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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>Proceedings of the Edinburgh Mathematical Society, 48, 479-484</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/5822">http://hdl.handle.net/2115/5822</a></td>
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<td>Type</td>
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<tr>
<td>File Information</td>
<td>PEMS48.pdf</td>
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HOKKAIDO UNIVERSITY
INVARIANT SUBSPACES AND HANKEL-TYPE OPERATORS
ON A BERGMAN SPACE

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(Received 29 April 2004)

Abstract Let $L^2 = L^2(D, r dr d\theta / \pi)$ be the Lebesgue space on the open unit disc $D$ and let $L^2_a = L^2 \cap \text{Hol}(D)$ be a Bergman space on $D$. In this paper, we are interested in a closed subspace $\mathcal{M}$ of $L^2$ which is invariant under the multiplication by the coordinate function $z$, and a Hankel-type operator from $L^2_a$ to $\mathcal{M}^\perp$. In particular, we study an invariant subspace $\mathcal{M}$ such that there does not exist a finite-rank Hankel-type operator except a zero operator.

Keywords: Bergman space; invariant subspace; Hankel-type operator

2000 Mathematics subject classification: Primary 47B35; 47A15

1. Introduction

Let $D$ be the open unit disc in $\mathbb{C}$ and $\text{Hol}(D)$ be the set of all holomorphic functions on $D$. Let $d\mu = r dr d\theta / \pi$ and $L^2 = L^2(D, d\mu)$ the Lebesgue space. The Bergman space $L^2_a$ on $D$ is defined by $L^2_a = L^2 \cap \text{Hol}(D)$. Then $L^2_a$ is the closed subspace of $L^2$. When $\mathcal{M}$ is a closed subspace of $L^2$ and $z\mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M}$ is called an invariant subspace. For $\varphi$ in $L^\infty = L^\infty(D, d\mu)$, a Hankel-type operator is defined by

$$H_\varphi^\mathcal{M} f = (I - P^\mathcal{M})(\varphi f) \quad (f \in L^2_a),$$

where $P^\mathcal{M}$ is the orthogonal projection from $L^2$ onto $\mathcal{M}$. When $\mathcal{M} = L^2_a$, $H_\varphi^\mathcal{M}$ is called a big Hankel operator and when $\mathcal{M} = (z \overline{L^2_a})^\perp$, $H_\varphi^\mathcal{M}$ is called a small Hankel operator. When $L^2_a \subseteq \mathcal{M} \subseteq (z \overline{L^2_a})^\perp$, $H_\varphi^\mathcal{M}$ is called an intermediate Hankel operator.

It is easy to see that there does not exist a finite-rank big Hankel operator except a zero one (see [3, 6]). On the other hand, there exist a lot of finite-rank non-zero small Hankel operators (see [6]). In fact, it is easy to see the results. Strouse [7] described completely all finite-rank intermediate Hankel operators for some invariant subspace. In the previous paper [6], we began to study finite-rank intermediate Hankel operators for arbitrary invariant subspace. In [6, Theorem 3.2], we gave three necessary and sufficient
conditions for $\mathcal{M}$ such that there does not exist a finite-rank intermediate Hankel operator except a zero one. In this paper, without the hypothesis on an invariant subspace $\mathcal{M}$, we give a new necessary and sufficient condition for $\mathcal{M}$ which have a finite-rank Hankel-type operator except a zero one.

For an invariant subspace $\mathcal{M}$ in $L^2$, $H_\varphi^\mathcal{M}$ denotes the kernel of $H_\varphi^\mathcal{M}$ and then $\ker H_\varphi^\mathcal{M} = \{ f \in L^2; \varphi f \in \mathcal{M} \}$. Hence $\ker H_\varphi^\mathcal{M}$ is also an invariant subspace in $L^2_a$. Thus each invariant subspace $\mathcal{M}$ in $L^2$ is related to an invariant subspace in $L^2_a$ by a Hankel-type operator. In this paper, the following property of invariant subspaces in $L^2$ is important.

**Definition 1.1.** Let $\mathcal{M}$ be an invariant subspace of $L^2$. $\mathcal{M}$ is called weakly divisible if whenever $f \in \mathcal{M}$ and $|f(z)| \leq \gamma |z - a|$ for some $a \in D$ and some $\gamma \geq 0$ then $f(z) = (z - a)g(z)$ and $g$ is a function in $\mathcal{M}$.

In §2, we generalize a theorem of Axler and Bourdon [1], which will be used later on. In §3, we show that there does not exist a finite-rank Hankel-type operator $H_\varphi^\mathcal{M}$ except a zero one if and only if $\mathcal{M}$ is weakly divisible. In §4, we give several examples of weakly divisible invariant subspaces.

In this paper $[S]_a$ denotes the weak$^*$ closed linear span of a subset $S$ in $L^\infty$ and $[S]_2$ denotes the closed linear span of a subset $S$ in $L^2$.

2. An invariant subspace and the index

In this section, for a given invariant subspace $\mathcal{M}$ we are interested in two invariant subspaces $\mathcal{M}'$ and $\mathcal{M}''$ such that $\mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{M}''$, $\dim \mathcal{M} \cap \mathcal{M}' < \infty$ and $\dim \mathcal{M}'' \cap \mathcal{M} < \infty$. Under some conditions on $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$, we describe $\mathcal{M}'$ and $\mathcal{M}''$ using $\mathcal{M}$. Corollary 2.4 will be used in §§3 and 4. Corollary 2.4 (i) is known from [1].

When $\mathcal{M}$ is an invariant subspace of $L^2$, for $a \in \mathbb{C}$ put $\text{ind}_a \mathcal{M} = \dim \{ \mathcal{M} \cap (z - a) \mathcal{M} \}$. $\text{ind}_a \mathcal{M}$ is called the index of $\mathcal{M}$ at $a$. It is known (cf. [1]) that for each $n$ ($0 \leq n \leq \infty$) and for any $a \in D$ there exists an invariant subspace $\mathcal{M}$ with $\text{ind}_a \mathcal{M} = n$.

**Theorem 2.1.** Let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces of $L^2$ and $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

(i) $\text{ind}_a \mathcal{M} = 0$ for any $a \notin D$.

(ii) If $\dim \mathcal{M}_2 \cap \mathcal{M}_1 < \infty$, then there exists a polynomial $b$ such that $b \mathcal{M}_2 \subseteq \mathcal{M}_1$, $Z(b) \subseteq D$ and the degree of $b \leq \dim \mathcal{M}_2 \cap \mathcal{M}_1$ and

$$\sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \geq \dim \mathcal{M}_2 \cap \mathcal{M}_1.$$

**Proof.** (i) If $|a| > 1$, then $(z - a)^{-1} \in H^\infty$ and $\mathcal{M} = (z - a) \mathcal{M}$. Hence $\text{ind}_a \mathcal{M} = 0$. If $|a| = 1$, then $(z - a) \mathcal{M} = (z - a) \mathcal{M} \{ z - a(1 + \varepsilon) \}^{-1} \mathcal{M}$. For any $f \in \mathcal{M}$, it is easy to see that

$$\int_D \left| \frac{z - a}{z - a(1 + \varepsilon)} f - f \right|^2 d\mu \to 0 \quad (\varepsilon \to 0)$$

by Lebesgue’s convergence theorem. This implies that $(z - a) \mathcal{M}$ is dense in $\mathcal{M}$ and so $\text{ind}_a \mathcal{M} = 0$ for $|a| = 1$. 
(ii) Put $N = M_2 \ominus M_1$ and $S_z = PM_z|N$, where $M_2$ is a multiplication operator on $L^2$ by the coordinate function $z$ and $P$ is the orthogonal projection from $L^2$ to $N$. If $n = \dim N < \infty$, then there exists a polynomial $b$ of degree $n$ such that $S_b = b(S_z) = 0$ and so $bM_2 \subseteq M_1$. By (i), we may assume that $Z(b) \subset D$. We will prove that $\sum (\ind_a M_2 ; a \in Z(b)) \geq n$. We can write that $b = a_0 \prod_{j=1}^n (z - a_j)$ and so $Z(b) = \{a_1, a_2, \ldots, a_n\}$, where $a_0 \in \mathbb{C}$. If $\sum (\ind_a M_2 ; a \in Z(b)) \leq n - 1$, then we may assume $\ind_a M_2 = 0$. Since $[(z - a_1)M_2]_2 = M_2$,
\[\prod_{j=2}^n (z - a_j)M_2 \subseteq M_1 \subset M_2.\]

Then it is easy to see that $\dim M_2 \ominus [\prod_{j=2}^n (z - a_j)M_2]_2 \leq n - 1$ because $\ind_a M_2 \leq 1$ for $2 \leq j \leq n$. This contradicts that $\dim M_2 \ominus M_1 = n$.

**Corollary 2.2.** Let $M_1$ and $M_2$ be invariant subspaces of $L^2$ and $M_1 \subseteq M_2$. If $\dim M_2 \ominus M_1 = 1$, then $(z - a)M_2 \subseteq M_1 \subset M_2$ for some $a \in D$ and $\ind_a M_2 \geq 1$. If $\ind_a M_1 = 1$ or $\ind_a M_2 = 1$, then $M_1 = [(z - a)M_2]_2$.

**Proof.** By Theorem 2.1, $(z - a)M_2 \subseteq M_1$ for some $a \in D$ and so $\ind_a M_2 \geq 1$. Since $(z - a)M_1 \subseteq (z - a)M_2 \subseteq M_1 \subset M_2$, $M_1 = [(z - a)M_2]_2$ if $\ind_a M_1 = 1$ or $\ind_a M_2 = 1$.

**Corollary 2.3.** Let $M_1$ and $M_2$ be invariant subspaces such that $M_1 \subseteq M_2$ and $\dim M_2 \ominus M_1 = n < \infty$. Suppose that $(z - a)M_j$ is closed for any $a \in D$ when $j = 1, 2$.
If $\ind_a M_1 = 1$ for any $a \in D$ or $\ind_a M_2 = 1$ for any $a \in D$, then $M_1 = bM_2$ and $M_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus M_1$, where $b = \prod_{j=1}^n (z - a_j), \{a_j\} \subset D$ and $\{f_j\} \subset M_1$.

**Proof.** By Theorem 2.1 there exists a polynomial $b$ such that $bM_2 \subseteq M_1$ and $Z(b) \subset D$ and the degree of $b \leq n$. Hence $b = \prod_{j=1}^\ell (z - a_j)$ and $\{a_j\} \subset D$ and $\ell \leq n$. When $\ind_a M_2 = 1$ for any $a \in D$, $\dim M_2 \ominus bM_2 = \ell$ because $(z - a_j)M_2$ is closed for $1 \leq j \leq \ell$ and so $\ell = n$. Hence $M_1 = bM_2$. If $\ind_a M_1 = 1$ for any $a \in D$, $\dim M_1 \ominus bM_1 = \ell$ by the same reason. Since $bM_1 \subseteq bM_2 \subseteq M_1$ and $\dim bM_2 \ominus bM_1 = n$, $\ell = n$ and so $M_1 = bM_2$. Put $M_2 = \langle \varphi_1, \ldots, \varphi_n \rangle \oplus M_1$, where $\{\varphi_j\}$ are orthogonal to $M_1$. What was just proved above, $bM_2 = M_1$ and so $bM_2 = \langle b\varphi_1, \ldots, b\varphi_n \rangle \oplus bM_1 = M_1$. Put $f_j = b\varphi_j$ for $j = 1, \ldots, n$, then $\{f_j\}$ are in $M_1$ and $M_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus M_1$.

**Corollary 2.4.** Let $M$ be an invariant subspace of $L^2$.

(i) If $\dim L^2_M = n < \infty$ and $n \neq 0$, then $M = bL^2_{a}$, where $b = \prod_{j=1}^n (z - a_j)$ and $\{a_j\} \subset D$.

(ii) If $\dim M \ominus L^2_{a} = n < \infty$, then $M = L^2_{a}$.

**Proof.** It is known that $\ind_a L^2_{a} = 1$ and $(z - a)L^2_{a}$ is closed for each $a \in D$. Hence we can apply Corollary 2.3 to $M_1 = L^2_{a}$ or $M_2 = L^2_{a}$. If $M_1 = M$ and $M_2 = L^2_{a}$, then (i) follows. If $M_1 = L^2_{a}$ and $M_2 = M$, then $M = \langle f_1/b, \ldots, f_n/b \rangle \oplus L^2_{a}$, where $b = \prod_{j=1}^n (z - a_j), \{a_j\} \subset D$ and $\{f_j\} \subset L^2_{a}$. For each $1 \leq \ell \leq n$, $f_{\ell}/b \in L^2$ and so
invariant subspaces.
In this section, we study the relation between finite-rank Hankel-type operators and invariant subspaces.

3. Finite-rank Hankel-type operators

In this section, we study the relation between finite-rank Hankel-type operators and invariant subspaces.

**Theorem 3.1.** Let $\mathcal{M}$ be an invariant subspace of $L^2$. Then there does not exist a finite-rank Hankel-type operator $H^\mathcal{M}_\varphi$ except a zero one if and only if $\mathcal{M}$ is weakly divisible.

**Proof.** Suppose $\mathcal{M}$ is weakly divisible. If $H^\mathcal{M}_\varphi$ is of finite rank, then $\ker H^\mathcal{M}_\varphi$ is an invariant subspace in $L^2$ and $\dim L^2 / \ker H^\mathcal{M}_\varphi < \infty$. By (i) of Corollary 2.4, $\ker H^\mathcal{M}_\varphi = bL^2$ for some polynomial $b$ with $Z(b) \subset D$ and so $b\varphi$ belongs to $\mathcal{M}$. Put $f = b\varphi$, then $|f(z)| \leq |b(z)| (z \in D)$, where $\gamma = \|\varphi\|_\infty$. Suppose $b(z) = a_0 \prod_{j=1}^n (z - a_j)$, where $\{a_j\} \subset D$. For any $\ell$ with $1 \leq \ell \leq n$,

$$
|f(z)| \leq |a_0| \prod_{j \neq \ell} |z - a_j| \quad (z \in D)
$$

and $f(z)/(z - a_\ell)$ belongs to $\mathcal{M}$ because $a_\ell \in D$ and $\mathcal{M}$ is weakly divisible. Thus $\varphi(z) = f(z)/b(z)$ belongs to $\mathcal{M}$. Hence $H^\mathcal{M}_\varphi = 0$.

Conversely, if $\mathcal{M}$ is not weakly divisible, then there exists a function $f$ in $\mathcal{M}$ and a point $a$ in $D$ such that $|f(z)| \leq |z - a| (z \in D)$ and $f(z)/(z - a)$ does not belong to $\mathcal{M}$. Put $\varphi = f(z)/(z - a)$, then $\varphi \in L^\infty$ and $H^\mathcal{M}_\varphi$ is not zero because $\varphi \notin \mathcal{M}$. On the other hand, $(z - a)\varphi \in \mathcal{M}$ and so the kernel of $H^\mathcal{M}_\varphi$ contains $(z - a)L^2$. This implies that $H^\mathcal{M}_\varphi$ is of rank one because $L^2 / (z - a)L^2 = \mathbb{C}$. $\square$

**Proposition 3.2.** If there exists a symbol $\varphi$ such that $r(H^\mathcal{M}_\varphi) = n \geq 1$, then there exists a symbol $\varphi_j$ such that $r(H^\mathcal{M}_{\varphi_j}) = j$ for any $j$ with $0 \leq j \leq n - 1$.

**Proof.** Suppose $1 \leq n = r(H^\mathcal{M}_\varphi) < \infty$. Then $\ker H^\mathcal{M}_{\varphi_j}$ is the kernel of $H^\mathcal{M}_{\varphi_j}$ is an invariant subspace of $L^2$ and $L^2 / \ker H^\mathcal{M}_{\varphi_j}$ is of finite dimension $n$. By Corollary 2.4, $\ker H^\mathcal{M}_{\varphi_j} = bL^2$, where $b = \prod_{j=1}^n (z - a_\ell)$ and $(a_\ell) \subset D$. Hence $b\varphi$ belongs to $\mathcal{M}$. Put

$$
\varphi_j = \varphi \prod_{\ell=j+1}^n (z - a_\ell) \quad \text{for } 1 \leq j \leq n - 1,
$$

then $\varphi_j \notin \mathcal{M}$ for $1 \leq j \leq n - 1$ and $\varphi_0 = b\varphi$. Since $\ker H^\mathcal{M}_{\varphi_j} = b_jL^2$ for $1 \leq j \leq n - 1$, where $b_j = \prod_{\ell=1}^n (z - a_\ell)$, $H^\mathcal{M}_{\varphi_j}$ is of finite rank $j$ for $0 \leq j \leq n - 1$. $\square$

**Corollary 3.3.** The following two expressions are equivalent for an invariant subspace $\mathcal{M}$.

(i) If $r(H^\mathcal{M}_\varphi) < \infty$, then $r(H^\mathcal{M}_{\varphi_j}) = 0$. 

(ii) If \( r(H^{M}_{\varphi}) \leq 1 \), then \( r(H^{M}_{\varphi}) = 0 \).

**Proof.** (i) \( \Rightarrow \) (ii). This is clear.

(ii) \( \Rightarrow \) (i). If (i) is not true, then there exists a symbol \( \varphi \) with \( r(H^{M}_{\varphi}) = n \geq 2 \). By Proposition 3.2 there exists a symbol \( \varphi_{1} \) such that \( r(H^{M}_{\varphi_{1}}) = 1 \). This contradicts (ii). \( \square \)

4. Weakly divisible invariant subspaces

For a function \( f \) in \( L^{2}_{\alpha} \), put \( Z(f) = \{ a \in D; f(a) = 0 \} \) and \( Z(G) = \cap \{ Z(f); f \in G \} \) for a subset \( G \) in \( L^{2}_{\alpha} \). For \( 1 \leq p \leq \infty \), if \( E \) is an open set in \( D \), \( H_{E}^{\infty} \) denotes the set of all functions in \( L^{p} \) that are analytic on \( E \). In Corollary 4.2, a weakly divisible invariant subspace \( M \) is described completely when \( M \) is in \( L^{2}_{\alpha} \). There exists a non-zero invariant subspace \( M \) in \( L^{2}_{\alpha} \) such that \( M \cap L^{\infty} = \{ 0 \} \). For it is known (see [5]) that there exists a non-zero function \( f \) in \( L^{2}_{\alpha} \) such that \( Z(f) \) does not satisfy the Blaschke condition.

**Theorem 4.1.** Let \( M \) be an invariant subspace of \( L^{2} \).

(i) If \( M \cap L^{\infty} \subseteq H^{\infty} \) and \( Z(M \cap L^{\infty}) = \emptyset \), then \( M \) is weakly divisible.

(ii) If \( M \cap L^{\infty} = H_{E}^{\infty} \) for some open set \( E \), then \( M \) is weakly divisible.

(iii) If \( M \cap L^{\infty} = \{ 0 \} \), then \( M \) is weakly divisible.

**Proof.** (i) If \( \{ f_{n} \} \) is a sequence in \( M \cap L^{\infty} \) which converges pointwise boundedly to \( f \), then \( f \in M \). By the Krein–Schmullian criterion (see [4, IV 2.1]), \( M \cap L^{\infty} \) is weakly closed. Hence, by a well-known theorem of Beurling [2] \( M \cap L^{\infty} = qH^{\infty} \) for some inner function \( q \). Hence if \( f \in M \) and \( |f(z)| = \gamma|z-a| \) \((z \in D)\) for some \( a \in D \), then \( f = qh \) for some \( h \in H^{\infty} \). Since \( Z(M \cap L^{\infty}) = \emptyset \), \( |q(z)| > 0 \) \((z \in D)\) and so \( h(a) = 0 \). Hence \( f(z)/(z-a) = q(z) \times (h(z)/(z-a)) \in qH^{\infty} \). Thus \( f(z)/(z-a) \) belongs to \( M \).

(ii) If \( f \in H_{E}^{\infty} \) and \( |f(z)| \leq \gamma|z-a| \) \((z \in D)\) for some \( a \in D \), then \( f(z)/(z-a) \in L^{\infty} \) and \( f(z)/(z-a) \) is analytic on \( E \). Hence \( f(z)/(z-a) \) belongs to \( H_{E}^{\infty} \) and so \( M \) is weakly divisible.

(iii) This is clear. \( \square \)

**Corollary 4.2.** Let \( M \) be an invariant subspace of \( L^{2}_{\alpha} \). Then \( M \) is weakly divisible if and only if \( M \cap L^{\infty} = \{ 0 \} \) or \( Z(M \cap L^{\infty}) = \emptyset \).

**Proof.** The part of ‘if’ is a result of (i) and (iii) of Theorem 4.1. Conversely, suppose that \( M \) is weakly divisible. If \( M \cap L^{\infty} \neq \{ 0 \} \), then by a theorem of Beurling there exists an inner function \( q \) with \( M \cap L^{\infty} = qH^{\infty} \). If \( q(a) = 0 \) for some \( a \in D \), then there exists a finite positive constant \( \gamma \) such that \( |q(z)| \leq \gamma|z-a| \) \((z \in D)\) and \( q/(z-a) \notin M \). This contradicts the weak divisibility of \( M \) and so \( Z(q) = Z(M \cap L^{\infty}) = \emptyset \). \( \square \)

**Corollary 4.3.** Let \( M \) be an invariant subspace of \( L^{2} \).

(i) If \( M \subseteq L^{2}_{\alpha} \) and \( \dim L^{2}_{\alpha}/M < \infty \), then \( M \) is not weakly divisible.
(ii) If $M \supseteq L^2_a$ and $\dim M/L^2_a < \infty$, then $M$ is weakly divisible.

**Proof.** (i) If $M \subset L^2_a$ and $\dim L^2_a/M = \ell < \infty$, then by (i) of Corollary 2.4 $M = bL^2_a$, where $b = \prod_{i=1}^{\ell} (z - a_j)$ and $a_j \in D$ (1 $\leq j \leq \ell$). Hence $Z(M \cap L^\infty) = Z(b) \neq \emptyset$ and so by Corollary 4.2 $M$ is not weakly divisible.

(ii) By (2) of Corollary 2.4 $M = L^2_a$ and so $M \cap L^\infty = H^\infty$. Hence (i) of Theorem 4.1 implies that $M$ is weakly divisible. \hfill \Box

**Corollary 4.4.** If $M = H^2_E$ for some open set $E$ in $D$, then $M$ is weakly divisible.

**Proof.** It is a result of (ii) of Theorem 4.1. \hfill \Box

**Proposition 4.5.** Suppose that $M_j$ is a weakly divisible invariant subspace of $L^2$ for $j = 1, 2, \ldots$ and $M_j \times M_\ell = \{fg; f \in M_j \text{ and } g \in M_\ell\} = \{0\}$ if $j \neq \ell$. If $M = \sum_{j=1}^\infty \oplus M_j$, then $M$ is a weakly divisible invariant subspace.

**Proof.** If $f \in M$, then $f = \sum_{j=1}^\infty f_j$ and $|f(z)| = \sum_{j=1}^\infty |f_j(z)|$ ($z \in D$) by hypothesis. This implies that $M$ is weakly divisible. \hfill \Box

**Corollary 4.6.** Let $1 \leq \ell \leq \infty$. Suppose $D_j$ is an open set in $D$ with $\mu(\partial D_j) = 0$ for $1 \leq j \leq \ell$, $D_i \cap D_j = \emptyset (i \neq j)$ and $D = \bigcup_{j=1}^{\ell} D_j$. Then $M = \sum_{j=1}^{\ell} \oplus L^2_a(D_j)$ is weakly divisible.

**Proof.** This is a result of Corollary 4.4 and Proposition 4.5. \hfill \Box

**Proposition 4.7.** If $M$ is a weakly divisible invariant subspace of $L^2$ and $\varphi$ is a unimodular function in $L^\infty$, then $\varphi M$ is a weakly divisible invariant subspace.

**Proof.** From the definition of weak divisibility, the proposition follows trivially. \hfill \Box

**Corollary 4.8.** If $\varphi$ is a unimodular function in $L^\infty$, then $\varphi L^2_a$ is weakly divisible.

**Acknowledgements.** This research was partly supported by Grant-in-Aid for Scientific Research, Ministry of Education of Japan.

**References**