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INVARIANT SUBSPACES AND HANKEL-TYPE OPERATORS ON A BERGMAN SPACE

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Abstract Let \( L^2 = L^2(\mathbb{D}, drd\theta / \pi) \) be the Lebesgue space on the open unit disc \( \mathbb{D} \) and let \( L^2_{a} = L^2 \cap \text{Hol}(\mathbb{D}) \) be a Bergman space on \( \mathbb{D} \). In this paper, we are interested in a closed subspace \( M \) of \( L^2 \) which is invariant under the multiplication by the coordinate function \( z \), and a Hankel-type operator from \( L^2_{a} \) to \( M^\perp \). In particular, we study an invariant subspace \( M \) such that there does not exist a finite-rank Hankel-type operator except a zero operator.

Keywords: Bergman space; invariant subspace; Hankel-type operator

2000 Mathematics subject classification: Primary 47B35; 47A15

1. Introduction

Let \( D \) be the open unit disc in \( \mathbb{C} \) and \( \text{Hol}(D) \) be the set of all holomorphic functions on \( D \). Let \( d\mu = rdrd\theta / \pi \) and \( L^2 = L^2(D, d\mu) \) the Lebesgue space. The Bergman space \( L^2_a \) on \( D \) is defined by \( L^2_a = L^2 \cap \text{Hol}(D) \). Then \( L^2_a \) is the closed subspace of \( L^2 \). When \( M \) is a closed subspace of \( L^2 \) and \( z \in M \), \( M \) is called an invariant subspace. For \( \varphi \) in \( L^\infty = L^\infty(D, d\mu) \), a Hankel-type operator is defined by

\[
H^M_{\varphi} f = (I - P^M)(\varphi f) \quad (f \in L^2_a),
\]

where \( P^M \) is the orthogonal projection from \( L^2 \) onto \( M \). When \( M = L^2_a \), \( H^M_{\varphi} \) is called a big Hankel operator and when \( M = (\overline{zL^2_a})^\perp \), \( H^M_{\varphi} \) is called a small Hankel operator. When \( L^2_a \subseteq M \subseteq (\overline{zL^2_a})^\perp \), \( H^M_{\varphi} \) is called an intermediate Hankel operator.

It is easy to see that there does not exist a finite-rank big Hankel operator except a zero one (see [3,6]). On the other hand, there exist a lot of finite-rank non-zero small Hankel operators (see [6]). In fact, it is easy to see the results. Strouse [7] described completely all finite-rank intermediate Hankel operators for some invariant subspace. In the previous paper [6], we began to study finite-rank intermediate Hankel operators for arbitrary invariant subspace. In [6, Theorem 3.2], we gave three necessary and sufficient
conditions for \( M \) such that there does not exist a finite-rank intermediate Hankel operator except a zero one. In this paper, without the hypothesis on an invariant subspace \( M \), we give a new necessary and sufficient condition for \( M \) which have a finite-rank Hankel-type operator except a zero one.

For an invariant subspace \( M \) in \( L^2 \), \( H^M \) denotes the kernel of \( H^M \) and then \( \ker H^M = \{ f \in L^2_a; \varphi f \in M \} \). Hence \( \ker H^M \) is also an invariant subspace in \( L^2_a \). Thus each invariant subspace \( M \) in \( L^2 \) is related to an invariant subspace in \( L^2_a \) by a Hankel-type operator. In this paper, the following property of invariant subspaces in \( L^2 \) is important.

**Definition 1.1.** Let \( M \) be an invariant subspace of \( L^2 \). \( M \) is called weakly divisible if whenever \( f \in M \) and \( |f(z)| \leq \gamma |z - a| \) for some \( a \in D \) and some \( \gamma \geq 0 \) then \( f(z) = (z - a)g(z) \) and \( g \) is a function in \( M \).

In §2, we generalize a theorem of Axler and Bourdon [1], which will be used later on. In §3, we show that there does not exist a finite-rank Hankel-type operator \( H^M \) except a zero one if and only if \( M \) is weakly divisible. In §4, we give several examples of weakly divisible invariant subspaces.

In this paper \([S]_a\) denotes the weak* closed linear span of a subset \( S \) in \( L^\infty \) and \([S]_2 \) denotes the closed linear span of a subset \( S \) in \( L^2 \).

### 2. An invariant subspace and the index

In this section, for a given invariant subspace \( M \) we are interested in two invariant subspaces \( M' \) and \( M'' \) such that \( M' \subseteq M \subseteq M'' \), \( \dim M \otimes M' < \infty \) and \( \dim M'' \otimes M < \infty \). Under some conditions on \( M \), \( M' \) and \( M'' \), we describe \( M' \) and \( M'' \) using \( M \). Corollary 2.4 will be used in §§3 and 4. Corollary 2.4 (i) is known from [1].

When \( M \) is an invariant subspace of \( L^2 \), for \( a \in \mathbb{C} \) put \( \text{ind}_a M = \dim \{ M \oplus (z - a)M \} \). \( \text{ind}_a M \) is called the index of \( M \) at \( a \). It is known (cf. [1]) that for each \( n \) (\( 0 \leq n \leq \infty \)) and for any \( a \) (\( \in D \)) there exists an invariant subspace \( M \) with \( \text{ind}_a M = n \).

**Theorem 2.1.** Let \( M, M_1 \) and \( M_2 \) be invariant subspaces of \( L^2 \) and \( M_1 \subseteq M_2 \).

(i) \( \text{ind}_a M = 0 \) for any \( a \notin D \).

(ii) If \( \dim M_2 \otimes M_1 < \infty \), then there exists a polynomial \( b \) such that \( bM_2 \subseteq M_1 \), \( Z(b) \subseteq D \) and the degree of \( b \leq \dim M_2 \otimes M_1 \) and

\[
\sum (\text{ind}_a M_2; a \in Z(b)) \geq \dim M_2 \otimes M_1.
\]

**Proof.** (i) If \( |a| > 1 \), then \( (z - a)^{-1} \in H^\infty \) and \( M = (z - a)M \). Hence \( \text{ind}_a M = 0 \). If \( |a| = 1 \), then \( (z - a)M = (z - a)\{ z - a(1 + \varepsilon) \}^{-1}M \). For any \( f \in M \), it is easy to see that

\[
\int_D \left| \frac{z - a}{z - a(1 + \varepsilon)} f - f \right|^2 d\mu \to 0 \quad (\varepsilon \to 0)
\]

by Lebesgue’s convergence theorem. This implies that \((z - a)M\) is dense in \( M \) and so \( \text{ind}_a M = 0 \) for \( |a| = 1 \).
(ii) Put $N = M_2 \ominus M_1$ and $S_z = PM_z|N$, where $M_2$ is a multiplication operator on $L^2$ by the coordinate function $z$ and $P$ is the orthogonal projection from $L^2$ to $N$. If $n = \dim N < \infty$, then there exists a polynomial $b$ of degree $n$ such that $S_b = b(S_z) = 0$ and so $bM_2 \subseteq M_1$. By (i), we may assume that $Z(b) \subset D$. We will prove that $\sum (\text{ind}_a M_2; a \in Z(b)) \geq n$. We can write that $b = a_0 \prod_{j=1}^n (z - a_j)$ and so $Z(b) = \{a_1, a_2, \ldots, a_n\}$, where $a_0 \in \mathbb{C}$. If $\sum (\text{ind}_a M_2; a \in Z(b)) \leq n - 1$, then we may assume $\text{ind}_a M_2 = 0$. Since $\sum_{j=2}^n (z - a_j)M_2 \subseteq M_1 \subset M_2$.

Then it is easy to see that $\dim M_2 \ominus \prod_{j=2}^n (z - a_j)M_2 \leq n - 1$ because $\text{ind}_a M_2 \leq 1$ for $2 \leq j \leq n$. This contradicts that $\dim M_2 \ominus M_1 = n$. 

Corollary 2.2. Let $M_1$ and $M_2$ be invariant subspaces of $L^2$ and $M_1 \subseteq M_2$. If $\dim M_2 \ominus M_1 = 1$, then $(z - a)M_2 \subseteq M_1 \subseteq M_2$ for some $a \in D$ and $\text{ind}_a M_2 \geq 1$. If $\text{ind}_a M_1 = 1$ or $\text{ind}_a M_2 = 1$, then $M_1 = [(z - a)M_2]_2$.

Proof. By Theorem 2.1, $(z - a)M_2 \subseteq M_1$ for some $a \in D$ and so $\text{ind}_a M_2 \geq 1$. Since $(z - a)M_1 \subseteq (z - a)M_2 \subseteq M_1 \subseteq M_2, M_1 = [(z - a)M_2]_2$ if $\text{ind}_a M_1 = 1$ or $\text{ind}_a M_2 = 1$. 

Corollary 2.3. Let $M_1$ and $M_2$ be invariant subspaces such that $M_1 \subseteq M_2$ and $\dim M_2 \ominus M_1 = n < \infty$. Suppose that $(z - a)M_j$ is closed for any $a \in D$ when $j = 1, 2$. If $\text{ind}_a M_1 = 1$ for any $a \in D$ or $\text{ind}_a M_2 = 1$ for any $a \in D$, then $M_1 = bM_2$ and $M_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus M_1$, where $b = \prod_{j=1}^n (z - a_j), \{a_j\} \subset D$ and $\{f_j\} \subset M_1$.

Proof. By Theorem 2.1 there exists a polynomial $b$ such that $bM_2 \subseteq M_1$ and $Z(b) \subset D$ and the degree of $b \leq n$. Hence $b = \prod_{j=1}^n (z - a_j)$ and $\{a_j\} \subset D$ and $b \leq n$. When $\text{ind}_a M_2 = 1$ for any $a \in D$, $\dim M_2 \ominus bM_2 = \ell$ because $(z - a_j)M_2$ is closed for $1 \leq j \leq \ell$ and so $\ell = n$. Hence $M_1 = bM_2$. When $\text{ind}_a M_1 = 1$ for any $a \in D$, $\dim M_1 \ominus bM_1 = \ell$ by the same reason. Since $bM_1 \subseteq bM_2 \subseteq M_1$ and $\dim bM_2 \ominus bM_1 = n, \ell = n$ and so $M_1 = bM_2$. Put $M_2 = \langle \varphi_1, \ldots, \varphi_n \rangle \oplus M_1$, where $\{\varphi_j\}$ are orthogonal to $M_1$. What was just proved above, $bM_2 = M_1$ and so $bM_2 = \langle b\varphi_1, \ldots, b\varphi_n \rangle \oplus bM_1 = M_1$. Put $f_j = b\varphi_j$ for $j = 1, \ldots, n$, then $\{f_j\}$ are in $M_1$ and $M_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus M_1$. 

Corollary 2.4. Let $M$ be an invariant subspace of $L^2$.

(i) If $\dim L^2_a \ominus M = n < \infty$ and $n \neq 0$, then $M = bL^2_a$, where $b = \prod_{j=1}^n (z - a_j)$ and $\{a_j\} \subset D$.

(ii) If $\dim M \cap L^2_a = n < \infty$, then $M = L^2_a$.

Proof. It is known that $\text{ind}_a L^2_a = 1$ and $(z - a)L^2_a$ is closed for each $a \in D$. Hence we can apply Corollary 2.3 to $M_1 = L^2_a$ or $M_2 = L^2_a$. If $M_1 = M$ and $M_2 = L^2_a$, then (i) follows. If $M_1 = L^2_a$ and $M_2 = M$, then $M = \langle f_1/b, \ldots, f_n/b \rangle \oplus L^2_a$, where $b = \prod_{j=1}^n (z - a_j), \{a_j\} \subset D$ and $\{f_j\} \subset L^2_a$. For each $1 \leq \ell \leq n$, $f_\ell/b \in L^2$ and so
3. Finite-rank Hankel-type operators

In this section, we study the relation between finite-rank Hankel-type operators and invariant subspaces.

**Theorem 3.1.** Let $\mathcal{M}$ be an invariant subspace of $L^2$. Then there does not exist a finite-rank Hankel-type operator $H^M_\varphi$ except a zero one if and only if $\mathcal{M}$ is weakly divisible.

**Proof.** Suppose $\mathcal{M}$ is weakly divisible. If $H^M_\varphi$ is of finite rank, then $\ker H^M_\varphi$ is an invariant subspace in $L^2_a$ and $\dim L^2_a/\ker H^M_\varphi < \infty$. By (i) of Corollary 2.4, $\ker H^M_\varphi = bL^2_a$ for some polynomial $b$ with $Z(b) \subset D$ and so $b\varphi$ belongs to $\mathcal{M}$. Put $f = b\varphi$, then $|f(z)| \leq \gamma |b(z)|$ ($z \in D$), where $\gamma = \|\varphi\|_\infty$. Suppose $b(z) = a_0 \prod_{j=1}^n (z - a_j)$, where $\{a_j\} \subset D$. For any $\ell$ with $1 \leq \ell \leq n$,

$$\left| \frac{f(z)}{z - a_\ell} \right| \leq \gamma |a_0| \prod_{j \neq \ell} |z - a_j| \quad (z \in D)$$

and $f(z)/(z - a_\ell)$ belongs to $\mathcal{M}$ because $a_\ell \in D$ and $\mathcal{M}$ is weakly divisible. Thus $\varphi(z) = f(z)/b(z)$ belongs to $\mathcal{M}$. Hence $H^M_\varphi = 0$.

Conversely, if $\mathcal{M}$ is not weakly divisible, then there exists a function $f$ in $\mathcal{M}$ and a point $a$ in $D$ such that $|f(z)| \leq \gamma |z - a|$ ($z \in D$) and $f(z)/(z - a)$ does not belong to $\mathcal{M}$. Put $\varphi = f(z)/(z - a)$, then $\varphi \in L^\infty$ and $H^M_\varphi$ is not zero because $\varphi \notin \mathcal{M}$. On the other hand, $(z - a)\varphi \in \mathcal{M}$ and so the kernel of $H^M_\varphi$ contains $(z - a)L^2_a$. This implies that $H^M_\varphi$ is of rank one because $L^2_a/(z - a)L^2_a = \mathbb{C}$.

**Proposition 3.2.** If there exists a symbol $\varphi$ such that $r(H^M_\varphi) = n \geq 1$, then there exists a symbol $\varphi_j$ such that $r(H^M_{\varphi_j}) = j$ for any $j$ with $0 \leq j \leq n - 1$.

**Proof.** Suppose $1 \leq n = r(H^M_\varphi) < \infty$. Then $\ker H^M_\varphi$ is the kernel of $H^M_\varphi$ is an invariant subspace of $L^2_a$ and $L^2_a/\ker H^M_\varphi$ is of finite dimension $n$. By Corollary 2.4, $\ker H^M_\varphi = bL^2_a$, where $b = \prod_{j=1}^n (z - a_\ell)$ and $(a_\ell) \subset D$. Hence $b\varphi$ belongs to $\mathcal{M}$. Put

$$\varphi_j = \varphi \prod_{\ell=1}^n (z - a_\ell) \quad \text{for } 1 \leq j \leq n - 1,$$

then $\varphi_j \notin \mathcal{M}$ for $1 \leq j \leq n - 1$ and $\varphi_0 = b\varphi$. Since $\ker H^M_{\varphi_j} = b_jL^2_a$ for $1 \leq j \leq n - 1$, where $b_j = \prod_{j=1}^n (z - a_\ell)$, $H^M_{\varphi_j}$ is of finite rank $j$ for $0 \leq j \leq n - 1$.

**Corollary 3.3.** The following two expressions are equivalent for an invariant subspace $\mathcal{M}$.

(i) If $r(H^M_\varphi) < \infty$, then $r(H^M_{\varphi_j}) = 0$. 


(ii) If \( r(H_\varphi^M) \leq 1 \), then \( r(H_\varphi^M) = 0 \).

**Proof.** (i) \( \Rightarrow \) (ii). This is clear.

(ii) \( \Rightarrow \) (i). If (i) is not true, then there exists a symbol \( \varphi \) with \( r(H_\varphi^M) = n \geq 2 \). By Proposition 3.2 there exists a symbol \( \varphi_1 \) such that \( r(H_{\varphi_1}^M) = 1 \). This contradicts (ii). \( \Box \)

4. Weakly divisible invariant subspaces

For a function \( f \) in \( L^2_a \), put \( Z(f) = \{ a \in D; f(a) = 0 \} \) and \( Z(G) = \cap \{ Z(f); f \in G \} \) for a subset \( G \) in \( L^2_a \). For \( 1 \leq p \leq \infty \), if \( E \) is an open set in \( D \), \( H^p_E \) denotes the set of all functions in \( L^p \) that are analytic on \( E \). In Corollary 4.2, a weakly divisible invariant subspace \( M \) is described completely when \( M \) is in \( L^2_a \). There exists a non-zero invariant subspace \( M \) in \( L^2_a \) such that \( M \cap L^\infty = \{ 0 \} \). For it is known (see [5]) that there exists a non-zero function \( f \) in \( L^2_a \) such that \( Z(f) \) does not satisfy the Blaschke condition.

**Theorem 4.1.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \).

(i) If \( \mathcal{M} \cap L^\infty \subseteq H^\infty \) and \( Z(\mathcal{M} \cap L^\infty) = \emptyset \), then \( \mathcal{M} \) is weakly divisible.

(ii) If \( \mathcal{M} \cap L^\infty = H^\infty_E \) for some open set \( E \), then \( \mathcal{M} \) is weakly divisible.

(iii) If \( \mathcal{M} \cap L^\infty = \{ 0 \} \), then \( \mathcal{M} \) is weakly divisible.

**Proof.** (i) If \( \{ f_n \} \) is a sequence in \( \mathcal{M} \cap L^\infty \) which converges pointwise boundedly to \( f \), then \( f \in \mathcal{M} \). By the Krein–Schmalian criterion (see [4, IV 2.1]), \( \mathcal{M} \cap L^\infty \) is weak* closed. Hence, by a well-known theorem of Beurling [2] \( \mathcal{M} \cap L^\infty = qH^\infty \) for some inner function \( q \). Hence if \( f \in \mathcal{M} \) and \( |f(z)| \leq \gamma |z - a| \) (\( z \in D \)) for some \( a \in D \), then \( f = qh \) for some \( h \in H^\infty \). Since \( Z(\mathcal{M} \cap L^\infty) = \emptyset \), \( |q(z)| > 0 \) (\( z \in D \)) and so \( h(a) = 0 \). Hence \( f(z)/(z - a) = q(z) \times (h(z)/(z - a)) \) \( \in qH^\infty \). Thus \( f(z)/(z - a) \) belongs to \( \mathcal{M} \).

(ii) If \( f \in H^\infty_E \) and \( |f(z)| \leq \gamma |z - a| \) (\( z \in D \)) for some \( a \in D \), then \( f(z)/(z - a) \in L^\infty \) and \( f(z)/(z - a) \) is analytic on \( E \). Hence \( f(z)/(z - a) \) belongs to \( H^\infty_E \) and so \( \mathcal{M} \) is weakly divisible.

(iii) This is clear. \( \Box \)

**Corollary 4.2.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2_a \). Then \( \mathcal{M} \) is weakly divisible if and only if \( \mathcal{M} \cap L^\infty = \{ 0 \} \) or \( Z(\mathcal{M} \cap L^\infty) = \emptyset \).

**Proof.** The part of ‘if’ is a result of (i) and (iii) of Theorem 4.1. Conversely, suppose that \( \mathcal{M} \) is weakly divisible. If \( \mathcal{M} \cap L^\infty \neq \{ 0 \} \), then by a theorem of Beurling there exists an inner function \( q \) with \( \mathcal{M} \cap L^\infty = qH^\infty \). If \( q(a) = 0 \) for some \( a \in D \), then there exists a finite positive constant \( \gamma \) such that \( |q(z)| \leq \gamma |z - a| \) (\( z \in D \)) and \( q/(z - a) \notin \mathcal{M} \). This contradicts the weak divisibility of \( \mathcal{M} \) and so \( Z(q) = Z(\mathcal{M} \cap L^\infty) = \emptyset \). \( \Box \)

**Corollary 4.3.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \).

(i) If \( \mathcal{M} \subseteq L^2_a \) and \( \dim L^2_a/\mathcal{M} < \infty \), then \( \mathcal{M} \) is not weakly divisible.
If $\mathcal{M} \supseteq L^2_\alpha$ and $\dim \mathcal{M}/L^2_\alpha < \infty$, then $\mathcal{M}$ is weakly divisible.

**Proof.** (i) If $\mathcal{M} \subseteq L^2_\alpha$ and $\dim L^2_\alpha/\mathcal{M} = \ell < \infty$, then by (i) of Corollary 2.4 $\mathcal{M} = bL^2_\alpha$, where $b = \prod_{j=1}^\ell (z - a_j)$ and $a_j \in D$ $(1 \leq j \leq \ell)$. Hence $Z(\mathcal{M} \cap L^\infty) = Z(b) \neq \emptyset$ and so by Corollary 4.2 $\mathcal{M}$ is not weakly divisible.

(ii) By (2) of Corollary 2.4 $\mathcal{M} = L^2_\alpha$ and so $\mathcal{M} \cap L^\infty = H^\infty$. Hence (i) of Theorem 4.1 implies that $\mathcal{M}$ is weakly divisible.

**Corollary 4.4.** If $\mathcal{M} = H^2_E$ for some open set $E$ in $D$, then $\mathcal{M}$ is weakly divisible.

**Proof.** It is a result of (ii) of Theorem 4.1.

**Proposition 4.5.** Suppose that $\mathcal{M}_j$ is a weakly divisible invariant subspace of $L^2$ for $j = 1, 2, \ldots$ and $\mathcal{M}_j \times \mathcal{M}_\ell = \{fg; f \in \mathcal{M}_j$ and $g \in \mathcal{M}_\ell\} = \{0\}$ if $j \neq \ell$. If $\mathcal{M} = \sum_{j=1}^\infty \oplus \mathcal{M}_j$, then $\mathcal{M}$ is a weakly divisible invariant subspace.

**Proof.** If $f \in \mathcal{M}$, then $f = \sum_{j=1}^\infty f_j$ and $|f(z)| = \sum_{j=1}^\infty |f_j(z)|$ $(z \in D)$ by hypothesis. This implies that $\mathcal{M}$ is weakly divisible.

**Corollary 4.6.** Let $1 \leq \ell \leq \infty$. Suppose $D_j$ is an open set in $D$ with $\mu(\partial D_j) = 0$ for $1 \leq j \leq \ell$, $D_i \cap D_j = \emptyset (i \neq j)$ and $D = \bigcup_{j=1}^\ell D_j$. Then $\mathcal{M} = \sum_{j=1}^\ell \oplus L^2_\alpha(D_j)$ is weakly divisible.

**Proof.** This is a result of Corollary 4.4 and Proposition 4.5.

**Proposition 4.7.** If $\mathcal{M}$ is a weakly divisible invariant subspace of $L^2$ and $\varphi$ is a unimodular function in $L^\infty$, then $\varphi \mathcal{M}$ is a weakly divisible invariant subspace.

**Proof.** From the definition of weak divisibility, the proposition follows trivially.

**Corollary 4.8.** If $\varphi$ is a unimodular function in $L^\infty$, then $\varphi L^2_\alpha$ is weakly divisible.

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