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**HOKKAIDO UNIVERSITY**
COMPACT TOEPLITZ OPERATORS WITH CONTINUOUS SYMBOLS ON WEIGHTED BERGMAN SPACES

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Abstract. Let \( L^2_a(D, d\sigma d\theta/2\pi) \) be a complete weighted Bergman space on the open unit disc \( D \), where \( d\sigma \) is a positive finite Borel measure on \([0, 1)\). We show the following: when \( \phi \) is a continuous function on the closed unit disc \( \overline{D} \), \( T_{\phi} \) is compact if and only if \( \hat{\phi} = 0 \) on \( \partial D \).

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Let \( D \) be the open unit disc and \( d\sigma \) a positive finite Borel measure on \([0, 1)\). Let \( L^2_a = L^2_a(D, d\sigma d\theta/2\pi) \) be a weighted Bergman space on \( D \); that is, \( L^2_a \) consists of analytic functions \( f \) in \( D \) with

\[
\|f\|_2^2 = \int_D |f(re^{\theta})|^2 d\sigma d\theta/2\pi < \infty.
\]

When \( L^2_a \) is closed, \( P \) denotes the orthogonal projection from \( L^2 = L^2(D, d\sigma d\theta/2\pi) \) onto \( L^2_a \). For \( \phi \) in \( L^\infty = L^\infty(D, d\sigma d\theta/2\pi) \), we consider the Toeplitz operator \( T_{\phi} : L^2_a \to L^2_a \) defined by \( T_{\phi}f = P(\phi)f, f \in L^2_a \). We prove the following theorem in this paper. For the Bergman space (that is, \( d\sigma = 2\pi dr \)), the Theorem is well known; see [5, p. 107] and [1]. When \( d\sigma = (1 - r^\alpha)^{\sigma} dr (-1 < \alpha < \infty) \), the Theorem is also true; see [3] and [4]. However, that argument does not work for the general situation. We need a new idea in order to prove the Theorem. Let \( H = H(D) \) denote the set of all analytic functions on \( D \).

**Theorem.** Suppose that \( L^2_a = L^2_a(D, d\sigma d\theta/2\pi) \) is complete. When \( \phi \) is a continuous function on the closed unit disc \( \overline{D} \), \( T_{\phi} \) is compact if and only if \( \hat{\phi} = 0 \) on \( \partial D \).

In order to prove the Theorem, we need three lemmas.

**Lemma 1.** \( L^2_a \) is complete if and only if \( \sigma([\varepsilon, 1)) > 0 \) for some \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \).

**Proof.** For \( a \in D \), put

\[
s(\mu, a) = \inf \left\{ \int_D |f|^2 d\mu : f \in H \text{ and } f(a) = 1 \right\},
\]

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where \( H \) is the set of all analytic functions on \( D \) and \( d\mu = d\sigma d\theta / 2\pi \). Statement (1) of Corollary 1 in [2] is valid for \( s(\mu, a) \) instead of \( S(\mu, a) \). When \( (\text{supp}\mu) \cap D \) is a uniqueness set for \( H \), by Statement (1) of Theorem 8 in [2], \( L^2_a \) is complete if and only if, for all compact sets \( K \) in \( D \), \( \int_K \log s(\mu, a) r d\sigma d\theta / \pi > -\infty \). If \( \sigma \) is not a zero measure, then \( (\text{supp}\mu) \cap D \) is a uniqueness set for \( H \). These statements suffice to prove the Lemma.

**Lemma 2.** If \( \sigma([\varepsilon, 1]) > 0 \) for every \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \), then

\[
\lim_{n \to \infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0 \quad (0 \leq \varepsilon < 1).
\]

**Proof.** When \( \delta \) is a positive constant with \( \varepsilon + \delta < 1 \), the following inequality holds.

\[
\frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} \leq \frac{\sigma([0, \varepsilon])}{\int_\varepsilon^1 (r/\varepsilon)^n d\sigma} \leq \frac{\sigma([0, \varepsilon])}{\int_{\varepsilon+\delta}^{1} (r/\varepsilon)^n d\sigma} \leq \frac{\sigma([0, \varepsilon])}{\left(\frac{\varepsilon + \delta}{\varepsilon}\right)^n \sigma([\varepsilon, 1])} \quad (0 < \varepsilon < 1).
\]

Since they are positive and \( \lim_{n \to \infty} ((\varepsilon + \delta)/\varepsilon)^n = \infty \), we have

\[
\lim_{n \to \infty} \left( \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} \right) = 0.
\]

**Lemma 3.** If for every \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \), we have

\[
\int_\varepsilon^1 r^n d\sigma > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0,
\]

then for any non-negative \( \ell \)

\[
\lim_{n \to \infty} \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} = 1.
\]
Proof. For every \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \), the following inequality holds.

\[
1 \geq \frac{\int_{0}^{1} r^{\alpha+\varepsilon} \, d\sigma}{\int_{0}^{1} r^{\alpha} \, d\sigma} = \frac{\int_{\varepsilon}^{1} r^{\alpha+\varepsilon} \, d\sigma + \int_{0}^{\varepsilon} r^{\alpha+\varepsilon} \, d\sigma}{\int_{\varepsilon}^{1} r^{\alpha} \, d\sigma + \int_{0}^{\varepsilon} r^{\alpha} \, d\sigma} \geq \frac{\varepsilon^{\alpha} \int_{0}^{1} r^{\alpha} \, d\sigma}{\int_{\varepsilon}^{1} r^{\alpha} \, d\sigma + \int_{0}^{\varepsilon} r^{\alpha} \, d\sigma} = \varepsilon^{\alpha} \left( 1 + \frac{\int_{0}^{\varepsilon} r^{\alpha} \, d\sigma}{\int_{\varepsilon}^{1} r^{\alpha} \, d\sigma} \right)^{-1}
\]

because \( \int_{\varepsilon}^{1} r^{\alpha} \, d\sigma > 0 \) and \( \varepsilon \geq 0 \). Thus \( \lim_{n \to \infty} \frac{\int_{0}^{1} r^{\alpha+\varepsilon} \, d\sigma}{\int_{0}^{1} r^{\alpha} \, d\sigma} \geq \varepsilon^{\alpha} \). Let \( \varepsilon \to 1 \) to prove the lemma.

Proof. Suppose that \( \phi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} \phi_{j}(r)e^{ij\theta} \) is continuous on \( \tilde{D} \), where

\[
\phi_{j}(r) = \int_{0}^{2\pi} \phi(re^{i\theta}) e^{-ij\theta} \, d\theta / 2\pi
\]

for \( j = 0, \pm 1, \pm 2, \ldots \). Then \( \phi_{j}(r) \) is continuous on \([0,1]\) for any \( j \). Put

\[
e_{n}(re^{i\theta}) = a_{n}r^{n}e^{in\theta}
\]

\[
e_{n}(re^{i\theta}) = r^{n}e^{in\theta} / \sqrt{\int_{0}^{1} r^{2n} \, d\sigma}
\]

for \( n \geq 0 \), then \( \{e_{n}\} \) is an orthonormal basis in \( L_{a}^{2} \). For each \( j \), put

\[
\Phi_{j}(re^{i\theta}) = r^{j}\|e^{i\theta}\| \phi(re^{i\theta}).
\]

Then \( T_{\Phi_{j}} = T_{r^{j}e^{i\theta}} T_{\Phi} \) for \( j \geq 0 \) and \( T_{\Phi_{j}} = T_{\Phi} T_{r^{j}e^{i\theta}} \) for \( j < 0 \). If \( T_{\Phi} \) is compact, then \( T_{\Phi_{j}} \) is also compact for any \( j \). For each \( j \), if \( n \geq 0 \), then

\[
|\langle T_{\Phi_{j}}e_{n}, e_{n} \rangle| \leq \|T_{\Phi_{j}}e_{n}\|_{2} \|e_{n}\|_{2} = \|T_{\Phi_{j}}e_{n}\|_{2}.
\]

Since \( T_{\Phi_{j}} \) is compact for each \( j \) and \( e_{n} \to 0(n \to \infty) \) weakly, \( \|T_{\Phi_{j}}e_{n}\|_{2} \to 0 \) \( (n \to \infty) \) and so \( \langle T_{\Phi_{j}}e_{n}, e_{n} \rangle \to 0 \) \( (n \to \infty) \). For each \( j \),
\[
\langle T_{\varphi_j}e_n, e_n \rangle = \int_0^{2\pi} \int_0^1 \phi(r e^{i\theta}) r^{|j|} e^{-ij\theta} a_n^2 r^{2n} d\sigma d\theta / 2\pi
\]

= \left. a_n^{-2} \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma \right.

and then \( \lim_{n \to \infty} a_n^{-2} \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma = 0 \). By Lemma 1, \( \sigma([\varepsilon, 1)) > 0 \) for some \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \) and hence \( \sigma([\varepsilon, 1)) > 0 \) for every \( \varepsilon < 1 \). Hence, by Lemma 2, we have

\[
\lim_{n \to \infty} \int_0^1 r^{2n} d\sigma = 0 \quad \text{for} \quad (0 \leq \varepsilon < 1).
\]

Then, by Lemma 3, for any integer \( j \) we have

\[
\lim_{n \to \infty} a_n^{-2} \int_0^1 r^{|j|+2n} d\sigma = 1.
\]

Since \( \phi_j(r) \) is continuous on \([0,1]\), we can approximate \( \phi_j(r) \) uniformly by polynomials \( \sum_{l=0}^{k} c_l r^l \). Since \( \lim_{n \to \infty} a_n^{-2} \int_0^1 r^{|j|+2n} d\sigma = 1 \) for any \( j \), we obtain

\[
\lim_{n \to \infty} a_n^{-2} \int_0^1 \left( \sum_{l=0}^{k} c_l r^l \right) r^{|j|+2n} d\sigma = \sum_{l=0}^{k} c_l
\]

and so

\[
\lim_{n \to \infty} a_n^{-2} \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma = \phi_j(1).
\]

Thus \( \phi_j(1) = 0 \) for any \( j \) because \( \lim_{n \to \infty} a_n^{-2} \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma = 0 \), and hence \( \phi = 0 \) on \( \partial D \).

Conversely suppose that \( \phi = 0 \) on \( \partial D \). Then we may assume that the support set of \( \phi \) is compact in \( D \). In order to show the compactness of \( T_{\phi} \), it is sufficient to show that if \( h_n \to 0 \) weakly \((n \to \infty)\) in \( L^2_a \) then \( h_n \to 0 \) uniformly on \( \text{supp} \phi \). By hypothesis on \( \sigma \), any point \( z \in D \) has a bounded point evaluation for \( L^2_a \) because Statement (1) of Corollary 1 in [2] is valid for \( s(\mu, a) \) instead of \( S(\mu, a) \) and \( r(\mu, a) s(\mu, a) = 1(a \in D) \). Hence \( h_n(z) \to 0 \). By the boundedness of analytic functions on \( \text{supp} \phi \) and the uniform boundedness principle, \( h_n \to 0 \) uniformly on \( \text{supp} \phi \).

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