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INVARIANT SUBSPACES AND HANKEL-TYPE OPERATORS
ON A BERGMAN SPACE

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Abstract Let \(L^2 = L^2(D, r dr d\theta/\pi)\) be the Lebesgue space on the open unit disc \(D\) and let \(L^2_a = L^2 \cap \operatorname{Hol}(D)\) be a Bergman space on \(D\). In this paper, we are interested in a closed subspace \(M\) of \(L^2\) which is invariant under the multiplication by the coordinate function \(z\), and a Hankel-type operator from \(L^2_a\) to \(M^\perp\). In particular, we study an invariant subspace \(M\) such that there does not exist a finite-rank Hankel-type operator except a zero operator.

Keywords: Bergman space; invariant subspace; Hankel-type operator

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1. Introduction

Let \(D\) be the open unit disc in \(\mathbb{C}\) and \(\operatorname{Hol}(D)\) be the set of all holomorphic functions on \(D\). Let \(d\mu = r dr d\theta/\pi\) and \(L^2 = L^2(D, d\mu)\) the Lebesgue space. The Bergman space \(L^2_a\) on \(D\) is defined by \(L^2_a = L^2 \cap \operatorname{Hol}(D)\). Then \(L^2_a\) is the closed subspace of \(L^2\). When \(M\) is a closed subspace of \(L^2\) and \(z M \subseteq M\), \(M\) is called an invariant subspace. For \(\varphi\) in \(L^\infty = L^\infty(D, d\mu)\), a Hankel-type operator is defined by

\[
H^M_\varphi f = (I - P^M)(\varphi f) \quad (f \in L^2_a),
\]

where \(P^M\) is the orthogonal projection from \(L^2\) onto \(M\). When \(M = L^2_a\), \(H^M_\varphi\) is called a big Hankel operator and when \(M = (\overline{z L^2_a})^\perp\), \(H^M_\varphi\) is called a small Hankel operator. When \(L^2_a \subseteq M \subseteq (\overline{z L^2_a})^\perp\), \(H^M_\varphi\) is called an intermediate Hankel operator.

It is easy to see that there does not exist a finite-rank big Hankel operator except a zero one (see [3, 6]). On the other hand, there exist a lot of finite-rank non-zero small Hankel operators (see [6]). In fact, it is easy to see the results. Strouse [7] described completely all finite-rank intermediate Hankel operators for some invariant subspace. In the previous paper [6], we began to study finite-rank intermediate Hankel operators for arbitrary invariant subspace. In [6, Theorem 3.2], we gave three necessary and sufficient
conditions for $\mathcal{M}$ such that there does not exist a finite-rank intermediate Hankel operator except a zero one. In this paper, without the hypothesis on an invariant subspace $\mathcal{M}$, we give a new necessary and sufficient condition for $\mathcal{M}$ which have a finite-rank Hankel-type operator except a zero one.

For an invariant subspace $\mathcal{M}$ in $L^2$, $\ker H_\varphi^\mathcal{M}$ denotes the kernel of $H_\varphi^\mathcal{M}$ and then $\ker H_\varphi^\mathcal{M} = \{ f \in L^2_a; \varphi f \in \mathcal{M} \}$. Hence $\ker H_\varphi^\mathcal{M}$ is also an invariant subspace in $L^2_a$. Thus each invariant subspace $\mathcal{M}$ in $L^2$ is related to an invariant subspace in $L^2_a$ by a Hankel-type operator. In this paper, the following property of invariant subspaces in $L^2$ is important.

**Definition 1.1.** Let $\mathcal{M}$ be an invariant subspace of $L^2$. $\mathcal{M}$ is called weakly divisible if whenever $f \in \mathcal{M}$ and $|f(z)| \leq \gamma |z - a|$ for some $a \in D$ and some $\gamma \geq 0$ then $f(z) = (z - a)g(z)$ and $g$ is a function in $\mathcal{M}$.

In §2, we generalize a theorem of Axler and Bourdon [1], which will be used later on. In §3, we show that there does not exist a finite-rank Hankel-type operator $H_\varphi^\mathcal{M}$ except a zero one if and only if $\mathcal{M}$ is weakly divisible. In §4, we give several examples of weakly divisible invariant subspaces.

In this paper $[S]_a$ denotes the weak$^\ast$ closed linear span of a subset $S$ in $L^\infty$ and $[S]_2$ denotes the closed linear span of a subset $S$ in $L^2$.

## 2. An invariant subspace and the index

In this section, for a given invariant subspace $\mathcal{M}$ we are interested in two invariant subspaces $\mathcal{M}'$ and $\mathcal{M}''$ such that $\mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{M}''$, dim $\mathcal{M} \cap \mathcal{M}' < \infty$ and dim $\mathcal{M}'' \cap \mathcal{M} < \infty$. Under some conditions on $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$, we describe $\mathcal{M}'$ and $\mathcal{M}''$ using $\mathcal{M}$. Corollary 2.4 will be used in §§3 and 4. Corollary 2.4 (i) is known from [1].

When $\mathcal{M}$ is an invariant subspace of $L^2$, for $a \in \mathbb{C}$ put $\text{ind}_a \mathcal{M} = \text{dim} \{ \mathcal{M} \cap (z - a)\mathcal{M} \}$. $\text{ind}_a \mathcal{M}$ is called the index of $\mathcal{M}$ at $a$. It is known (cf. [1]) that for each $n \ (0 \leq n \leq \infty)$ and for any $a \ (\in D)$ there exists an invariant subspace $\mathcal{M}$ with $\text{ind}_a \mathcal{M} = n$.

**Theorem 2.1.** Let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces of $L^2$ and $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

(i) $\text{ind}_a \mathcal{M} = 0$ for any $a \notin D$.

(ii) If dim $\mathcal{M}_2 \cap \mathcal{M}_1 < \infty$, then there exists a polynomial $b$ such that $b\mathcal{M}_2 \subseteq \mathcal{M}_1$, $Z(b) \subseteq D$ and the degree of $b$ is $\leq \text{dim} \mathcal{M}_2 \cap \mathcal{M}_1$ and

$$\sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \geq \text{dim} \mathcal{M}_2 \cap \mathcal{M}_1.$$ 

**Proof.** (i) If $|a| > 1$, then $(z - a)^{-1} \in H^\infty$ and $\mathcal{M} = (z - a)\mathcal{M}$. Hence $\text{ind}_a \mathcal{M} = 0$. If $|a| = 1$, then $(z - a)\mathcal{M} = (z - a)(z - a(1 + \varepsilon))^{-1}\mathcal{M}$. For any $f \in \mathcal{M}$, it is easy to see that

$$\int_D \left| \frac{z - a}{z - a(1 + \varepsilon)} f - f \right|^2 d\mu \to 0 \quad (\varepsilon \to 0)$$

by Lebesgue’s convergence theorem. This implies that $(z - a)\mathcal{M}$ is dense in $\mathcal{M}$ and so $\text{ind}_a \mathcal{M} = 0$ for $|a| = 1$. 

(ii) Put \( \mathcal{N} = \mathcal{M}_2 \ominus \mathcal{M}_1 \) and \( S_z = PM_z|\mathcal{N} \), where \( \mathcal{M}_2 \) is a multiplication operator on \( L^2 \) by the coordinate function \( z \) and \( P \) is the orthogonal projection from \( L^2 \) to \( \mathcal{N} \). If \( n = \dim \mathcal{N} < \infty \), then there exists a polynomial \( b \) of degree \( n \) such that \( S_b = b(S_z) = 0 \) and so \( b\mathcal{M}_2 \subseteq \mathcal{M}_1 \). By (i), we may assume that \( Z(b) \subset D \). We will prove that \( \sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \geq n \). We can write that \( b = a_0 \prod_{j=1}^{n}(z - a_j) \) and so \( Z(b) = \{a_1, a_2, \ldots, a_n\} \), where \( a_0 \in \mathbb{C} \). If \( \sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \leq n - 1 \), then we may assume \( \text{ind}_a \mathcal{M}_2 = 0 \). Since \( [(z-a_1)\mathcal{M}_2]_2 = \mathcal{M}_2 \),

\[
\prod_{j=2}^{n}(z-a_j)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subset \mathcal{M}_2.
\]

Then it is easy to see that \( \dim \mathcal{M}_2 \ominus \prod_{j=2}^{n}(z-a_j)\mathcal{M}_2)_2 \leq n - 1 \) because \( \text{ind}_a \mathcal{M}_2 \leq 1 \) for \( 2 \leq j \leq n \). This contradicts that \( \dim \mathcal{M}_2 \ominus \mathcal{M}_1 = n \).

**Corollary 2.2.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be invariant subspaces of \( L^2 \) and \( \mathcal{M}_1 \subseteq \mathcal{M}_2 \). If \( \dim \mathcal{M}_2 \ominus \mathcal{M}_1 = 1 \), then \( (z - a)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subset \mathcal{M}_2 \) for some \( a \in D \) and \( \text{ind}_a \mathcal{M}_2 = 1 \). If \( \text{ind}_a \mathcal{M}_1 = 1 \) or \( \text{ind}_a \mathcal{M}_2 = 1 \), then \( \mathcal{M}_1 = [(z - a)\mathcal{M}_2]_2 \).

**Proof.** By Theorem 2.1, \( (z - a)\mathcal{M}_2 \subseteq \mathcal{M}_1 \) for some \( a \in D \) and so \( \text{ind}_a \mathcal{M}_2 \geq 1 \). Since \( (z-a)\mathcal{M}_1 \subseteq (z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1 \not\subset \mathcal{M}_2 \), \( \mathcal{M}_1 = [(z-a)\mathcal{M}_2]_2 \) if \( \text{ind}_a \mathcal{M}_1 = 1 \) or \( \text{ind}_a \mathcal{M}_2 = 1 \).

**Corollary 2.3.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be invariant subspaces such that \( \mathcal{M}_1 \subseteq \mathcal{M}_2 \) and \( \dim \mathcal{M}_2 \ominus \mathcal{M}_1 = n < \infty \). Suppose that \( (z - a)\mathcal{M}_j \) is closed for any \( a \in D \) when \( j = 1, 2 \). If \( \text{ind}_a \mathcal{M}_1 = 1 \) for any \( a \in D \) or \( \text{ind}_a \mathcal{M}_2 = 1 \) for any \( a \in D \), then \( \mathcal{M}_1 = b\mathcal{M}_2 \) and \( \mathcal{M}_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus \mathcal{M}_1 \), where \( b = \prod_{j=1}^{n}(z - a_j), \{a_j\} \subset D \) and \( \{f_j\} \subset \mathcal{M}_1 \).

**Proof.** By Theorem 2.1 there exists a polynomial \( b \) such that \( b\mathcal{M}_2 \subseteq \mathcal{M}_1 \) and \( Z(b) \subset D \) and the degree of \( b \leq n \). Hence \( b = \prod_{j=1}^{\ell}(z - a_j) \) and \( \{a_j\} \subset D \) and \( \ell \leq n \). When \( \text{ind}_a \mathcal{M}_2 = 1 \) for any \( a \in D \), \( \dim \mathcal{M}_2 \ominus b\mathcal{M}_2 = \ell \) because \( (z-a)\mathcal{M}_2 \) is closed for \( 1 \leq j \leq \ell \) and so \( \ell = n \). Hence \( \mathcal{M}_1 = b\mathcal{M}_2 \). But \( \mathcal{M}_1 = b\mathcal{M}_1 \). When \( \text{ind}_a \mathcal{M}_2 = 1 \) for any \( a \in D \), \( \dim \mathcal{M}_1 \ominus b\mathcal{M}_1 = \ell \) by the same reason. Since \( b\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_1 \) and \( \dim b\mathcal{M}_2 \ominus b\mathcal{M}_1 = n, \ell = n \) and so \( \mathcal{M}_1 = b\mathcal{M}_2 \). Put \( \mathcal{M}_2 = \langle \varphi_1, \ldots, \varphi_n \rangle \oplus \mathcal{M}_1 \), where \( \{\varphi_j\} \) are orthogonal to \( \mathcal{M}_1 \). What was just proved above, \( b\mathcal{M}_2 = \mathcal{M}_1 \) and so \( b\mathcal{M}_2 = \langle b\varphi_1, \ldots, b\varphi_n \rangle \oplus \mathcal{M}_1 = \mathcal{M}_1 \). Put \( f_j = b\varphi_j \) for \( j = 1, \ldots, n \), then \( \{f_j\} \) are in \( \mathcal{M}_1 \) and \( \mathcal{M}_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus \mathcal{M}_1 \).

**Corollary 2.4.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \).

(i) If \( \dim \mathcal{L}_a^2 \ominus \mathcal{M} = n < \infty \) and \( n \neq 0 \), then \( \mathcal{M} = b\mathcal{L}_a^2 \), where \( b = \prod_{j=1}^{n}(z - a_j) \) and \( \{a_j\} \subset D \).

(ii) If \( \dim \mathcal{M} \ominus \mathcal{L}_a^2 = n < \infty \), then \( \mathcal{M} = \mathcal{L}_a^2 \).

**Proof.** It is known that \( \text{ind}_a \mathcal{L}_a^2 = 1 \) and \( (z-a)\mathcal{L}_a^2 \) is closed for each \( a \in D \). Hence we can apply Corollary 2.3 to \( \mathcal{M}_1 = \mathcal{L}_a^2 \) or \( \mathcal{M}_2 = \mathcal{L}_a^2 \). If \( \mathcal{M}_1 = \mathcal{M} \) and \( \mathcal{M}_2 = \mathcal{L}_a^2 \), then (i) follows. If \( \mathcal{M}_1 = \mathcal{L}_a^2 \) and \( \mathcal{M}_2 = \mathcal{M} \), then \( \mathcal{M} = \langle f_1/b, \ldots, f_n/b \rangle \oplus \mathcal{L}_a^2 \), where \( b = \prod_{j=1}^{n}(z - a_j), \{a_j\} \subset D \) and \( \{f_j\} \subset \mathcal{L}_a^2 \). For each \( 1 \leq \ell \leq n \), \( f_\ell/b \in L^2 \) and so
In this section, we study the relation between finite-rank Hankel-type operators and invariant subspaces.

**Theorem 3.1.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \). Then there does not exist a finite-rank Hankel-type operator \( H^M_{\varphi} \) except a zero one if and only if \( \mathcal{M} \) is weakly divisible.

**Proof.** Suppose \( \mathcal{M} \) is weakly divisible. If \( H^M_{\varphi} \) is of finite rank, then \( \ker H^M_{\varphi} \) is an invariant subspace in \( L^2_a \) and \( \dim L^2_a / \ker H^M_{\varphi} < \infty \). By (i) of Corollary 2.4, \( \ker H^M_{\varphi} = bL^2_a \) for some polynomial \( b \) with \( Z(b) \subset D \) and so \( b\varphi \) belongs to \( \mathcal{M} \). Put \( f = b\varphi \), then \( |f(z)| \leq |b(z)| (z \in D) \), where \( \gamma = \| \varphi \|_\infty \). Suppose \( b(z) = a_0 \prod_{j=1}^a (z - a_j) \), where \( \{a_j\} \subset D \). For any \( \ell \) with \( 1 \leq \ell \leq n \),

\[
|f(z)| \leq |a_0| \prod_{j \neq \ell} |z - a_j| \quad (z \in D)
\]

and \( f(z)/(z - a_\ell) \) belongs to \( \mathcal{M} \) because \( a_\ell \in D \) and \( \mathcal{M} \) is weakly divisible. Thus \( \varphi(z) = f(z)/b(z) \) belongs to \( \mathcal{M} \). Hence \( H^M_{\varphi} = 0 \).

Conversely, if \( \mathcal{M} \) is not weakly divisible, then there exists a function \( f \) in \( \mathcal{M} \) and a point \( a \) in \( D \) such that \( |f(z)| \leq |z - a| (z \in D) \) and \( f(z)/(z - a) \) does not belong to \( \mathcal{M} \). Put \( \varphi = f(z)/(z - a) \), then \( \varphi \in L^\infty \) and \( H^M_{\varphi} \) is not zero because \( \varphi \notin \mathcal{M} \). On the other hand, \( (z - a)\varphi \in \mathcal{M} \) and so the kernel of \( H^M_{\varphi} \) contains \( (z - a)L^2_a \). This implies that \( H^M_{\varphi} \) is of rank one because \( L^2_a/(z - a)L^2_a = \mathbb{C} \). \( \square \)

**Proposition 3.2.** If there exists a symbol \( \varphi \) such that \( r(H^M_{\varphi}) = n \geq 1 \), then there exists a symbol \( \varphi_j \) such that \( r(H^M_{\varphi_j}) = j \) for any \( j \) with \( 0 \leq j \leq n - 1 \).

**Proof.** Suppose \( 1 \leq n = r(H^M_{\varphi}) < \infty \). Then \( \ker H^M_{\varphi} = \ker H^M_{\varphi_j} \) is the kernel of \( H^M_{\varphi} \) is an invariant subspace of \( L^2_a \) and \( L^2_a / \ker H^M_{\varphi} \) is of finite dimension \( n \). By Corollary 2.4, \( \ker H^M_{\varphi} = bL^2_a \), where \( b = \prod_{j=1}^n (z - a_j) \) and \( \{a_j\} \subset D \). Hence \( b\varphi \) belongs to \( \mathcal{M} \). Put

\[
\varphi_j = \varphi \prod_{\ell = j+1}^n (z - a_\ell) \quad \text{for} \ 1 \leq j \leq n - 1,
\]

then \( \varphi_j \notin \mathcal{M} \) for \( 1 \leq j \leq n - 1 \) and \( \varphi_0 = b\varphi \). Since \( \ker H^M_{\varphi_j} = b_jL^2_a \) for \( 1 \leq j \leq n - 1 \), where \( b_j = \prod_{\ell=1}^n (z - a_\ell) \), \( H^M_{\varphi_j} \) is of finite rank \( j \) for \( 0 \leq j \leq n - 1 \). \( \square \)

**Corollary 3.3.** The following two expressions are equivalent for an invariant subspace \( \mathcal{M} \).

(i) If \( r(H^M_{\varphi}) < \infty \), then \( r(H^M_{\varphi}) = 0 \).
(ii) If \( r(H_\varphi^M) \leq 1 \), then \( r(H_\varphi^M) = 0 \).

**Proof.** (i) ⇒ (ii). This is clear.

(ii) ⇒ (i). If (i) is not true, then there exists a symbol \( \varphi \) with \( r(H_\varphi^M) = n \geq 2 \). By Proposition 3.2 there exists a symbol \( \varphi_1 \) such that \( r(H_\varphi^M) = 1 \). This contradicts (ii). □

4. Weakly divisible invariant subspaces

For a function \( f \) in \( L_a^2 \), put \( Z(f) = \{ a \in D; f(a) = 0 \} \) and \( Z(G) = \cap \{ Z(f); f \in G \} \) for a subset \( G \) in \( L_a^2 \). For \( 1 \leq p \leq \infty \), if \( E \) is an open set in \( D \), \( H_\varphi^E \) denotes the set of all functions in \( L^p \) that are analytic on \( E \). In Corollary 4.2, a weakly divisible invariant subspace \( \mathcal{M} \) is described completely when \( \mathcal{M} \) is in \( L_a^2 \). There exists a non-zero invariant subspace \( \mathcal{M} \) in \( L_a^2 \) such that \( \mathcal{M} \cap L^\infty = (0) \). For it is known (see [5]) that there exists a non-zero function \( f \) in \( L_a^2 \) such that \( Z(f) \) does not satisfy the Blaschke condition.

**Theorem 4.1.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \).

(i) If \( \mathcal{M} \cap L^\infty \subseteq H^\infty \) and \( Z(\mathcal{M} \cap L^\infty) = \emptyset \), then \( \mathcal{M} \) is weakly divisible.

(ii) If \( \mathcal{M} \cap L^\infty = H_\varphi^E \) for some open set \( E \), then \( \mathcal{M} \) is weakly divisible.

(iii) If \( \mathcal{M} \cap L^\infty = (0) \), then \( \mathcal{M} \) is weakly divisible.

**Proof.** (i) If \( \{ f_n \} \) is a sequence in \( \mathcal{M} \cap L^\infty \) which converges pointwise boundedly to \( f \), then \( f \in \mathcal{M} \). By the Krein–Schmulian criterion (see [4, IV 2.1]), \( \mathcal{M} \cap L^\infty \) is weak∗ closed. Hence, by a well-known theorem of Beurling [2] \( \mathcal{M} \cap L^\infty = qH^\infty \) for some inner function \( q \). Hence if \( f \in \mathcal{M} \) and \( |f(z)| \leq |z - a| \) \( (z \in D) \) for some \( a \in D \), then \( f = qh \) for some \( h \in H^\infty \). Since \( Z(\mathcal{M} \cap L^\infty) = \emptyset \), \( |q(z)| > 0 \) \((z \in D)\) and so \( h(a) = 0 \). Hence \( f(z)/(z - a) = \langle h(z)/(z - a) \rangle \in qH^\infty \). Thus \( f(z)/(z - a) \) belongs to \( \mathcal{M} \).

(ii) If \( f \in H_\varphi^E \) and \( |f(z)| \leq |z - a| \) \( (z \in D) \) for some \( a \in D \), then \( f(z)/(z - a) \in L^\infty \) and \( f(z)/(z - a) \) is analytic on \( E \). Hence \( f(z)/(z - a) \) belongs to \( H_\varphi^E \) and so \( \mathcal{M} \) is weakly divisible.

(iii) This is clear. □

**Corollary 4.2.** Let \( \mathcal{M} \) be an invariant subspace of \( L_a^2 \). Then \( \mathcal{M} \) is weakly divisible if and only if \( \mathcal{M} \cap L^\infty = (0) \) or \( Z(\mathcal{M} \cap L^\infty) = \emptyset \).

**Proof.** The part of ‘if’ is a result of (i) and (iii) of Theorem 4.1. Conversely, suppose that \( \mathcal{M} \) is weakly divisible. If \( \mathcal{M} \cap L^\infty \neq (0) \), then by a theorem of Beurling there exists an inner function \( q \) with \( \mathcal{M} \cap L^\infty = qH^\infty \). If \( q(a) = 0 \) for some \( a \in D \), then there exists a finite positive constant \( \gamma \) such that \( |q(z)| \leq \gamma |z - a| \) \( (z \in D) \) and \( q/(z - a) \notin \mathcal{M} \). This contradicts the weak divisibility of \( \mathcal{M} \) and so \( Z(q) = Z(\mathcal{M} \cap L^\infty) = \emptyset \). □

**Corollary 4.3.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \).

(i) If \( \mathcal{M} \subseteq L_a^2 \) and \( \dim L_a^2/\mathcal{M} < \infty \), then \( \mathcal{M} \) is not weakly divisible.
(ii) If $M \supseteq L^2_0$ and $\dim M/L^2_0 < \infty$, then $M$ is weakly divisible.

Proof. (i) If $M \subseteq L^2_0$ and $\dim L^2_0/M = \ell < \infty$, then by (i) of Corollary 2.4 $M = bL^2_0$, where $b = \prod_{j=1}^{\ell} (z - a_j)$ and $a_j \in D$ ($1 \leq j \leq \ell$). Hence $Z(M \cap L^\infty) = Z(b) \neq \emptyset$ and so by Corollary 4.2 $M$ is not weakly divisible.

(ii) By (2) of Corollary 2.4 $M = L^2_0$ and so $M \cap L^\infty = H^\infty$. Hence (i) of Theorem 4.1 implies that $M$ is weakly divisible. \hfill $\Box$

Corollary 4.4. If $M = H^2_E$ for some open set $E$ in $D$, then $M$ is weakly divisible.

Proof. It is a result of (ii) of Theorem 4.1. \hfill $\Box$

Proposition 4.5. Suppose that $M_j$ is a weakly divisible invariant subspace of $L^2$ for $j = 1, 2, \ldots$ and $M_j \times M_\ell = \{fg; f \in M_j \text{ and } g \in M_\ell\} = \{0\}$ if $j \neq \ell$. If $M = \sum_{j=1}^{\ell} \oplus M_j$, then $M$ is a weakly divisible invariant subspace.

Proof. If $f \in M$, then $f = \sum_{j=1}^{\ell} f_j$ and $|f(z)| = \sum_{j=1}^{\ell} |f_j(z)|$ ($z \in D$) by hypothesis. This implies that $M$ is weakly divisible. \hfill $\Box$

Corollary 4.6. Let $1 \leq \ell \leq \infty$. Suppose $D_j$ is an open set in $D$ with $\mu(\partial D_j) = 0$ for $1 \leq j \leq \ell$, $D_i \cap D_j = \emptyset (i \neq j)$ and $D = \bigcup_{j=1}^{\ell} D_j$. Then $M = \sum_{j=1}^{\ell} \oplus L^2_0(D_j)$ is weakly divisible.

Proof. This is a result of Corollary 4.4 and Proposition 4.5. \hfill $\Box$

Proposition 4.7. If $M$ is a weakly divisible invariant subspace of $L^2$ and $\varphi$ is a unimodular function in $L^\infty$, then $\varphi M$ is a weakly divisible invariant subspace.

Proof. From the definition of weak divisibility, the proposition follows trivially. \hfill $\Box$

Corollary 4.8. If $\varphi$ is a unimodular function in $L^\infty$, then $\varphi L^2_0$ is weakly divisible.

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