COMPACT TOEPLITZ OPERATORS WITH CONTINUOUS SYMBOLS ON WEIGHTED BERGMAN SPACES

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Abstract. Let \(L^2_a(D, d\sigma d\theta/2\pi)\) be a complete weighted Bergman space on the open unit disc \(D\), where \(d\sigma\) is a positive finite Borel measure on \([0, 1)\). We show the following: when \(\phi\) is a continuous function on the closed unit disc \(\overline{D}\), \(T_\phi\) is compact if and only if \(\hat{\phi} = 0\) on \(\partial D\).

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Let \(D\) be the open unit disc and \(d\sigma\) a positive finite Borel measure on \([0, 1)\). Let \(L^2_a = L^2_a(D, d\sigma d\theta/2\pi)\) be a weighted Bergman space on \(D\); that is, \(L^2_a\) consists of analytic functions \(f\) in \(D\) with

\[\|f\|_2^2 = \int_D |f(re^{i\theta})|^2 d\sigma d\theta/2\pi < \infty.\]

When \(L^2_a\) is closed, \(P\) denotes the orthogonal projection from \(L^2 = L^2(D, d\sigma d\theta/2\pi)\) onto \(L^2_a\). For \(\phi\) in \(L^\infty = L^\infty(D, d\sigma d\theta/2\pi)\), we consider the Toeplitz operator \(T_\phi : L^2_a \to L^2_a\) defined by \(T_\phi f = P(\phi f), f \in L^2_a\). We prove the following theorem in this paper. For the Bergman space (that is, \(d\sigma = 2r dr\)), the Theorem is well known; see [5, p. 107] and [1]. When \(d\sigma = (1 - r^2)\alpha dr(-1 < \alpha < \infty)\), the Theorem is also true; see [3] and [4]. However, that argument does not work for the general situation. We need a new idea in order to prove the Theorem. Let \(H = H(D)\) denote the set of all analytic functions on \(D\).

**Theorem.** Suppose that \(L^2_a = L^2_a(D, d\sigma d\theta/2\pi)\) is complete. When \(\phi\) is a continuous function on the closed unit disc \(\overline{D}\), \(T_\phi\) is compact if and only if \(\hat{\phi} = 0\) on \(\partial D\).

In order to prove the Theorem, we need three lemmas.

**Lemma 1.** \(L^2_a\) is complete if and only if \(\sigma([\varepsilon, 1)) > 0\) for some \(\varepsilon\) with \(0 \leq \varepsilon < 1\).

**Proof.** For \(a \in D\), put

\[s(\mu, a) = \inf \left\{ \int_D |f|^2 d\mu : f \in H \text{ and } f(a) = 1 \right\},\]

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where $H$ is the set of all analytic functions on $D$ and $d\mu = d\sigma d\theta/2\pi$. Statement (1) of Corollary 1 in [2] is valid for $s(\mu, \alpha)$ instead of $S(\mu, \alpha)$. When $(\text{supp}\mu) \cap D$ is a uniqueness set for $H$, by Statement (1) of Theorem 8 in [2], $L^2_\alpha$ is complete if and only if, for all compact sets $K$ in $D$, \[ \int_K \log s(\mu, \alpha) r dr d\theta / \pi > -\infty. \] If $\sigma$ is not a zero measure, then $(\text{supp}\mu) \cap D$ is a uniqueness set for $H$. These statements suffice to prove the Lemma.

**Lemma 2.** If $\sigma([\epsilon, 1]) > 0$ for every $\epsilon$ with $0 \leq \epsilon < 1$, then

$$
\lim_{n \to \infty} \frac{\int_0^\epsilon r^n d\sigma}{\int_\epsilon^1 r^n d\sigma} = 0 \quad (0 \leq \epsilon < 1).
$$

**Proof.** When $\delta$ is a positive constant with $\epsilon + \delta < 1$, the following inequality holds.

$$
\frac{\int_0^\epsilon r^n d\sigma}{\int_\epsilon^1 r^n d\sigma} \leq \frac{\sigma([0, \epsilon])}{\int_\epsilon^1 \left(\frac{r}{\epsilon}\right)^n d\sigma} \leq \frac{\sigma([0, \epsilon])}{\int_{\epsilon+\delta}^1 \left(\frac{r}{\epsilon}\right)^n d\sigma} \leq \frac{\sigma([0, \epsilon])}{\frac{\epsilon + \delta}{\epsilon} \sigma([\epsilon, \epsilon + \delta, 1])} \quad (0 < \epsilon < 1).
$$

Since they are positive and $\lim_{n \to \infty} \{[(\epsilon + \delta)/\epsilon]^n\} = \infty$, we have

$$
\lim_{n \to \infty} \left(\int_0^\epsilon r^n d\sigma / \int_\epsilon^1 r^n d\sigma\right) = 0.
$$

**Lemma 3.** If for every $\epsilon$ with $0 \leq \epsilon < 1$, we have

$$
\int_\epsilon^1 r^n d\sigma > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\int_0^\epsilon r^n d\sigma}{\int_\epsilon^1 r^n d\sigma} = 0,
$$

then for any non-negative $\ell$

$$
\lim_{n \to \infty} \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} = 1.
$$
Proof. For every \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \), the following inequality holds.

\[
1 \geq \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} = \frac{\int_0^\varepsilon r^{n+\ell} d\sigma + \int_\varepsilon^1 r^{n+\ell} d\sigma}{\int_0^\varepsilon r^n d\sigma + \int_\varepsilon^1 r^n d\sigma} \\
\geq \frac{\varepsilon^\ell \int_\varepsilon^1 r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma + \int_0^\varepsilon r^n d\sigma} \\
= \varepsilon^\ell \left( 1 + \frac{\int_\varepsilon^1 r^n d\sigma}{\int_0^\varepsilon r^n d\sigma} \right)^{-1}
\]

because \( \int_\varepsilon^1 r^n d\sigma > 0 \) and \( \ell \geq 0 \). Thus \( \lim_{n \to \infty} \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} \geq \varepsilon^\ell \). Let \( \varepsilon \to 1 \) to prove the lemma.

Proof. Suppose that \( \phi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta} \) is continuous on \( \tilde{D} \), where

\[
\phi_j(r) = \int_0^{2\pi} \phi(re^{i\theta})e^{-ij\theta} d\theta / 2\pi
\]

for \( j = 0, \pm 1, \pm 2, \cdots \). Then \( \phi_j(r) \) is continuous on \( [0,1] \) for any \( j \). Put

\[
e_n(re^{i\theta}) = a_n r^n e^{in\theta}
\]

for \( n \geq 0 \), then \( \{e_n\} \) is an orthonormal basis in \( L^2_\alpha \). For each \( j \), put

\[
\Phi_j(re^{i\theta}) = r^{|j|} e^{-ij\theta} \phi(re^{i\theta}).
\]

Then \( T_{\phi_j} = T_{r^j e^{-i\theta}} T_{\phi} \) for \( j \geq 0 \) and \( T_{\phi_j} = T_{\phi} T_{r^j e^{-i\theta}} \) for \( j < 0 \). If \( T_{\phi} \) is compact, then \( T_{\phi_j} \) is also compact for any \( j \). For each \( j \), if \( n \geq 0 \), then

\[
|\langle T_{\phi_j} e_n, e_n \rangle| \leq \|T_{\phi_j} e_n\|_2 \|e_n\|_2 = \|T_{\phi_j} e_n\|_2.
\]

Since \( T_{\phi_j} \) is compact for each \( j \) and \( e_n \to 0(n \to \infty) \) weakly, \( \|T_{\phi_j} e_n\|_2 \to 0 \) \( (n \to \infty) \) and so \( \langle T_{\phi_j} e_n, e_n \rangle \to 0 \) \( (n \to \infty) \). For each \( j \),
\[
\langle T_{\Phi_j}e_n, e_n \rangle = \int_0^{2\pi} \int_0^1 \phi(re^{i\theta}) r^{|j|} e^{-i\theta} a^2 r^{2n} d\sigma d\theta / 2\pi
\]

\[
= a^2 \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma
\]

and then \( \lim_{n \to \infty} a^2 \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma = 0 \). By Lemma 1, \( \sigma(\varepsilon, 1) > 0 \) for some \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \) and hence \( \sigma(\varepsilon, 1) > 0 \) for every \( \varepsilon < 1 \). Hence, by Lemma 2, we have

\[
\lim_{n \to \infty} \frac{\int_0^e r^{2n} d\sigma}{\int_0^1 r^{2n} d\sigma} = 0 \text{ for } (0 \leq \varepsilon < 1).
\]

Then, by Lemma 3, for any integer \( j \) we have

\[
\lim_{n \to \infty} a^2 \int_0^1 r^{|j|+2n} d\sigma = 1.
\]

Since \( \phi_j(r) \) is continuous on \([0,1]\), we can approximate \( \phi_j(r) \) uniformly by polynomials \( \sum_{i=0}^k c_i r^i \). Since \( \lim_{n \to \infty} a^2 \int_0^1 r^{|j|+2n} d\sigma = 1 \) for any \( j \), we obtain

\[
\lim_{n \to \infty} a^2 \int_0^1 \left( \sum_{i=0}^k c_i r^i \right) r^{|j|+2n} d\sigma = \sum_{i=0}^k c_i
\]

and so

\[
\lim_{n \to \infty} a^2 \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma = \phi_j(1).
\]

Thus \( \phi_j(1) = 0 \) for any \( j \) because \( \lim_{n \to \infty} \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma = 0 \), and hence \( \phi = 0 \) on \( \partial D \).

Conversely suppose that \( \phi = 0 \) on \( \partial D \). Then we may assume that the support set of \( \phi \) is compact in \( D \). In order to show the compactness of \( T_{\Phi_j} \), it is sufficient to show that if \( h_n \to 0 \) weakly \( (n \to \infty) \) in \( L^2_D \), then \( h_n \to 0 \) uniformly on \( \text{supp} \, \phi \). By hypothesis on \( \sigma \), any point \( z \in \partial D \) has a bounded point evaluation for \( L^2_D \) because Statement (1) of Corollary 1 in \([2]\) is valid for \( s(\mu, a) \) instead of \( S(\mu, a) \) and \( r(\mu, a)s(\mu, a) = 1(a \in D) \). Hence \( h_n(z) \to 0 \). By the boundedness of analytic functions on \( \text{supp} \, \phi \) and the uniform boundedness principle, \( h_n \to 0 \) uniformly on \( \text{supp} \, \phi \).

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