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Dynamical systems in the variational formulation of the Fokker-Planck equation by the Wasserstein metric

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ABSTRACT
R. Jordan, D. Kinderlehrer and F. Otto proposed the discrete-time approximation of the Fokker-Planck equation by the variational formulation. It is determined by the Wasserstein metric, an energy functional and the Gibbs-Boltzmann entropy functional. In this paper we study the asymptotic behavior of the dynamical systems which describe their approximation of the Fokker-Planck equation and characterize the limit as a solution to a class of variational problems.

MSC: primary 60F15; secondary 60H30.

Key words: Fokker-Planck equation, Wasserstein metric, energy functional, Gibbs-Boltzmann entropy functional, dynamical systems, variational problem.

Abbreviated title: Dynamical systems for Fokker-Planck equation

1. Introduction.

Let us consider a nonnegative solution of the following Fokker-Planck equation:

$$\partial p(t,x)/\partial t = \triangle_x p(t,x) + div_x(\nabla\Psi(x)p(t,x)) \quad (t>0, x\in \mathbf{R}^d), \qquad (1.1).$$

$$\int_{\mathbf{R}^d} p(t,x)dx = 1 \quad (t \geq 0). \qquad (1.2).$$

Here $\Psi(x)$ is a function from $\mathbf{R}^d$ to $\mathbf{R}$, and we put $\triangle_x \equiv \sum_{i=1}^d \partial^2/\partial x_i^2$, $\nabla \equiv (\partial/\partial x_i)_{i=1}^d$, and $div_x(\cdot) \equiv < \nabla, \cdot >$. In Nelson's stochastic mechanics (see [18, 19]), it is crucial to construct a Markov process $\{\xi(t)\}_{t\geq 0}$, so called Nelson process, such that for $t \geq 0$

$$P(\xi(t)\in dx) = p(t,x)dx,$$

$$\xi(t) = \xi(0) - \int_0^t \nabla\Psi(\xi(s))ds + 2^{1/2}W(t),$$

where $W(t)$ denotes a d-dimensional Wiener process (see [26]).

For $\varepsilon > 0$, by (1.1),

$$\partial p(t,x)/\partial t = \varepsilon \triangle_x p(t,x)/2 + div_x\{((1-\varepsilon/2)\nabla_x \log p(t,x) + \nabla\Psi(x))p(t,x)\}. \quad (1.3).$$

Suppose that $\nabla_x \log p(t,x)$ and $\nabla\Psi(x)$ are continuously differentiable in x and that $(1+|x|)^{-1}\nabla_x \log p(t,x)$ and $(1+|x|)^{-1}\nabla\Psi(x)$ are bounded. Then there exists a unique solution to the following stochastic integral equation: for $t \geq 0$ and $x \in \mathbf{R}^d$,

$$\xi^\varepsilon(t,x) = x - \int_0^t \{(1-\varepsilon/2)\nabla_x \log p(s,\xi^\varepsilon(s,x)) + \nabla\Psi(\xi^\varepsilon(s,x))\}ds + \varepsilon^{1/2}W(t) \quad (1.4).$$

such that

$$\int_{\mathbf{R}^d} p_0(y)dyP(\xi^\varepsilon(t,y)\in dz) = p(t,z)dz \qquad (1.5).$$

(see [2 and 26, and also 3, 14, 16, 21, 27]). Moreover for any $T > 0$, $(1-\varepsilon/2)\nabla_x \log p(t,x) + \nabla\Psi(x)$ is the unique minimizer of

$$\int_0^T \int_{\mathbf{R}^d} |b(t,x)|^2 p(t,x)dxdt \qquad (1.6).$$

over all $b(t,x)$ for which

$$\partial p(t,x)/\partial t = \varepsilon\triangle_x p(t,x)/2 - div_x(b(t,x)p(t,x)) \quad (0<t<T, x\in\mathbf{R}^d). \qquad (1.7).$$

(This can be shown in the same way as in (6.1)-(6.2), by replacing $\log p(t,x)$ by $(1-\varepsilon/2)\log p(t,x)$ in (6.2).) By the standard argument (see [8]), one can show the following: for any $x \in \mathbf{R}^d$,

$$P(\lim_{\varepsilon\to 0}\sup_{0\le t\le T}|\xi^0(t,x)-\xi^\varepsilon(t,x)|=0)=1. \qquad (1.8).$$

This means that $\xi^0(t,x)$ can be considered as the semiclassical limit of the Nelson processes $\xi^\varepsilon(t,x)$ with small fluctuation. The minimum of (1.6) over all $b(t,x)$ for which (1.7) hold converges, as $\varepsilon \to 0$, to

$$\int_0^T \int_{\mathbf{R}^d} |d\xi^0(t,x)/dt|^2 p(0,x)dxdt. \qquad (1.9).$$

In this paper we show that ξ^0 also plays a crucial role in the construction, by way of the Wasserstein metric, of the solution to (1.1)-(1.2) (see [12]). We also characterize ξ^0 as the solution to a class of variational problems. The importance of the consideration in (1.3)-(1.9) will be discussed again in the end of section 2.

Let d denote the Wasserstein metric (or distance) defined by the following (see [22] or [4], [5], [10]): for Borel probability measures P, Q on R^d, put

$$\begin{aligned} d(P,Q) \equiv \inf\{(\int_{\mathbf{R}^d\times\mathbf{R}^d} |x-y|^2\mu(dxdy))^{1/2} :\\ \mu(dx\times\mathbf{R}^d)=P(dx), \mu(\mathbf{R}^d\times dy)=Q(dy)\}. \end{aligned} \qquad (1.10).$$

In particular, we put $d(p,q) \equiv d(P,Q)$ when $P(dx)=p(x)dx$ and $Q(dx)=q(x)dx$. Next we introduce the assumption used by R. Jordan, D. Kinderlehrer and F. Otto in [12].
(A.1). $\Psi \in C^\infty(\mathbf{R}^d;[0,\infty))$ and $\sup_{x\in\mathbf{R}^d}\{|\nabla\Psi(x)|/(\Psi(x)+1)\}$ is finite.
(A.2). $p_0(x)$ is a probability density function on $\mathbf{R}^d$ and the following holds:

$$M(p_0) \equiv \int_{\mathbf{R}^d}|x|^2 p_0(x)dx < \infty,$$

$$F(p_0) \equiv \int_{\mathbf{R}^d}(\log p_0(x)+\Psi(x))p_0(x)dx < \infty.$$

Under (A.1)-(A.2), for $h>0$, we can define, a sequence of probability density functions $\{p_h^n\}_{n\ge 0}$ on $\mathbf{R}^d$, inductively, by the following: put $p_h^0 = p_0$, and for p_h^n, determine p_h^{n+1} as the minimizer of

$$d(p_h^n, p)^2/2 + hF(p) \qquad (1.11).$$

over all probability density functions p for which $M(p)$ is finite (see [12, Proposition 4.1]). For a probability density function p on $\mathbf{R}^d$, put

$$E(p) \equiv \int_{\mathbf{R}^d}\Psi(x)p(x)dx, \quad S(p) \equiv \int_{\mathbf{R}^d}\log p(x)p(x)dx, \qquad (1.12).$$

and for $h \in (0,1)$, $t \geq 0$ and $x \in \mathbf{R}^d$, put

$$p_h(t,x) \equiv p_h^{[t/h]}(x), \tag{1.13.}$$

where $[r]$ denotes the integer part of $r \in \mathbf{R}$. Then the following is known (see [20] and the references therein for an application to physics).

Theorem 1.1. *([12, Theorem 5.1]). Suppose that (A.1)-(A.2) hold. Then for any $T > 0$, as $h \to 0$, $p_h(T,\cdot)$ converges to $p(T,\cdot)$ weakly in $L^1(\mathbf{R}^d; dx)$, and p_h converges to p strongly in $L^1([0,T] \times \mathbf{R}^d; dtdx)$, where $p(t,x) \in C^\infty((0,\infty) \times \mathbf{R}^d; [0,\infty))$ is the unique solution of (1.1)-(1.2) with an initial condition*

$$p(t,\cdot) \to p_0, \quad \text{strongly in } L^1(\mathbf{R}^d; dx), \text{ as } t \to 0, \tag{1.14.}$$

and $M(p(t,\cdot))$, $E(p(t,\cdot))$ and $S(p(t,\cdot))$ belong to $L^\infty([0,T]; dt)$.

For $p_h^n(x)$ and $p_h^{n+1}(x)$, there exists a lower semicontinuous convex function $\varphi_h^{n+1}(x)$ such that

$$p_h^n(x)\delta_{\nabla \varphi_h^{n+1}(x)}(dy)dx \tag{1.15.}$$

is the minimizer of $d(p_h^n, p_h^{n+1})$. $\nabla \varphi_h^{n+1}$ is called Monge function for $d(p_h^n, p_h^{n+1})$. On the probability space $(\mathbf{R}^d, \mathbf{B}(\mathbf{R^d}), P_0(dx) \equiv p_0(x)dx)$, put for $h \in (0,1)$, $t \geq 0$ and $x \in \mathbf{R}^d$,

$$\begin{aligned} X^h(0,x) &= \nabla \varphi_h^0(x) \equiv x, \\ X^h(t,x) &= \nabla \varphi_h^{[t/h]}(X^h(\max([t/h]-1,0)h,x)). \end{aligned} \tag{1.16.}$$

In this paper we first give a stochastic representation for $p(t,x)$ (see Theorem 2.1) from which we give the estimate for $\nabla_x \log p(t,x)$ (see Theorem 2.2) . In the proof, we use exponential estimates on large deviations and the idea in [25] where they gave estimates for the derivatives of the transition probability density functions of diffusion processes (see section 4). By this estimate and an assumption on Ψ (see section 2), we can construct the solution to the following: for $x \in \mathbf{R}^d$,

$$\begin{aligned} dX(t,x)/dt &= -\nabla_x \log p(t, X(t,x)) - \nabla \Psi(X(t,x)) \quad (t > 0), \\ X(0,x) &= x \end{aligned} \tag{1.17.}$$

(From now on, we use the notation $X(t,x)$ instead of $\xi^0(t,x)$.) We also show that $X^h(t,x)$ converges to $X(t,x)$, as $h \to 0$. In particular, it can be shown that $P_0^{X(t,\cdot)^{-1}}(dx) = p(t,x)dx$ for $t \geq 0$ (see Theorem 2.3). (Recall that $P_0^{X(t,\cdot)^{-1}}(B) = P_0(\{x \in \mathbf{R}^d : X(t,x) \in B\})$ for $B \in \mathbf{B}(\mathbf{R^d})$.) This is conjecturable by the Euler equation to (1.11). It can be written, formally, as the following: for $n \geq 0$,

$$\begin{aligned} &X^h((n+1)h,x) - X^h(nh,x) \\ &= -h\{\nabla \log p_h^{n+1}(X^h((n+1)h,x)) + \nabla \Psi(X^h((n+1)h,x))\} \end{aligned} \tag{1.18.}$$

(see Lemma 5.3 in section 5 for the exact statement of (1.18)).

Let us give two examples.

Example 1.1 (One-dimensional case (see [22, Chap. 3], or [17], [23], [24])). Put, for $n \geq 0$, $h \in (0,1)$ and $x \in \mathbf{R}$,

$$F_h^n(x) = \int_{(-\infty,x]} p_h^n(y)dy. \tag{1.19}$$

For a distribution function F on $\mathbf{R}$, put

$$F^{-1}(u) \equiv \sup\{x \in \mathbf{R} : F(x) < u\} \quad \text{for } 0 < u < 1. \tag{1.20}$$

Then for $n \geq 0$, $h \in (0,1)$, $x \in \mathbf{R}$ and $t \geq 0$,

$$\begin{aligned} \nabla \varphi_h^{n+1}(x) &= (F_h^{n+1})^{-1}(F_h^n(x)), \\ X^h(t,x) &= (F_h^{[t/h]})^{-1}(F_0(x)). \end{aligned} \tag{1.21}$$

Example 1.2 (Gaussian case). If $\Psi(x) = 0$ and $p_0(x) = (4\pi)^{-d/2}\exp(-|x|^2/4)$, then

$$p(t,x) = (4\pi(t+1))^{-d/2}\exp(-|x|^2/\{4(t+1)\}), \quad X(t,x) = (t+1)^{1/2}x. \tag{1.22}$$

In section 2, we state our result which will be proved in sections 3-6.

2. Convergence and characterization of dynamical systems.

In this section we state our main result. Let us recall that $P_0(dx) = p_0(x)dx$. The following is an additional assumption in this paper.

(A.3). $\Psi \in C^4(\mathbf{R}^d; \mathbf{R})$ and has bounded second, third and fourth derivatives.

(A.4). $p_0(\cdot)$ is a probability density function on $\mathbf{R}^d$, and is twice continuously differentiable, with bounded derivatives up to the second order.

$(A.5)$. $$-\infty < -C_1 \equiv \inf_{x\in\mathbf{R}^d}\{(|x|^2+1)^{-1}\log p_0(x)\}.$$

$(A.6)$. $$\infty > C_2 \equiv \sup_{x\in\mathbf{R}^d}\{(|x|+1)^{-1}|\nabla \log p_0(x)|\}.$$

For $t \geq 0$ and $y \in \mathbf{R}^d$, let $\{Y(s,(t,y))\}_{s\geq t}$ be the solution to the following stochastic integral equation:

$$Y(s,(t,y)) = y + \int_t^s \nabla\Psi(Y(u,(t,y)))du + 2^{1/2}(W(s) - W(t)). \tag{2.1}$$

(2.1) has a unique strong solution under (A.3) (see [9], [13], or [26]). We also put, for the sake of simplicity,

$$Y(s,y) \equiv Y(s,(0,y)). \tag{2.2}$$

It is known that $\{Y(s,(t,y))\}_{s\geq t}$ has the same probability law as that of $\{Y(s,y)\}_{s\geq 0}$.

The following theorem gives a stochastic representation for $p(t,x)$.

Theorem 2.1. *Suppose that (A.1)-(A.4) hold. Then for any $T > 0$, $p(t,x)$ is continuously differentiable in t and has bounded, continuous derivatives up to the second order in x on $[0,T] \times \mathbf{R}^d$, and for any $t > 0$ and $x \in \mathbf{R}^d$,*

$$p(t,x) = E[p_0(Y(t,x)) \exp(\int_0^t \triangle\Psi(Y(s,x))ds)]. \qquad (2.3).$$

By Theorem 2.1, we obtain the following result.

Theorem 2.2. *Suppose that (A.1)-(A.6) hold. Then for any $T > 0$,*

$$\sup_{x \in \mathbf{R}^d, 0 \le t \le T} \{(|x|+1)^{-1} |\nabla_x \log p(t,x)|\} < \infty. \qquad (2.4).$$

In particular, (1.17) has a unique solution.
REMARK 2.1. In Theorems 2.1 and 2.2, we assumed (A.1)-(A.2) only to use the fact that $p(t,x)$ is a smooth solution to (1.1)-(1.2) with (1.14).

By Theorems 2.1-2.2, we obtain the following.

Theorem 2.3. *Suppose that (A.1)-(A.6) hold. Then for any $T > 0$ and $\delta > 0$,*

$$\lim_{h \to 0} P_0(\sup_{0 \le t \le T} |X(t,x) - X^h(t,x)| \ge \delta) = 0. \qquad (2.5).$$

In particular, for $t \ge 0$,

$$P_0^{X(t,\cdot)^{-1}}(dy) = p(t,y)dy. \qquad (2.6).$$

Put, for $T > 0$,

$$\begin{aligned} A^T \equiv \{ \{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d} ; P_0^{S(t,\cdot)^{-1}}(dx) = p(t,x)dx (0 \le t \le T), \qquad (2.7). \\ \{S(t,x)\}_{0 \le t \le T} \text{ is absolutely continuous, } P_0 - a.s. \}. \end{aligned}$$

The following result is a version of [14] in the case the stochastic processes under consideration do not have random time evolution.

Theorem 2.4. *Suppose that (A.1)-(A.6) hold. Then for any $T > 0$ and any $\{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d} \in A^T$,*

$$E_0[\int_0^T |dX(t,x)/dt|^2 dt] \le E_0[\int_0^T |dS(t,x)/dt|^2 dt], \qquad (2.8).$$

where the equality holds if and only if $dS(t,x)/dt = dX(t,x)/dt \ dtP_0(dx)$-a.e..

For $h \in (0,1)$ and $n \ge 0$, let $\nabla \tilde{\varphi}_h^{n+1}$ be the Monge function for $d(p(nh,\cdot), p((n+1)h,\cdot))$ (see section 1). On the probability space $(\mathbf{R}^d, \mathbf{B}(\mathbf{R^d}), P_0)$, put for $h \in (0,1)$, $t \ge 0$ and $x \in \mathbf{R}^d$,

$$\tilde{X}^h(0,x) = \nabla\tilde{\varphi}_h^0(x) \equiv x, \quad \tilde{X}^h((k+1)h,x) = \nabla\tilde{\varphi}_h^{k+1}(\tilde{X}^h(kh,x)) \quad (k \geq 0),$$
$$\tilde{X}^h(t,x) = \tilde{X}^h([t/h]h,x) + (t - [t/h]h) \times (\tilde{X}^h(([t/h]+1)h,x) - \tilde{X}^h([t/h]h,x))/h. \qquad (2.9).$$

Put also for $h \in (0,1)$ and $T > 0$,

$$A_h^T \equiv \{\{S(t,x)\}_{0\leq t\leq T, x\in\mathbf{R}^d}; P_0^{S(t,\cdot)^{-1}}(dx) = p(t,x)dx(t = 0, h, \cdots, [T/h]h), \{S(t,x)\}_{0\leq t\leq T} \text{ is absolutely continuous, } P_0 - a.s.\}. \qquad (2.10).$$

Then the following holds.

Theorem 2.5. *Suppose that (A.1)-(A.6) hold. Then for any $h \in (0,1)$ and $T \geq h$, $\{\tilde{X}^h(t,x)\}_{0\leq t\leq T, x\in\mathbf{R}^d}$ is the unique minimizer of*

$$\int_0^{[T/h]h} E_0[|dS(t,x)/dt|^2]dt \qquad (2.11).$$

over all $\{S(t,x)\}_{0\leq t\leq T, x\in\mathbf{R}^d} \in A_h^T$, and the following holds: for any $T > 0$ and $\delta > 0$,

$$\lim_{h\to 0} P_0(\sup_{0\leq t\leq T} |X(t,x) - \tilde{X}^h(t,x)| \geq \delta) = 0. \qquad (2.12).$$

For Borel probability density functions $p_0(x)$ and $p_1(x)$ on $\mathbf{R}^d$, the Markov diffusion process $\{\tilde{\xi}(t)\}_{0\leq t\leq 1}$ with a drift vector $b^{\tilde{\xi}}(t,x)$ and with an identity diffusion matrix is called the h-pass process with the initial and terminal distributions $p_0(x)dx$ and $p_1(x)dx$, respectively, if and only if $P(\tilde{\xi}(t) \in dx) = p_t(x)dx$ $(t = 0,1)$ and if $\int_0^1 E[|b^{\tilde{\xi}}(t,\tilde{\xi}(t))|^2]dt$ is the minimum of $\int_0^1 \int_{\mathbf{R}^d} |b(t,x)|^2 q(t,x)dxdt$ over all $(b(t,x), q(t,x))$ for which $q(t,x)$ satisfies (1.7) with $\varepsilon = 1$ and with p replaced by q on $(0,1) \times \mathbf{R}^d$ and for which $q(t,x) = p_t(x)$ $(t = 0,1)$. Theorem 2.5 implies that $\tilde{X}^1(t,x)$ on $(\mathbf{R}^d, \mathbf{B}(\mathbf{R^d}), P_0)$ plays a similar role to that of the h-path process (see [14]), when diffusion matrices vanish. If the similar result to (1.3)-(1.9) holds for $\tilde{X}^1(t,x)$ and the h-pass process with a diffusion matrix $= \varepsilon Id$, then one can consider Theorem 2.5 as a zero noise limit of stochastic control problems. This implies that one might be able to treat the Monge-Kantorovich problem in the frame work of stochastic control problems. This is our future problem.

3. Proof of Theorem 2.1.

In this section we prove Theorem 2.1. The proof is devided into four lemmas.

For a m-dimensional vector function $f(x) = (f^i(x))_{i=1}^m$ $(x \in \mathbf{R}^d)$, put

$$Df(x) \equiv (\partial f(x)/\partial x_i)_{i=1}^d, \quad |f|_\infty \equiv \sup_{x\in\mathbf{R}^d} (\sum_{i=1}^m |f^i(x)|^2)^{1/2}. \tag{3.1}$$

The following lemma can be proved by the standard argument, making use of Itô's formula (see e.g. [9]) and of Gronwall's inequality (see [11]), and we omit the proof (see also [9, p. 120, Theorem 5.3]).

Lemma 3.1. *Suppose that (A.3) holds. Then (2.1) has a unique strong solution, and there exist positive constants C_3 and $\{C(m)\}_{m\geq 1}$ which depends only on $|\nabla\Psi|_\infty$ and $|D^2\Psi|_\infty$ such that for $t \geq 0$ and $y \in \mathbf{R}^d$,*

$$\begin{aligned} &E[|Y(t,y)|^{2m}] \leq C(m)(\sum_{k=1}^m |y|^{2k} + t)\exp(C(m)t) \quad (m \geq 1), \\ &|\partial Y^i(t,y)/\partial y_j| \leq C_3 \exp(C_3 t), \quad P-a.s. \quad (i,j=1,\cdots,d). \end{aligned} \tag{3.2}$$

For $t \geq 0$ and $y \in \mathbf{R}^d$, put

$$q(t,y) = E[p_0(Y(t,y))\exp(\int_0^t \triangle\Psi(Y(s,y))ds)]. \tag{3.3}$$

Then the following can be proved in the same way as in [9, Chap. 5, Theorems 5.5 and 6.1] and the proof is omitted.

Lemma 3.2. *Suppose that (A.3)-(A.4) hold. Then for any $T \geq 0$, $q(t,y)$ has bounded, continuous derivatives in y up to the second order, and is continuously differentiable in t on $[0,T]\times\mathbf{R}^d$, and is a solution to (1.1).*

By Lemmas 3.1 and 3.2, we get the following lemma.

Lemma 3.3. *Suppose that (A.1)-(A.4) hold. Then for $t \geq 0$ and $x \in \mathbf{R}^d$,*

$$p(t,x) \geq q(t,x). \tag{3.4}$$

(Proof). For $R > 0$ and $x \in \mathbf{R}^d$, put

$$\sigma_R(x) = \inf\{t > 0 : |Y(t,x)| > R\}. \tag{3.5}$$

By Itô's formula, if $R > |x|$ and $0 < s < t$, then one can easily show that the following is true:

$$p(t,x) = E[p(t-\min(\sigma_R(x),s), Y(\min(\sigma_R(x),s),x)) \times \exp(\int_0^{\min(\sigma_R(x),s)} \triangle\Psi(Y(u,x))du)] \geq E[p(t-s,Y(s,x))\exp(\int_0^s \triangle\Psi(Y(u,x))du);\sigma_R(x)\geq t] \to q(t,x), \quad (3.6).$$

as $s\to t$ and then $R\to\infty$. Indeed, by (A.3), Lemma 3.1, and the Cameron-Martin-Maruyama-Girsanov formula (see [13]),

$$\begin{aligned} &E[|p(t-s,Y(s,x))-p(0,Y(s,x))|\exp(\int_0^s \triangle\Psi(Y(u,x))du);\sigma_R(x)\geq t] \quad (3.7).\\ &= E[|p(t-s,x+2^{1/2}W(s))-p(0,x+2^{1/2}W(s))| \\ &\quad\times \exp(\int_0^t < \nabla\Psi(x+2^{1/2}W(u)), 2^{-1/2}dW(u) > - \int_0^t |\nabla\Psi(x+2^{1/2}W(u))|^2 du/4 \\ &\quad + \int_0^s \triangle\Psi(x+2^{1/2}W(u))du); \sup_{0\leq s\leq t}|x+2^{1/2}W(u)|\leq R] \\ &\leq \int_{\mathbf{R}^d} |p(t-s,x+2^{1/2}y)-p(0,x+2^{1/2}y)|dy \\ &\quad\times (2\pi s)^{-d/2}\exp(\sup_{|z|\leq R}\Psi(z)/2 + t|\triangle\Psi|_\infty/2) \to 0, \end{aligned}$$

as $s\to t$, by Theorem 1.1. Here we used the following: by Itô's formula,

$$\begin{aligned} &\int_0^t < \nabla\Psi(x+2^{1/2}W(u)), 2^{-1/2}dW(u) > \\ &= \{\Psi(x+2^{1/2}W(t)) - \Psi(x) - \int_0^t \triangle\Psi(x+2^{1/2}W(u))du\}/2. \end{aligned}$$

Q. E. D.

The following lemma together with Lemma 3.2 completes the proof of Theorem 2.1.

Lemma 3.4. *Suppose that (A.1)-(A.4) hold. Then for* $t\geq 0$ *and* $x\in\mathbf{R}^d$,

$$p(t,x) = E[p_0(Y(t,x))\exp(\int_0^t \triangle\Psi(Y(s,x))ds)]. \quad (3.8).$$

(Proof). By Lemma 3.2, $q(t,x)$ is a solution to (1.1) with $q(0,x)=p_0(x)$. Hence for $t\geq 0$,

$$\int_{\mathbf{R}^d} q(t,x)dx = \int_{\mathbf{R}^d} p_0(x)dx = 1 \quad (3.9).$$

by Lemma 3.3. (3.9) together with (1.2), (3.4) and the continuity of p and q completes the proof.

Q. E. D.

4. Proof of Theorem 2.2.

In this section we prove Theorem 2.2. We put $C_4 = |\nabla\Psi|_\infty + |D^2\Psi|_\infty$.

We first state and prove six technical lemmas.

Lemma 4.1. *Suppose that (A.1)-(A.5) hold. Then there exists a positive constant C_5 which depends only on $|\nabla\Psi|_\infty$, $|D^2\Psi|_\infty$ and $|p_0|_\infty$ such that for $t \geq 0$ and $x \in \mathbf{R}^d$,*

$$\exp(-C_5(|x|^2 + 1 + t)\exp(C_5 t)) \leq p(t,x) \leq C_5 \exp(C_5 t). \tag{4.1.}$$

(Proof). By Lemma 3.4,

$$p(t,x) \leq |p_0|_\infty \exp(t|\triangle\Psi|_\infty), \tag{4.2.}$$

and by Jensen's inequality (see [1]),

$$\begin{aligned} p(t,x) &\geq \exp(E[\log p_0(Y(t,x)) + \int_0^t \triangle\Psi(Y(s,y))ds]) \\ &\geq \exp(-E[C_1(|Y(t,x)|^2 + 1)] - t|\triangle\Psi|_\infty) \end{aligned} \tag{4.3.}$$

by (A.5), which completes the proof by Lemma 3.1.

Q. E. D.

For t and T for which $0 \leq t < T$ and $z \in \mathbf{R}^d$, let $\{Z^T(s,(t,z))\}_{t \leq s \leq T}$ be the solution to the following stochastic integral equation: for $s \in [t,T]$,

$$\begin{aligned} Z^T(s,(t,z)) = z + \int_t^s &\{2\nabla_x \log p(t+T-u, Z^T(u,(t,z))) \\ &+ \nabla\Psi(Z^T(u,(t,z)))\}du + 2^{1/2}\{W(s) - W(t)\}, \end{aligned} \tag{4.4.}$$

up to the explosion time (see [26]).

The following lemma shows that (4.4) has a nonexplosive strong solution.

Lemma 4.2. *Suppose that (A.1)-(A.5) hold. Then for t and T for which $0 \leq t < T$, (4.4) has a unique nonexplosive strong solution and there exists a positive constant C_6 which depends only on $|\nabla\Psi|_\infty$, $|D^2\Psi|_\infty$ and $|p_0|_\infty$ such that for $z \in \mathbf{R}^d$,*

$$\begin{aligned} &C_6 \exp(C_6 T)(|z|^2 + 1 + T) \\ &\geq \sup_{t \leq s \leq T} E[|Z^T(s,(t,z))|^2] + E[\int_t^T |\nabla_x \log p(T+t-s, Z^T(s,(t,z)))|^2 ds]. \end{aligned} \tag{4.5.}$$

(Proof). For $R > 0$, put

$$\tau_R^T(t,z) = \inf\{\min(s,T) > t : |Z^T(s,(t,z))| > R\}. \qquad (4.6).$$

Then by Lemma 4.1,

$$E[\int_t^{\tau_R^T(t,z)} |\nabla_x \log p(T+t-s, Z^T(s,(t,z)))|^2 ds] \qquad (4.7).$$
$$\leq \log C_5 + C_5 T + C_5(|z|^2 + 1 + T)\exp(C_5 T) + T|\triangle\Psi|_\infty.$$

This can be shown by applying Itô's formula to $\log p(T+t-s, Z^T(s,(t,z)))$, and by the following: by (1.1),

$$\partial \log p(t,x)/\partial t = \triangle_x \log p(t,x) + < 2\nabla_x \log p(t,x) + \nabla\Psi(x), \nabla_x \log p(t,x) > \qquad (4.8).$$
$$+ \triangle\Psi(x) - |\nabla_x \log p(t,x)|^2.$$

The following also can be shown, making use of Itô's formula and Gronwall's inequality, by tha standard argument: for $s \in [t,T]$,

$$E[|Z^T(\min(s,\tau_R^T(t,z)),(t,z))|^2] \qquad (4.9).$$
$$\leq (E[\int_t^{\tau_R^T(t,z)} |2\nabla_x \log p(T+t-u, Z^T(u,(t,z)))|^2 du]$$
$$+ |z|^2 + 2(T-t)(d + C_4^2))\exp(2(C_4^2+1)(s-t)).$$

Let $R \to \infty$ in (4.7) and (4.9) and then the proof is over.

Q. E. D.

The following lemma can be proved easily and we only sketch the proof.

Lemma 4.3. *Suppose that (A.3) holds. Then for $T \in (0, 1/(2C_4))$ and $y \in \mathbf{R}^d$,*

$$\limsup_{R\to\infty} R^{-2} \log P(\sup_{0\leq t\leq T} |Y(t,y)| \geq R) \leq -(1 - 2C_4 T)^2/(16T). \qquad (4.10).$$

(Proof). Put

$$r = (R^2 - |y|^2 - 2Td - 2TC_4 R(R+1))/(8TR^2) \qquad (4.11).$$

which is positive for sufficiently large $R > 0$. Then by (3.5) and by applying Itô's formula to $|Y(t,y)|^2$, and by the Cameron-Martin-Maruyama-Girsanov formula,

$$
\begin{aligned}
&P(\sup_{0\le t\le T}|Y(t,y)|\ge R) \qquad (4.12).\\
&=\exp(-rR^2+r|y|^2)E[\exp(r|Y(\sigma_R(y),y)|^2-r|y|^2);\sigma_R(y)\le T]\\
&=\exp(-rR^2+r|y|^2)E[\exp(r2^{3/2}\int_0^{\min(T,\sigma_R(y))}<Y(s,y),dW(s)>\\
&\qquad -4r^2\int_0^{\min(T,\sigma_R(y))}|Y(s,y)|^2ds+\int_0^{\min(T,\sigma_R(y))}(4r^2|Y(s,y)|^2\\
&\qquad +r<2Y(s,y),\nabla\Psi(Y(s,y))>+2rd)ds);\sigma_R(y)\le T]\\
&\le \exp(-rR^2+r|y|^2+T(4r^2R^2+2rC_4R(R+1)+2rd))\\
&=\exp(-R^2(1-(|y|^2+2Td)/R^2-2TC_4(1+1/R))^2/(16T))
\end{aligned}
$$

by (4.11), which completes the proof.

Q. E. D.

Lemma 4.4. *Suppose that (A.1)-(A.5) hold. Then for t and T for which $0\le t<T$ and $z\in\mathbf{R}^d$, the probability law of $\{Z^T(s,(t,z))\}_{t\le s\le T}$ is absolutely continuous with respect to that of $\{Y(s,(t,z))\}_{t\le s\le T}$ and on $C([t,T];\mathbf{R}^d)$,*

$$
\begin{aligned}
&(dP^{Z^T(\cdot,(t,z))^{-1}}/dP^{Y(\cdot,(t,z))^{-1}})(Y(\cdot,(t,z))) \qquad (4.13).\\
&=[p(t,Y(T,(t,z)))/p(T,z)]\exp(\int_t^T\triangle\Psi(Y(s,(t,z)))ds).
\end{aligned}
$$

Moreover if $T-t<1/(2C_4)$, then

$$
\begin{aligned}
&\limsup_{R\to\infty}R^{-2}\log P(\sup_{t\le s\le T}|Z^T(s,(t,z))|\ge R) \qquad (4.14).\\
&\le -(1-2C_4(T-t))^2/(16(T-t)).
\end{aligned}
$$

(Proof). First we prove (4.13). By Lemma 4.2, $P^{Z^T(\cdot,(t,z))^{-1}}$ is absolutely continuous with respect to $P^{Y(\cdot,(t,z))^{-1}}$ on $C([t,T];\mathbf{R}^d)$, and

$$
\begin{aligned}
&(dP^{Z^T(\cdot,(t,z))^{-1}}/dP^{Y(\cdot,(t,z))^{-1}})(Y(\cdot,(t,z))) \qquad (4.15).\\
&=\exp(2^{1/2}\int_t^T<\nabla_x\log p(T+t-s,Y(s,(t,z))),dW(s)>\\
&\qquad -\int_t^T|\nabla_x\log p(T+t-s,Y(s,(t,z)))|^2ds)
\end{aligned}
$$

on $C([t,T];\mathbf{R}^d)$ (see [13, Chap. 7]). Applying Itô's formula to $\log p(T+t-s,Y(s,(t,z)))$, we get (4.13).

Next we prove (4.14). By (4.13),

$$P(\sup_{t\le s\le T}|Z^T(s,(t,z))|\ge R) \tag{4.16}$$
$$= E[(p(t,Y(T,(t,z)))/p(T,z))\exp(\int_t^T \triangle\Psi(Y(s,(t,z)))ds); \sup_{t\le s\le T}|Y(s,(t,z))|\ge R]$$
$$\le C_5\exp(C_5t+C_5(|z|^2+1+T)\exp(C_5T)+(T-t)|\triangle\Psi|_\infty)$$
$$\times P(\sup_{t\le s\le T}|Y(s,(t,z))|\ge R)$$

by Lemma 4.1. This and Lemma 4.3 completes the proof (see below (2.2)).

Q. E. D.

Put $\partial_i=\partial/\partial_{x_i}$. We obtain the following lemma.

Lemma 4.5. *Suppose that (A.1)-(A.6) hold. Then for any* $T>0$,

$$\limsup_{R\to\infty} R^{-2}\log\{\sup_{|x|=R,0\le t\le T}|\partial_i\log p(t,x)|\}\le 0. \tag{4.17}$$

(Proof). For $t\in[0,T]$ and $y\in\mathbf{R}^d$, by (A.6) (see [9, p.122, Theorem 5.5]),

$$|\partial_i\log p(t,y)|\le E[\{C_2(|Y(t,y)|+1)+t|\nabla(\triangle\Psi)|_\infty\}\sup_{0\le s\le t}|\partial Y(s,y)/\partial y_i| \tag{4.18}$$
$$\times p_0(Y(t,y))\exp(\int_0^t\triangle\Psi(Y(s,y))ds)]/p(t,y)$$
$$\le d^{1/2}C_3\exp(C_3t)\{C_2+t|\nabla(\triangle\Psi)|_\infty$$
$$+C_2E[|Y(t,y)|p_0(Y(t,y))\exp(\int_0^t\triangle\Psi(Y(s,y))ds)]/p(t,y)\}$$

by Lemma 3.1. We only have to consider the second part on the last part of (4.18): for $m\in\mathbf{N}$, by Hölder's inequality

$$E[|Y(t,y)|p_0(Y(t,y))\exp(\int_0^t\triangle\Psi(Y(s,y))ds)]/p(t,y) \tag{4.19}$$
$$\le E[|Y(t,y)|^{2m}p_0(Y(t,y))\exp(\int_0^t\triangle\Psi(Y(s,y))ds)]^{1/(2m)}p(t,y)^{-1/(2m)}$$
$$\le\{|p_0|_\infty\exp(t|\triangle\Psi|_\infty)\}^{1/(2m)}E[|Y(t,y)|^{2m}]^{1/(2m)}$$
$$\times\exp(C_5(|y|^2+t+1)\exp(C_5t)/(2m))$$

by Lemma 4.1. By Lemma 3.1, (4.18) and (4.19), as $m\to\infty$,

$$\limsup_{R\to\infty}R^{-2}\log\{\sup_{|x|=R,0\le t\le T}|\partial_i\log p(t,x)|\}\le C_5\exp(C_5T)/(2m)\to 0. \tag{4.20}$$

Q. E. D.

Lemma 4.6. *Suppose that (A.1)-(A.6) hold and that (2.4) holds with $T = T_0$ for some $T_0 \geq 0$. Then for $T \in (T_0, T_0 + 1/(2C_4))$ and $z \in \mathbf{R}^d$,*

$$\begin{aligned}&\lim_{R\to\infty} E[\partial_i \log p(T + T_0 - \tau_R^T(T_0, z), Z^T(\tau_R^T(T_0, z), (T_0, z)))] \\ &= E[\partial_i \log p(T_0, Z^T(T, (T_0, z)))].\end{aligned} \quad (4.21).$$

(Proof). For $T \in (T_0, T_0 + 1/(2C_4))$ and $z \in \mathbf{R}^d$, by (4.6),

$$\begin{aligned}&E[\partial_i \log p(T + T_0 - \tau_R^T(T_0, z), Z^T(\tau_R^T(T_0, z), (T_0, z)))] \\ &= E[\partial_i \log p(T_0, Z^T(T, (T_0, z))); \tau_R^T(T_0, z) = T] \\ &\quad + E[\partial_i \log p(T + T_0 - \tau_R^T(T_0, z), Z^T(\tau_R^T(T_0, z), (T_0, z))); \tau_R^T(T_0, z) < T].\end{aligned} \quad (4.22).$$

The second part on the right hand side of (4.22) converges to 0 as $R \to \infty$, by Lemmas 4.4 and 4.5. The first part on the right hand side of (4.22) converges to $E[\partial_i \log p(T_0, Z^T(T, (T_0, z)))]$ as $R \to \infty$, by Lemma 4.2 and the assumption on induction.

Q. E. D.

Finally we prove Theorem 2.2.

Proof of Theorem 2.2.

Suppose that (2.4) holds for $T = T_0 \geq 0$. Then for $z \in \mathbf{R}^d$ and $T \in (T_0, T_0+1/(2C_4))$, by Itô's formula,

$$\begin{aligned}&E[\partial_i \log p(T + T_0 - \tau_R^T(T_0, z), Z^T(\tau_R^T(T_0, z), (T_0, z)))] - \partial_i \log p(T, z) \\ &= -E[\int_{T_0}^{\tau_R^T(T_0,z)} [\partial_i \triangle \Psi(Z^T(u, (T_0, z))) + < \partial_i \nabla \Psi(Z^T(u, (T_0, z))) \\ &\quad , \nabla_x \log p(T + T_0 - u, Z^T(u, (T_0, z))) >] du],\end{aligned} \quad (4.23).$$

since $p(t, x)$ is smooth by Theorem 1.1, and since

$$\begin{aligned}&\partial[\partial_i \log p(t, x)]/\partial t \\ &= \triangle_x[\partial_i \log p(t, x)] + < 2\nabla_x \log p(t, x) + \nabla \Psi(x), \nabla_x[\partial_i \log p(t, x)] > \\ &\quad + \partial_i \triangle \Psi(x) + < \partial_i \nabla \Psi(x), \nabla_x \log p(t, x) >\end{aligned} \quad (4.24).$$

from (4.8). Let $R \to \infty$ in (4.23). Then by (A.3), Lemmas 4.2 and 4.6,

$$E[\partial_i \log p(T_0, Z^T(T,(T_0,z)))] - \partial_i \log p(T,z) \qquad (4.25).$$
$$= -E[\int_{T_0}^{T} [\partial_i \triangle \Psi(Z^T(u,(T_0,z))) + < \partial_i \nabla \Psi(Z^T(u,(T_0,z)))$$
$$, \nabla_x \log p(T+T_0-u, Z^T(u,(T_0,z))) >] du].$$

(4.25) and Lemma 4.2 show that (2.4) is true for $T \in (T_0, T_0 + 1/(2C_4))$ by (A.3) and the assumption on induction. Inductively, one can show that (2.4) is true.

Q. E. D.

5. Proof of Theorem 2.3.

In this section, we prove Theorem 2.3. Throughout this section, we assume that $h \in (0,1)$ and fix $T > 0$.

Let us first state and prove technical lemmas.

Lemma 5.1. *(see [12, p. 12, (45)]) Suppose that (A.1)-(A.2) hold. Then the following holds:*

$$\sup_{0<h\le 1} \sum_{k=0}^{[T/h]} E_0[|X^h((k+1)h,x) - X^h(kh,x)|^2]/h < \infty. \qquad (5.1).$$

Put, for $t \ge 0$ and $x \in \mathbf{R}^d$,

$$\begin{aligned} \overline{X}^h(t,x) &= X^h([t/h]h,x) \\ &\qquad + (t-[t/h]h)\{X^h(([t/h]+1)h,x) - X^h([t/h]h,x)\}/h, \\ b(t,x) &\equiv -\nabla_x \log p(t,x) - \nabla\Psi(x), \\ C(b,R) &\equiv \sup\{|b(s,x)-b(s,y)|/|x-y| : 0 \le s \le T, \\ &\qquad x \ne y, |x|, |y| \le R\}, \quad (R>0), \\ C(b) &\equiv \sup\{|b(t,x)/(|x|+1) : 0 \le t \le T, x \in \mathbf{R}^d\} \end{aligned} \qquad (5.2).$$

(see Theorems 2.1 and 2.2). Then we obtain the following.

Lemma 5.2. *Suppose that (A.1)-(A.6) hold. For $R_1 > 0$, suppose that*

$$|X(0,x)| = |\overline{X}^h(0,x)| < R_1, \quad \sum_{k=0}^{[T/h]} |X^h((k+1)h,x) - X^h(kh,x)|^2/h < R_1. \quad (5.3).$$

Then for $R > \max((R_1 + C(b)(T+1))\exp(C(b)(T+1)), R_1 + ((T+1)R_1)^{1/2})$, the following holds: for $t \in [0,T]$,

$$
\begin{aligned}
&|X(t,x)-\overline{X}^h(t,x)| \qquad (5.4).\\
&\le (T\sup\{|b(s,y)-b(s+h,z)| : 0\le s\le T, |y|,|z|\le R, |y-z|^2\le hR_1\}\\
&\quad +\int_0^T |b(s+h,\overline{X}^h(([s/h]+1)h,x))\\
&\quad -(X^h(([s/h]+1)h,x)-X^h([s/h]h,x))/h|ds)\exp(tC(b,R)).
\end{aligned}
$$

(Proof). By Gronwall's inequality,

$$
\sup_{0\le t\le([T/h]+1)h,|x|\le R_1}\max(|X(t,x)|,|\overline{X}^h(t,x)|)\le R, \qquad (5.5).
$$

since

$$
|X(t,x)|=|x+\int_0^t b(s,X(s,x))ds|\le |x|+\int_0^t C(b)(|X(s,x)|+1)ds,
$$

and since

$$
|\overline{X}^h(t,x)|\le |x|+[([t/h]+1)\sum_{k=0}^{[t/h]}|X^h((k+1)h,x)-X^h(kh,x)|^2]^{1/2}.
$$

By (5.5) and Gronwall's inequality, we can show that (5.4) is true, since for $t\in[0,T]$,

$$
\begin{aligned}
&X(t,x)-\overline{X}^h(t,x) \qquad (5.6).\\
&=\int_0^t (b(s,X(s,x))-b(s,\overline{X}^h(s,x)))ds\\
&\quad +\int_0^t (b(s,\overline{X}^h(s,x))-b(s+h,\overline{X}^h(([s/h]+1)h,x)))ds\\
&\quad +\int_0^t (b(s+h,\overline{X}^h(([s/h]+1)h,x))\\
&\quad -(X^h(([s/h]+1)h,x)-X^h([s/h]h,x))/h)ds,
\end{aligned}
$$

and since for $s\in[0,T]$,

$$
|\overline{X}^h(s,x)-\overline{X}^h(([s/h]+1)h,x)|^2\le \sum_{k=0}^{[T/h]}|X^h((k+1)h,x)-X^h(kh,x)|^2.
$$

Q. E. D.

Lemma 5.3. *(see [12, p. 11, (40)]) Suppose that (A.1)-(A.2) hold. Then for any* $f \in C_o^\infty(\mathbf{R}^d : \mathbf{R}^d)$ *and* $k \geq 0$,

$$\begin{aligned} &E_0[< f(X^h((k+1)h,x)), X^h((k+1)h,x) - X^h(kh,x) >] \qquad (5.7). \\ &= -hE_0[< \nabla\Psi(X^h((k+1)h,x)), f(X^h((k+1)h,x)) > -divf(X^h((k+1)h,x))]. \end{aligned}$$

For $R > 0$, take $\phi_R \in C_o^\infty(\mathbf{R}^d : [0,\infty))$ such that

$$\sup_{x\in\mathbf{R}^d} |\nabla\phi_R(x)| \leq 1/R,$$

$$\phi_R(x) = \begin{cases} 1; \text{ if } |x| \leq R, \\ \in [0,1]; \text{ if } R \leq |x| \leq 2R+1, \\ 0; \text{ if } 2R+1 \leq |x|, \end{cases} \qquad (5.8).$$

and put

$$b_R(t,x) = \phi_R(x)b(t,x). \qquad (5.9).$$

The following lemma can be easily shown by Theorem 1.1 and Lemma 5.3, and the proof is omitted.

Lemma 5.4. *Suppose that (A.1)-(A.5) hold. Then for any* $R > 0$, *the following holds.*

$$\begin{aligned} &\lim_{h\to 0} E_0[\int_0^T |b_R(s+h, \overline{X}^h(([s/h]+1)h,x))|^2 ds] \qquad (5.10). \\ &= \int_0^T ds \int_{\mathbf{R}^d} |b_R(s,y)|^2 p(s,y)dy. \end{aligned}$$

$$\begin{aligned} &\lim_{h\to 0} E_0[\int_0^T < b_R(s+h, \overline{X}^h(([s/h]+1)h,x)) \qquad (5.11). \\ &\quad , (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h > ds] \\ &= \int_0^T ds \int_{\mathbf{R}^d} < b_R(s,y), b(s,y) > p(s,y)dy. \end{aligned}$$

For $k \geq 0$, $s \geq kh$, $x \in \mathbf{R}^d$ and $R > 0$, put

$$\begin{aligned} \Phi_{h,R}^k(s,x) &= x + (s-kh)b_R(kh,x), \\ D\Phi_{h,R}^k(s,x)(= D_x\Phi_{h,R}^k(s,x)) &= Identity + (s-kh)(\partial b_R^i(kh,x)/\partial x_j)_{i,j=1}^d, \qquad (5.12). \\ q_{h,R}^k(x)dx &= (p_h^k(x)dx)^{\Phi_{h,R}^k((k+1)h,\cdot)^{-1}}, \end{aligned}$$

provided that it exists. Then we obtain the following.

Lemma 5.5. *Suppose that (A.1)-(A.5) hold. Then for* $R > 0$ *and* $k = 0, \cdots, [T/h] - 1$*, there exist mappings* $\{\Phi_{h,R}^k(s,\cdot)^{-1}\}_{kh \le s \le (k+1)h}$ *for sufficiently small* $h > 0$ *depending only on* T *and* R*, and the following holds:*

$$\lim_{h\to 0} \sum_{k=0}^{[T/h]-1} E_0[\log q_{h,R}^k(\Phi_{h,R}^k((k+1)h, X^h(kh,x))) - \log p_h^k(X^h(kh,x))] \quad (5.13).$$
$$= -\int_0^T ds \int_{\mathbf{R}^d} div_x b_R(s,y)p(s,y)dy,$$

$$\lim_{h\to 0} \sum_{k=0}^{[T/h]-1} E_0[\Psi(\Phi_{h,R}^k((k+1)h, X^h(kh,x))) - \Psi(X^h(kh,x))] \quad (5.14).$$
$$= \int_0^T ds \int_{\mathbf{R}^d} < \nabla\Psi(y), b_R(s,y) > p(s,y)dy.$$

(Proof). Take $h \in (0,1)$ sufficiently small so that

$$h \sup\{(\sum_{i,j=1}^d |\partial b_R^i(s,x)/\partial x_j|^2)^{1/2} : 0 \le s \le T, x \in \mathbf{R}^d\} < 1, \quad (5.15).$$

which is possible from Theorem 2.1 and Lemma 4.1. By (5.15), the proof of the first part is trivial (see [11]).

Let us prove (5.13). Since

$$q_{h,R}^k(x) = p_h^k(\Phi_{h,R}^k((k+1)h,\cdot)^{-1}(x))det(D\Phi_{h,R}^k((k+1)h,\cdot)^{-1}(x))$$

for $k = 0, \cdots, [T/h] - 1$ and $x \in \mathbf{R}^d$, we have

$$E_0[\log q_{h,R}^k(\Phi_{h,R}^k((k+1)h, X^h(kh,x))) - \log p_h^k(X^h(kh,x))] \quad (5.16).$$
$$= -\int_{kh}^{(k+1)h} \int_{\mathbf{R}^d} \sum_{\sigma\in S_d} sgn\sigma \sum_{i=1}^d D_y b_R(kh,y)^{i\sigma(i)}$$
$$\times \Pi_{j\ne i} D\Phi_{h,R}^k(s,y)^{j\sigma(j)}\{det(D\Phi_{h,R}^k(s,y))\}^{-1} p_h(s,y)dyds.$$

Here S_d denotes a permutation group on $\{1, \cdots, d\}$. Hence we obtain (5.13) by Theorem 1.1, the smoothness of b_R, and the bounded convergence theorem since $D\Phi_{h,R}^{[s/h]}(s,y)$ is bounded and converges to an identity matrix as $h \to 0$.

Next we prove (5.14). For $k = 0, \cdots, [T/h] - 1$,

$$E_0[\Psi(\Phi^k_{h,R}((k+1)h, X^h(kh,x))) - \Psi(X^h(kh,x))] \qquad (5.17).$$
$$= \int_{kh}^{(k+1)h} \int_{\mathbf{R}^d} [< \nabla\Psi(\Phi^k_{h,R}(s,y)), b_R(kh,y) >] p_h(s,y)dyds,$$

which completes the proof by Theorem 1.1.

Q. E. D.

Lemma 5.6. *(see [12, p. 6, (15)]) For any $\alpha \in (d/(d+2), 1)$, there exists a positive constant C such that the following holds: for any $R > 0$ and any probability density function ρ on $\mathbf{R}^d$ for which $M(\rho) < \infty$ (see (A.2)),*

$$\int_{|x|\geq R, \rho(x)<1} |\rho(x)\log\rho(x)|dx \leq C(R^2+1)^{(-(2+d)\alpha+d)/2}(M(\rho)+1)^{\alpha}. \qquad (5.18).$$

Lemma 5.7. *Suppose that (A.1)-(A.2) hold. Then the following holds:*

$$\liminf_{h\to 0} F(p_h^{[T/h]}) \geq F(p(T,\cdot)). \qquad (5.19).$$

(Proof).

$$F(p_h^{[T/h]}) \geq \int_{p_h^{[T/h]}(x)<1, |x|\geq R} p_h^{[T/h]}(x)\log p_h^{[T/h]}(x)dx \qquad (5.20).$$
$$+ \int_{|x|<R} (\log p_h^{[T/h]}(x) + \Psi(x))p_h^{[T/h]}(x)dx.$$

The first integral on the right hand side of (5.20) can be shown to converges to zero as $h \to 0$ and then $R \to \infty$ by Lemmas 5.1 and 5.6, and (A.2), since

$$M(p_h^{[T/h]}) \leq 2([T/h]) \sum_{k=0}^{[T/h]-1} E_0[|X^h((k+1)h,x) - X^h(kh,x)|^2] + 2E_0[|x|^2]. \quad (5.21).$$

The following together with Theorem 1.1 completes the proof: by Jensen's inequality,

$$\int_{|x|<R} p_h^{[T/h]}(x)\log p_h^{[T/h]}(x)dx \qquad (5.22).$$
$$\geq \int_{|x|<R} p_h^{[T/h]}(x)\log p(T,x)dx$$
$$- \int_{|x|<R} p_h^{[T/h]}(x)dx \log\left(\int_{|x|<R} p(T,x)dx \Big/ \int_{|x|<R} p_h^{[T/h]}(x)dx\right).$$

Q. E. D.

Lemma 5.8. *Suppose that (A.1)-(A.6) hold. Then*

$$\limsup_{h\to 0}\sum_{k=0}^{[T/h]-1} E_0[|X^h((k+1)h,x)-X^h(kh,x)|^2]/h \qquad (5.23).$$
$$\le \int_0^T ds\int_{\mathbf{R}^d}|b(s,y)|^2 p(s,y)dy.$$

(Proof). For $k=0,\cdots,[T/h]-1$ and $R>0$,

$$E_0[|X^h((k+1)h,x)-X^h(kh,x)|^2]/h = d(p_h^k,p_h^{k+1})^2/h \qquad (5.24).$$
$$\le E_0[|\Phi_{h,R}^k((k+1)h,X^h(kh,x))-X^h(kh,x)|^2/h]+2F(q_{h,R}^k)-2F(p_h^{k+1})$$
$$=2F(q_{h,R}^k)-2F(p_h^k)+E[h|b_R(kh,X^h(kh,x))|^2]-2F(p_h^{k+1})+2F(p_h^k)$$

(see (1.11), (1.15)-(1.16) and (5.12)). By Lemmas 5.5 and 5.7, we only have to show the following:

$$-F(p(T,\cdot))+F(p(0,\cdot))=\int_0^T ds\int_{\mathbf{R}^d}|b(s,x)|^2p(s,x)dx. \qquad (5.25).$$

For s and t for which $0\le t<s<t+1/(2C_4)$,

$$-F(p(s,\cdot))+F(p(t,\cdot))=\int_t^s du\int_{\mathbf{R}^d}|b(u,x)|^2p(u,x)dx. \qquad (5.26).$$

This is true, since

$$\int_{\mathbf{R}^d}p(s,z)dzP(Z^s(u,(t,z))\in dx)=p(t+s-u,x)dx \quad (t\le u\le s) \qquad (5.27).$$

by (4.4) (see [7] or [15]), and henceforth by applying Itô's formula to $\log p(t+s-\tau_R^s(t,z),Z^s(\tau_R^s(t,z),(t,z)))+\Psi(Z^s(\tau_R^s(t,z),(t,z)))$ $(z\in\mathbf{R}^d, R>0)$,

$$-F(p(s,\cdot))+F(p(t,\cdot)) \qquad (5.28).$$
$$=\int_{\mathbf{R}^d}p(s,z)dzE[\log p(t,Z^s(s,(t,z)))+\Psi(Z^s(s,(t,z)))-\log p(s,z)-\Psi(z)]$$
$$=\int_t^s du\int_{\mathbf{R}^d}p(t+s-u,x)dx|b(t+s-u,x)|^2$$

by (4.8), Lemmas 4.2 and 4.4, Theorem 2.2 and (A.3).

Q. E. D.

Let us finally prove Theorem 2.3.

Proof of Theorem 2.3.

For $R_1 > 0$ and $\varepsilon > (hR_1)^{1/2}$,

$$\begin{aligned}
&P_0(\sup_{0\le t\le T} |X(t,x) - X^h(t,x)| \ge 2\varepsilon) \qquad (5.29).\\
&\le P_0(\sum_{k=0}^{[T/h]} |X^h((k+1)h,x) - X^h(kh,x)|^2/h \ge R_1)\\
&\quad + P_0(|X(0,x)| = |\overline{X}^h(0,x)| \ge R_1)\\
&\quad + P_0(\sum_{k=0}^{[T/h]} |X^h((k+1)h,x) - X^h(kh,x)|^2/h < R_1,\\
&\quad |X(0,x)| = |\overline{X}^h(0,x)| < R_1, \sup_{0\le t\le T} |X(t,x) - \overline{X}^h(t,x)| \ge \varepsilon).
\end{aligned}$$

This is true, since for $t \in [0,T]$

$$|\overline{X}^h(t,x) - X^h(t,x)| \le \{\sum_{i=0}^{[T/h]} |X^h((i+1)h,x) - X^h(ih,x)|^2\}^{1/2}.$$

The first and the second probabilities on the right hand side of (5.29) converge to zero as $h \to 0$ and then $R_1 \to \infty$ by Lemma 5.1 and Chebychev's inequality. Let us show that the third probability on the right hand side of (5.29) converges to zero as $h \to 0$. By Lemma 5.2 and Chebychev's inequality, we only have to show the following:

$$\begin{aligned}
0 = \lim_{h\to 0} E_0[\int_0^T &|b(s+h, \overline{X}^h(([s/h]+1)h,x)) \qquad (5.30).\\
&- (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h|ds].
\end{aligned}$$

Let us prove (5.30). For $R' > 0$,

$$\begin{aligned}
&E_0[\int_0^T |b(s+h, \overline{X}^h(([s/h]+1)h,x)) \qquad (5.31).\\
&\quad - (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h)|ds]\\
&\le E_0[\int_0^T |b(s+h, \overline{X}^h(([s/h]+1)h,x))\\
&\quad - b_{R'}(s+h, \overline{X}^h(([s/h]+1)h,x))|ds]\\
&\quad + (TE_0[\int_0^T |b_{R'}(s+h, \overline{X}^h(([s/h]+1)h,x))\\
&\quad - (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h)|^2 ds])^{1/2}
\end{aligned}$$

(see (5.8)-(5.9)).

The first part on the right hand side of (5.31) can be shown to converge to zero as follows: by (5.2) and Chebychev's inequality,

$$
\begin{aligned}
&E_0[\int_0^T |b(s+h,\overline{X}^h(([s/h]+1)h,x)) \qquad (5.32).\\
&\quad - b_{R'}(s+h,\overline{X}^h(([s/h]+1)h,x))|ds]\\
&\le \int_0^T E_0[C(b)(|X^h(([s/h]+1)h,x)|+1);|X^h(([s/h]+1)h,x)|\ge R']ds\\
&\le 2C(b)T(\sup_{0\le s\le T+h} M(p_h(s,\cdot))+1)/(R'+1),
\end{aligned}
$$

which converges to zero as $h\to 0$ and then $R'\to\infty$ by Lemma 5.1 and (5.21).

By Lemmas 5.4 and 5.8, the second part on the right hand side of (5.31) converges to zero as $h\to 0$ and then $R'\to\infty$.

Q. E. D.

6. Proof of Theorems 2.4 and 2.5.

In this section we prove Theorems 2.4 and 2.5. We fix $T>0$.

Let us first prove Theorem 2.4.

Proof of Theorem 2.4.

For $\{S(t,x)\}_{0\le t\le T, x\in \mathbf{R}^d}\in A^T$,

$$
\begin{aligned}
&E_0[\int_0^T |dS(t,x)/dt|^2dt] \qquad (6.1).\\
&\ge 2E_0[\int_0^T < b(t,S(t,x)),dS(t,x)/dt > dt]-E_0[\int_0^T |b(t,S(t,x))|^2dt],
\end{aligned}
$$

and

$$
\begin{aligned}
&E_0[\int_0^T < b(t,S(t,x)),dS(t,x)/dt > dt] \qquad (6.2).\\
&=-E_0[\log p(T,S(T,x))+\Psi(S(T,x))-\log p(0,S(0,x))-\Psi(S(0,x))]\\
&\quad +E_0[\int_0^T \partial \log p(s,S(s,x))/\partial s ds]\\
&=E_0[\int_0^T < b(s,X(s,x)),dX(s,x)/ds > ds]=E_0[\int_0^T |b(s,X(s,x))|^2ds]
\end{aligned}
$$

by Theorem 2.3. Here we used the following:

$$\int_0^T ds \int_{\mathbf{R}^d} |\partial \log p(s,y)/\partial s| p(s,y) dy < \infty. \qquad (6.3).$$

Let us prove (6.3) to complete the proof. By (4.8), (A.3) and Theorem 2.2, we only have to show the following:

$$\int_0^T ds \int_{\mathbf{R}^d} |\triangle_x \log p(s,x)| p(s,x) dx < \infty, \qquad (6.4).$$

since by Theorem 1.1,

$$\sup_{0 \le s \le T} M(p(s,\cdot)) < \infty. \qquad (6.5).$$

(6.4) can be shown by the following: by (5.27), for $i = 1, \cdots, d$, in the same way as in (4.23),

$$\begin{aligned}
&\int_0^T ds \int_{\mathbf{R}^d} |\partial^2 \log p(s,x)/\partial x_i^2|^2 p(s,x) dx \qquad (6.6).\\
&\le \int_{\mathbf{R}^d} p(T,z) dz E[(\int_0^T < \partial_i \nabla_x \log p(T-t, Z^T(t,(0,z))), dW(t) >)^2]\\
&= \int_{\mathbf{R}^d} p(T,z) dz E[(\partial_i \log p(0, Z^T(T,(0,z))) - \partial_i \log p(T,z)\\
&\quad + \int_0^T [\partial_i \triangle \Psi(Z^T(t,(0,z))) + < \partial_i \nabla \Psi(Z^T(t,(0,z)))\\
&\quad , \nabla_x \log p(T-t, Z^T(t,(0,z))) >] dt)^2] < \infty,
\end{aligned}$$

by (A.3), (A.6), Theorem 2.2, (6.5) and Lemma 4.2.

Q. E. D.

The proof of Theorem 2.5 can be done almost in the same way as in Theorem 2.3. The following lemma plays a similar role to that of Lemma 5.1.

Lemma 6.1. *Suppose that (A.1)-(A.6) hold. Then the following holds: for* $h \in (0,1)$

$$\sum_{k=0}^{[T/h]} E_0[|\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|^2]/h \le \int_0^{T+h} E_0[|b(s,X(s,x))|^2 ds] < \infty. \quad (6.7).$$

(Proof). The proof is done by the following: for any $k = 0, \cdots, [T/h]$,

$$\begin{aligned}
&E_0[|\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|^2] \qquad (6.8).\\
&\le E_0[|X((k+1)h,x) - X(kh,x)|^2] \le h \int_{kh}^{(k+1)h} E_0[|b(s,X(s,x))|^2 ds]
\end{aligned}$$

(see (2.9)) by Schwartz's inequality.

Q. E. D.

Let us finally prove Theorem 2.5.

Proof of Theorem 2.5.

Let us prove the first part of Theorem 2.5. For $\{S(t,x)\}_{0\le t\le T, x\in \mathbf{R}^d} \in A_h^T$,

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[|d\tilde{X}^h(t,x)/dt|^2]dt \qquad (6.9).\\
&= \sum_{k=0}^{[T/h]-1} E_0[|\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|^2]/h\\
&\le \sum_{k=0}^{[T/h]-1} E_0[|S((k+1)h,x) - S(kh,x)|^2]/h \le \int_0^{[T/h]h} E_0[|dS(t,x)/dt|^2]dt,
\end{aligned}$$

where the equality holds if and only if $dS(t,x)/dt = d\tilde{X}^h(t,x)/dt\ dtP_0(dx)$-a.e. by definition (see (2.9)).

Let us prove the rest part of Theorem 2.5. In the same way as in (5.29)-(5.32), by Lemma 6.1, we only have to show the following:

$$\int_0^T E_0[|b_{R'}(s,\tilde{X}^h(s,x)) - (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h|^2]ds \to 0, \quad (6.10).$$

as $h \to 0$ and then $R' \to \infty$. Let us prove (6.10).

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x)) - (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h|^2]ds \qquad (6.11).\\
&= \int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x))|^2]ds + \sum_{k=0}^{[T/h]} E_0[|\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|^2]/h\\
&\quad - 2\int_0^{[T/h]h} E_0[< b_{R'}(s,\tilde{X}^h(s,x)), (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h >]ds,
\end{aligned}$$

and by Lemma 6.1, we only have to show the following:

$$\int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x))|^2]ds \to \int_0^T E_0[|b(s,X(s,x))|^2]ds, \qquad (6.12).$$

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[< b_{R'}(s,\tilde{X}^h(s,x)), (\tilde{X}^h(([s/h]+1)h,x) \qquad (6.13).\\
&\qquad - \tilde{X}^h([s/h]h,x))/h >]ds \to \int_0^T E_0[|b(s,X(s,x))|^2]ds,
\end{aligned}$$

as $h \to 0$, and then $R' \to \infty$.

(6.12) can be shown as follows:

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[|b_{R'}(s, \tilde{X}^h(s,x))|^2]ds \qquad (6.14).\\
&= \int_0^{[T/h]h} E_0[|b_{R'}(s, \tilde{X}^h(s,x))|^2 - |b_{R'}(s, \tilde{X}^h([s/h]h,x))|^2]ds\\
&\quad + \int_0^{[T/h]h} E_0[|b_{R'}(s, \tilde{X}^h([s/h]h,x))|^2]ds.
\end{aligned}$$

By the continuity of $p(t,x)$, we only have to show that the first part on the right hand side of (6.14) converges to 0 as $h \to 0$, which can be done as follows:

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[|b_{R'}(s, \tilde{X}^h(s,x))|^2 - |b_{R'}(s, \tilde{X}^h([s/h]h,x))|^2]ds \qquad (6.15).\\
&\le 2 \sup_{0\le s\le T} |b_{R'}(s,\cdot)|_\infty \sup_{0\le s\le T} |D_z b_{R'}(s,\cdot)|_\infty\\
&\quad \times \int_0^{[T/h]h} E_0[|\tilde{X}^h(s,x) - \tilde{X}^h([s/h]h,x)|]ds \to 0 \quad (\text{as } h \to 0)
\end{aligned}$$

by Lemma 6.1 (see below (5.6)).

Let us prove (6.13). By the continuity of $p(t,x)$ and (6.3), we only have to show that

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[< \nabla\phi_{R'}(\tilde{X}^h(s,x)), (\tilde{X}^h(([s/h]+1)h,x) \qquad (6.16).\\
&\quad - \tilde{X}^h([s/h]h,x))/h > \{\log p(s, \tilde{X}^h(s,x)) + \Psi(\tilde{X}^h(s,x))\}]ds \to 0,
\end{aligned}$$

as $h \to 0$ and then $R' \to \infty$. This is true, since by (5.8)-(5.9),

$$\begin{aligned}
&- \int_0^{[T/h]h} E_0[< b_{R'}(s, \tilde{X}^h(s,x)), (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h >]ds\\
&= E_0[\phi_{R'}(\tilde{X}^h([T/h]h,x))\{\log p([T/h]h, \tilde{X}^h([T/h]h,x)) + \Psi(\tilde{X}^h([T/h]h,x))\}\\
&\quad - \phi_{R'}(x)\{\log p(0,x) + \Psi(x)\}] - \int_0^{[T/h]h} E_0[\phi_{R'}(\tilde{X}^h(s,x))\partial \log p(s, \tilde{X}^h(s,x))/\partial s\\
&\quad + < \nabla\phi_{R'}(\tilde{X}^h(s,x)), (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h >\\
&\quad \times \{\log p(s, \tilde{X}^h(s,x)) + \Psi(\tilde{X}^h(s,x))\}]ds.
\end{aligned}$$

Let us prove (6.16),

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[< \nabla\phi_{R'}(\tilde{X}^h(s,x)), (\tilde{X}^h(([s/h]+1)h,x) \qquad (6.17).\\
&\quad - \tilde{X}^h([s/h]h,x))/h > \{\log p(s,\tilde{X}^h(s,x)) + \Psi(\tilde{X}^h(s,x))\}]ds\\
&= \int_0^{[T/h]h} E_0[< \nabla\phi_{R'}(\tilde{X}^h(s,x))\{\log p(s,\tilde{X}^h(s,x)) + \Psi(\tilde{X}^h(s,x))\}\\
&\quad - \nabla\phi_{R'}(\tilde{X}^h([s/h]h,x))\{\log p(s,\tilde{X}^h([s/h]h,x)) + \Psi(\tilde{X}^h([s/h]h,x))\}\\
&\quad , (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h >]ds\\
&\quad + \int_0^{[T/h]h} E_0[< \nabla\phi_{R'}(\tilde{X}^h([s/h]h,x))\{\log p(s,\tilde{X}^h([s/h]h,x))\\
&\quad + \Psi(\tilde{X}^h([s/h]h,x))\}, (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h >]ds.
\end{aligned}$$

The first part on the right hand side of (6.17) can be shown to converge to zero as $h \to 0$ in the same way as in (6.15), by Lemma 6.1. The second part can be shown to converge to zero, as $h \to 0$ and $R' \to \infty$ by Lemma 6.1, the continuity of p, (5.8), (5.32), (A.3) and Theorem 2.2, since for $y \in \mathbf{R}^d$

$$\begin{aligned}
&|\nabla\phi_{R'}(y)\{\log p(s,y) + \Psi(y)\}|\\
&\le I_{[R',2R'+1]}(y)(R')^{-1}(1 + (2R'+1)^2)|\log p(s,y) + \Psi(y)|/(1+|y|^2),
\end{aligned}$$

and since $M(p(t,\cdot)) \in L^\infty([0,T];dt)$ by Theorem 1.1.

Q. E. D.

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