The result shows the effect of the introduction of noise into a controller. When the noise is absent in input term with the initial values $x_1 = x_2 = 0$ and $x_3 \neq 0$, the state variables will stay at the initial values. Figure 3.7 shows the values of x_1 and x_2 around the initial time, which are shown in Fig.3.6. The noise has the effect to change the value of the state variable at $x_1 = x_2 = 0$ and $x_3 \neq 0$, then the state can escape from the initial values. Finally, the state converges to the equilibrium.

Figure 3.8 and 3.9 show the time responses of an ordinary feedback term $v(t)$ and the diffusion coefficient $B(x(t))$. As seen in the previous example, the diffusion coefficient has nonzero value around the initial time, and the noise is effective to stabilize the equilibrium.

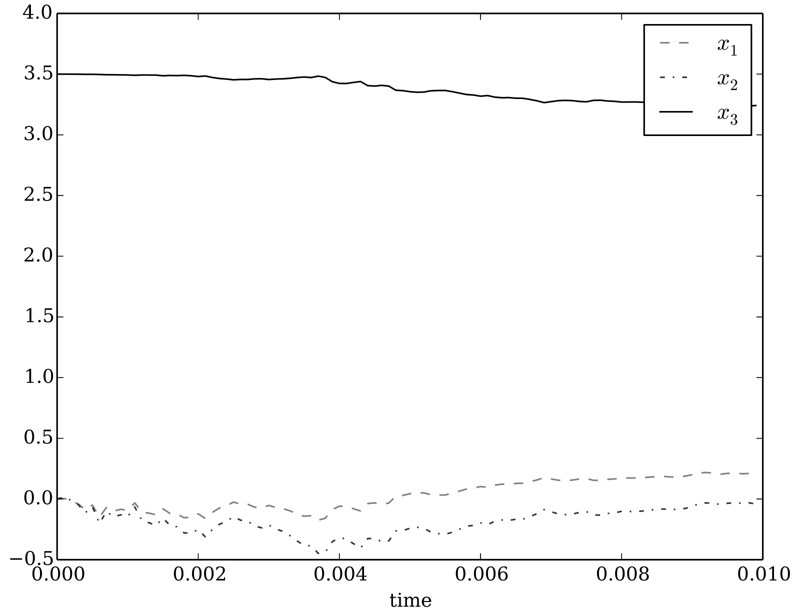

Fig. 3.7: Time responses of state variables around initial time

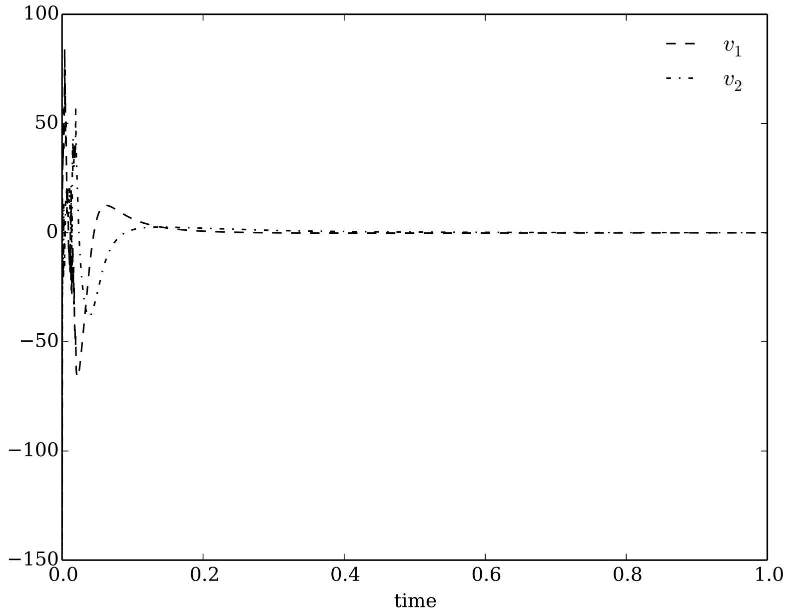

Fig. 3.8: Time responses of the ordinary input term

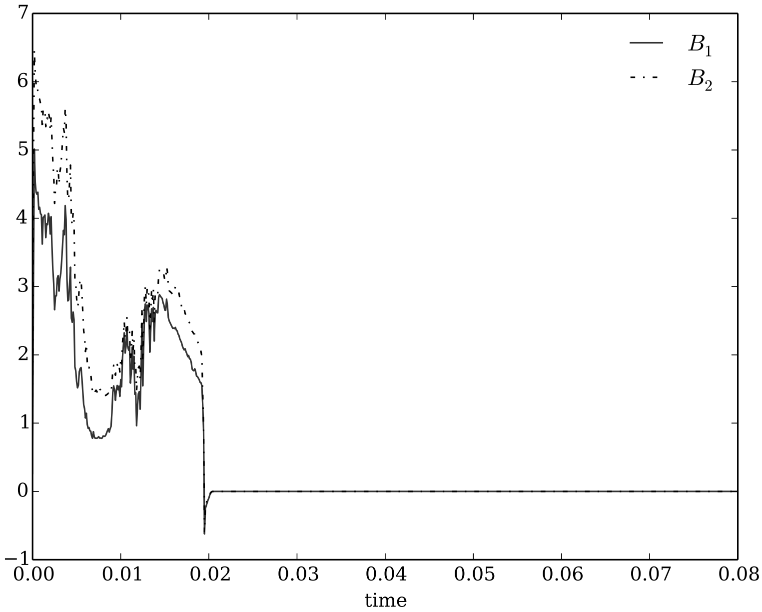

Fig. 3.9: Time responses of the diffusion coefficient

Rigid-Body Control

The proposed method can be used for the global attitude stabilization. The following discussion is based on the results of [15, 14].

The attitude of the rigid body is represented as the element of $SO(3)$. We denote the attitude of the rigid body relative to a reference frame as $R \in SO(3)$, and the angular velocity of the rigid body relative to a reference frame as $\omega \in \mathbb{R}^3$. Then, the time rate of change of R is given by

$$\dot{R} = R\tilde{\omega}, \tag{3.34}$$

where

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

In the following, consider the stabilization of $R = I$, and consider the angular velocity ω as the input. According to the proposed method, we consider the feedback in the form

$$\omega_i dt = v_i dt + B_i(x) \circ dw, \qquad \text{for } i = 1, 2, 3,$$

where w is a one-dimensional standard Wiener process. The candidate of a stochastic control Lyapunov function is given by

$$V(R) = \mathrm{trace}(A - AR^T), \tag{3.35}$$

where

$$A = \mathrm{diag}(a_1, a_2, a_3)$$

with positive integers a_1, a_2, a_3. Then, the derivative of $V(R)$ along the vector fields of (3.34) becomes

$$DV(R) = -\Omega_a(R)^T \omega,$$

where

$$\Omega_a(R) = \sum_{i=1}^{3} a_i e_i \times R e_i,$$

and $[e_1, e_2, e_3]$ is the identity matrix. The elements of $SO(3)$,

$$\begin{aligned} R_1 = \mathrm{diag}(1,1,1),\ R_2 = \mathrm{diag}(-1,1,-1),\\ R_3 = \mathrm{diag}(1,-1,-1),\ R_4 = \mathrm{diag}(-1,-1,1), \end{aligned}$$

satisfy $\Omega_a(R) = 0$ [14]. Thus, the function $DV(R)$ cannot be negative definite.

Denoting the elements of R as $r_{i,j}$ $(i, j = 1, 2, 3)$, the function $H(R)$ is given as

$$H(R) = \begin{bmatrix} -a_2 r_{2,2} - a_3 r_{3,3} & \frac{1}{2}(a_2 r_{2,1} - a_1 r_{1,2}) & \frac{1}{2}(a_1 r_{1,3} - a_3 r_{3,1}) \\ \frac{1}{2}(a_2 r_{2,1} - a_1 r_{1,2}) & a_1 r_{1,1} + a_3 r_{3,3} & \frac{1}{2}(-a_2 r_{2,3} - a_3 r_{3,2}) \\ \frac{1}{2}(a_1 r_{1,3} - a_3 r_{3,1}) & \frac{1}{2}(-a_2 r_{2,3} - a_3 r_{3,2}) & a_1 r_{1,1} + a_2 r_{2,2} \end{bmatrix}.$$

For R_i $(i = 2, 3, 4)$, we obtain

$$\begin{aligned} H(R_2) &= \begin{bmatrix} a_3 - a_2 & 0 & 0 \\ 0 & -a_1 - a_3 & 0 \\ 0 & 0 & a_2 - a_1 \end{bmatrix}, \\ H(R_3) &= \begin{bmatrix} a_2 + a_3 & 0 & 0 \\ 0 & a_1 - a_3 & 0 \\ 0 & 0 & a_1 - a_2 \end{bmatrix}, \\ H(R_4) &= \begin{bmatrix} a_2 - a_3 & 0 & 0 \\ 0 & a_3 - a_1 & 0 \\ 0 & 0 & -a_1 - a_2 \end{bmatrix}. \end{aligned}$$

Thus, $H(R)$ has negative eigenvalues for R_i $(i = 2, 3, 4)$ if $a_1 < a_2$ or $a_1 < a_3$ holds. Therefore, $V(R)$ of (3.35) is a stochastic control Lyapunov function for (3.34).

We can construct the stabilizing feedback with noise by using the stochastic control Lyapunov function with

$$\begin{aligned} \xi(\lambda) &= \begin{cases} 0.001 \tanh(-\lambda^3) & \text{if } \lambda < 0 \\ 0, & \text{if } \lambda \geq 0 \end{cases} \\ \alpha(R) &= \mathrm{trace}(A - AR), \end{aligned}$$

$a_1 = 5$, $a_2 = 1$ and $a_3 = 6$, and we set

$$v' = -k(x)\Omega_a(R),$$

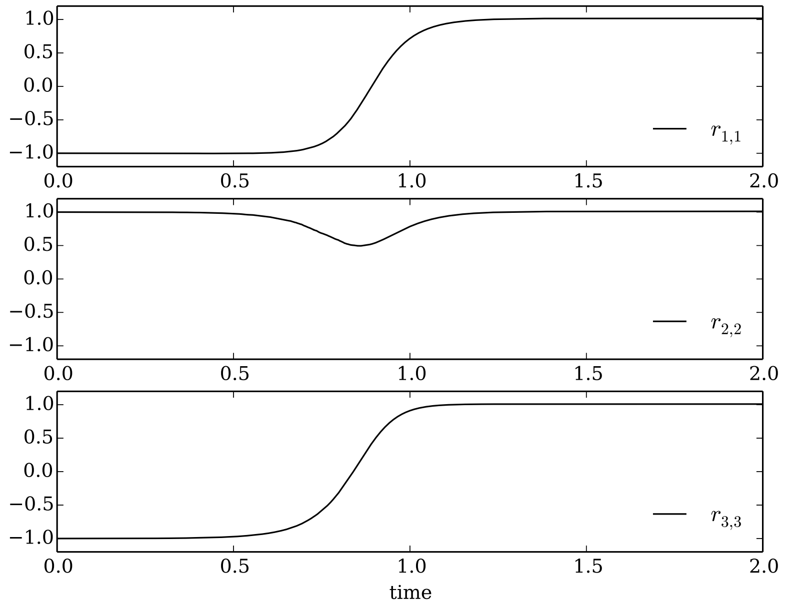

Fig. 3.10: Time responses of the diagonal elements of $R(t)$ with the proposed controller

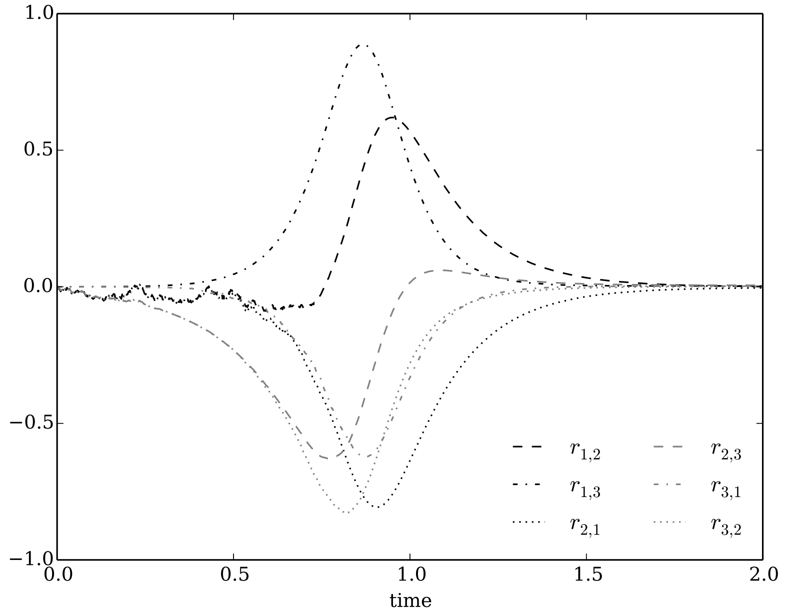

Fig. 3.11: Time responses of the off-diagonal elements of $R(t)$ with the proposed controller

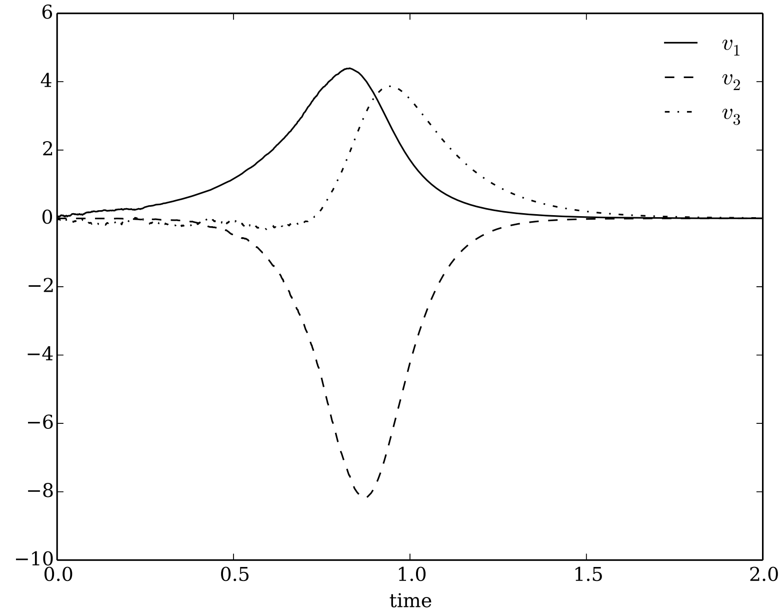

Fig. 3.12: Time responses of v in the proposed controller

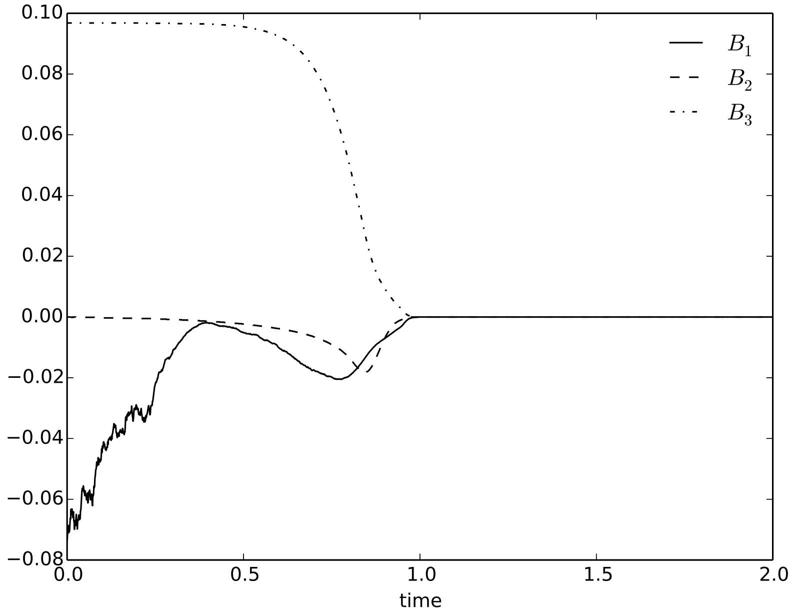

Fig. 3.13: Time responses of $B(x)$ in the proposed controller

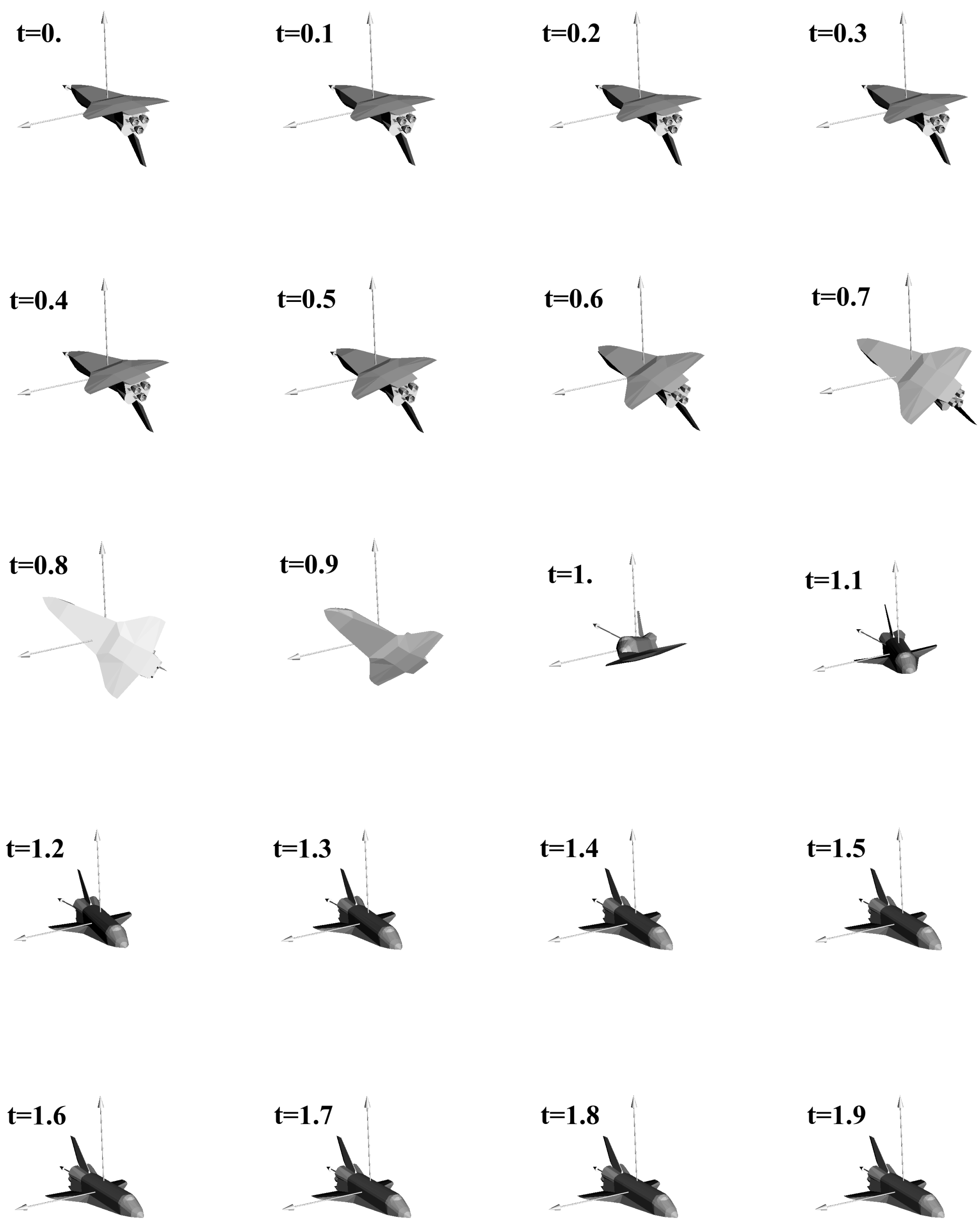

Fig. 3.14: Visualization of rotation by $R(t)$ in Fig. 3.10 and 3.11 with 3D space shuttle model

in (3.17).

Figure 3.10 and 3.11 show the stabilization results with $R(0) = R_3$. When the noise is absent, R_3 is an equilibrium of the closed-loop system, and the value of $R(t)$ will stay at R_3. The diagonal elements of $R(t)$ in Fig.3.10 converge to one, and the off-diagonal elements of $R(t)$ in Fig.3.11 converge to zero. That is, the value of $R(t)$ converges to the identity matrix. Figure 3.12 and 3.13 show the time responses of v and $B(x)$ of the controller, respectively. The values of v and $B(x)$ converge to zero. For better understanding, Fig. 3.14 shows the rotation of a 3D space shuttle model[1] by the time responses of $R(t)$ in Fig. 3.10 and 3.11.

3.5 Inverse Optimality and Robustness of Stabilizing Controllers

This section presents an inverse optimality and robustness of the proposed controller. In general, controllers obtained by Sontag's formula satisfy the inverse optimality [61]. Since a proposed noise-based controller is designed by using Sontag's formula, we expect that the controller satisfies the inverse optimality, and we show the inverse optimatlity of noise-based controller. Moreover, inverse optimality often implies the robustness of a controller. We also show the robustness of the noise-based controller, which is derived from the inverse optimality in this section. We restrict our discussion to the case in which the state space is a Euclidean space, that is, $M = \mathbb{R}^n$.

3.5.1 Inverse Optimality of the Proposed Controller

First, this subsection shows the inverse optimality of the proposed controller. The inverse optimality of the Sontag-type controller for deterministic systems has been studied by Sepulchre et al.[61], and Freeman and Kokotović [23] Further, the inverse optimality of the Sontag-type controller for stochastic systems also has been studied by Deng and Krstić [18]. In contrast to [18], the noise-based stabilization method designs the diffusion coefficient in addition to the ordinary feedback term. Thus, we cannot apply their results [18] directly, and it is not apparent that the proposed controller possesses inverse optimality. This subsection discusses the inverse optimality of the proposed noise-based controller below. In the following discussions, the gain function of the controller $k(x)$ is replaced with

$$k(x) = \begin{cases} c_0 + \dfrac{L_f V(x) + \sqrt{(L_f V(x))^2 + (G(x)G(x)^T)^2}}{G(x)G(x)^T} & (G(x) \neq 0) \\ 0 & (G(x) = 0) \end{cases}, \tag{3.36}$$

where c_0 is a positive parameter.

[1] The example of Fig. 3.14 is made by using Mathematica® and the 3D model in the software.

The inverse optimality of $z = \mu(x)$ of (3.23) is shown.

Theorem 3.4
Let $V(x)$ be a stochastic control Lyapunov function of (3.11) satisfying the small control property. The feedback $z = \mu(x)$ of (3.23) with (3.36) minimizes the cost functional

$$J(z) = \mathrm{E}\left\{\int_0^\infty \left(Q(x) + \frac{1}{4}z^T R(x) z\right) d\tau\right\}, \tag{3.37}$$

where

$$Q(x) = G(x)R^{-1}(x)G(x)^T - L_f V(x), \tag{3.38}$$

$$R(x) = \frac{2}{k(x)}I, \tag{3.39}$$

and I is the $2m \times 2m$ identity matrix.

Proof
First, we prove that the function $Q(x)$ is positive definite. Substituting (3.39) into (3.38), we have

$$\begin{aligned} Q(x) &= \frac{1}{2}k(x)G(x)G(x)^T - L_f V(x) \\ &= \frac{1}{2}\sqrt{(L_f V(x))^2 + (G(x)G(x)^T)^2} - \frac{1}{2}L_f V(x) + c_0 G(x)G(x)^T \\ &> 0 \quad \text{for } x \neq 0. \end{aligned} \tag{3.40}$$

If $G(x) = 0$, $k(x) = 0$ and $-L_f V(x) > 0$ for $x \neq 0$ hold according to the nature of the stochastic control Lyapunov function. Thus, the function $Q(x)$ is positive definite.

We show the optimality of $z = \mu(x)$ for the cost functional in the following. The equation

$$\mathrm{E}\left\{V(x(0)) - V(x(t)) + \int_0^t \mathcal{L}V(x(\tau))d\tau\right\} = 0 \tag{3.41}$$

holds according to Dynkin's formula. Using (3.41), we obtain

$$\begin{aligned} J(z) &= \mathrm{E}\left\{\int_0^\infty \left(Q(x(\tau)) + \frac{1}{4}z^T R(x(\tau))z\right) d\tau\right\} \\ &\quad + \mathrm{E}\left\{V(x(0)) - \lim_{t\to\infty} V(x(t)) + \int_0^\infty \mathcal{L}V(x(\tau))d\tau\right\} \\ &= \mathrm{E}V(x(0)) - \lim_{t\to\infty} \mathrm{E}V(x(t)) \\ &\quad + \mathrm{E}\left\{\int_0^\infty \left(G(x(\tau))z + G(x(\tau))R^{-1}(x(\tau))G(x(\tau))^T + \frac{1}{4}z^T R(x(\tau))z\right) d\tau\right\} \\ &= \mathrm{E}V(x(0)) - \lim_{t\to\infty} \mathrm{E}V(x(t)) \\ &\quad + \mathrm{E}\left\{\int_0^\infty \frac{1}{4}\left|R^{1/2}(x(\tau))z + 2R^{-1/2}(x(\tau))G(x(\tau))^T\right|^2 d\tau\right\}. \end{aligned}$$

Therefore, we see that

$$z = -2R^{-1}(x)G(x)^T = -k(x)G(x)^T = \mu(x) \tag{3.42}$$

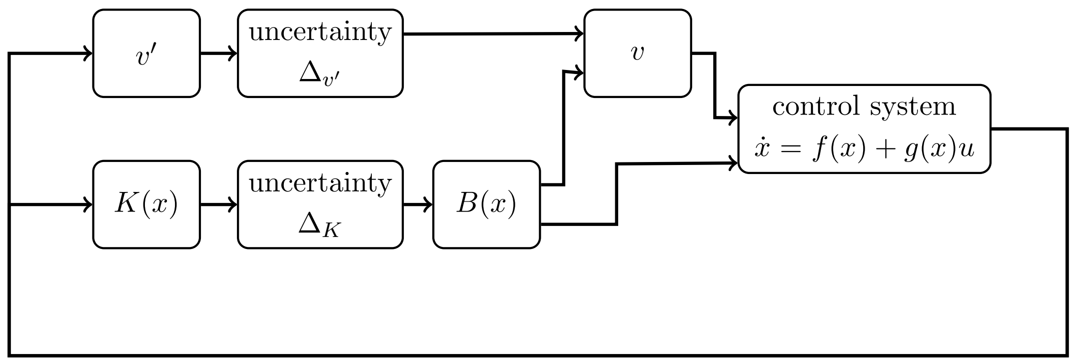

Fig. 3.15: Control system with input uncertainty

minimizes the cost functional (3.37). Since z globally asymptotically stabilizes the system, we obtain

$$\lim_{t\to\infty} \mathrm{E}V(x(t)) = 0.$$

This completes the proof.

3.5.2 Stability Margin of the Proposed Controller

The inverse optimality derives robustness of the controller. The robustness of the noise-based controller is shown, which is called the stability margin, based the inverse optimality shown above. Although the inverse optimality of the proposed controller requires the condition $c_0 > 0$ in (3.36) to ensure the local boundedness of $R(x)$, the following discussion on the stability margin holds when $c_0 = 0$ in (3.36).

The effect of input uncertainties of $z = \mu(X)$ is modeled by following [61]. Figure. 3.15 shows the illustration of the uncertainties, $\Delta_{v'}$ and Δ_K having the effects on the input v' and the function $K(x)$. The effects of the uncertainties are modeled by the functions given in the following. Let $\phi : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ be a map

$$\phi(\mu(x)) = \begin{pmatrix} \phi_{v'}(\beta(x)) \\ \phi_K(K(x)) \end{pmatrix},$$

$$\phi_{v'}(\beta(x)) = \begin{pmatrix} \phi_{v'_1}(\beta_1(x)) \\ \vdots \\ \phi_{v'_m}(\beta_m(x)) \end{pmatrix}, \quad \phi_K(K(x)) = \begin{pmatrix} \phi_{K_1}(K_1(x)) \\ \vdots \\ \phi_{K_m}(K_m(x)) \end{pmatrix}$$

where functions $\phi_{v'_i}, \phi_{K_i} : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$ satisfy, for each $s \neq 0$,

$$\begin{aligned} \epsilon s^2 < s\phi_{v'_i}(s) < \delta s^2, \\ \epsilon s^2 < s\phi_{K_i}(s) < \delta s^2, \end{aligned} \tag{3.43}$$

where ϵ, δ are given parameters satisfying $0 < \epsilon < \delta$, and $\phi_{v'_i}(0) = 0$, $\phi_{K_i}(0) = 0$. This means that these functions are linearly bounded. With these conditions, the function ϕ is said to belong to a sector (ϵ, δ) [61].

Definition 3.6
Let $\mu(x)$ be a function given in (3.23) for the system (3.11). A function $\mu(x)$ is said to satisfy a sector margin (ϵ, δ) if the function $\phi(\mu(x))$ stabilizes the system (3.11) in the sense of global asymptotic stability in probability.

Theorem 3.5
Consider the system (3.11). Let $V(x)$ be a stochastic control Lyapunov function of (3.11) satisfying the small control property. Let a function $z = \mu(x)$ be given by (3.23). Then, the function $z = \mu(x)$ achieves the sector margin $(1/2, \infty)$.

Proof
The proof of this theorem follows that of [61]. Let ϕ be the function belonging to a sector $(1/2, \infty)$. The infinitesimal generator of the stochastic control Lyapunov function $V(x)$ becomes

$$\begin{aligned}\mathcal{L}V(x) &= L_f V(x) + G(x)\phi(z) \\ &= L_f V(x) + G(x)z + G(x)\phi(z) - G(x)z \\ &= -Q(x) + \frac{1}{4}z^T R(x) z - \frac{1}{2}z^T R(x)\phi(z).\end{aligned} \tag{3.44}$$

The last equality follows by (3.38) and (3.42). Further, with (3.23), (3.39), and (3.43), we obtain

$$\begin{aligned}\mathcal{L}V(x) &= -Q(x) + \frac{1}{2}z^T R(x)\left\{\frac{1}{2}z - \phi(z)\right\} \\ &= -Q(x) + \frac{1}{k(x)}\sum_{i=1}^{m}\left\{\frac{1}{2}z_i^2 - z_i\phi_i(z_i)\right\} \\ &= -Q(x) + \sum_{i=1}^{2m}\left\{\frac{1}{2}k(x)G_i(x)^2 - G_i(x)\phi_i(k(x)G_i(x))\right\} \\ &\le -Q(x).\end{aligned} \tag{3.45}$$

The last inequality follows from (3.43). When $k(x) = 0$, since

$$\frac{1}{2}k(x)G_i(x)^2 - G_i(x)\phi_i(k(x)G_i(x)) = 0,$$

holds, the last inequlity in (3.45) also holds. In addition, $Q(x)$ is positive definite even when $c_0 = 0$ from the defintion. Thus, we can conclude that the globally asymptotic stability in probability holds even when the control law $\phi(\mu(x))$ is used.

The above sector margin does not imply a sector margin of the diffusion coefficient $B(x)$. Although we can use the sector margin for the input v' directly, it seems difficult to introduce a sector margin of the diffusion coefficient $B(x)$ for the nonlinear uncertainty because $B(x)$ includes the matrix $P(x)$ to orthogonalize the matrix $H(x)$ in (3.18). To introduce robustness of the diffusion coefficient directly, we consider simpler input uncertainties. We define a function $\gamma_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, m)$ to model the uncertainties such that

$$\phi_K(K(x)) = \mathrm{diag}(\gamma_1(K_1(x)), \ldots, \gamma_m(K_m(x)))K(x),$$

where $0 \le \epsilon \le \gamma_i(s) \le \delta$ for $s \in \mathbb{R}$ $(i = 1, \dots, m)$. When $\phi_K(K(x))$ is given, the diffusion coefficient is transformed into $\mathrm{diag}\left(\sqrt{\gamma_1(K_1(x))}, \dots, \sqrt{\gamma_m(K_m(x))}\right) B(x)$. Then, a gain margin of the diffusion coefficient can be defined.

Definition 3.7
Let $z = \mu(x)$ be a function of (3.23). An input $v' = \beta(x)$ is said to satisfy a sector margin (ϵ, δ) and a diffusion coefficient $B(x)$ is said to satisfy a gain margin $\left(\sqrt{\epsilon}, \sqrt{\delta}\right)$ if the system (3.11) is stabilized in the sense of global asymptotic stability in probability by $z = (\phi_{v'}(\beta(x)), \phi_K(K(x)))^T$ for all $\phi_{v'}$ and ϕ_K satisfying (3.43).

Under this definition, the following result is obtained immediately from Theorem 3.5.

Theorem 3.6
Consider the system (3.11). Let $V(x)$ be a stochastic control Lyapunov function of (3.11) satisfying small control property. Let the control law $z = \mu(x)$ be given by (3.23). Then, the input $v' = \beta(x)$ achieves the sector margin $(1/2, \infty)$ and the diffusion coefficient $B(x)$ achieves the gain margin $(\sqrt{1/2}, \infty)$.

3.5.3 Numerical Example of Inverse Optimality

We confirm the stability margin of the controller designed by the proposed method. Consider the Brockett integrator given by (3.32). Further, the same stochastic control Lyapunov function given by (3.33) is used to construct a stabilizing controller. To describe a input uncertainty to the designed controller, we use the function

$$\mu(x) = \begin{pmatrix} \phi_{v'}(\beta(x)) \\ \varphi_K(K(x)) \end{pmatrix}, \tag{3.46}$$

where

$$\begin{aligned} \phi_{v'}(\beta(x)) &= \begin{pmatrix} \beta_1(x) + 0.2\sin(\beta_1(x)) \\ \beta_2(x) + 0.3\sin(\beta_2(x)) \end{pmatrix} \\ \varphi_K(K(x)) &= 2.0K(x). \end{aligned}$$

Then, we substitute the function $\mu(x)$ into (3.23), and construct the controller perturbed by the input uncertainty.

Figure 3.16 shows the time responses of the closed-loop system with the input uncertainty. The result shows that the states of the closed-loop system converge to the origin, and that the designed controller provides the robustness to the input uncertainty.

3.6 Summary

This section presented the constructive design method of the noise-based controller for the stabilization of systems such as nonholonomic systems and non-Euclidean systems. The design method was developed as a generalization of the result in [55], and

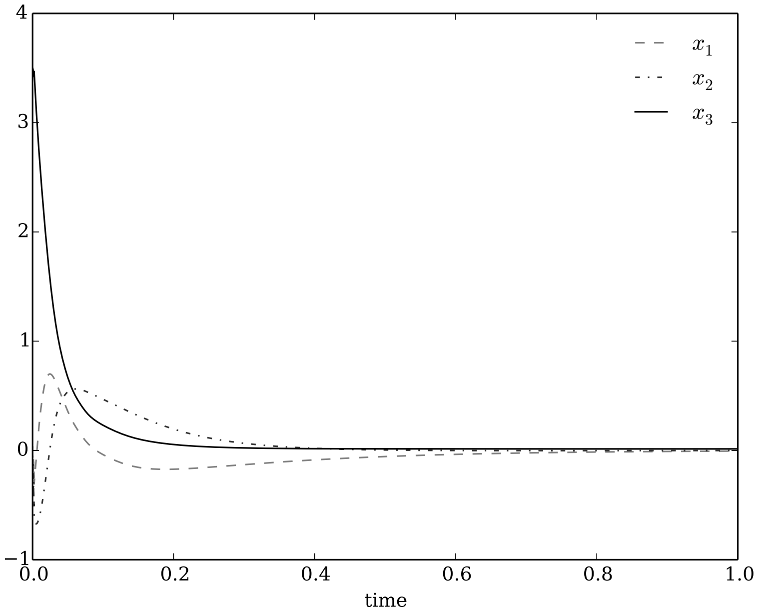

Fig. 3.16: Time responses of state variables of (3.32) with input uncertainty (3.46)

the stochastic control Lyapunov function is defined for the case where the closed-loop system with the noise-based controller is modeled by a Stratonovich stochastic differential equation. The main result in this chapter is that the noise-based controller given by the proposed design method globally asymptotically stabilizes the equilibrium of a given system if a stochastic control Lyapunov function exists for the system. Moreover, in this chapter, the proposed method was applied to the stabilization of the non-Euclidean systems and the nonholnomic systems. These numerical examples showed the effectiveness of the proposed method for the stabilization of the systems that cannot be stabilized by any smooth feedback controllers.

Moreover, this section also showed the stability margin of the controller given by the design method. The stability margin is due to the fact that the stochastic control Lyapunov function is a smooth strict control Lyapunov function in the sense of stochastic systems. The stability margin due to strict control Lyapunov function can be an advantage over other stabilization method for systems that cannot be stabilized by any smooth feedback controllers. Since time-varying feedback methods use non-strict Lyapunov functions in the stability analysis, the stability margin might not be guaranteed. Moreover, since the control Lyapunov function is smooth in this method, the proposed method can be superior to discontinuous feedback methods. Non-smoothness of control Lyapunov functions in discontinuous feedback methods often causes the difficulty in the calculation of the time derivative of the control Lyapunov functions. The smoothness of the control Lyapunov function leads the simplicity in the analysis. On the other hand, the design of the control Lyapunov function for this method becomes difficult in some cases. The design method of the control Lyapunov function is included in the future works.

Since the noise-based stabilization method is a variant of the time-varying feedback methods, slow convergence sometimes occurs in the stabilization. To address the slow convergence, Chapter 4 studies stochastic homogeneous systems and shows the noise-based stabilization method which can guarantee the stability similar to exponential stability for driftless systems.

Chapter 4. Stochastic Homogeneous Systems

4.1 Introduction

Homogeneous systems are key gradients for the analysis of nonlinear dynamical systems [7, 60, 62, 9, 26, 24, 34, 35, 10]. In a short, homogeneous systems are systems that preserve some properties with respect to a scaling operation in the state space. The systems have many interesting features, and therefore have many applications.

One of these is that an index, called a homogeneous degree, implies the convergence speed of asymptotically stable systems. Bhat and Bernstein [10] have shown the relations between homogeneous systems and finite-time stability. Regarding the Lyapunov stability theory, Rosier [60] has shown the converse Lyapunov theorem of homogeneous systems. By this converse Lyapunov theorem, it can be shown that asymptotically stable homogeneous systems can show exponential convergence, or finite-time convergence, depending on their homogeneous degrees. Further, M'closkey and Murray [50] showed the design of homogeneous feedbacks to improve the convergence speed of time-varying feedbacks for nonlinear driftless systems including nonholonomic systems.

Further, homogeneous systems are useful for the analysis of general nonlinear systems. In the analysis of nonlinear systems, homogeneous approximation has been developed (See [25]). When a linearization of a nonlinear system is insufficient to analyze the original system, the approximation can be used. Using higher order terms, this method approximates the original systems with homogeneous systems. For the stability analysis, it has been shown that, if an approximated system is asymptotically stable, the original system is also locally asymptotically stable, as the case in linearization. The same is true for small-time local controllability in this approximation.

This chapter presents homogeneous stochastic systems and their stability [31]. To the best of the author's knowledge, except for [22], homogeneous stochastic systems have not been studied. Florchinger [22] have shown the stabilization of homogeneous stochastic systems defined by the standard dilation. However, recent studies on homogeneous systems deal with a larger class of systems, which is given by weighted dilations including the standard dilation. Thus, this chapter considers homogeneous stochastic systems with weighted dilation, which can be seen as an extension of deterministic homogeneous systems with weighted dilations to stochastic systems. Further, as seen in the literature of deterministic homogeneous systems, this chapter investigates the relations between their stability and their homogeneity. this chapter shows that stable homogeneous systems whose degree is zero exhibit exponential convergence, and those whose degree is negative converge to the origin in finite time. Further, this chapter considers homogeneous stochastic control systems and shows their stabilization. The analysis and synthesis are shown by using stochastic Lyapunov theory.

The possible application of the results on homogeneous stochastic systems is the noise-based stabilization. In the noise-based stabilization method, the closed-loop system sometimes shows slow convergence. As seen in the study of M'closkey and Murray [50], the homogeneity can be used to improve the convergence of the stabilization. Thus, this chapter presents a method for designing a controller which guarantees almost sure ρ-exponential stability. The controller is obtained by redesigning the stochastic controller given by the method described in Chapter 3.

The remainder of this chapter is organized as follows. Section 4.2 introduces a result on stochastic systems having non-Lipschitz vector fields. Subsequently, Section 4.3 shows the definition of homogeneous functions and homogeneous vector fields, and their properties. Then, Section 4.4 presents the definition of homogeneous stochastic systems in this thesis and the relation between the homogeneity of systems and the convergence speed of asymptotic stable systems. Section 4.5 presents the stabilization of homogeneous stochastic control systems. This section shows a feedback controller that preserves the homogeneity of the control systems. Section 4.6 shows the improvement method of the convergence of the stochastic feedback controller given by the method in Chapter 3 for homogeneous driftless systems.

4.2 non-Lipschitz Stochastic Systems

This section introduces a sufficient condition of the existence of a unique solution of stochastic differential equation whose vector fields are non-Lipschitz. It is well known that the uniqueness of the solution of a stochastic differential equation is guaranteed when its vector fields are Lipschitz functions. However, a homogeneous vector field is sometimes non-Lipschitz. Further, the finite-time convergence of the solution of (4.1) occurs only if a vector field of a system is a non-Lipschitz [71].

Consider a stochastic system given by

$$dx = F(x)dt + \Sigma(x)dw, \quad x(0) = x_0, \tag{4.1}$$

where $x \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\Sigma : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $F(0) = 0$ and $\Sigma(0) = 0$, and w is a one-dimensional standard Wiener process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We show the result in the case of time-invariant systems, which is fitted with our problem setting. For time-varying systems, see [59].

Theorem 4.1 ([59])
Suppose that F and Σ are continuous in x. Assume that for $N = 1, 2, \dots,$

$$\begin{aligned}
&\|F(x)\| \le c(1 + \|x\|),\\
&\|\Sigma(x)\|^2 \le c(1 + \|x\|^2),\\
&2\langle x_1 - x_2, F(x_1) - F(x_2)\rangle + \|\Sigma(x_1) - \Sigma(x_2)\|^2\\
&\ \le \rho^N(\|x_1 - x_2\|^2),
\end{aligned}$$

for $\|x_i\| \le N$, $i = 1, 2$, where c is a positive constant. The function $\rho^N(u) \ge 0$ $(u \ge 0)$ is assumed to be nonrandom, strictly increasing, continuous, and concave such that $\int_{0+} du/\rho^N(u) = \infty$. Then, for any given $x_0 \in \mathbb{R}^n$, (4.1) has a pathwise unique strong solution.

Note that we consider only forward solutions of stochastic differential equations and this theorem does not guarantee the uniqueness of backward solutions.

4.3 Homogeneity

This section introduces a homogeneous function, a homogeneous vector field, and their properties. Homogeneous systems are defined by homogeneous vector fields, and their properties will be the basis of the later results. We consider the homogeneity with weighted dilation, which is a generalization of the homogeneity with standard dilation.

Let $x = (x_1, \dots, x_n)$ be an n-dimensional vector, and consider n positive real numbers $r = (r_1, r_2, \dots r_n)$.

Definition 4.1 (dilation)
Let λ be a positive parameter. A dilation $\Delta_\lambda^r : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping such that

$$\Delta_\lambda^r x = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n), \quad \lambda > 0. \tag{4.2}$$

When $r_i = 1$ for $i = 1, \dots, n$, the dilation is called a standard dilation, which gives the standard homogeneity.

Given $x \in \mathbb{R}^n$ and $r = (r_1, \dots, r_m)$, the function of $\lambda \in \mathbb{R}^+$, $\Delta_\cdot^r x : \mathbb{R} \to \mathbb{R}^n$, draws a curve in $\mathbb{R}^n$, which is called a homogeneous ray. In another view, x is mapped to $\Delta_\lambda^r x$ by the scaling operation with respect to λ. Figure 4.1 shows an example of homogeneous rays in the plane by a dilation $\Delta_\lambda^r x$ with $r = (1, 3)$.

Then, the definition of a homogeneous function is introduced.

Definition 4.2 (homogeneous function)
A function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be a homogeneous function of degree l with respect to a dilation Δ_λ^r if

$$h(\Delta_\lambda^r x) = \lambda^l h(x) \tag{4.3}$$

holds.

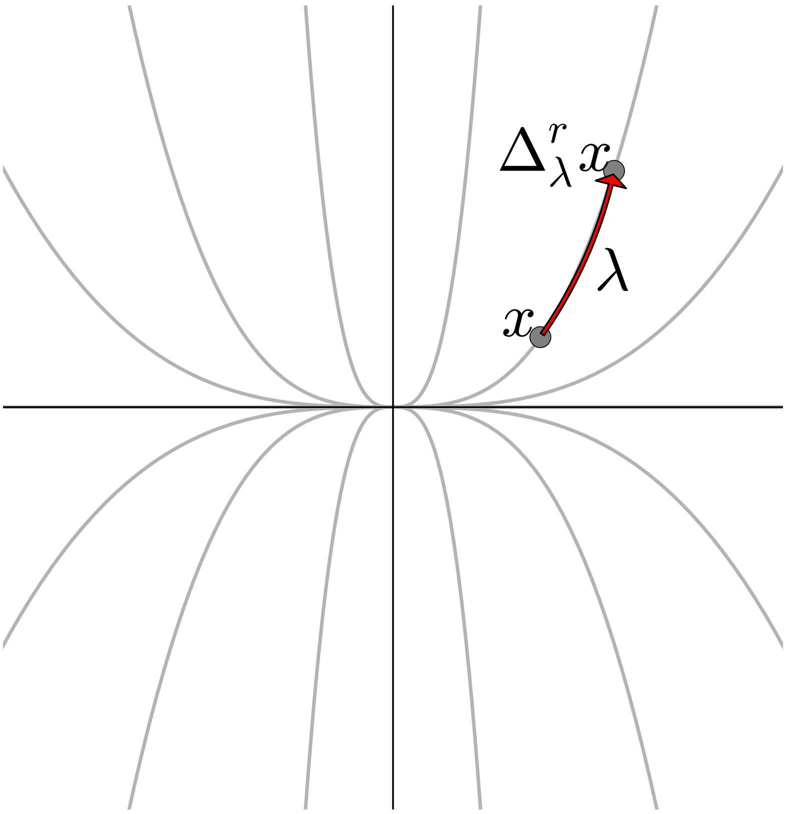

Fig. 4.1: Example of homogeneous ray

In a nutshell, a homogeneous function is scaled by the dilation along a homogeneous ray passing through x with a parameter λ.

We show an example of a homogeneous function.

Example 4.1
Consider a function $h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x_1, x_2) = x_1^4 + x_2^2. \tag{4.4}$$

The function is a homogeneous of degree four with $r = (1, 2)$. Indeed,

$$h(\Delta_\lambda^r(x_1, x_2)) = (\lambda x_1)^4 + (\lambda^2 x_2)^2 = \lambda^4(x_1^4 + x_2^2) = \lambda^4 h(x_1, x_2) \tag{4.5}$$

holds.

Definition 4.3 (homgeneous norm, [50])
A function $\rho : \mathbb{R}^n \to \mathbb{R}$ is called a homogeneous norm with respect to a dilation Δ_λ^r if ρ is a homogeneous function of degree one with respect to Δ_λ^r such that $\rho(0) = 0$ and $\rho(x) > 0$ for $x \neq 0$.

In general, the homogeneous norm does not satisfy the norm axiom, and this is a pseudo norm.

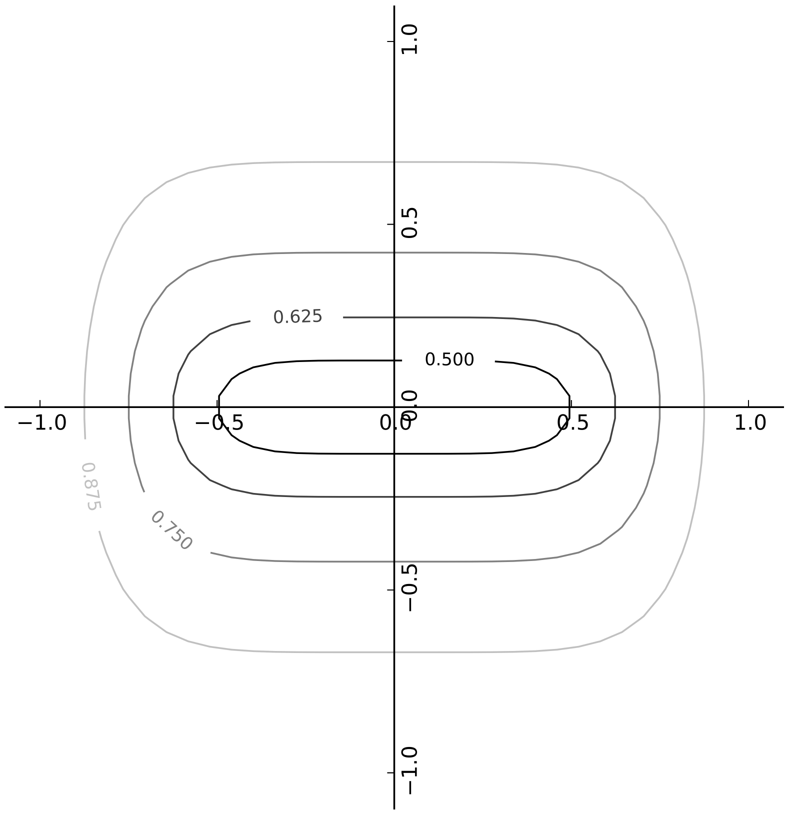

Fig. 4.2: Example of homogeneous norm

Example 4.2
An example of a homogeneous norm is given by

$$\rho(x) = \left(|x_1|^{\frac{c}{r_1}} + |x_2|^{\frac{c}{r_2}} + \cdots + |x_n|^{\frac{c}{r_n}} \right)^{\frac{1}{c}}, \tag{4.6}$$

where $c \geq 1$.

Figure 4.2 shows the level sets of a homogeneous norm with $r = (1, 3)$ and $c = 6$, which is given by

$$\rho(x) = \left(|x_1|^6 + |x_2|^2 \right)^{\frac{1}{6}}.$$

Then, the properties of homogeneous functions are shown.

Proposition 4.1 ([60])
Let $h : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function of degree l with respect to a dilation Δ_λ^r. Then, for $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$,

$$\frac{\partial h}{\partial x_i}(\Delta_\lambda^r x) = \lambda^{l-r_i} \frac{\partial h}{\partial x_i}(x) \tag{4.7}$$

holds. Similarly,

$$\frac{\partial^2 h}{\partial x_i x_j}(\Delta_\lambda^r x) = \lambda^{l-(r_i+r_j)} \frac{\partial h}{\partial x_i x_j}(x) \tag{4.8}$$

is also true for $i, j = 1, \dots, n$.

Proof
Since the function $h(x)$ is homogeneous of degree l,

$$h(\Delta_\lambda^r x) = \lambda^l h(x) \tag{4.9}$$

holds. Differentiating the both sides of (4.9) with respect to x_i, we obtain

$$\lambda^{r_i} \frac{\partial h}{\partial x_i}(\Delta_\lambda^r x) = \lambda^l \frac{\partial h}{\partial x_i}(x) \tag{4.10}$$

by the chain rule. Then, we obtain the equation (4.7) from the equation (4.10) immediately. The equation (4.8) is obtained in a similar way.

Proposition 4.2 ([10])
Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous and homogeneous function with respect to a dilation Δ_λ^r. If h is positive (negative) definite, then h is radially unbounded. Furthermore, if $n > 1$ and h is proper, then h is positive (negative) definite.

The next proposition is used in the stability analysis for homogeneous stochastic systems in this chapter.

Proposition 4.3 ([10])
Suppose that $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$ are continuous and homogeneous with degree l_1 and l_2, respectively, with respect to a dilation Δ_λ^r, and h_1 is positive definite. Then, for all $x \in \mathbb{R}^n$,

$$k_1 h_1(x)^{\frac{l_2}{l_1}} \leq h_2(x) \leq k_2 h_1(x)^{\frac{l_2}{l_1}} \tag{4.11}$$

holds, where $k_1 = \min_{\{z:h_1(z)=1\}} h_2(z)$ and $k_2 = \max_{\{z:h_1(z)=1\}} h_2(z)$.

Proof
Choosing a parameter λ as

$$\lambda = (h_1(x))^{-\frac{1}{l_1}},$$

the equation

$$h_1(\Delta_\lambda^r x) = \lambda^{l_1} h_1(x) = 1$$

holds. Then, consider the level set $h_1^{-1}(\{1\})$. Since $h_1(x)$ is a continuous and homogeneous, then, by Proposition 4.2, the level set $h_1^{-1}(\{1\})$ is compact. Thus, we obtain that

$$\min_{\{z:h_1(z)=1\}} h_2(z) \leq h_2(\Delta_\lambda^r x) \leq \max_{\{z:h_1(z)=1\}} h_2(z). \tag{4.12}$$

Recalling that

$$h_2(\Delta_\lambda^r x) = \lambda^{l_2} h_2(x) = (h_1(x))^{-\frac{l_2}{l_1}} h_2(x), \tag{4.13}$$

we obtain the inequality (4.11) from (4.12) and (4.13). This completes the proof.

We introduce a homogeneous vector field, which is used to define homogeneous systems.

Definition 4.4 (homogeneous vector field)
A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be homogeneous with degree l with respect to a dilation Δ_λ^r if each f_i, $i = 1, \ldots, n$, is a homogeneous function of degree $l + r_i$, i.e.,

$$f_i(\Delta_\lambda^r x) = \lambda^{r_i + l} f_i(x).$$

We can obtain the following proposition from the definition of homogeneous vector fields and Proposition 4.1.

Proposition 4.4
If a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homogeneous vector field of degree l with respect to Δ_λ^r and $h : \mathbb{R}^n \to \mathbb{R}^n$ is a homogeneous function of degree m with respect to Δ_λ^r, then $L_f h(x)$ is a homogeneous function of degree $l + m$ with respect to Δ_λ^r.

4.4 Homogeneous Stochastic System and its Stability

This section defines a homogeneous stochastic system, using homogeneous vector fields, and shows the relations between the homogeneity and the stability of homogeneous stochastic systems. Further, examples of asymptotically stable homogeneous systems are also shown.

4.4.1 Definition of Homogeneous Stochastic System

Homogeneous stochastic systems are defined as follows.

Definition 4.5 (homogeneous Itô stochastic system)
Consider a stochastic system given by an Itô stochastic differential equation

$$dx = f(x)dt + \sigma(x)dw, \tag{4.14}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^n$, and w is a one-dimensional standard Wiener process. The system (4.14) is said to be a homogeneous Itô stochastic system of degree l with respect to a dilation Δ_λ^r if the vector fields f and σ are homogeneous with respect to the dilation Δ_λ^r of degree l and $l/2$, respectively.

Definition 4.6 (homogeneous Stratonovich stochastic system)
Consider a stochastic system given by a Stratonovich stochastic differential equation

$$dx = f(x)dt + \sigma(x) \circ dw, \tag{4.15}$$

where $x \in \mathbb{R}^n$, and f, g, σ, w are same as Definition 4.5. The system (4.15) is said to be a homogeneous Stratonovich stochastic system of degree l with respect to a dilation Δ_λ^r if the vector fields f and σ are homogeneous with respect to the dilation Δ_λ^r of degree l and $l/2$, respectively.

Remark 4.1
Definition 4.5 with a weighted dilation is a generalization of a homogeneous stochastic system with standard dilation as described in [22]. This thesis also considers homogeneous stochastic systems defined by a Stratonovich stochastic differential equation.

Remark 4.2
In the following, we mainly consider homogeneous Itô stochastic homogeneous systems. Further, the term, a homogeneous stochastic system, means a homogeneous Itô stochastic system unless mentioned otherwise. Note that the following results for homogeneous Itô stochastic homogeneous systems also hold for homogeneous Stratonovich stochastic homogeneous systems.

4.4.2 Stability of Stochastic Homogeneous System

This subsection presents relations between the homogeneity of stable systems and their convergence speed. This subsection first provides two definitions of stability concerned with convergence speed of systems, almost surely ρ-exponential stability and finite-time stability in probability. Then, this subsection shows that stable homogeneous systems can exhibit these stabilities depending on their degree of homogeneity. The results presented here can be seen as counterparts to existing results of homogeneous deterministic systems [7, 10].

Definitions of Stability

First, two definitions of stability are presented, which relate to the convergence of speed of systems. The following definitions are not restricted to homogeneous systems, thus, the definitions are stated for the general stochastic system (4.1).

Definition 4.7 (almost sure ρ-exponential stability)
The origin of (4.1) is said to be almost surely ρ-exponentially stable if there exists $\gamma > 0$ such that for $\forall x_0 \in \mathbb{R}^n$,

$$\rho(x(t)) < K_{x_0} e^{-\gamma t} \text{ for } \forall t \geq 0, \text{ a.s.}$$

holds, where $\rho(x)$ is a homogeneous norm with respect to a dilation Δ_λ^r, and K_{x_0} is an almost-surely finite random variable.

The stability for stochastic systems is a counterpart to ρ-exponential stability of deterministic systems [50]. This stability also can be seen as a generalization of almost sure exponential stability with homogeneous norms [37].

Then, we define the finite-time stability in probability.

Definition 4.8 (finite-time stability in probability [71])
The origin of (4.1) is said to be finite-time stable in probability if the origin is stable in

probability, and for any initial value $x_0 \in \mathbb{R}^n \setminus \{0\}$,

$$T_{x_0} = \inf\{t : x(t) = 0\}$$

is finite almost surely.

Results on Stability of Homogeneous System and Homogeneity

We suppose that f and σ in (4.14) are C^0 and satisfy $f(0) = 0$ and $\sigma(0) = 0$ in the following.

To discuss the relation between the stability of homogeneous stochastic systems and their homogeneity, we define homogeneous Lyapunov functions.

Definition 4.9 (homogeneous Lyapunov function)
A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a (stochastic) homogeneous Lyapunov function of the system (4.1) if $V(x)$ is a global Lyapunov function of (4.1) in the sense of Definition 2.25 and is also a homogeneous function with respect to a dilation Δ_λ^r.

We can obtain the following result on the stability of homogeneous systems whose degree is zero.

Theorem 4.2
Consider a homogeneous stochastic systems of degree zero with respect to a dilation Δ_λ^r, which is given by (4.14). Further, assume that there exists a homogeneous Lyapunov function $V(x)$ of the system (4.14) of degree m with respect to Δ_λ^r, and $\mathcal{L}V(x)$ is continuous. Then, the origin of the system (4.14) is almost surely ρ-exponentially stable.

To prove this theorem, we use the following lemma in the proof.

Lemma 4.1
Assume that there exists a Lyapunov function $V(x)$ of a stochastic system (4.1) such that

$$k_1\rho(x)^p \le V(x) \le k_2\rho(x)^p, \tag{4.16}$$

$$\mathcal{L}V(x) \le -k_3\rho(x)^p, \tag{4.17}$$

where $k_1, k_2, k_3 > 0$, $p > 0$, and $\rho(x)$ is a homogeneous norm with respect to the dilation Δ_λ^r. Then, the origin of the system (4.1) is almost surely ρ-exponentially stable.

Remark 4.3
This lemma can be seen as a generalization of Theorem 5.15 in [37] on almost sure exponential stability to almost surely ρ-exponential stability.

Proof
First, define a function

$$W(t, x) = V(x)\exp\left(\frac{k_3 t}{k_2}\right). \tag{4.18}$$

From (4.16) and (4.17), we have

$$\begin{aligned}\mathcal{L}W(t,x) &= \frac{k_3}{k_2}\exp\left(\frac{k_3 t}{k_2}\right)V(x) + \exp\left(\frac{k_3 t}{k_2}\right)\mathcal{L}V(x)\\ &\leq 0.\end{aligned}$$

This implies that $W(t,x)$ is a supermartingale. Noting that $W(t,x) > 0$, $x \neq 0$, for all $x_0 \in \mathbb{R}^n$, according to the supermartingale convergence theorem (Theorem 2.2), $W(t,x(t))$ converges to a finite limit almost surely. Thus, there exists an almost surely finite random variable $A'_{x_0} > 0$ such that

$$A'_{x_0} = \sup_t W(t,x(t)) < \infty. \tag{4.19}$$

It follows from (4.18) and (4.19) that

$$V(x(t)) \leq A'_{x_0}\exp\left(-\frac{k_3 t}{k_2}\right).$$

Using (4.16) and (4.17) again, there exist $A_{x_0} > 0$ and $\gamma > 0$ such that

$$\rho(x(t)) \leq A_{x_0}\exp(-\gamma t),$$

which completes the proof.

We give the proof of Theorem 4.2.

Proof
Since the function $V(x)$ is homogeneous of degree m and continuous, it follows from Proposition 4.3 that

$$k_1\rho(x)^m \leq V(x) \leq k_2\rho(x)^m,$$

where $k_1 = \min_{\{z:\rho(z)=1\}} V(z)$ and $k_2 = \max_{\{z:\rho(z)=1\}} V(z)$. Further, since the system is homogeneous of degree zero, according to Proposition 4.4, $\mathcal{L}V(x)$ is a homogeneous function of degree m. In addition, from the condition of the theorem, $\mathcal{L}V(x)$ is continuous and negative definite. Thus, according to Proposition 4.3, since the homogeneous norm $\rho(x)$ is also continuous, there exists a constant $k_3 = -\max_{\{z:\rho(z)=1\}} \mathcal{L}V(z) > 0$, and

$$\mathcal{L}V(x) \leq -k_3\rho(x)^m$$

holds. Then, it follows from Lemma 4.1 that the origin is almost surely ρ-exponentially stable.

Theorem 4.3
Consider a homogeneous system whose degree is negative, $l < 0$, with respect to a dilation Δ^r_λ, which is given by (4.14). Assume that there exists a homogeneous Lyapunov function $V(x)$ of (4.14) of degree m $(> -l)$ with respect to the dilation Δ^r_λ, and $\mathcal{L}V(x)$ is continuous. Then, the origin of the system (4.14) is finite-time stable in probability.

We use the following lemma in the proof of Theorem 4.3.

Lemma 4.2 ([71])
Suppose that there exists a Lyapunov function of the system (4.1), $c > 0$, and $0 < q < 1$ such that

$$\mathcal{L}V(x) \leq -cV(x)^q.$$

Then, the origin of the system (4.1) is finite-time stable in probability.

Then, the proof of Theorem 4.3 are presented.

Proof
Since the homogeneous degree of the system (4.14) is l (< 0) and the homogeneous degree of $V(x)$ is m ($> -l$), it follows from Propositions 4.1 and 4.4 that $\mathcal{L}V(x)$ is a homogeneous function of degree $l + m$. Recalling that $\mathcal{L}V(x)$ is continuous, according to Proposition 4.3, there exists a constant $k_4 = -\max_{\{z:V(z)=1\}} \mathcal{L}V(z) > 0$ and

$$\mathcal{L}V(x) \leq -k_4 V(x)^{\frac{l+m}{m}}$$

holds. Because $0 < (l + m)/m < 1$, according to Lemma 4.2, the origin is finite-time stable in probability.

4.4.3 Examples of Almost Sure ρ-Exponentially Stable System and Finite-Time Stable System

This subsection shows examples of systems that are almost sure ρ-exponentially stable or finite-time stable in probability.

First, a simple example of a one-dimensional system is shown, which can exhibit both stabilities depending on its parameter.

Example 4.3
Consider a system

$$dx = -\text{sgn}(x)|x|^\alpha dt + 0.4|x|^\beta dw, \tag{4.20}$$

where $x \in \mathbb{R}$ and w is a one-dimensional standard Wiener process. For the system (4.20), there exists a Lyapunov function candidate $V : \mathbb{R} \to \mathbb{R}$

$$V(x) = \frac{1}{2}x^2.$$

Assume that $\alpha = 1/2$ and $\beta = 3/4$. Then, the system becomes homogeneous with degree $-1/2$ with repect to the standard dilation, because f and σ have the degree of $-1/2$ and $-1/4$, respectively. Thus,

$$\mathcal{L}V(x) = -0.92|x|^{\frac{3}{2}}$$

holds. According to Theorem 4.3, the origin of this system is finite-time stable in probability. In this case, although the vector fields are not Lipschitz, a unique solution

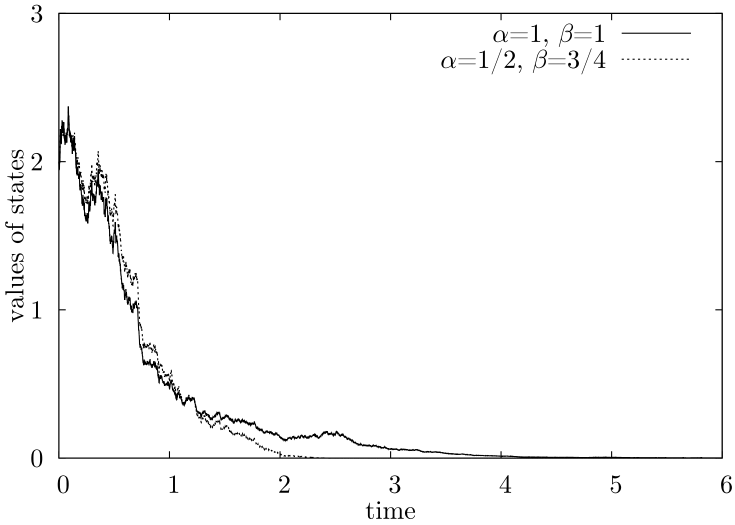

Fig. 4.3: Time responses of states of (4.20) in Example 4.3

exists according to Theorem 4.1. In addition, consider the case when $\alpha = 1$ and $\beta = 1$. In this case, the system (4.20) becomes homogeneous of degree zero with respect to the standard dilation. Thus, according to Theorem 4.2, the origin is almost surely ρ-exponentially stable when $\alpha = 1$ and $\beta = 1$. Figure 4.3 shows sample trajectories of $x(t)$ for the both cases with $x(0) = 2$.

Then, a system exhibiting almost sure ρ-exponential stability is presented in the next example.

Example 4.4
Consider a system given by

$$\begin{aligned} dx_1 &= -x_1 dt + 0.1 x_1 dw, \\ dx_2 &= (x_1^3 - x_2)dt + 0.5 x_1^3 dw, \end{aligned}$$

where $(x_1, x_2) \in \mathbb{R}^2$ and w is a one-dimensional standard Wiener process. This system is a homogeneous system of degree zero with respect to $r = (1, 3)$. There exists a homogeneous Lyapunov function given by

$$V(x_1, x_2) = x_1^6 - x_1^3 x_2 + x_2^2.$$

Therefore, the origin of this system is almost surely ρ-exponentially stable. A sample trajectory with $(x_1(0), x_2(0)) = (2, 1)$ is shown in Fig. 4.4.

Although the following example is somewhat artificial, the system exhibits finite time stability in probability.

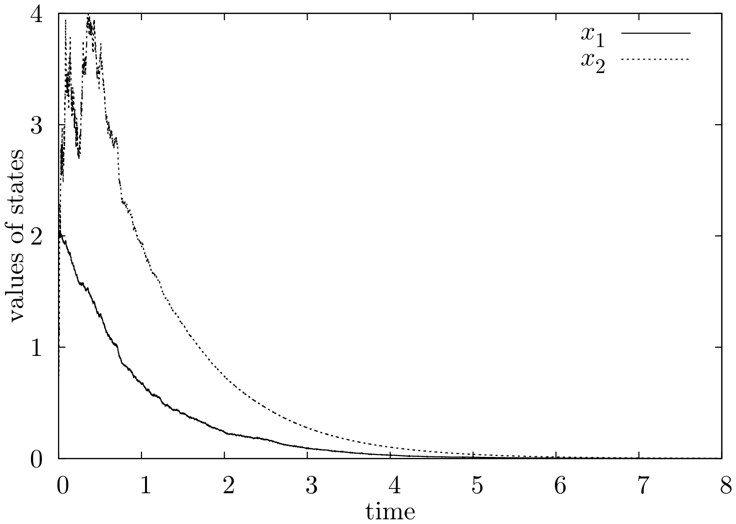

Fig. 4.4: ρ-exponential stable system in Example 4.4

Example 4.5
Consider a homogeneous systems whose degree is $-1/2$ for $r = (1, 1)$, which is given by

$$\begin{aligned} dx_1 &= -\operatorname{sgn}(x_1 - x_2)|x_1 - x_2|^{\frac{1}{2}}dt + 0.3|-x_1 + 2x_2|^{\frac{3}{4}}dw, \\ dx_2 &= -\operatorname{sgn}(-x_1 + 2x_2)|-x_1 + 2x_2|^{\frac{1}{2}}dt + 0.2|x_1 - x_2|^{\frac{3}{4}}dw, \end{aligned} \tag{4.21}$$

where $(x_1, x_2) \in \mathbb{R}^2$ and w is a one-dimensional standard Wiener process. We can obtain a homogeneous Lyapunov function of this system, given by

$$V(x_1, x_2) = x_1^2 - 2x_1x_2 + 2x_2^2.$$

This function is homogeneous of degree two. Since the degree of system is negative and a homogeneous Lyapunov function exists, according to Theorem 4.3, the origin of this system is finite-time stable in probability. The state x converges to the origin in finite time with $(x_1(0), x_2(0)) = (1, 1.2)$ as shown in Fig. 4.5.

4.5 Stabilization of Homogeneous Stochastic System

By the results on the stability of homogeneous stochastic systems in the previous section, we can consider the stabilization of homogeneous stochastic control systems. In the following results, we show a stabilizing controller preserving homogeneity of systems. By the homogeneity, closed-loop systems with the controller can exhibit almost surely ρ-exponential stabilization or finite-time stabilization in probability depending on the homogeneous degree.

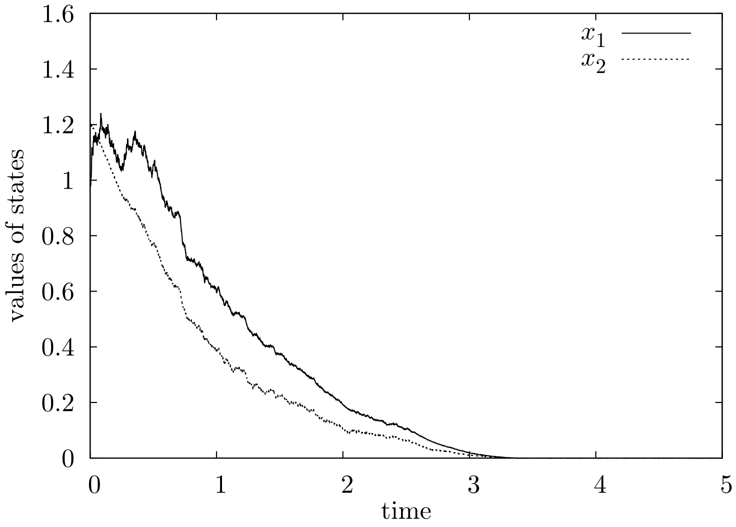

Fig. 4.5: Time responses of states of (4.21) in Example 4.5

4.5.1 Definition of Homogeneous Stochastic Control System

First, homogeneous stochastic control systems are defined.

Definition 4.10 (homogeneous stochastic control system)
Consider a stochastic control system

$$dx = f(x)dt + g(x)udt + \sigma(x)dw, \tag{4.22}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ is a input, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^n$, and w is a one-dimensional standard Wiener process. The system (4.22) is said to be a homogeneous Itô stochastic control system of degree (l, k) with respect to a dilation Δ_λ^r if the vector fields f, g, and σ are homogeneous with respect to the dilation Δ_λ^r of degree of l, k, and $l/2$, respectively. With the same settings, a stochastic system given by a Stratonovich stochastic system

$$dx = f(x)dt + g(x)udt + \sigma(x) \circ dw, \tag{4.23}$$

is said to be a homogeneous Stratonvich stochastic system.

In the remainder of this section, as in Section 4.4, we mainly consider homogeneous Itô stochastic control systems, and the word, a homogeneous stochastic control system, means homogeneous Itô stochastic control systems unless mentioned otherwise.

4.5.2 Results on Stabilization of Homogeneous Stochastic System

In this subsection and the following subsections, we consider a homogeneous stochastic control system given by (4.22), with the vector fields f, g, and σ are C^0 and satisfy $f(0) = 0$ and $\sigma(0) = 0$.

To discuss the stabilization of homogeneous control systems, we define homogeneous control Lyapunov functions.

Definition 4.11 (homogeneous control Lyapunov function)
A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a homogeneous control Lyapunov function of the stochastic control system (4.22) if $V(x)$ is homogeneous with respect to a dilation Δ_λ^r, C^2, positive definite, and radially unbounded, and if it satisfies

$$\mathcal{L}_0 V(x) < 0 \quad \text{if } L_g V(x) = 0,$$

where

$$\mathcal{L}_0 V(x) = L_f V(x) + \frac{1}{2}\sigma(x)^T \frac{\partial}{\partial x}\left[\frac{\partial V}{\partial x}\right]^T \sigma(x).$$

The function is a control Lyapunov function which is homogeneous. Although homogeneous stochastic systems can be stabilized by using general control Lyapunov functions, homogeneous systems can be stabilized in the sense of almost surely ρ-exponentially stable or finite-time stable, depending on their homogeneous degree, by using homogeneous control Lyapunov function. Then, we can obtain the following result.

Theorem 4.4
Consider a stochastic homogeneous control system given by (4.22) where the degrees of the vector fields f, g, and σ are l, k $(< l)$, and $l/2$, respectively, with respect to a dilation Δ_λ^r. Suppose that there exists a homogeneous control Lyapunov function of (4.22) of degree m with respect to the dilation Δ_λ^r. Then, there exists a constant $\beta > 0$ such that the feedback law

$$u = \alpha(x) = -\beta\rho(x)^p L_g V(x),$$

where $p = l - m - 2k$, stabilizes the origin of the system of (4.22) in the sense of global asymptotic stability in probability. Further, if $l = 0$, the origin of the closed-loop system of (4.22) becomes almost surely ρ-exponentially stable. If $l < 0$, the origin of the closed-loop system of (4.22) becomes finite-time stable in probability.

Remark 4.4
The feedback law $\alpha(x)$ in Theorem 4.4 is continuous since it is homogeneous of degree $l - k$ (> 0).

Proof
Considering a feedback law given by

$$u = \alpha(x) = -\beta\rho(x)^p L_g V(x),$$

where $\beta > 0$, the infinitesimal generator of a homogeneous control Lyapunov function $V(x)$ for the closed-loop system becomes

$$\mathcal{L}V(x) = \mathcal{L}_0 V(x) - \beta\rho(x)^p L_g V(x)^2. \tag{4.24}$$

If $\mathcal{L}_0V(x)$ is negative definite, for any value of $\beta > 0$, the origin of the closed-loop system is globally asymptotically stable in probability. In the following, we consider the case that $\mathcal{L}_0V(x)$ is not negative definite. By the condition of the theorem, $\mathcal{L}_0V(x)$ and $\rho(x)^pL_gV(x)^2$ are homogeneous functions of degree $l+m$. According to Proposition 4.3,

$$\mathcal{L}_0V(x) \leq k_1\rho(x)^{m+l}$$

hold, where $k_1 = \max_{\{z:\rho(z)=1\}} \mathcal{L}_0V(z)$. In addition, there exists a negative constant c such that

$$\max_{\{z:\rho(z)=1 \text{ and } L_gV(z)=0\}} \mathcal{L}_0V(z) < c < 0.$$

Note that $\max_{\{z:\rho(z)=1 \text{ and } L_gV(z)=0\}} \mathcal{L}_0V(z) < 0$ holds by the definition of a homogeneous control Lyapunov function. Further, there exists a positive constant

$$k_2 = \min_{\{z:\rho(z)=1 \text{ and } \mathcal{L}_0V(z)\geq c\}} \rho(z)^p(L_gV(z))^2.$$

Thus, there exists the parameter $\beta > k_1/k_2$ such that $\mathcal{L}V(x)$ becomes negative definite on the compact set $\{x : \rho(x) = 1\}$. It follows from the nature of homogeneous functions that for any $x \in \mathbb{R}^n \setminus \{0\}$, $\mathcal{L}V(x) < 0$ with $\beta > k_1/k_2$, and the origin of the closed-loop system is globally asymptotically stable in probability owing to Theorem 2.8. The second part of the theorem follows from Theorems 4.2 and 4.3, and the proof is complete.

4.5.3 Example of Stabilization of Homogeneous System

We show two examples of the stabilization of homogeneous stochastic control systems by using the feedback law presented above.

Example 4.6
Consider a control system given by

$$\begin{aligned} dx_1 &= (x_1 - 14x_2^3)dt, \\ dx_2 &= udt + 0.5x_2dw, \end{aligned} \tag{4.25}$$

where $(x_1, x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}$ is the input, and w is a one-dimensional standard Wiener process. This system is homogeneous with degree $(0, -1)$ for $r = (3, 1)$. A homogeneous control Lyapunov function is given by

$$V(x_1, x_2) = 3x_1^{\frac{4}{3}} - 2x_1x_2 + 4x_2^4.$$

Then, we obtain a feedback law

$$u = \alpha(x) = -\beta\frac{(-2x_1 + 16x_2^3)}{|x_1|^{\frac{2}{3}} + x_2^2}, \tag{4.26}$$

where $\beta = 145$. Under the feedback law, the origin becomes almost surely ρ-exponentially stable. Fig. 4.6 shows that $x(t)$ of the closed-loop system converge to the origin when $(x_1(0), x_2(0)) = (1.0, 0.5)$.

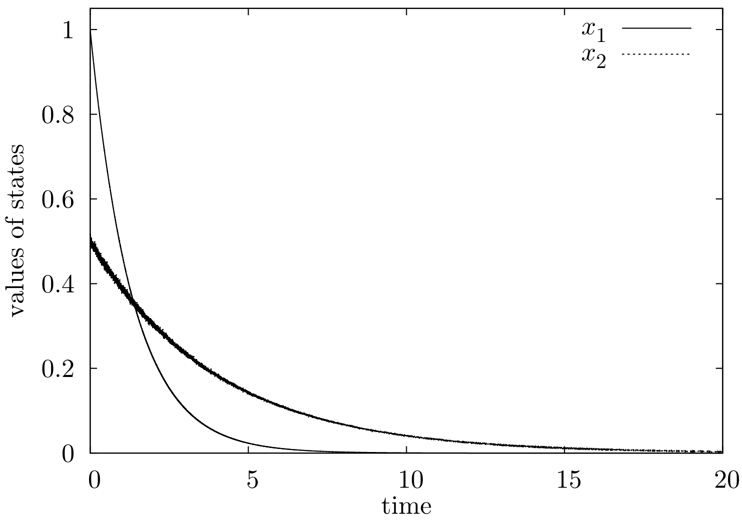

Fig. 4.6: Time responses of states of (4.25) with (4.26) in Example 4.6

Remark 4.5
The system in example 4.6 is obtained by adding a diffusion term to a system in [27]. A system without a diffusion term in (4.25) is often discussed in the literature of homogeneous deterministic systems [34].

Then, the example of finite-time stabilization is shown.

Example 4.7
In this example, we consider a control system given by

$$\begin{aligned} dx_1 &= |x_1|^{\frac{3}{4}}dt + udt + 0.1|x_1|^{\frac{7}{8}}dw \\ dx_2 &= \left(0.5\mathrm{sgn}(x_1)|x_1|^{\frac{1}{4}} - \mathrm{sgn}(x_2)|x_2|^{\frac{1}{2}}\right)dt + 0.1|x_2|^{\frac{3}{4}}dw \end{aligned} \tag{4.27}$$

where $(x_1, x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}$ is the input, and w is a one-dimensional standard Wiener process. The system (4.27) is degree $(-1/2, -1/4)$ homogeneous system with respect to $r = (2, 1)$. A homogeneous control Lyapunov function for (4.27) is a function given by

$$V(x_1, x_2) = x_1^2 - 2x_1x_2^2 + 2x_2^4,$$

whose degree is four. Since a degree of the system (4.27) is less than zero, we can obtain a finite-time stabilizing controller,

$$\alpha(x) = \frac{4(2x_1 - 2x_2^2)}{(|x_1| + |x_2|^2)^{\frac{1}{4}}}$$

Figure 4.7 shows the finite-time stabilization of the closed-loop system with the initial value $(x_1(0), x_2(0)) = (1.0, -1.5)$.

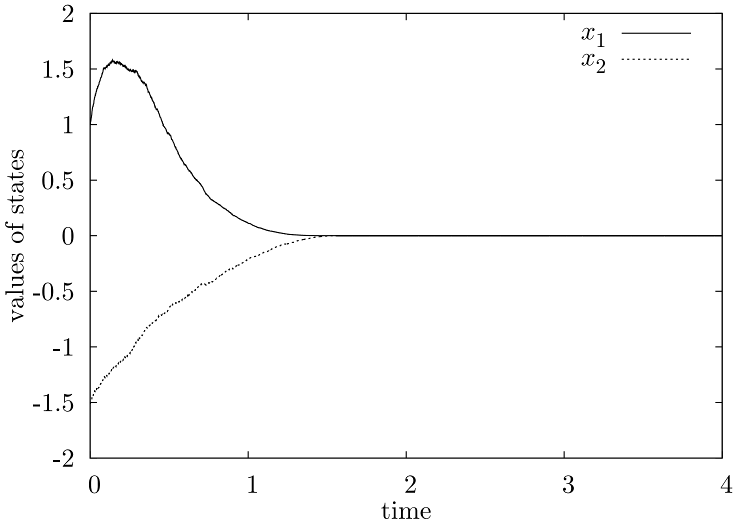

Fig. 4.7: Time responses of (4.27) with 4.7 in Example 4.7

4.6 Almost Sure ρ-Exponential Stabilization Using Homogeneity

This section presents a method for designing stabilizing controllers with noise for nonlinear driftless systems, which guarantee almost sure ρ-exponential stability by using homogeneity, for the noise-based stabilization method in Chapter 3. For time-varying feedback controllers, M'Closkey and Murray [50] showed a method for exponential stabilization of nonlinear driftless systems to improve the convergence speed. In their method, a stabilizing feedback controller is converted to a homogeneous feedback controller. As in the case of time-varying feedback controllers, stochastic feedback controllers described in Chapter 3 sometimes show slow convergence. Therefore, this section presents a design method of stochastic feedback controllers guaranteeing the almost sure ρ-exponential stability based on the method of [50].

In this section, we consider the stabilization of a homogeneous driftless system

$$\dot{x} = g(x)u, \tag{4.28}$$

where $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, each column of $g(x)$ is homogeneous vector field of degree -1 with respect to Δ_λ^r, and $u \in \mathbb{R}^m$. A class of nonholonomic systems such as the Brockett integrator and chained systems is modeled by (4.28). As in Chapter 3, we consider the stabilization of (4.28) by using the feedback controller with noise

$$udt = v(x)dt + B(x) \circ dw, \tag{4.29}$$

where $v : \mathbb{R}^n \to \mathbb{R}^m$, $B : \mathbb{R}^n \to \mathbb{R}^m$, and w is a one-dimensional standard Wiener process. The closed-loop system is given by

$$dx = g(x)v(x)dt + g(x)B(x) \circ dw, \tag{4.30}$$

and the functions $v(x)$ and $B(x)$ is designed to almost surely ρ-exponentially stabilize the origin of (4.30)

To show the design method, we make the following assumption.

Assumption 4.1
There exists a stochastic control Lyapunov function $V(x)$ for the system (4.28) such that, for each x in some level surface of $V(x)$, $L_\nu V(x) > 0$ holds.

Almost surely ρ-exponentially stabilizing controller is designed as follows. We first obtain a feedback controller with noise by using (3.10) with (3.17), (3.19), and (3.23) in Chapter 3. Then, using obtained functions $v' = \beta(x)$ and $B(x)$ in (3.10), we obtain the following functions

$$\tilde{\beta}(x) = \begin{cases} \phi(x)\beta(\Delta^r_{\gamma(x)}x) & x \neq 0 \\ 0 & x = 0 \end{cases}, \tag{4.31}$$

$$\tilde{B}(x) = \begin{cases} \phi(x)B(\Delta^r_{\gamma(x)}x) & x \neq 0 \\ 0 & x = 0, \end{cases} \tag{4.32}$$

where $\gamma(x)$ is obtained by solving the following equation

$$F(\lambda, x) = V(\Delta^r_\lambda x) - C = 0, \tag{4.33}$$

with respect to λ, and $\phi(x)$ is defined as

$$\phi(x) = \begin{cases} \frac{1}{\gamma(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}. \tag{4.34}$$

Furthermore, we use the notation

$$\bar{x} = \Delta_{\gamma(x)}x,$$

in the following. Then, the feedback guaranteeing almost sure ρ-exponential stability is given as

$$\begin{aligned} udt &= \tilde{v}(x)dt + \tilde{B}(x) \circ dw, \\ \tilde{v}(x) &= \tilde{\beta}(x) + R(x), \\ R(x) &= \frac{1}{2}\left(-\frac{1}{L_\nu V(\bar{x})}L_{(g\cdot B)}V(\bar{x}) - \frac{L_\nu L_\nu V(\bar{x})}{L_\nu V(\bar{x})^2}L_{(g\cdot B)}V(\bar{x}) + \frac{L_{(g\cdot B)}L_\nu V(\bar{x})}{L_\nu V(\bar{x})}\right. \\ &\quad \left. + \frac{L_\nu L_{(g\cdot B)}V(\bar{x})}{L_\nu V(\bar{x})}\right)\tilde{B}(x) - \frac{1}{2}\phi(x)\frac{\partial B}{\partial x}(\bar{x})g(\bar{x})B(\bar{x}). \end{aligned} \tag{4.35}$$

Using the feedback given by (4.35), we obtain the following the result.

Theorem 4.5
If the system (4.28) satisfies Assumption 4.1, the feedback (4.35) with (4.31) and (4.32) almost surely ρ-exponentially stabilizes the origin.

Proof

We first show that $\phi(x)$ is twice continuously differentiable except at the origin, positive definite, and proper, as shown in [50]. From the equation (4.33) and Assumption 4.1, we obtain

$$\frac{\partial F}{\partial \lambda}(\lambda, x) = \frac{1}{\lambda} L_\nu V(\bar{x}) > 0. \tag{4.36}$$

Since $V(x)$ is twice continuously differentiable, by using the implicit function theorem, it is shown that $\gamma(x)$ is twice continuously differentiable except at the origin. Thus, we can conclude that $\phi(x)$ is twice continuously differentiable except at the origin.

Then, we show the positive definiteness and radial unboundedness of $\phi(x)$. By the definition of dilation, $\gamma(x) > 0$ for $x \neq 0$. Since $\phi(x)$ is given by (4.34), $\phi(x)$ is positive definite. Moreover, according to Proposition 4.2, $\phi(x)$ is radially unbounded.

Next, we show the homogeneity of $\phi(x)$ by the homogeneity of $\gamma(x)$. Assume that, for some $x \in \mathbb{R}^n$, $\gamma(x)$ is obtained by solving

$$F(\lambda, x) = 0,$$

that is, $\lambda = \gamma(x)$. Then, for some $\sigma > 0$, we consider $x' \in \mathbb{R}^n$ so that $x' = \Delta^r_\sigma x$. Solving

$$F(\lambda', x') = 0,$$

we obtain $\lambda' = \gamma(x')$. When we set $\bar{x} = \Delta^r_\lambda x$, we also have $\bar{x} = \Delta^r_{\lambda'} x' = \Delta^r_{\lambda'} \Delta^r_\sigma x$. Thus, we obtain

$$\begin{aligned} \Delta^r_\lambda x &= \Delta^r_{\lambda'} \Delta^r_\sigma x \\ &= \Delta^r_{\lambda' \cdot \sigma} x, \end{aligned}$$

and $\lambda = \lambda' \cdot \sigma$. This shows

$$\gamma(\Delta^r_\sigma x) = \sigma^{-1} \gamma(x). \tag{4.37}$$

By using (4.37), we can conclude that $\gamma(x)$ is homogeneous of degree -1, and that $\phi(x)$ is homogeneous of degree one.

Finally, we show that $\tilde{\mathcal{L}}\phi(x)$ becomes negative definite and homogeneous of degree one, where $\tilde{\mathcal{L}}$ denotes the infinitesimal generator of the closed-loop system. By using the feedback (4.35), we obtain

$$\tilde{\mathcal{L}}\phi(x) = \frac{\phi(x)}{L_\nu V(\bar{x})} \left(L_g V(\bar{x}) \beta(\bar{x}) + \frac{1}{2} B(\bar{x})^T H(\bar{x}) B(\bar{x}) \right).$$

According to Theorem 3.3,

$$L_g V(\bar{x}) \beta(\bar{x}) + \frac{1}{2} B(\bar{x})^T H(\bar{x}) B(\bar{x}) < 0$$

holds. Thus, we can conclude that $\tilde{\mathcal{L}}\phi(x)$ is negative definite. Moreover, since the functions in (4.35) are homogeneous of degree one, the closed-loop system becomes homogeneous of degree zero. Therefore, the origin is almost surely ρ-exponentially stable according to Theorem 4.2. This completes the proof.

A numerical example is shown for almost sure ρ-exponential stabilization of Brockett integrator (3.32) by using the design method. The vector fields of the Brockett integrator is homogeneous of degree -1 with respect to $r = (1, 1, 2)$. Using the stochastic control Lyapunov function given by (3.33), we obtain the stabilizing feedback (3.10) with (3.17), (3.19), and (3.23). To obtain the feedback, we use the functions

$$\xi(\alpha)^2 = \begin{cases} \tanh(-\lambda^3), & \text{if } \lambda < 0 \\ 0, & \text{if } \lambda \geq 0 \end{cases}$$
$$\alpha(x)^2 = x_1^2 + x_2^2 + x_3^2.$$

Finally, by using this feedback, we obtain the feedback (4.35) with (4.31) and (4.32).

Figure 4.8 shows the numerical result using the designed feedback. In Fig. 4.8, the state variables converge to the origin when $x(0) = (0, 0, 1.5)$. For comparison, the stabilization results by (3.10) is shown in Fig. 4.9 with the same settings. In this case, the state variables show the slow convergence. Thus, we can confirm the effectiveness of the feedback by the design method. Figure 4.10 shows the time responses of $\tilde{v}(x)$ and $\tilde{B}(x)$, and these values converge to zero.

4.7 Summary

This chapter considered the homogeneous stochastic systems, and showed the relation between the homogeneous degree and the convergence speed of stable homogeneous systems. The homogeneous stochastic systems are defined as the counterpart of the homogeneous deterministic systems. As a main result, this chapter showed that asymptotically stable systems whose homogeneous degree is zero converge to the origin in the sense of almost surely ρ-exponentially stable, and those whose degree is negative converge to the origin in finite time.

Moreover, this chapter also presented the stabilization of homogeneous stochastic control systems by using a feedback law that preserves the homogeneity. By the stability results, the feedback law can guarantee the convergence speed, such as almost surely ρ-exponentially stable and finite-time stability in probability.

Finally, Section 4.6 presented the design method of a stochastic feedback controller for driftless homogeneous systems, which guarantees the almost sure ρ-exponential stability. The design method converts a stochastic feedback controller given by the method in Chapter 3 into a controller using the homogeneity, based on the method of M'Closkey and Murray [50]. This method was developed to improve the convergence speed when a stochastic feedback controller given by the method in Chapter 3 provides slow convergence. The numerical example of the stabilization of the Brockett integrator showed the effectiveness of the proposed method compared with the method in Chapter 3.

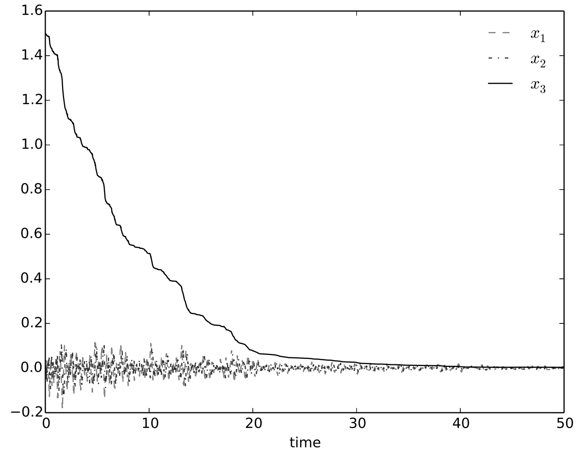

Fig. 4.8: Time responses of state variables of (3.32) with (4.35)

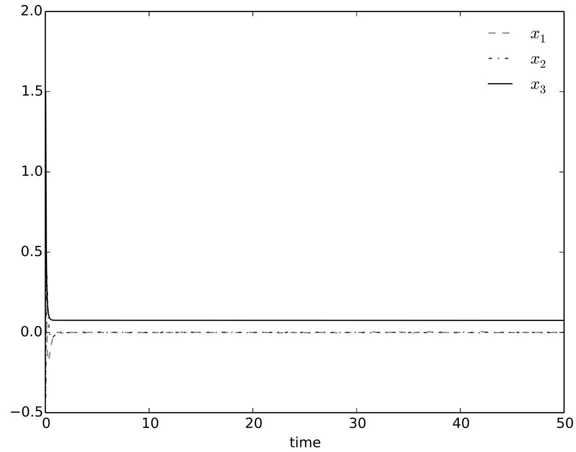

Fig. 4.9: Time responses of state variables of (3.32) with (3.10)

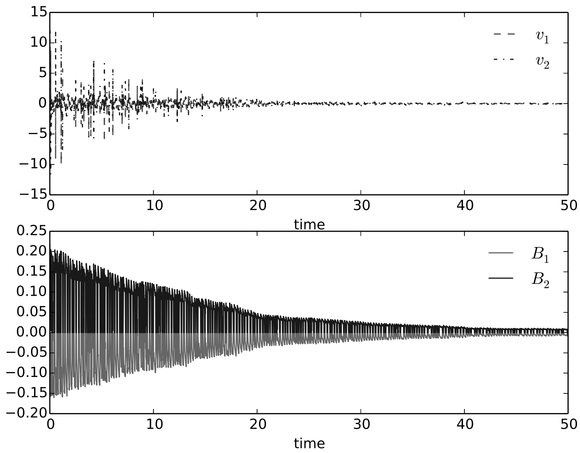

Fig. 4.10: Time responses of input $\tilde{\beta}(x)$ and diffusion coefficient $\tilde{B}(x)$

Chapter 5. Stochastic Extremum Seeking

5.1 Introduction

This chapter shows a stochastic extremum seeking method as a noise-based method [30, 29].

Extremum seeking method is a non-model based method for real-time optimization [2, 53, 67]. The method solves the optimization problem in continuous time. The extremum seeking method is a non-model based optimization method, that is, the method estimates the optimum value of objective functions using the input values and the output values of the systems. Since the exact models are not required, the method can be applied to the real-time optimization. The method can also be applied to the optimization with uncertainties, such as parameter changes due to the the aging of the system. Although extremum seeking method is an old-established method [11], it has again received attention in the literature of the control engineering after the stability analysis by Krstić and Wang [38]. The methods add dither signal to systems to approximate the gradients of the objective functions. Previous methods use the periodic signals to approximate the gradients as a dither signal. However, the estimation of the gradients might become difficult because the interaction between periodic signals when the number of optimization parameters becomes large.

In recent years, stochastic extremum seeking methods have been studied [40, 44]. The first study of a stochastic extremum seeking was related to discrete systems [44] and a discrete extremum seeking algorithm in stochastic environments was applied to mobile networks in other studies [65]. An extremum seeking algorithm for continuous time systems with stochastic dither signal is also proposed in [40]. The stochastic extremum seeking methods are expected to deal with the optimization with many optimization parameters. In addition to the motivation, another motivation is the use of stochastic noise associated with systems for extremum seeking algorithms. The use of noise associated with systems leads to the simplification of the implementation of extremum seeking algorithms.

We propose a stochastic extremum seeking method that can guarantee the convergence of the estimation variable of the optimum. In previous extremum seeking methods, the residuals remain even when the estimation variables approach the optimum sufficiently. In this study, we use a Wiener process to approximate the gradients of the objective function. Further, we introduce a state-dependent parameter into the update mechanism in the proposed method based on the stochastic Lyapunov stability analysis. Thus, the proposed method can guarantee the convergence of the estimation variable.

In the remainder of this chapter, followed by the problem setting of the optimization, the proposed method is shown. The three schemes are proposed, the basic scheme, the annealing scheme, and the high-pass filter scheme. The basic scheme is a fundamental scheme of other two schemes, and the high-pass filter scheme can guarantee the convergence. Then, Section 5.5 presents the convergence analysis based on the stochastic Lyapunov stability theory. Further, Section 5.6 extends the proposed method for the multivariate problems. Section 5.7 shows numerical examples, and Section 5.8 gives the conclusion.

5.2 Problem Setting

This chapter deal with the optimization problem. In the extremum seeking methods, the optimum value of a unknown objective function is estimated by adding the dither signal. We deal with single parameter problems in the following, and deal with multivariate problem in Section 5.6. Moreover, in the following, we deal with the optimization of static systems, while previous extremum seeking methods can deal with the optimization of dynamical systems.

Consider a parameter $\theta \in \mathbb{R}$, and an objective function $\varphi(\theta) : \mathbb{R} \to \mathbb{R}$. Suppose that the objective function has an optimum value θ^*, that is,

$$\theta^* = \max_{\theta \in \mathbb{R}} \varphi(\theta),$$

and we make the following assumption.

Assumption 5.1
The objective function $\varphi(\theta)$ is twice continuously differentiable, and moreover,

$$\frac{\partial \varphi(\theta^*)}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial \varphi(\theta)}{\partial \theta} \neq 0 \quad \text{for } \theta \neq \theta^* \tag{5.1}$$

hold.

The objective of extremum seeking methods is to estimate the optimum value θ^* by updating the estimation parameter $\hat{\theta}$ based on the approximation of the gradient of the objective function $\varphi(\theta)$.

Remark 5.1
These assumptions are natural, as required in hill-climbing-like methods. Although hill-climbing methods are used in optimization problems in discrete time, the proposed method is developed for problems in continuous time.

5.3 Previous Extremum Seeking Algorithms

This section reviews the previous extremum seeking methods for the comparison to the proposed method. Although these methods can be applied to dynamical systems, methods for the static systems are shown for the comparison to the proposed method.

A standard deterministic extremum seeking method [2] is summarized as follows. This method uses periodic signals to approximate the gradients of objective functions. In this method, an update law of the estimation variable $\hat{\theta}$ is given by

$$\frac{d\hat{\theta}}{dt} = K_1 \left(\varphi(\hat{\theta} + K_2 \sin(\omega t)) - P\right) \sin(\omega t),$$
$$\frac{dP}{dt} = -aP + a\varphi\left(\hat{\theta} + K_2 \sin(\omega t)\right),$$

where $K_1, K_2, a, \omega > 0$. The variable P acts as a variable of a high-pass filter in this update law, which is introduced to approximate the gradients more accurately.

A stochastic extremum seeking method [40] is described as follows. In contrast with the deterministic algorithms, this method uses a stochastic process to estimate the gradients of objective functions. The update law of the estimation variable in the stochastic extremum seeking method of [40] is given by

$$\frac{d\hat{\theta}}{dt} = K_1 \left(\varphi(\hat{\theta} + K_2 \sin(\eta)) - P\right) \sin(\eta),$$
$$\frac{dP}{dt} = -aP + a\varphi(\hat{\theta} + K_2 \sin(\eta)),$$

where $K_1, K_2, a > 0$, and P is a variable of a high-pass filter as with the above. The dither signal η is the Ornstein-Uhlenbeck process which is the solution of the equation

$$\epsilon d\eta = -\eta dt + \sqrt{\epsilon} q dW$$

with $\epsilon, q > 0$, and W is a one-dimensional standard Wiener process. In the following, an update mechanism is shown, which uses a state-dependent parameter in a high-pass filter to guarantee the convergence of the estimation variable, in contrast with the above methods.

5.4 Proposed Extremum Seeking Algorithms

This section presents there schemes, which are a basic scheme, an annealing parameter scheme, and a high-pass filter (HPF) scheme. Extremum seeking methods approximates the gradient of objective functions by using dither signal. To approximate the gradient, the proposed methods use Wiener processes. The basic scheme is a simple scheme to approximate the gradient, and is a basis of the other two schemes. Moreover, the annealing parameter scheme is developed by introducing an annealing parameter to the simple scheme in order to reduce the residuals of the estimation parameter. The HPF scheme can guarantee the convergence of a estimation parameter to the optimum value by using a high-pass filter with variable parameters.

5.4.1 Basic Scheme

The system of the basic scheme is shown in Figure 5.1. In this method, we use a one-dimensional standard Wiener process W to extract the gradient of the objective function. The update law of the estimation variable $\hat{\theta}$ is given by

$$d\hat{\theta} = K_1\varphi(\hat{\theta} + K_2 dW)dW, \tag{5.2}$$

where $K_1, K_2 > 0$ are design parameters, and W is a one-dimensional standard Wiener process.

The following indicates that the update law can extract the gradients of the objective functions. We obtain the Taylor expansion of (5.2) as

$$d\hat{\theta} = K_1\varphi(\hat{\theta})dW + K_1K_2 \left.\frac{\partial\varphi}{\partial\theta}\right|_{\theta=\hat{\theta}} (dW)^2 + \cdots . \tag{5.3}$$

Moreover, we have

$$d\hat{\theta} = K_1K_2 \left.\frac{\partial\varphi}{\partial\theta}\right|_{\theta=\hat{\theta}} dt + K_1\varphi(\hat{\theta})dW, \tag{5.4}$$

using Itô's rules ($(dW)^2 = dt$ and $(dW)(dt) = 0$). Thus, we have the stochastic differential equation (5.4) from the parameter update law.

Equation (5.4) implies the approximation of the gradient of the objective function $\varphi(\theta)$. The first term on the right hand side of (5.4), called the drift term, consists of the gradient the objective function. The second term, called the diffusion term, consists of the objective function. The drift term drives the estimation parameter $\hat{\theta}$ to the optimum. Thus, this implies that the estimation variable is updated based on the value of the gradient of the objective function.

Further, note that the parameter K_1 should be small and the parameter K_2 should be large. In (5.4), since the diffusion term determines the effect of noise, the update of the estimation variable is disturbed by its value. Thus, the parameters K_1, K_2 should be determined so that the diffusion term has little effect on the update law because the function $\varphi(\hat{\theta})$ is assumed to be unknown.

Although the estimation variable $\hat{\theta}$ does not converged to the optimum in general, we can obtain the probability density of $\hat{\theta}$ instead. The Fokker-Planck equation of (5.4) is given by

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial\theta}\left(K_1K_2\frac{\partial\varphi}{\partial\theta}p(t,\hat{\theta})\right) + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\left(\left(K_1\varphi(\hat{\theta})\right)^2 p(t,\hat{\theta})\right). \tag{5.5}$$

Remark 5.2

The update law (5.2) is considered as the limit of the following discrete time system. For an interval of time $[a, b]$ and its partition $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} = b$, the parameter update law in the discrete time system is given by

$$\Delta\hat{\theta}_i = K_1\varphi(\hat{\theta}_i + K_2\Delta W_i)\Delta W_i, \tag{5.6}$$

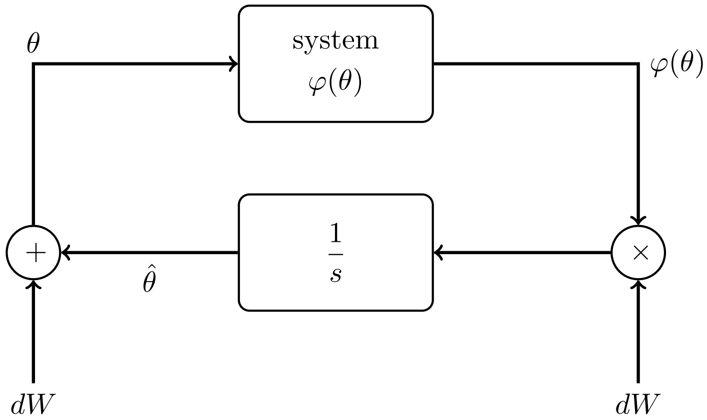

Fig. 5.1: Basic Scheme of Extremum Seeking Method

where $\Delta\hat{\theta}_i = \hat{\theta}(t_{i+1}) - \hat{\theta}(t_i)$ and $\Delta W_i = W(t_{i+1}) - W(t_i)$. We see that the parameter update law (5.2) is the limit of (5.6) as the limit $n \to \infty$ so that $\max_i(t_{i+1} - t_i) \to 0$.

5.4.2 Annealing Parameter Scheme

We developed an annealing parameter scheme to reduce fluctuations in the basic scheme. The parameter K_1 in (5.2), which was a constant, is the time-varying parameter decreasing with time in the annealing parameter scheme. Decreasing the value of K_1 makes the effect of noise small in time in the update law.

Figure 5.2 shows the system of the annealing parameter scheme. The parameter update law of the estimation parameter $\hat{\theta}$ in the annealing parameter scheme is given by

$$d\hat{\theta} = K_1\varphi(\hat{\theta} + K_2 dW)dW, \tag{5.7}$$

$$\frac{dK_1}{dt} = -\epsilon K_1, \tag{5.8}$$

where K_2, ϵ are the constant parameters. The parameter ϵ determines the attenuation rate of the value K_1, and it should be much smaller than K_1, K_2 to avoid the early suspension of the estimation variable. However, since the function $\varphi(\theta)$ is unknown, we do not have an adequate criteria to determine the value of ϵ.

5.4.3 High-Pass Filter Scheme

Extremum seeking methods often use a high-pass filter for the better approximation of the gradient of the objective function, because the high-pass filter washes out the DC component of the output. Although the high-pass filter leads the better approximation, the previous methods often show residuals after the sufficient estimation time has passed.

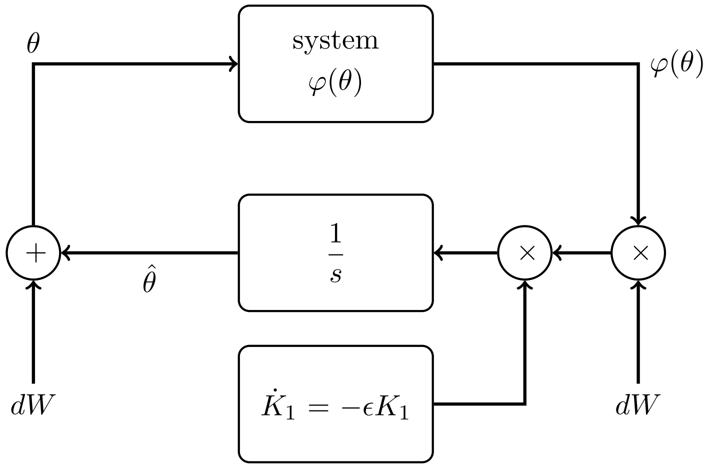

Fig. 5.2: Annealing Scheme of Extremum Seeking Method

The HPF shceme in this study is proposed to guarantee the convergence of the estimation parameter $\hat{\theta}$, introducing a state-dependent parameter in a high-pass filter.

We make the following assumption to ensure the convergence.

Assumption 5.2
There exists a constant $M > 0$ such that

$$\left|\frac{\partial^2\varphi}{\partial\theta^2}(\theta)\right| \leq M \text{ for } \forall\theta \in \mathbb{R}. \tag{5.9}$$

Furthermore, there exist positive constants α, β such that

$$-\alpha(\theta-\theta^*)^2 < \varphi(\theta) - \varphi(\theta^*) < -\beta(\theta-\theta^*)^2 \text{ for } \forall\theta \in \mathbb{R}\setminus\{0\} \tag{5.10}$$

We assume that the constants M, α, and β are obtained a priori, and that we know the upper/lower bounds of θ^* and $\varphi(\theta^*)$, i.e.,

$$\hat{\theta}_l \leq \theta^* \leq \hat{\theta}_u, \quad P_l \leq \varphi(\theta^*) \leq P_u. \tag{5.11}$$

We have

$$\left|\frac{\partial\varphi}{\partial\theta}(\theta)\right| \leq M\,|\theta-\theta^*| \tag{5.12}$$

from Assumptions 5.1 and 5.2.

The proposed HPF scheme is presented below. Figure 5.3 shows the system of the HPF scheme. The parameter update law of the HPF scheme is given by

$$d\hat{\theta} = K_1(\varphi(\hat{\theta}+K_2dW) - P)dW \tag{5.13}$$

$$dP = -a(\hat{\theta},P)Pdt + a(\hat{\theta},P)\varphi(\hat{\theta}+K_2dW)dt, \tag{5.14}$$

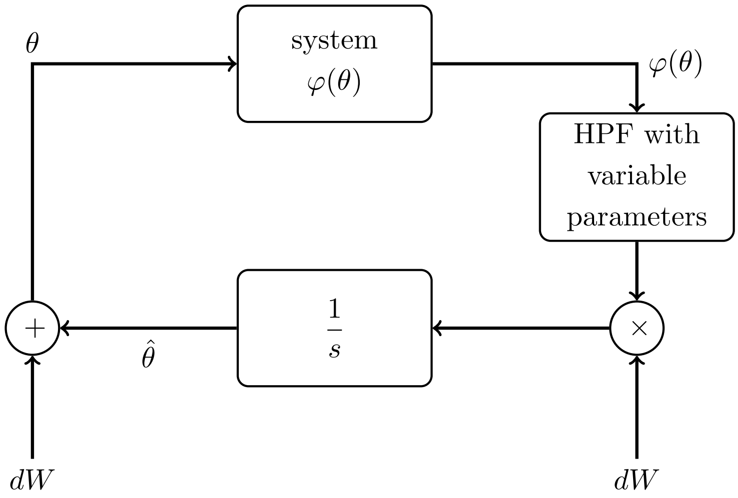

Fig. 5.3: HPF Scheme of Extremum Seeking Method

where P is the HPF variable and the function $a(\hat{\theta}, P)$ is the state-dependent coefficient of the HPF. The function $a(\hat{\theta}, P)$ is given by

$$a(\hat{\theta}, P) = a_0 + \frac{K_1^2}{2}\left[\left\{\frac{K_1 K_2 \eta^3}{2a_0\xi}(\rho+\eta)+\eta^2\right\} f(\hat{\theta}) + \eta g(\hat{\theta}, P)\right], \tag{5.15}$$

where a_0 is an arbitrary positive constant, η, ρ, and ξ are constant parameters such that $\rho \geq \alpha$, $\eta \geq M$, and $\xi \leq \beta$, and the functions f and g are defined as

$$f(\hat{\theta}) = 2\left(\hat{\theta} - \frac{\hat{\theta}_u + \hat{\theta}_l}{2}\right)^2 + 2\left(\frac{\hat{\theta}_u - \hat{\theta}_l}{2}\right)^2 + 1 \tag{5.16}$$

$$g(\hat{\theta}, P) = \frac{\left(P - \frac{\varphi_u(\hat{\theta})+\varphi_l(\hat{\theta})}{2}\right)^2}{\sqrt{1+\left(P - \frac{\varphi_u(\hat{\theta})+\varphi_l(\hat{\theta})}{2}\right)^2}} + \frac{\varphi_u(\hat{\theta}) - \varphi_l(\hat{\theta})}{2} + 1. \tag{5.17}$$

The functions $\varphi_l(\hat{\theta})$ and $\varphi_u(\hat{\theta})$ in (5.17) are given by

$$\begin{aligned} \varphi_l(\hat{\theta}) &= -\rho f(\hat{\theta}) + P_l, \\ \varphi_u(\hat{\theta}) &= -\xi h(\hat{\theta}) + P_u, \end{aligned} \tag{5.18}$$

where the function $h(\hat{\theta})$ is defined as

$$h(\hat{\theta}) = \frac{1}{2}\left(\hat{\theta} - \frac{\hat{\theta}_u + \hat{\theta}_l}{2}\right)^2 - \left(\frac{\hat{\theta}_u - \hat{\theta}_l}{2}\right)^2 - 1. \tag{5.19}$$

By using a state-dependent coefficient in the HPF, the HPF scheme can guarantee the convergence because the the cutoff frequency of the HPF is automatically tuned.

The criteria in the determination of the free parameters are stated as follows. The sufficiently large value of the parameter η and ρ and the sufficiently small values of the parameter ξ can guarantee the convergence if the tight bounds of M, α, and β are unknown. Similarly, the sufficiently wide ranges of $\hat{\theta}_l$, $\hat{\theta}_u$ and P_l, P_u will guarantee the convergence even if the tight upper/lower bounds of θ and $\varphi(\theta^*)$ are unknown.

The functions in (5.15)- (5.19) are determined from the Lyapunov stability analysis. The next section shows the analysis.

5.5 Convergence Analysis of Proposed Scheme

This section presents the proof of the convergence of the proposed HPF scheme by using the stochastic Lyapunov stability theory.

We first derive the error system as the difference between $(\hat{\theta}, P)$ and $(\theta^*, \varphi(\hat{\theta}))$. Consider the error variables given by

$$\tilde{\theta} = \hat{\theta} - \theta^*, \tag{5.20}$$

$$\tilde{P} = P - \varphi(\hat{\theta}). \tag{5.21}$$

By expanding (5.13) and (5.14) using Itô's rule, we obtain

$$d\hat{\theta} = K_1 K_2 \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} dt + K_1(\varphi(\hat{\theta}) - P)dW, \tag{5.22}$$

$$dP = -a(\hat{\theta}, P)\left(P - \varphi(\hat{\theta})\right) dt. \tag{5.23}$$

Then, by using the Itô rule again, the error dynamics is obtained as

$$d\tilde{\theta} = K_1 K_2 \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} dt - K_1 \tilde{P} dW, \tag{5.24}$$

$$d\tilde{P} = -\left(a(\hat{\theta}, P)\tilde{P} + K_1 K_2 \left(\frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}}\right)^2 + \frac{1}{2} K_1^2 \frac{\partial^2 \varphi(\hat{\theta})}{\partial \hat{\theta}^2} \tilde{P}^2\right) dt + K_1 \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \tilde{P} dW. \tag{5.25}$$

Then, the convergence result is shown.

Theorem 5.1
If Assumptions 5.1 and 5.2 hold, then the origin of the system defined by (5.24), (5.25) with (5.15) is globally asymptotically stable in probability.

In the proof of Theorem 5.1, we use the following lemma.

Lemma 5.1
Under Assumption 5.2, both

$$f(\hat{\theta}) > \tilde{\theta}^2, \tag{5.26}$$

$$g(\hat{\theta}, P) > \tilde{P} \tag{5.27}$$

hold.

This lemma is proved in Appendix B.

Now, we prove Theorem 5.1.

Proof

Consider $a_1(\hat{\theta}, P) = a(\hat{\theta}, P) - a_0$, and it satisfies

$$\begin{aligned} a_1(\hat{\theta}, P) &:= a(\hat{\theta}, P) - a_0 \\ &> \frac{K_1^2}{2}\left[\left\{\frac{K_1K_2M^3}{2a_0\beta}(\alpha+M) + M^2\right\} f(\hat{\theta}) + Mg(\hat{\theta}, P)\right]. \end{aligned} \tag{5.28}$$

Then, a Lyapunov function candidate is given as

$$V(\tilde{\theta}, \tilde{P}) = \Big(\varphi(\tilde{\theta}+\theta^*) - \varphi(\theta^*)\Big)^2 + \lambda\tilde{P}^2, \tag{5.29}$$

where $\lambda = (2a_0\beta)/(K_1K_2M^2)$. The infinitesimal generator of the error dynamics given by (5.24) and (5.25) for the Lyapunov function candidate $V(\tilde{\theta}, \tilde{P})$ becomes

$$\begin{aligned} \mathcal{L}V(\tilde{\theta}, \tilde{P}) &= 2K_1K_2\left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta})\right)^2(\varphi(\tilde{\theta}+\theta^*) - \varphi(\theta^*)) \\ &\quad + K_1^2\tilde{P}^2\left\{\left(\frac{\partial^2\varphi}{\partial\hat{\theta}^2}(\hat{\theta})\right)\Big(\varphi(\tilde{\theta}+\theta^*) - \varphi(\theta^*)\Big) + \left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta})\right)^2\right\} \\ &\quad - 2\lambda a(\hat{\theta}, P)\tilde{P}^2 - 2\lambda K_1K_2\left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta})\right)^2\tilde{P} - \lambda K_1^2\frac{\partial^2\varphi}{\partial\hat{\theta}^2}(\hat{\theta})\tilde{P}^3 \\ &\quad + \lambda K_1^2\left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta})\right)^2\tilde{P}^2 \\ &= 2K_1K_2\left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta})\right)^2(\varphi(\tilde{\theta}+\theta^*) - \varphi(\theta^*)) + \frac{\lambda K_1^2K_2^2}{2a_0}\left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta})\right)^4 \\ &\quad - 2\lambda a_0\left(\tilde{P} + \frac{K_1K_2}{2a_0}\left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta})\right)^2\right)^2 - 2\lambda a_1(\hat{\theta}, P)\tilde{P}^2 \\ &\quad + K_1^2\tilde{P}^2\left\{\left(\frac{\partial^2\varphi}{\partial\hat{\theta}^2}(\hat{\theta})\right)\Big(\varphi(\tilde{\theta}+\theta^*) - \varphi(\theta^*)\Big) + \left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta})\right)^2\right\} \\ &\quad - \lambda K_1^2\frac{\partial^2\varphi}{\partial\hat{\theta}^2}(\hat{\theta})\tilde{P}^3 + \lambda K_1^2\left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta})\right)^2\tilde{P}^2. \end{aligned} \tag{5.30}$$

Since $\varphi(\tilde{\theta}+\theta^*) < \varphi(\theta^*)$ for $\tilde{\theta} \neq 0$, the first term on the right-hand side of (5.30) is negative definite with respect to $\tilde{\theta}$. Then, the sum of the first two terms on the right-hand side of (5.30) can be evaluated as

$$\begin{aligned} &2K_1K_2\left(\frac{\partial\varphi(\hat{\theta})}{\partial\hat{\theta}}\right)^2(\varphi(\tilde{\theta}+\theta^*) - \varphi(\theta^*)) + \frac{\lambda K_1^2K_2^2}{2a_0}\left(\frac{\partial\varphi(\hat{\theta})}{\partial\hat{\theta}}\right)^4 \\ &= 2K_1K_2\left(\frac{\partial\varphi(\hat{\theta})}{\partial\hat{\theta}}\right)^2\left\{(\varphi(\tilde{\theta}+\theta^*) - \varphi(\theta^*)) + \frac{\beta}{2M^2}\left(\frac{\partial\varphi(\hat{\theta})}{\partial\hat{\theta}}\right)^2\right\} \\ &< 0 \text{ for } \hat{\theta} \neq \theta^* \end{aligned} \tag{5.31}$$

by (5.10), (5.12), and $\lambda = (2a_0\beta)/(K_1K_2M^2)$.

For the third term in the right-hand side of (5.30), it is obvious that

$$-2\lambda a_0 \left(\tilde{P} + \frac{K_1K_2}{2a_0} \left(\frac{\partial\varphi(\hat{\theta})}{\partial\hat{\theta}} \right)^2 \right)^2 \le 0. \tag{5.32}$$

Finally, for the last four terms in the right-hand side of (5.30), we show that the term $-2\lambda a_1(\hat{\theta}, P)\tilde{P}^2$ cancels the last three terms. By (5.9), (5.10), (5.26), and (5.27), the last four terms are evaluated as

$$\begin{aligned}
&- 2\lambda a_1(\hat{\theta}, P)\tilde{P}^2 + K_1^2\tilde{P}^2 \left\{ \left(\frac{\partial^2\varphi}{\partial\hat{\theta}^2}(\hat{\theta}) \right) \left(\varphi(\tilde{\theta} + \theta^*) - \varphi(\theta^*) \right) + \left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta}) \right)^2 \right\} \\
&\quad - \lambda K_1^2 \frac{\partial^2\varphi}{\partial\hat{\theta}^2}(\hat{\theta})\tilde{P}^3 + \lambda K_1^2 \left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta}) \right)^2 \tilde{P}^2 \\
&< K_1^2 \left[- \left(M\alpha + M^2 \right) f(\hat{\theta}) + \left(\frac{\partial^2\varphi}{\partial\hat{\theta}^2}(\hat{\theta}) \right) \left(\varphi(\tilde{\theta} + \theta^*) - \varphi(\theta^*) \right) + \left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta}) \right)^2 \right. \\
&\quad \left. + \lambda \left\{ -M^2 f(\hat{\theta}) + \left(\frac{\partial\varphi}{\partial\hat{\theta}}(\hat{\theta}) \right)^2 - Mg(\hat{\theta}, P) - \frac{\partial^2\varphi}{\partial\hat{\theta}^2}(\hat{\theta})\tilde{P} \right\} \right] \tilde{P}^2 \\
&< 0 \quad \text{for } \tilde{P} \neq 0.
\end{aligned} \tag{5.33}$$

Thus, the last four terms in the right-hand side of (5.30) are semi-negative definite. Consequently, we can conclude that

$$\mathcal{L}V < 0 \quad \text{for } (\tilde{\theta}, P) \neq 0. \tag{5.34}$$

from (5.30), (5.31), (5.32), and (5.33). Thus, the origin of the system is globally asymptotically stable in probability.

This theorem implies the convergence of $\hat{\theta}$ and P in (5.13) and (5.14) to the optimum value θ^* and the value of $\varphi(\hat{\theta})$.

5.6 Proposed Method for Multivariate Problem

This section shows a generalization of the proposed stochastic extremum seeking algorithms to the multivariate problem. One of the motivations of the study on stochastic extremum seeking algorithms is to avoid the difficulty of deterministic algorithms in multivariate optimization problems, which comes from the interaction between dither signals. Stochastic extremum seeking algorithms are expected to overcome the problem. In this section, a multivariate HPF scheme and the convergence analysis are shown.

The problem setting of the multivariate problem is stated as follows. Let $\theta \in \mathbb{R}^n$ denote the parameter, and let $\varphi(\theta) : \mathbb{R}^n \to \mathbb{R}$ denote the objective function. As seen in the previous section, the stochastic extremum seeking algorithms estimate the optimum of the objective function. In this section, the following assumptions are made.

Assumption 5.3
There exists a unique variable θ^* such that

$$\theta^* = \max_{\theta \in \mathbb{R}^n} \varphi(\theta).$$

Moreover,

$$\frac{\partial \varphi(\theta^*)}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial \varphi(\theta)}{\partial \theta} \neq 0 \text{ for } \theta \neq \theta^* \tag{5.35}$$

hold.

In the following, the norm of a matrix is defined as its maximum singular value.

Assumption 5.4
There exists a supremum of the norm of the Hessian matrix of $\varphi(\theta)$, M, that is,

$$\left\| \frac{\partial}{\partial \theta} \left(\frac{\partial \varphi(\theta)}{\partial \theta} \right)^T \right\| \leq M. \tag{5.36}$$

Moreover, the function $\varphi(\theta)$ satisfies

$$-\alpha \|\theta - \theta^*\|^2 < \varphi(\theta) - \varphi(\theta^*) < -\beta \|\theta - \theta^*\|^2 \quad \text{for } \theta \neq \theta^*. \tag{5.37}$$

Assume that we know the upper/lower bounds of $\varphi(\theta^*)$, P_l and P_u, satisfying

$$P_l \leq \varphi(\theta^*) \leq P_u. \tag{5.38}$$

and that we know the range of θ^*, that is, we know the center $c \in \mathbb{R}^n$ and the radius $r \in \mathbb{R}$ of the pre-estimated range of θ^* satisfying

$$\|\theta^* - c\| \leq r. \tag{5.39}$$

We can obtain that

$$\left\| \frac{\partial \varphi}{\partial \theta}(\theta) \right\| \leq M \|\theta - \theta^*\| \tag{5.40}$$

from Assumption 5.3 and (5.36),

Denoting $\hat{\theta} \in \mathbb{R}^n$ as the estimation variable for θ^*, the parameter update law for multivariate problems is given by

$$d\hat{\theta} = K_1 \left(\varphi \left(\hat{\theta} + K_2 dW \right) - P \right) dW, \tag{5.41}$$

$$dP = -a(\hat{\theta}, P) \left(P - \varphi \left(\hat{\theta} + K_2 dW \right) \right) dt, \tag{5.42}$$

where K_1, K_2 are constant parameters, $P \in \mathbb{R}$ is the HPF variable, and W is the n-dimensional standard Wiener process whose components are mutually independent. As in the one-dimensional case, the state-dependent parameter a in the HPF scheme for multivariate problems is defined as

$$a(\hat{\theta}, P) = a_0 + \frac{K_1^2}{2}\left[\left\{\frac{K_1 K_2 \eta^3}{2a_0\xi}(n\rho+\eta)+\eta^2\right\}p(\hat{\theta})+n\eta g(\hat{\theta}, P)\right], \tag{5.43}$$

where a_0 is an arbitrary positive parameter, and η, ρ, and ξ are constant parameters satisfying $\eta \geq M$, $\rho \geq \alpha$, and $\xi \leq \beta$. The function $p(\hat{\theta})$ is given by

$$p(\hat{\theta}) = 2\left\|\hat{\theta} - c\right\|^2 + 2r^2 + 1, \tag{5.44}$$

and $g(\hat{\theta}, P)$ is similar to the single-variable case of (5.17) except that

$$\begin{aligned} \varphi_l(\hat{\theta}) &= -\rho p(\hat{\theta}) + P_l, \\ \varphi_u(\hat{\theta}) &= -\xi q(\hat{\theta}) + P_u, \end{aligned} \tag{5.45}$$

where

$$q(\hat{\theta}) = \frac{1}{2}\left\|\hat{\theta} - c\right\|^2 - r^2 - 1. \tag{5.46}$$

Then, the convergence analysis is presented. Define the error variables as

$$\tilde{\theta} = \hat{\theta} - \theta^*, \tag{5.47}$$

$$\tilde{P} = P - \varphi(\hat{\theta}). \tag{5.48}$$

From (5.41) and (5.42), the update law is converted to

$$d\hat{\theta} = K_1 K_2 \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} dt + K_1\left(\varphi(\hat{\theta}) - P\right) dW, \tag{5.49}$$

$$dP = -a(\hat{\theta}, P)\left(P - \varphi(\hat{\theta})\right) dt. \tag{5.50}$$

These are obtained by taking the Taylor expansion around $\hat{\theta}$ and using Itô's rule, $(dW_i)^2 = dt, (dW_i)dt = 0, dW_i dW_j = 0$. Thus, the error dynamics is obtained as

$$d\tilde{\theta} = K_1 K_2 \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} dt - K_1 \tilde{P} dW \tag{5.51}$$

$$d\tilde{P} = \left\{-a(\hat{\theta}, P)\tilde{P} - \sum_i \left(K_1 K_2 \left(\frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}_i}\right)^2 + \frac{K_1^2 \tilde{P}^2}{2}\frac{\partial^2 \varphi(\hat{\theta})}{\partial \hat{\theta}_i^2}\right)\right\} dt + \sum_i K_1 \tilde{P} \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}_i} dW_i, \tag{5.52}$$

by again using Itô's formula.

Then, the convergence result is stated as follows.

Theorem 5.2
If Assumptions 5.3 and 5.4 hold, then the origin of the system given by (5.51), (5.52) with (5.43) is globally asymptotically stable in probability.

Proof

Consider a Lyapunov function candidate $V(\tilde{\theta}, \tilde{P})$ given by

$$V(\tilde{\theta}, \tilde{P}) = \Big(\varphi(\tilde{\theta} + \theta^*) - \varphi(\theta^*)\Big)^2 + \lambda \tilde{P}^2, \tag{5.53}$$

where $\lambda = (2a_0\beta)/(K_1K_2M^2)$. The infinitesimal generator of the error dynamics for the Lyapunov function candidate $V(\tilde{\theta}, \tilde{P})$ is calculated as

$$\begin{aligned}
\mathcal{L}V(\tilde{\theta}, \tilde{P}) = {} & 2K_1K_2 \left\| \frac{\partial \varphi}{\partial \hat{\theta}}(\hat{\theta}) \right\|^2 (\varphi(\tilde{\theta} + \theta^*) - \varphi(\theta^*)) \\
& + K_1^2 \left\{ \Big(\varphi(\tilde{\theta} + \theta^*) - \varphi(\theta^*)\Big) \sum_i^n \frac{\partial^2 \varphi(\hat{\theta})}{\partial \hat{\theta}_i^2} + \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^2 \right\} \tilde{P}^2 \\
& - 2\lambda a(\hat{\theta}, P)\tilde{P}^2 - 2\lambda K_1K_2 \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^2 \tilde{P} - \lambda K_1^2 \tilde{P}^3 \sum_i \frac{\partial^2 \varphi(\hat{\theta})}{\partial \hat{\theta}_i^2} + \lambda K_1^2 \tilde{P}^2 \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^2 \\
= {} & 2K_1K_2 \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^2 (\varphi(\tilde{\theta}+\theta^*)-\varphi(\theta^*)) + \frac{\lambda K_1^2 K_2^2}{2a_0} \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^4 \\
& - 2\lambda a_0 \left(\tilde{P} - \frac{K_1K_2}{2a_0} \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^2 \right)^2 - 2\lambda a_1(\hat{\theta}, P)\tilde{P}^2 \\
& - \lambda K_1^2 \tilde{P}^3 \sum_i \frac{\partial^2 \varphi(\hat{\theta})}{\partial \hat{\theta}_i^2} + (\lambda + 1) K_1^2 \tilde{P}^2 \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^2 \\
& + K_1^2 \Big(\varphi(\tilde{\theta}+\theta^*)-\varphi(\theta^*)\Big) \tilde{P}^2 \sum_i^n \frac{\partial^2 \varphi(\hat{\theta})}{\partial \hat{\theta}_i^2},
\end{aligned} \tag{5.54}$$

where $a_1(\hat{\theta}, P) = a(\hat{\theta}, P) - a_0$. Since the inequality

$$-2K_1K_2 \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^2 (\varphi(\tilde{\theta} + \theta^*) - \varphi(\theta^*)) > \lambda \frac{K_1^2 K_2^2}{2a_0} \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^4 \tag{5.55}$$

holds for $\hat{\theta} \neq \theta^*$ according to (5.37), (5.40), and the definition of λ, the inequality with respect to the first two terms of the right-hand side of (5.54)

$$\begin{aligned}
& 2K_1K_2 \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^2 (\varphi(\tilde{\theta} + \theta^*) - \varphi(\theta^*)) + \lambda \frac{K_1^2 K_2^2}{2a_0} \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}} \right\|^4 \\
& < 0 \quad \text{for } \theta \neq \theta^*.
\end{aligned} \tag{5.56}$$

holds. This implies negative definiteness of the first two terms of the right-hand side of (5.54) with respect to $\hat{\theta}$.

We have the inequalities

$$\begin{aligned}
& K_1^2 \left\{ \Big(\varphi(\tilde{\theta}+\theta^*)-\varphi(\theta^*)\Big) \sum_i^n \frac{\partial^2 \varphi(\hat{\theta})}{\partial \hat{\theta}_i^2} + (\lambda + 1) \left\| \frac{\partial \varphi(\hat{\theta})}{\partial \theta} \right\|^2 \right\} \\
& < \lambda K_1^2 \left\{ \frac{K_1K_2M^3}{2a_0\beta} (n\alpha + M) + M^2 \right\} p(\hat{\theta})
\end{aligned} \tag{5.57}$$

and

$$\lambda K_1^2 \tilde{P} \sum_i \frac{\partial^2 \varphi(\hat{\theta})}{\partial \hat{\theta}_i^2} \le \lambda K_1^2 n M g(\hat{\theta}, P), \tag{5.58}$$

according to (5.36), (5.37), and the definition of $p(\hat{\theta})$. According to the definition of $a(\hat{\theta}, P)$, $a_1(\hat{\theta}, P)$ satisfies

$$a_1(\hat{\theta}, P) \ge \frac{K_1^2}{2} \left[\left\{ \frac{K_1 K_2 M^3}{2a_0\beta}(n\alpha + M) + M^2 \right\} p(\hat{\theta}) + nMg(\hat{\theta}, P) \right]. \tag{5.59}$$

Thus, by (5.54), (5.56), (5.57), (5.58), $\mathcal{L}V(\tilde{\theta}, \tilde{P})$ is negative definite with respect to $(\tilde{\theta}, \tilde{P})$. Therefore, we can conclude that the origin of (5.51), (5.52) is globally asymptotically stable in probability. This completes the proof.

5.7 Numerical Examples

The numerical examples of the proposed method are shown. The basic scheme and the annealing scheme show that we can obtain the approximate value of the optimum of a given objective function. Further, the HPF scheme shows the convergence of the estimation variable to the optimum solution. In addition to the single parameter optimization problem, the multivariate optimization problem is shown.

5.7.1 Simulation Settings

In the next three subsections, we deal with a maximization problem of the objective function

$$\varphi(\theta) = -2(\theta - 0.5)^2 + 5(\cos(\theta - 0.5) - 1) + 1. \tag{5.60}$$

This objective function has a maximum value of $\varphi(\theta^*) = 1$ at $\theta^* = 0.5$.

5.7.2 Results Using the Basic Scheme

First, the basic scheme is applied to this example problem.

Ten sample solutions of the estimation variable by the update law (5.2) are shown in Fig. 5.4. The values of free parameters, $K_1 = 0.01, K_2 = 2.0$, are used in the simulation. And the initial value of the estimation variable is set as $\hat{\theta} = 0.35$. Figure 5.5 shows the value of $\varphi(\hat{\theta})$ for a sample path. In Fig. 5.4, the solutions of the estimation parameter $\hat{\theta}$ approach the optimum value $\theta^* = 0.5$. However, some fluctuations remain because the effect of dither signals does not vanish at the optimum. The annealing parameter scheme and the HPF scheme are proposed to reduce such fluctuation.

We can estimate the probability density by using (5.5). Figure 5.6 shows the temporal transition of the probability density under the same settings with the initial density

$p(0, \hat{\theta}) = \delta(0.35)$, where δ denotes the Dirac delta function. The probability density becomes steady state whose center is located at the optimum value $\theta^* = 0.5$. This corresponds to the result in Fig. 5.4.

5.7.3 Results Using the Annealing Scheme

This subsection shows the numerical example of the annealing scheme. The annealing scheme is developed to reduce the residual between the estimation variable and the optimum value.

Figure 5.7 shows ten sample solutions obtained by using the annealing scheme. The initial values of $\hat{\theta}$ and K_1 are set as 0.35, and 0.01, respectively. The values of the free parameters are $K_2 = 2.0$ and $\epsilon = 0.05$. Figure 5.8 shows the time response of $\varphi(\hat{\theta})$ for a solution.

Figure 5.7 shows that each estimation parameter approaches the optimum value. They have less residual between the solution and the optimum value than in the basic scheme. However, in general, the problem of early convergence might happen in the optimization process. That is, when the annealing parameter becomes too small for the solution $\hat{\theta}$ to converge to the optimal point in the early stage of the estimation, the update law stops the estimation. Some sample solutions shows the early convergence in Fig. 5.7. Despite this disadvantage, the introduction of annealing parameters is a convenient way to reduce the residual error.

5.7.4 Results Using the HPF Scheme

The HPF scheme guarantees the convergence of the estimation variable, unlike the other schemes. Figure 5.9 shows ten sample solutions of $\hat{\theta}$ whose initial values are $\theta = 0.35$. Figures 5.10 and 5.11 also give the evolution of $\varphi(\hat{\theta})$ and P where the initial value of P is -0.5. The free parameters are set to $K_1 = 0.01$, $K_2 = 2.0$, $a_0 = 0.5$, $\rho = 5.0$, $\eta = 20.0$, $\xi = 0.5$, $\hat{\theta}_l = 0.0$, $\hat{\theta}_u = 2.0$, $P_l = 0.0$, and $P_u = 2.0$. Figure 5.9 shows the convergence of the estimation variable $\hat{\theta}$ to the optimum value $\theta^* = 0.5$.

5.7.5 Results for the Multivariate Problem

This subsection shows the results of the numerical example of the multivariate problem. A two-parameter problem is considered, and the objective function is given by

$$\varphi(\theta) = -0.5((\theta_1 - 0.1)^2 + \theta_2^2) + 2.5\,(\cos((\theta_1 - 0.1) + \theta_2) - 1) + 1,$$

where $\theta = (\theta_1, \theta_2)$ denotes the parameter vector. The objective function has the a maximum value at a unique optimal point $\theta^* = (0.1, 0)$. A solution obtained by the proposed method is shown in the trajectory of a solution from $t = 0$ to $t = 100$ in Fig. 5.12. In the simulation, the initial values of our variables are $\hat{\theta} = (0.4, 0.6)$ and

$P = 0$, and the design parameters are chosen as $K_1 = 0.01$, $K_2 = 2.0$, $a_0 = 0.5$, $\rho = 5.0$, $\eta = 4.0$, $\xi = 0.1$, $c = (0.5, 0.5)$, $r = 4.0$, $P_l = 0.0$, and $P_u = 2.0$. We can see that the solution converges to the optimum value in the case of the multivariable optimization problem.

5.8 Summary

This section presented the extremum seeking algorithms that include the basic scheme, the annealing scheme, and the high-pass filter scheme. The basic scheme can estimate the gradient of a given objective function which is unknown to the optimizer. The annealing scheme is proposed to reduce the fluctuations in the basic scheme. In the proposed algorithms, the HPF scheme can guarantee the convergence of the estimation variable. In the HPF scheme, we incorporate the state-dependent parameter in the high-pass filter, which is determined from the Lyapunov stability analysis unlike the previous extremum seeking methods. Previous extremum seeking methods can obtain the approximate optimum value of a given objective function. However, the estimation variable does not converge to the optimum. On the other hand, the proposed HPF scheme can guarantee the convergence of the estimation variable. Moreover, this section extended the proposed HPF scheme to multivariate problems. As in the case of the single-variable problems, the convergence of the estimation parameters is guaranteed.

The future work includes the investigation of the determination method of the free parameters, the extension to dynamical systems, and the extension to the optimization of multimodal objective functions. Since the proposed HPF scheme has a lot of free parameters, it might require the adjustment of these parameters. Thus, we need the more detailed criteria in the determination of the free parameters. Moreover, since other extremum seeking methods can be applied to the dynamical systems, another future work is to develop a method that can make the estimation variable to converge to the optimum in the optimization in dynamical systems by using noise for the dither signal. Since the original motivation of this study is to develop the extremum seeking method for the global optimization of the multimodal objective function as in an extremum seeking method [68], we will address to develop the stochastic extremum seeking method such that the method uses the stochastic effect to obtain the global optimum as seen in the simulated annealing.

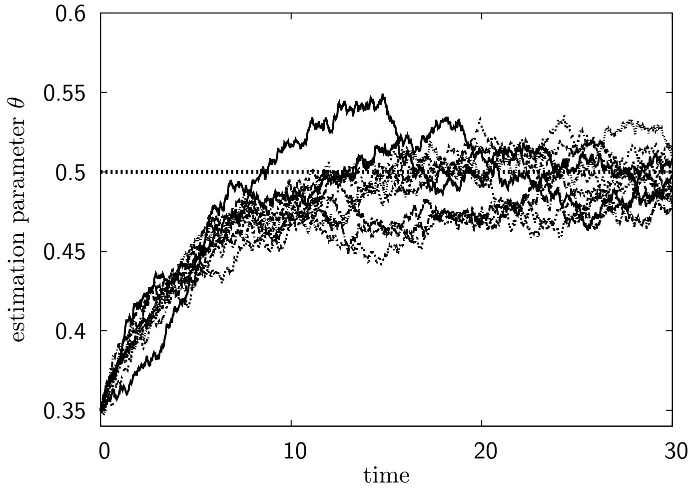

Fig. 5.4: Values of $\hat{\theta}$ in the proposed extremum-seeking scheme

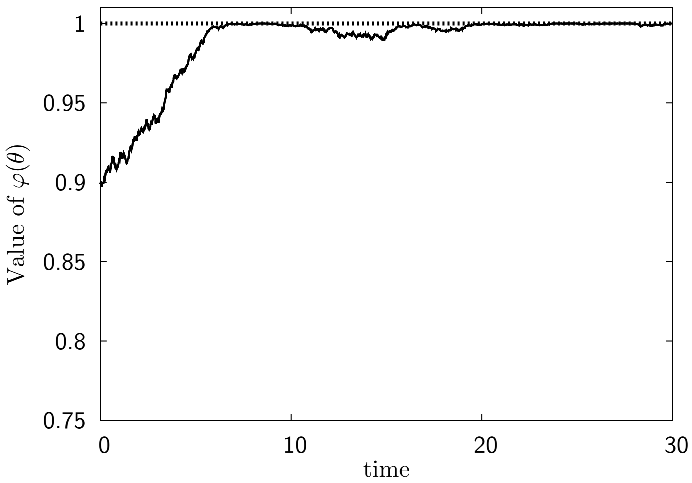

Fig. 5.5: Value of $\varphi(\hat{\theta})$ in the proposed extremum-seeking scheme

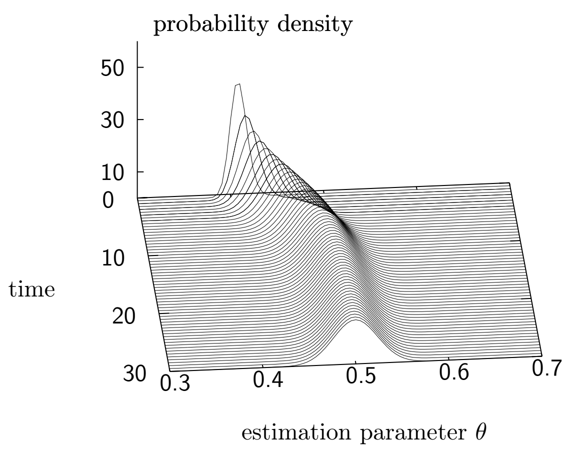

Fig. 5.6: Time evolution of the probability density function $p(t, \hat{\theta})$

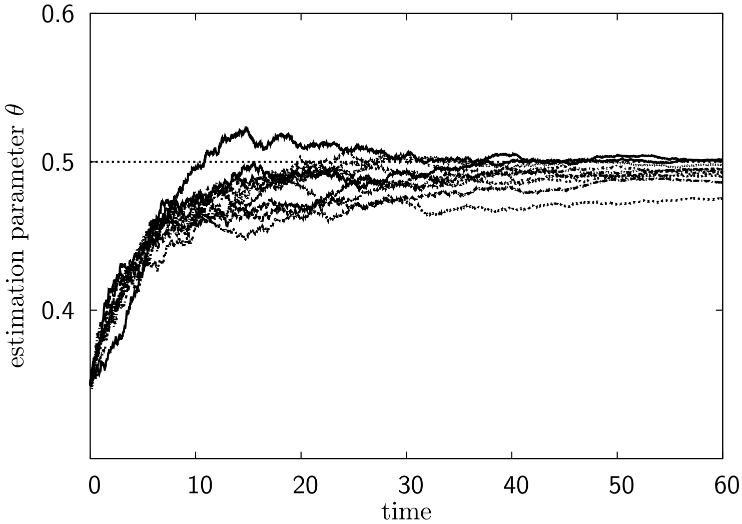

Fig. 5.7: Values of $\hat{\theta}$ in the proposed annealing parameter scheme

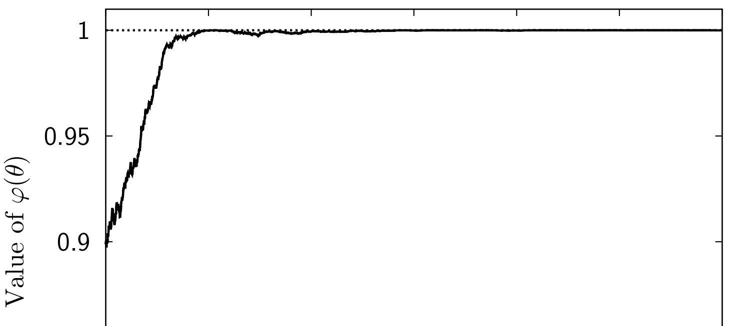

Fig. 5.8: Value of $\varphi(\hat{\theta})$ in the proposed annealing parameter scheme

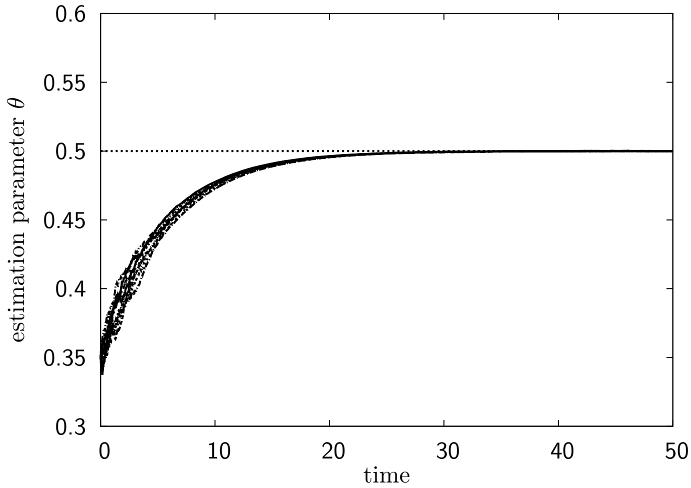

Fig. 5.9: Values of $\hat{\theta}$ in the proposed HPF method

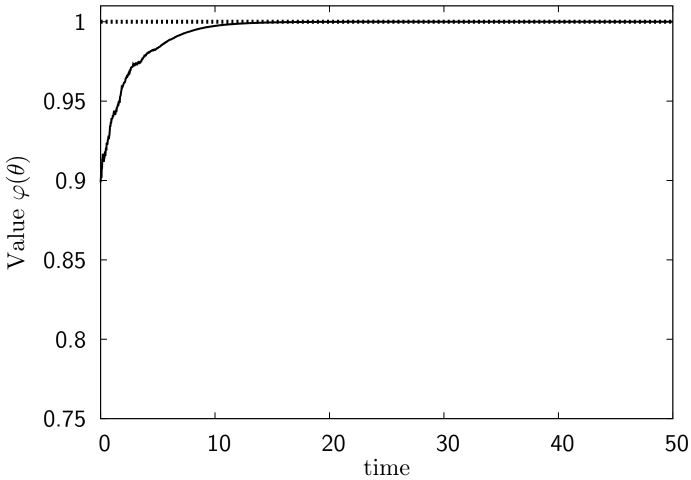

Fig. 5.10: Value of $\varphi(\hat{\theta})$ in the proposed HPF method

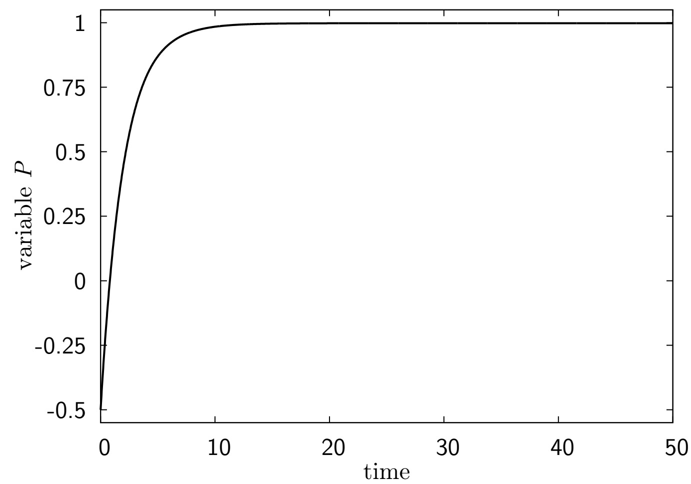

Fig. 5.11: Value of P in the proposed HPF method

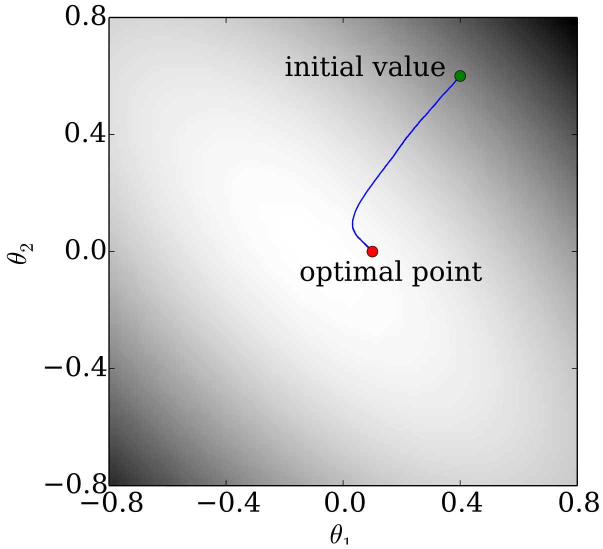

Fig. 5.12: Value of $\hat{\theta}$ in the proposed multivariate HPF method

Chapter 6. Conclusion

This thesis showed the noise-based methods for the global asymptotic stabilization and the optimization problem in control engineering. These methods were proposed to show the new insight for the control engineering from the point of the effective use of noise.

In the global asymptotic stabilization, the noise-based stabilization method was employed for the stabilization of nonholonomic systems and non-Euclidean systems. This thesis addressed the design problem of noise-based controllers when closed-loop systems are given by Stratonovich stochastic differential equations. The design method uses the stochastic control Lyapunov function, and gives a controller which is a generalization of Sontag-type controller. The one of the main results of this thesis is that if there exists a stochastic control Lyapunov function satisfying small control property, the designed controller globally asymptotically stabilizes a given system in the sense of global asymptotic stability in probability. Further, the proposed stochastic control Lyapunov function is a strict control Lyapunov function in the sense of stochastic system, which can be an advantage over other stabilization methods, such as time-varying feedback methods and discontinuous feedback methods. This implies the robustness of the controller given by the proposed design method. In this thesis, the stability margin of the designed controllers was shown from the inverse optimality of the controllers. This thesis also studied homogeneous stochastic systems, and showed the relations between the homogeneity and the convergence speed of the asymptotically stable homogeneous systems. Moreover, the method improving the convergence of the noise-based stabilization was developed by using the homogeneity. As a summary of the above discussion, this thesis presented the constructive design method of the noise-based controller, investigates the robustness, and shows the improvement of the convergence for driftless systems, for the noise-based stabilization.

In optimization problems, this thesis dealt with a stochastic extremum seeking method. In this thesis, we proposed a stochastic extremum seeking methods for static systems. The proposed method uses Wiener processes to approximate the gradients of objective functions. One of the proposed schemes, the HPF scheme, ensures the convergence of the estimation variables to the optimum value. The scheme uses the high-pass filter with state-dependent parameters obtained from the stochastic Lyapunov stability analysis. Based on the Lyapunov analysis, we show the global convergence of the estimation variable with probability one.

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Appendix A Proof of Proposition 3.1

We give the proof of the lemma in local coordinates. Considering the characteristic polynomial of $H(x)$

$$\rho(x, \widetilde{\lambda}) = \det\left(\widetilde{\lambda} I - H(x)\right),$$

an eigenvalue $\lambda_i(x)$ is given as a solution of the equation

$$\rho(x, \lambda) = 0.$$

From the assumption, each eigenvalue is simple, and we have

$$\frac{\partial \rho}{\partial \lambda}(x, \lambda_i(x)) \neq 0.$$

Thus, according to the implicit function theorem, we can conclude the continuous differentiability of the eigenvalue $\lambda_i(x)$.

Then, we show the differentiability of the eigenvectors of the matrix $H(x)$. Consider a vector-valued function

$$\phi_i(x, z) = \begin{bmatrix} (\lambda_i(x) I - H(x))\, z \\ z^T z - 1 \end{bmatrix}.$$

For a eigenvalue $\lambda_i(x)$, the normalized orthogonal eigenvector of $H(x)$ is a solution of $\phi_i(x, z) = 0$ with respect to z. Let $z(x)$ be the solution of $\phi_i(x, z) = 0$. Then, since z is orthogonal to all rows of $\lambda_i(x) I - H(x)$ and λ_i is a simple eigenvalue,

$$\operatorname{rank}\left[\frac{\partial \phi_i}{\partial z}(x, z)\right] = \operatorname{rank}\begin{bmatrix} \lambda_i(x) I - H(x) \\ 2z^T \end{bmatrix} = m$$

holds. We use the implicit function theorem again, and this implies the continuous differentiability of the eigenvectors. This completes the proof.

Appendix B Proof of Lemma 5.1

To show the inequality $f(\hat{\theta}) - \tilde{\theta}^2 > 0$, we obtain

$$
\begin{aligned}
f(\hat{\theta}) - \tilde{\theta}^2 &= 2\left(\hat{\theta} - \frac{\hat{\theta}_u + \hat{\theta}_l}{2}\right)^2 + 2\left(\frac{\hat{\theta}_u - \hat{\theta}_l}{2}\right)^2 + 1 - \tilde{\theta}^2 \\
&= 2\left(\tilde{\theta} - \left(\frac{\hat{\theta}_u + \hat{\theta}_l}{2} - \theta^*\right)\right)^2 + 2\left(\frac{\hat{\theta}_u - \hat{\theta}_l}{2}\right)^2 + 1 - \tilde{\theta}^2 \\
&= \left(\tilde{\theta} - 2\left(\frac{\hat{\theta}_u + \hat{\theta}_l}{2} - \theta^*\right)\right)^2 + 2\left\{\left(\frac{\hat{\theta}_u - \hat{\theta}_l}{2}\right)^2 - \left(\frac{\hat{\theta}_u + \hat{\theta}_l}{2} - \theta^*\right)^2\right\} + 1.
\end{aligned}
$$

According to the fact that $\hat{\theta}_u$ and $\hat{\theta}_l$ satisfy $\hat{\theta}_l \leq \theta^* \leq \hat{\theta}_u$, we have

$$\left(\frac{\hat{\theta}_u - \hat{\theta}_l}{2}\right)^2 - \left(\frac{\hat{\theta}_u + \hat{\theta}_l}{2} - \theta^*\right)^2 \geq 0. \tag{B.1}$$

According to (B.1), we obtain

$$f(\hat{\theta}) - \tilde{\theta}^2 > 0. \tag{B.2}$$

Then, we give the proof of the inequality $g(\theta, P) > \tilde{P}$. The function $h(\hat{\theta})$ in (5.19) satisfies

$$h(\hat{\theta}) < \tilde{\theta}^2. \tag{B.3}$$

We can prove this inequality in a similar way to the proof of $f(\hat{\theta}) - \tilde{\theta}^2 > 0$. According to (5.18), (B.2), (B.3), and Assumption 5.2, we obtain

$$\varphi_l(\hat{\theta}) \leq \varphi(\hat{\theta}) \leq \varphi_u(\hat{\theta}). \tag{B.4}$$

Then, using (B.4), we have

$$\begin{aligned}
g(\hat{\theta}, P) - \tilde{P} &= \frac{\left(P - \frac{\varphi_u(\hat{\theta}) + \varphi_l(\hat{\theta})}{2}\right)^2}{\sqrt{1 + \left(P - \frac{\varphi_u(\hat{\theta}) + \varphi_l(\hat{\theta})}{2}\right)^2}} + \frac{\varphi_u(\hat{\theta}) - \varphi_l(\hat{\theta})}{2} + 1 - \tilde{P} \\
&= 1 - \frac{1}{\sqrt{1 + \left(P - \frac{\varphi_u(\hat{\theta}) + \varphi_l(\hat{\theta})}{2}\right)^2}} + \sqrt{1 + \left(P - \frac{\varphi_u(\hat{\theta}) + \varphi_l(\hat{\theta})}{2}\right)^2} \\
&\quad + \frac{\varphi_u(\hat{\theta}) - \varphi_l(\hat{\theta})}{2} - \tilde{P} \\
&> \sqrt{1 + \left(P - \frac{\varphi_u(\hat{\theta}) + \varphi_l(\hat{\theta})}{2}\right)^2} + \frac{\varphi_u(\hat{\theta}) - \varphi_l(\hat{\theta})}{2} - \tilde{P} \\
&> \left| P - \frac{\varphi_u(\hat{\theta}) + \varphi_l(\hat{\theta})}{2} \right| + \frac{\varphi_u(\hat{\theta}) - \varphi_l(\hat{\theta})}{2} - \tilde{P} \\
&= \left| P - \frac{\varphi_u(\hat{\theta}) + \varphi_l(\hat{\theta})}{2} \right| + \frac{\varphi_u(\hat{\theta}) - \varphi_l(\hat{\theta})}{2} - \left(P - \varphi(\hat{\theta})\right) \\
&\geq \frac{\varphi_u(\hat{\theta}) - \varphi_l(\hat{\theta})}{2} - \left(\frac{\varphi_u(\hat{\theta}) + \varphi_l(\hat{\theta})}{2} - \varphi(\hat{\theta})\right) \geq 0.
\end{aligned}$$

Thus, we have $g(\theta, P) > \tilde{P}$.