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COMPACT TOEPLITZ OPERATORS WITH CONTINUOUS SYMBOLS ON WEIGHTED BERGMAN SPACES

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Abstract. Let $L^2_a(D, d\sigma d\theta/2\pi)$ be a complete weighted Bergman space on the open unit disc $D$, where $d\sigma$ is a positive finite Borel measure on $[0, 1)$. We show the following: when $\phi$ is a continuous function on the closed unit disc $\overline{D}$, $T_\phi$ is compact if and only if $\phi = 0$ on $\partial D$.

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Let $D$ be the open unit disc and $d\sigma$ a positive finite Borel measure on $[0, 1)$. Let $L^2_a = L^2_0(D, d\sigma d\theta/2\pi)$ be a weighted Bergman space on $D$; that is, $L^2_a$ consists of analytic functions $f$ in $D$ with

$$\|f\|_2^2 = \int_D |f(re^{i\theta})|^2 d\sigma d\theta/2\pi < \infty.$$ 

When $L^2_a$ is closed, $P$ denotes the orthogonal projection from $L^2 = L^2(D, d\sigma d\theta/2\pi)$ onto $L^2_a$. For $\phi$ in $L^\infty = L^\infty(D, d\sigma d\theta/2\pi)$, we consider the Toeplitz operator $T_\phi : L^2_a \to L^2_a$ defined by $T_\phi f = P(\phi f), f \in L^2_a$. We prove the following theorem in this paper. For the Bergman space (that is, $d\sigma = 2rdr$), the Theorem is well known; see [5, p. 107] and [1]. When $d\sigma = (1 - r^2)^\alpha dr(-1 < \alpha < \infty)$, the Theorem is also true; see [3] and [4]. However, that argument does not work for the general situation. We need a new idea in order to prove the Theorem. Let $H = H(D)$ denote the set of all analytic functions on $D$.

**Theorem.** Suppose that $L^2_a = L^2_0(D, d\sigma d\theta/2\pi)$ is complete. When $\phi$ is a continuous function on the closed unit disc $\overline{D}$, $T_\phi$ is compact if and only if $\phi = 0$ on $\partial D$.

In order to prove the Theorem, we need three lemmas.

**Lemma 1.** $L^2_a$ is complete if and only if $\sigma([\varepsilon, 1)) > 0$ for some $\varepsilon$ with $0 < \varepsilon < 1$.

**Proof.** For $a \in D$, put

$$s(\mu, a) = \inf \left\{ \int_D |f|^2 d\mu : f \in H \text{ and } f(a) = 1 \right\},$$

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where $\mathcal{H}$ is the set of all analytic functions on $D$ and $d\mu = d\sigma d\theta / 2\pi$. Statement (1) of Corollary 1 in [2] is valid for $s(\mu, a)$ instead of $S(\mu, a)$. When $(\text{supp} \mu) \cap D$ is a uniqueness set for $\mathcal{H}$, by Statement (1) of Theorem 8 in [2], $L^2_{\mu}$ is complete if and only if, for all compact sets $K$ in $D$,

$$\int_K \log s(\mu, a)rdrd\theta/\pi > -\infty.$$ 

If $\sigma$ is not a zero measure, then $(\text{supp} \mu) \cap D$ is a uniqueness set for $\mathcal{H}$. These statements suffice to prove the Lemma.

**Lemma 2.** If $\sigma([\varepsilon, 1)) > 0$ for every $\varepsilon$ with $0 \leq \varepsilon < 1$, then

$$\lim_{n \to \infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0 \quad (0 \leq \varepsilon < 1).$$

**Proof.** When $\delta$ is a positive constant with $\varepsilon + \delta < 1$, the following inequality holds.

$$\frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} \leq \frac{\sigma([0, \varepsilon])}{\int_{\varepsilon+\delta}^1 (\frac{r}{\varepsilon})^n d\sigma} \leq \frac{\sigma([0, \varepsilon])}{\varepsilon + \delta} \leq \frac{\varepsilon}{\varepsilon + \delta} \leq \frac{1}{\varepsilon + \delta} \leq 1.$$ 

Since they are positive and $\lim_{n \to \infty} \{(\varepsilon + \delta)/\varepsilon\}^n = \infty$, we have

$$\lim_{n \to \infty} \left( \int_0^\varepsilon r^n d\sigma / \int_\varepsilon^1 r^n d\sigma \right) = 0.$$

**Lemma 3.** If for every $\varepsilon$ with $0 \leq \varepsilon < 1$, we have

$$\int_\varepsilon^1 r^n d\sigma > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0,$$

then for any non-negative $\ell$

$$\lim_{n \to \infty} \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} = 1.$$
Proof. For every $\varepsilon$ with $0 \leq \varepsilon < 1$, the following inequality holds.

\[
1 \geq \frac{\int_0^1 r^{n+\ell} \, d\sigma}{\int_0^1 r^n \, d\sigma} = \frac{\int_0^\varepsilon r^{n+\ell} \, d\sigma + \int_\varepsilon^1 r^{n+\ell} \, d\sigma}{\int_0^\varepsilon r^n \, d\sigma + \int_\varepsilon^1 r^n \, d\sigma} \\
\geq \frac{\varepsilon^\ell \int_\varepsilon^1 r^n \, d\sigma}{\int_\varepsilon^1 r^n \, d\sigma + \int_0^\varepsilon r^n \, d\sigma} = \varepsilon^\ell \left(1 + \frac{\int_0^\varepsilon r^n \, d\sigma}{\int_\varepsilon^1 r^n \, d\sigma}\right)^{-1}
\]

because $\int_\varepsilon^1 r^n \, d\sigma > 0$ and $\ell \geq 0$. Thus $\lim_{n \to \infty} \frac{\int_0^1 r^{n+\ell} \, d\sigma}{\int_0^1 r^n \, d\sigma} \geq \varepsilon^\ell$. Let $\varepsilon \to 1$ to prove the lemma.

Proof. Suppose that $\phi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta}$ is continuous on $\bar{D}$, where

\[
\phi_j(r) = \int_0^{2\pi} \phi(re^{i\theta}) e^{-ij\theta} \, d\theta / 2\pi
\]

for $j = 0, \pm 1, \pm 2, \cdots$. Then $\phi_j(r)$ is continuous on $[0,1]$ for any $j$. Put

\[
e_n(re^{i\theta}) = a_n r^n e^{in\theta} = r^n e^{in\theta} / \sqrt{\int_0^1 r^{2n} \, d\sigma}
\]

for $n \geq 0$, then $\{e_n\}$ is an orthonormal basis in $L_2^2$. For each $j$, put

\[
\Phi_j(re^{i\theta}) = r^{||e||} e^{-ij\theta} \phi(re^{i\theta}).
\]

Then $T_{\Phi_j} = T_{re^{i\theta}} T_{\Phi}$ for $j \geq 0$ and $T_{\Phi_j} = T_{\Phi} T_{re^{i\theta}}$ for $j < 0$. If $T_{\Phi}$ is compact, then $T_{\Phi_j}$ is also compact for any $j$. For each $j$, if $n \geq 0$, then

\[
|\langle T_{\Phi_j} e_n, e_n \rangle| \leq \|T_{\Phi_j} e_n\|_2 \|e_n\|_2 = \|T_{\Phi_j} e_n\|_2.
\]

Since $T_{\Phi_j}$ is compact for each $j$ and $e_n \to 0 (n \to \infty)$ weakly, $\|T_{\Phi_j} e_n\|_2 \to 0 (n \to \infty)$ and so $\langle T_{\Phi_j} e_n, e_n \rangle \to 0 (n \to \infty)$. For each $j$. 

\[
\langle T_{\Phi_j} e_n, e_n \rangle = \int_0^{2\pi} \int_0^1 \phi(re^{i\theta}) r^{j|+2n} e^{-i\theta} a_n^2 r^{2n} d\sigma d\theta / 2\pi
\]
\[
= a_n^2 \int_0^1 \phi_j(r) r^{j|+2n} d\sigma
\]

and then \( \lim_{n \to \infty} a_n^2 \int_0^1 \phi_j(r) r^{j|+2n} d\sigma = 0. \) By Lemma 1, \( \sigma([\varepsilon, 1]) > 0 \) for some \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \) and hence \( \sigma([\varepsilon, 1]) > 0 \) for every \( \varepsilon < 1. \) Hence, by Lemma 2, we have

\[
\lim_{n \to \infty} \int_0^1 \frac{r^{2n} d\sigma}{\int_0^1 r^{2n} d\sigma} = 0 \text{ for } (0 \leq \varepsilon < 1).
\]

Then, by Lemma 3, for any integer \( j \) we have

\[
\lim_{n \to \infty} a_n^2 \int_0^1 r^{j|+2n} d\sigma = 1.
\]

Since \( \phi_j(r) \) is continuous on \([0,1], \) we can approximate \( \phi_j(r) \) uniformly by polynomials \( \sum_{l=0}^{k} c_l r^l. \) Since \( \lim_{n \to \infty} a_n^2 \int_0^1 r^{j|+2n} d\sigma = 1 \) for any \( j, \) we obtain

\[
\lim_{n \to \infty} a_n^2 \int_0^1 \left( \sum_{l=0}^{k} c_l r^l \right) r^{j|+2n} d\sigma = \sum_{l=0}^{k} c_l
\]

and so

\[
\lim_{n \to \infty} a_n^2 \int_0^1 \phi_j(r) r^{j|+2n} d\sigma = \phi_j(1).
\]

Thus \( \phi_j(1) = 0 \) for any \( j \) because \( \lim_{n \to \infty} a_n^2 \int_0^1 \phi_j(r) r^{j|+2n} d\sigma = 0, \) and hence \( \phi = 0 \) on \( \partial D. \)

Conversely suppose that \( \phi = 0 \) on \( \partial D. \) Then we may assume that the support set of \( \phi \) is compact in \( D. \) In order to show the compactness of \( T_{\phi}, \) it is sufficient to show that if \( h_n \to 0 \) weakly \( (n \to \infty) \) in \( L^2_D \) then \( h_n \to 0 \) uniformly on \( \text{supp } \phi. \) By hypothesis on \( \sigma, \) any point \( z \in D \) has a bounded point evaluation for \( L^2_D \) because Statement (1) of Corollary 1 in [2] is valid for \( s(\mu, a) \) instead of \( S(\mu, a) \) and \( r(\mu, a)s(\mu, a) = 1(a \in D). \) Hence \( h_n(z) \to 0. \) By the boundedness of analytic functions on \( \text{supp } \phi \) and the uniform boundedness principle, \( h_n \to 0 \) uniformly on \( \text{supp } \phi. \)

REFERENCES


