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Equivalent conditions on the central limit theorem  
for a sequence of probability measures on $R$

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ABSTRACT

In this paper we give equivalent conditions on the central limit theorem in total variation norm for a sequence of probability measures on $R$. This generalizes Cacoullos, Papathanasiou and Utev’s central limit theorem in $L^1$-norm for a sequence of probability density functions on $R$. We also give equivalent conditions on the central limit theorem in weak convergence and those on the local limit theorem.

Key words and phrases; central limit theorem, total variation norm.

1. Introduction.

Let $f$ be a probability density function on $R$ such that $\int_R yf(y)dy = \mu$ and $\int_R (y - \mu)^2 f(y)dy = \sigma^2 < \infty$. Cacoullos and Papathanasiou (1989) introduced the following function, called a covariance kernel or $\omega$-function of $f$, to study the characterization of probability distributions:

$$
\omega(x) \equiv \int_{-\infty}^{x} (\mu - y)f(y)dy/[\sigma^2 f(x)] \tag{1}
$$

on the set \{y \in R : f(y) > 0\}.

It is known that $\omega(x)f(x)$ is also a probability density function on $R$ and that $f(x)$ is normal iff $\omega(x) \equiv 1$ (see Cacoullos, Papathanasiou and Utev, 1992).

Put

$$
\phi(x) \equiv (2\pi)^{-1/2} \exp(-x^2/2),
\Phi(x) \equiv \int_{-\infty}^{x} \phi(y)dy. \tag{2}
$$

Then the following was given in Cacoullos, Papathanasiou and Utev, 1994 (Theorem 1.2), which can also be stated as follows.

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Theorem 1.1. Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of probability density functions on \( R \) with \( \int_R y f_n(y) dy = 0 \) and \( \int_R y^2 f_n(y) dy = 1 \) \((n \geq 1)\) and with interval supports. Denoting by \( \omega_n \) the \( \omega \)-function of \( f_n \) \((n \geq 1)\), the following holds:

\[
\lim_{n \to \infty} \int_R |\phi(x) - f_n(x)| dx = 0 \tag{3}
\]

iff \( \lim_{n \to \infty} \int_R \sum_{|\omega_n(x)|-1>\delta} f_n(x) dx = 0 \) for any \( \delta > 0 \).

Remark 1.1. It is easy to see that the following is also true (see Cacoullos, Papathanasiou and Utev, 1994, and the proof of Theorem 1.2 below):

\[
\lim_{n \to \infty} \int_R |\omega_n(x) - 1| f_n(x) dx = 0 \tag{4}
\]

iff \( \lim_{n \to \infty} \int_R \sum_{|\omega_n(x)|-1>\delta} f_n(x) dx = 0 \) for any \( \delta > 0 \).

In this paper we extend Theorem 1.1 to a sequence of Borel probability measures on \((R, \mathcal{B}(R))\), using a different method from Cacoullos, Papathanasiou and Utev (CPU), 1994. We also give equivalent conditions on central and local limit theorems.

In section 2 we state our results which will be proved in section 3.

2. Main results.

In this section we state our results.

Let us first introduce our assumption

(A.1) \( \{P_n\}_{n=1}^{\infty} \) is a sequence of probability measures on \((R, \mathcal{B}(R))\) such that \( \int_R x P_n(dx) = 0 \) and \( \int_R x^2 P_n(dx) = 1 \) \((n \geq 1)\).

Before we state Theorem 2.1, we give the following definition.

Definition 2.1. For a probability measure \( P \) on \((R, \mathcal{B}(R))\) such that \( \int_R y P(dy) = 0 \) and \( \int_R y^2 P(dy) = 1 \) put for \( x \in R \)

\[
W(P)(x) \equiv \int_{-\infty}^{x} -y P(dy) \tag{5}
\]

(cf. \( f^* \) in Lemma 2.2, CPU, 1994).

It is easy to see that \( W(P)(x) \) is a probability density function on \( R \) since \( \int_R y P(dy) = 0 \) and \( \int_R y^2 P(dy) = 1 \). We would also like to point out that \( W(P)(\cdot) \) is defined on the whole real line, though \( \omega(\cdot) \) is not; in fact this necessitated the restriction to interval support of \( f \) in CPU’s proofs of the CLT.

Theorem 2.1. Suppose that (A.1) holds. Then (I), (II) and (III) are equivalent.

(I) \( \lim_{n \to \infty} P_n(dx) = \Phi(dx) \) weakly.

(II) \( W(P_n)(\cdot) \) converges to \( \phi(\cdot) \), as \( n \to \infty \), uniformly on \( R \).

(III) For any \( g \in C_0^\infty(R; R) \).
\[
\lim_{n \to \infty} \int_R g(x)(W(P_n)(x)dx - P_n(dx)) = 0.
\]

For two probability measures \( P(dx) \) and \( Q(dx) \) on \((R, \mathcal{B}(R))\), let

\[
\rho(P, Q) \equiv \sup_{A \in \mathcal{B}(R)} |P(A) - Q(A)|
\]

(6).

denote the total variation distance. For the sake of simplicity, we also write

\[
\rho(P, W(P)) \equiv \sup_{A \in \mathcal{B}(R)} |P(A) - \int_A W(P)(x)dx|.
\]

(7).

The following result plays a crucial role in the proof of Theorem 2.3.

**Proposition 2.2.** For any probability measures \( P(dx) \) such that \( \int_R xP(dx) = 0 \) and that \( \int_R x^2P(dx) < \infty \) and \( Q(dx) = q(x)dx \) on \((R, \mathcal{B}(R))\), and \( r > 0 \),

\[
|\rho(P, W(P)) - \rho(P, Q)| \leq 2r \sup_{|x| \leq r} |g(x) - W(P)(x)| + \int_{|x| \geq r} q(x)dx + \int_{|x| \geq r} x^2P(dx).
\]

(8).

Finally we state our main result. It turns out that the following equivalence can be shown via the weak convergence on the central limit theorem by way of Proposition 2.2.

**Theorem 2.3.** Suppose that (A.1) holds. Then (I) and (II) are equivalent.

(I). \( \lim_{n \to \infty} \rho(P_n, W(P_n)) = 0 \).

(II). \( \lim_{n \to \infty} \rho(\Phi, P_n) = 0 \).

**Remark 2.1.** Theorem 2.3 generalizes Theorem 1.1 in view of the following: for any probability density functions \( f \) and \( g \) on \( R \)

\[
2 \sup_{A \in \mathcal{B}(R)} |\int_A f(x)dx - \int_A g(x)dx| = \int_R |f(y) - g(y)|dy.
\]

Let us state another assumption and definition to state our final result.

(A.2). \( P_n(dx) = f_n(x)dx \) for \( n \geq 1 \).

**Definition 2.2.** For a probability density function \( f \) on \( R \) such that \( \int_R yf(y)dy = 0 \) and \( \int_R y^2f(y)dy = 1 \), put for \( x \in R \),

\[
W(f)(x) \equiv \int_{-\infty}^x -yf(y)dy.
\]

(9).

\( W(f) \) is defined on \( R \) and \( W(f)(x) = \omega(x)f(x) \) on the set \( \{ y \in R : f(y) > 0 \} \) (cf. (1)).

**Proposition 2.4.** Suppose that (A.1)-(A.2) hold. Then (I) and (II) are equivalent.

(I). \( \lim_{n \to \infty} f_n(\cdot) = \phi(\cdot) \), uniformly on every compact subsets of \( R \).

(II). \( \lim_{n \to \infty} W(f_n)(\cdot)/f_n(\cdot) = 1 \), uniformly on every compact subsets of \( R \).
3. Proof of results in section 2.

Let us first prove Theorem 2.1.

Proof of Theorem 2.1.

(Proof of (II) from (I)). For \( x \in R \), take \( r \) for which \( r > |x| \). Then

\[
W(P_n)(x) = \int_{-r}^{x} -yP_n(dy) + \int_{-\infty}^{-r} -yP_n(dy) \equiv I_{n,r} + II_{n,r}.
\]

From (I),

\[
I_{n,r} \to \int_{-r}^{x} -y\phi(y)dy \quad \text{as } n \to \infty
\]

\[
\to \int_{-\infty}^{x} -y\phi(y)dy = \phi(x) \quad \text{as } r \to \infty;
\]

and from (A.1), by Chebychev’s inequality

\[
II_{n,r} \leq 1/r \to 0 \quad \text{as } r \to \infty.
\]

The convergence of \( W(P_n) \) at each point implies the uniform convergence of \( W(P_n) \) on \( R \), since \( W(P)(\cdot) \) is nondecreasing on \( (-\infty, 0] \) and nonincreasing on \( [0, \infty) \), and since \( \phi(x) \) is continuous on \( R \), and since \( \lim_{|x| \to \infty} \phi(x) = 0 \).

Q. E. D.

(Proof of (I) from (II)). By (A.1), \( \{P_n\}_{n \geq 1} \) is tight. Therefore there exist a probability measure \( Q \) on \( (R, B(R)) \) and a subsequence \( \{P_{n_k}\}_{k \geq 1} \) which converges weakly to \( Q \) as \( k \to \infty \).

In the same way as in Proof of (II) from (I), for \( x \in R \), \( W(P_{n_k})(x) \) converges to \( W(Q)(x) \) as \( k \to \infty \), and

\[
W(Q)(x) = \int_{-\infty}^{x} -yQ(dy) = \phi(x),
\]

from (II). This completes the proof.

Q. E. D.

(Proof of (III) from (I)). (II) implies that \( W(P_n)(x)dx \) converges to \( \phi(x)dx \) as \( n \to \infty \), weakly. Since (I) and (II) are equivalent to each other, the proof is over.

Q. E. D.

(Proof of (I) from (III)). For any \( \varphi \in C_{0}^{\infty}(R; R) \)

\[
\int_{R} [\varphi''(x) - x\varphi'(x)]P_n(dx) = \int_{R} \varphi''(x)[P_n(dx) - W(P_n)(x)dx] \to 0
\]

as \( n \to \infty \), from (III). This implies (I), since \( \{P_n\}_{n \geq 1} \) is tight from (A.1) and since \( \phi(x) \) is a unique solution of the following: for any \( g \in C_{0}^{\infty}(R; R) \)
\[ \int_R \left[ g''(x) - xg'(x) \right] f(x) dx = 0, \]
\[ \int_R f(x) dx = 1. \]

Q. E. D.

Next let us prove Proposition 2.2.

Proof of Proposition 2.2.

For \( r > 0 \), put \( U_r(\omega) \equiv \{ y \in R; |y| \leq r \} \). Then we only have to prove the following:

\[ \rho(W(P), Q) \leq 2r \sup_{|x| \leq r} |W(P)(x) - q(x)| + \int_{|x| \geq r} x^2 P(dx) + Q(U_r(\omega)^c), \tag{10} \]

since

\[ |\rho(P, W(P)) - \rho(P, Q)| \leq \rho(W(P), Q). \]

Let us prove (10). For any \( A \in \mathcal{B}(R) \),

\[ \int_A W(P)(x) dx - Q(A) = \int_{A \cap U_r(\omega)} (W(P)(x) - q(x)) dx + \int_{A \cap U_r(\omega)^c} W(P)(x) dx - Q(A \cap U_r(\omega)^c), \]

and from the assumption on \( P \),

\[ \int_{|x| > r} W(P)(x) dx = \int_{|x| > r} \left( \int_{-\infty}^{x} -yP(dy) \right) dx \]
\[ = \int_{r}^{\infty} \left( \int_{x}^{\infty} yP(dy) \right) dx + \int_{-\infty}^{-r} \left( \int_{-\infty}^{x} -yP(dy) \right) dx \]
\[ = \int_{r}^{\infty} y(y - r)P(dy) + \int_{-\infty}^{-r} y(y + r)P(dy) \]
\[ \leq \int_{r}^{\infty} y^2 P(dy) + \int_{-\infty}^{-r} y^2 P(dy). \]

Q. E. D.

Finally we prove Theorem 2.3.

Proof of Theorem 2.3.

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(I) and (II) implies (III) and (I) in Theorem 2.1, respectively, and henceforth (II) in Theorem 2.1.

This completes the proof. In fact, by Proposition 2.2, from (A.1), for \( r > 0 \)

\[
|\rho(P_n, W(P_n)) - \rho(P_n, \Phi)|
\]

\[
\leq r \sup_{|x| \leq r} |\phi(x) - W(P_n)(x)| + \int_{|x| \geq r} |\phi(x)|dx + 1 - \int_{|x| \leq r} x^2 P_n(dx)
\]

\[
\rightarrow \int_{|x| \geq r} |\phi(x)|dx + 1 - \int_{|x| \leq r} x^2 |\phi(x)|dx
\]

(as \( n \rightarrow \infty \), from Theorem 2.1, (I) and (II))

\[
\rightarrow 0 \text{ (as } r \rightarrow \infty)\).
\]

Q. E. D.

Finally we prove Proposition 2.4.

Proof of Proposition 2.4.

(Proof of (II) from (I)). From (I), by Theorem 2.1, \( \lim_{n \rightarrow \infty} W(f_n)(\cdot) = \phi(\cdot) \), uniformly on \( R \). This and (I) implies (II).

Q. E. D.

(Proof of (I) from (II)). By (II), for any \( g \in C_0^\infty(R; R) \),

\[
\lim_{n \rightarrow \infty} \int_R g(x)[W(f_n)(x) - f_n(x)]dx = 0.
\]

Therefore by Theorem 2.1, \( \lim_{n \rightarrow \infty} W(f_n)(\cdot) = \phi(\cdot) \), uniformly on \( R \). Hence as \( n \rightarrow \infty \),

\[
f_n(\cdot) = W(f_n)(\cdot)\{W(f_n)(\cdot)/f_n(\cdot)\}^{-1} \rightarrow \phi(\cdot),
\]

uniformly on every compact subsets of \( R \) from (II).

Q. E. D.

References


Cacoullos, T., Papathanasiou, V. and Utev, S. A. (1992), Another characterization of the normal law and a proof of the central limit theorem connected with it, Theory Probab. Appl. 37, 581-588.