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Author(s)	Mikami, Toshio
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# Dynamical systems in the variational formulation of the Fokker-Planck equation by the Wasserstein metric

Toshio Mikami Department of Mathematics Hokkaido University Sapporo 060-0810, Japan mikami@math.sci.hokudai.ac.jp

### ABSTRACT

R. Jordan, D. Kinderlehrer and F. Otto proposed the discrete-time approximation of the Fokker-Planck equation by the variational formulation. It is determined by the Wasserstein metric, an energy functional and the Gibbs-Boltzmann entropy functional. In this paper we study the asymptotic behavior of the dynamical systems which describe their approximation of the Fokker-Planck equation and characterize the limit as a solution to a class of variational problems.

MSC: primary 60F15; secondary 60H30.

Key words: Fokker-Planck equation, Wasserstein metric, energy functional, Gibbs-Boltzmann entropy functional, dynamical systems, variational problem.

Abbreviated title: Dynamical systems for Fokker-Planck equation

#### 1. Introduction.

Let us consider a nonnegative solution of the following Fokker-Planck equation:

$$\partial p(t,x)/\partial t = \Delta_x p(t,x) + div_x (\nabla \Psi(x)p(t,x)) \quad (t > 0, x \in \mathbf{R}^d), \qquad (1.1).$$

$$\int_{\mathbf{R}^d} p(t, x) dx = 1 \quad (t \ge 0).$$
(1.2).

Here  $\Psi(x)$  is a function from  $\mathbf{R}^d$  to  $\mathbf{R}$ , and we put  $\Delta_x \equiv \sum_{i=1}^d \partial^2 / \partial x_i^2$ ,  $\nabla \equiv (\partial / \partial x_i)_{i=1}^d$ , and  $div_x(\cdot) \equiv \langle \nabla, \cdot \rangle$ . In Nelson's stochastic mechanics (see [18, 19]), it is crucial to construct a Markov process  $\{\xi(t)\}_{t\geq 0}$ , so called Nelson process, such that for  $t \geq 0$ 

$$\begin{aligned} P(\xi(t) \in dx) &= p(t, x) dx, \\ \xi(t) &= \xi(0) - \int_0^t \nabla \Psi(\xi(s)) ds + 2^{1/2} W(t), \end{aligned}$$

where W(t) denotes a d-dimensional Wiener process (see [26]).

For  $\varepsilon > 0$ , by (1.1),

$$\partial p(t,x)/\partial t = \varepsilon \Delta_x p(t,x)/2 + div_x \{ ((1-\varepsilon/2)\nabla_x \log p(t,x) + \nabla \Psi(x))p(t,x) \}.$$
(1.3).

Suppose that  $\nabla_x \log p(t, x)$  and  $\nabla \Psi(x)$  are continuously differentiable in x and that  $(1 + |x|)^{-1} \nabla_x \log p(t, x)$  and  $(1 + |x|)^{-1} \nabla \Psi(x)$  are bounded. Then there exists a unique solution to the following stochastic integral equation: for  $t \ge 0$  and  $x \in \mathbf{R}^d$ ,

$$\xi^{\varepsilon}(t,x) = x - \int_0^t \{(1 - \varepsilon/2)\nabla_x \log p(s,\xi^{\varepsilon}(s,x)) + \nabla\Psi(\xi^{\varepsilon}(s,x))\} ds + \varepsilon^{1/2}W(t) \quad (1.4).$$

such that

$$\int_{\mathbf{R}^d} p_0(y) dy P(\xi^{\varepsilon}(t,y) \in dz) = p(t,z) dz$$
(1.5).

(see [2 and 26, and also 3, 14, 16, 21, 27]). Moreover for any T > 0,  $(1-\varepsilon/2)\nabla_x \log p(t, x) + \nabla \Psi(x)$  is the unique minimizer of

$$\int_{0}^{T} \int_{\mathbf{R}^{d}} |b(t,x)|^{2} p(t,x) dx dt$$
 (1.6)

over all b(t, x) for which

$$\partial p(t,x)/\partial t = \varepsilon \Delta_x p(t,x)/2 - div_x(b(t,x)p(t,x)) \quad (0 < t < T, x \in \mathbf{R}^d).$$
(1.7)

(This can be shown in the same way as in (6.1)-(6.2), by replacing  $\log p(t, x)$  by  $(1 - \varepsilon/2) \log p(t, x)$  in (6.2).) By the standard argument (see [8]), one can show the following: for any  $x \in \mathbf{R}^d$ ,

$$P(\lim_{\varepsilon \to 0} \sup_{0 \le t \le T} |\xi^0(t, x) - \xi^\varepsilon(t, x)| = 0) = 1.$$
(1.8).

This means that  $\xi^0(t, x)$  can be considered as the semiclassical limit of the Nelson processes  $\xi^{\varepsilon}(t, x)$  with small fluctuation. The minimum of (1.6) over all b(t, x) for which (1.7) hold converges, as  $\varepsilon \to 0$ , to

$$\int_0^T \int_{\mathbf{R}^d} |d\xi^0(t,x)/dt|^2 p(0,x) dx dt.$$
(1.9).

In this paper we show that  $\xi^0$  also plays a crucial role in the construction, by way of the Wasserstein metric, of the solution to (1.1)-(1.2) (see [12]). We also characterize  $\xi^0$ as the solution to a class of variational problems. The importance of the consideration in (1.3)-(1.9) will be discussed again in the end of section 2.

Let d denote the Wasserstein metric (or distance) defined by the following (see [22] or [4], [5], [10]): for Borel probability measures P, Q on  $\mathbb{R}^d$ , put

$$d(P,Q) \equiv \inf\{\left(\int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^2 \mu(dxdy)\right)^{1/2}:$$
  
$$\mu(dx \times \mathbf{R}^d) = P(dx), \mu(\mathbf{R}^d \times dy) = Q(dy)\}.$$
 (1.10).

In particular, we put  $d(p,q) \equiv d(P,Q)$  when P(dx) = p(x)dx and Q(dx) = q(x)dx. Next we introduce the assumption used by R. Jordan, D. Kinderlehrer and F. Otto in [12].

(A.1).  $\Psi \in C^{\infty}(\mathbf{R}^d; [0, \infty))$  and  $\sup_{x \in \mathbf{R}^d} \{ |\nabla \Psi(x)| / (\Psi(x) + 1) \}$  is finite. (A.2).  $p_0(x)$  is a probability density function on  $\mathbf{R}^d$  and the following holds:

$$M(p_0) \equiv \int_{\mathbf{R}^d} |x|^2 p_0(x) dx < \infty,$$
  
$$F(p_0) \equiv \int_{\mathbf{R}^d} (\log p_0(x) + \Psi(x)) p_0(x) dx < \infty$$

Under (A.1)-(A.2), for h > 0, we can define, a sequence of probability density functions  $\{p_h^n\}_{n\geq 0}$  on  $\mathbf{R}^d$ , inductively, by the following: put  $p_h^0 = p_0$ , and for  $p_h^n$ , determine  $p_h^{n+1}$  as the minimizer of

$$d(p_h^n, p)^2/2 + hF(p)$$
(1.11).

over all probability density functions p for which M(p) is finite (see [12, Proposition 4.1]). For a probability density function p on  $\mathbf{R}^d$ , put

$$E(p) \equiv \int_{\mathbf{R}^d} \Psi(x) p(x) dx, \quad S(p) \equiv \int_{\mathbf{R}^d} \log p(x) p(x) dx, \quad (1.12).$$

and for  $h \in (0, 1), t \ge 0$  and  $x \in \mathbf{R}^d$ , put

$$p_h(t,x) \equiv p_h^{[t/h]}(x),$$
 (1.13).

where [r] denotes the integer part of  $r \in \mathbf{R}$ . Then the following is known (see [20] and the references therein for an application to physics).

**Theorem 1.1.** ([12, Theorem 5.1]). Suppose that (A.1)-(A.2) hold. Then for any T > 0, as  $h \to 0$ ,  $p_h(T, \cdot)$  converges to  $p(T, \cdot)$  weakly in  $L^1(\mathbf{R}^d; dx)$ , and  $p_h$  converges to p strongly in  $L^1([0,T] \times \mathbf{R}^d; dtdx)$ , where  $p(t,x) \in C^{\infty}((0,\infty) \times \mathbf{R}^d; [0,\infty))$  is the unique solution of (1.1)-(1.2) with an initial condition

$$p(t, \cdot) \to p_0$$
, strongly in  $L^1(\mathbf{R}^d; dx)$ , as  $t \to 0$ , (1.14).

and  $M(p(t, \cdot))$ ,  $E(p(t, \cdot))$  and  $S(p(t, \cdot))$  belong to  $L^{\infty}([0, T]; dt)$ .

For  $p_h^n(x)$  and  $p_h^{n+1}(x)$ , there exists a lower semicontinuous convex function  $\varphi_h^{n+1}(x)$  such that

$$p_h^n(x)\delta_{\nabla\varphi_{\cdot}^{n+1}(x)}(dy)dx \tag{1.15}.$$

is the minimizer of  $d(p_h^n, p_h^{n+1})$ .  $\nabla \varphi_h^{n+1}$  is called Monge function for  $d(p_h^n, p_h^{n+1})$ . On the probability space  $(\mathbf{R}^d, \mathbf{B}(\mathbf{R}^d), P_0(dx) \equiv p_0(x)dx)$ , put for  $h \in (0, 1), t \geq 0$  and  $x \in \mathbf{R}^d$ ,

$$X^{h}(0,x) = \nabla \varphi_{h}^{0}(x) \equiv x,$$
  

$$X^{h}(t,x) = \nabla \varphi_{h}^{[t/h]}(X^{h}(\max([t/h] - 1, 0)h, x)).$$
(1.16).

In this paper we first give a stochastic representation for p(t, x) (see Theorem 2.1) from which we give the estimate for  $\nabla_x \log p(t, x)$  (see Theorem 2.2). In the proof, we use exponential estimates on large deviations and the idea in [25] where they gave estimates for the derivatives of the transition probability density functions of diffusion processes (see section 4). By this estimate and an assumption on  $\Psi$  (see section 2), we can construct the solution to the following: for  $x \in \mathbf{R}^d$ ,

$$\frac{dX(t,x)}{dt} = -\nabla_x \log p(t, X(t,x)) - \nabla \Psi(X(t,x)) \quad (t > 0),$$
  

$$X(0,x) = x$$
(1.17).

(From now on, we use the notation X(t,x) instead of  $\xi^0(t,x)$ .) We also show that  $X^h(t,x)$  converges to X(t,x), as  $h \to 0$ . In particular, it can be shown that  $P_0^{X(t,\cdot)^{-1}}(dx) = p(t,x)dx$  for  $t \ge 0$  (see Theorem 2.3). (Recall that  $P_0^{X(t,\cdot)^{-1}}(B) =$  $P_0(\{x \in \mathbf{R}^d : X(t,x) \in B\})$  for  $B \in \mathbf{B}(\mathbf{R}^d)$ .) This is conjecturable by the Euler equation to (1.11). It can be written, formally, as the following: for  $n \ge 0$ ,

$$X^{h}((n+1)h, x) - X^{h}(nh, x)$$

$$= -h\{\nabla \log p_{h}^{n+1}(X^{h}((n+1)h, x)) + \nabla \Psi(X^{h}((n+1)h, x))\}$$
(1.18).

(see Lemma 5.3 in section 5 for the exact statement of (1.18)).

Let us give two examples.

Example 1.1 (One-dimensional case (see [22, Chap. 3], or [17], [23], [24])). Put, for  $n \ge 0, h \in (0, 1)$  and  $x \in \mathbf{R}$ ,

$$F_h^n(x) = \int_{(-\infty,x]} p_h^n(y) dy.$$
 (1.19).

For a distribution function F on  $\mathbf{R}$ , put

$$F^{-1}(u) \equiv \sup\{x \in \mathbf{R} : F(x) < u\} \quad \text{for } 0 < u < 1.$$
(1.20).

Then for  $n \ge 0$ ,  $h \in (0, 1)$ ,  $x \in \mathbf{R}$  and  $t \ge 0$ ,

$$\nabla \varphi_h^{n+1}(x) = (F_h^{n+1})^{-1}(F_h^n(x)),$$
  

$$X^h(t,x) = (F_h^{[t/h]})^{-1}(F_0(x)).$$
(1.21).

Example 1.2 (Gaussian case). If  $\Psi(x) = 0$  and  $p_0(x) = (4\pi)^{-d/2} \exp(-|x|^2/4)$ , then

$$p(t,x) = (4\pi(t+1))^{-d/2} \exp(-|x|^2/\{4(t+1)\}), \quad X(t,x) = (t+1)^{1/2}x. \quad (1.22).$$

In section 2, we state our result which will be proved in sections 3-6.

### 2. Convergence and characterization of dynamical systems.

In this section we state our main result. Let us recall that  $P_0(dx) = p_0(x)dx$ . The following is an additional assumption in this paper.

(A.3).  $\Psi \in C^4(\mathbf{R}^d; \mathbf{R})$  and has bounded second, third and fourth derivatives.

(A.4).  $p_0(\cdot)$  is a probability density function on  $\mathbf{R}^d$ , and is twice continuously differentiable, with bounded derivatives up to the second order.

(A.5). 
$$-\infty < -C_1 \equiv \inf_{x \in \mathbf{R}^d} \{ (|x|^2 + 1)^{-1} \log p_0(x) \}.$$

(A.6). 
$$\infty > C_2 \equiv \sup_{x \in \mathbf{R}^d} \{ (|x|+1)^{-1} |\nabla \log p_0(x)| \}.$$

For  $t \ge 0$  and  $y \in \mathbf{R}^d$ , let  $\{Y(s, (t, y))\}_{s \ge t}$  be the solution to the following stochastic integral equation:

$$Y(s,(t,y)) = y + \int_{t}^{s} \nabla \Psi(Y(u,(t,y))) du + 2^{1/2} (W(s) - W(t)).$$
(2.1).

(2.1) has a unique strong solution under (A.3) (see [9], [13], or [26]). We also put, for the sake of simplicity,

$$Y(s,y) \equiv Y(s,(0,y)).$$
 (2.2).

It is known that  $\{Y(s, (t, y))\}_{s \ge t}$  has the same probability law as that of  $\{Y(s, y)\}_{s \ge 0}$ . The following theorem gives a stochastic representation for p(t, y).

The following theorem gives a stochastic representation for p(t, x).

**Theorem 2.1.** Suppose that (A.1)-(A.4) hold. Then for any T > 0, p(t, x) is continuously differentiable in t and has bounded, continuous derivatives up to the second order in x on  $[0, T] \times \mathbf{R}^d$ , and for any t > 0 and  $x \in \mathbf{R}^d$ ,

$$p(t,x) = E[p_0(Y(t,x)) \exp(\int_0^t \Delta \Psi(Y(s,x)) ds)].$$
 (2.3).

By Theorem 2.1, we obtain the following result.

**Theorem 2.2.** Suppose that (A.1)-(A.6) hold. Then for any T > 0,

$$\sup_{x \in \mathbf{R}^d, 0 \le t \le T} \{ (|x|+1)^{-1} |\nabla_x \log p(t,x)| \} < \infty.$$
(2.4).

In particular, (1.17) has a unique solution.

REMARK 2.1. In Theorems 2.1 and 2.2, we assumed (A.1)-(A.2) only to use the fact that p(t, x) is a smooth solution to (1.1)-(1.2) with (1.14).

By Theorems 2.1-2.2, we obtain the following.

**Theorem 2.3.** Suppose that (A.1)-(A.6) hold. Then for any T > 0 and  $\delta > 0$ ,

$$\lim_{h \to 0} P_0(\sup_{0 \le t \le T} |X(t,x) - X^h(t,x)| \ge \delta) = 0.$$
(2.5).

In particular, for  $t \ge 0$ ,

$$P_0^{X(t,\cdot)^{-1}}(dy) = p(t,y)dy.$$
(2.6).

Put, for T > 0,

$$A^{T} \equiv \{\{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^{d}}; P_{0}^{S(t,\cdot)^{-1}}(dx) = p(t,x)dx(0 \le t \le T), \qquad (2.7).$$
$$\{S(t,x)\}_{0 < t < T} \text{ is absolutely continuous, } P_{0} - a.s.\}.$$

The following result is a version of [14] in the case the stochastic processes under consideration do not have random time evolution.

**Theorem 2.4.** Suppose that (A.1)-(A.6) hold. Then for any T > 0 and any  $\{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d} \in A^T$ ,

$$E_0\left[\int_0^T |dX(t,x)/dt|^2 dt\right] \le E_0\left[\int_0^T |dS(t,x)/dt|^2 dt\right],\tag{2.8}$$

where the equality holds if and only if  $dS(t, x)/dt = dX(t, x)/dt dtP_0(dx)$ -a.e..

For  $h \in (0, 1)$  and  $n \ge 0$ , let  $\nabla \tilde{\varphi}_h^{n+1}$  be the Monge function for  $d(p(nh, \cdot), p((n+1)h, \cdot))$  (see section 1). On the probability space  $(\mathbf{R}^d, \mathbf{B}(\mathbf{R}^d), P_0)$ , put for  $h \in (0, 1)$ ,  $t \ge 0$  and  $x \in \mathbf{R}^d$ ,

$$\tilde{X}^{h}(0,x) = \nabla \tilde{\varphi}^{0}_{h}(x) \equiv x, \quad \tilde{X}^{h}((k+1)h,x) = \nabla \tilde{\varphi}^{k+1}_{h}(\tilde{X}^{h}(kh,x)) \quad (k \ge 0), \\
\tilde{X}^{h}(t,x) = \tilde{X}^{h}([t/h]h,x) + (t - [t/h]h) \\
\times (\tilde{X}^{h}(([t/h] + 1)h,x) - \tilde{X}^{h}([t/h]h,x))/h.$$
(2.9)

Put also for  $h \in (0, 1)$  and T > 0,

$$A_h^T \equiv \{\{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d}; P_0^{S(t,\cdot)^{-1}}(dx) = p(t,x)dx(t=0,h,\cdots,[T/h]h)(2.10).$$
  
$$\{S(t,x)\}_{0 \le t \le T} \text{ is absolutely continuous, } P_0 - a.s.\}.$$

Then the following holds.

**Theorem 2.5.** Suppose that (A.1)-(A.6) hold. Then for any  $h \in (0,1)$  and  $T \ge h$ ,  $\{\tilde{X}^h(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d}$  is the unique minimizer of

$$\int_{0}^{[T/h]h} E_0[|dS(t,x)/dt|^2]dt \qquad (2.11).$$

over all  $\{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d} \in A_h^T$ , and the following holds: for any T > 0 and  $\delta > 0$ ,

$$\lim_{h \to 0} P_0(\sup_{0 \le t \le T} |X(t,x) - \tilde{X}^h(t,x)| \ge \delta) = 0.$$
(2.12).

For Borel probability density functions  $p_0(x)$  and  $p_1(x)$  on  $\mathbf{R}^d$ , the Markov diffusion process  $\{\tilde{\xi}(t)\}_{0 \le t \le 1}$  with a drift vector  $b^{\tilde{\xi}}(t,x)$  and with an identity diffusion matrix is called the h-pass process with the initial and terminal distributions  $p_0(x)dx$  and  $p_1(x)dx$ , respectively, if and only if  $P(\tilde{\xi}(t) \in dx) = p_t(x)dx$ (t = 0, 1) and if  $\int_0^1 E[|b^{\tilde{\xi}}(t, \tilde{\xi}(t))|^2]dt$  is the minimum of  $\int_0^1 \int_{\mathbf{R}^d} |b(t,x)|^2 q(t,x) dx dt$ over all (b(t,x), q(t,x)) for which q(t,x) satisfies (1.7) with  $\varepsilon = 1$  and with p replaced by q on  $(0,1) \times \mathbf{R}^d$  and for which  $q(t,x) = p_t(x)$  (t = 0,1). Theorem 2.5 implies that  $\tilde{X}^1(t,x)$  on  $(\mathbf{R}^d, \mathbf{B}(\mathbf{R}^d), P_0)$  plays a similar role to that of the h-path process (see [14]), when diffusion matrices vanish. If the similar result to (1.3)-(1.9) holds for  $\tilde{X}^1(t,x)$  and the h-pass process with a diffusion matrix =  $\varepsilon Id$ , then one can consider Theorem 2.5 as a zero noise limit of stochastic control problems. This implies that one might be able to treat the Monge-Kantorovich problem in the frame work of stochastic control problems. This is our future problem.

#### 3. Proof of Theorem 2.1.

In this section we prove Theorem 2.1. The proof is devided into four lemmas. For a m-dimensional vector function  $f(x) = (f^i(x))_{i=1}^m \ (x \in \mathbf{R}^d)$ , put

$$Df(x) \equiv (\partial f(x)/\partial x_i)_{i=1}^d, \quad |f|_{\infty} \equiv \sup_{x \in \mathbf{R}^d} (\sum_{i=1}^m |f^i(x)|^2)^{1/2}.$$
 (3.1).

The following lemma can be proved by the standard argument, making use of Itô's formula (see e.g. [9]) and of Gronwall's inequality (see [11]), and we omit the proof (see also [9, p. 120, Theorem 5.3]).

**Lemma 3.1.** Suppose that (A.3) holds. Then (2.1) has a unique strong solution, and there exist positive constants  $C_3$  and  $\{C(m)\}_{m\geq 1}$  which depends only on  $|\nabla\Psi|_{\infty}$ and  $|D^2\Psi|_{\infty}$  such that for  $t \geq 0$  and  $y \in \mathbf{R}^d$ ,

$$E[|Y(t,y)|^{2m}] \le C(m)(\sum_{k=1}^{m} |y|^{2k} + t) \exp(C(m)t) \quad (m \ge 1),$$
  
$$\partial Y^{i}(t,y)/\partial y_{j}| \le C_{3} \exp(C_{3}t), \quad P-a.s. \quad (i,j = 1, \cdots, d).$$
(3.2).

For  $t \geq 0$  and  $y \in \mathbf{R}^d$ , put

$$q(t,y) = E[p_0(Y(t,y))\exp(\int_0^t \Delta \Psi(Y(s,y))ds)].$$
(3.3).

Then the following can be proved in the same way as in [9, Chap. 5, Theorems 5.5 and 6.1] and the proof is omitted.

**Lemma 3.2.** Suppose that (A.3)-(A.4) hold. Then for any  $T \ge 0$ , q(t, y) has bounded, continuous derivatives in y up to the second order, and is continuously differentiable in t on  $[0, T] \times \mathbf{R}^d$ , and is a solution to (1.1).

By Lemmas 3.1 and 3.2, we get the following lemma.

**Lemma 3.3.** Suppose that (A.1)-(A.4) hold. Then for  $t \ge 0$  and  $x \in \mathbf{R}^d$ ,

$$p(t,x) \ge q(t,x). \tag{3.4}$$

(Proof). For R > 0 and  $x \in \mathbf{R}^d$ , put

$$\sigma_R(x) = \inf\{t > 0 : |Y(t, x)| > R\}.$$
(3.5).

By Itô's formula, if R > |x| and 0 < s < t, then one can easily show that the following is true:

$$p(t,x) = E[p(t - \min(\sigma_R(x), s), Y(\min(\sigma_R(x), s), x))$$

$$\times \exp(\int_0^{\min(\sigma_R(x), s)} \bigtriangleup \Psi(Y(u, x)) du)]$$

$$\geq E[p(t - s, Y(s, x)) \exp(\int_0^s \bigtriangleup \Psi(Y(u, x)) du); \sigma_R(x) \ge t] \to q(t, x),$$
(3.6)

as  $s \to t$  and then  $R \to \infty$ . Indeed, by (A.3), Lemma 3.1, and the Cameron-Martin-Maruyama-Girsanov formula (see [13]),

$$\begin{split} E[|p(t-s,Y(s,x)) - p(0,Y(s,x))| \exp(\int_{0}^{s} \bigtriangleup \Psi(Y(u,x))du); \sigma_{R}(x) \ge t] \quad (3.7). \\ &= E[|p(t-s,x+2^{1/2}W(s)) - p(0,x+2^{1/2}W(s))| \\ &\quad \times \exp(\int_{0}^{t} < \nabla \Psi(x+2^{1/2}W(u)), 2^{-1/2}dW(u) > -\int_{0}^{t} |\nabla \Psi(x+2^{1/2}W(u))|^{2}du/4 \\ &\quad + \int_{0}^{s} \bigtriangleup \Psi(x+2^{1/2}W(u))du); \sup_{0 \le s \le t} |x+2^{1/2}W(u)| \le R] \\ &\leq \int_{\mathbf{R}^{d}} |p(t-s,x+2^{1/2}y) - p(0,x+2^{1/2}y)|dy \\ &\quad \times (2\pi s)^{-d/2} \exp(\sup_{|z| \le R} \Psi(z)/2 + t|\bigtriangleup \Psi|_{\infty}/2) \to 0, \end{split}$$

as  $s \to t$ , by Theorem 1.1. Here we used the following: by Itô's formula,

$$\int_0^t < \nabla \Psi(x+2^{1/2}W(u)), 2^{-1/2}dW(u) >$$
  
= { $\Psi(x+2^{1/2}W(t)) - \Psi(x) - \int_0^t \bigtriangleup \Psi(x+2^{1/2}W(u))du$ }/2.

The following lemma together with Lemma 3.2 completes the proof of Theorem 2.1.

**Lemma 3.4.** Suppose that (A.1)-(A.4) hold. Then for  $t \ge 0$  and  $x \in \mathbf{R}^d$ ,

$$p(t,x) = E[p_0(Y(t,x))\exp(\int_0^t \Delta \Psi(Y(s,x))ds)].$$
(3.8).

(Proof). By Lemma 3.2, q(t, x) is a solution to (1.1) with  $q(0, x) = p_0(x)$ . Hence for  $t \ge 0$ ,

$$\int_{\mathbf{R}^{d}} q(t,x)dx = \int_{\mathbf{R}^{d}} p_{0}(x)dx = 1$$
(3.9).

Q. E. D.

by Lemma 3.3. (3.9) together with (1.2), (3.4) and the continuity of p and q completes the proof.

Q. E. D.

#### 4. Proof of Theorem 2.2.

In this section we prove Theorem 2.2. We put  $C_4 = |\nabla \Psi|_{\infty} + |D^2 \Psi|_{\infty}$ . We first state and prove six technical lemmas.

**Lemma 4.1.** Suppose that (A.1)-(A.5) hold. Then there exists a positive constant  $C_5$  which depends only on  $|\nabla \Psi|_{\infty}$ ,  $|D^2 \Psi|_{\infty}$  and  $|p_0|_{\infty}$  such that for  $t \ge 0$  and  $x \in \mathbf{R}^d$ ,

$$\exp(-C_5(|x|^2 + 1 + t)\exp(C_5t)) \le p(t, x) \le C_5\exp(C_5t).$$
(4.1).

(Proof). By Lemma 3.4,

$$p(t,x) \le |p_0|_{\infty} \exp(t| \triangle \Psi|_{\infty}), \tag{4.2}$$

and by Jensen's inequality (see [1]),

$$p(t,x) \ge \exp(E[\log p_0(Y(t,x)) + \int_0^t \Delta \Psi(Y(s,y))ds])$$

$$\ge \exp(-E[C_1(|Y(t,x)|^2 + 1)] - t|\Delta \Psi|_{\infty})$$
(4.3).

by (A.5), which completes the proof by Lemma 3.1.

Q. E. D.

For t and T for which  $0 \le t < T$  and  $z \in \mathbf{R}^d$ , let  $\{Z^T(s, (t, z))\}_{t \le s \le T}$  be the solution to the following stochastic integral equation: for  $s \in [t, T]$ ,

$$Z^{T}(s,(t,z)) = z + \int_{t}^{s} \{2\nabla_{x} \log p(t+T-u, Z^{T}(u,(t,z))) + \nabla \Psi(Z^{T}(u,(t,z)))\} du + 2^{1/2} \{W(s) - W(t)\},$$

$$(4.4)$$

up to the explosion time (see [26]).

The following lemma shows that (4.4) has a nonexplosive strong solution.

**Lemma 4.2.** Suppose that (A.1)-(A.5) hold. Then for t and T for which  $0 \le t < T$ , (4.4) has a unique nonexplosive strong solution and there exists a positive constant  $C_6$  which depends only on  $|\nabla \Psi|_{\infty}$ ,  $|D^2 \Psi|_{\infty}$  and  $|p_0|_{\infty}$  such that for  $z \in \mathbf{R}^d$ ,

$$C_{6} \exp(C_{6}T)(|z|^{2} + 1 + T)$$

$$\geq \sup_{t \leq s \leq T} E[|Z^{T}(s, (t, z))|^{2}] + E[\int_{t}^{T} |\nabla_{x} \log p(T + t - s, Z^{T}(s, (t, z)))|^{2} ds].$$
(4.5).

(Proof). For R > 0, put

$$\tau_R^T(t,z) = \inf\{\min(s,T) > t : |Z^T(s,(t,z))| > R\}.$$
(4.6).

Then by Lemma 4.1,

$$E\left[\int_{t}^{\tau_{R}^{T}(t,z)} |\nabla_{x} \log p(T+t-s, Z^{T}(s,(t,z)))|^{2} ds\right]$$

$$\leq \log C_{5} + C_{5}T + C_{5}(|z|^{2}+1+T) \exp(C_{5}T) + T|\Delta \Psi|_{\infty}.$$
(4.7)

This can be shown by applying Itô's formula to  $\log p(T + t - s, Z^T(s, (t, z)))$ , and by the following: by (1.1),

$$\frac{\partial \log p(t,x)}{\partial t} = \Delta_x \log p(t,x) + \langle 2\nabla_x \log p(t,x) + \nabla \Psi(x), \nabla_x \log p(t,x) \rangle (4.8).$$
$$+ \Delta \Psi(x) - |\nabla_x \log p(t,x)|^2.$$

The following also can be shown, making use of Itô's formula and Gronwall's inequality, by the standard argument: for  $s \in [t, T]$ ,

$$E[|Z^{T}(\min(s,\tau_{R}^{T}(t,z)),(t,z))|^{2}]$$

$$\leq (E[\int_{t}^{\tau_{R}^{T}(t,z)}|2\nabla_{x}\log p(T+t-u,Z^{T}(u,(t,z)))|^{2}du]$$

$$+|z|^{2}+2(T-t)(d+C_{4}^{2}))\exp(2(C_{4}^{2}+1)(s-t)).$$

$$(4.9)$$

Let  $R \to \infty$  in (4.7) and (4.9) and then the proof is over.

Q. E. D.

The following lemma can be proved easily and we only sketch the proof.

**Lemma 4.3.** Suppose that (A.3) holds. Then for  $T \in (0, 1/(2C_4))$  and  $y \in \mathbf{R}^d$ ,

$$\limsup_{R \to \infty} R^{-2} \log P(\sup_{0 \le t \le T} |Y(t, y)| \ge R) \le -(1 - 2C_4 T)^2 / (16T).$$
(4.10).

(Proof). Put

$$r = (R^2 - |y|^2 - 2Td - 2TC_4R(R+1))/(8TR^2)$$
(4.11).

which is positive for sufficiently large R > 0. Then by (3.5) and by applying Itô's formula to  $|Y(t, y)|^2$ , and by the Cameron-Martin-Maruyama-Girsanov formula,

$$P(\sup_{0 \le t \le T} |Y(t,y)| \ge R)$$

$$= \exp(-rR^{2} + r|y|^{2})E[\exp(r|Y(\sigma_{R}(y),y)|^{2} - r|y|^{2}); \sigma_{R}(y) \le T]$$

$$= \exp(-rR^{2} + r|y|^{2})E[\exp(r2^{3/2} \int_{0}^{\min(T,\sigma_{R}(y))} < Y(s,y), dW(s) >$$

$$-4r^{2} \int_{0}^{\min(T,\sigma_{R}(y))} |Y(s,y)|^{2} ds + \int_{0}^{\min(T,\sigma_{R}(y))} (4r^{2}|Y(s,y)|^{2} + r < 2Y(s,y), \nabla\Psi(Y(s,y)) > +2rd) ds); \sigma_{R}(y) \le T]$$

$$\le \exp(-rR^{2} + r|y|^{2} + T(4r^{2}R^{2} + 2rC_{4}R(R+1) + 2rd))$$

$$= \exp(-R^{2}(1 - (|y|^{2} + 2Td)/R^{2} - 2TC_{4}(1 + 1/R))^{2}/(16T))$$

$$(4.12)$$

by (4.11), which completes the proof.

Q. E. D.

**Lemma 4.4.** Suppose that (A.1)-(A.5) hold. Then for t and T for which  $0 \le t < T$ and  $z \in \mathbf{R}^d$ , the probability law of  $\{Z^T(s, (t, z))\}_{t \le s \le T}$  is absolutely continuous with respect to that of  $\{Y(s, (t, z))\}_{t \le s \le T}$  and on  $C([t, T]; \mathbf{R}^d)$ ,

$$(dP^{Z^{T}(\cdot,(t,z))^{-1}}/dP^{Y(\cdot,(t,z))^{-1}})(Y(\cdot,(t,z)))$$

$$= [p(t,Y(T,(t,z)))/p(T,z)] \exp(\int_{t}^{T} \Delta \Psi(Y(s,(t,z)))ds).$$
(4.13).

Moreover if  $T - t < 1/(2C_4)$ , then

$$\limsup_{R \to \infty} R^{-2} \log P(\sup_{t \le s \le T} |Z^T(s, (t, z))| \ge R)$$

$$\leq -(1 - 2C_4(T - t))^2 / (16(T - t)).$$
(4.14).

(Proof). First we prove (4.13). By Lemma 4.2,  $P^{Z^{T}(\cdot,(t,z))^{-1}}$  is absolutely continuous with respect to  $P^{Y(\cdot,(t,z))^{-1}}$  on  $C([t,T]; \mathbf{R}^{d})$ , and

$$(dP^{Z^{T}(\cdot,(t,z))^{-1}}/dP^{Y(\cdot,(t,z))^{-1}})(Y(\cdot,(t,z)))$$

$$= \exp(2^{1/2} \int_{t}^{T} < \nabla_{x} \log p(T+t-s,Y(s,(t,z))), dW(s) >$$

$$- \int_{t}^{T} |\nabla_{x} \log p(T+t-s,Y(s,(t,z)))|^{2} ds)$$

$$(4.15).$$

on  $C([t,T]; \mathbf{R}^d)$  (see [13, Chap. 7]). Applying Itô's formula to  $\log p(T+t-s, Y(s, (t, z)))$ , we get (4.13).

Next we prove (4.14). By (4.13),

$$P(\sup_{t \le s \le T} |Z^{T}(s, (t, z))| \ge R)$$

$$= E[(p(t, Y(T, (t, z)))/p(T, z)) \exp(\int_{t}^{T} \bigtriangleup \Psi(Y(s, (t, z))) ds); \sup_{t \le s \le T} |Y(s, (t, z))| \ge R]$$

$$\le C_{5} \exp(C_{5}t + C_{5}(|z|^{2} + 1 + T) \exp(C_{5}T) + (T - t)|\bigtriangleup \Psi|_{\infty})$$

$$\times P(\sup_{t \le s \le T} |Y(s, (t, z))| \ge R)$$

$$(4.16)$$

by Lemma 4.1. This and Lemma 4.3 completes the proof (see below (2.2)).

Q. E. D. Put  $\partial_i = \partial/\partial_{x_i}$ . We obtain the following lemma.

**Lemma 4.5.** Suppose that (A.1)-(A.6) hold. Then for any T > 0,

$$\limsup_{R \to \infty} R^{-2} \log\{ \sup_{|x|=R, 0 \le t \le T} |\partial_i \log p(t, x)| \} \le 0.$$
(4.17)

(Proof). For  $t \in [0, T]$  and  $y \in \mathbf{R}^d$ , by (A.6) (see [9, p.122, Theorem 5.5]),

$$\begin{aligned} |\partial_{i} \log p(t,y)| &\leq E[\{C_{2}(|Y(t,y)|+1) + t |\nabla(\bigtriangleup\Psi)|_{\infty}\} \sup_{0 \leq s \leq t} |\partial Y(s,y)/\partial y_{i}| \quad (4.18). \\ &\times p_{0}(Y(t,y)) \exp(\int_{0}^{t} \bigtriangleup\Psi(Y(s,y))ds)]/p(t,y) \\ &\leq d^{1/2}C_{3} \exp(C_{3}t)\{C_{2} + t |\nabla(\bigtriangleup\Psi)|_{\infty} \\ &+ C_{2}E[|Y(t,y)|p_{0}(Y(t,y)) \exp(\int_{0}^{t} \bigtriangleup\Psi(Y(s,y))ds)]/p(t,y)\} \end{aligned}$$

by Lemma 3.1. We only have to consider the second part on the last part of (4.18): for  $m \in \mathbf{N}$ , by Hölder's inequality

$$E[|Y(t,y)|p_{0}(Y(t,y))\exp(\int_{0}^{t} \bigtriangleup \Psi(Y(s,y))ds)]/p(t,y)$$

$$\leq E[|Y(t,y)|^{2m}p_{0}(Y(t,y))\exp(\int_{0}^{t} \bigtriangleup \Psi(Y(s,y))ds)]^{1/(2m)}p(t,y)^{-1/(2m)}$$

$$\leq \{|p_{0}|_{\infty}\exp(t|\bigtriangleup \Psi|_{\infty})\}^{1/(2m)}E[|Y(t,y)|^{2m}]^{1/(2m)}$$

$$\times \exp(C_{5}(|y|^{2}+t+1)\exp(C_{5}t)/(2m))$$

$$(4.19)$$

by Lemma 4.1. By Lemma 3.1, (4.18) and (4.19), as  $m \to \infty$ ,

$$\limsup_{R \to \infty} R^{-2} \log \{ \sup_{|x|=R, 0 \le t \le T} |\partial_i \log p(t, x)| \} \le C_5 \exp(C_5 T) / (2m) \to 0.$$
(4.20).  
Q. E. D.

**Lemma 4.6.** Suppose that (A.1)-(A.6) hold and that (2.4) holds with  $T = T_0$  for some  $T_0 \ge 0$ . Then for  $T \in (T_0, T_0 + 1/(2C_4))$  and  $z \in \mathbf{R}^d$ ,

$$\lim_{R \to \infty} E[\partial_i \log p(T + T_0 - \tau_R^T(T_0, z), Z^T(\tau_R^T(T_0, z), (T_0, z)))]$$
(4.21).  
=  $E[\partial_i \log p(T_0, Z^T(T, (T_0, z)))].$ 

(Proof). For  $T \in (T_0, T_0 + 1/(2C_4))$  and  $z \in \mathbf{R}^d$ , by (4.6),

$$E[\partial_i \log p(T + T_0 - \tau_R^T(T_0, z), Z^T(\tau_R^T(T_0, z), (T_0, z)))]$$

$$= E[\partial_i \log p(T_0, Z^T(T, (T_0, z))); \tau_R^T(T_0, z) = T]$$

$$+ E[\partial_i \log p(T + T_0 - \tau_R^T(T_0, z), Z^T(\tau_R^T(T_0, z), (T_0, z))); \tau_R^T(T_0, z) < T].$$
(4.22).

The second part on the right hand side of (4.22) converges to 0 as  $R \to \infty$ , by Lemmas 4.4 and 4.5. The first part on the right hand side of (4.22) converges to  $E[\partial_i \log p(T_0, Z^T(T, (T_0, z)))]$  as  $R \to \infty$ , by Lemma 4.2 and the assumption on induction.

Finally we prove Theorem 2.2.

#### Proof of Theorem 2.2.

Suppose that (2.4) holds for  $T = T_0 \ge 0$ . Then for  $z \in \mathbf{R}^d$  and  $T \in (T_0, T_0 + 1/(2C_4))$ , by Itô's formula,

$$E[\partial_{i} \log p(T + T_{0} - \tau_{R}^{T}(T_{0}, z), Z^{T}(\tau_{R}^{T}(T_{0}, z), (T_{0}, z)))] - \partial_{i} \log p(T, z) \quad (4.23).$$

$$= -E[\int_{T_{0}}^{\tau_{R}^{T}(T_{0}, z)} [\partial_{i} \triangle \Psi(Z^{T}(u, (T_{0}, z))) + \langle \partial_{i} \nabla \Psi(Z^{T}(u, (T_{0}, z))) \rangle, \nabla_{x} \log p(T + T_{0} - u, Z^{T}(u, (T_{0}, z))) \rangle] du],$$

since p(t, x) is smooth by Theorem 1.1, and since

$$\partial [\partial_i \log p(t, x)] / \partial t$$

$$= \Delta_x [\partial_i \log p(t, x)] + \langle 2\nabla_x \log p(t, x) + \nabla \Psi(x), \nabla_x [\partial_i \log p(t, x)] \rangle$$

$$+ \partial_i \Delta \Psi(x) + \langle \partial_i \nabla \Psi(x), \nabla_x \log p(t, x) \rangle$$

$$(4.24)$$

from (4.8). Let  $R \to \infty$  in (4.23). Then by (A.3), Lemmas 4.2 and 4.6,

$$E[\partial_{i} \log p(T_{0}, Z^{T}(T, (T_{0}, z)))] - \partial_{i} \log p(T, z)$$

$$= -E[\int_{T_{0}}^{T} [\partial_{i} \triangle \Psi(Z^{T}(u, (T_{0}, z))) + \langle \partial_{i} \nabla \Psi(Z^{T}(u, (T_{0}, z)))$$

$$, \nabla_{x} \log p(T + T_{0} - u, Z^{T}(u, (T_{0}, z))) > ]du].$$
(4.25).

(4.25) and Lemma 4.2 show that (2.4) is true for  $T \in (T_0, T_0 + 1/(2C_4))$  by (A.3) and the assumption on induction. Inductively, one can show that (2.4) is true.

Q. E. D.

### 5. Proof of Theorem 2.3.

In this section, we prove Theorem 2.3. Throughout this section, we assume that  $h \in (0, 1)$  and fix T > 0.

Let us first state and prove technical lemmas.

**Lemma 5.1.** (see [12, p. 12, (45)]) Suppose that (A.1)-(A.2) hold. Then the following holds:

$$\sup_{0 < h \le 1} \sum_{k=0}^{[T/h]} E_0[|X^h((k+1)h, x) - X^h(kh, x)|^2]/h < \infty.$$
(5.1).

Put, for  $t \ge 0$  and  $x \in \mathbf{R}^d$ ,

$$\overline{X}^{h}(t,x) = X^{h}([t/h]h,x) + (t - [t/h]h)\{X^{h}(([t/h] + 1)h,x) - X^{h}([t/h]h,x)\}/h,$$
  

$$b(t,x) \equiv -\nabla_{x} \log p(t,x) - \nabla \Psi(x),$$
  

$$C(b,R) \equiv \sup\{|b(s,x) - b(s,y)|/|x - y| : 0 \le s \le T,$$
  

$$x \ne y, |x|, |y| \le R\}, \quad (R > 0),$$
  

$$C(b) \equiv \sup\{|b(t,x)/(|x| + 1) : 0 \le t \le T, x \in \mathbf{R}^{d}\}$$
(5.2)

(see Theorems 2.1 and 2.2). Then we obtain the following.

**Lemma 5.2.** Suppose that (A.1)-(A.6) hold. For  $R_1 > 0$ , suppose that

$$|X(0,x)| = |\overline{X}^{h}(0,x)| < R_{1}, \quad \sum_{k=0}^{[T/h]} |X^{h}((k+1)h,x) - X^{h}(kh,x)|^{2}/h < R_{1}. \quad (5.3).$$

Then for  $R > \max((R_1 + C(b)(T+1)) \exp(C(b)(T+1)), R_1 + ((T+1)R_1)^{1/2})$ , the following holds: for  $t \in [0, T]$ ,

$$\begin{aligned} |X(t,x) - \overline{X}^{h}(t,x)| & (5.4). \\ &\leq (T \sup\{|b(s,y) - b(s+h,z)| : 0 \le s \le T, |y|, |z| \le R, |y-z|^{2} \le hR_{1}\} \\ &+ \int_{0}^{T} |b(s+h, \overline{X}^{h}(([s/h]+1)h, x)) \\ &- (X^{h}(([s/h]+1)h, x) - X^{h}([s/h]h, x))/h|ds) \exp(tC(b,R)). \end{aligned}$$

(Proof). By Gronwall's inequality,

$$\sup_{0 \le t \le ([T/h]+1)h, |x| \le R_1} \max(|X(t,x)|, |\overline{X}^h(t,x)|) \le R,$$
(5.5).

since

$$|X(t,x)| = |x + \int_0^t b(s, X(s,x))ds| \le |x| + \int_0^t C(b)(|X(s,x)| + 1)ds,$$

and since

$$|\overline{X}^{h}(t,x)| \le |x| + [([t/h] + 1)\sum_{k=0}^{[t/h]} |X^{h}((k+1)h,x) - X^{h}(kh,x)|^{2}]^{1/2}.$$

By (5.5) and Gronwall's inequality, we can show that (5.4) is true, since for  $t \in [0, T]$ ,

$$X(t,x) - \overline{X}^{h}(t,x)$$

$$= \int_{0}^{t} (b(s,X(s,x)) - b(s,\overline{X}^{h}(s,x))) ds$$

$$+ \int_{0}^{t} (b(s,\overline{X}^{h}(s,x)) - b(s+h,\overline{X}^{h}(([s/h]+1)h,x))) ds$$

$$+ \int_{0}^{t} (b(s+h,\overline{X}^{h}(([s/h]+1)h,x))$$

$$- (X^{h}(([s/h]+1)h,x) - X^{h}([s/h]h,x))/h) ds,$$
(5.6).

and since for  $s \in [0, T]$ ,

$$|\overline{X}^{h}(s,x) - \overline{X}^{h}(([s/h]+1)h,x)|^{2} \leq \sum_{k=0}^{[T/h]} |X^{h}((k+1)h,x) - X^{h}(kh,x)|^{2}.$$
Q. E. D.

**Lemma 5.3.** (see [12, p. 11, (40)]) Suppose that (A.1)-(A.2) hold. Then for any  $f \in C_o^{\infty}(\mathbf{R}^d : \mathbf{R}^d)$  and  $k \ge 0$ ,

$$E_0[< f(X^h((k+1)h, x)), X^h((k+1)h, x) - X^h(kh, x) >]$$

$$= -hE_0[< \nabla \Psi(X^h((k+1)h, x)), f(X^h((k+1)h, x)) > -divf(X^h((k+1)h, x))].$$
(5.7)

For R > 0, take  $\phi_R \in C_o^{\infty}(\mathbf{R}^d : [0, \infty))$  such that

$$\sup_{x \in \mathbf{R}^{d}} |\nabla \phi_{R}(x)| \leq 1/R,$$

$$\phi_{R}(x) = \begin{cases} 1; \text{ if } |x| \leq R, \\ \in [0,1]; \text{ if } R \leq |x| \leq 2R+1, \\ 0; \text{ if } 2R+1 \leq |x|, \end{cases}$$
(5.8).

and put

$$b_R(t,x) = \phi_R(x)b(t,x).$$
 (5.9).

The following lemma can be easily shown by Theorem 1.1 and Lemma 5.3, and the proof is omitted.

**Lemma 5.4.** Suppose that (A.1)-(A.5) hold. Then for any R > 0, the following holds.

$$\lim_{h \to 0} E_0 \left[ \int_0^T |b_R(s+h, \overline{X}^h(([s/h]+1)h, x))|^2 ds] \right]$$
(5.10).  
=  $\int_0^T ds \int_{\mathbf{R}^d} |b_R(s, y)|^2 p(s, y) dy.$ 

$$\lim_{h \to 0} E_0 \left[ \int_0^T \langle b_R(s+h, \overline{X}^h(([s/h]+1)h, x)) \rangle \right]$$
(5.11).  
$$(X^h(([s/h]+1)h, x) - X^h([s/h]h, x))/h > ds ]$$
$$= \int_0^T ds \int_{\mathbf{R}^d} \langle b_R(s, y), b(s, y) > p(s, y) dy.$$

For  $k \ge 0, s \ge kh, x \in \mathbf{R}^d$  and R > 0, put

$$\Phi_{h,R}^{k}(s,x) = x + (s - kh)b_{R}(kh,x),$$
  

$$D\Phi_{h,R}^{k}(s,x) (= D_{x}\Phi_{h,R}^{k}(s,x)) = Identity + (s - kh)(\partial b_{R}^{i}(kh,x)/\partial x_{j})_{i,j=1}^{d}, \quad (5.12),$$
  

$$q_{h,R}^{k}(x)dx = (p_{h}^{k}(x)dx)^{\Phi_{h,R}^{k}((k+1)h,\cdot)^{-1}},$$

provided that it exists. Then we obtain the following.

**Lemma 5.5.** Suppose that (A.1)-(A.5) hold. Then for R > 0 and  $k = 0, \dots, [T/h] -$ 1, there exist mappings  $\{\Phi_{h,R}^{k}(s,\cdot)^{-1}\}_{kh\leq s\leq (k+1)h}$  for sufficiently small h > 0 depending only on T and R, and the following holds:

$$\lim_{h \to 0} \sum_{k=0}^{[T/h]-1} E_0[\log q_{h,R}^k(\Phi_{h,R}^k((k+1)h, X^h(kh, x))) - \log p_h^k(X^h(kh, x))] (5.13).$$
  
=  $-\int_0^T ds \int_{\mathbf{R}^d} div_x b_R(s, y) p(s, y) dy,$ 

$$\lim_{h \to 0} \sum_{k=0}^{[T/h]-1} E_0[\Psi(\Phi_{h,R}^k((k+1)h, X^h(kh, x))) - \Psi(X^h(kh, x))] \qquad (5.14).$$
$$= \int_0^T ds \int_{\mathbf{R}^d} \langle \nabla \Psi(y), b_R(s, y) \rangle p(s, y) dy.$$

(Proof). Take  $h \in (0, 1)$  sufficiently small so that

$$h\sup\{(\sum_{i,j=1}^{d} |\partial b_{R}^{i}(s,x)/\partial x_{j}|^{2})^{1/2}: 0 \le s \le T, x \in \mathbf{R}^{d}\} < 1,$$
(5.15).

which is possible from Theorem 2.1 and Lemma 4.1. By (5.15), the proof of the first part is trivial (see [11]).

Let us prove (5.13). Since

$$q_{h,R}^{k}(x) = p_{h}^{k}(\Phi_{h,R}^{k}((k+1)h, \cdot)^{-1}(x))det(D\Phi_{h,R}^{k}((k+1)h, \cdot)^{-1}(x))$$

for  $k = 0, \dots, [T/h] - 1$  and  $x \in \mathbf{R}^d$ , we have

$$E_{0}[\log q_{h,R}^{k}(\Phi_{h,R}^{k}((k+1)h, X^{h}(kh, x))) - \log p_{h}^{k}(X^{h}(kh, x))]$$
(5.16).  
$$= -\int_{kh}^{(k+1)h} \int_{\mathbf{R}^{d}} \sum_{\sigma \in S_{d}} sgn\sigma \sum_{i=1}^{d} D_{y}b_{R}(kh, y)^{i\sigma(i)}$$
$$\times \Pi_{j \neq i} D\Phi_{h,R}^{k}(s, y)^{j\sigma(j)} \{det(D\Phi_{h,R}^{k}(s, y))\}^{-1} p_{h}(s, y) dyds.$$

Here  $S_d$  denotes a permutation group on  $\{1, \dots, d\}$ . Hence we obtain (5.13) by Theorem 1.1, the smoothness of  $b_R$ , and the bounded convergence theorem since  $D\Phi_{h,R}^{[s/h]}(s,y)$  is bounded and converges to an identity matrix as  $h \to 0$ . Next we prove (5.14). For  $k = 0, \dots, [T/h] - 1$ ,

$$E_{0}[\Psi(\Phi_{h,R}^{k}((k+1)h, X^{h}(kh, x))) - \Psi(X^{h}(kh, x))]$$

$$= \int_{kh}^{(k+1)h} \int_{\mathbf{R}^{d}} [\langle \nabla \Psi(\Phi_{h,R}^{k}(s, y)), b_{R}(kh, y) \rangle] p_{h}(s, y) dy ds,$$
(5.17)

which completes the proof by Theorem 1.1.

Q. E. D.

**Lemma 5.6.** (see [12, p. 6, (15)]) For any  $\alpha \in (d/(d+2), 1)$ , there exists a positive constant C such that the following holds: for any R > 0 and any probability density function  $\rho$  on  $\mathbf{R}^d$  for which  $M(\rho) < \infty$  (see (A.2)),

$$\int_{|x| \ge R, \rho(x) < 1} |\rho(x) \log \rho(x)| dx \le C(R^2 + 1)^{(-(2+d)\alpha + d)/2} (M(\rho) + 1)^{\alpha}.$$
(5.18).

**Lemma 5.7.** Suppose that (A.1)-(A.2) hold. Then the following holds:

$$\liminf_{h \to 0} F(p_h^{[T/h]}) \ge F(p(T, \cdot)).$$
(5.19).

(Proof).

$$F(p_h^{[T/h]}) \ge \int_{p_h^{[T/h]}(x) < 1, |x| \ge R} p_h^{[T/h]}(x) \log p_h^{[T/h]}(x) dx \qquad (5.20).$$
$$+ \int_{|x| < R} (\log p_h^{[T/h]}(x) + \Psi(x)) p_h^{[T/h]}(x) dx.$$

The first integral on the right hand side of (5.20) can be shown to converges to zero as  $h \to 0$  and then  $R \to \infty$  by Lemmas 5.1 and 5.6, and (A.2), since

$$M(p_h^{[T/h]}) \le 2([T/h]) \sum_{k=0}^{[T/h]-1} E_0[|X^h((k+1)h, x) - X^h(kh, x)|^2] + 2E_0[|x|^2].$$
(5.21).

The following together with Theorem 1.1 completes the proof: by Jensen's inequality,

$$\begin{split} &\int_{|x|< R} p_h^{[T/h]}(x) \log p_h^{[T/h]}(x) dx & (5.22). \\ &\geq \int_{|x|< R} p_h^{[T/h]}(x) \log p(T, x) dx & \\ &\quad - \int_{|x|< R} p_h^{[T/h]}(x) dx \log(\int_{|x|< R} p(T, x) dx / \int_{|x|< R} p_h^{[T/h]}(x) dx). \end{split}$$

$$Q. \text{ E. D.}$$

**Lemma 5.8.** Suppose that (A.1)-(A.6) hold. Then

$$\limsup_{h \to 0} \sum_{k=0}^{[T/h]-1} E_0[|X^h((k+1)h, x) - X^h(kh, x)|^2]/h$$

$$\leq \int_0^T ds \int_{\mathbf{R}^d} |b(s, y)|^2 p(s, y) dy.$$
(5.23)

(Proof). For  $k = 0, \dots, [T/h] - 1$  and R > 0,

$$E_{0}[|X^{h}((k+1)h,x) - X^{h}(kh,x)|^{2}]/h = d(p_{h}^{k},p_{h}^{k+1})^{2}/h$$

$$\leq E_{0}[|\Phi_{h,R}^{k}((k+1)h,X^{h}(kh,x)) - X^{h}(kh,x)|^{2}/h] + 2F(q_{h,R}^{k}) - 2F(p_{h}^{k+1})$$

$$= 2F(q_{h,R}^{k}) - 2F(p_{h}^{k}) + E[h|b_{R}(kh,X^{h}(kh,x))|^{2}] - 2F(p_{h}^{k+1}) + 2F(p_{h}^{k})$$
(5.24).

(see (1.11), (1.15)-(1.16) and (5.12)). By Lemmas 5.5 and 5.7, we only have to show the following:

$$-F(p(T,\cdot)) + F(p(0,\cdot)) = \int_0^T ds \int_{\mathbf{R}^d} |b(s,x)|^2 p(s,x) dx.$$
(5.25).

For s and t for which  $0 \le t < s < t + 1/(2C_4)$ ,

$$-F(p(s,\cdot)) + F(p(t,\cdot)) = \int_{t}^{s} du \int_{\mathbf{R}^{d}} |b(u,x)|^{2} p(u,x) dx.$$
 (5.26).

This is true, since

$$\int_{\mathbf{R}^d} p(s,z) dz P(Z^s(u,(t,z)) \in dx) = p(t+s-u,x) dx \quad (t \le u \le s)$$
(5.27).

by (4.4) (see [7] or [15]), and henceforth by applying Itô's formula to  $\log p(t + s - \tau_R^s(t,z), Z^s(\tau_R^s(t,z), (t,z))) + \Psi(Z^s(\tau_R^s(t,z), (t,z))) \ (z \in \mathbf{R}^d, R > 0),$ 

$$- F(p(s, \cdot)) + F(p(t, \cdot))$$

$$= \int_{\mathbf{R}^d} p(s, z) dz E[\log p(t, Z^s(s, (t, z))) + \Psi(Z^s(s, (t, z))) - \log p(s, z) - \Psi(z)]$$

$$= \int_t^s du \int_{\mathbf{R}^d} p(t + s - u, x) dx |b(t + s - u, x)|^2$$
(5.28)

by (4.8), Lemmas 4.2 and 4.4, Theorem 2.2 and (A.3).

Q. E. D.

Let us finally prove Theorem 2.3.

# Proof of Theorem 2.3.

For  $R_1 > 0$  and  $\varepsilon > (hR_1)^{1/2}$ ,

$$P_{0}(\sup_{0 \le t \le T} |X(t,x) - X^{h}(t,x)| \ge 2\varepsilon)$$

$$\leq P_{0}(\sum_{k=0}^{[T/h]} |X^{h}((k+1)h,x) - X^{h}(kh,x)|^{2}/h \ge R_{1})$$

$$+ P_{0}(|X(0,x)| = |\overline{X}^{h}(0,x)| \ge R_{1})$$

$$+ P_{0}(\sum_{k=0}^{[T/h]} |X^{h}((k+1)h,x) - X^{h}(kh,x)|^{2}/h < R_{1},$$

$$|X(0,x)| = |\overline{X}^{h}(0,x)| < R_{1}, \sup_{0 \le t \le T} |X(t,x) - \overline{X}^{h}(t,x)| \ge \varepsilon).$$
(5.29)

This is true, since for  $t \in [0, T]$ 

$$|\overline{X}^{h}(t,x) - X^{h}(t,x)| \le \{\sum_{i=0}^{[T/h]} |X^{h}((i+1)h,x) - X^{h}(ih,x)|^{2}\}^{1/2}.$$

The first and the second probabilities on the right hand side of (5.29) converge to zero as  $h \to 0$  and then  $R_1 \to \infty$  by Lemma 5.1 and Chebychev's inequality. Let us show that the third probability on the right hand side of (5.29) converges to zero as  $h \to 0$ . By Lemma 5.2 and Chebychev's inequality, we only have to show the following:

$$0 = \lim_{h \to 0} E_0 \left[ \int_0^T |b(s+h, \overline{X}^h(([s/h]+1)h, x)) - (X^h(([s/h]+1)h, x) - X^h([s/h]h, x))/h|ds] \right].$$
(5.30)

Let us prove (5.30). For R' > 0,

$$E_{0}\left[\int_{0}^{T} |b(s+h, \overline{X}^{h}(([s/h]+1)h, x)) - (X^{h}(([s/h]+1)h, x)) - X^{h}([s/h]h, x))/h)|ds\right]$$

$$\leq E_{0}\left[\int_{0}^{T} |b(s+h, \overline{X}^{h}(([s/h]+1)h, x)) - b_{R'}(s+h, \overline{X}^{h}(([s/h]+1)h, x))|ds\right]$$

$$+ (TE_{0}\left[\int_{0}^{T} |b_{R'}(s+h, \overline{X}^{h}(([s/h]+1)h, x)) - (X^{h}(([s/h]+1)h, x))/h)|^{2}ds\right])^{1/2}$$
(5.31)

(see (5.8)-(5.9)).

The first part on the right hand side of (5.31) can be shown to converge to zero as follows: by (5.2) and Chebychev's inequality,

$$E_{0}\left[\int_{0}^{T} |b(s+h, \overline{X}^{h}(([s/h]+1)h, x)) - b_{R'}(s+h, \overline{X}^{h}(([s/h]+1)h, x))|ds] \\ \leq \int_{0}^{T} E_{0}[C(b)(|X^{h}(([s/h]+1)h, x)|+1); |X^{h}(([s/h]+1)h, x)| \geq R']ds \\ \leq 2C(b)T(\sup_{0 \leq s \leq T+h} M(p_{h}(s, \cdot)) + 1)/(R'+1),$$
(5.32).

which converges to zero as  $h \to 0$  and then  $R' \to \infty$  by Lemma 5.1 and (5.21).

By Lemmas 5.4 and 5.8, the second part on the right hand side of (5.31) converges to zero as  $h \to 0$  and then  $R' \to \infty$ .

# 6. Proof of Theorems 2.4 and 2.5.

In this section we prove Theorems 2.4 and 2.5. We fix T > 0. Let us first prove Theorem 2.4.

#### Proof of Theorem 2.4.

For  $\{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d} \in A^T$ ,

$$E_{0}\left[\int_{0}^{T} |dS(t,x)/dt|^{2} dt\right]$$

$$\geq 2E_{0}\left[\int_{0}^{T} \langle b(t,S(t,x)), dS(t,x)/dt \rangle dt\right] - E_{0}\left[\int_{0}^{T} |b(t,S(t,x))|^{2} dt\right],$$
(6.1)

and

$$E_{0}\left[\int_{0}^{T} \langle b(t, S(t, x)), dS(t, x)/dt \rangle dt\right]$$

$$= -E_{0}\left[\log p(T, S(T, x)) + \Psi(S(T, x)) - \log p(0, S(0, x)) - \Psi(S(0, x))\right]$$

$$+ E_{0}\left[\int_{0}^{T} \partial \log p(s, S(s, x))/\partial s ds\right]$$

$$= E_{0}\left[\int_{0}^{T} \langle b(s, X(s, x)), dX(s, x)/ds \rangle ds\right] = E_{0}\left[\int_{0}^{T} |b(s, X(s, x))|^{2} ds\right]$$
(6.2)

by Theorem 2.3. Here we used the following:

$$\int_{0}^{T} ds \int_{\mathbf{R}^{d}} |\partial \log p(s, y) / \partial s| p(s, y) dy < \infty.$$
(6.3).

Let us prove (6.3) to complete the proof. By (4.8), (A.3) and Theorem 2.2, we only have to show the following:

$$\int_0^T ds \int_{\mathbf{R}^d} |\Delta_x \log p(s, x)| p(s, x) dx < \infty, \tag{6.4}$$

since by Theorem 1.1,

$$\sup_{0 \le s \le T} M(p(s, \cdot)) < \infty.$$
(6.5).

(6.4) can be shown by the following: by (5.27), for  $i = 1, \dots, d$ , in the same way as in (4.23),

$$\int_{0}^{T} ds \int_{\mathbf{R}^{d}} |\partial^{2} \log p(s, x) / \partial x_{i}^{2}|^{2} p(s, x) dx$$

$$\leq \int_{\mathbf{R}^{d}} p(T, z) dz E[(\int_{0}^{T} < \partial_{i} \nabla_{x} \log p(T - t, Z^{T}(t, (0, z)), dW(t) >)^{2}] \\
= \int_{\mathbf{R}^{d}} p(T, z) dz E[(\partial_{i} \log p(0, Z^{T}(T, (0, z))) - \partial_{i} \log p(T, z) \\
+ \int_{0}^{T} [\partial_{i} \Delta \Psi(Z^{T}(t, (0, z))) + < \partial_{i} \nabla \Psi(Z^{T}(t, (0, z))) \\
, \nabla_{x} \log p(T - t, Z^{T}(t, (0, z))) >] dt)^{2}] < \infty,$$
(6.6).

by (A.3), (A.6), Theorem 2.2, (6.5) and Lemma 4.2.

Q. E. D.

The proof of Theorem 2.5 can be done almost in the same way as in Theorem 2.3. The following lemma plays a similar role to that of Lemma 5.1.

**Lemma 6.1.** Suppose that (A.1)-(A.6) hold. Then the following holds: for  $h \in (0, 1)$ 

$$\sum_{k=0}^{[T/h]} E_0[|\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|^2]/h \le \int_0^{T+h} E_0[|b(s,X(s,x))|^2ds] < \infty.$$
(6.7).

(Proof). The proof is done by the following: for any  $k = 0, \dots, [T/h]$ ,

$$E_{0}[|\tilde{X}^{h}((k+1)h,x) - \tilde{X}^{h}(kh,x)|^{2}]$$

$$\leq E_{0}[|X((k+1)h,x) - X(kh,x)|^{2}] \leq h \int_{kh}^{(k+1)h} E_{0}[|b(s,X(s,x))|^{2}ds]$$
(6.8)

(see (2.9)) by Schwartz's inequality.

Q. E. D.

Let us finally prove Theorem 2.5.

## Proof of Theorem 2.5.

Let us prove the first part of Theorem 2.5. For  $\{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d} \in A_h^T$ ,

$$\int_{0}^{[T/h]h} E_{0}[|d\tilde{X}^{h}(t,x)/dt|^{2}]dt$$

$$= \sum_{k=0}^{[T/h]-1} E_{0}[|\tilde{X}^{h}((k+1)h,x) - \tilde{X}^{h}(kh,x)|^{2}]/h$$

$$\leq \sum_{k=0}^{[T/h]-1} E_{0}[|S((k+1)h,x) - S(kh,x)|^{2}]/h \leq \int_{0}^{[T/h]h} E_{0}[|dS(t,x)/dt|^{2}]dt,$$
(6.9)

where the equality holds if and only if  $dS(t,x)/dt = d\tilde{X}^h(t,x)/dt \ dtP_0(dx)$ -a.e. by definition (see (2.9)).

Let us prove the rest part of Theorem 2.5. In the same way as in (5.29)-(5.32), by Lemma 6.1, we only have to show the following:

$$\int_0^T E_0[|b_{R'}(s,\tilde{X}^h(s,x)) - (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h|^2]ds \to 0, \ (6.10).$$

as  $h \to 0$  and then  $R' \to \infty$ . Let us prove (6.10).

$$\int_{0}^{[T/h]h} E_{0}[|b_{R'}(s,\tilde{X}^{h}(s,x)) - (\tilde{X}^{h}(([s/h]+1)h,x) - \tilde{X}^{h}([s/h]h,x))/h|^{2}]ds \quad (6.11).$$

$$= \int_{0}^{[T/h]h} E_{0}[|b_{R'}(s,\tilde{X}^{h}(s,x))|^{2}]ds + \sum_{k=0}^{[T/h]} E_{0}[|\tilde{X}^{h}((k+1)h,x) - \tilde{X}^{h}(kh,x)|^{2}]/h$$

$$- 2\int_{0}^{[T/h]h} E_{0}[\langle b_{R'}(s,\tilde{X}^{h}(s,x)), (\tilde{X}^{h}(([s/h]+1)h,x) - \tilde{X}^{h}([s/h]h,x))/h \rangle]ds,$$

and by Lemma 6.1, we only have to show the following:

$$\int_{0}^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x))|^2]ds \to \int_{0}^{T} E_0[|b(s,X(s,x))|^2]ds, \qquad (6.12).$$

$$\int_{0}^{[T/h]h} E_0[\langle b_{R'}(s, \tilde{X}^h(s, x)), (\tilde{X}^h(([s/h] + 1)h, x))$$
(6.13).

$$-\tilde{X}^{h}([s/h]h,x))/h > ]ds \to \int_{0}^{T} E_{0}[|b(s,X(s,x))|^{2}]ds,$$

as  $h \to 0$ , and then  $R' \to \infty$ .

(6.12) can be shown as follows:

$$\int_{0}^{[T/h]h} E_{0}[|b_{R'}(s,\tilde{X}^{h}(s,x))|^{2}]ds$$

$$= \int_{0}^{[T/h]h} E_{0}[|b_{R'}(s,\tilde{X}^{h}(s,x))|^{2} - |b_{R'}(s,\tilde{X}^{h}([s/h]h,x))|^{2}]ds$$

$$+ \int_{0}^{[T/h]h} E_{0}[|b_{R'}(s,\tilde{X}^{h}([s/h]h,x))|^{2}]ds.$$
(6.14).

By the continuity of p(t, x), we only have to show that the first part on the right hand side of (6.14) converges to 0 as  $h \to 0$ , which can be done as follows:

$$\int_{0}^{[T/h]h} E_{0}[|b_{R'}(s,\tilde{X}^{h}(s,x))|^{2} - |b_{R'}(s,\tilde{X}^{h}([s/h]h,x))|^{2}]ds \qquad (6.15).$$

$$\leq 2 \sup_{0 \leq s \leq T} |b_{R'}(s,\cdot)|_{\infty} \sup_{0 \leq s \leq T} |D_{z}b_{R'}(s,\cdot)|_{\infty} \\
\times \int_{0}^{[T/h]h} E_{0}[|\tilde{X}^{h}(s,x) - \tilde{X}^{h}([s/h]h,x)|]ds \to 0 \quad (\text{as } h \to 0)$$

by Lemma 6.1 (see below (5.6)).

Let us prove (6.13). By the continuity of p(t, x) and (6.3), we only have to show that

$$\int_{0}^{[T/h]h} E_{0}[\langle \nabla \phi_{R'}(\tilde{X}^{h}(s,x)), (\tilde{X}^{h}(([s/h]+1)h,x) - \tilde{X}^{h}([s/h]h,x))/h \rangle \{\log p(s,\tilde{X}^{h}(s,x)) + \Psi(\tilde{X}^{h}(s,x))\}]ds \to 0,$$
(6.16)

as  $h \to 0$  and then  $R' \to \infty$ . This is true, since by (5.8)-(5.9),

$$\begin{split} &-\int_{0}^{[T/h]h} E_{0}[\langle b_{R'}(s,\tilde{X}^{h}(s,x)), (\tilde{X}^{h}(([s/h]+1)h,x) - \tilde{X}^{h}([s/h]h,x))/h \rangle] ds \\ &= E_{0}[\phi_{R'}(\tilde{X}^{h}([T/h]h,x))\{\log p([T/h]h,\tilde{X}^{h}([T/h]h,x)) + \Psi(\tilde{X}^{h}([T/h]h,x))\} \\ &- \phi_{R'}(x)\{\log p(0,x) + \Psi(x)\}] - \int_{0}^{[T/h]h} E_{0}[\phi_{R'}(\tilde{X}^{h}(s,x))\partial \log p(s,\tilde{X}^{h}(s,x))/\partial s \\ &+ \langle \nabla \phi_{R'}(\tilde{X}^{h}(s,x)), (\tilde{X}^{h}(([s/h]+1)h,x) - \tilde{X}^{h}([s/h]h,x))/h \rangle \\ &\times \{\log p(s,\tilde{X}^{h}(s,x)) + \Psi(\tilde{X}^{h}(s,x))\}] ds. \end{split}$$

Let us prove (6.16),

$$\int_{0}^{[T/h]h} E_{0}[\langle \nabla \phi_{R'}(\tilde{X}^{h}(s,x)), (\tilde{X}^{h}(([s/h]+1)h,x)$$

$$(6.17)$$

$$- \tilde{X}^{h}([s/h]h,x))/h > \{\log p(s, \tilde{X}^{h}(s,x)) + \Psi(\tilde{X}^{h}(s,x))\}]ds$$

$$= \int_{0}^{[T/h]h} E_{0}[\langle \nabla \phi_{R'}(\tilde{X}^{h}(s,x))\{\log p(s, \tilde{X}^{h}(s,x)) + \Psi(\tilde{X}^{h}(s,x))\}$$

$$- \nabla \phi_{R'}(\tilde{X}^{h}([s/h]h,x))\{\log p(s, \tilde{X}^{h}([s/h]h,x)) + \Psi(\tilde{X}^{h}([s/h]h,x))\}$$

$$, (\tilde{X}^{h}(([s/h]+1)h,x) - \tilde{X}^{h}([s/h]h,x))/h > ]ds$$

$$+ \int_{0}^{[T/h]h} E_{0}[\langle \nabla \phi_{R'}(\tilde{X}^{h}([s/h]h,x))\{\log p(s, \tilde{X}^{h}([s/h]h,x))$$

$$+ \Psi(\tilde{X}^{h}([s/h]h,x))\}, (\tilde{X}^{h}(([s/h]+1)h,x) - \tilde{X}^{h}([s/h]h,x))/h > ]ds.$$

The first part on the right hand side of (6.17) can be shown to converge to zero as  $h \to 0$  in the same way as in (6.15), by Lemma 6.1. The second part can be shown to converge to zero, as  $h \to 0$  and  $R' \to \infty$  by Lemma 6.1, the continuity of p, (5.8), (5.32), (A.3) and Theorem 2.2, since for  $y \in \mathbf{R}^d$ 

$$\begin{aligned} |\nabla \phi_{R'}(y) \{ \log p(s, y) + \Psi(y) \} | \\ &\leq I_{[R', 2R'+1]}(y) (R')^{-1} (1 + (2R'+1)^2) |\log p(s, y) + \Psi(y)| / (1 + |y|^2), \end{aligned}$$

and since  $M(p(t, \cdot)) \in L^{\infty}([0, T]; dt)$  by Theorem 1.1.

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