COMPACT TOEPLITZ OPERATORS WITH CONTINUOUS SYMBOLS ON WEIGHTED BERGMAN SPACES

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Abstract. Let \( L^2_a(D, d\sigma d\theta/2\pi) \) be a complete weighted Bergman space on the open unit disc \( D \), where \( d\sigma \) is a positive finite Borel measure on \([0, 1)\). We show the following: when \( \phi \) is a continuous function on the closed unit disc \( \bar{D} \), \( T_\phi \) is compact if and only if \( \hat{\phi} \) is in \( @D \).

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Let \( D \) be the open unit disc and \( d\sigma \) a positive finite Borel measure on \([0, 1)\). Let \( L^2_a = L^2_a(D, d\sigma d\theta/2\pi) \) be a weighted Bergman space on \( D \); that is, \( L^2_a \) consists of analytic functions \( f \) in \( D \) with

\[
\|f\|^2_2 = \int_D |f(re^{i\theta})|^2 d\sigma d\theta/2\pi < \infty.
\]

When \( L^2_\sigma \) is closed, \( P \) denotes the orthogonal projection from \( L^2 = L^2(D, d\sigma d\theta/2\pi) \) onto \( L^2_\sigma \). For \( \phi \) in \( L^\infty = L^\infty(D, d\sigma d\theta/2\pi) \), we consider the Toeplitz operator \( T_\phi : L^2_\sigma \rightarrow L^2_\sigma \) defined by \( T_\phi f = P(\phi f), f \in L^2_\sigma \). We prove the following theorem in this paper. For the Bergman space (that is, \( d\sigma = 2rdr \)), the Theorem is well known; see [5, p. 107] and [1]. When \( d\sigma = (1 - r^\alpha)\sigma dr(-1 < \alpha < \infty) \), the Theorem is also true; see [3] and [4]. However, that argument does not work for the general situation. We need a new idea in order to prove the Theorem. Let \( H = H(D) \) denote the set of all analytic functions on \( D \).

**Theorem.** Suppose that \( L^2_a = L^2_a(D, d\sigma d\theta/2\pi) \) is complete. When \( \phi \) is a continuous function on the closed unit disc \( \bar{D} \), \( T_\phi \) is compact if and only if \( \hat{\phi} = 0 \) on \( @D \).

In order to prove the Theorem, we need three lemmas.

**Lemma 1.** \( L^2_a \) is complete if and only if \( \sigma([\varepsilon, 1)) > 0 \) for some \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \).

**Proof.** For \( a \in D \), put

\[
s(\mu, a) = \inf \left\{ \int_D |f|^2 d\mu : f \in H \text{ and } f(a) = 1 \right\}.
\]

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where $H$ is the set of all analytic functions on $D$ and $d\mu = d\sigma d\theta / 2\pi$. Statement (1) of Corollary 1 in [2] is valid for $s(\mu, a)$ instead of $S(\mu, a)$. When $(\text{supp}\mu) \cap D$ is a uniqueness set for $H$, by Statement (1) of Theorem 8 in [2], $L^2_a$ is complete if and only if, for all compact sets $K$ in $D$, $\int_K \log s(\mu, a) r d\sigma d\theta / \pi > -\infty$. If $\sigma$ is not a zero measure, then $(\text{supp}\mu) \cap D$ is a uniqueness set for $H$. These statements suffice to prove the Lemma.

**Lemma 2.** If $\sigma([\varepsilon, 1]) > 0$ for every $\varepsilon$ with $0 \leq \varepsilon < 1$, then

$$\lim_{n \to \infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0 \quad (0 \leq \varepsilon < 1).$$

**Proof.** When $\delta$ is a positive constant with $\varepsilon + \delta < 1$, the following inequality holds.

$$\frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} \leq \frac{\sigma([0, \varepsilon])}{\int_\varepsilon^1 (\frac{r}{\varepsilon})^n d\sigma} \leq \frac{\sigma([0, \varepsilon])}{\int_{\varepsilon+\delta}^1 (\frac{r}{\varepsilon})^n d\sigma} \leq \frac{\sigma([0, \varepsilon])}{(\varepsilon + \delta)^n \sigma([\varepsilon + \delta, 1])} \quad (0 < \varepsilon < 1).$$

Since they are positive and $\lim_{n \to \infty} (\varepsilon + \delta)/\varepsilon^n = \infty$, we have

$$\lim_{n \to \infty} \left(\frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma}\right) = 0.$$

**Lemma 3.** If for every $\varepsilon$ with $0 \leq \varepsilon < 1$, we have

$$\int_\varepsilon^1 r^n d\sigma > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0,$$

then for any non-negative $\ell$

$$\lim_{n \to \infty} \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} = 1.$$
Proof. For every $\varepsilon$ with $0 \leq \varepsilon < 1$, the following inequality holds.

\[
1 \geq \frac{\int_{0}^{1} r^{n+\ell} d\sigma}{\int_{0}^{1} r^{n} d\sigma} = \frac{\int_{0}^{\varepsilon} r^{n+\ell} d\sigma + \int_{\varepsilon}^{1} r^{n+\ell} d\sigma}{\int_{0}^{\varepsilon} r^{n} d\sigma + \int_{\varepsilon}^{1} r^{n} d\sigma} \\
\geq \frac{\varepsilon \int_{\varepsilon}^{1} r^{n} d\sigma}{\int_{\varepsilon}^{1} r^{n} d\sigma + \int_{\varepsilon}^{1} r^{n} d\sigma} \\
= \varepsilon \left( 1 + \frac{\int_{\varepsilon}^{1} r^{n} d\sigma}{\int_{\varepsilon}^{1} r^{n} d\sigma} \right)^{-1}
\]

because $\int_{\varepsilon}^{1} r^{n} d\sigma > 0$ and $\ell \geq 0$. Thus $\lim_{n \to \infty} \frac{\int_{0}^{1} r^{n+\ell} d\sigma}{\int_{0}^{1} r^{n} d\sigma} \geq \varepsilon^\ell$. Let $\varepsilon \to 1$ to prove the lemma.

Proof. Suppose that $\phi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta}$ is continuous on $\tilde{D}$, where

\[
\phi_j(r) = \int_{0}^{2\pi} \phi(re^{i\theta}) e^{-ij\theta} d\theta / 2\pi
\]

for $j = 0, \pm 1, \pm 2, \ldots$. Then $\phi_j(r)$ is continuous on $[0,1]$ for any $j$. Put

\[
e_n(re^{i\theta}) = a_ne^{i\theta}e^{in\theta} = r^ne^{i\theta} \sqrt{\int_{0}^{1} r^{2n} d\sigma}
\]

for $n \geq 0$, then $\{e_n\}$ is an orthonormal basis in $L^2_\alpha$. For each $j$, put

\[
\Phi_j(re^{i\theta}) = r^{j\ell} e^{-ij\theta} \phi(re^{i\theta}).
\]

Then $T_{\Phi_j} = T_{re^{i\theta}}T_\phi$ for $j \geq 0$ and $T_{\Phi_j} = T_\phi T_{re^{i\theta}}$ for $j < 0$. If $T_\phi$ is compact, then $T_{\Phi_j}$ is also compact for any $j$. For each $j$, if $n \geq 0$, then

\[
|\langle T_{\Phi_j}e_n, e_n \rangle| \leq \|T_{\Phi_j}e_n\|_2 \|e_n\|_2 = \|T_{\Phi_j}e_n\|_2.
\]

Since $T_{\Phi_j}$ is compact for each $j$ and $e_n \to 0 (n \to \infty)$ weakly, $\|T_{\Phi_j}e_n\|_2 \to 0 (n \to \infty)$ and so $\langle T_{\Phi_j}e_n, e_n \rangle \to 0 (n \to \infty)$. For each $j$,
\[
\langle T_\Phi e_n, e_n \rangle = \int_0^{2\pi} \int_0^1 \phi(re^{i\theta})r^{|j|+2n}d\sigma d\theta / 2\pi
= a_n^2 \int_0^1 \phi_j(r)r^{|j|+2n}d\sigma
\]

and then \( \lim_{n \to \infty} a_n^2 \int_0^1 \phi_j(r)r^{|j|+2n}d\sigma = 0 \). By Lemma 1, \( \sigma([\varepsilon, 1]) > 0 \) for some \( \varepsilon \) with \( 0 \leq \varepsilon < 1 \) and hence \( \sigma([\varepsilon, 1]) > 0 \) for every \( \varepsilon < 1 \). Hence, by Lemma 2, we have

\[
\lim_{n \to \infty} \int_0^\varepsilon r^{2n}d\sigma = 0 \quad \text{for } (0 \leq \varepsilon < 1).
\]

Then, by Lemma 3, for any integer \( j \) we have

\[
\lim_{n \to \infty} a_n^2 \int_0^1 r^{|j|+2n}d\sigma = 1.
\]

Since \( \phi_j(r) \) is continuous on \([0,1]\), we can approximate \( \phi_j(r) \) uniformly by polynomials \( \sum_{r=0}^k c_r r^r \). Since \( \lim_{n \to \infty} a_n^2 \int_0^1 r^{|j|+2n}d\sigma = 1 \) for any \( j \), we obtain

\[
\lim_{n \to \infty} a_n^2 \int_0^1 \left( \sum_{r=0}^k c_r r^r \right) r^{|j|+2n}d\sigma = \sum_{r=0}^k c_r
\]

and so

\[
\lim_{n \to \infty} a_n^2 \int_0^1 \phi_j(r)r^{|j|+2n}d\sigma = \phi_j(1).
\]

Thus \( \phi_j(1) = 0 \) for any \( j \) because \( \lim_{n \to \infty} a_n^2 \int_0^1 \phi_j(r)r^{|j|+2n}d\sigma = 0 \), and hence \( \phi = 0 \) on \( \partial D \).

Conversely suppose that \( \phi = 0 \) on \( \partial D \). Then we may assume that the support set of \( \phi \) is compact in \( D \). In order to show the compactness of \( T_\phi \), it is sufficient to show that if \( h_n \to 0 \) weakly \((n \to \infty)\) in \( L^2_a \) then \( h_n \to 0 \) uniformly on supp \( \phi \). By hypothesis on \( \sigma \), any point \( z \in D \) has a bounded point evaluation for \( L^2_a \) because Statement (1) of Corollary 1 in [2] is valid for \( s(\mu, a) \) instead of \( S(\mu, a) \) and \( r(\mu, a)s(\mu, a) = 1(a \in D) \). Hence \( h_n(z) \to 0 \). By the boundedness of analytic functions on supp \( \phi \) and the uniform boundedness principle, \( h_n \to 0 \) uniformly on supp \( \phi \).

REFERENCES


