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Bosonic representation of one-dimensional Heisenberg ferrimagnets

Shoji Yamamoto
Division of Physics, Hokkaido University, Sapporo 060-0810, Japan
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We present a comparative study of bosonic languages to describe one-dimensional Heisenberg ferrimagnets. The ferrimagnetic Schwinger-boson mean-field theory demonstrated by Wu et al., the antiferromagnetic modified spin-wave theory designed by Takahashi, and its ferrimagnetic variant proposed by Yamamoto et al. are employed to calculate the energy structure and the thermodynamics of various ferrimagnets. A modified spin-wave scheme, which introduces a Lagrange multiplier keeping the native energy structure free from temperature and thus differs from the original Takahashi scheme, is particularly stressed as a useful tool to investigate one-dimensional quantum ferrimagnetism. The antiferromagnetic limit of these descriptions is also considered.

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I. INTRODUCTION

Significant efforts have been devoted to synthesizing low-dimensional ferrimagnets and understanding their quantum behavior in recent years. The first example of one-dimensional ferrimagnets, MnCu(S₂C₂O₄)₂(THF)₄·4.5H₂O, was synthesized by Gleizes and Verdaguer and followed by a series of ordered bimetallic chain compounds in an attempt to design molecule-based ferrimagnets. Caneschi et al. demonstrated another approach to alternating-spin chains hybridizing manganese complexes and nitronyl nitroxide radicals. The inorganic-organic hybrid strategy realized more complicated alignments of mixed spins. There also exists an attempt at stacking novel triradicals into a purely organic ferrimagnet. Monospin chains can be ferrimagnetic with polymerized exchange interactions. An example of such ferrimagnets is the ferromagnetic-antiferromagnetic bond-alternating copper tetramer chain compound Cu(C₅H₄NCl)₂(N₃)₂. The trimeric intertwining double-chain compound Ca₂Cu₃(PO₄)₄ is another solution to homometallic one-dimensional ferrimagnets, where the noncompensation of sublattice magnetizations is of topological origin. Besides one-dimensional ferrimagnets, metal-ion magnetic clusters such as [Mn₃O₂(CH₃COO)₆(H₂O)₄] (Ref. 9) and [Fe₈(N₃C₆H₁₅)₆O₆(OH)₁₂] for which resonant magnetization tunneling was observed, are also worth mentioning as zero-dimensional ferrimagnets.

The discovery of ordered bimetallic chain compounds stimulated extensive theoretical interest in (quasi-)one-dimensional quantum ferrimagnets. Early efforts were devoted to numerically diagonalizing alternating-spin Heisenberg chains. Numerical diagonalization, combined with the Lanczos algorithm and a scaling technique, further contributed to studying modern topics such as phase transitions of the Kosterlitz-Thouless type and quantized magnetization plateaux. Alternating-spin chains were further investigated by density-matrix renormalization-group methods and quantum Monte Carlo methods in an attempt to illuminate dual features of ferrimagnetic excitations. More general mixed-spin chains were analyzed via the nonlinear σ model with particular emphasis on the competition between massive and massless phases. Quasi-one-dimensional mixed-spin systems were also investigated in order to explain the inelastic-neutron-scattering findings for the rare-earth nickelates R₂BaNiO₅.

In order to complement numerical tools and to achieve further understanding of the magnetic double structure of ferrimagnetism, several authors have recently begun to construct bosonic theories of low-dimensional quantum ferrimagnets. The conventional spin-wave description of the ground-state properties, a modified spin-wave scheme for the low-temperature properties, and the Schwinger-boson representation of the low-energy structure and the thermodynamics, all reveal the potential of bosonic languages for various ferrimagnetic systems. However, considering the global argument and total understanding over the bosonic theory of ferromagnets and antiferromagnets, ferrimagnets are still undeveloped in this context especially in one dimension. In such circumstances, we represent one-dimensional Heisenberg ferrimagnets in terms of the Schwinger bosons and the Holstein-Primakoff spin waves. Based on a mean-field ansatz, the local constraints on the Schwinger bosons are relaxed and imposed only on the average. The conventional antiferromagnetic spin-wave formalism is modified, on the one hand following the Takahashi scheme which was originally proposed for ferrimagnets, while on the other hand introducing a slightly different strategy. The Schwinger bosons and the modified spin waves both interpret the low-energy properties fairly well identifying the ferrimagnetic long-range order with a Bose condensation, while the two languages are qualitatively distinguished in describing the thermodynamics. We demonstrate that the modified spin-wave scheme is much better than the others at describing one-dimensional ferrimagnets.

II. FORMALISM

A practical model for one-dimensional ferrimagnets is two kinds of spins, S and s (S > s), alternating on a ring with antiferromagnetic exchange coupling between nearest neighbors, as described by the Hamiltonian,
develop the analytic argument in more detail. Hereafter, the distance between neighboring spins is represented by $a$. The simplest case, $(S,s) = (1, \frac{1}{2})$, has so far been discussed fairly well using the matrix-product formalism,\textsuperscript{31} a modified spin-wave scheme,\textsuperscript{30} the Schwinger-boson representation,\textsuperscript{36} and modern numerical techniques.\textsuperscript{22–25} We make further explorations into higher-spin systems and develop the analytic argument in more detail.

A. Schwinger-boson mean-field theory

Let us describe each spin variable in terms of two kinds of bosons as

$$S_+^n = a_n^\dagger a_n, \quad S_-^n = \frac{1}{2} (a_n^\dagger + a_n - a_n^\dagger a_n),$$

$$s_+^n = b_n^\dagger b_n, \quad \bar{s}_-^n = \frac{1}{2} (b_n^\dagger + b_n - b_n^\dagger b_n),$$

where the constraints

$$\sum_{\sigma = \pm} a_{n\sigma}^\dagger a_{n\sigma} = 2S, \quad \sum_{\sigma = \pm} b_{n\sigma}^\dagger b_{n\sigma} = 2s,$$

are imposed on the bosons. Relaxing the constraints (2.3) as

$$\sum_{n=1}^N \sum_{\sigma = \pm} a_{n\sigma}^\dagger a_{n\sigma} = 2NS, \quad \sum_{n=1}^N \sum_{\sigma = \pm} b_{n\sigma}^\dagger b_{n\sigma} = 2Ns,$$

and assuming the thermal average of the short-range antiferromagnetic order to be uniform and static as

$$\langle a_n^\dagger b_n^\dagger - a_n b_n \rangle_T = 2\Omega,$$

with $\bar{n} = n, n - 1$, we obtain the mean-field Hamiltonian in the momentum space as

$$\mathcal{H}_{MF} = 2NJSs + 4NJO^2 - 2NJ\lambda(2S+1) - 2NJ\mu(2s+1) + 2J\sum_k \omega_k + \sum_{\sigma = \pm} (\omega_{k\sigma}^+ \alpha_{k\sigma}^\dagger \alpha_{k\sigma} + \omega_{k\sigma}^- \beta_{k\sigma}^\dagger \beta_{k\sigma}),$$

(2.9)

where

$$\omega_{k\sigma}^\pm = \omega_{k\pm} = \omega_{k\pm}(\mu - \lambda),$$

$$\omega_{k\pm} = \sqrt{(\lambda + \mu)^2 - 4\Omega^2 \cos^2 ak}.$$  

(2.10)

$\lambda$, $\mu$, and $\Omega$ are determined through a set of equations

$$\sum_k (\bar{n}_{k\sigma}^- \cosh^2 \theta_k + \bar{n}_{k\sigma}^+ \sinh^2 \theta_k + \sinh^2 \theta_k) = NS,$$

(2.11)

$$\sum_k (\bar{n}_{k\sigma}^- \sinh^2 \theta_k + \bar{n}_{k\sigma}^+ \cosh^2 \theta_k + \sinh^2 \theta_k) = Ns,$$

(2.12)

$$\sum_k (\bar{n}_{k\sigma}^- + \bar{n}_{k\sigma}^+) \cosh \theta_k \sinh \theta_k = N\Omega,$$

(2.13)

where the thermal distribution functions $\bar{n}_{k\sigma}^\pm = \langle \alpha_{k\sigma}^\dagger \alpha_{k\sigma} \rangle_T$ and $\bar{n}_{k\sigma}^\pm = \langle \beta_{k\sigma}^\dagger \beta_{k\sigma} \rangle_T$ are required to minimize the free energy and given by

$$\bar{n}_{k\sigma}^\pm = \frac{1}{e^{\omega_{k\sigma}^+/T} - 1}.$$  

(2.14)

The magnetic susceptibility is expressed as

$$\chi = \frac{(g\mu_B)^2}{4k_BT} \sum_{k\sigma} \sum_{\tau = \pm} \sum_{\sigma' = \pm} \bar{n}_{k\tau\sigma}^\dagger (\bar{n}_{k\tau\sigma'} + 1),$$  

(2.15)

where we have set the $g$ factors of spins $S$ and $s$ both equal to $g$. The internal energy should be given by

$$E = \frac{1}{2} (E_{MF} + 2NJSs) - 2NJSs,$$  

(2.16)

where

$$E_{MF} = 2NJSs + 4NJO^2 - 2NJ(2\lambda S + 2\mu s + \lambda + \mu) + 2J\sum_k \omega_k + \sum_{\sigma = \pm} \sum_{\tau = \pm} \bar{n}_{k\tau\sigma} \omega_{k\sigma}^\dagger.$$

(2.17)

Arovas and Auerbach\textsuperscript{44} pointed out that relaxing the original constraints (2.3) into Eq. (2.4) leads to double counting the number of independent boson degrees of freedom. Therefore, in Eq. (2.16), we have corrected the mean-field artifact reducing the overestimated quantum fluctuation.

B. Modified spin-wave theory: Takahashi scheme

Next we consider a single-component bosonic representation of each spin variable at the cost of the rotational symmetry. We start from the Holstein-Primakoff transformation
Treating $S$ and $s$ as $O(S) = O(s)$, we can expand the Hamiltonian with respect to $1/S$ as

$$
\mathcal{H} = -2NJSs + E_1 + E_0 + \mathcal{H}_1 + \mathcal{H}_0 + O(S^{-1}),
$$

where $E_i$ and $\mathcal{H}_i$ give the $O(S^i)$ quantum corrections to the ground-state energy and the dispersion relations, respectively. Via the Bogoliubov transformation

$$
a_k = a_\epsilon \cosh \theta_k - \beta_\epsilon \sinh \theta_k,
$$

$$
b_k = \beta_\epsilon \cosh \theta_k - a_\epsilon \sinh \theta_k,
$$

they are written as

$$
E_1 = -2NJ[2\sqrt{S}S\Gamma - (S + s)A],
$$

$$
E_0 = -2NJ[I^2 + A^2 - \left( \sqrt{\frac{S}{S}} + \sqrt{\frac{S}{S}} \right) \Gamma A],
$$

$$
\mathcal{H}_i = J \sum_k \left[ \omega_i^-(k) a_\epsilon^\dagger a_\epsilon + \omega_i^+(k) \beta_\epsilon^\dagger \beta_\epsilon + \gamma_i(k) (a_\epsilon^\dagger \beta_\epsilon + a_\epsilon \beta_\epsilon^\dagger) \right],
$$

where

$$
\Gamma = \frac{1}{2N} \sum_k \cos ak \sinh 2\theta_k,
$$

$$
\Lambda = \frac{1}{2N} \sum_k (\cosh 2\theta_k - 1),
$$

$$
\omega_1^+(k) = (S + s)\cosh 2\theta_k - 2\sqrt{S}\cos ak \sinh 2\theta_k \pm (S - s),
$$

$$
\omega_0^+(k) = \left( \sqrt{\frac{S}{S}} + \sqrt{\frac{S}{S}} \right) \Gamma - 2\Lambda - \left( \sqrt{\frac{S}{S}} \right) \cosh 2\theta_k - \left( \sqrt{\frac{S}{S}} \right) \sinh 2\theta_k \pm (S - s),
$$

$$
\gamma_1(k) = 2\sqrt{S} \cos ak \cosh 2\theta_k - (S + s) \sinh 2\theta_k,
$$

$$
\gamma_0(k) = \left[ 2\Gamma - \left( \sqrt{\frac{S}{S}} + \sqrt{\frac{S}{S}} \right) \Lambda \right] \cos ak \cosh 2\theta_k
$$

$$
- \left[ \left( \sqrt{\frac{S}{S}} + \sqrt{\frac{S}{S}} \right) \Gamma - 2\Lambda \right] \sinh 2\theta_k.
$$

$\mathcal{H}_0$ originally consists of quartic bosonic terms and has been decomposed by means of the Wick theorem. The residual two-body interactions have been neglected so as to keep the ferromagnetic excitation branch gapless.

The conventional spin-wave scheme naively diagonalize the Hamiltonian (2.19) and ends up with the number of bosons diverging with increasing temperature. In order to suppress this thermal divergence, Takahashi\textsuperscript{38} considered optimizing the bosonic distribution functions under zero magnetization and obtained an excellent description of the low-temperature thermodynamics for low-dimensional Heisenberg ferromagnets. For ferrimagnets, this idea is still useful\textsuperscript{34,50} but never applies away from the low-temperature region as it is. The zero-magnetization constraint plays a role of keeping the number of bosons finite under ferromagnetic interactions but does not work so under antiferromagnetic interactions. Takahashi\textsuperscript{38} and Hirsch et al.\textsuperscript{39} proposed constraining the staggered magnetization, instead of the uniform magnetization, to be zero as the antiferromagnetic version of the modified spin-wave theory. Their scheme was applied to extensive antiferromagnets in both two\textsuperscript{38,39,52-55} and one\textsuperscript{40,41} dimensions. The conventional spin-wave procedure assumes that spins on one sublattice point predominantly up, while those on the other predominantly down. The modified spin-wave treatment restores the sublattice symmetry. We consider the naivest generalization of the antiferromagnetic modified spin-wave scheme to ferrimagnets.

The constraint of zero staggered magnetization reads

$$
\sum_n (a_n^\dagger a_n + b_n^\dagger b_n) = N(S + s).
$$

In order to enforce this condition, we first introduce a Lagrange multiplier and diagonalize the effective Hamiltonian

$$
\mathcal{\tilde{H}} = \mathcal{H} + J \nu \sum_n (a_n^\dagger a_n + b_n^\dagger b_n).
$$

Then the ground-state energy and the dispersion relations are obtained as

$$
E_\kappa = -2NJSs + \tilde{E}_1, \quad \tilde{E}_1 = E_1 + 4NJ \Lambda \nu,
$$

$$
\omega_\kappa^+ = \tilde{\omega}_1^+(k), \quad \tilde{\omega}_0^+ = \omega_1^+(k) + 2\nu \cosh 2\theta_k,
$$

keeping only the bilinear terms and as

$$
E_\kappa^- = -2NJSs + \tilde{E}_1 + E_0, \quad \tilde{\omega}_1^- = \tilde{\omega}_0^-(k) + \omega_0^-(k),
$$

considering the $O(S^0)$ interactions as well. In terms of the spin-wave distribution functions

$$
\tilde{n}_\kappa^\pm = \frac{1}{e^{\omega^\kappa / k_BT} - 1},
$$

the internal energy and the magnetic susceptibility are expressed as\textsuperscript{56}
\[ E = E_g + \sum_k \sum_{\tau = \pm} \bar{n}_k \omega^\tau_k, \quad (2.34) \]

\[ \chi = \frac{\langle \mu_B \rangle^2}{3k_B T} \sum_k \sum_{\tau = \pm} \bar{n}_k^\tau (\bar{n}_k^{\bar{\tau}} + 1). \quad (2.35) \]

\[ \theta_k, \text{ defining the Bogoliubov transformation (2.20), is determined through} \]

\[ \gamma_1(k) - 2 \sinh 2 \theta_k = \frac{\bar{n}_k^\tau}{\bar{n}_k^{\bar{\tau}} + 1} = 0, \quad (2.36) \]

provided we treat \( \mathcal{H}_0 \) as a perturbation to \( \mathcal{H}_1 \).

### C. Modified spin-wave theory: A different scheme

Although the Takahashi scheme overcomes the difficulty of sublattice magnetizations diverging thermally, the obtained thermodynamics is still far from satisfactory (see Fig. 2 later on). Within the conventional spin-wave theory, the quantum spin reduction, that is, the quantum fluctuation of the ground-state sublattice magnetization per unit cell, reads

\[ \langle a^\dagger_k a_n \rangle_{T = 0} = \langle b^\dagger_n b_n \rangle_{T = 0} = \delta \]

\[ = \int_0^\pi \frac{S + s}{\sqrt{(S-s)^2 + 4 S s \sin^2(k/2)}} \frac{dk}{2 \pi} - \frac{1}{2}, \]

and diverges at \( S = s \). The Takahashi scheme settles this quantum divergence as well as the thermal divergence. However, the number of bosons does not diverge in the ferrimagnetic ground state. Without quantum divergence, it is not necessary to modify the dispersion relations (2.25a) into the temperature-dependent form (2.30). While the thermodynamics should be modified, the quantum mechanics may be left as it is.

Such an idea leads to the Bogoliubov transformation free from temperature replacing Eq. (2.36) by \( \gamma_1(k) = 0 \), that is,

\[ \tanh 2 \theta_k = \frac{2 \sqrt{S s \cos ak}}{S + s}. \quad (2.38) \]

The ground-state energy and the dispersion relations are simply given by

\[ E_g = -2NJSs + E_1, \quad \omega_k^\tau = \omega_1^\tau(k), \quad (2.39) \]

within the up-to-\( O(S^1) \) treatment and by

\[ E_g = -2NJSs + E_1 + E_0, \quad \omega_k^\tau = \omega_1^\tau(k) + \omega_0^\tau(k), \quad (2.40) \]

in the up-to-\( O(S^0) \) treatment. They are nothing but the \( T = 0 \) findings in the Takahashi scheme.

At finite temperatures we replace \( \alpha_k^\dagger \alpha_k \) and \( \beta_k^\dagger \beta_k \) in the spin-wave Hamiltonian (2.22) by

\[ \bar{n}_k^\tau = \sum_{n = -n^+}^{\infty} n^\tau P_k(n^- n^+), \quad (2.41) \]

where \( P_k(n^- n^+) \) is the probability of \( n^- \) ferromagnetic and \( n^+ \) antiferromagnetic spin waves appearing in the \( k \)-momentum state and satisfies

\[ \sum_{n^-, n^+} P_k(n^-, n^+) = 1 \quad (2.42) \]

for all \( k \)'s. Then the free energy is written as

\[ F = E_g + J \sum_{k} \sum_{n^+, n^+} P_k(n^- n^+) \sum_{\tau = \pm} n^\tau \omega_k^\tau \]

\[ + k_B T \sum_{k} \sum_{n^-, n^+} P_k(n^-, n^+) \ln P_k(n^-, n^+). \quad (2.43) \]

We minimize the free energy with respect to \( P_k(n^-, n^+) \) enforcing a condition

\[ \langle S_n^z - S_n^z \rangle_T + 2 \delta = \langle c_n^\dagger c_n \rangle_T \]

\[ = S + s - \frac{S + s}{N} \sum_k \sum_{\tau = \pm} \bar{n}_k^\tau / \omega_k = 0, \quad (2.44) \]

as well as the trivial constraints (2.42). In the second-side compact expression, the normal ordering is taken with respect to both operators \( \alpha \) and \( \beta \). Equation (2.44) claims that the thermal fluctuation \( (S + s) \sum_k (n^+_k + n^-_k) / \omega_k \) should cancel the full, or classical, Néel order \( (S + s)N \) rather than the quantum-mechanically reduced one \( (S + s - 2 \delta)N \). Without consideration of the quantum fluctuation \( 2 \delta \), which is absent from ferromagnets but peculiar to ferrimagnets, the present

| TABLE I. The Schwinger-boson (SB), linear-modified-spin-wave (LMSW), perturbational interacting-modified-spin-wave (PIMSW), and numerical diagonalization (exact) calculations of the ground-state energy \( E_g \) and the zero-temperature antiferromagnetic excitation gap \( \Delta_0 \) for the spin-(\( S, s \)) ferrimagnetic Heisenberg chains. |
|---|---|---|---|---|---|
| (\( S, s = \frac{1}{2}, \frac{1}{2} \)) | (\( S, s = \frac{3}{2}, \frac{1}{2} \)) | (\( S, s = \frac{3}{2}, \frac{1}{2} \)) |
| Approach | \( E_g/NJ \) | \( \Delta_0/J \) | \( E_g/NJ \) | \( \Delta_0/J \) | \( E_g/NJ \) | \( \Delta_0/J \) |
| SB | -1.45525 | 1.77804 | -1.96755 | 2.84973 | -3.86270 | 1.62152 |
| LMSW | -1.43646 | 1 | -1.95804 | 2 | -3.82807 | 1 |
| PIMSW | -1.46084 | 1.67566 | -1.96983 | 2.80253 | -3.86758 | 1.52139 |
| Exact | -1.4541(1) | 1.759(1) | -1.9672(1) | 2.842(1) | -3.861(1) | 1.615(5) |
In terms of the optimum distribution functions, whereas the modified theory reveals gives no quantitative information on the magnetic susceptibility, the free energy at the thermal equilibrium is written as with Eq. \(\text{Takahashi scheme with Eq. (2.27)}\) and the different scheme with Eq. (2.44) indeed keep unchanged the conventional spin-wave findings

\[
\frac{C}{Nk_B} \sim \frac{3}{4} \sqrt{\frac{S-s}{Ss}} \frac{\zeta(z)}{\sqrt{2\pi}} t^{1/2} (T \rightarrow 0),
\]

where \(t = k_B T/J\) within the up-to-\(O(S^4)\) treatment, while \(t = k_B T/\gamma J\) with \(\gamma = 1 + T/\sqrt{Ss-(S+s)/Ss}\) in the up-to-\(O(S^0)\) treatment. The conventional spin-wave approach gives no quantitative information on the magnetic susceptibility, whereas the modified theory reveals

\[
\frac{\chi J}{N(g \mu_B)^2} \sim \frac{Ss(S-s)^2}{3} t^{-2} (T \rightarrow 0).
\]

In terms of the optimum distribution functions

\[
\tilde{n}_k = \frac{1}{e^{[J\omega_k - \nu(S+s)/\omega_k]/k_BT} - 1},
\]

the free energy at the thermal equilibrium is written as

\[
F = E_g + \nu(S+s)N - \frac{k_B T}{T} \sum_k \sum_{\tau = \pm} \ln(1 + \tilde{n}_k^\tau),
\]

where \(\nu\) is the Lagrange multiplier due to the constraint (2.44).

\section{III. RESULTS}

First we calculate the ground-state energy \(E_g\) and the antiferromagnetic excitation gap \(\omega_{\pm}^0\) and compare them with numerical findings in Table I. At \(T = 0\), the Takahashi scheme and the different scheme both give \(\nu = 0\) and lead to the conventional spin-wave findings. Higher-order spin-wave calculation\(^{27}\) of the ground-state properties is feasible, while more sophisticated series-expansion technique\(^{58}\) has recently been proposed. We are fully convinced that the spin-wave treatment better works for larger spins. Table I further shows that the spin-wave approach is better justified with increasing \(S/s\) as well as \(S_s\), which is because the quantity \(S-s\) fills the role of suppressing the divergence in Eq. (2.37). On the other hand, the Schwinger-boson approach constantly gives highly precise estimates of the low-energy properties. Figure 1 further demonstrates that the Schwinger-boson mean-field theory is highly successful in describing the low-lying excitations. Both the bosonic languages fully interpret the ferromagnetic excitations, whereas the linear spin waves considerably underestimate the antiferromagnetic excitation energies. The quantum correlation has much effect on the antiferromagnetic excitation mode and such an effect is well included into the Schwinger-boson calculation even at the mean-field level.

Next we calculate the thermodynamic properties. Figure 2 shows the temperature dependence of the specific heat. The Schwinger-boson mean-field theory is still highly successful...
the inclusion of the antiferromagnetic excitation gap is significantly improved by the scheme. Since the modified spin-wave theory based on the different scheme succeeds in interpreting the Schottky peak, we can fully investigate such polymeric chain compounds as well.

Above this temperature, \( \Omega \) sticks at zero suggesting no antiferromagnetic correlation in the system. The onset of the paramagnetic phase at a finite temperature is a mean-field artifact and the particular temperature (3.1) is an increasing function of \( S \) and \( s \). The modified spin-wave theory based on the Takahashi scheme also fails to describe the Schottky peak. Because of the Lagrange multiplier \( \nu \), which turns out a monotonically increasing function of temperature, the dispersion relations (2.30) lead to endlessly increasing energy and thus nonvanishing specific heat at high temperatures. Only the modified spin-wave theory based on the different scheme succeeds in interpreting the Schottky peak. Since the antiferromagnetic excitation gap is significantly improved by the inclusion of the \( O(S^0) \) correlation, the interacting modified spin waves reproduce the location of the Schottky peak fairly well. Mixed-spin trimeric chain ferrimagnets have recently been synthesized and their low-temperature thermal properties were well elucidated by the modified spin-wave theory. However, it was unfortunate that the additional constraint was imposed on the uniform magnetization and therefore the higher-temperature properties were much less illuminated. Controlling the staggered magnetization instead based on the different scheme, we can fully investigate such polymeric chain compounds as well.

In the Schwinger representation and the modified spin-wave treatment based on the Takahashi scheme, the energy spectrum depends on temperature. Since the low-energy band structure is well reflected in the thermal behavior and can directly be observed through inelastic-neutron-scattering measurements, we investigate the ferromagnetic \( (\omega_{v=0}^-) \) and antiferromagnetic \( (\omega_{v=0}^+) \) excitation gaps as functions of temperature in Fig. 3. The Schwinger-boson mean-field theory claims that the antiferromagnetic gap should first decrease and then increase with increasing temperature, while the modified spin-wave theory predicts that the antiferromagnetic gap should decrease with increasing temperature, which turns out the two languages applied to ladder ferrimagnets. In the case of Haldane-gap antiferromagnets, both the Schwinger-boson and modified-spin-wave findings, together with the nonlinear-\( \sigma \)-model calculations, commonly suggest that the Haldane gap is a simply activated function of temperature. Extensive measurements on spin-1 antiferromagnetic Heisenberg chain compounds also report that the Haldane massive mode is shifted upward with increasing temperature. Neutron-scattering experiments on ferrimagnetic chain compounds may solve the present disagreement between the Schwinger-boson and modified-spin-wave calculations of the antiferromagnetic excitation gap as a function of temperature.

Figure 4 shows the temperature dependence of the magnetic-susceptibility-temperature product, which elucidates ferromagnetic and antiferromagnetic features coexisting in ferrimagnets. \( \chi T \) diverges at low temperatures in a ferromagnetic fashion but approaches the high-temperature
paramagnetic behavior showing an antiferromagnetic increase. The modified spin waves much better describe the magnetic behavior than the Schwinger bosons. The spin waves modified along with the Takahashi scheme better work at high temperatures, while those along with the different scheme precisely reproduce the low-temperature behavior. Both calculations converge into the paramagnetic behavior \( \chi k_B T/N (g \mu_B)^2 = [S(S+1) + s(s+1)]/3 \) at high temperatures, whereas the Schwinger-boson mean-field theory again breaks down at the particular temperature \( \sim \). However, no parameter assignment has yet succeeded in interpreting all the observations consistently. There are further chemical attempts to synthesize novel ferrimagnets. Organic ferrimagnets are free from magnetic anisotropy and thus suitable for analyzing in terms of the modified spin waves.

**IV. SUMMARY AND DISCUSSION**

We have demonstrated the Schwinger-boson mean-field representation and the modified spin-wave treatment of one-dimensional Heisenberg ferrimagnets. The Schwinger bosons form an excellent language at low temperatures but rapidly lose their validity with increasing temperature. The modified spin-wave theory is more reliable in totality provided the number of bosons is controlled without modifying the native energy structure. On the other hand, the Schwinger-boson representation can be extended to anisotropic systems more reasonably because it is rotationally invariant in contrast to the modified spin-wave theory. While the temperature dependence of the antiferromagnetic excitation gap \( \omega_{k=0} \) is left to solve experimentally, we are now convinced that the bosonic languages remain effective in low dimensions and may be applied to extensive ferrimagnets. Besides ground-state properties and thermodynamics, quantum spin dynamics\(^{70,71}\) can be investigated through the modified spin-wave scheme.

We further mention our findings in the antiferromagnetic limit with the view of realizing the close relation between the two bosonic languages. We equalize \( s \) with \( S \) and set \( 2N \), the number of spins, equal to \( L \) for the Hamiltonian (2.1). At \( S = s \), the ground-state sublattice magnetization (2.37) diverges and therefore the different modified spin-wave scheme is no more applicable. We have to settle the quantum, as well as thermal, divergence inevitably employing the

**TABLE II.** The Schwinger-boson (SB), linear-modified-spin-wave (LMSW), perturbational interacting-modified-spin-wave (PIMSW), full-diagonalization interacting-modified-spin-wave (FDIMS), and quantum Monte Carlo (QMC)\(^2\) calculations of the ground-state energy \( E_g \) and the lowest excitation gap \( \Delta_0 \) for the spin-\( S \) antiferromagnetic Heisenberg chains (Ref. 72).

<table>
<thead>
<tr>
<th>Approach</th>
<th>( S = 1 )</th>
<th>( S = 2 )</th>
<th>( S = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( E_g/J )</td>
<td>( \Delta_0/J )</td>
<td>( E_g/J )</td>
</tr>
<tr>
<td>SB</td>
<td>-1.396148</td>
<td>0.08507</td>
<td>-4.759769</td>
</tr>
<tr>
<td>LMSW</td>
<td>-1.361879</td>
<td>0.07200</td>
<td>-4.726749</td>
</tr>
<tr>
<td>PIMSW</td>
<td>-1.394853</td>
<td>0.07853</td>
<td>-4.759760</td>
</tr>
<tr>
<td>FDIMS</td>
<td>-1.394617</td>
<td>0.08507</td>
<td>-4.759759</td>
</tr>
<tr>
<td>QMC</td>
<td>-1.401481(4)</td>
<td>0.41048(6)</td>
<td>-4.761249(6)</td>
</tr>
</tbody>
</table>
Takahashi scheme. Besides the perturbational treatment of \(\mathcal{H}_0\), we may consider the full diagonalization of \(\mathcal{H}_1 + \mathcal{H}_0\), where the ground-state energy and the dispersion relations are still given by Eqs. (2.31) and (2.32), respectively, but with \(\theta_k\) satisfying

\[
\overline{\gamma}_1(k) + \gamma_0(k) = 0. \tag{4.1}
\]

Such an idea applied to ferrimagnets ends in gapped ferromagnetic excitations and misreads the low-energy physics. The perturbational series-expansion approach is highly successful in the case of ferrimagnets.\(^{32,33}\) Focussing our interest on Haldane-gap antiferromagnets, we list the bosonic calculations of the ground-state properties in Table II. The bosonic languages interpret the ground-state correlation very well but underestimate the Haldane gap considerably. Indeed they cannot detect the topological terms responsible for vanishing gap,\(^{73}\) but they are still qualitatively consistent with the nonlinear-\(\sigma\)-model quantum field theory, yielding the low-temperature limiting behavior \(\omega_k = -\Delta_0e^{-\Delta_0/kT}\) (Refs. 41, 61) and the large-spin asymptotic behavior \(\Delta_0 \propto e^{-\pi s^4}\)\(^{44,46,73}\) The Schwinger-boson mean-field theory and the full-diagonalization interacting modified spin-wave treatment give the same estimate of the Haldane gap. The Schwinger-boson dispersion relation (2.10) indeed coincides analytically with that of the full-diagonalization interacting modified spin waves at zero temperature. This is interesting but not so surprising, because the Holstein-Primakoff bosons (2.18) are obtained by replacing both \(a_n^\dagger (b_n^\dagger)\) and \(a_n (b_n^\dagger)\) by \(\sqrt{2S - a_n a_n^\dagger} / (\sqrt{2S - b_n^\dagger b_n})\) in the transformation (2.2).

Figure 6 shows the thermodynamic calculations for the spin-1 antiferromagnetic Heisenberg chain. We learn that the Schwinger-boson mean-field theory does not work at all for spin-gapped antiferromagnets at finite temperatures, which is in contrast with its fairly good representation of the low-temperature thermodynamics for ferrimagnetic chains. On the other hand, the modified spin-wave treatment maintains its validity to a certain extent. Indeed the Takahashi scheme still fails to reproduce the antiferromagnetic Schottky-type peak of the specific heat, but it describes the susceptibility very well except for the low-temperature findings attributable to the underestimate of the Haldane gap. We may expect the modified spin wave treatments to efficiently depict the dynamic, as well as static, susceptibility for extensive spin-gapped antiferromagnets including spin ladders.\(^{74}\) As for the thermal properties of one-dimensional antiferromagnets, whether spin gapped or not, there is a possibility of a fermionic language\(^{75,76}\) which is in principle compact, being superior to any bosonic representation.

In the case of ferromagnets, the Holstein-Primakoff bosons are already diagonal in the momentum space,\(^{37,56}\) suggesting no quantum fluctuation in the ground state, and therefore the present scheme turns out equivalent to the Takahashi scheme. The different-scheme-based modified spin-wave theory is the very method for low-dimensional ferromagnets and is ready for extensive explorations.

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**References**
