The lifespan of solutions to initial value problem for nonlinear wave equations

(非線形波動方程式に対する初期値問題の解の最大存在時間)

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Chapter 1

Introduction

In this thesis, we are concerned with the initial value problem for nonlinear wave equations,

$$\begin{cases} u_{tt} - \Delta u = F(u) & \text{in} \quad \mathbf{R}^n \times [0, \infty), \\ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x), & \text{for } x \in \mathbf{R}^n \end{cases} \tag{1.0.1}$$

where $n \geq 1$, $f, g \in C_0^\infty(\mathbf{R}^n)$, $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$ with $p > 1$, and $\varepsilon > 0$ is a parameter.

For this initial value problem, Strauss [30] conjectured the following. Let $n \geq 2$ and $p_0(n)$ is the positive root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$. Then the solution of (1.0.1) exists globally in time for small ε if $p > p_0(n)$, and the solution blows up in finite time if $1 < p \leq p_0(n)$. This is first proved by F. John [13] in the case of $n = 3$, except for $p = p_0(3)$, and after that, many contributions for this conjecture has been done. (See the section 1 in Chapter 2 for details.)

The first topic of this thesis is due to Takamura & Wakasa [33]. We shall briefly introduce the result.

We consider an integral equation related to (1.0.1):

$$u(x,t) = u^0(x,t) + L(F(u))(x,t) \quad \text{for } (x,t) \in \mathbf{R}^n \times [0, \infty), \tag{1.0.2}$$

where u^0 is a solution of the homogeneous wave equations, that is the solution of (1.0.1) with $H \equiv 0$, and

$$L(F(u))(x,t) = \frac{1}{2m-1} \int_0^t (t-\tau) M(F(u((\cdot,\tau)))|x, t-\tau) d\tau,$$

and

$$M(\phi|x,r) = \begin{cases} \dfrac{1}{\omega_n} \displaystyle\int_{|\omega|=1} \phi(x + r\omega) dS_\omega & \text{for } n = 2m+1, \\ \dfrac{2}{\omega_{n+1}} \displaystyle\int_{|\xi| \leq 1} \dfrac{\phi(x + r\xi)}{\sqrt{1 - |\xi|^2}} d\xi & \text{for } n = 2m, \end{cases}$$

for $\phi \in C(\mathbf{R}^n)$ and $m \in \mathbf{N}$, where $r = |x|$ and ω_n is the area of the unit sphere in $\mathbf{R}^n$.

When $n \geq 4$ and $F(u) = |u|^p$ in (1.0.2), Agemi & Kubota & Takamura [1] showed that the solution of (1.0.2) exists globally in time if $p > p_0(n)$. In [33], we proved that the solution of (1.0.2) blows up in finite time if $1 < p \leq p_0(n)$, and we have obtained some estimates for the lifespan of solution $u \in C^1(\mathbf{R}^n \times [0,T])$ to (1.0.2) which is the maximal existence time of solutions. The estimate of the lifespan is the same order as in the one for (1.0.1) with $F(u) = |u|^p$ and $1 < p \leq p_0(n)$, with respect to ε. We note that if we assume that u is the C^2 solution, and impose appropriate regularity on f, g, and F, then u satisfies some nonlinear wave equations. See Remark 2.3.1 below.

In the second topic of this thesis is due to Wakasa [34]. We consider the following initial value problem for nonlinear wave equations:

$$\begin{cases} u_{tt} - u_{xx} = \dfrac{|u|^{p-1}u}{(1+x^2)^{(1+a)/2}}, & (x,t) \in \mathbf{R} \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), & x \in \mathbf{R}, \end{cases} \tag{1.0.3}$$

where $a \geq -1$, $\varepsilon > 0$ and $p > 1$. Suzuki [31] showed that (1.0.3) has a global in-time solution for $p > (1+\sqrt{5})/2$ and $pa > 1$ if f and g are odd functions and ε is small enough, and Kubo & Osaka & Yazici [20] have obtained the same conclusion for any $p > 1$ satisfying $pa > 1$. On the other hand, they showed that the solution of (1.0.3) blows up in finite time if f and g are not odd functions. In addition, [20] obtained an upper bound of the lifespan of solution to (1.0.3). In [34], we have improved the upper bound of the lifespan and derive its lower bound which shows the optimality of our new upper bound.

In Appendix, we derive a representation formula for the following homogeneous wave equation:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = 0,\ u_t(x,0) = g(x), & x \in \mathbf{R}^n. \end{cases}$$

The proof of the formula divided into two steps. First, we show the representation formula by using Fourier transform in odd space dimensions. Next, we use Hadamard's method of descent to get the representation formula in even space dimensions from the case of odd space dimensions. The argument is based on Courant & Hilbert [4] and Yajima [35].

Chapter 2

Almost global solutions of semilinear wave equations with the critical exponent in high dimensions

In this chapter, we introduce the result of Takamura & Wakasa [33].

2.1 Historical Background

In this section, we review known results for the following initial value problem for fully nonlinear wave equations:

$$\begin{cases} u_{tt} - \Delta u = H(u, Du, D_x Du) \quad \text{in} \quad \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \ u_t(x, 0) = \varepsilon g(x), \end{cases} \tag{2.1.1}$$

where $u = u(x, t)$ is a scalar unknown function of space-time variables,

$$\begin{aligned} &Du = (u_{x_0}, u_{x_1}, \cdots, u_{x_n}), \ x_0 = t, \\ &D_x Du = (u_{x_i x_j}, \ i, j = 0, 1, \cdots, n, \ i + j \geq 1), \end{aligned}$$

$f, g \in C_0^\infty(\mathbf{R}^n)$ and $\varepsilon > 0$ is a "small" parameter.

Let

$$\widehat{\lambda} = (\lambda; \ (\lambda_i), i = 0, 1, \cdots, n; \ (\lambda_{ij}), i, j = 0, 1, \cdots, n, \ i + j \geq 1).$$

Suppose that the nonlinear term $H = H(\widehat{\lambda})$ is a sufficiently smooth function with

$$H(\widehat{\lambda}) = O(|\widehat{\lambda}|^{1+\alpha})$$

in a neighborhood of $\widehat{\lambda} = 0$, where $\alpha \geq 1$ is an integer.

Let us define the lifespan $\widetilde{T}_\varepsilon$ of classical solutions of (2.1.1) by

$$\widetilde{T}_\varepsilon \equiv \widetilde{T}_\varepsilon(f,g) := \sup\{T \in [0,\infty) \ : \ \begin{array}{l}\text{There exists a unique solution} \\ u \in C^2(\mathbf{R} \times [0,T)) \text{ of (2.1.1)}\}\end{array}$$

with arbitrarily fixed (f,g). When $\widetilde{T}_\varepsilon = \infty$, the problem (2.1.1) admits a global-in-time solution, while we only have a local-in-time solution on $[0, \widetilde{T}_\varepsilon)$ when $\widetilde{T}_\varepsilon < \infty$. From now on, we omit "-in-time" and simply use "global" and "local".

We are interested in "small" initial data, because there exist different cases in which we have $\widetilde{T}_\varepsilon < \infty$, even if ε is "small enough". We note that when the initial data is "large", there are blow-up results (For example, see Glassey [7], Levine [21], or Sideris [29]). The meaning of blow-up is that

$$\sup_{x \in \mathbf{R}^n} |u(x,t)| \to \infty \quad \text{if } t \to \widetilde{T}_\varepsilon - 0$$

holds for the classical solution u of (2.1.1).

Therefore, in what follows, we assume that ε is "small" enough. As is shown in Chapter 2 of Li and Chen [22], we have long histories on the estimate for $\widetilde{T}_\varepsilon$. Various lower bounds of $\widetilde{T}_\varepsilon$ are summarized in the following table, where $a = a(\varepsilon)$ satisfies

$$a^2 \varepsilon^2 \log(a+1) = 1 \tag{2.1.2}$$

and c stands for a positive constant independent of ε.

$\widetilde{T}_\varepsilon \geq$	$\alpha = 1$	$\alpha = 2$	$\alpha \geq 3$
$n = 1$	$\varepsilon^{-\alpha/2}$ in general, $c\varepsilon^{-\alpha(1+\alpha)/(2+\alpha)}$ if $\int_{\mathbf{R}} g(x)dx = 0$, $c\varepsilon^{-\alpha}$ if $\partial_u^\beta H(0) = 0$ for $1+\alpha \leq \forall\beta \leq 2\alpha$.	same as $(n,\alpha) = (1,1)$	same as $(n,\alpha) = (1,1)$
$n = 2$	$ca(\varepsilon)$ in general , $c\varepsilon^{-1}$ if $\int_{\mathbf{R}^2} g(x)dx = 0$, $c\varepsilon^{-2}$ if $\partial_u^2 H(0) = 0$	$c\varepsilon^{-6}$ in general, $c\varepsilon^{-18}$ if $\partial_u^3 H(0) = 0$, $\exp(c\varepsilon^{-2})$ if $\partial_u^3 H(0) = \partial_u^4 H(0) = 0$	∞
$n = 3$	$c\varepsilon^{-2}$ in general, $\exp(c\varepsilon^{-1})$ if $\partial_u^2 H(0) = 0$	∞	∞
$n = 4$	$\exp(c\varepsilon^{-2})$ in general, ∞ if $\partial_u^2 H(0) = 0$	∞	∞
$n \geq 5$	∞	∞	∞

We would not like to present references for each results. But we note that two parts in this table are different from the original one in Li and Chen [22]. One is the general case in $(n,\alpha) = (4,1)$. In this part, the lower bound of $\widetilde{T}_\varepsilon$ is $\exp(c\varepsilon^{-1})$ in Li and Chen [22]. But later, it has been improved by Li and Zhou [23]. The other is the case for $\partial_u^3 H(0) = 0$ in $(n,\alpha) = (2,2)$. This part is due to Katayama [16]. But this result was missed in Li and Chen [22].

Sharpness of these lower bounds is established if one could prove that there is no possibility to improve the lower bound of $\widetilde{T}_\varepsilon$ in sense of order of ε by showing a blow-up result for some special equations and special data. We note that the sharpness for the general case in $(n,\alpha) = (4,1)$ has been open more than 20 years. The answer was obtained by our previous work, Takamura and Wakasa [32], by studying $H = u^2$ with some special data. It is interesting to notice that this equation is in the critical level concerning the Strauss conjecture, since $p_0(4) = 2$. We mention whole histories for the semilinear equation in the next section. We also note that Zhou and Han

[41] have obtained the sharpness in $(n, \alpha) = (2, 2)$ by studying $H = u_t^2 u + u^4$. (note that $\partial_u^3 H(0) = 0$ but $\partial_u^4 H(0) \neq 0$).

Remark 2.1.1 *It is remarkable that when $(n, \alpha) = (3, 1)$, Klainerman [18] showed the global existence by using a special structure on $H = H(Du, D_x Du)$. It is so-called "null condition". (Christodoulou [3] also showed the global existence by using the conformal mapping method.) When $(n, \alpha) = (2, 2)$, the null condition has been also established independently by Godin [10] for $H = H(Du)$ and Katayama [15] for $H = H(Du, D_x Du)$. Later, Professor R.Agemi conjectured global existence result under "non-positive condition" in this case which is weaker than the null condition with $(n, \alpha) = (2, 2)$. This conjecture has been verified by Hoshiga [11] and Kubo [19] independently.*

2.2 Semilinear Case

We consider the following semilinear wave equations:

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & \text{in} \quad \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \ u_t(x, 0) = \varepsilon g(x), \end{cases} \tag{2.2.1}$$

assuming that $p > 1$ and $\varepsilon > 0$ is "small". Let us define the lifespan $\widehat{T}_\varepsilon$ of a solution of (2.2.1) by

$$\widehat{T}_\varepsilon \equiv \widehat{T}_\varepsilon(f, g) := \sup\{T \in [0, \infty) \ : \ \text{There exists a unique weak solution } u(x, t) \text{ of (2.2.1)}\}$$

with arbitrarily fixed (f, g).

Remark 2.2.1 *In the definition of the lifespan, we consider weak solutions which is a C^1-solution of associated integral equations to (2.2.1). However, when $p \geq 2$, we can obtain the classical solution.*

When $n = 1$, we have $\widehat{T}_\varepsilon < \infty$ for any power $p > 1$ by Kato [17]. When $n \geq 2$, the following conjecture was proposed by Strauss [30]. Let $p_0(n)$ be the positive root of the quadratic equation,

$$\gamma(p, n) = 2 + (n + 1)p - (n - 1)p^2 = 0. \tag{2.2.2}$$

That is,

$$p_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}. \tag{2.2.3}$$

Then we have

$$\begin{array}{lll} \widehat{T}_\varepsilon = \infty & \text{if } p > p_0(n) \text{ and } \varepsilon \text{ is "small"} & \text{(global existence)}, \\ \widehat{T}_\varepsilon < \infty & \text{if } 1 < p \le p_0(n) & \text{(blow-up in finite time)}. \end{array}$$

We note that $p_0(n)$ is monotonously decreasing in n, and $p_0(4) = 2$. This conjecture had been verified by many authors in different cases. All the references on the final result in each part can be summarized in the following table.

	$1 < p < p_0(n)$	$p = p_0(n)$	$p > p_0(n)$
$n = 2$	Glassey [8]	Schaeffer [27]	Glassey [9]
$n = 3$	John [13]	Schaeffer [27]	John [13]
$n \ge 4$	Sideris [28]	Yordanov & Zhang [36] Zhou [40], indep.	Georgiev & Lindblad & Sogge [5]

We are interested in the estimate of the lifespan $\widehat{T}_\varepsilon$ in the blow-up case. From now on, c and C stand for positive constants independent of ε. When $n = 1$, we have the following estimate of the lifespan $\widehat{T}_\varepsilon$ for any $p > 1$:

$$\begin{cases} c\varepsilon^{-(p-1)/2} \le \widehat{T}_\varepsilon \le C\varepsilon^{-(p-1)/2} & \text{if} \quad \displaystyle\int_{\mathbf{R}} g(x)dx \ne 0, \\ c\varepsilon^{-p(p-1)/(p+1)} \le \widehat{T}_\varepsilon \le C\varepsilon^{-p(p-1)/(p+1)} & \text{if} \quad \displaystyle\int_{\mathbf{R}} g(x)dx = 0. \end{cases} \tag{2.2.4}$$

This result has been obtained by Zhou [37]. Moreover, Lindblad [24] has obtained a more precise result for $p = 2$,

$$\begin{cases} \exists \lim\limits_{\varepsilon \to +0} \varepsilon^{1/2}\widehat{T}_\varepsilon > 0 & \text{for} \quad \displaystyle\int_{\mathbf{R}} g(x)dx \ne 0, \\ \exists \lim\limits_{\varepsilon \to +0} \varepsilon^{2/3}\widehat{T}_\varepsilon > 0 & \text{for} \quad \displaystyle\int_{\mathbf{R}} g(x)dx = 0. \end{cases} \tag{2.2.5}$$

Similarly to this, Lindblad [24] has also obtained the following result for $(n, p) = (2, 2)$.

$$\begin{cases} \exists \lim\limits_{\varepsilon \to +0} a(\varepsilon)^{-1}\widehat{T}_\varepsilon > 0 & \text{for} \quad \displaystyle\int_{\mathbf{R}^2} g(x)dx \ne 0, \\ \exists \lim\limits_{\varepsilon \to +0} \varepsilon\widehat{T}_\varepsilon > 0 & \text{for} \quad \displaystyle\int_{\mathbf{R}^2} g(x)dx = 0, \end{cases} \tag{2.2.6}$$

where $a(\varepsilon)$ is the one in (2.1.2).

When $1 < p < p_0(n)$ $(n \ge 3)$ or $2 < p < p_0(2)$ $(n = 2)$, we have

$$c\varepsilon^{-2p(p-1)/\gamma(p,n)} \le \widehat{T}_\varepsilon \le C\varepsilon^{-2p(p-1)/\gamma(p,n)}, \tag{2.2.7}$$

where $\gamma(p, n)$ is defined by (2.2.2), as is summarized in the following table.

	lower bound of $\widehat{T}_\varepsilon$	upper bound of $\widehat{T}_\varepsilon$
$n=2$	Zhou [39]	Zhou [39]
$n=3$	Lindblad [24]	Lindblad [24]
$n \geq 4$	Lindblad & Sogge [25] : $n \leq 8$ or radially symmetric sol.	Georgiev & Takamura & Zhou [6]

On the other hand, when $p = p_0(n)$, we have

$$\exp\left(c\varepsilon^{-p(p-1)}\right) \leq \widehat{T}_\varepsilon \leq \exp\left(C\varepsilon^{-p(p-1)}\right), \tag{2.2.8}$$

as is summarized in the following table.

	lower bound of $\widehat{T}_\varepsilon$	upper bound of $\widehat{T}_\varepsilon$
$n=2$	Zhou [39]	Zhou [39]
$n=3$	Zhou [38]	Zhou [38]
$n \geq 4$	Lindblad & Sogge [25] : $n \leq 8$ or radially symm. sol.	Takamura & Wakasa [32]

2.3 Main theorems

In this section, we introduce the main results of [33]. From now on, we assume that $n \geq 2$ and write $n = 2m,\ 2m+1$ $(m = 1, 2, 3, \cdots)$. For a function $\phi \in C(\mathbf{R})$, we define

$$M(\phi|x,r) = \begin{cases} \dfrac{1}{\omega_n}\displaystyle\int_{|\omega|=1} \phi(x+r\omega)dS_\omega & \text{for } n = 2m+1, \\ \dfrac{2}{\omega_{n+1}}\displaystyle\int_{|\xi|\leq 1} \dfrac{\phi(x+r\xi)}{\sqrt{1-|\xi|^2}}d\xi & \text{for } n = 2m, \end{cases} \tag{2.3.1}$$

where ω_n is the area of the unit sphere in $\mathbf{R}^n$. We consider the following integral equation,

$$u(x,t) = \varepsilon u^0(x,t) + L(F(u))(x,t) \quad \text{for } (x,t) \in \mathbf{R}^n \times [0,\infty), \tag{2.3.2}$$

where F is a given function,

$$L(F(u))(x,t) = \frac{1}{2m-1}\int_0^t (t-\tau)M(F(u(\cdot,\tau))|x,t-\tau)d\tau, \tag{2.3.3}$$

and

$$u^0(x,t) = \partial_t R(f|x,t) + R(g|x,t). \tag{2.3.4}$$

Here

$$R(\phi|x,t) = \frac{1}{(2m-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\left\{t^{2m-1}M(\phi|x,t)\right\}. \tag{2.3.5}$$

Note that (2.3.4) is a representation formula of the solution to the homogeneous wave equation. This fact is proved in Appendix below.

Remark 2.3.1 *Assume that u is a C^2 solution of (2.3.2) with $f \in C^{m+3}(\mathbf{R}^n)$, $g \in C^{m+2}(\mathbf{R}^n)$ and $F \in C^2(\mathbf{R})$, then u satisfies*

$$\begin{cases} u_{tt} - \Delta u = & F(u) - G(u) \\ & -\dfrac{2(m-1)}{2m-1}M(F(\varepsilon f)|x,t) & \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), & & x \in \mathbf{R}^n \end{cases} \tag{2.3.6}$$

where

$$G(u) = \frac{2(m-1)}{2m-1}\int_0^t M(F'(u(\cdot,\tau))u_t(\cdot,\tau)|x,t-\tau)d\tau. \tag{2.3.7}$$

We note that $G \equiv 0$ when $n = 2, 3$ $(m = 1)$. Moreover, assuming $M(F(f)) \in C^{m+1}(\mathbf{R}^n \times [0,\infty))$, and replacing $\varepsilon u^0(x,t)$ by a classical solution $v = v(x,t)$ of

$$\begin{cases} v_{tt} - \Delta v = \dfrac{2(m-1)}{2m-1}M(F(\varepsilon f)|x,t) & \text{in } \mathbf{R}^n \times [0,\infty), \\ v(x,0) = \varepsilon f(x),\ v_t(x,0) = \varepsilon g(x) & x \in \mathbf{R}^n \end{cases} \tag{2.3.8}$$

in (2.3.2), we have that a C^2 solution u of

$$u(x,t) = v(x,t) + L(F(u))(x,t) \quad \text{for } (x,t) \in \mathbf{R}^n \times [0,\infty) \tag{2.3.9}$$

satisfies

$$\begin{cases} u_{tt} - \Delta u = F(u) - G(u) & \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), & x \in \mathbf{R}^n. \end{cases} \tag{2.3.10}$$

We shall explain some relation between (2.3.6) and the following initial value problem:

$$\begin{cases} u_{tt} - \Delta u = F(u) \quad \text{in} \quad \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), \end{cases} \tag{2.3.11}$$

where $f \in C^{m+3}(\mathbf{R}^n)$, $g \in C^{m+2}(\mathbf{R}^n)$ and $F \in C^{m+1}(\mathbf{R})$. The solution of (2.3.11) is written as

$$u(x,t) = \varepsilon u^0(x,t) + \int_0^t R(F(u(\cdot,\tau))|x,t-\tau)d\tau, \tag{2.3.12}$$

where u^0 and R are defined in (2.3.4) and (2.3.5), respectively. Making use of the result of Proposition A.2 and (2.3.5), we see that the second term of the right hand side of (2.3.12) is equal to

$$\begin{aligned}
&\int_0^t R(F(u(\cdot,\tau))|x,t-\tau)d\tau \\
&= \int_0^t (t-\tau)M(F(u(\cdot,\tau))|x,t-\tau)d\tau \\
&\qquad + \int_0^t \sum_{k=1}^{m-1} \alpha_{m-1,k}(t-\tau)^{k+1}\frac{d^k(t-\tau)}{d(t-\tau)^k}\left\{M(F(u(\cdot,\tau))|x,t-\tau)\right\}d\tau,
\end{aligned}$$

where $\alpha_{m-1,k}$ are suitable constants. If we put the coefficient, $(2m-1)^{-1}$ in the first term, and remove the second term of the right hand side of above, then (2.3.12) coincides with (2.3.2). By the argument in 254-255pp. in Agemi, Kubota and Takamura [1], we see that if u is the C^2 solution of (2.3.2), then u satisfies (2.3.6).

From now on, let $F(u) = |u|^p$ and let us define a lifespan $\overline{T}_\varepsilon$ by

$$\overline{T}_\varepsilon \equiv \overline{T}_\varepsilon(f,g) := \sup\{T \in [0,\infty) \ : \ \text{There exists a unique weak solution } u \text{ of (2.3.2)}\}$$

with arbitrarily fixed (f,g). Agemi, Kubota and Takamura [1] have obtained that $\overline{T}_\varepsilon = \infty$ for $p > p_0(n)$ if ε is small enough, where $p_0(n)$ is Strauss' exponent. Their theorem is written for (2.3.9) only, but it is available also for (2.3.2).

Our purpose is to establish the same estimates for $\overline{T}_\varepsilon$ as in (2.2.7) and (2.2.8) when $n \geq 4$ and $1 < p \leq p_0(n)$. They are divided into two theorems below. We note that one can expect to get a C^2 solution only for $n = 4$ and $F(u) = u^2$.

We shall state our main result. We assume on the data that

$$\begin{cases} \text{both } f \in C_0^{m+3}(\mathbf{R}^n) \text{ and } g \in C_0^{m+2}(\mathbf{R}^n) \text{ have} \\ \text{compact support contained in } \{x \in \mathbf{R}^n \ : \ |x| \leq k\} \\ \text{with some constant } k > 0. \end{cases} \tag{2.3.13}$$

Then, we have the following existence theorem for large time interval.

Theorem 2.3.1 *Let $n \geq 4$ and $F(u) = |u|^p$ with $1 < p \leq p_0(n)$. Suppose that (2.3.13) is fulfilled. Then there exists a positive constant $\varepsilon_0 = \varepsilon_0(f,g,n,p,k)$ such that the lifespan $\overline{T}_\varepsilon$ satisfies*

$$\begin{aligned} &\overline{T}_\varepsilon \geq c\varepsilon^{-2p(p-1)/\gamma(p,n)} && \text{if } 1 < p < p_0(n), \\ &\overline{T}_\varepsilon \geq \exp\left(c\varepsilon^{-p(p-1)}\right) && \text{if } p = p_0(n) \end{aligned} \tag{2.3.14}$$

for any ε with $0 < \varepsilon \leq \varepsilon_0$, where c is a positive constant independent of ε.

In the proof of Theorem 2.3.1, we shall employ a similar weighted L^∞ iteration method to Agemi, Kubota and Takamura [1]. Such an argument had been firstly introduced by John [13] in the case of $n = 3$.

For the counter part, the following assumptions on the data are required.

$$\begin{cases} \text{Let } f \equiv 0,\ g(x) = g(|x|) \text{ and } g \in C_0^1([0,\infty)) \text{ satisfy that there} \\ \text{exist positive constants } k_0 \text{ and } k_1 \text{ with } 0 < k_0 < k_1 < k \\ \text{such that the following three conditions hold.} \\ \text{(i)} \quad \text{supp } g \subset \{x \in \mathbf{R}^n \ : \ |x| \le k\} \\ \text{(ii)} \quad g(|x|) \ge 0 \text{ for } k_0 < |x| < k \text{ and } \displaystyle\int_{(k_1+k)/2}^{k} \lambda^{[n/2]} g(\lambda) d\lambda > 0, \\ \text{(iii)} \quad k_0 \text{ is sufficiently close to } k \text{ so that} \\ \qquad P_m(z) > \dfrac{1}{2} \text{ and } T_m(z) > \dfrac{1}{2} \text{ for all } z > \dfrac{k_0}{k}, \\ \qquad \text{where } P_m \text{ or } T_m \text{ denote Legendre or Tschebyscheff} \\ \qquad \text{polynomials of degree } m \text{ respectively.} \end{cases} \tag{2.3.15}$$

Then, we have the following blow-up theorem.

Theorem 2.3.2 *Let $n \ge 4$ and $F(u) = |u|^p$ with $1 < p \le p_0(n)$. Assume that* (2.3.15) *holds. Then there exist a positive constant $\varepsilon_1 = \varepsilon_1(g, n, p, k)$ such that the lifespan $\overline{T}_\varepsilon$ satisfies*

$$\begin{aligned} &\overline{T}_\varepsilon \le C\varepsilon^{-2p(p-1)/\gamma(p,n)} \quad \text{if } 1 < p < p_0(n), \\ &\overline{T}_\varepsilon \le \exp\left(C\varepsilon^{-p(p-1)}\right) \quad \text{if } p = p_0(n) \end{aligned} \tag{2.3.16}$$

for any ε with $0 < \varepsilon \le \varepsilon_1$, where C is a positive constant independent of ε.

Remark 2.3.2 *The uniqueness of the solution of (2.3.6) is an open problem. Although in Agemi, Kubota and Takamura [1] the uniqueness is claimed in the below of (1.8) at 242p, it seems difficult to apply the uniqueness theorem in Appendix 1 in John [14] because (99a) in [14] fails.*

The proof of Theorem 2.3.2 is done by an iteration argument concerning point-wise estimates from below which is basically introduced by John [13] for $n = 3$. But in the critical case, we have to reduce the proof to the argument of Zhou [38] in which the solution is compared with a blowing-up solution of a nonlinear ordinary differential equation of the second order. We also have to employ the slicing method of the blow-up set which is introduced by Agemi, Kurokawa and Takamura [2] due to technical difficulties in high dimensions. See the sections 2.8, 2.9, 2.10 and 2.11 below.

2.4 Weighted L^∞ space

First, we shall state the following two lemmas for v of (2.3.8) which play key roles in proofs of Theorem 2.3.1 and Theorem 2.3.2. The first one is Huygens' principle in odd space dimensions.

Lemma 2.4.1 (Agemi, Kubota and Takamura [1]) *Let $n = 5, 7, 9, \cdots$. Under the same assumption as in Theorem 2.3.1, there exists a classical solution v of* (2.3.8) *which satisfies*

$$\text{supp } v \subset \{x \in \mathbf{R}^n \ : \ t - k \le |x| \le t + k\}, \tag{2.4.1}$$

where k is the one in (2.3.13).

See 253p. in [1] for the proof of this lemma.

Next, we shall introduce the decay estimate for v. First, we write v in the form

$$v = v_0 + v_1. \tag{2.4.2}$$

Here, $v_0 = \varepsilon u^0$ is in (2.3.4) and v_1 is a solution to the inhomogeneous wave equation

$$\begin{cases} (v_1)_{tt} - \Delta v_1 = \dfrac{2(m-1)}{2m-1} M(F(\varepsilon f)|x,t) & \text{in } \mathbf{R}^n \times [0,\infty), \\ v_1(x,0) = (v_1)_t(x,0) = 0, & x \in \mathbf{R}^n \end{cases} \tag{2.4.3}$$

where M is defined in (2.3.1). Then we have the following lemma.

Lemma 2.4.2 (Agemi, Kubota and Takamura [1]) *Under the same assumption as in Theorem 2.3.1, there exists positive constants $C_{n,k,f,g}$ and C_{n,k,f^p} such that v_0 and v_1 satisfies*

$$\sum_{|\alpha| \le 1} |\nabla_x^\alpha v_0(x,t)| \le \frac{C_{n,k,f,g}\varepsilon}{(t + |x| + 2k)^{(n-1)/2}} \tag{2.4.4}$$

and

$$\sum_{|\alpha| \le 1} |\nabla_x^\alpha v_1(x,t)| \le \frac{C_{n,k,f^p}\varepsilon^p}{(t + |x| + 2k)^{(n-1)/2}} \tag{2.4.5}$$

for $n = 5, 7, 9, \cdots$, and

$$\sum_{|\alpha| \le 2} |\nabla_x^\alpha v_0(x,t)| \le \frac{C_{n,k,f,g}\varepsilon}{(t + |x| + 2k)^{(n-1)/2}(t - |x| + 2k)^{(n-1)/2}} \tag{2.4.6}$$

and

$$\sum_{|\alpha| \le 2} |\nabla_x^\alpha v_1(x,t)| \le \frac{C_{n,k,f^p}\varepsilon^p}{(t + |x| + 2k)^{(n-1)/2}(t - |x| + 2k)^{(n-3)/2}} \tag{2.4.7}$$

for $n = 4, 6, 8, \cdots$.

This lemma directly follows from Lemma 3.2 and Lemma 3.4 in [1]. We omit the proof here.

It is obvious that v is a global classical solution of (2.3.8). Therefore our unknown function shall be $U = u - v$. Then, (2.3.9) can be rewritten into

$$U = L(F(U + v)). \tag{2.4.8}$$

In what follows, we shall construct a solution of (2.4.8) in a weighted L^∞ space. For this purpose, define a sequence of functions $\{U_l\}_{l\in\mathbf{N}}$ by

$$U_l = L(F(U_{l-1} + U_0)), \quad U_0 = v, \tag{2.4.9}$$

where L is the one in (2.3.3).

Taking into account of these Lemma 2.4.2, we shall introduce a weighted L^∞ space as follows. We denote a weighted L^∞ norm of U by

$$\|U\| = \sup_{(x,t)\in\mathbf{R}^n\times[0,T]} \{w(|x|,t)|U(x,t)|\} \tag{2.4.10}$$

with a weighted function

$$w(r,t) = \begin{cases} \tau_+(r,t)^{(n-1)/2}\tau_-(r,t)^q & \text{if} \quad p > \dfrac{n+1}{n-1}, \\ \tau_+(r,t)^{(n-1)/2}\left(\log 4\dfrac{\tau_+(r,t)}{\tau_-(r,t)}\right)^{-1} & \text{if} \quad p = \dfrac{n+1}{n-1}, \\ \tau_+(r,t)^{(n-1)/2+q} & \text{if} \quad 1 < p < \dfrac{n+1}{n-1}, \end{cases} \tag{2.4.11}$$

where we set

$$\tau_+(r,t) = \frac{t+r+2k}{k}, \quad \tau_-(r,t) = \frac{t-r+2k}{k}$$

and q is defined by

$$q = \frac{n-1}{2}p - \frac{n+1}{2}. \tag{2.4.12}$$

In order to get a C^1 solution of (2.4.8), we shall show the convergence of $\{U_l\}_{l\in\mathbf{N}}$ in a function space X defined by

$$X = \big\{U \in C^1(\mathbf{R}^n \times [0,T]) \ : \ \begin{array}{l} \|U\|_X < \infty, \\ \text{supp } U(x,t) \subset \{|x| \le t+k\}\big\} \end{array}$$

which equips a norm

$$\|U\|_X = \sum_{|\alpha|\le 1} \|\nabla_x^\alpha U\|.$$

In view of (2.3.9), we note that $\partial U/\partial t$ can be expressed in terms of $\nabla_x U$. Hence we consider spatial derivatives of U only. Moreover, we see that X is a Banach space for any fixed $T > 0$. It follows from the definition of the norm (2.4.10) that there exists a positive constant C_T depending on T such that

$$\|U\| \geq C_T |U(x,t)|, \quad t \in [0,T].$$

Later, we shall make use of Hölder's inequality

$$\||U_1|^a |U_2|^b\| \leq \|U_1\|^a \|U_2\|^b, \quad a+b=1, \quad a,b \in [0,1]. \tag{2.4.13}$$

Furthermore, for v_0 and v_1 in (2.4.2), we define

$$U_{00} = v_0 \quad \text{and} \quad U_{01} = v_1.$$

We also denote $\partial/\partial x_i$ by ∂_i for $i = 1, 2, \cdots, n$, and set

$$\begin{aligned} &\partial W_l = \max_{|\alpha| \leq 1} \{|\nabla_x^\alpha U_l|, |\nabla_x^\alpha U_{l-1}|\}, \quad W_l = \max\{|U_l|, |U_{l-1}|\}, \\ &\partial W_0 = \max_{|\alpha| \leq 1} \{|\nabla_x^\alpha U_0|\}, \quad \partial W_{0a} = \max_{|\alpha| \leq 1} \{|\nabla_x^\alpha U_{0a}|\} \ (a = 0, 1). \end{aligned}$$

2.5 A priori estimates

In this section, we show *a priori* estimates which plays a key role in the contraction mapping argument. The following lemmas are the essential estimates.

Lemma 2.5.1 *Let L be the linear integral operator defined by* (2.3.3). *Assume that $U \in C^0(\mathbf{R}^n \times [0,T])$ with* supp $U \subset \{(x,t) \in \mathbf{R}^n \times [0,T] : |x| \leq t + k\}$ *and $\|U\| < \infty$. Then, there exists a positive constant C independent of k and T such that*

$$\|L(|U|^p)\| \leq C k^2 \|U\|^p D(T), \tag{2.5.1}$$

where $D(T)$ is defined by

$$D(T) = \begin{cases} 1 & \text{if } p > p_0(n), \\ \log \dfrac{2T+3k}{k} & \text{if } p = p_0(n), \\ \left(\dfrac{2T+3k}{k}\right)^{\gamma(p,n)/2} & \text{if } 1 < p < p_0(n) \end{cases} \tag{2.5.2}$$

and $\gamma(p,n)$ is the one in (2.2.2).

Lemma 2.5.2 *Let $n = 5, 7, 9, \cdots$ and L be the linear integral operator defined by* (2.3.3). *Assume that $U, U_0 \in C^0(\mathbf{R}^n \times [0,T])$ with* $\operatorname{supp} U \subset \{(x,t) \in \mathbf{R}^n \times [0,T] : |x| \le t+k\}$, $\operatorname{supp} U_0 \subset \{(x,t) \in \mathbf{R}^n \times [0,T] : t-k \le |x| \le t+k\}$ *and* $\|U\|, \|\tau_+^{(n-1)/2} U_0 w^{-1}\| < \infty$. *Then, there exists a positive constant $C_{n,\nu,p}$ depending on n, ν and p such that*

$$\|L(|U_0|^{p-\nu}|U|^\nu)\| \le C_{n,\nu,p} k^2 \left\| \frac{\tau_+^{(n-1)/2}}{w} U_0 \right\|^{p-\nu} \|U\|^\nu E_\nu(T), \tag{2.5.3}$$

where $0 \le \nu \le p$. $E_\nu(T)$ is defined by

$$E_\nu(T) = \begin{cases} 1 & \text{if } p > \dfrac{n+1}{n-1}, \\ \left(\dfrac{2T+3k}{k}\right)^{\nu\delta} & \text{if } p = \dfrac{n+1}{n-1}, \\ \left(\dfrac{2T+3k}{k}\right)^{-\nu q} & \text{if } p < \dfrac{n+1}{n-1} \end{cases} \quad \text{for } 0 \le \nu < p, \tag{2.5.4}$$

where q is the one in (2.4.12) and δ is a small positive constant. When $\nu = p$, (2.5.3) coincides with (2.5.1) as $E_p(T) = D(T)$ and $C_{n,p,p} = C$.

Lemma 2.5.3 *Let $n = 4, 6, 8, \cdots$ and L be the linear integral operator defined by (2.3.3). Assume that $U, U_{00}, U_{01} \in C^0(\mathbf{R}^n \times [0,T])$ with* $\operatorname{supp}(U, U_{00}, U_{01}) \subset \{(x,t) \in \mathbf{R}^n \times [0,T] : |x| \le t+k\}$ *and* $\|U\|$, $\|(\tau_+\tau_-)^{(n-1)/2} U_{0a} (w\tau_-^a)^{-1}\| < \infty$ $(a = 0, 1)$. *Then, there exists a positive constant $C_{n,\nu,p}$ depending on n, ν, and p such that*

$$\|L(|U_{0a}|^{p-\nu}|U|^\nu)\| \le C_{n,\nu,p} k^2 \left\| (\tau_+\tau_-)^{(n-1)/2} \frac{U_{0a}}{w\tau_-^a} \right\|^{p-\nu} \|U\|^\nu E_{\nu,a}(T), \tag{2.5.5}$$

where $0 \le \nu \le p$ and $a = 0, 1$. When $0 \le \nu < p$, $E_{\nu,a}(T)$ is defined by

$$E_{\nu,a}(T) = \begin{cases} 1 & \text{if } \mu < -1, \\ \log \dfrac{2T+3k}{k} & \text{if } \mu = -1, \\ \left(\dfrac{2T+3k}{k}\right)^{1+\mu} & \text{if } \mu > -1 \end{cases} \quad \text{for } p > \frac{n+1}{n-1}, \tag{2.5.6}$$

where $\mu = (p-\nu)\left(a - \dfrac{n-1}{2}\right) - \nu q$ *and* q *is the one in (2.4.12), and*

$$E_{\nu,a}(T) = \begin{cases} \log \dfrac{2T+3k}{k} & \text{if } \sigma = -1,\ \nu = 0, \\ \left(\dfrac{2T+3k}{k}\right)^{1+\sigma} & \text{if } \sigma > -1, \\ \left(\dfrac{2T+3k}{k}\right)^{\nu\delta} & \text{otherwise} \end{cases} \qquad \text{for } p = \frac{n+1}{n-1}, \tag{2.5.7}$$

where $\sigma = (p-\nu)\left(a - \dfrac{n-1}{2}\right)$ *and* δ *stands for any positive constant, and*

$$E_{\nu,a}(T) = \begin{cases} \left(\dfrac{2T+3k}{k}\right)^{-\nu q} & \text{if } \sigma < -1, \\ \log \dfrac{2T+3k}{k}\left(\dfrac{2T+3k}{k}\right)^{-\nu q} & \text{if } \sigma = -1, \\ \left(\dfrac{2T+3k}{k}\right)^{1+\mu} & \text{if } \sigma > -1 \end{cases} \quad \text{for } p < \frac{n+1}{n-1}. \tag{2.5.8}$$

When $\nu = p$*, (2.5.5) coincides with (2.5.1) as* $E_{p,a}(T) = D(T)$ *for* $a = 0, 1$ *and* $C_{n,p,p} = C$*.*

Lemma 2.5.4 *Suppose that the same assumption as in Lemma 2.5.3 is fulfilled. Then, there exists a positive constant* $C_{n,\nu,p}$ *depending on* n*,* ν*, and* p *such that*

$$\begin{aligned} &\|L(|U_{00}|^{p-\nu}|U_{01}|^{\nu})\| \\ &\quad \le C_{n,\nu,p}k^2 \left\|(\tau_+\tau_-)^{(n-1)/2}\frac{U_{00}}{w}\right\|^{p-\nu} \left\|(\tau_+\tau_-)^{(n-1)/2}\frac{U_{01}}{w\tau_-}\right\|^{\nu} F_\nu(T), \end{aligned} \tag{2.5.9}$$

where $0 \le \nu \le p$*. When* $0 < \nu < p$*,* $F_\nu(T)$ *is defined by*

$$F_\nu(T) = \begin{cases} 1 & \text{if } \kappa < -1, \\ \log \dfrac{2T+3k}{k} & \text{if } \kappa = -1, \\ \left(\dfrac{2T+3k}{k}\right)^{1+\kappa} & \text{if } \kappa > -1, \end{cases} \tag{2.5.10}$$

where $\kappa = \nu - \dfrac{n-1}{2}p$*. When* $\nu = 0$ *or* $\nu = p$*, (2.5.9) coincides with (2.5.5) as* $F_0(T) = E_{0,0}(T)$ *for* $a = \nu = 0$ *and* $F_p(T) = E_{0,1}(T)$ *for* $a = 1$*,* $\nu = p$*.*

Four lemmas above follow from the following basic estimate.

Lemma 2.5.5 (Basic estimate) *Let L be the linear integral operator defined by* (2.3.3) *and $a_1 \geq 0$, $a_2 \in \mathbf{R}$ and $a_3 \geq 0$. Then, there exists a positive constant C_{n,p,a_1,a_2,a_3} such that*

$$\begin{aligned} &L\left\{\tau_+^{-(n-1)p/2+a_1}\tau_-^{a_2}\left(\log(4\tau_+/\tau_-)\right)^{a_3}\right\}(x,t) \\ &\leq C_{n,p,a_1,a_2,a_3}k^2 w(r,t)^{-1}\left(\frac{2T+3k}{k}\right)^{a_1} E_{a_1,a_2,a_3}(T) \end{aligned} \tag{2.5.11}$$

for $|x| \leq t+k$, $t \in [0,T]$, where $E_{a_1,a_2,a_3}(T)$ is defined by

$$E_{a_1,a_2,a_3}(T) = \begin{cases} 1 & \text{if } a_2 < -1 \text{ and } a_3 = 0, \\ \log\dfrac{2T+3k}{k} & \text{if } a_2 = -1 \text{ and } a_3 = 0, \\ \left(\dfrac{2T+3k}{k}\right)^{\delta a_3} & \text{if } a_2 \leq -1 \text{ and } a_3 > 0, \\ \left(\dfrac{2T+3k}{k}\right)^{1+a_2} & \text{if } a_2 > -1, \end{cases} \tag{2.5.12}$$

where δ stands for any positive constant.

Proof. First we introduce the following fundamental identity for spherical means.

Lemma 2.5.6 (John [12]) *Let $b \in C([0,\infty))$. Then, the identity*

$$\int_{|\omega|=1} b(|x+\rho\omega|)dS_\omega = 2^{3-n}\omega_{n-1}(r\rho)^{2-n}\int_{|\rho-r|}^{\rho+r}\lambda h(\lambda,\rho,r)b(\lambda)d\lambda \tag{2.5.13}$$

holds for $x \in \mathbf{R}^n$, $r = |x|$ and $\rho > 0$, where

$$h(\lambda,\rho,r) = \{\lambda^2-(\rho-r)^2\}^{(n-3)/2}\{(\rho+r)^2-\lambda^2\}^{(n-3)/2}. \tag{2.5.14}$$

See [12] for the proof of this lemma.

In order to continue the proof of Lemma 2.5.5, we need radially symmetric versions of L which follows from Lemma 2.5.6. From now on, a positive constant C depending only on n and p may change from line to line.

Lemma 2.5.7 *Let L be the linear integral operator defined by* (2.3.3) *and $\Psi = \Psi(|x|,t) \in C([0,\infty)^2)$, $x \in \mathbf{R}^n$. Then,*

$$L(\Psi)(x,t) = L_{odd}(\Psi)(r,t), \ r = |x| \tag{2.5.15}$$

holds for $n = 5, 7, 9 \cdots$ *and*

$$L(\Psi)(x,t) = L_{even,1}(\Psi)(r,t) + L_{even,2}(\Psi)(r,t), \ r = |x| \tag{2.5.16}$$

holds for $n = 4, 6, 8 \cdots$, *where* $L_{odd}(\Psi)$ *is defined by*

$$\begin{aligned} &L_{odd}(\Psi)(r,t) \\ &= Cr^{2-n} \int_0^t (t-\tau)^{3-n} d\tau \int_{|t-\tau-r|}^{t-\tau+r} \lambda h(\lambda, t-\tau, r) \Psi(\lambda, \tau) d\lambda \end{aligned} \tag{2.5.17}$$

and each $L_{even,i}(\Psi)$ $(i = 1, 2)$ *is defined by*

$$\begin{aligned} L_{even,1}(\Psi)(r,t) = \ & Cr^{2-n} \int_0^t (t-\tau)^{2-n} d\tau \int_{|t-r-\tau|}^{t+r-\tau} \lambda \Psi(\lambda, \tau) d\lambda \times \\ & \times \int_{|\lambda - r|}^{t-\tau} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho, \end{aligned} \tag{2.5.18}$$

$$\begin{aligned} L_{even,2}(\Psi)(r,t) = \ & Cr^{2-n} \int_0^{(t-r)_+} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda \Psi(\lambda, \tau) d\lambda \times \\ & \times \int_{|\lambda - r|}^{\lambda + r} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho. \end{aligned} \tag{2.5.19}$$

Here the usual notation $a_+ = \max\{a, 0\}$ *is used.*

Proof. (2.5.15) immediately follows from Lemma 2.5.6. For (2.5.16), we make use of changing variables by $y - x = (t-\tau)\xi$ in (2.3.3). Then, we obtain

$$L(\Psi)(x,t) = C \int_0^t (t-\tau)^{2-n} d\tau \int_{|y-x| \le t-\tau} \frac{\Psi(|y|, \tau)}{\sqrt{(t-\tau)^2 - |y-x|^2}} dy.$$

Introducing polar coordinates, we have

$$\begin{aligned} L(\Psi)(x,t) \ &= C \int_0^t (t-\tau)^{2-n} d\tau \int_0^{t-\tau} \frac{\rho^{n-1} d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \ \times \\ & \times \int_{|\omega|=1} \Psi(|x + \rho\omega|, \tau) dS_\omega. \end{aligned}$$

Thus Lemma 2.5.6 yields

$$\begin{aligned} L(\Psi)(x,t) \ &= Cr^{2-n} \int_0^t (t-\tau)^{2-n} d\tau \int_0^{t-\tau} \frac{\rho d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \times \\ & \times \int_{|\rho - r|}^{\rho + r} \lambda \Psi(\lambda, \tau) h(\lambda, \rho, r) d\lambda. \end{aligned} \tag{2.5.20}$$

Therefore, (2.5.16) follows from inverting the order of (ρ, λ)-integral in (2.5.20). □

In order to estimate the kernel $h(\lambda, \rho, r)$, we need the following lemma.

Lemma 2.5.8 (Agemi, Kubota and Takamura [1]) *Let $h(\lambda, \rho, r)$ be the one in (2.5.14). Suppose that $|\rho - r| \leq \lambda \leq \rho + r$ and $\rho \geq 0$. Then, the inequality*

$$|\lambda - r| \leq \rho \leq \lambda + r \tag{2.5.21}$$

holds. Moreover, the following three estimates are available.

$$\begin{aligned} h(\lambda, \rho, r) &\leq Cr^{n-3}\lambda^{n-3}, && (2.5.22)\\ h(\lambda, \rho, r) &\leq C\rho^{n-3}r^{(n-3)/2}\lambda^{(n-3)/2}, && (2.5.23)\\ h(\lambda, \rho, r) &\leq Cr^{n-3}\rho^{n-3}. && (2.5.24) \end{aligned}$$

See 257-258pp. in [1] for the proof of this lemma.

Let us continue the proof of Lemma 2.5.5. For simplicity, we set

$$\begin{aligned} I_{odd}(r,t) &= L_{odd}\left\{\tau_+^{-(n-1)p/2+a_1}\tau_-^{a_2}\left(\log(4\tau_+/\tau_-)\right)^{a_3}\right\}(r,t),\\ I_{even,i}(r,t) &= L_{even,i}\left\{\tau_+^{-(n-1)p/2+a_1}\tau_-^{a_2}\left(\log(4\tau_+/\tau_-)\right)^{a_3}\right\}(r,t)\ (i=1,2). \end{aligned}$$

Estimates for I_{odd} and $I_{even,1}$. We shall estimate I_{odd} and $I_{even,1}$ on the following three domains.

$$\begin{aligned} D_1 &= \{(r,t) \mid r \geq t-r > -k \text{ and } r \geq 2k\},\\ D_2 &= \{(r,t) \mid r \geq t-r > -k \text{ and } r \leq 2k\},\\ D_3 &= \{(r,t) \mid t-r \geq r\}. \end{aligned}$$

(i) Estimate in D_1,

Making use of (2.5.23), we get

$$\begin{aligned} I_{odd}(r,t) &\leq Cr^{-(n-1)/2}\int_0^t d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda^{(n-1)/2}\tau_-(\lambda,\tau)^{a_2}\times\\ &\quad\times\tau_+(\lambda,\tau)^{-(n-1)p/2+a_1}\left(\log 4\frac{\tau_+(\lambda,\tau)}{\tau_-(\lambda,\tau)}\right)^{a_3} d\lambda \end{aligned} \tag{2.5.25}$$

and

$$\begin{aligned} I_{even,1}(r,t) &\leq Cr^{-(n-1)/2}\int_0^t (t-\tau)^{2-n}d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda^{(n-1)/2}\times\\ &\quad\times\tau_+(\lambda,\tau)^{-(n-1)p/2+a_1}\tau_-(\lambda,\tau)^{a_2}\times\\ &\quad\times\left(\log 4\frac{\tau_+(\lambda,\tau)}{\tau_-(\lambda,\tau)}\right)^{a_3} d\lambda \int_{|\lambda-r|}^{t-\tau}\frac{\rho^{n-2}}{\sqrt{(t-\tau)^2-\rho^2}}d\rho. \end{aligned} \tag{2.5.26}$$

If one apply the simple inequality

$$\int_{|\lambda-r|}^{t-\tau} \frac{\rho^{n-2}}{\sqrt{(t-\tau)^2-\rho^2}} d\rho \le (t-\tau)^{n-2} \quad \text{for } 0 \le \tau \le t \tag{2.5.27}$$

to the right-hand side of (2.5.26), the same quantity as the right-hand side of (2.5.25) appears. Hence, we shall estimate for I_{odd} only from now on.

Changing variables in (2.5.25) by

$$\alpha = \tau + \lambda, \ \beta = \tau - \lambda, \tag{2.5.28}$$

we get

$$\begin{aligned} I_{odd}(r,t) \ &\le Cr^{-(n-1)/2} \int_{-k}^{t-r} \left(\frac{\beta+2k}{k}\right)^{a_2} d\beta \int_{|t-r|}^{t+r} (\alpha-\beta)^{(n-1)/2} \times \\ &\times \left(\frac{\alpha+2k}{k}\right)^{-(n-1)p/2+a_1} \left(\log 4\frac{\alpha+2k}{\beta+2k}\right)^{a_3} d\alpha. \end{aligned}$$

It follows from

$$\frac{r}{k} = \frac{2r+r+r}{4k} \ge \frac{\tau_+(r,t)}{4}$$

that

$$\begin{aligned} &I_{odd}(r,t) \\ &\le C\tau_+(r,t)^{-(n-1)/2} \left(\frac{t+r+2k}{k}\right)^{a_1} \int_{-k}^{t-r} \left(\frac{\beta+2k}{k}\right)^{a_2} d\beta \times \\ &\times \int_{t-r}^{t+r} \left(\frac{\alpha+2k}{k}\right)^{-1-q} \left(\log 4\frac{\alpha+2k}{\beta+2k}\right)^{a_3} d\alpha. \end{aligned} \tag{2.5.29}$$

When $a_3 = 0$, α-integral in (2.5.29) is dominated by

$$\begin{cases} Ck\tau_-^{-q} & \text{if } p > \dfrac{n+1}{n-1}, \\ k\log\dfrac{\tau_+}{\tau_-} & \text{if } p = \dfrac{n+1}{n-1}, \\ Ck\tau_+^{-q} & \text{if } p < \dfrac{n+1}{n-1} \end{cases}$$

and β-integral in (2.5.29) is dominated by

$$\begin{cases} \dfrac{k}{-(1+a_2)} & \text{if } a_2 < -1, \\ k\log\dfrac{t-r+2k}{k} & \text{if } a_2 = -1, \\ \dfrac{k}{(1+a_2)}\left(\dfrac{t-r+2k}{k}\right)^{1+a_2} & \text{if } a_2 > -1. \end{cases}$$

(2.5.11) is now established for $a_3 = 0$.

When $a_3 > 0$, we employ the following simple lemma.

Lemma 2.5.9 *Let $\delta > 0$ be any given constant. Then, we have*

$$\log X \le \frac{X^\delta}{\delta} \ for \ X \ge 1. \tag{2.5.30}$$

The proof of this lemma follows from elementary computation. We shall omit it. Then, it follows from Lemma 2.5.9 that

$$\begin{aligned} I_{odd}(r,t) \ &\le C(4\delta^{-1})^{a_3}\tau_+(r,t)^{-(n-1)/2}\left(\frac{t+r+2k}{k}\right)^{a_1+\delta a_3} \times \\ &\times \int_{-k}^{t-r}\left(\frac{\beta+2k}{k}\right)^{a_2-\delta a_3} d\beta \int_{t-r}^{t+r}\left(\frac{\alpha+2k}{k}\right)^{-1-q} d\alpha. \end{aligned}$$

The α-integral above can be estimated by the same manner in the case of $a_3 = 0$. The β-integral is dominated by

$$\begin{cases} \dfrac{-k}{1+a_2-\delta a_3} & \text{if } a_2 \le -1, \\ \dfrac{k}{1+a_2-\delta a_3}\left(\dfrac{t-r+2k}{k}\right)^{1+a_2-\delta a_3} & \text{if } a_2 > -1 \end{cases} \tag{2.5.31}$$

with $\delta > 0$ satisfying $1 + a_2 - \delta a_3 > 0$. Therefore I_{odd} and $I_{even,1}$ are bounded in D_1 by the quantity in the right-hand side of (2.5.11) as desired.

(ii) Estimate in D_2.

By making use of (2.5.24), we have

$$\begin{aligned} I_{odd}(r,t) \ &\le Cr^{-1}\int_0^t d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda\tau_+(\lambda,\tau)^{-(n-1)p/2+a_1} \times \\ &\times \tau_-(\lambda,\tau)^{a_2}\left(\log 4\frac{\tau_+(\lambda,\tau)}{\tau_-(\lambda,\tau)}\right)^{a_3} d\lambda \end{aligned} \tag{2.5.32}$$

and

$$\begin{aligned} I_{even,1}(r,t) \ &\le Cr^{-1}\int_0^t (t-\tau)^{-1} d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda\tau_+(\lambda,\tau)^{-(n-1)p/2+a_1} \times \\ &\times \tau_-(\lambda,\tau)^{a_2}\left(\log 4\frac{\tau_+(\lambda,\tau)}{\tau_-(\lambda,\tau)}\right)^{a_3} d\lambda \int_{|\lambda-r|}^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} d\rho. \end{aligned}$$

Similarly to the estimate in D_1, the simple inequality

$$\int_{|\lambda-r|}^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} d\rho \le t-\tau \quad \text{for } 0 \le \tau \le t$$

helps us to estimate I_{odd} only. Changing variables by (2.5.28), we get

$$\begin{aligned} I_{odd}(r,t) \ &\leq Ckr^{-1}\left(\frac{t+r+2k}{k}\right)^{a_1}\int_{-k}^{t-r}\left(\frac{\beta+2k}{k}\right)^{a_2}d\beta\times \\ &\times\int_{t-r}^{t+r}\left(\frac{\alpha+2k}{k}\right)^{1-(n-1)p/2}\left(\log 4\frac{\alpha+2k}{\beta+2k}\right)^{a_3}d\alpha. \end{aligned} \tag{2.5.33}$$

Note that both τ_+ and τ_- are numerical constants in this domain, and that the integrand of both α-integral and β-integral in (2.5.33) is numerical constant C_{a_1,a_2,a_3} depending on a_1, a_2 and a_3. Hence we have

$$I_{odd}(r,t) \leq CC_{a_1,a_2,a_3}kr^{-1}\int_{-k}^{t-r}d\beta\int_{t-r}^{t+r}d\alpha \leq CC_{a_1,a_2,a_3}k^2. \tag{2.5.34}$$

This is the desired estimate in D_2.
(iii) Estimate in D_3.

By the same reason, we have to estimate I_{odd} in (2.5.33) only. In D_3, since $1-(n-1)p/2<0$ is trivial, we get

$$\left(\frac{\alpha+2k}{k}\right)^{1-(n-1)p/2} \leq \left(\frac{t-r+2k}{k}\right)^{1-(n-1)p/2} \leq Cw(r,t)^{-1}$$

because $t-r\geq r$ is equivalent to $3(t-r)\geq t+r$. Hence, when $a_3=0$, we obtain

$$I_{odd}(r,t) \leq Ckw(r,t)^{-1}\left(\frac{t+r+2k}{k}\right)^{a_1}\int_{-k}^{t-r}\left(\frac{\beta+2k}{k}\right)^{a_2}d\beta.$$

When $a_3>0$, due to Lemma 2.5.9, we have

$$\begin{aligned} I_{odd}(r,t) \leq \ & C(4\delta^{-1})^{a_3}kw(r,t)^{-1}\times \\ & \times\left(\frac{t+r+2k}{k}\right)^{a_1+\delta a_3}\int_{-k}^{t-r}\left(\frac{\beta+2k}{k}\right)^{a_2-\delta a_3}d\beta. \end{aligned}$$

Therefore, in view of (2.5.31), I_{odd} and $I_{even,1}$ are bounded in D_3 by the quantity in the right-hand side of (2.5.11).

Estimates for $I_{even,2}$. We shall estimate $I_{even,2}$ on the following three domains.

$$\begin{aligned} D_4 &= \{(r,t) \mid 0<t-r\leq k \text{ and } t\leq 2k\}, \\ D_5 &= \{(r,t) \mid 0<t-r\leq k \text{ and } t\geq 2k\}, \\ D_6 &= \{(r,t) \mid t-r\geq k\}. \end{aligned}$$

(iv) Estimate in D_4,

Note that τ_+ and τ_- are numerical constants in this case. By virtue of (2.5.21) and (2.5.22), we get

$$\begin{aligned} I_{even,2}(r,t) &\le CC_{a_1,a_2,a_3} r^{-1} \int_0^{t-r} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda^{n-2} d\lambda \\ &\quad \times \int_{|\lambda-r|}^{\lambda+r} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} d\rho. \end{aligned}$$

It follows from

$$\int_{|\lambda-r|}^{\lambda+r} \frac{\rho d\rho}{\sqrt{(t-\tau)^2-\rho^2}} \le \frac{2r\lambda}{\sqrt{t-\tau+\lambda+r}\sqrt{t-\tau-\lambda-r}} \tag{2.5.35}$$

that

$$\begin{aligned} &I_{even,2}(r,t) \\ &\le CC_{a_1,a_2,a_3} \int_0^{t-r} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} d\lambda. \end{aligned}$$

Noticing that

$$\lambda \le t-r-\tau \le t-\tau \quad \text{for } \tau \ge 0,$$

we obtain

$$I_{even,2}(r,t) = CC_{a_1,a_2,a_3} \int_0^{t-r} (t-\tau)^{1/2} d\tau \int_0^{t-r-\tau} \frac{d\lambda}{\sqrt{t-\tau-\lambda-r}}.$$

Making use of (2.5.28), we have

$$\begin{aligned} I_{even,2}(r,t) &\le CC_{a_1,a_2,a_3} k^{1/2} \int_{-k}^{t-r} d\beta \int_{\beta}^{t-r} \frac{d\alpha}{\sqrt{t-r-\alpha}} \\ &\le CC_{a_1,a_2,a_3} k^2. \end{aligned}$$

This is the desired estimate in D_4.
(v) Estimate in D_5.

In this domain, (2.5.11) follows from

$$I_{even,2}(r,t) \le Ck^2 C_{a_1,1_2,a_3} \tau_+(r,t)^{-(n-1)/2}.$$

To see this, we shall employ (2.5.21) and (2.5.23). Then we have

$$\begin{aligned} I_{even,2}(r,t) &\le Cr^{-(n-1)/2} \int_0^{t-r} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda^{(n-1)/2} \times \\ &\quad \times \tau_+(\lambda,\tau)^{-(n-1)p/2+a_1} \tau_-(\lambda,\tau)^{a_2} \times \\ &\quad \times \left(\log 4 \frac{\tau_+(\lambda,\tau)}{\tau_-(\lambda,\tau)}\right)^{a_3} d\lambda \int_{|\lambda-r|}^{\lambda+r} \frac{\rho^{n-2}}{\sqrt{(t-\tau)^2-\rho^2}} d\rho. \end{aligned}$$

Similarly to (2.5.34), it follows from (2.5.27) that

$$I_{even,2}(r,t) \le CC_{a_1,a_2,a_3} r^{-(n-1)/2} k^2.$$

In D_5, we have $r \ge k$ which implies $r/k \ge C\tau_+(r,t)$. Hence, we obtain the desired estimate.
(vi) Estimate in D_6.

By virtue of (2.5.22) and (2.5.35), we get

$$\begin{aligned} &I_{even,2}(r,t) \\ &\le C \int_0^{t-r} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} \times \\ &\quad \times \tau_+(\lambda,\tau)^{-(n-1)p/2+a_1} \tau_-(\lambda,\tau)^{a_2} \left(\log 4 \frac{\tau_+(\lambda,\tau)}{\tau_-(\lambda,\tau)} \right)^{a_3} d\lambda. \end{aligned}$$

Then, we divide the integral of the right-hand side above as

$$I_{even,3}(r,t) + I_{even,4}(r,t),$$

where

$$\begin{aligned} I_{even,3}(r,t) &= C \int_0^{(t-r)/2} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} \times \\ &\quad \times \tau_+(\lambda,\tau)^{-(n-1)p/2+a_1} \tau_-(\lambda,\tau)^{a_2} \left(\log 4 \frac{\tau_+(\lambda,\tau)}{\tau_-(\lambda,\tau)} \right)^{a_3} d\lambda \end{aligned}$$

and

$$\begin{aligned} I_{even,4}(r,t) &= C \int_{(t-r)/2}^{t-r} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} \times \\ &\quad \times \tau_+(\lambda,\tau)^{-(n-1)p/2+a_1} \tau_-(\lambda,\tau)^{a_2} \left(\log 4 \frac{\tau_+(\lambda,\tau)}{\tau_-(\lambda,\tau)} \right)^{a_3} d\lambda. \end{aligned}$$

First, we shall estimate $I_{even,3}$. It follows from (2.5.28) that

$$\begin{aligned} &I_{even,3}(r,t) \\ &\le Ck^{1/2} \left(\frac{t+r+2k}{k} \right)^{3/2-n} \left(\frac{t-r+2k}{k} \right)^{n-1-(n-1)p/2+a_1} \times \\ &\quad \times \int_{-k}^{t-r} \left(\frac{\beta+2k}{k} \right)^{a_2} d\beta \int_\beta^{t-r} \left(\log 4 \frac{\alpha+2k}{\beta+2k} \right)^{a_3} \frac{d\alpha}{\sqrt{t-r-\alpha}} \end{aligned}$$

because of $n-1-(n-1)p/2 \ge 0$ for $p \le 2$. When $a_3 = 0$, we get

$$\begin{aligned} I_{even,3}(r,t) \le \; & Ck \left(\frac{t+r+2k}{k} \right)^{3/2-n} \times \\ & \times \left(\frac{t-r+2k}{k} \right)^{-q+(n-2)/2+a_1} \int_{-k}^{t-r} \left(\frac{\beta+2k}{k} \right)^{a_2} d\beta. \end{aligned}$$

Hence the desired estimate follows from

$$\frac{\tau_-(r,t)^{-q+(n-2)/2}}{\tau_+(r,t)^{n-3/2}} \le Cw(r,t)^{-1} \tag{2.5.36}$$

in this case. When $a_3 > 0$, Lemma 2.5.9 yields that

$$\begin{aligned} I_{even,3}(r,t) \le \; & Ck(4\delta^{-1})^{a_3}\left(\frac{t+r+2k}{k}\right)^{3/2-n} \times \\ & \left(\frac{t-r+2k}{k}\right)^{-q+(n-2)/2+a_1+\delta a_3} \int_{-k}^{t-r}\left(\frac{\beta+2k}{k}\right)^{a_2-\delta a_3} d\beta. \end{aligned}$$

Hence the desired estimate follows from (2.5.31) and (2.5.36).

Next, we shall estimate $I_{even,4}$. If $r \ge t-r \ge k$, (2.5.28) yields that

$$\begin{aligned} I_{even,4}(r,t) \le \; & \frac{Ck^{n-1}}{r^{n-3/2}}\left(\frac{t-r+2k}{k}\right)^{n-1-(n-1)p/2+a_1} \times \\ & \times \int_{-k}^{t-r}\left(\frac{\beta+2k}{k}\right)^{a_2} d\beta \int_{\beta}^{t-r}\frac{d\alpha}{\sqrt{t-r-\alpha}}\left(\log 4\frac{\alpha+2k}{\beta+2k}\right)^{a_3}. \end{aligned}$$

In this case, we have $r/k \ge C\tau_+(r,t)$, so that (2.5.11) follows from the same argument as for $I_{even,3}$. On the other hand, if $t-r \ge r$ and $t-r \ge k$, we have

$$\tau+\lambda+2k \ge \frac{t-r}{2}+2k \ge \frac{t+r+2k}{6} \text{ for } \tau \ge \frac{t-r}{2},\ \lambda \ge 0.$$

Hence (2.5.35) yields that

$$\begin{aligned} I_{even,4}(r,t) \le \; & Ck^{1/2}\left(\frac{t+r+2k}{k}\right)^{1/2-(n-1)p/2+a_1} \times \\ & \times \int_0^{t-r} d\tau \int_0^{t-r-\tau}\frac{\tau_-(\lambda,\tau)^{a_2}}{\sqrt{t-\tau-\lambda-r}}\left(\log 4\frac{\tau_+(\lambda,\tau)}{\tau_-(\lambda,\tau)}\right)^{a_3} d\lambda. \end{aligned}$$

Changing variables by (2.5.28), we have

$$\begin{aligned} I_{even,4}(r,t) \le \; & Ck^{1/2}\left(\frac{t+r+2k}{k}\right)^{1/2-(n-1)p/2+a_1} \times \\ & \times \int_{-k}^{t-r}\left(\frac{\beta+2k}{k}\right)^{a_2} d\beta \int_{\beta}^{t-r}\left(\log 4\frac{\alpha+2k}{\beta+2k}\right)^{a_3}\frac{d\alpha}{\sqrt{t-r-\alpha}}. \end{aligned}$$

Therefore, applying the simple inequality

$$\tau_+(r,t)^{1-(n-1)p/2} \le Cw(r,t)^{-1},$$

we obtain the desired estimates by the same argument as for $I_{even,3}$. The proof of Lemma 2.5.5 is now completed. □

Proof of Lemma 2.5.1. Since (2.3.3) and (2.4.10) yield that

$$L(|U|^p)(x,t) \le \|U\|^p L(w^{-p})(x,t),$$

it is enough to show the inequality

$$w(r,t)L(w^{-p})(x,t) \le Ck^2 D(T).$$

This is established by (2.5.11) with setting

$$\begin{cases} a_1 = a_3 = 0,\ a_2 = -pq & \text{if } p > \dfrac{n+1}{n-1}, \\ a_1 = a_2 = 0,\ a_3 = p & \text{if } p = \dfrac{n+1}{n-1}, \\ a_1 = -pq,\ a_2 = a_3 = 0 & \text{if } p < \dfrac{n+1}{n-1}. \end{cases}$$

□

Proof of Lemma 2.5.2. Due to Huygens' principle in Lemma 2.4.1, one can replace τ_- by $\tau_-\chi_{\{-k\le t-r\le k\}}$ in (2.5.11). Then, the integral with respect to the variable of $\beta = \tau - \lambda$ is bounded. In order to establish (2.5.3), it is enough to show the inequality

$$w(r,t)L\left(\tau_+^{-(n-1)(p-\nu)/2} w^{-\nu}\chi_{\{-k\le t-r\le k\}}\right) \le C_{n,\nu,p}k^2 E_\nu(T).$$

This is established by (2.5.11) with setting

$$\begin{cases} a_1 = a_3 = 0,\ a_2 = -\nu q & \text{if } p > \dfrac{n+1}{n-1}, \\ a_1 = a_2 = 0,\ a_3 = \nu & \text{if } p = \dfrac{n+1}{n-1}, \\ a_1 = -\nu q,\ a_2 = a_3 = 0 & \text{if } p < \dfrac{n+1}{n-1}. \end{cases}$$

□

Proof of Lemma 2.5.3 and Lemma 2.5.4. In order to prove (2.5.5) and (2.5.9), it is enough to show inequalities

$$w(r,t)L\left(\tau_+^{-(n-1)(p-\nu)/2}\tau_-^{\sigma} w^{-\nu}\right) \le C_{n,\nu,p}k^2 E_{\nu,a}(T)$$

and

$$w(r,t)L\left(\tau_+^{-(n-1)p/2}\tau_-^{\kappa}\right) \le C_{n,\nu,p}k^2 F_\nu(T).$$

If we set

$$\begin{cases} a_1 = a_3 = 0,\ a_2 = \mu & \text{if } p > \dfrac{n+1}{n-1}, \\ a_1 = 0,\ a_2 = \sigma,\ a_3 = \nu & \text{if } p = \dfrac{n+1}{n-1}, \\ a_1 = -\nu q,\ a_2 = \sigma,\ a_3 = 0 & \text{if } p < \dfrac{n+1}{n-1} \end{cases}$$

in (2.5.11), we have (2.5.5). If we set $a_1 = a_3 = 0$, $a_2 = \kappa$ in (2.5.11), we have (2.5.9). □

2.6 Lower bound in odd space dimensions

In this section, we prove Theorem 2.3.1 in odd space dimensions. It is obviously enough for this to show the following proposition.

Proposition 2.6.1 *Let $n = 5, 7, 9, \cdots$. Assume* (3.1.5)*. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$ such that each of* (2.3.2) *and* (2.3.9) *admits a unique solution $u \in C^1(\mathbf{R}^n \times [0, T])$ as far as T satisfies*

$$T \le \begin{cases} c\varepsilon^{-2p(p-1)/\gamma(p,n)} & \text{if } 1 < p < p_0(n) \\ \exp\left(c\varepsilon^{-p(p-1)}\right) & \text{if } p = p_0(n) \end{cases} \tag{2.6.1}$$

for $0 < \varepsilon \le \varepsilon_0$, where c is a positive constant independent of ε.

Our purpose is to construct a solution of the integral equation (2.4.8) as a limit of $\{U_l\}_{l \in \mathbf{N}}$ in X. To end this, we define a closed subspace Y_0 in X by

$$Y_0 = \{U \in X \ : \ \|\nabla_x^\alpha U\| \le 2M_0\varepsilon^p \ (|\alpha| \le 1)\},$$

where we set

$$M_0 = 2^p p C_{n,0,p} k^2 C_0^p > 0.$$

Recall that $C_{n,0,p}$ is the one in (2.5.3). We note that there exists a positive constant C_0 independent of ε which satisfies that

$$\left\| \frac{\tau_+^{(n-1)/2}}{w} \partial W_0 \right\| \le C_0 \varepsilon \quad \text{for } 0 < \varepsilon \le 1. \tag{2.6.2}$$

It is easy to check that this fact follows from the definition of the weight function in (2.4.11) and the decay estimate for $U_0 = v$ in (2.4.4) and (2.4.5).

Proof of Proposition 2.6.1. First of all, we assume that

$$0 < \varepsilon \le 1 \tag{2.6.3}$$

to make use of (2.6.2). We shall show the convergence of $\{U_l\}_{l\in\mathbf{N}}$. The boundedness of $\{U_l\}_{l\in\mathbf{N}}$,

$$\|U_l\| \le 2M_0\varepsilon^p \quad (l \in \mathbf{N}), \tag{2.6.4}$$

can be obtained by induction with respect to l as follows. Recall that L is a positive and linear operator by its definition, (2.3.3). It follows from (2.4.9), (2.5.3) with $\nu = 0$ and (2.6.2) that

$$\begin{aligned} \|U_1\| &= 2^p\|L(|U_0|^p)\| \\ &\le 2^pC_{n,0,p}k^2\|\tau_+^{(n-1)/2}w^{-1}U_0\|^pE_0(T) \le M_0\varepsilon^p, \end{aligned} \tag{2.6.5}$$

where we have used $E_0(T) = 1$ for $p > 1$. Assume that $\|U_{l-1}\| \le 2M_0\varepsilon^p$ ($l \ge 2$). It follows from the simple estimate

$$\begin{aligned} |U_l| &= L(|U_{l-1} + U_0|^p) \\ &\le 2^p\{L(|U_{l-1}|^p) + L(|U_0|^p)\}, \end{aligned} \tag{2.6.6}$$

(2.5.1) and (2.5.3) with $\nu = 0$ that

$$\|U_l\| \le 2^pk^2\{C\|U_{l-1}\|^pD(T) + C_{n,0,p}\|\tau_+^{(n-1)/2}w^{-1}U_0\|^pE_0(T)\}.$$

Hence (2.6.2) and the assumption of the induction yield that

$$\|U_l\| \le 2^pCk^2(2M_0\varepsilon^p)^pD(T) + M_0\varepsilon^p.$$

This inequality shows (2.6.4) provided

$$2^pCk^2(2M_0)^p\varepsilon^{p^2}D(T) \le M_0\varepsilon^p. \tag{2.6.7}$$

Next we shall estimate the differences of $\{U_l\}_{l\in\mathbf{N}}$ under the conditions, (2.6.3) and (2.6.7) ensuring the boundedness (2.6.4). Then there exists a $\theta \in (0, 1)$ such that

$$\begin{aligned} |U_{l+1} - U_l| &= |L\{F'(U_{l-1} + U_0 + \theta(U_l - U_{l-1}))(U_l - U_{l-1})\}| \\ &\le pL\{|U_{l-1} + U_0 + \theta(U_l - U_{l-1})|^{p-1}|U_l - U_{l-1}|\}. \end{aligned} \tag{2.6.8}$$

Hence we have

$$|U_{l+1} - U_l| \le 2^{p-1}pL\{(|3W_l|^{p-1} + |U_0|^{p-1})|U_l - U_{l-1}|\}. \tag{2.6.9}$$

Hölder's inequality (2.4.13) and *a priori* estimate (2.5.1) yield that

$$\begin{aligned} &\|L(|3W_l|^{p-1}|U_l - U_{l-1}|)\| \\ &= \|L\{(|3W_l|^{(p-1)/p}|U_l - U_{l-1}|^{1/p})^p\}\| \\ &\le Ck^2\||3W_l|^{(p-1)/p}|U_l - U_{l-1}|^{1/p}\|^pD(T) \\ &\le Ck^2\|3W_l\|^{p-1}D(T)\|U_l - U_{l-1}\|. \end{aligned} \tag{2.6.10}$$

We note that (2.6.4) implies $\|W_l\| \le 2M_0\varepsilon^p$ ($l \in \mathbf{N}$). Moreover, (2.5.3) with $\nu = 1$ implies that

$$\begin{aligned}&\|L(|U_0|^{p-1}|U_l - U_{l-1}|)\| \\ &\le C_{n,1,p}k^2\|\tau_+^{(n-1)/2}w^{-1}U_0\|^{p-1}\|U_l - U_{l-1}\|E_1(T).\end{aligned} \tag{2.6.11}$$

Since (2.6.4) implies that $\|W_l\| \le 2M_0\varepsilon^p$ for $l \ge 2$, the convergence of $\{U_l\}_{l\in\mathbf{N}}$ follows from

$$\|U_{l+1} - U_l\| \le \frac{1}{2}\|U_l - U_{l-1}\| \quad \text{for } l \ge 2$$

provided

$$2^{p-1}pk^2\{C(6M_0\varepsilon^p)^{p-1}D(T) + C_{n,1,p}(C_0\varepsilon)^{p-1}E_1(T)\} \le \frac{1}{2}, \tag{2.6.12}$$

In fact, we obtain

$$\|U_{l+1} - U_l\| \le \frac{1}{2^{l-1}}\|U_2 - U_1\| \quad \text{for } l \ge 2 \tag{2.6.13}$$

which implies the convergence of $\{U_l\}_{l\in\mathbf{N}}$.

Now we shall show the convergence of $\{\partial_i U_l\}_{l\in\mathbf{N}}$ for $i = 1, 2, \cdots, n$ under the conditions, (2.6.3), (2.6.7) and (2.6.12) which ensure the convergence of $\{U_l\}_{l\in\mathbf{N}}$. As before, the boundedness of $\{\partial_i U_l\}_{l\in\mathbf{N}}$ for $i = 1, 2, \cdots, n$,

$$\|\partial_i U_l\| \le 2M_0\varepsilon^p \quad (l \in \mathbf{N},\ i = 1, 2, \cdots, n), \tag{2.6.14}$$

can be obtained by induction as follows. Similarly to (2.6.5), we have

$$\begin{aligned}\|\partial_i U_1\| &= 2^p p\|L(|U_0|^{p-1}|\partial_i U_0|)\| \\ &\le 2^p p\|L(|\partial W_0|^p)\| \le 2^p pC_{n,0,p}k^2\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|^p E_0(T).\end{aligned}$$

Hence (2.6.2) and $E_0(T) = 1$ for $p > 1$ implies $\|\partial_i U_1\| \le M_0\varepsilon^p$. Assume that $\|\partial_i U_{l-1}\| \le 2M_0\varepsilon^p$ ($l \ge 2$). We note that this means $\|\partial W_{l-1}\| \le 2M_0\varepsilon^p$ ($l \ge 2$). It follows from (2.4.9) that

$$\begin{aligned}|\partial_i U_l| &= p|L(|U_{l-1} + U_0|^{p-1}|\partial_i(U_{l-1} + U_0)|)| \\ &\le 2^{p-1}pL\{(|U_{l-1}|^{p-1} + |U_0|^{p-1})(|\partial_i U_{l-1}| + |\partial_i U_0|)\}.\end{aligned}$$

Similarly to (2.6.10) and (2.6.11), we obtain that

$$\begin{aligned}&\|L(|U_{l-1}|^{p-1}|\partial_i U_{l-1}|)\| \le Ck^2\|U_{l-1}\|^{p-1}\|\partial_i U_{l-1}\|D(T), \\ &\|L(|U_{l-1}|^{p-1}|\partial_i U_0|)\| \le C_{n,p-1,p}k^2\|U_{l-1}\|^{p-1}\|\tau_+^{(n-1)/2}w^{-1}\partial_i U_0\|E_{p-1}(T), \\ &\|L(|U_0|^{p-1}|\partial_i U_{l-1}|)\| \le C_{n,1,p}k^2\|\tau_+^{(n-1)/2}U_0 w^{-1}\|^{p-1}\|\partial_i U_{l-1}\|E_1(T), \\ &\|L(|U_0|^{p-1}||\partial_i U_0|)\| \le C_{n,0,p}k^2\|\tau_+^{(n-1)/2}\partial W_0 w^{-1}\|^p E_0(T).\end{aligned}$$

Hence, we get

$$
\begin{aligned}
\|\partial_i U_l\| \le\ & 2^{p-1}pk^2\{C\|\partial W_{l-1}\|^p D(T) \\
& \quad +C_{n,p-1,p}\|\partial W_{l-1}\|^{p-1}\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|E_{p-1}(T) \\
& +C_{n,1,p}\|\partial W_{l-1}\|\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|^{p-1}E_1(T) \\
& \quad +C_{n,0,p}\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|^p\} \\
\le\ & 2^{p-1}pk^2\{C(2M_0\varepsilon^p)^p D(T) + C_{n,p-1,p}(2M_0\varepsilon^p)^{p-1}C_0\varepsilon E_{p-1}(T) \\
& +C_{n,1,p}(2M_0\varepsilon^p)(C_0\varepsilon)^{p-1}E_1(T) + C_{n,0,p}(C_0\varepsilon)^p\}.
\end{aligned}
$$

This inequality shows (2.6.14) provided

$$
\begin{aligned}
(3/2)M_0\varepsilon^p \ge\ & 2^{p-1}pk^2\{C(2M_0)^p\varepsilon^{p^2}D(T) \\
& +C_{n,p-1,p}(2M_0)^{p-1}C_0\varepsilon^{p^2-p+1}E_{p-1}(T) \\
& +C_{n,1,p}2M_0C_0^{p-1}\varepsilon^{2p-1}E_1(T)\}.
\end{aligned} \tag{2.6.15}
$$

Next we shall estimate the differences of $\{\partial_i U_l\}_{l\in\mathbf{N}}$ for $i = 1, 2, \cdots, n$, under the conditions, (2.6.3), (2.6.7), (2.6.12) and (2.6.15) which ensure the convergence of $\{U_l\}_{l\in\mathbf{N}}$ and the boundedness of $\{\partial W_l\}_{l\in\mathbf{N}}$. (2.4.9) implies that

$$
\begin{aligned}
& |\partial_i U_{l+1} - \partial_i U_l| \\
& = p|L\{|U_l + U_0|^{p-1}(\partial_i U_l + \partial_i U_0) - |U_{l-1} + U_0|^{p-1}(\partial_i U_{l-1} + \partial_i U_0)\}| \\
& \le pL\{|U_l + U_0|^{p-1}|\partial_i U_l - \partial_i U_{l-1}|\} \\
& \qquad +pL\{|U_l - U_{l-1}|^{p-1}|\partial_i U_{l-1} + \partial_i U_0|\} \\
& \le 2^{p-1}pL\{(|U_l|^{p-1} + |U_0|^{p-1})|\partial_i U_l - \partial_i U_{l-1}|\} \\
& \qquad +pL\{|U_l - U_{l-1}|^{p-1}(|\partial_i U_{l-1}| + |\partial_i U_0|)\}.
\end{aligned}
$$

Similarly to the proof of the convergence of $\{U_l\}_{l\in\mathbf{N}}$, we obtain that

$$
\begin{aligned}
& \|L\{|U_l|^{p-1}|\partial_i U_l - \partial_i U_{l-1}|\}\| \le Ck^2\|U_l\|^{p-1}\|\partial_i U_l - \partial_i U_{l-1}\|D(T), \\
& \|L\{|U_0|^{p-1}|\partial_i U_l - \partial_i U_{l-1}|\}\| \\
& \quad \le C_{n,1,p}k^2\|\tau_+^{(n-1)/2}w^{-1}U_0\|^{p-1}\|\partial_i U_l - \partial_i U_{l-1}\|E_1(T), \\
& \|L\{|U_l - U_{l-1}|^{p-1}|\partial_i U_{l-1}|\}\| \le Ck^2\|U_l - U_{l-1}\|^{p-1}\|\partial_i U_{l-1}\|D(T), \\
& \|L\{|U_l - U_{l-1}|^{p-1}|\partial_i U_0|\}\| \\
& \quad \le C_{n,p-1,p}k^2\|U_l - U_{l-1}\|^{p-1}\|\tau_+^{(n-1)/2}w^{-1}\partial_i U_0\|E_{p-1}(T).
\end{aligned}
$$

Therefore, due to (2.6.13), all the assumptions imply that

$$
\begin{aligned}
& \|\partial_i U_{l+1} - \partial_i U_l\| \\
& \le 2^{p-1}pk^2\{C\|W_l\|^{p-1}D(T) + C_{n,1,p}\|\tau_+^{(n-1)/2}w^{-1}U_0\|^{p-1}E_1(T)\}\times \\
& \quad \times\|\partial_i U_l - \partial_i U_{l-1}\| \\
& \quad +pk^2\{C\|\partial W_{l-1}\|D(T) + C_{n,p-1,p}\|\tau_+^{(n-1)/2}w^{-1}\partial W_0\|E_{p-1}(T)\}\times \\
& \quad \times\|U_l - U_{l-1}\|^{p-1} \\
& \le 2^{p-1}pk^2\{C(2M_0\varepsilon^p)^{p-1}D(T) + C_{n,1,p}(C_0\varepsilon)^{p-1}E_1(T)\}\|\partial_i U_l - \partial_i U_{l-1}\| \\
& \quad +pk^2\{C(2M_0\varepsilon^p)D(T) + C_{n,p-1,p}C_0\varepsilon E_{p-1}(T)\}(\|U_2 - U_1\|2^{-(l-1)})^{p-1}.
\end{aligned}
$$

This inequality yields

$$\|\partial_i U_{l+1} - \partial_i U_l\| \le \frac{1}{2}\|\partial_i U_l - \partial_i U_{l-1}\| + \frac{N_0(\varepsilon, T)}{2^{(l-1)(p-1)}}, \tag{2.6.16}$$

where we set

$$N_0(\varepsilon, T) = pk^2\|U_2 - U_1\|^{p-1}\{C(2M_0\varepsilon^p)D(T) + C_{n,p-1,p}C_0\varepsilon E_{p-1}(T)\}$$

provided

$$2^{p-1}pk^2\{C(2M_0)^{p-1}\varepsilon^{p(p-1)}D(T) + C_{n,1,p}(C_0)^{p-1}\varepsilon^{p-1}E_1(T)\} \le \frac{1}{2}. \tag{2.6.17}$$

We note that (2.6.16) implies

$$\|\partial_i U_{l+1} - \partial_i U_l\| \le \frac{1}{2^{l-1}}\|\partial_i U_2 - \partial_i U_1\| + \frac{N_0(\varepsilon, T)}{2^{(l-1)(p-1)}}\sum_{\nu=0}^{l-2}\frac{1}{2^\nu} \quad \text{for } l \ge 2.$$

The convergence of $\{\partial_i U_l\}_{l\in\mathbf{N}}$ $(i = 1, 2, 3, \cdots, n)$ follows from this estimate.

In this way, the convergence of $\{U_l\}_{l\in\mathbf{N}}$ in $Y_0 \subset X$ can be established if all the five conditions, (2.6.3), (2.6.7), (2.6.12), (2.6.15), (2.6.17), are satisfied. In order to complete the proof of Proposition 2.6.1, we shall fix $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$ and c in the statement of Proposition 2.6.1. First, we propose a sufficient condition to (2.6.3), (3.4.7) as well as related factors in (2.6.12), (2.6.15), (2.6.17) to $D(T)$ by

$$2^{2p}3^{p-1}pCk^2M_0^{p-1}\varepsilon^{p(p-1)}D(T) \le 1. \tag{2.6.18}$$

Next, we propose sufficient conditions to related factors in (2.6.12), (2.6.15), (2.6.17) to $E_1(T)$ and $E_{p-1}(T)$ according to p by the following.

In the case of $1 < p < \dfrac{n+1}{n-1}$, such conditions are

$$2^{p+1}pk^2C_{n,1,p}C_0^{p-1}\varepsilon^{p-1}\left(\frac{2T+3k}{k}\right)^{-q} \le 1 \tag{2.6.19}$$

and

$$1 \ge 2^{2p-1}pk^2C_{n,p-1,p}M_0^{p-2}C_0\varepsilon^{(p-1)^2}\left(\frac{2T+3k}{k}\right)^{-(p-1)q}. \tag{2.6.20}$$

If we put

$$\begin{aligned} c = \min \quad & \{(2^{2p}3^{p-1}pCk^2(M_0)^{p-1})^{-1}, (2^{p+1}pk^2C_{n,1,p}C_0^{p-1})^{-p}, \\ & (2^{2p-1}pk^2C_{n,p-1,p}M_0^{p-2}C_0)^{-p/(p-1)}\} > 0, \end{aligned}$$

the inequality $\varepsilon^{p(p-1)}D(T) \le c$ implies (2.6.18), (2.6.19) and (2.6.20), because of $-pq < \gamma(n,p)/2$. Furthermore, one can readily check that $\varepsilon_0 = 1$ by (2.6.3).

In the case of $p = \dfrac{n+1}{n-1}$, let us fix δ with

$$0 < \delta < \frac{1}{p}. \tag{2.6.21}$$

Then, similarly to the above, our conditions are

$$2^{p+1} p C_{n,1,p} k^2 C_0^{p-1} \varepsilon^{p-1} \left(\frac{2T+3k}{k} \right)^{\delta} \le 1 \tag{2.6.22}$$

and

$$1 \ge 2^{2p-1} p k^2 C_{n,p-1,p} M_0^{p-2} C_0 \varepsilon^{(p-1)^2} \left(\frac{2T+3k}{k} \right)^{\delta(p-1)}. \tag{2.6.23}$$

If we put

$$\begin{aligned} c = \min \quad & \{(2^{2p}3^{p-1} p C k^2 (M_0)^{p-1})^{-1}, (2^{p+1} p k^2 C_{n,1,p} C_0^{p-1})^{-p}, \\ & (2^{2p-1} p k^2 C_{n,p-1,p} M_0^{p-2} C_0)^{-p/(p-1)}\} > 0, \end{aligned}$$

the inequality $\varepsilon^{p(p-1)}D(T) \le c$ implies (2.6.18), (2.6.22) and (2.6.23), because of (2.6.21). Furthermore, one can readily check that $\varepsilon_0 = 1$ by (2.6.3).

Finally, in the case of $p > \dfrac{n+1}{n-1}$, our conditions are

$$2^{p+1} p k^2 C_{n,1,p} C_0^{p-1} \varepsilon^{p-1} \le 1 \tag{2.6.24}$$

and

$$2^{2p-1} p k^2 C_{n,p-1,p} M_0^{p-2} C_0 \varepsilon^{(p-1)^2} \le 1. \tag{2.6.25}$$

If we put

$$c = (2^{2p}3^{p-1} p C k^2 (M_0)^{p-1})^{-1},$$

the inequality $\varepsilon^{p(p-1)}D(T) \le c$ implies (2.6.18). Furthermore, one can find that

$$\begin{aligned} \varepsilon_0 = \min \quad & \{1, (2^{p+1} p k^2 C_{n,1,p})^{-1/(p-1)} C_0^{-1}, \\ & (2^{2p-1} p k^2 M_0^{p-2} C_{n,p-1,p} C_0)^{-1/(p-1)^2}\} > 0, \end{aligned}$$

by (2.6.3), (2.6.24) and (2.6.25). Therefore, the proof of proposition 2.6.1 is completed. □

2.7 Lower bound in even space dimensions

Similarly to the previous section, we investigate the lower bound of the lifespan in the even dimensional case. Our purpose is to show the following proposition.

Proposition 2.7.1 *Let $n = 4, 6, 8, \cdots$. Suppose that the same assumption in Theorem 2.3.1,* (3.1.5) *is fulfilled. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$ such that each of* (2.3.6) *and* (2.3.10) *admits a unique classical solution $u \in C^2(\mathbf{R}^4 \times [0, T])$ if $n = 4$ and $p = p_0(4) = 2$, or each of* (2.3.2) *and* (2.3.9) *admits a unique solution $u \in C^1(\mathbf{R}^n \times [0, T])$ otherwise, as far as T satisfies*

$$T \leq \begin{cases} c\varepsilon^{-2p(p-1)/\gamma(p,n)} & \text{if } 1 < p < p_0(n) \\ \exp\left(c\varepsilon^{-p(p-1)}\right) & \text{if } p = p_0(n) \end{cases} \tag{2.7.1}$$

for $0 < \varepsilon \leq \varepsilon_0$ and c is a positive constant independent of ε.

Employing the similar argument to odd dimensions, we shall construct a C^1 solution of the integral equation (2.4.8) as a limit of $\{U_l\}_{l \in \mathbf{N}}$ in X. We also remark that it is possible to construct a C^2 solution if and only if $(n, p) = (4, 2)$ in our problem. However, its construction is almost the same as for C^1 solution. Therefore we shall omit it. Now, define a closed subspace Y_1 in X by

$$Y_1 = \{U \in X : \|\nabla_x^\alpha U\| \leq M_1 \varepsilon^p \ (|\alpha| \leq 1)\},$$

where M_1 is defined by

$$M_1 = 2^{2p} p k^2 (C_{n,0,p} C_1^p + C_{n,1,p} C_1^{p-1} C_2 + C_{n,p-1,p} C_1 C_2^{p-1} + C_{n,0,p} C_2^p) > 0.$$

Recall that $C_{n,0,p}$, $C_{n,1,p}$ and $C_{n,p-1,p}$ are the one in (2.5.5) and (2.5.9). C_1 and C_2 are positive constants independent of ε which satisfy that

$$\left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{\partial W_{00}}{w} \right\| \leq C_1 \varepsilon, \quad \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{\partial W_{01}}{w \tau_-} \right\| \leq C_2 \varepsilon^p. \tag{2.7.2}$$

The existence of C_1 and C_2 is trivial by the definition of the weight function in (2.4.11) and the decay estimate for $U_{00} = v_0$ and $U_{01} = v_1$ in (2.4.6) and (2.4.7).

Proof of the Proposition 2.7.1. First of all, we shall show the convergence of $\{U_l\}_{l \in \mathbf{N}}$. The boundedness of $\{U_l\}_{l \in \mathbf{N}}$ in Y_1,

$$\|U_l\| \leq 2 M_1 \varepsilon^p \ (l \in \mathbf{N}), \tag{2.7.3}$$

can be obtained by induction with respect to l as follows. Similarly to (2.6.5), it follows from (2.4.9) and (2.5.5) of Lemma 2.5.3 with $\nu = 0$ and $a = 0, 1$ that

$$\begin{aligned} \|U_1\| \leq \; & 2^{2p} k^2 C_{n,0,p} \left(\left\| (\tau_+\tau_-)^{(n-1)/2} \frac{U_{00}}{w} \right\|^p E_{0,0}(T) + \right. \\ & \left. + \left\| (\tau_+\tau_-)^{(n-1)/2} \frac{U_{01}}{w\tau_-} \right\|^p E_{0,1}(T) \right). \end{aligned} \tag{2.7.4}$$

By the definition of (2.5.6), (2.5.7) and (2.5.8) with $\nu = a = 0$, we get

$$E_{0,0}(T) = 1. \tag{2.7.5}$$

It follows from (2.7.2) that

$$\|U_1\| \leq 2^{2p} k^2 C_{n,0,p} \varepsilon^p (C_1^p + C_2^p \varepsilon^{p(p-1)} E_{0,1}(T)).$$

This inequality show $\|U_1\| \leq M_1 \varepsilon^p$ provided

$$\varepsilon^{p(p-1)} E_{0,1}(T) \leq 1. \tag{2.7.6}$$

Assume that $\|U_{l-1}\| \leq 2M_1\varepsilon^p$ $(l \geq 2)$. Making use of (2.6.6), the assumption of the induction and (2.5.1) yield that

$$\begin{aligned} \|U_l\| \; & \leq 2^p k^2 \left\{ C\|U_{l-1}\|^p D(T) + C_{n,0,p} \left\| (\tau_+\tau_-)^{(n-1)/2} \frac{U_{00}}{w} \right\|^p E_{0,0}(T) \right. \\ & \quad \left. + C_{n,0,p} \left\| (\tau_+\tau_-)^{(n-1)/2} \frac{U_{01}}{w\tau_-} \right\|^p E_{0,1}(T) \right\} \\ & \leq 2^p C k^2 (2M_1\varepsilon^p)^p D(T) + M_1\varepsilon^p. \end{aligned}$$

This inequality shows (2.7.3) provided

$$2^p C k^2 (2M_1)^p \varepsilon^{p^2} D(T) \leq M_1 \varepsilon^p. \tag{2.7.7}$$

Next we shall estimate the differences under (2.7.6) and (2.7.7) ensuring the boundedness (2.7.3). We note that (2.7.3) implies $\|W_l\| \leq 2M_1\varepsilon^p$ $(l \in \mathbf{N})$. Making use of the inequalities (2.6.8) and (2.6.9), we have

$$|U_{l+1} - U_l| \leq 2^{p-1} pL\{\{|3W_l|^{p-1} + 2^{p-1}(|U_{00}|^{p-1} + |U_{01}|^{p-1})\}|U_l - U_{l-1}|\}.$$

Applying (2.5.5) of Lemma 2.5.3 with $\nu = 1$ and $a = 0, 1$, we get

$$\begin{aligned} & \|L(|U_{0a}|^{p-1}|U_l - U_{l-1}|)\| \\ & \leq k^2 C_{n,1,p} E_{1,a}(T) \left\| (\tau_-\tau_+)^{(n-1)/2} \frac{U_{0a}}{w\tau_-^a} \right\|^{p-1} \|U_l - U_{l-1}\| \end{aligned} \tag{2.7.8}$$

for $a = 0, 1$. Since (2.7.3) implies that $\|W_l\| \le 2M_1\varepsilon^p$ for $l \ge 2$, the convergence of $\{U_l\}_{l\in\mathbf{N}}$ follows from

$$\|U_{l+1} - U_l\| \le \frac{1}{2}\|U_l - U_{l-1}\| \quad \text{for } l \ge 2$$

provided

$$\begin{aligned} &2^{p-1}pk^2\{C(6M_1\varepsilon^p)^{p-1}D(T) + 2^{p-1}C_{n,1,p}(C_1\varepsilon)^{p-1}E_{1,0}(T) \\ &\quad +2^{p-1}C_{n,1,p}C_2^{p-1}E_{1,1}(T)\varepsilon^{p(p-1)}\} \le \frac{1}{2}, \end{aligned} \tag{2.7.9}$$

Thus, we obtain

$$\|U_{l+1} - U_l\| \le \frac{1}{2^{l-1}}\|U_2 - U_1\| \quad \text{for } l \ge 2 \tag{2.7.10}$$

which implies the convergence of $\{U_l\}_{l\in\mathbf{N}}$.

Now we shall show the convergence of $\{\partial_i U_l\}_{l\in\mathbf{N}}$ for $i = 1, 2, \cdots, n$ under the conditions, (2.7.6), (2.7.7), (2.7.9) which ensure the convergence of $\{U_l\}_{l\in\mathbf{N}}$. As before, the boundedness

$$\|\partial_i U_l\| \le 2M_1\varepsilon^p \quad (l \in \mathbf{N},\ i = 1, 2, \cdots, n) \tag{2.7.11}$$

can be obtained by induction as follows. Similarly to (2.7.4), we have

$$\begin{aligned} \|\partial_i U_1\| &= 2^p p\|L(|U_0|^{p-1}|\partial_i U_0|)\| \\ &\le 2^{2p-1}p\|L((|U_{00}|^{p-1} + |U_{01}|^{p-1})(|\partial_i U_{00}| + |\partial_i U_{01}|))\| \\ &\le 2^{2p-1}p\|L(\partial W_{00}^p + U_{00}^{p-1}\partial W_{01} + U_{01}^{p-1}\partial W_{00} + \partial W_{01}^p)\|. \end{aligned}$$

Applying (2.5.9) of Lemma 2.5.3 with $\kappa = 1$ and $\kappa = p - 1$, we get

$$\begin{aligned} &\|L(U_{00}^{p-1}\partial W_{01})\| \\ &\quad \le C_{n,1,p}k^2 \left\|(\tau_-\tau_+)^{(n-1)/2}\frac{U_{00}}{w}\right\|^{p-1} \left\|(\tau_-\tau_+)^{(n-1)/2}\frac{\partial W_{01}}{w\tau_-}\right\| F_1(T) \end{aligned}$$

and

$$\begin{aligned} &\|L(U_{01}^{p-1}\partial W_{00})\| \\ &\le C_{n,p-1,p}k^2 \left\|(\tau_-\tau_+)^{(n-1)/2}\frac{U_{01}}{w\tau_-}\right\|^{p-1} \left\|(\tau_-\tau_+)^{(n-1)/2}\frac{\partial W_{00}}{w}\right\| F_{p-1}(T). \end{aligned}$$

By virtue of (2.7.2) and (2.7.5), we obtain

$$\begin{aligned} \|\partial_i U_1\| &\le 2^{2p-1}pk^2\varepsilon^p \left\{C_{n,0,p}C_1^p + C_{n,1,p}C_1^{p-1}C_2\varepsilon^{p-1}F_1(T)\right. \\ &\quad +C_{n,p-1,p}C_2^{p-1}C_1\varepsilon^{(p-1)^2}F_{p-1}(T) \\ &\quad \left.+C_{n,0,p,}C_2^p\varepsilon^{p(p-1)}E_{0,1}(T)\right\}. \end{aligned}$$

This inequality shows $\|\partial_i U_1\| \le M_1\varepsilon^p$ provided (2.7.6),

$$\varepsilon^{p-1}F_1(T) \le 1 \tag{2.7.12}$$

and

$$\varepsilon^{(p-1)^2}F_{p-1}(T) \le 1 \tag{2.7.13}$$

hold.

Assume that $\|\partial_i U_{l-1}\| \le 2M_1\varepsilon^p$ $(l \ge 2)$. Then we get

$$\begin{aligned}
|\partial_i U_l| &= p|L(|U_{l-1}+U_0|^{p-1}|\partial_i(U_{l-1}+U_0)|)| \\
&\le 2^{p-1}pL\{(|U_{l-1}|^{p-1}+|U_0|^{p-1})(|\partial_i U_{l-1}|+|\partial_i U_0|)\} \\
&\le 2^{p-1}pL\{(|U_{l-1}|^{p-1}+2^{p-1}(|U_{00}|^{p-1}+|U_{01}|^{p-1})) \\
&\quad \times(|\partial_i U_{l-1}|+|\partial_i U_{00}|+|\partial_i U_{01}|)\}.
\end{aligned}$$

Similarly to (2.7.8), we obtain that

$$\begin{aligned}
&\|L(|U_{l-1}|^{p-1}|\partial_i U_{0a}|)\| \\
&\quad \le C_{n,p-1,p}k^2\|U_{l-1}\|^{p-1}\left\|(\tau_-\tau_+)^{(n-1)/2}\frac{\partial_i U_{0a}}{w\tau_-^a}\right\| E_{p-1,a}(T), \\
&\|L(|U_{0a}|^{p-1}|\partial_i U_{l-1}|)\| \\
&\quad \le C_{n,1,p}k^2\|\partial_i U_{l-1}\|\left\|(\tau_-\tau_+)^{(n-1)/2}\frac{U_{0a}}{w\tau_-^a}\right\|^{p-1} E_{1,a}(T)
\end{aligned}$$

for $a = 0, 1$. Making use of (2.7.2) and (2.7.5), we get

$$\begin{aligned}
&\|\partial_i U_l\| \\
&\le 2^{p-1}pk^2[C(2M_1\varepsilon^p)^p D(T) + C_{n,p-1,p}(2M_1\varepsilon^p)^{p-1}C_1\varepsilon E_{p-1,0}(T) \\
&\quad +C_{n,p-1,p}(2M_1\varepsilon^p)^{p-1}C_2\varepsilon^p E_{p-1,1}(T) \\
&\quad +2^{p-1}\{C_{n,1,p}(C_1\varepsilon)^{p-1}(2M_1\varepsilon^p)E_{1,0}(T) + C_{n,0,p}C_1^p\varepsilon^p \\
&\quad +C_{n,1,p}(C_1\varepsilon)^{p-1}C_2\varepsilon^p F_1(T) + C_{n,1,p}(C_2\varepsilon^p)^{p-1}(2M_1\varepsilon^p)E_{1,1}(T) \\
&\quad +C_{n,p-1,p}(C_2\varepsilon^p)^{p-1}C_1\varepsilon F_{p-1}(T) + C_{n,0,p}(C_2\varepsilon^p)^p E_{0,1}(T)\}].
\end{aligned}$$

Under the assumptions (2.7.6), (2.7.12) and (2.7.13), this inequality shows (2.7.11) provided

$$\begin{aligned}
&(7/4)M_1\varepsilon^p \\
&\ge 2^{p-1}pk^2[C(2M_1)^p\varepsilon^{p^2}D(T) \\
&\quad +C_{n,p-1,p}(2M_1)^{p-1}C_1\varepsilon^{p^2-p+1}E_{p-1,0}(T) \\
&\quad +C_{n,p-1,p}(2M_1)^{p-1}C_2\varepsilon^{p^2}E_{p-1,1}(T) + 2^{p-1}\left\{C_{n,1,p}C_1^{p-1}\varepsilon^{2p-1}\right. \\
&\quad \left.\times 2M_1E_{1,0}(T) + C_{n,1,p}C_2^{p-1}2M_1\varepsilon^{p^2}E_{1,1}(T)\right\}].
\end{aligned} \tag{2.7.14}$$

Next we shall estimate the differences under the conditions, (2.7.6), (2.7.7), (2.7.9), (2.7.12), (2.7.13), (2.7.14) which ensure the convergence of $\{U_l\}_{l\in\mathbf{N}}$

and the boundedness of $\{\partial U_l\}_{l\in\mathbf{N}}$. Similarly to odd dimensions, we have

$$\begin{aligned}
&|\partial_i U_{l+1} - \partial_i U_l| \\
&\le 2^{p-1} pL\{\{|U_l|^{p-1} + 2^{p-1}\{|U_{00}|^{p-1} + |U_{01}|^{p-1}\}\}|\partial_i U_l - \partial_i U_{l-1}|\} \\
&\quad + pL\{|U_l - U_{l-1}|^{p-1}(|\partial_i U_{l-1}| + |\partial_i U_{00}| + |\partial_i U_{01}|)\}.
\end{aligned}$$

Applying (2.5.5) of Lemma 2.5.3 with $\nu = 1$, $\nu = p-1$ and $a = 0, 1$, we get

$$\begin{aligned}
&\|L\{|U_{0a}|^{p-1}|\partial_i U_l - \partial_i U_{l-1}|\}\| \\
&\quad \le C_{n,1,p} k^2 \left\| (\tau_-\tau_+)^{(n-1)/2} \frac{U_{0a}}{w\tau_-^a} \right\|^{p-1} \|\partial_i U_l - \partial_i U_{l-1}\| E_{1,a}(T), \\
&\|L\{|U_l - U_{l-1}|^{p-1}|\partial_i U_{0a}|\}\| \\
&\quad \le C_{n,p-1,p} k^2 \|U_l - U_{l-1}\|^{p-1} \left\| (\tau_-\tau_+)^{(n-1)/2} \frac{\partial_i U_{0a}}{w\tau_-^a} \right\| E_{p-1,a}(T)
\end{aligned}$$

for $a = 0, 1$. Then, all the assumptions imply that

$$\begin{aligned}
&\|\partial_i U_{l+1} - \partial_i U_l\| \\
&\le 2^{p-1} pk^2[\{C(2M_1\varepsilon^p)^{p-1} D(T) + 2^{p-1}\{C_{n,1,p}(C_1\varepsilon)^{p-1} E_{1,0}(T) \\
&\quad + C_{n,1,p}(C_2\varepsilon^p)^{p-1} E_{1,1}(T)\}]\|\partial_i U_l - \partial_i U_{l-1}\| \\
&\quad + pk^2\{C(2M_1\varepsilon^p) D(T) + C_{n,p-1,p} C_1\varepsilon E_{p-1,0}(T) \\
&\quad + C_{n,p-1,p} C_2\varepsilon^p E_{p-1,1}(T)\}(\|U_2 - U_1\| 2^{-(l-1)})^{p-1}.
\end{aligned}$$

This inequality yields

$$\|\partial_i U_{l+1} - \partial_i U_l\| \le \frac{1}{2}\|\partial_i U_l - \partial_i U_{l-1}\| + \frac{N_1(\varepsilon, T)}{2^{(l-1)(p-1)}}, \tag{2.7.15}$$

where we set

$$\begin{aligned}
N_1(\varepsilon, T) \quad &= pk^2\{C(2M_1\varepsilon^p) D(T) + C_{n,p-1,p} C_1\varepsilon E_{p-1,0}(T) \\
&\quad + C_{n,p-1,p} C_2\varepsilon^p E_{p-1,1}(T)\}
\end{aligned}$$

provided

$$\begin{aligned}
&2^{p-1} pk^2[\{C(2M_1)^{p-1}\varepsilon^{p(p-1)} D(T) \\
&\quad + 2^{p-1}\{C_{n,1,p}(C_1)^{p-1}\varepsilon^{p-1} E_{1,0}(T) \\
&\quad + C_{n,1,p} C_2^{p-1}\varepsilon^{p(p-1)} E_{1,1}(T)\}] \le \frac{1}{2}.
\end{aligned} \tag{2.7.16}$$

Hence, the convergence of $\{\partial_i U_l\}_{l\in\mathbf{N}}$ $(i = 1, 2, 3, \cdots, n)$ follows from this estimate.

In this way, the convergence of $\{U_l\}_{l\in\mathbf{N}}$ in $Y_0 \subset X$ can be established if all the seven conditions, (2.7.6), (2.7.7), (2.7.9), (2.7.12), (2.7.13), (2.7.14), (2.7.16) are satisfied. In order to complete the proof of Proposition 2.7.1, we shall fix $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$ and c in the statement of Proposition 2.7.1.

First, we propose a sufficient condition to (2.7.7) as well as related factors in (2.7.9), (2.7.14), (2.7.16) to $D(T)$ as

$$2^{2p-1}3^p pCk^2 M_1^{p-1}\varepsilon^{p(p-1)}D(T) \le 1. \tag{2.7.17}$$

Next we propose sufficient conditions to related factors in (2.7.6), (2.7.9), (2.7.12), (2.7.13), (2.7.14) and (2.7.16) to

$$E_{0,1}(T),\ E_{1,0}(T),\ E_{1,1}(T),\ F_1(T),\ F_{p-1}(T),\ E_{p-1,0}(T),\ E_{p-1,1}(T)$$

up to p.

Conditions in the case of $1 < p < \dfrac{n+1}{n-1}$.

(i) Conditions from $E_{0,1}(T)$.

In (2.5.8), setting $\nu = 0$ and $a = 1$, we have that $\sigma = -(n-3)p/2 < -1$ when $n \ge 6$ and $\sigma = \mu = -p/2 > -1$ when $n = 4$, which imply

$$E_{0,1}(T) = \begin{cases} 1 & \text{if } n \ge 6, \\ \left(\dfrac{2T+3k}{k}\right)^{1-p/2} & \text{if } n = 4. \end{cases} \tag{2.7.18}$$

Since $E_{0,1}(T)$ appears in (2.7.6), the conditions are

$$\varepsilon^{p(p-1)} \le 1 \quad \text{if } n \ge 6, \tag{2.7.19}$$

$$\left(\frac{2T+3k}{k}\right)^{1-p/2} \varepsilon^{p(p-1)} \le 1 \quad \text{if } n = 4. \tag{2.7.20}$$

(ii) Condition from $E_{1,0}(T)$.

In (2.5.8), setting $\nu = 1$ and $a = 0$, we have that $\sigma = -(n-1)(p-1)/2 > -1$ and $\mu = -2q - 1$, which imply

$$E_{1,0}(T) = \left(\frac{2T+3k}{k}\right)^{-2q}.$$

Since $E_{1,0}(T)$ appears in (2.7.9), (2.7.14) and (2.7.16), the condition is

$$2^{2p-1}3pk^2 C_{n,1,p} C_1^{p-1}\varepsilon^{p-1}\left(\frac{2T+3k}{k}\right)^{-2q} \le 1. \tag{2.7.21}$$

(iii) Condition from $E_{1,1}(T)$.

In (2.5.8), setting $\nu = a = 1$, we have that $\sigma = -(n-3)(p-1)/2 > -1$ and $\mu - 1 - (n-2)p$, which imply

$$E_{1,1}(T) = \left(\frac{2T+3k}{k}\right)^{n-(n-2)p}.$$

Since $E_{1,1}(T)$ appears in (2.7.9), (2.7.14) and (2.7.16), the condition is

$$2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1}\varepsilon^{p(p-1)}\left(\frac{2T+3k}{k}\right)^{n-(n-2)p} \leq 1. \tag{2.7.22}$$

(iv) Conditions from $F_1(T)$.

In (2.5.10), setting $\nu = 1$, we have that

$$\begin{aligned}
&\kappa = 1 - \frac{n-1}{2}p < -1 && \text{if } n \geq 6 \text{ , or } n = 4 \text{ and } p > \frac{4}{3},\\
&\kappa = -1 && \text{if } n = 4 \text{ and } p = \frac{4}{3},\\
&\kappa > -1 && \text{if } n = 4 \text{ and } 1 < p < \frac{4}{3},
\end{aligned}$$

which imply

$$F_1(T) = \begin{cases} 1 & \text{if } n \geq 6 \text{ , or } n = 4 \text{ and } p > \frac{4}{3},\\ \log\dfrac{2T+3k}{k} & \text{if } n = 4 \text{ and } p = \frac{4}{3},\\ \left(\dfrac{2T+3k}{k}\right)^{2-3p/2} & \text{if } n = 4 \text{ and } 1 < p < \frac{4}{3}. \end{cases} \tag{2.7.23}$$

Since $F_1(T)$ appears in (2.7.12), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \quad \text{if} \quad n \geq 6 \text{ , or } n = 4 \text{ and } p > \frac{4}{3}, \tag{2.7.24}$$

$$\log\frac{2T+3k}{k}\varepsilon^{p(p-1)} \leq 1 \quad \text{if} \quad n = 4 \text{ and } p = \frac{4}{3}, \tag{2.7.25}$$

$$\left(\frac{2T+3k}{k}\right)^{2-3p/2}\varepsilon^{p(p-1)} \leq 1 \quad \text{if} \quad n = 4 \text{ and } 1 < p < \frac{4}{3}. \tag{2.7.26}$$

(v) Condition from $F_{p-1}(T)$.

In (2.5.10), setting $\nu = p-1$, we have $\kappa = -(n-3)p/2 - 1 < -1$, which implies $F_{p-1}(T) = 1$. Since $F_{p-1}(T)$ appears in (2.7.13), the condition is

$$\varepsilon^{p(p-1)} \leq 1. \tag{2.7.27}$$

(vi) Condition from $E_{p-1,0}(T)$.

In (2.5.8), setting $\nu = p-1$ and $a = 0$, we have $\sigma = -(n-1)/2 < -1$, which implies

$$E_{p-1,0}(T) = \left(\frac{2T+3k}{k}\right)^{-(p-1)q}.$$

Since $E_{p-1,0}(T)$ appears in (2.7.14), the condition is

$$2^{2p}5\cdot 7^{-1}pk^2C_{n,p-1,p}C_1M_1^{p-2}\varepsilon^{(p-1)^2}\left(\frac{2T+3k}{k}\right)^{-(p-1)q}\le 1. \tag{2.7.28}$$

(vii) Conditions from $E_{p-1,1}(T)$.

In (2.5.8), setting $\nu=p-1$ and $a=1$, we have that $\sigma=-(n-3)/2<-1$ when $n\ge 6$ and $\sigma>-1, \mu=-1/2-(p-1)q$ when $n=4$, which imply

$$E_{p-1,1}(T)=\begin{cases}\left(\dfrac{2T+3k}{k}\right)^{-(p-1)q} & \text{if } n\ge 6,\\ \left(\dfrac{2T+3k}{k}\right)^{1/2-(p-1)q} & \text{if } n=4.\end{cases} \tag{2.7.29}$$

Since $E_{p-1,1}(T)$ appear in (2.7.14), the conditions are

$$\begin{aligned}&2^{2p}5\cdot 7^{-1}pk^2C_{n,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)}\\ &\times\left(\frac{2T+3k}{k}\right)^{-(p-1)q}\le 1 \qquad \text{if } n\ge 6,\end{aligned} \tag{2.7.30}$$

$$\begin{aligned}&2^{2p}5\cdot 7^{-1}pk^2C_{4,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)}\\ &\times\left(\frac{2T+3k}{k}\right)^{1/2-(p-1)q}\le 1 \qquad \text{if } n=4.\end{aligned} \tag{2.7.31}$$

Now, we are in a position to summarize all the conditions in (i)-(vii) above. Set

$$\varepsilon_0=1 \tag{2.7.32}$$

Then, (2.7.32) implies (2.7.19), (2.7.24) and (2.7.27). In order to make that (2.7.17) includes (2.7.25), we employ Lemma 2.5.9 with $\delta=\gamma(p,4)/2>0$ and $X=(2T+3k)/k>1$. Then, if we set

$$\begin{aligned}c=\min\ &\{(2^{2p-1}3^ppCk^2(M_1)^{p-1})^{-1},1,(2^{2p-1}3pk^2C_{n,1,p}C_1^{p-1})^{-p},\\ &(2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1})^{-1},\gamma(4/3,4)/2,\\ &(2^{2p}5\cdot 7^{-1}pk^2C_{n,p-1,p}C_1M_1^{p-2})^{-p/(p-1)},\\ &(2^{2p}5\cdot 7^{-1}pk^2C_{n,p-1,p}C_2M_1^{p-2})^{-1}\}>0,\end{aligned}$$

the inequality $\varepsilon^{p(p-1)}D(T)\le c$ implies (2.7.17), (2.7.20), (2.7.21), (2.7.22), (2.7.25), (2.7.26), (2.7.28), (2.7.30) and (2.7.31) because of $1-p/2\le\gamma(p,4)/2$ in (2.7.20), $-2pq\le\gamma(p,n)/2$ in (2.7.21), $n-(n-2)p\le\gamma(p,n)/2$ in (2.7.22), $2-3p/2\le\gamma(p,4)/2$ in (2.7.26), $-pq\le\gamma(p,n)/2$ in (2.7.28), $-(p-1)q\le\gamma(p,n)/2$ in (2.7.30) and $1/2-(p-1)q\le\gamma(p,4)/2$ in (2.7.31).

Conditions in the case of $p = \dfrac{n+1}{n-1}$.

(i) Conditions from $E_{0,1}(T)$.

In (2.5.7), setting $\nu = 0$ and $a = 1$, we have that $\sigma = -(n-3)(n+1)/2(n-1) < -1$ when $n \geq 6$ and $\sigma > -1$ when $n = 4$, which imply

$$E_{0,1}(T) = \begin{cases} 1 & \text{if } n \geq 6, \\ \left(\dfrac{2T+3k}{k}\right)^{1-p/2} & \text{if } n = 4. \end{cases} \tag{2.7.33}$$

Since $E_{0,1}(T)$ appears in (2.7.6), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \text{ if } n \geq 6, \tag{2.7.34}$$

$$\left(\frac{2T+3k}{k}\right)^{1-p/2} \varepsilon^{p(p-1)} \leq 1 \text{ if } n = 4. \tag{2.7.35}$$

(ii) Condition from $E_{1,0}(T)$.

In (2.5.7), setting $\nu = 1$ and $a = 0$, we have $\sigma = -(n-1)(p-1)/2 = -1$, which implies

$$E_{1,0}(T) = \left(\frac{2T+3k}{k}\right)^{\delta}.$$

Since $E_{1,0}(T)$ appears in (2.7.9), (2.7.14) and (2.7.16), the conditions is

$$2^{2p-1}3pk^2 C_{n,1,p} C_1^{p-1} \varepsilon^{p-1} \left(\frac{2T+3k}{k}\right)^{\delta} \leq 1. \tag{2.7.36}$$

(iii) Condition from $E_{1,1}(T)$.

In (2.5.7), setting $\nu = a = 1$, we have $\sigma = -(n-3)/(n-1) > -1$, which implies

$$E_{1,1}(T) = \left(\frac{2T+3k}{k}\right)^{2/(n-1)}.$$

Since $E_{1,1}(T)$ appears in (2.7.9) (2.7.14) and (2.7.16), the condition is

$$2^{2p-1}3pk^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \left(\frac{2T+3k}{k}\right)^{2/(n-1)} \leq 1. \tag{2.7.37}$$

(iv) Condition from $F_1(T)$.

In (2.5.10), setting $\nu = 1$, we have $\kappa < -1$, which implies $F_1(T) = 1$. Since, $F_1(T)$ appears in (2.7.12), the condition is

$$\varepsilon^{p(p-1)} \leq 1. \tag{2.7.38}$$

(v) Condition from $F_{p-1,0}(T)$.

In (2.5.10), setting $\nu = p-1$, we have $\kappa < -1$, which implies $F_{p-1}(T) = 1$. Since, $F_{p-1}(T)$ appears in (2.7.13), the condition is (2.7.38).

(vi) Condition from $E_{p-1,0}(T)$.

In (2.5.7), setting $\nu = p-1$ and $a = 0$, we have $\sigma = -(n-1)/2 < -1$, which implies

$$E_{p-1,0}(T) = \left(\frac{2T+3k}{k}\right)^{(p-1)\delta}.$$

Since $E_{p-1,0}(T)$ appears in (2.7.14), the condition is

$$2^{2p}5 \cdot 7^{-1}pk^2 C_{n,p-1,p} C_1 M_1^{p-2} \varepsilon^{(p-1)^2} \left(\frac{2T+3k}{k}\right)^{(p-1)\delta} \leq 1. \tag{2.7.39}$$

(vii) Conditions from $E_{p-1,1}(T)$.

In (2.5.7), setting $\nu = p-1$ and $a = 1$, we have that $\sigma = -(n-3)/2 < -1$ when $n \geq 6$ and $\sigma > -1$ when $n = 4$, which imply

$$E_{p-1,1}(T) = \begin{cases} \left(\dfrac{2T+3k}{k}\right)^{(p-1)\delta} & \text{if } n \geq 6, \\ \left(\dfrac{2T+3k}{k}\right)^{1/2} & \text{if } n = 4. \end{cases} \tag{2.7.40}$$

Since $E_{p-1,1}(T)$ appears in (2.7.16), the conditions are

$$\begin{aligned} &2^{2p}5 \cdot 7^{-1}pk^2 C_{n,p-1,p} C_2 M_1^{p-2} \varepsilon^{p(p-1)} \\ &\times \left(\frac{2T+3k}{k}\right)^{(p-1)\delta} \leq 1 \qquad \text{if } n \geq 6, \end{aligned} \tag{2.7.41}$$

$$\begin{aligned} &2^{2p}5 \cdot 7^{-1}pk^2 C_{4,p-1,p} C_2 M_1^{p-2} \varepsilon^{p(p-1)} \\ &\times \left(\frac{2T+3k}{k}\right)^{1/2} \leq 1. \qquad \text{if } n = 4. \end{aligned} \tag{2.7.42}$$

Now, we are in a position to summarize all the conditions in (i)-(vii) above. First we note that (2.7.32) implies (2.7.34) and (2.7.38). Then, if we assume (2.6.21) and set

$$\begin{aligned} c = \min \quad &\{(2^{2p-1}3^p pCk^2(M_1)^{p-1})^{-1}, 1, (2^{2p-1}3pk^2 C_{n,1,p} C_1^{p-1})^{-p}, \\ &(2^{2p-1}3pk^2 C_{n,1,p} C_2^{p-1})^{-1}, \\ &(2^{2p}5 \cdot 7^{-1}pk^2 C_{n,p-1,p} C_1 M_1^{p-2})^{-p/(p-1)}, \\ &(2^{2p}5 \cdot 7^{-1}pk^2 C_{n,p-1,p} C_2 M_1^{p-2})^{-1}\} > 0, \end{aligned}$$

the inequality $\varepsilon^{p(p-1)}D(T) \le c$ implies (2.7.17), (2.7.35), (2.7.36), (2.7.37), (2.7.39), (2.7.41) and (2.7.42) because of $1-p/2 \le 1$ in (2.7.35), $p\delta < 1$ in (2.7.36) and (2.7.39), $2/(n-1) < 1$ in (2.7.37) and $(p-1)\delta < 1$ in (2.7.41).

Conditions in the case of $p > \dfrac{n+1}{n-1}$.

(i) Conditions from $E_{0,1}(T)$.

In (2.5.6), setting $\nu = 0$ and $a = 1$, we have that $\mu = -(n-3)p/2 < -1$ when $n \ge 6$ and $\mu = -p/2 \ge -1$ when $n = 4$, which imply

$$E_{0,1}(T) = \begin{cases} 1 & \text{if } n \ge 6, \\ \left(\dfrac{2T+3k}{k}\right)^{1-p/2} & \text{if } n = 4,\ p < 2, \\ \log \dfrac{2T+3k}{k} & \text{if } n = 4,\ p = 2. \end{cases} \tag{2.7.43}$$

Since $E_{0,1}(T)$ appear (2.7.6), the conditions are

$$\varepsilon^{p(p-1)} \le 1 \quad \text{if} \quad n \ge 6, \tag{2.7.44}$$

$$\left(\frac{2T+3k}{k}\right)^{1-p/2} \varepsilon^{p(p-1)} \le 1 \quad \text{if} \quad n = 4,\ p < 2, \tag{2.7.45}$$

$$\log \frac{2T+3k}{k} \varepsilon^{p(p-1)} \le 1 \quad \text{if} \quad n = 4,\ p = 2. \tag{2.7.46}$$

(ii) Condition from $E_{1,0}(T)$.

In (2.5.6), setting $\nu = 1$ and $a = 0$, we have $\mu = -(n-1)(p-1)/2 > -1$, which implies $E_{1,0}(T) = 1$. Since $E_{1,0}(T)$ appears in (2.7.9), (2.7.14) and (2.7.16), the condition is

$$2^{2p-1}3pk^2 C_{n,1,p} C_1^{p-1} \varepsilon^{p-1} \le 1. \tag{2.7.47}$$

(iii) Conditions from $E_{1,1}(T)$.

In (2.5.6), setting $\nu = a = 1$, we have $\mu = -(n-3)(p-1)/2 - q - 1 - (n-2)p$, which implies

$$E_{1,1}(T) = \begin{cases} 1 & \text{if } p > \dfrac{n}{n-2}, \\ \log \dfrac{2T+3k}{k} & \text{if } p = \dfrac{n}{n-2}, \\ \left(\dfrac{2T+3k}{k}\right)^{n-(n-2)p} & \text{if } p < \dfrac{n}{n-2}. \end{cases} \tag{2.7.48}$$

Since $E_{1,1}(T)$ appears in (2.7.9), (2.7.14) and (2.7.16), the conditions are

$$\left.\begin{aligned} &2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1}\varepsilon^{p(p-1)}\times \\ &\times\left(\frac{2T+3k}{k}\right)^{n-(n-2)p}\end{aligned}\right\}\le 1 \quad \text{if } p<\frac{n}{n-2}, \tag{2.7.49}$$

$$2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1}\varepsilon^{p(p-1)}\log\frac{2T+3k}{k}\le 1 \quad \text{if } p=\frac{n}{n-2}, \tag{2.7.50}$$

$$2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1}\varepsilon^{p(p-1)}\le 1 \quad \text{if } p>\frac{n}{n-2}. \tag{2.7.51}$$

(iv) Condition from $F_1(T)$ and $F_{p-1}(T)$.

In (2.5.10), setting $\nu=1$ and $\nu=p-1$, we have $\kappa<-1$, which implies $F_1(T)=F_{p-1}(T)=1$. Since $F_1(T)$ or $F_{p-1}(T)$ appears in (2.7.12) or (2.7.13) respectively, the condition is

$$\varepsilon^{p(p-1)}\le 1. \tag{2.7.52}$$

(v) Condition from $E_{p-1,0}(T)$.

In (2.5.6), setting $\nu=p-1$ and $a=0$, we have $\mu=-(n-1)/2-q(p-1)<-1$, which implies $E_{p-1,0}(T)=1$. Since $E_{p-1,0}(T)$ appears in (2.7.14), the condition is

$$2^{2p}5\cdot 7^{-1}pk^2C_{n,p-1,p}C_1M_1^{p-2}\varepsilon^{(p-1)^2}\le 1. \tag{2.7.53}$$

(vi) Conditions from $E_{p-1,1}(T)$.

In (2.5.6), setting $\nu=p-1$ and $a=1$, we have that $\mu=-(n-3)/2-(p-1)q<-1$ when $n\ge 6$ and $\mu>-1$ when $n=4$, which imply

$$E_{p-1,1}(T)=\begin{cases} 1 & \text{if } n\ge 6,\\ \left(\dfrac{2T+3k}{k}\right)^{1/2-(p-1)q} & \text{if } n=4,\ p<2,\\ \log\dfrac{2T+3k}{k} & \text{if } n=4,\ p=2.\end{cases} \tag{2.7.54}$$

Since $E_{p-1,1}(T)$ appears in (2.7.16), the condition are

$$2^{2p}5\cdot 7^{-1}pk^2C_{n,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)}\le 1 \quad \text{if} \quad n\ge 6, \tag{2.7.55}$$

$$\left.\begin{aligned} &2^{2p}5\cdot 7^{-1}pk^2C_{4,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)}\times \\ &\times\left(\frac{2T+3k}{k}\right)^{1/2-(p-1)q}\end{aligned}\right\}\le 1 \quad \text{if} \quad n=4,\ p<2, \tag{2.7.56}$$

$$\left.\begin{aligned} &2^{2p}5\cdot 7^{-1}pk^2C_{4,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)}\times \\ &\times\log\frac{2T+3k}{k}\end{aligned}\right\}\le 1 \quad \text{if} \quad n=4,\ p=2. \tag{2.7.57}$$

Now, we are in a position to summarize all the conditions in (i)-(vi) above. Set

$$\begin{aligned}\varepsilon_0 = \min \quad & \{1, (2^{2p-1}3pk^2C_{n,1,p}C_1^{p-1})^{-1/(p-1)}, \\ & \{2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1}\}^{-1/p(p-1)}, \\ & \{2^{2p}5\cdot 7^{-1}pk^2C_{n,p-1,p}C_1M_1^{p-2}\}^{-1/(p-1)^2}, \\ & \{2^{2p}5\cdot 7^{-1}pk^2C_{n,p-1,p}C_2M_1^{p-2}\}^{-1/p(p-1)}\} > 0.\end{aligned} \tag{2.7.58}$$

Then, (2.7.58) implies (2.7.44), (2.7.47), (2.7.51), (2.7.52), (2.7.53) and (2.7.55). In order to make that (2.7.50) includes (2.7.17) when $n \geq 6$, we employ the lemma 2.5.9 with $\delta = \gamma(p,n)/2 > 0$ and $X = (2T+3k)/k > 1$. Then, if we set

$$\begin{aligned}c = \min \quad & \{(2^{2p-1}3^ppCk^2(M_1)^{p-1})^{-1}, 1, (2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1})^{-1}, \\ & (2^{2p-1}3pk^2C_{n,1,p}C_2^{p-1})^{-1}\gamma(n/(n-2),n)/2, \\ & (2^{2p}5\cdot 7^{-1}pk^2C_{4,p-1,p}C_2M_1^{p-2})^{-1}\} > 0,\end{aligned}$$

the inequality $\varepsilon^{p(p-1)}D(T) \leq c$ implies (2.7.17), (2.7.45), (2.7.46), (2.7.49), (2.7.50) and (2.7.56) and (2.7.57), because of $1-p/2 \leq \gamma(p,4)/2$ in (2.7.45), $n-(n-2)p \leq \gamma(p,n)/2$ in (2.7.49), and $1/2-(p-1)q \leq \gamma(p,4)/2$ in (2.7.56). Therefore the proof of proposition 2.7.1 is now completed. □

2.8 Upper bound of the lifespan for the critical case in odd dimensions

In this section, we prove Theorem 2.3.2 for the critical case in odd space dimensions. The proof is divided into three steps. In the first step, we get a point-wise estimate of the linear term u^0 from below by means of the representation formula due to Rammaha [26]. In the second step, we employ the comparison argument between the solution of integral equations (2.3.9) and the blowing-up solution of ODE, basically introduced by Zhou [38], in order to overcome the difficulty in the critical case. In the last step, we also employ the slicing method introduced by Agemi, Kurokawa and Takamura [2].

Proposition 2.8.1 *Suppose that the assumptions of Theorem 2.3.2 are fulfilled. Let u be a C^0-solution of (2.3.9) in $\mathbf{R}^n \times [0,T]$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(g,n,p,k)$ such that T cannot be taken as*

$$T > \exp\left(c\varepsilon^{-p(p-1)}\right) \ \text{ if } p = p_0(n) \tag{2.8.1}$$

for $0 < \varepsilon \leq \varepsilon_0$, where c is a positive constant independent of ε.

Proof. First we note that one may assume that the solution of (2.3.9) is radially symmetric without loss of the generality. To see this, we employ the spherical mean which is defined by

$$\widetilde{v}(r,t) = \frac{1}{\omega_n}\int_{|\omega|=1} v(r\omega,t)dS_\omega \ (r>0),$$

for $v \in C(\mathbf{R}^n \times [0,\infty))$. If we take the spherical mean of (2.3.9), we get

$$\widetilde{u} = \varepsilon\widetilde{u^0} + L(\widetilde{|u|^p}).$$

Thanks to the fundamental identity for iterated spherical means, we have

$$L(\widetilde{|u|^p}) = L_{odd}(\widetilde{|u|^p}),$$

where L_{odd} is the one in (2.5.17). See 78-81pp. of John [12] for details. Thus, it follows from Jensen's inequality $\widetilde{|u|^p} \geq |\widetilde{u}|^p$ $(p>1)$ and the positivity of L_{odd} that

$$\widetilde{u} \geq \varepsilon\widetilde{u^0} + L_{odd}(|\widetilde{u}|^p).$$

We estimate $\widetilde{u}$ from below all the time in this section, so that we may assume that the equality holds here.

Let $u = u(r,t)$ be a C^0-solution of

$$u = \varepsilon u^0 + L_{odd}(|u|^p) \quad \text{in} \quad (0,\infty)\times[0,T] \tag{2.8.2}$$

which is associated by (2.3.9). Note that $u^0 = u^0(r,t)$ is a solution of

$$\begin{cases} u^0_{tt} - \dfrac{n-1}{r}u^0_r - u^0_{rr} = 0 & \text{in } (0,\infty)\times[0,\infty) \\ u^0(r,0) = 0,\ u^0_t(r,0) = g(r), & r\in(0,\infty). \end{cases} \tag{2.8.3}$$

[The 1st step] Estimate of u^0.

We have the following representation of u^0.

Lemma 2.8.1 (Rammaha [26]) *Let $n = 5,7,9\cdots$ and u^0 be a solution of (2.8.3). Then, u^0 is represented by*

$$u^0(r,t) = \frac{1}{2r^{(n-1)/2}}\int_{|r-t|}^{r+t} \lambda^{(n-1)/2} g(\lambda) P_{(n-3)/2}\left(\frac{\lambda^2+r^2-t^2}{2r\lambda}\right)d\lambda,$$

where P_k is Legendre polynomial of degree k defined by

$$P_k(z) = \frac{1}{2^k k!}\frac{d^k}{dz^k}(z^2-1)^k.$$

See (6a) on 681p. in [26] for the proof. This lemma implies the following estimate.

Lemma 2.8.2 (Rammaha [26]) *Let $n = 5, 7, 9, \cdots$. Assume (3.1.3). Then there exists a positive constant C_g such that for $t+k_0 < r < t+k_1$ and $t \geq k_2$,*

$$u^0(r,t) \geq \frac{C_g}{r^{(n-1)/2}}, \tag{2.8.4}$$

where $k_2 = k - k_0$.

See Lemma 2 on 682p. in [26] for the proof.

In order to prove the blow up result, we employ the iteration argument originally introduced by John [13]. Our frame in the argument is obtained by the following lemma.

Lemma 2.8.3 *Let u be a C^0-solution of (2.8.2). Assume (3.1.3). Then u in $\Sigma_0 = \{(r,t) : \ 2k \leq t - r \leq r\}$ satisfies*

$$\begin{aligned} u(r,t) \geq & \frac{C2^{(n-3)/2}(t-r)^{(n-1)/2}}{r^{(3n-7)/2}} \times \\ \times & \iint_{R(r,t)} \{(t-r-\tau+\lambda)(t+r-\tau-\lambda)\}^{(n-3)/2} |u(\lambda,\tau)|^p d\lambda d\tau + \\ & + \frac{E_1(t-r)^{(3n-5)/2-(n-1)p/2}}{r^{(3n-7)/2}} \varepsilon^p, \end{aligned} \tag{2.8.5}$$

where C is the one in (2.5.17),

$$E_1 = \frac{CC_g^p(k_1 - k_0)}{(n-1)2^{(n-1)p/2-(3n-11)/2}}$$

and

$$R(r,t) = \{(\lambda,\tau) : \ t - r \leq \lambda, \tau + \lambda \leq t + r, 2k \leq \tau - \lambda \leq t - r\}.$$

Proof. By virtue of Huygens' principle on u^0 and (2.5.17), we have

$$u \geq I_1 + I_2 \quad \text{in } \Sigma_0,$$

where we set

$$\begin{aligned} I_1(r,t) &= Cr^{2-n} \int_{R(r,t)} (t-\tau)^{3-n} h(\lambda, t-\tau, r) \lambda |u(\lambda,\tau)|^p d\lambda d\tau, \\ I_2(r,t) &= Cr^{2-n} \int_{S(r,t)} (t-\tau)^{3-n} h(\lambda, t-\tau, r) \lambda |u(\lambda,\tau)|^p d\lambda d\tau, \\ S(r,t) &= \{(\lambda,\tau) : \ t - r \leq \lambda, \tau + \lambda \leq t + r, -k_1 \leq \tau - \lambda \leq -k_0\}. \end{aligned}$$

Changing variable by (2.5.28) in I_1, we have

$$\begin{aligned} I_1(r,t) \geq \ & \frac{Cr^{2-n}}{4}\int_{2k}^{t-r}\{(t-r-\beta)(t+r-\beta)\}^{(n-3)/2}\,d\beta\times \\ & \times\int_{2(t-r)+\beta}^{t+r}\{(\alpha-(t-r))(t+r-\alpha)\}^{(n-3)/2}\times \\ & \times(t-(\alpha+\beta)/2)^{3-n}(\alpha-\beta)|u(\lambda,\tau)|^p d\alpha \end{aligned}$$

in Σ_0. Noticing that

$$\begin{aligned} & t+r-\beta \geq 2r,\ t-\frac{\alpha+\beta}{2}\leq r, \\ & \alpha-\beta\geq 2(t-r) \text{ and } \alpha-(t-r)\geq t-r \end{aligned}$$

hold in the domain of the integral above, we have

$$\begin{aligned} I_1(r,t) \geq \ & \frac{C2^{(n-5)/2}(t-r)^{(n-1)/2}}{r^{(3n-7)/2}}\int_{2k}^{t-r}(t-r-\beta)^{(n-3)/2}d\beta\times \\ & \times\int_{2(t-r)+\beta}^{t+r}(t+r-\alpha)^{(n-3)/2}|u(\lambda,\tau)|^p d\alpha \end{aligned}$$

in Σ_0. Hence, we have the first term of the right-hand side of (2.8.5).

Next, we shall show the second term of (2.8.5). Similarly to the above, we have

$$\begin{aligned} I_2(r,t) \geq \ & \frac{Cr^{2-n}}{4}\int_{-k_1}^{-k_0}\{(t-r-\beta)(t+r-\beta)\}^{(n-3)/2}\,d\beta\times \\ & \times\int_{2(t-r)+\beta}^{t+r}\{(\alpha-(t-r))(t+r-\alpha)\}^{(n-3)/2}\times \\ & \times(t-(\alpha+\beta)/2)^{3-n}(\alpha-\beta)|u(\lambda,\tau)|^p d\alpha \end{aligned}$$

in Σ_0. Note that

$$\begin{aligned} & t+r-\beta\geq r,\ t-\frac{\alpha+\beta}{2}\leq r-\beta\leq 2r, \\ & \alpha-(t-r)\geq t-r+\beta\geq t-r-k \text{ and } t-r-\beta\geq t-r \end{aligned}$$

hold in the domain of the integral above. By making use of (2.8.4), we have

$$\begin{aligned} I_2(r,t) \ & \geq \frac{CC_g^p(t-r)^{(n-3)/2}(t-r-k)^{(n-3)/2}}{2^{n-1-(n-1)p/2}r^{(3n-7)/2}}\varepsilon^p\int_{-k_1}^{-k_0}d\beta\times \\ & \times\int_{2(t-r)+\beta}^{t+r}(\alpha-\beta)^{1-(n-1)p/2}(t+r-\alpha)^{(n-3)/2}d\alpha \\ & \geq \frac{CC_g^p(t-r)^{(n-2)-(n-1)p/2}}{2^{(n-1)p/2-(3n-9)/2}r^{(3n-7)/2}}\varepsilon^p\int_{-k_1}^{-k_0}d\beta\times \\ & \times\int_{2(t-r)+\beta}^{3(t-r)}\{3(t-r)-\alpha\}^{(n-3)/2}d\alpha \end{aligned}$$

in Σ_0. The second term of the right-hand side of (2.8.5) follows from this inequality. Therefore, the proof of Lemma 2.8.3 is ended. $\square$

[The 2nd Step] Comparison argument.

Let us consider a solution w of

$$\begin{aligned} w(t-r) = \ & \frac{C2^{(n-5)/2}(t-r)^{(n-1)/2}}{r^{(3n-7)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-3)/2} d\beta \\ & \times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-3)/2} |w(\beta)|^p d\alpha \\ & + \frac{E_1(t-r)^{(3n-5)/2-(n-1)p/2}}{2r^{(3n-7)/2}} \varepsilon^p. \end{aligned} \tag{2.8.6}$$

Then we have the following comparison lemma.

Lemma 2.8.4 *Let u be a solution of (2.8.2) and w be a solution of (2.8.6). Then, u and w satisfy*

$$u > w \quad \text{in } \Sigma_0.$$

Proof. Fix a point $(r_0, t_0) \in \Sigma_0$. Define

$$\Lambda(r,t) = \{(\lambda, \tau) \in D(r,t) : \ 2k \le \tau - \lambda \le \lambda\},$$

where we set

$$D(r,t) = \{(\lambda, \tau) : \ t - r \le \tau + \lambda \le t + r, -k \le \tau - \lambda \le t - r\}$$

which is the domain of the integral in (2.5.17). Let us consider u and v in $\Lambda(r_0, t_0)$. Note that $u > w$ on $\tau - \lambda = 2k$ and at $(2k, 4k)$ which is an edge of Σ_0. By compactness of the closure of $\Lambda(r_0, t_0)$, we have $u > w$ in a neighborhood of $\tau - \lambda = 2k$ and $\lambda \ge 2k$.

Assume that there exist a point (r_1, t_1) with $u(r_1, t_1) = w(t_1 - r_1)$, which is nearest to $(2k, 4k)$ in such a neighborhood. Since $u > w$ in $R(r_1, t_1)$, we have

$$\begin{aligned} & \frac{C2^{(n-3)/2}(t_1-r_1)^{(n-1)/2}}{r_1^{(3n-7)/2}} \iint_{R(r_1,t_1)} (t_1 - r_1 - \tau + \lambda)^{(n-3)/2} \times \\ & \times (t_1 + r_1 - \tau - \lambda)^{(n-3)/2} |u(\lambda, \tau)|^p d\lambda d\tau + \frac{E_1(t_1-r_1)^{(3n-5)/2-(n-1)p/2}}{{r_1}^{(3n-7)/2}} \varepsilon^p \\ & > \frac{C2^{(n-3)/2}(t_1-r_1)^{(n-1)/2}}{r_1^{(3n-7)/2}} \iint_{R(r_1,t_1)} (t_1 - r_1 - \tau + \lambda)^{(n-3)/2} \\ & \times (t_1 + r_1 - \tau - \lambda)^{(n-3)/2} |w(\tau - \lambda)|^p d\lambda d\tau + \frac{E_1(t_1-r_1)^{(3n-5)/2-(n-1)p/2}}{2{r_1}^{(3n-7)/2}} \varepsilon^p, \end{aligned}$$

In view of (2.8.5) and (2.8.6), this inequality implies that $u > w$ at (r_1, t_1), which is a contradiction to the definition of (r_1, t_1). Therefore, we have $u > w$ in $\Lambda(r_0, t_0)$. (r_0, t_0) stands for any point in Σ_0, so that $\Lambda(r_0, t_0)$ covers all of Σ_0. The proof is completed. $\square$

We note that Lemma 2.8.4 implies that the lifespan of w is greater than the one of u, so that it is sufficient to look for the lifespan of w in Σ_0. By definition of w in (2.8.6), we have

$$\begin{aligned} w(\xi) \geq \ & \frac{C\xi^{3-n}}{2^{n-1}} \int_{2k}^{\xi} (\xi - \beta)^{(n-3)/2} |w(\beta)|^p d\beta \\ & \times \int_{2\xi+\beta}^{3\xi} (3\xi - \alpha)^{(n-3)/2} d\alpha + \frac{E_1}{2^{(3n-5)/2}} \xi^{-q-(n-1)/2} \varepsilon^p \end{aligned}$$

in Γ_0, where we set

$$\xi = \frac{r}{2}, \ \Gamma_0 = \{t - r = \xi, r \geq 4k\}.$$

Hence we obtain that

$$w(\xi) \geq \frac{C\xi^{3-n}}{2^{n-2}(n-1)} \int_{2k}^{\xi} (\xi - \beta)^{n-2} |w(\beta)|^p d\beta + \frac{E_1 \xi^{-q-(n-1)/2}}{2^{(3n-5)/2}} \varepsilon^p$$

for $\xi \geq 2k$. Then, it follows from the setting

$$W(\xi) = \xi^{q+(n-1)/2} w(\xi)$$

that

$$W(\xi) \geq D_n \xi^{q-(n-5)/2} \int_{2k}^{\xi} \frac{(\xi - \beta)^{n-2} |W(\beta)|^p d\beta}{\beta^{(n-1)p/2+pq}} + E_2 \varepsilon^p \quad \text{for } \xi \geq 2k, \quad (2.8.7)$$

where we set

$$D_n = \frac{C}{2^{n-3}(n-1)}, \ E_2 = \frac{E_1}{2^{(3n-5)/2}}.$$

Therefore we obtain the iteration frame in this section,

$$W(\xi) \geq D_n \int_{2k}^{\xi} \left(\frac{\xi - \beta}{\xi} \right)^{n-2} \frac{|W(\beta)|^p}{\beta^{pq}} d\beta + E_2 \varepsilon^p \quad \text{for } \xi \geq 2k. \qquad (2.8.8)$$

[The 3rd step] Slicing method in the iteration.

Let us define a blow-up domain as follows. Let us set

$$\Gamma_j = \{\xi \geq l_j k\}, \ l_j = 2 + \frac{1}{2} + \cdots + \frac{1}{2^j} \ (j \in \mathbf{N}).$$

We shall use the fact that a sequence $\{l_j\}$ is monotonously increasing and bounded, $2 < l_j < 3$, so that $\Gamma_{j+1} \subset \Gamma_j$. Assume an estimate of the form

$$W(\xi) \geq C_j \left(\log \frac{\xi}{l_j k}\right)^{a_j} \quad \text{in } \Gamma_j \tag{2.8.9}$$

where $a_j \geq 0$ and $C_j > 0$. Putting (2.8.9) into (2.8.8) and recalling that $pq = 1$, we get an estimate in Γ_{j+1} such as

$$W(\xi) \;\; \geq \;\; D_n C_j^p \int_{l_j k}^{\xi} \left(\frac{\xi - \beta}{\xi}\right)^{n-2} \left(\log \frac{\beta}{l_j k}\right)^{pa_j} \frac{d\beta}{\beta}.$$

Noting that $\dfrac{l_j}{l_{j+1}}\xi \geq l_j k$ in Γ_{j+1}, we have

$$\begin{aligned} W(\xi) \;\; &\geq \;\; D_n C_j^p \int_{l_j k}^{l_j \xi / l_{j+1}} \left(\frac{\xi - \beta}{\xi}\right)^{n-2} \left(\log \frac{\beta}{l_j k}\right)^{pa_j} \frac{d\beta}{\beta} \\ &\geq \;\; D_n C_j^p \left(1 - \frac{l_j}{l_{j+1}}\right)^{n-2} \int_{l_j k}^{l_j \xi / l_{j+1}} \left(\log \frac{\beta}{l_j k}\right)^{pa_j} \frac{d\beta}{\beta} \\ &= \;\; \frac{D_n C_j^p}{pa_j + 1} \left(1 - \frac{l_j}{l_{j+1}}\right)^{n-2} \left(\log \frac{\xi}{l_{j+1} k}\right)^{pa_j + 1}. \end{aligned}$$

By monotonicity of $\{l_j\}$ and

$$1 - \frac{l_j}{l_{j+1}} = \frac{l_{j+1} - l_j}{l_{j+1}} = \frac{1}{2^{j+1} l_{j+1}} \geq \frac{1}{3 \cdot 2^{j+1}},$$

we finally obtain

$$W(\xi) \geq C_{j+1} \left(\log \frac{\xi}{l_{j+1} k}\right)^{pa_j + 1} \quad \text{in } \Gamma_{j+1}, \tag{2.8.10}$$

where we set

$$C_{j+1} = \frac{D_n C_j^p}{3^{n-2} \cdot 2^{(j+1)(n-2)} (pa_j + 1)}.$$

Now, we are in a position to define sequences in the iteration. In view of (2.8.8), the first estimate is $W(\xi) \geq E_2 \varepsilon^p$, so that, with the help of (2.8.9) and (2.8.10), a sequence $\{a_j\}$ should be defined by

$$a_1 = 0, \;\; a_{j+1} = pa_j + 1 \;\; (j \in \mathbf{N}).$$

Also a sequence $\{C_j\}$ should be defined by

$$C_1 = E_2 \varepsilon^p, \;\; C_{j+1} = \frac{D_n C_j^p}{3^{n-2} \cdot 2^{(j+1)(n-2)} (pa_j + 1)} \;\; (j \in \mathbf{N}).$$

One can easily check that

$$a_j = \frac{p^{j-1}-1}{p-1} \ (j \in \mathbf{N}),$$

which gives us

$$\frac{1}{pa_j+1} \geq \frac{p-1}{p^j}.$$

Thus one can find that

$$C_{j+1} \geq E\frac{C_j^p}{(2^{n-2}p)^j} \ (j \in \mathbf{N}),$$

where E is a positive constant defined by

$$E = \frac{D_n(p-1)}{6^{n-2}}.$$

Hence we inductively obtain, for $j \geq 2$, that

$$\log C_j \geq p^{j-1}\left\{\log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log E - (j-1-k)p^k \log(2^{n-2}p)}{p^{j-1}}\right\}.$$

The sum part of above inequality converges as $j \to \infty$ by d'Alembert's criterion. It follows from this fact that there exist a constant S independent of j such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates above and making use of the monotonicity of Γ_j, we have the final inequality

$$\begin{aligned} W(\xi) &\geq \exp\{p^{j-1}(\log C_1 + S)\}\left(\log\frac{\xi}{3k}\right)^{(p^{j-1}-1)/(p-1)} \\ &= \exp\{p^{j-1}I(\xi)\}\left(\log\frac{\xi}{3k}\right)^{-1/(p-1)} \end{aligned}$$

in $\Gamma_\infty = \{\xi \geq 3k\}$, where we set

$$I(\xi) = \log\left(e^S E_2 \varepsilon^p \left(\log\frac{\xi}{3k}\right)^{1/(p-1)}\right).$$

If there exist a point $\xi_0 \in \Gamma_\infty \subset \Gamma_j$ $(j \geq 1)$ such that $I(\xi_0) > 0$, we get $W(\xi_0) \to \infty$ as $j \to \infty$. Note that $I(\xi_0) > 0$ is equivalent to

$$\xi_0 > 3k\exp\{(e^S E_2)^{-(p-1)}\varepsilon^{-p(p-1)}\}.$$

It is trivial that there exists a positive constant $\varepsilon_0 = \varepsilon_0(g, n, p, k)$ such that

$$\exp\{(e^S E_2)^{-(p-1)} \varepsilon^{-p(p-1)}\} \geq 1 \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

Since there exists $(r_0, t_0) \in \Sigma_0$ such that $t_0 - r_0 = \xi_0 > 3k$, we obtain the desired conclusion;

$$T > 3k \exp\{(e^S E_2)^{-(p-1)} \varepsilon^{-p(p-1)}\}.$$

Therefore the proof of the critical case is now completed with a minor modification on ε_0. □

2.9 Upper bound of the lifespan for the subcritical case in odd dimensions

By a similar argument to the previous section, we prove the blow-up result for (2.3.9) in the subcritical case in odd space dimensions. Note that we do not have to make use of the slicing method.

Proposition 2.9.1 *Suppose that the assumptions of Theorem 2.3.2 are fulfilled. Let u be a C^0-solution of (2.3.9) in $\mathbf{R}^n \times [0, T]$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(g, n, p, k)$ such that T cannot be taken as*

$$T > c\varepsilon^{-2p(p-1)/\gamma(p,n)} \text{ if } 1 < p < p_0(n) \tag{2.9.1}$$

for $0 < \varepsilon \leq \varepsilon_0$, where c is a positive constant independent of ε.

Proof. Because of the fact that $-(n-1)p/2 - pq < 0$ for $n \geq 5$, (2.8.7) yields

$$W(\xi) \geq D_n \xi^{-(n-2)-pq} \int_{2k}^{\xi} (\xi - \beta)^{n-2} |W(\beta)|^p d\beta + E_2 \varepsilon^p \quad \text{for } \xi \geq 2k. \tag{2.9.2}$$

This is our iteration frame in this case.

Assume an estimate of the form

$$W(\xi) \geq C_j \frac{(\xi - 2k)^{a_j}}{\xi^{b_j}} \text{ in } \Gamma_0, \tag{2.9.3}$$

where $a_j, b_j \geq 0$ and $C_j > 0$. Then, putting (2.9.3) into (2.9.2), we get

$$W(\xi) \geq \frac{D_n C_j^p}{\xi^{(n-2)+pq+pb_j}} \int_{2k}^{\xi} (\xi - \beta)^{n-2} (\beta - 2k)^{pa_j} d\beta \quad \text{in } \Gamma_0.$$

Applying the integration by parts to β-integral $(n-2)$ times, we obtain that

$$\begin{aligned}
&\frac{n-2}{pa_j+1}\int_{2k}^{\xi}(\xi-\beta)^{n-3}(\beta-2k)^{pa_j+1}d\beta \\
&\geq \frac{(n-2)(n-3)}{(pa_j+2)^2}\int_{2k}^{\xi}(\xi-\beta)^{n-4}(\beta-2k)^{pa_j+2}d\beta \\
&\cdots \\
&\geq \frac{(n-2)!}{(pa_j+n-1)^{n-1}}(\xi-2k)^{pa_j+n-1}.
\end{aligned}$$

Therefore we finally get

$$W(\xi) \geq C_{j+1}\frac{(\xi-2k)^{pa_j+n-1}}{\xi^{pb_j+n-2+pq}} \text{ in } \Gamma_0, \tag{2.9.4}$$

where we set

$$C_{j+1} = \frac{D_n C_j^p (n-2)!}{(pa_j+n-1)^{n-1}}.$$

Now, we are in a position to define sequences in the iteration. In view of (2.9.2), the first estimate is $W(\xi) \geq E_2\varepsilon^p$, so that, with the help of (2.9.3) and (2.9.4), sequences $\{a_j\}$ and $\{b_j\}$ should be defined by

$$a_1 = 0,\ a_{j+1} = pa_j + n - 1\ (j \in \mathbf{N})$$

and

$$b_1 = 0,\ b_{j+1} = pb_j + n - 2 + pq\ (j \in \mathbf{N}).$$

Also a sequence $\{C_j\}$ should be defined by

$$C_1 = E_2\varepsilon^p,\ C_{j+1} = \frac{D_n C_j^p (n-2)!}{(pa_j+n-1)^{n-1}}\ (j \in \mathbf{N}).$$

One can readily check that

$$a_j = \frac{n-1}{p-1}(p^{j-1}-1),\ b_j = \frac{pq+n-2}{p-1}(p^{j-1}-1)\ (j \geq \mathbf{N}),$$

which gives us

$$\frac{1}{pa_j+n-1} \geq \frac{p-1}{p^j(n-1)}.$$

Hence one can find that

$$C_{j+1} \geq F\frac{C_j^p}{p^{(n-1)j}}\ (j \in \mathbf{N}),$$

where F is a positive constant defined by

$$F = \frac{D_n(n-2)!(p-1)^{n-1}}{(n-1)^{n-1}}.$$

Due to the induction argument again, we obtain, for $j \geq 2$, that

$$\log C_j \geq p^{j-1}\left\{\log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log F - (j-1-k)p^k \log(p^{n-1})}{p^{j-1}}\right\}.$$

As before, this inequality yields that there exist a constant S independent of j such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates above, we reach the the final inequality

$$W(\xi) \geq \exp\{p^{j-1}I(\xi)\}\frac{\xi^{(pq+n-2)/(p-1)}}{(\xi-2k)^{(n-1)/(p-1)}} \text{ for } \xi \geq 2k,$$

where we set

$$I(\xi) = \log\left(E_2 e^S \varepsilon^p (\xi-2k)^{(n-1)/(p-1)} \xi^{-(pq+n-2)/(p-1)}\right).$$

Note that

$$\frac{n-1}{p-1} - \frac{pq+n-2}{p-1} = \frac{1-pq}{p-1}.$$

If there exist a point $\xi_0 \in \{\xi \geq 4k\} \subset \{\xi \geq 2k\}$ such that $I(\xi_0) > 0$, the desired conclusion can be established by the same argument as in the previous section. $I(\xi_0) > 0$ is equivalent to

$$\xi_0 > 2^{(n-1)/(1-pq)}\left(e^S E_2\right)^{-(p-1)/(1-pq)} \varepsilon^{-2p(p-1)/\gamma(p,n)}$$

in this case, so that the proof of the subcritical case is now completed. □

2.10 Upper bound of the lifespan for the critical case in even dimensions

In this section, we prove the blow-up theorem in the critical case in even dimensions. The proof is based on the one in odd dimensional case. However, Huygens' principle for u^0 is no longer available. Therefore the blow-up domain to ensure the positivity of the linear part is modified.

Proposition 2.10.1 *Suppose that of the assumption of Theorem 2.3.2 are fulfilled. Let u be a C^0-solution of (2.3.9) if $n > 4$ and $p = p_0(n)$, or u be a classical solution of (2.3.10) if $n = 4$ and $p = p_0(4)$ in $\mathbf{R}^n \times [0, T]$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(g, n, p, k)$ such that T cannot be taken as*

$$T > \exp\left(c\varepsilon^{-p(p-1)}\right) \quad \text{if } p = p_0(n) \tag{2.10.1}$$

for $0 < \varepsilon \le \varepsilon_0$, where c is a positive constant independent of ε.

Proof. Similarly to the odd dimensional case, we may assume that the solution of (2.3.9) is radially symmetric without loss of the generality. Let $u = u(r, t)$ be a C^0-solution of

$$u \ge \varepsilon u^0 + L_{even,1}(|u|^p) \quad \text{in} \quad (0, \infty) \times [0, T], \tag{2.10.2}$$

where $L_{even,1}$ is defined by (2.5.18) and $u^0 = u^0(r, t)$ is a solution of (2.8.3).

[The 1st step] Estimate of u^0.

We shall employ the following representation of u^0.

Lemma 2.10.1 (Rammaha [26]) *Let $n = 4, 6, 8 \cdots$ and u^0 be a solution of (2.8.3). Then, u^0 is represented by*

$$u^0(r,t) = \frac{2}{\pi r^{(n-2)/2}} \int_0^t \frac{\rho d\rho}{\sqrt{t^2 - \rho^2}} \times$$
$$\times \int_{|r-\rho|}^{r+\rho} \frac{\lambda^{(n-2)/2} g(\lambda) T_{(n-4)/2}\left((\lambda^2 + r^2 - \rho^2)/(2r\lambda)\right) d\lambda}{\sqrt{\lambda^2 - (r-\rho)^2}\sqrt{(r+\rho)^2 - \lambda^2}},$$

where T_k is Tschebyscheff polynomials of degree k defined by

$$T_k(z) = \frac{(-1)^k}{(2k-1)!!}(1 - z^2)^{1/2} \frac{d^k}{dz^k}(1 - z^2)^{k-(1/2)}.$$

See (6b) on 681p. in [26] for the proof. This lemma implies the following estimate.

Lemma 2.10.2 (Rammaha [26]) *Let $n = 4, 6, 8, \cdots$. Assume (3.1.3). Then there exists a positive constant C_g such that, for $t + k_0 < r < t + k_1$ and $t \ge k_2$,*

$$u^0(r,t) \ge \frac{C_g}{r^{(n-1)/2}}, \tag{2.10.3}$$

where $k_2 = k - k_0$.

See Lemma 2 on 682p. in [26] for the proof.

Our frame in the iteration argument is obtained by the following lemma.

Lemma 2.10.3 *Let u be a C^0-solution of (2.10.2). Assume (3.1.3). Then u in $\Sigma_0 = \{(r,t) \ : \ 2k \le t-r \le r\}$ satisfies*

$$\begin{aligned} u(r,t) \ge & \frac{C2^{(n-3)/2}(t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \times \\ \times & \iint_{R(r,t)} \{(t-r-\tau+\lambda)(t+r-\tau-\lambda)\}^{(n-2)/2}|u(\lambda,\tau)|^p d\lambda d\tau + \\ & + \frac{F_1(t-r)^{(3n-3)/2-(n-1)p/2}}{r^{(3n-5)/2}}\varepsilon^p + \varepsilon u^0(r,t), \end{aligned} \tag{2.10.4}$$

where C is the one in (2.5.18) and

$$F_1 = \frac{CC_g^p(k_1-k_0)2^{(9-3n)/2-(n-1)p/2}}{n(n-1)}.$$

Proof. In view of (2.5.18), we have that

$$\begin{aligned} L_{even,1}(|u|^p)(r,t) \ & \ge \frac{C}{r^{n-2}} \int_0^t (t-\tau)^{2-n} d\tau \int_{|t-r-\tau|}^{t+r-\tau} \lambda |u(\lambda,\tau)|^p d\lambda \times \\ & \times \int_{|\lambda-r|}^{t-\tau} \frac{\rho h(\lambda,\rho,r)}{\sqrt{(t-\tau)^2-\rho^2}} d\rho \end{aligned}$$

in Σ_0. Noticing that $(\lambda+r)^2-\rho^2 \ge (\lambda+r)^2-(t-\tau)^2$ for $\rho \le t-\tau$, we get

$$\begin{aligned} L_{even,1}(|u|^p)(r,t) \ge & \frac{C}{r^{n-2}} \int_0^t (t-\tau)^{2-n} d\tau \times \\ & \times \int_{|t-r-\tau|}^{t+r-\tau} \frac{\lambda |u(\lambda,\tau)|^p \{(\lambda+r)^2-(t-\tau)^2\}^{(n-3)/2} d\lambda}{\{(t-\tau)^2-(\lambda-r)^2\}^{1/2}} \times \\ & \times \int_{|\lambda-r|}^{t-\tau} \rho \{\rho^2-(\lambda-r)^2\}^{(n-3)/2} d\rho \end{aligned}$$

in Σ_0. Since the ρ-integral above is

$$\frac{1}{n-1}\{(t-\tau)^2-(\lambda-r)^2\}^{(n-1)/2},$$

we obtain that

$$\begin{aligned} L_{even,1}(|u|^p)(r,t) \ & \ge \frac{C}{r^{n-2}(n-1)} \int_0^t (t-\tau)^{2-n} d\tau \times \\ & \times \int_{|t-r-\tau|}^{t+r-\tau} \{(\lambda+r)^2-(t-\tau)^2\}^{(n-3)/2} \times \\ & \times \{(t-\tau)^2-(\lambda-r)^2\}^{(n-2)/2} \lambda |u(\lambda,\tau)|^p d\lambda \\ & \ge J_1 + J_2 \end{aligned}$$

in Σ_0, where we set

$$J_1(r,t) = \frac{C}{(n-1)r^{n-2}} \int_{R(r,t)} (t-\tau)^{2-n}\{(\lambda+r)^2-(t-\tau)^2\}^{(n-3)/2}\times$$
$$\times\{(t-\tau)^2-(\lambda-r)^2\}^{(n-2)/2}\lambda|u(\lambda,\tau)|^p d\lambda d\tau,$$

$$J_2(r,t) = \frac{C}{(n-1)r^{n-2}} \int_{S(r,t)} (t-\tau)^{2-n}\{(\lambda+r)^2-(t-\tau)^2\}^{(n-3)/2}\times$$
$$\times\{(t-\tau)^2-(\lambda-r)^2\}^{(n-2)/2}\lambda|u(\lambda,\tau)|^p d\lambda d\tau.$$

Changing variables by (2.5.28) in J_1, we have that

$$J_1(r,t) \geq \frac{C}{2(n-1)r^{n-2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-2)/2}(t+r-\beta)^{(n-3)/2} d\beta\times$$
$$\times \int_{2(t-r)+\beta}^{t+r} \{\alpha-(t-r)\}^{(n-3)/2}(t+r-\alpha)^{(n-2)/2}\times$$
$$\times\{t-(\alpha+\beta)/2\}^{2-n}(\alpha-\beta)|u(\lambda,\tau)|^p d\alpha$$

in Σ_0. Note that

$$t+r-\beta \geq 2r,\ t-\frac{\alpha+\beta}{2} \leq r,$$
$$\alpha-\beta \geq 2(t-r),\ \alpha-(t-r) \geq t-r+\beta \geq t-r$$

hold in the domain of the integral above. Hence we get

$$J_1(r,t) \geq \frac{C2^{(n-5)/2}(t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-2)/2} d\beta\times$$
$$\times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-2)/2}|u(\lambda,\tau)|^p d\alpha$$

in Σ_0. Therefore, we obtain the first term of the right-hand side in (2.10.4).

Similarly to the above, $J_2(r,t)$ is bounded from below by

$$\frac{C}{4(n-1)r^{n-2}} \int_{-k_1}^{-k_0} (t-r-\beta)^{(n-2)/2}(t+r-\beta)^{(n-3)/2} d\beta$$
$$\times \int_{2(t-r)+\beta}^{t+r} \{\alpha-(t-r)\}^{(n-3)/2}(t+r-\alpha)^{(n-2)/2}$$
$$\times\{t-(\alpha+\beta)/2\}^{2-n}(\alpha-\beta)|u(\lambda,\tau)|^p d\alpha$$

in Σ_0. Note that

$$t+r-\beta \geq r,\ t-\frac{\alpha+\beta}{2} \leq 2r,\ \alpha-(t-r) \geq t-r-k$$
$$\text{and } t-r-\beta \geq t-r$$

hold in the domain of the integral above. Hence (2.10.3) yields that $J_2(r,t)$ in Σ_0 is estimated from below by

$$
\begin{aligned}
&\geq \frac{\varepsilon^p C C_g^p 2^{(n-1)p/2-n}(t-r)^{(n-2)/2}(t-r-k)^{(n-3)/2}}{(n-1)r^{(3n-5)/2}} \int_{-k_1}^{-k_0} d\beta \\
&\quad \times \int_{2(t-r)+\beta}^{t+r} (\alpha-\beta)^{1-(n-1)p/2}(t+r-\alpha)^{(n-2)/2} d\alpha \\
&\geq \frac{\varepsilon^p C C_g^p 2^{2-n-(n-1)p}(t-r)^{n/2-(n-1)p/2}(t-r-k)^{(n-3)/2}}{(n-1)r^{(3n-5)/2}} \int_{-k_1}^{-k_0} d\beta \times \\
&\quad \times \int_{2(t-r)+\beta}^{3(t-r)} \{3(t-r)-\alpha\}^{(n-2)/2} d\alpha.
\end{aligned}
$$

The second term of the right-hand side of (2.10.4) follows from this inequality. Therefore, the proof of Lemma 2.10.3 is ended. □

Next, we shall show the positivity of the right-hand side of (2.10.4). Under the condition (3.1.3), (2.4.6) yields that

$$
\begin{aligned}
\varepsilon u^0(r,t) &\geq \frac{-C_{n,k,0,g}\varepsilon}{(t+r+2k)^{(n-1)/2}(t-r+2k)^{(n-1)/2}} \\
&\geq \frac{-C_{n,k,0,g}\varepsilon}{r^{(n-1)/2}(t-r)^{(n-1)/2}}
\end{aligned}
$$

for $t-r>0$. Let us define a domain

$$
\Sigma_1 = \left\{ (r,t) \in (0,\infty)^2 : \ r \geq t-r \geq \frac{r}{2}, \ r \geq K\varepsilon^{-L} \right\},
$$

where we set

$$
\begin{aligned}
K &= \left(2^{2n-(n-1)p/2-1} F_1^{-1} C_{n,k,0,g}\right)^{1/(n-(n-1)p/2)}, \\
L &= \frac{p-1}{n-(n-1)p/2} > 0.
\end{aligned}
$$

Taking ε to satisfy

$$
K\varepsilon^{-L} \geq 4k
$$

and setting

$$
A(r,t) = \frac{F_1(t-r)^{(3n-3)/2-(n-1)p/2}}{r^{(3n-5)/2}} > 0,
$$

we obtain that, in Σ_1,

$$
\begin{aligned}
&\frac{A(r,t)}{2}\varepsilon^p + \varepsilon u^0(r,t) \\
&\geq \frac{F_1 \varepsilon^p (t-r)^{2n-2-(n-1)p/2} r^{(n-1)/2} - 2C_{n,k,0,g}\varepsilon r^{(3n-5)/2}}{2r^{2n-3}(t-r)^{(n-1)/2}} \geq 0.
\end{aligned}
$$

Making use of this inequality, we obtain that

$$u(r,t) \geq \frac{C2^{(n-5)/2}(t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-2)/2} d\beta \times$$
$$\times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda,\tau)|^p d\alpha + \frac{A(r,t)}{2}\varepsilon^p$$

in Σ_1. Cutting the domain of the integral, we get

$$u(r,t) > \frac{C2^{(n-5)/2}(t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L}/2}^{t-r} (t-r-\beta)^{(n-2)/2} d\beta \times$$
$$\times \int_{3(t-r)}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda,\tau)|^p d\alpha + \frac{A(r,t)}{4}\varepsilon^p$$

in Σ_1. Here we introduce a change of variables (α,β) to (ξ,η) by

$$\xi = \alpha,\ \eta = \frac{\alpha+\beta}{2} - \frac{3}{2}\cdot\frac{\alpha-\beta}{2} = \frac{5\beta-\alpha}{4}.$$

Then, cutting the domain of the integral again, we get

$$u(r,t) > \frac{C2^{(n-1)/2}(t-r)^{(n-1)/2}}{5(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L}/2}^{t-3r/2} \left\{ t-r-\left(\frac{4\eta+\xi}{5}\right)\right\}^{(n-2)/2} d\eta \times$$
$$\times \int_{3(t-r)}^{t+r} (t+r-\xi)^{(n-2)/2} |u(\lambda,\tau)|^p d\xi + B(r,t)\varepsilon^p$$

in Σ_2, where we set

$$\Sigma_2 = \left\{ (r,t) \in (0,\infty)^2 :\ \frac{r}{2} \geq t - \frac{3}{2}r \geq \frac{K\varepsilon^{-L}}{2} \right\}$$

and

$$B(r,t) = \frac{A(r,t)}{4}.$$

Therefore we obtain that, in Σ_2,

$$u(r,t) > \frac{C2^{(3n-5)/2}(t-r)^{(n-1)/2}}{5^{n/2}(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L}/2}^{t-3r/2} \left(t-\frac{3}{2}r-\eta\right)^{(n-2)/2} d\eta \times$$
$$\times \int_{3(t-r)}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda,\tau)|^p d\alpha + B(r,t)\varepsilon^p. \tag{2.10.5}$$

[The 2nd Step] Comparison argument.

Let us consider a solution y of

$$y\left(t-\frac{3}{2}r\right)=\frac{C2^{(3n-3)/2}(t-r)^{(n-1)/2}}{5^{n/2}(n-1)r^{(3n-5)/2}}\int_{K\varepsilon^{-L}/2}^{t-3r/2}\left(t-\frac{3}{2}r-\eta\right)^{(n-2)/2}d\eta\times$$
$$\times\int_{3(t-r)}^{t+r}(t+r-\alpha)^{(n-2)/2}|y(\eta)|^p d\alpha+B(r,t)\varepsilon^p. \tag{2.10.6}$$

Then we have the following comparison lemma.

Lemma 2.10.4 *Let u be a solution of (2.10.2) and y be a solution of (2.10.6). Then, u and y satisfy*

$$u>y \quad \text{in } \Sigma_2.$$

Proof. Fix a point for any $(r_0,t_0)\in\Sigma_2$. Define

$$\Lambda(r,t)=\left\{(\lambda,\tau)\in D(r,t):\ \frac{K\varepsilon^{-L}}{2}\le\tau-\frac{3}{2}\lambda\le\frac{\lambda}{2}\right\},$$

where

$$D(r,t)=\{(\lambda,\tau):\ t-r\le\tau+\lambda\le t+r,-k\le\tau-\lambda\le t-r\}.$$

Let us consider u and y in $\Lambda(r_0,t_0)$. Note that $u>y$ on $\tau-\frac{3}{2}\lambda=\frac{K\varepsilon^{-L}}{2}$ and at $(K\varepsilon^{-L},2K\varepsilon^{-L})$ which is an edge point of Σ_2. By compactness of the closure of $\Lambda(r_0,t_0)$, we have $u>y$ in a neighborhood of $\tau-\frac{3}{2}\lambda=\frac{K\varepsilon^{-L}}{2}$ and $\lambda\ge K\varepsilon^{-L}$.

Assume that there exist a point (r_1,t_1) with $u(r_1,t_1)=y(t_1-3r_1/2)$ which is nearest to $(K\varepsilon^{-L},2K\varepsilon^{-L})$ in such a neighborhood. Since $u>y$ in $R'(r_1,t_1)$, we have

$$\begin{aligned}&\frac{C2^{(3n-7)/2}(t_1-r_1)^{(n-1)/2}}{5^{(n-2)/2}(n-1)r_1^{(3n-5)/2}}\iint_{R'(r_1,t_1)}\left(t_1-\frac{3}{2}r_1-\tau+\frac{3}{2}\lambda\right)^{(n-2)/2}\times\\&\quad\times(t_1+r_1-\tau-\lambda)^{(n-2)/2}|u(\lambda,\tau)|^p d\lambda d\tau+B(r_1,t_1)\varepsilon^p\\&>\frac{C2^{(3n-7)/2}(t_1-r_1)^{(n-1)/2}}{5^{(n-2)/2}(n-1)r_1^{(3n-5)/2}}\iint_{R'(r_1,t_1)}\left(t_1-\frac{3}{2}r_1-\tau+\frac{3}{2}\lambda\right)^{(n-2)/2}\times\\&\quad\times\left(t_1+r_1-\tau-\lambda\right)^{(n-2)/2}\left|y\left(\tau-\frac{3}{2}\lambda\right)\right|^p d\lambda d\tau+B(r_1,t_1)\varepsilon^p,\end{aligned}$$

where we set

$$R'(r,t)=\left\{(\lambda,\tau):\ 3(t-r)\le\tau+\lambda\le t+r,\frac{K\varepsilon^{-L}}{2}\le\tau-\frac{3}{2}\lambda\le t-\frac{3}{2}r\right\}.$$

In view of (2.10.5) and (2.10.6), this inequality yield that $u > y$ at (r_1, t_1), which is a contradiction to the definition of (r_1, t_1). Therefore, the proof of Lemma 2.10.4 is now established by the same argument as the one for Lemma 2.8.4. $\square$

We note that Lemma 2.10.4 implies that the lifespan of y is greater than the lifespan of u, so that it is sufficient to look for the lifespan of y in Σ_2. By definition y in (2.10.6), we have

$$\begin{aligned} y(\xi) = \ & \frac{C3^{(n-1)/2}\xi^{2-n}}{2^{(3n-5)/2}5^{n/2}(n-1)} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi-\eta)^{(n-2)/2}|y(\eta)|^p d\eta \\ & \times \int_{9\xi}^{11\xi} (11\xi-\alpha)^{(n-2)/2} d\alpha + \frac{\varepsilon^p F_1 \xi^{-q-(n-1)/2}}{2^{(3n-3)}3^{(n-1)p/2-(3n-3)/2}} \end{aligned}$$

in Γ_1, where we set

$$\xi = \frac{r}{4},\ \Gamma_1 = \left\{ t - \frac{3}{2}r = \xi, r \geq 2K\varepsilon^{-L} \right\} \subset \Sigma_2.$$

Hence, we obtain that

$$\begin{aligned} y(\xi) \geq \ & \frac{C3^{(n-1)/2}\xi^{(4-n)/2}}{2^{(2n-7)/2}5^{n/2}n(n-1)} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi-\eta)^{(n-2)/2}|y(\eta)|^p d\eta \\ & + \frac{\varepsilon^p F_1 \xi^{-q-(n-1)/2}}{2^{(3n-3)}3^{(n-1)p/2-(3n-3)/2}} \end{aligned}$$

for $\xi \geq K\varepsilon^{-L}/2$. Then it follows from the setting

$$Y(\xi) = \xi^{q+(n-1)/2} y(\xi)$$

that

$$Y(\xi) \geq E_n \xi^{q+3/2} \int_{K\varepsilon^{-L}/2}^{\xi} \frac{(\xi-\eta)^{(n-2)/2}|Y(\eta)|^p d\eta}{\eta^{(n-1)p/2+pq}} + F_2\varepsilon^p, \tag{2.10.7}$$

where we set

$$E_n = \frac{C3^{(n-1)/2}}{2^{(2n-7)/2}5^{n/2}n(n-1)},\ F_2 = \frac{F_1}{2^{(3n-3)}3^{(n-1)p/2-(3n-3)/2}}.$$

Therefore we obtain the iteration frame in this section,

$$Y(\xi) \geq E_n \int_{K\varepsilon^{-L}/2}^{\xi} \left(\frac{\xi-\eta}{\xi}\right)^{(n-2)/2} \frac{|Y(\eta)|^p}{\eta^{pq}} d\eta + F_2\varepsilon^p \quad \text{for } \xi \geq \frac{K\varepsilon^{-L}}{2}. \tag{2.10.8}$$

[The 3rd step] Slicing method with the iteration.

Let us define a blow-up domain as follows. Let us set

$$\Gamma_j = \{\xi \geq l_j K\varepsilon^{-L}\}, \quad l_j = \frac{1}{2} + \cdots + \frac{1}{2^j} \ (j \in \mathbf{N}).$$

We shall use the fact that a sequence $\{l_j\}$ is monotonously increasing and bounded as $\frac{1}{2} < l_j < 1$, so that $\Gamma_{j+1} \subset \Gamma_j$. Assume an estimate of the form

$$Y(\xi) \geq C_j \left(\log \frac{\xi}{l_j K\varepsilon^{-L}}\right)^{a_j} \text{ in } \Gamma_j, \tag{2.10.9}$$

where $a_j \geq 0$ and $C_j > 0$. Putting (2.10.9) into (2.10.8) and recalling $pq = 1$, we get an estimate in Γ_{j+1} such as

$$Y(\xi) \geq E_n C_j^p \int_{l_j K\varepsilon^{-L}}^{\xi} \left(\frac{\xi - \eta}{\xi}\right)^{(n-2)/2} \left(\log \frac{\eta}{l_j K\varepsilon^{-L}}\right)^{pa_j} \frac{d\eta}{\eta}.$$

Noting that $\frac{l_j}{l_{j+1}}\xi \geq l_j K\varepsilon^{-L}$ in Γ_{j+1}, we have

$$\begin{aligned} Y(\xi) &\geq E_n C_j^p \left(1 - \frac{l_j}{l_{j+1}}\right)^{(n-2)/2} \int_{l_j K\varepsilon^{-L}}^{l_j\xi/l_{j+1}} \left(\log \frac{\eta}{l_j k}\right)^{pa_j} \frac{d\eta}{\eta} \\ &= \frac{E_n C_j^p}{pa_j + 1} \left(1 - \frac{l_j}{l_{j+1}}\right)^{(n-2)/2} \left(\log \frac{\xi}{l_{j+1} K\varepsilon^{-L}}\right)^{pa_j+1}. \end{aligned}$$

By monotonicity of $\{l_j\}$ and

$$1 - \frac{l_j}{l_{j+1}} = \frac{l_{j+1} - l_j}{l_{j+1}} = \frac{1}{2^{j+1} l_{j+1}} \geq \frac{1}{2^{j+1}},$$

we finally obtain

$$Y(\xi) \geq C_{j+1} \left(\log \frac{\xi}{l_{j+1} K\varepsilon^{-L}}\right)^{pa_j+1} \text{ in } \Gamma_{j+1}, \tag{2.10.10}$$

where we set

$$C_{j+1} = \frac{E_n C_j^p}{2^{(n-2)(j+1)/2}(pa_j + 1)}.$$

Now, we are in a position to define sequences in the iteration. In view of (2.10.8), the first estimate is $Y(\xi) \geq F_2\varepsilon^p$, so that, with the help of (2.10.9) and (2.10.10), a sequence $\{a_j\}$ should be defined by

$$a_1 = 0, \ a_{j+1} = pa_j + 1 \ (j \in \mathbf{N}).$$

Also a sequence $\{C_j\}$ should be defined by

$$C_1 = F_2\varepsilon^p,\ C_{j+1} = \frac{E_n C_j^p}{2^{(n-2)(j+1)/2}(pa_j+1)}\ (j \in \mathbf{N}).$$

One can easily check that

$$a_j = \frac{p^{j-1}-1}{p-1}\ (j \in \mathbf{N})$$

which gives us

$$\frac{1}{pa_j+1} \geq \frac{p-1}{p^j}.$$

Thus one can find that

$$C_{j+1} \geq E\frac{C_j^p}{(2^{(n-2)/2}p)^j}\ (j \in \mathbf{N}),$$

where E is a positive constant defined by

$$E = \frac{E_n(p-1)}{2^{(n-2)/2}}.$$

Hence, we inductively obtain, for $j \geq 2$, that

$$\log C_j \geq p^{j-1}\left\{\log C_1 + \sum_{k=0}^{j-2}\frac{p^k \log E - (j-1-k)p^k \log(2^{(n-2)/2}p)}{p^{j-1}}\right\}.$$

This inequality yields that there exist a constant S independent of j such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates above and making use of the monotonicity of Γ_j, we obtain the final inequality,

$$\begin{aligned} Y(\xi) &\geq \exp\{p^{j-1}(\log C_1 + S)\}\left(\log\frac{\xi}{K\varepsilon^{-L}}\right)^{(p^{j-1}-1)/(p-1)} \\ &= \exp\{p^{j-1}I(\xi)\}\left(\log\frac{\xi}{K\varepsilon^{-L}}\right)^{-1/(p-1)}. \end{aligned}$$

in $\Gamma_\infty = \{\xi \geq K\varepsilon^{-L}\}$, where we set

$$I(\xi) = \log\left(e^S F_2\varepsilon^p\left(\log\frac{\xi}{K\varepsilon^{-L}}\right)^{1/(p-1)}\right).$$

If there exist a point $\xi_0 \in \Gamma_\infty \subset \Gamma_j$ $(j \geq 1)$ such that $I(\xi_0) > 0$, we get the desired conclusion by the same argument in the end of section 7. In this case, $I(\xi_0) > 0$ is equivalent to

$$\xi_0 > \exp\{(e^S F_2)^{-(p-1)} \varepsilon^{-p(p-1)}\} K \varepsilon^{-L}.$$

Therefore the proof of the critical case is now completed. □

2.11 Upper bound of the lifespan for the subcritical case in even dimensions

Similarly to the previous section, we prove the blow-up result of solution for (2.3.9) in the subcritical case in even space dimensions. Note that we do not have to make use of the slicing method.

Proposition 2.11.1 *Suppose that the same assumption of Theorem 2.3.2 are fulfilled. Let u be a C^0-solution of (2.3.9) $\mathbf{R}^n \times [0, T]$. Then there exists a positive constant $\varepsilon_0 = \varepsilon_0(g, n, p, k)$ such that T cannot be taken as*

$$T > c\varepsilon^{-2p(p-1)/\gamma(p,n)} \text{ if } 1 < p < p_0(n) \tag{2.11.1}$$

for $0 < \varepsilon \leq \varepsilon_0$, where c is a positive constant independent of ε.

Proof. Because of the fact that $-(n-1)p/2 - pq < 0$ for $n \geq 4$, (2.10.7) yields

$$Y(\xi) \geq E_n \xi^{-(n-2)/2-pq} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} |Y(\eta)|^p d\eta + F_2 \varepsilon^p \tag{2.11.2}$$

for $\xi \geq K\varepsilon^{-L}/2$. This is our iteration frame in this case.

Assume an estimate of the form

$$Y(\xi) \geq C_j \frac{\left(\xi - \dfrac{K\varepsilon^{-L}}{2}\right)^{a_j}}{\xi^{b_j}} \text{ in } \Gamma_1, \tag{2.11.3}$$

where $a_j, b_j \geq 0$ and $C_j > 0$. Then, putting (2.11.3) into (2.11.2), we get the following estimate in Γ_1 of the form

$$Y(\xi) \geq E_n C_j^p \xi^{-(n-2)/2-pq-pb_j} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} \left(\eta - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j} d\eta.$$

Applying the integration by parts to β-integral $(n-2)/2$ times, we obtain that

$$\begin{aligned}
&\frac{(n-2)}{2(pa_j+1)}\int_{K\varepsilon^{-L}/2}^{\xi}(\xi-\eta)^{(n-4)/2}\left(\eta-\frac{K\varepsilon^{-L}}{2}\right)^{pa_j+1}d\eta \\
&\geq \frac{(n-2)(n-4)}{(pa_j+2)^2 2\cdot 2}\int_{K\varepsilon^{-L}/2}^{\xi}(\xi-\eta)^{(n-6)/2}\left(\eta-\frac{K\varepsilon^{-L}}{2}\right)^{pa_j+2}d\eta \\
&\cdots \\
&\geq \frac{((n-2)/2)!}{\left(pa_j+\frac{n}{2}\right)^{n/2}}\left(\xi-\frac{K\varepsilon^{-L}}{2}\right)^{pa_j+n/2}.
\end{aligned}$$

Therefore we finally get

$$Y(\xi)\geq \frac{C_{j+1}\left(\xi-\frac{K\varepsilon^{-L}}{2}\right)^{pa_j+n/2}}{\xi^{pb_j+(n-2)/2+pq}} \text{ in } \Gamma_1, \tag{2.11.4}$$

where

$$C_{j+1}=\frac{((n-2)/2)!E_nC_j^p}{\left(pa_j+\frac{n}{2}\right)^{n/2}}.$$

Now, we are in a position to define sequences in the iteration. In view of (2.11.2), the first estimate is $Y(\xi)\geq F_2\varepsilon^p$, so that, with the help of (2.11.3) and (2.11.4), sequences $\{a_j\}$ and $\{b_j\}$ should be defined by

$$a_1=0,\ a_{j+1}=pa_j+\frac{n}{2}\ (j\in\mathbf{N})$$

and

$$b_1=0,\ b_{j+1}=pb_j+pq+\frac{n-2}{2}\ (j\in\mathbf{N}).$$

Also a sequence $\{C_j\}$ should be defined by

$$C_1=F_2\varepsilon^p,\ C_{j+1}=\frac{((n-2)/2)!E_nC_j^p}{\left(pa_j+\frac{n}{2}\right)^{n/2}}\ (j\in\mathbf{N}).$$

One can readily check that

$$a_j=\frac{n}{2(p-1)}(p^{j-1}-1),\ b_j=\frac{pq+(n-2)/2}{p-1}(p^{j-1}-1)\ (j\in\mathbf{N}),$$

which gives us

$$\frac{1}{pa_j + n/2} \geq \frac{2(p-1)}{p^j n}.$$

Hence one can find that

$$C_{j+1} \geq F \frac{C_j^p}{p^{(n/2)j}} \ (j \in \mathbf{N}),$$

where F is a positive constant defined by

$$F = \frac{E_n((n-2)/2)!\{2(p-1)\}^{n/2}}{n^{n/2}}.$$

Due to the induction argument again, we obtain, for $j \geq 2$, that

$$\log C_j \geq p^{j-1} \left\{ \log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log F - (j-1-k)p^k \log(p^{n/2})}{p^{j-1}} \right\}.$$

This inequality yields that there exist a constant S independent of j such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates, we reach the final inequality

$$Y(\xi) \geq \exp\{p^{j-1} I(\xi)\} \frac{\xi^{(pq+(n-2)/2)/(p-1)}}{\left(\xi - \dfrac{K\varepsilon^{-L}}{2}\right)^{n/2(p-1)}} \quad \text{for } \xi \geq \frac{K\varepsilon^{-L}}{2},$$

where we set

$$I(\xi) = \log \left(F_2 e^S \varepsilon^p \left(\xi - \frac{K\varepsilon^{-L}}{2}\right)^{n/2(p-1)} \xi^{-(pq+(n-2)/2)/(p-1)} \right).$$

Note that

$$\frac{n}{2(p-1)} - \frac{pq + (n-2)/2}{p-1} = \frac{1-pq}{p-1}.$$

If there exist a point $\xi_0 \in \{\xi \geq K\varepsilon^{-L}\} \subset \{\xi \geq (K\varepsilon^{-L})/2\}$ such that $I(\xi_0) > 0$, we get the desired conclusion as before. $I(\xi_0) > 0$ is equivalent to

$$\xi_0 > 2^{n/2(1-pq)} \left(e^S F_2\right)^{-(1-p)/(1-pq)} \varepsilon^{-2p(p-1)/\gamma(p,n)}.$$

Therefore the proof of the subcritical case is completed. □

Chapter 3

The lifespan of classical solutions to one dimensional wave equations with weighted nonlinear terms

In this chapter, we introduce the result of Wakasa [34].

3.1 Introduction

In this chapter, we are concerned with the estimates of the lifespan of classical solutions to the following Cauchy problem:

$$\begin{cases} u_{tt} - u_{xx} = H(x, u(x,t)), & (x,t) \in \mathbf{R} \times [0, \infty), \\ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x), & x \in \mathbf{R}, \end{cases} \tag{3.1.1}$$

where $u = u(x,t)$ is a scalar unknown function of space-time variables, $(f,g) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$ and $\varepsilon > 0$ is a "small" parameter. The nonlinear term, H is given by

$$H(x,u) = \frac{F(u(x,t))}{(1+x^2)^{(1+a)/2}}, \tag{3.1.2}$$

where $a \geq -1$, and $F(u) = |u|^p$ or $|u|^{p-1}u$ with $p > 1$. Let us define the lifespan T_ε of C^2-solution (classical solution) of (3.1.1) by

$$T_\varepsilon \equiv T_\varepsilon(f,g) := \sup\{T \in [0,\infty) \ : \ \text{There exists a unique solution } u \in C^2(\mathbf{R} \times [0,T]) \text{ of (3.1.1)}\}$$

with arbitrarily fixed (f, g).

Let us recall known results for the case $a = -1$ in general spatial dimensions from the section 2.2 in Chapter 2:

$$\begin{cases} u_{tt} - \Delta u = |u|^p \quad \text{in} \quad \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \ u_t(x, 0) = \varepsilon g(x), \quad \text{for } x \in \mathbf{R}^n, \end{cases}$$

where $n \geq 1$. When $n \geq 2$, let $p_0(n)$ is a positive root of (2.2.2). Then, $T_\varepsilon = \infty$ holds for "small" ε with compact support if $p > p_0(n)$, and $T_\varepsilon < \infty$ holds for "positive" (f, g) if $1 < p \leq p_0(n)$.

On the other hand, when $n = 1$, and (f, g) has a compact support and satisfies some positivity assumption, Kato [17] showed that $T_\varepsilon < \infty$ for any $p > 1$. The difference between the cases $n \geq 2$ and $n = 1$ comes from the fact that the solutions to the homogeneous wave equations has a decay estimate, $|u(x, t)| \leq (t + 1)^{-(n-1)/2}$. Especially, the solution does not have decay property when $n = 1$.

The result due to [17] motivates one to introduce a weight function $(1 + x^2)^{-(1+a)/2}$ in the nonlinearity for getting a global solution. Actually, Suzuki [31] showed that $T_\varepsilon = \infty$ with $F(u) = |u|^{p-1}u$ for $p > (1+\sqrt{5})/2$ and $pa > 1$ if f and g are odd functions and ε is small enough, and Kubo & Osaka & Yazici [20] have obtained the same conclusion for any $p > 1$ satisfying $pa > 1$. On the other hand, they showed that $T_\varepsilon < \infty$ for $F(u) = |u|^p$ with $p > 1$ and $a \geq -1$ if (f, g) satisfies $f \equiv 0$, $g(x) \geq 0$ for $x \in \mathbf{R}$, and $\int_{\delta/2}^{\delta} g(y)dy > 0$ with some $0 < \delta < 1$. Also, they obtained an upper bound of the lifespan, $T_\varepsilon \leq C\varepsilon^{-p^2}$, where C is a positive constant independent of ε. However, this estimate is not sharp at least in the case of $a = -1$ by Zhou's result (2.2.4).

Our purpose is to extend Zhou's result to the case where $a > -1$. To obtain a blow-up result, we require the following assumptions on the data:

$$\begin{aligned} &\text{Let } f \equiv 0 \text{ and } g \in C^1(\mathbf{R}) \text{ does not vanish identically.} \\ &\text{Assume } g(x) \geq 0 \text{ for all } x \in \mathbf{R} \text{ and } \int_{-1}^{1} g(y)dy > 0. \end{aligned} \tag{3.1.3}$$

Then, we have the following blow-up theorem.

Theorem 3.1.1 *Let $a \geq -1$ and $F(u) = |u|^{p-1}u$ or $|u|^p$ with $p > 1$. Assume (3.1.3). Then, there exist positive constants $\varepsilon_0 = \varepsilon_0(g, a, p)$ and $C = C(g, a, p)$ such that*

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-(p-1)/(1-a)} & \text{if } -1 \leq a < 0, \\ \phi^{-1}(C\varepsilon^{-(p-1)}) & \text{if } a = 0, \\ C\varepsilon^{-(p-1)} & \text{if } a > 0, \end{cases} \tag{3.1.4}$$

holds for any ε with $0 < \varepsilon \le \varepsilon_0$, where $\phi = \phi(s)$ is a function defined by $\phi(s) = s\log(2+s)$ for $s \ge 0$.

The proof of this theorem done by an iteration argument concerning pointwise estimates. Such kind of framework was introduced by John [13] in three space dimensions. The first step of the iteration argument comes from the linear estimate of the solution to the homogeneous wave equation from below. Kubo & Osaka & Yazici [20] obtained such an estimate only in a strip domain, $\{0 \le x - t \le \delta/2\}$, where $0 < \delta < 1$ is a constant. On the other hand, we are able to show a similar estimate in unbounded domain, $\{t - x \ge 1\}$. This improvement enable us to establish sharp upper bound of T_ε. See Lemma 3.3.2 and Remark 3.3.1 for details.

To show the optimality of the upper bounds in Theorem 3.1.1, we require the following assumptions on (f, g)

$$\begin{aligned} &f \in C^2(\mathbf{R}) \text{ and } g \in C^1(\mathbf{R}) \text{ satisfy } \|f\|_{L^\infty(\mathbf{R})} < \infty \\ &\text{and } \|g\|_{L^1(\mathbf{R})} < \infty. \end{aligned} \tag{3.1.5}$$

Then, we have the following theorem.

Theorem 3.1.2 *Let $a \ge -1$ and $F(u) = |u|^{p-1}u$ or $|u|^p$ with $p > 1$. Assume (3.1.5). Then, there exists a positive constant $c = c(f, g, a, p)$ such that*

$$T_\varepsilon \ge \begin{cases} c\varepsilon^{-(p-1)/(1-a)} & \text{if } -1 \le a < 0, \\ \phi^{-1}(c\varepsilon^{-(p-1)}) & \text{if } a = 0, \\ c\varepsilon^{-(p-1)} & \text{if } a > 0, \end{cases} \tag{3.1.6}$$

holds for $\varepsilon > 0$, where ϕ is the function in Theorem 3.1.1.

Remark 3.1.1 *One can easily generalize the assumption on F in Theorem 3.1.2 as follows:*

$$\begin{aligned} &F \in C^1(\mathbf{R}) \text{ satisfies } F(0) = F'(0) = 0 \text{ and} \\ &|F'(s)| \le pA|s|^{p-1} \text{ for } s \in \mathbf{R}, \text{ where } p > 1 \text{ and } A > 0. \end{aligned} \tag{3.1.7}$$

This chapter is organized as follows. In the next section, we prepare some notations. The upper bounds of the lifespan and lower bounds of the lifespan are obtained in Section 3.2 and Section 3.3, respectively.

3.2 Notations

In this section, we give some notations and definitions.

We define

$$u^0(x,t) = \frac{1}{2}\{f(x+t) + f(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy \tag{3.2.1}$$

and

$$L(V)(x,t) = \frac{1}{2}\iint_{D(x,t)} V(y,s)dyds \tag{3.2.2}$$

for $V \in C(\mathbf{R} \times [0,\infty))$, where

$$D(x,t) = \{(y,s) \in \mathbf{R} \times [0,\infty) : 0 \le s \le t, x-t+s \le y \le x+t-s\}.$$

For $(f,g) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$, if $u \in C(\mathbf{R} \times [0,\infty))$ is a solution of

$$u(x,t) = \varepsilon u^0(x,t) + L(H(\cdot,u))(x,t), \quad (x,t) \in \mathbf{R} \times [0,\infty), \tag{3.2.3}$$

then $u \in C^2(\mathbf{R} \times [0,\infty))$ is the solution to the initial value problem (3.1.1).

For $T > 0$, we define the following domains:

$$\begin{aligned} \Gamma_1 &= \{(x,t) \in [0,\infty) \times [0,T] : t-x \ge 1\}, \\ \Gamma_2 &= \{(x,t) \in [0,\infty) \times [0,T] : x \ge t-x \ge 1\}, \\ \Sigma_j &= \{(x,t) \in [0,\infty) \times [0,T] : t-x \ge l_j\}, \end{aligned} \tag{3.2.4}$$

where

$$\begin{cases} l_1 = 3 \\ l_j = l_1 + \displaystyle\sum_{k=1}^{j-1} 2^{-(k-1)} = l_1 + 2\left(1 - \frac{1}{2^{j-1}}\right) \quad \text{for } j \ge 2. \end{cases} \tag{3.2.5}$$

3.3 Upper bound of the lifespan

In this section, we prove Theorem 3.1.1. It is sufficient to show that the solution to the integral equation,

$$u(x,t) = \varepsilon u^0(x,t) + \frac{1}{2}\iint_{D(x,t)} \frac{|u(y,s)|^p dyds}{(1+y^2)^{(1+a)/2}}, \quad (x,t) \in \mathbf{R} \times [0,\infty), \tag{3.3.1}$$

blows up in finite time. Because, if $u \in C(\mathbf{R} \times [0,\infty))$ is a solution of (3.3.1), then u satisfies $u(x,t) \ge 0$ for $(x,t) \in \mathbf{R} \times [0,\infty)$ by the assumptions in (3.1.3). Therefore, this u must solve the equation (3.2.3) with $F(u) = |u|^{p-1}u$ by the uniqueness of solutions to (3.1.1).

Before proving Theorem 3.1.1, we prepare the following lemmas:

Lemma 3.3.1 *Let $p > 1$, $a \geq -1$ and let us define a sequence*

$$\begin{cases} C_{a,j} = \exp\{p^{j-1}(\log(C_{a,1}F_{p,a}^{-S_j}E_{p,a}^{1/(p-1)})) - \log E_{p,a}^{1/(p-1)}\} \ (j \geq 2), \\ C_{a,1} = c_0^p k_a \varepsilon^p, \end{cases} \tag{3.3.2}$$

where

$$E_{p,a} = \begin{cases} (p-1)^2/(2^{a+5}p^2), & \text{if } -1 \leq a < 0, \\ (p-1)^2/(2p^2), & \text{if } a = 0, \\ (p-1)/(2^{a+2}p). & \text{if } a > 0, \end{cases} \tag{3.3.3}$$

$$F_{p,a} = \begin{cases} p^2, & \text{if } -1 \leq a \leq 0, \\ 2p & \text{if } a > 0, \end{cases} \tag{3.3.4}$$

$$k_a = \begin{cases} 2^{-(a+4)}, & \text{if } -1 \leq a < 0, \\ 2^{-1}, & \text{if } a = 0, \\ 2^{-(a+2)}. & \text{if } a > 0, \end{cases} \tag{3.3.5}$$

and

$$S_j = \sum_{i=1}^{j-1} \frac{i}{p^i}. \tag{3.3.6}$$

Then, we have the following relation:

$$C_{a,j+1} = \frac{C_{a,j}^p E_{p,a}}{F_{p,a}^j} \quad (j \in \mathbf{N}). \tag{3.3.7}$$

Proof. First, we shall show (3.3.7) for $j = 1$. One can easily get

$$\begin{aligned} \log\left(\frac{C_{a,1}^p E_{p,a}}{F_{p,a}}\right) &= p\log(C_{a,1}F_{p,a}^{-1/p}) + \log E_{p,a} \\ &= p\log(C_{a,1}F_{p,a}^{-1/p}E_{p,a}^{1/(p-1)}) - \log E_{p,a}^{1/(p-1)} = \log C_{a,2}. \end{aligned}$$

Hence (3.3.7) holds for $j = 1$. Next, we shall show (3.3.7) for $j \geq 2$. Note that (3.3.7) is equivalent to

$$\log C_{a,j+1} = p\log C_{a,j} - j\log F_{p,a} + \log E_{p,a}.$$

By (3.3.2) and the expression of S_j in (3.3.6), the right-hand side of this identity is equal to

$$\begin{aligned} &p^j\{\log(C_{a,1}F_{p,a}^{-S_j}E_{p,a}^{1/(p-1)})\} - p\log E_{p,a}^{1/(p-1)} - j\log F_{p,a} + \log E_{p,a} \\ &= p^j\{\log(C_{a,1}F_{p,a}^{-S_{j+1}}E_{p,a}^{1/(p-1)})\} + p^j\log F_{p,a}^{j/p^j} - j\log F_{p,a} - \log E_{p,a}^{1/(p-1)} \\ &= p^j\{\log(C_{a,1}F_{p,a}^{-S_{j+1}}E_{p,a}^{1/(p-1)})\} - \log E_{p,a}^{1/(p-1)}. \end{aligned}$$

Hence, we obtain (3.3.7) by (3.3.2) with j replaced by $j+1$. This completes the proof. □

Next, we derive a lower bound of the solution to (3.3.1) which is a starting point of our iteration argument.

Lemma 3.3.2 *Suppose that the assumptions in Theorem 3.1.1 are fulfilled. Let $u \in C(\mathbf{R} \times [0,T])$ be the solution of (3.3.1). Then, u satisfies*

$$u(x,t) \geq \varepsilon c_0 \quad \textit{for} \quad (x,t) \in \Gamma_1, \tag{3.3.8}$$

where $c_0 = \dfrac{1}{2}\displaystyle\int_{-1}^{1} g(y)dy > 0$ *and* $\Gamma_1 (= \{(x,t) \in [0,\infty) \times [0,T] : t - x \geq 1\})$ *is the one in (3.2.4).*

Proof. By (3.1.3) and (3.2.1), we get

$$\varepsilon u^0(x,t) = \frac{\varepsilon}{2}\int_{x-t}^{x+t} g(y)dy \geq \varepsilon c_0 \quad \text{for} \quad (x,t) \in \Gamma_1.$$

Making use of the positivity of the second term of right-hand side in (3.3.1), we have (3.3.8). This completes the proof. □

Remark 3.3.1 *In three space dimensions, the following estimate which is necessary to get the first step of the iteration argument was obtained by John [13] in a strip domain: For $(x,t) \in S$, we have*

$$u^0(x,t) \geq Cr^{-1},$$

where $r = |x|$, C is a positive constant and $S = \{(r,t) \in (0,\infty) \times [0,\infty) : \delta \leq t - r \leq \delta'\}$, with some δ', δ $(\delta' > \delta > 0)$.

On the contrary, our estimate holds in some domain without any restriction of upper bound for $t - x$. This is the key point to obtain sharp upper bound of T_ε.

Our iteration argument will be done by using the following estimates.

Proposition 3.3.1 *Suppose that the assumptions in Theorem 3.1.1 are fulfilled. Let $j \in \mathbf{N}$ and let $u \in C(\mathbf{R} \times [0,T])$ be the solution of (3.3.1). Then, u satisfies*

$$u(x,t) \geq C_{a,j}\{(t-x)^{-(a+1)}(t-x-1)^2\}^{a_j} \quad \textit{if } -1 \leq a < 0, \tag{3.3.9}$$

for $(x,t) \in \Gamma_2$,

$$u(x,t) \geq C_{0,j}\{(t-x-1)\log(1+x)\}^{a_j} \quad \textit{if } a = 0, \tag{3.3.10}$$

for $(x,t) \in \Gamma_1$, and

$$u(x,t) \geq C_{a,j}(t-x-l_j)^{a_j} \quad \textit{if } a > 0, \tag{3.3.11}$$

for $(x,t) \in \Sigma_j$, where Γ_1, Γ_2 and Σ_j are defined in (3.2.4). Here $C_{a,j}$ is the one in (3.3.2) with $c_0 = \dfrac{1}{2}\displaystyle\int_{-1}^{1} g(y)dy > 0$ and a_j is defined by

$$a_j = \frac{p^j - 1}{p - 1} \quad (j \in \mathbf{N}). \tag{3.3.12}$$

Proof. We shall show (3.3.9), (3.3.10) and (3.3.11) by induction. Noticing that $u^0(x,t) \geq 0$ for $(x,t) \in \mathbf{R} \times [0,\infty)$ and $(1+y^2)^{1/2} \leq 1 + |y|$, we get

$$u(x,t) \geq \frac{1}{2} \iint_{D(x,t)} \frac{|u(y,s)|^p}{(1+|y|)^{1+a}} dyds \quad \text{in } \mathbf{R} \times [0,\infty). \tag{3.3.13}$$

(i) Estimate in the case of $-1 \leq a < 0$.
Let $(x,t) \in \Gamma_2$. Define

$$T_1(x,t) := \{(y,s) \in D(x,t) : 1 \leq s - y \leq t - x \leq y, s + y \leq t + x\}.$$

Changing the variables in the integral of (3.3.13) by

$$\alpha = s + y, \ \beta = s - y, \tag{3.3.14}$$

(Jacobian, $\partial(s,y)/\partial(\alpha,\beta)$ is 1/2.) and replacing the domain of integration by $T_1(x,t)$, we get

$$u(x,t) \geq \frac{1}{4} \int_1^{t-x} d\beta \int_{2(t-x)+\beta}^{t+x} \frac{|u(y,s)|^p}{\{1 + (\alpha - \beta)/2\}^{1+a}} d\alpha \quad \text{in } \Gamma_2. \tag{3.3.15}$$

Making use of (3.3.8) and $T_1(x,t) \subset \Gamma_1$ for $(x,t) \in \Gamma_2$, we have

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x} d\beta \int_{2(t-x)+\beta}^{t+x} \frac{d\alpha}{\{1 + (\alpha - \beta)/2\}^{1+a}} \quad \text{in } \Gamma_2.$$

Note that $x \geq t - x$ is equivalent to $t + x \geq 3(t - x)$, we get

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{4} \int_1^{t-x} d\beta \int_{2(t-x)+\beta}^{3(t-x)} \frac{d\alpha}{\{1 + (\alpha - \beta)/2\}^{1+a}} \quad \text{in } \Gamma_2.$$

It follows from

$$1 + \frac{\alpha - \beta}{2} \leq 1 + \frac{3(t-x) - 1}{2} \leq 2(t - x) \tag{3.3.16}$$

for $\alpha \le 3(t-x)$, $\beta \ge 1$ and $t-x \ge 1$ that

$$u(x,t) \ge \frac{c_0^p \varepsilon^p}{2^{a+3}(t-x)^{1+a}} \int_1^{t-x} (t-x-\beta) d\beta = C_{a,1} \frac{(t-x-1)^2}{(t-x)^{1+a}} \quad \text{in } \Gamma_2.$$

Therefore, (3.3.9) holds for $j=1$.

Assume that (3.3.9) holds. Noticing that $T_1(x,t) \subset \Gamma_2$ for $(x,t) \in \Gamma_2$ and putting (3.3.9) into (3.3.15), we have

$$u(x,t) \ge \frac{C_{a,j}^p}{4} \int_1^{t-x} \frac{(\beta-1)^{2pa_j}}{\beta^{p(a+1)a_j}} d\beta \int_{2(t-x)+\beta}^{t+x} \frac{d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Gamma_2.$$

It follows from (3.3.16) that

$$u(x,t) \ge \frac{C_{a,j}^p}{2^{a+3}(t-x)^{a+1}} \int_1^{t-x} \frac{(\beta-1)^{2pa_j}}{\beta^{p(a+1)a_j}} d\beta \int_{2(t-x)+\beta}^{3(t-x)} d\alpha.$$

Noticing that $\beta \le t-x$, we have

$$\begin{aligned} u(x,t) &\ge \frac{C_{a,j}^p}{2^{a+3}(t-x)^{(a+1)(pa_j+1)}} \int_1^{t-x} (\beta-1)^{2pa_j} d\beta \int_{2(t-x)+\beta}^{3(t-x)} d\alpha \\ &= \frac{C_{a,j}^p}{2^{a+3}(t-x)^{(a+1)(pa_j+1)}} \int_1^{t-x} (\beta-1)^{2pa_j} (t-x-\beta) d\beta \end{aligned}$$

in Γ_2. Making use of integration by parts to the integral above, we have

$$\begin{aligned} \int_1^{t-x} (\beta-1)^{2pa_j}(t-x-\beta) d\beta &= \frac{1}{2pa_j+1} \int_1^{t-x} (\beta-1)^{2pa_j+1} d\beta \\ &= \frac{(t-x-1)^{2(pa_j+1)}}{2(2pa_j+1)(pa_j+1)} \ge \frac{(t-x-1)^{2(pa_j+1)}}{2^2(pa_j+1)^2}. \end{aligned}$$

Hence we get

$$u(x,t) \ge \frac{C_{a,j}^p (t-x-1)^{2(pa_j+1)}}{2^{a+5}(pa_j+1)^2(t-x)^{(a+1)(pa_j+1)}} \quad \text{in } \Gamma_2.$$

Recalling the definition of a_j, we have

$$a_{j+1} = pa_j + 1 \le \frac{p^{j+1}}{p-1}. \tag{3.3.17}$$

Making use of (3.3.7), we get

$$u(x,t) \ge \frac{C_{a,j}^p (p-1)^2}{2^{a+5} p^{2(j+1)}} \cdot \frac{(t-x-1)^{2a_{j+1}}}{(t-x)^{(a+1)a_{j+1}}} = C_{a,j+1} \frac{(t-x-1)^{2a_{j+1}}}{(t-x)^{(a+1)a_{j+1}}}$$

in Γ_2. Therefore, (3.3.9) holds for all $j \in \mathbf{N}$.

(ii) Estimate in the case of $a = 0$.

Let $(x,t) \in \Gamma_1$. Define

$$T_2(x,t) := \{(y,s) \in D(x,t) : 1 \le s-y \le t-x, s+y \le t+x, y \ge 0\}.$$

Changing the variables by (3.3.14) in the integral of (3.3.13) and replacing the domain of integration by $T_2(x,t)$, we get

$$u(x,t) \ge \frac{1}{4}\int_1^{t-x} d\beta \int_\beta^{t+x} \frac{|u(y,s)|^p}{1+(\alpha-\beta)/2} d\alpha \quad \text{in } \Gamma_1. \tag{3.3.18}$$

By making use of (3.3.8) and $T_2(x,t) \subset \Gamma_1$ for $(x,t) \in \Gamma_1$, we get

$$u(x,t) \ge \frac{c_0^p \varepsilon^p}{4}\int_1^{t-x} d\beta \int_\beta^{t+x} \frac{d\alpha}{1+(\alpha-\beta)/2} \quad \text{in } \Gamma_1.$$

Noticing that

$$\int_\beta^{t+x} \frac{d\alpha}{1+(\alpha-\beta)/2} = 2\log\left(1+\frac{t+x-\beta}{2}\right) \ge 2\log(1+x),$$

for $\beta \le t-x$, we obtain

$$u(x,t) \ge \frac{c_0^p \varepsilon^p}{2}\log(1+x)\int_1^{t-x} d\beta = C_{0,1}(t-x-1)\log(1+x) \quad \text{in } \Gamma_1.$$

Therefore, (3.3.10) holds for $j = 1$.

Assume that (3.3.10) holds. Noticing that $T_2(x,t) \subset \Gamma_1$ for $(x,t) \in \Gamma_1$ and putting (3.3.10) into (3.3.18), we have

$$u(x,t) \ge \frac{C_{0,j}^p}{4}\int_1^{t-x} (\beta-1)^{pa_j} d\beta \int_\beta^{t+x} \frac{\{\log(1+(\alpha-\beta)/2)\}^{pa_j}\, d\alpha}{1+(\alpha-\beta)/2} \quad \text{in } \Gamma_1.$$

The α-integral yields

$$\int_\beta^{t+x} \frac{\{\log(1+(\alpha-\beta)/2)\}^{pa_j}\, d\alpha}{1+(\alpha-\beta)/2} = \frac{2}{pa_j+1}\left\{\log\left(1+\frac{t+x-\beta}{2}\right)\right\}^{pa_j+1}.$$

Then, we get

$$\begin{aligned} u(x,t) &\ge \frac{C_{0,j}^p}{2(pa_j+1)}\int_1^{t-x} (\beta-1)^{pa_j}\left\{\log\left(1+\frac{t+x-\beta}{2}\right)\right\}^{pa_j+1} d\beta \\ &\ge \frac{C_{0,j}^p\{\log(1+x)\}^{pa_j+1}}{2(pa_j+1)}\int_1^{t-x} (\beta-1)^{pa_j} d\beta \end{aligned}$$

in Γ_1. It follows from (3.3.17) and (3.3.7) that

$$\begin{aligned} u(x,t) &\geq \frac{C_{0,j}^p (p-1)^2}{2p^{2(j+1)}} \cdot \{(t-x-1)\log(1+x)\}^{a_{j+1}} \\ &= C_{0,j+1}\{(t-x-1)\log(1+x)\}^{a_{j+1}} \end{aligned}$$

in Γ_1. Therefore, (3.3.10) holds for all $j \in \mathbf{N}$.

(iii) Estimate in the case of $a > 0$.

Let $(x,t) \in \Sigma_1$. Define

$$L_1(x,t) := \{(y,s) \in D(x,t) : 1 \leq s-y \leq t-x-2, 0 \leq y \leq 1\}.$$

Changing the variables by (3.3.14) in the integral of (3.3.13) and replacing the domain of integration by $L_1(x,t)$, we get

$$u(x,t) \geq \frac{1}{4}\int_1^{t-x-2} d\beta \int_\beta^{2+\beta} \frac{|u(y,s)|^p d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Sigma_1.$$

By making use of (3.3.8) and $L_1(x,t) \subset \Gamma_1$ for $(x,t) \in \Sigma_1$, we have

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{4}\int_1^{t-x-2} d\beta \int_\beta^{2+\beta} \frac{d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Sigma_1.$$

It follows from $1+(\alpha-\beta)/2 \leq 2$ for $\alpha \leq 2+\beta$ that

$$u(x,t) \geq \frac{c_0^p \varepsilon^p}{2^{a+2}}\int_1^{t-x-2} d\beta = C_{a,1}(t-x-3) \quad \text{in } \Sigma_1.$$

Therefore, (3.3.11) holds for $j = 1$.

Assume that (3.3.11) holds. Let $(x,t) \in \Sigma_{j+1}$. Define

$$L_j(x,t) := \{(y,s) \in D(x,t) : l_j \leq s-y \leq t-x-2^{-(j-1)}, 0 \leq y \leq 2^{-j}\}$$

for $j \geq 1$, where l_j is defined in (3.2.5). Making use of (3.3.14) and replacing the domain of integration in (3.3.13) by $L_j(x,t)$, we have

$$u(x,t) \geq \frac{1}{4}\int_{l_j}^{t-x-2^{-(j-1)}} d\beta \int_\beta^{2^{-(j-1)}+\beta} \frac{|u(y,s)|^p d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}} \quad \text{in } \Sigma_{j+1}.$$

Noticing that $L_j(x,t) \subset \Sigma_j$ for $(x,t) \in \Sigma_{j+1}$ and putting (3.3.11) into the integral above, we have

$$u(x,t) \geq \frac{C_{a,j}^p}{4}\int_{l_j}^{t-x-2^{-(j-1)}} (\beta-l_j)^{pa_j} d\beta \int_\beta^{2^{-(j-1)}+\beta} \frac{d\alpha}{\{1+(\alpha-\beta)/2\}^{1+a}}$$

in Σ_{j+1}. Note that

$$1+\frac{\alpha-\beta}{2}\le 1+\frac{1}{2^j}\le 2$$

for $\alpha\le 2^{-(j-1)}+\beta$, we get

$$u(x,t)\ge \frac{C_{a,j}^p}{2^{a+2+j}}\int_{l_j}^{t-x-2^{-(j-1)}}(\beta-l_j)^{pa_j}d\beta \quad \text{in } \Sigma_{j+1}.$$

It follows from $l_j+2^{-(j-1)}=l_{j+1}$, (3.3.17) and (3.3.7) that

$$u(x,t)\ge \frac{(p-1)C_{a,j}^p}{2^{a+2+j}p^{j+1}}\cdot(t-x-l_{j+1})^{a_{j+1}}=C_{a,j+1}(t-x-l_{j+1})^{a_{j+1}}$$

in Σ_{j+1}. Therefore, (3.3.11) holds for all $j\in\mathbf{N}$. The proof of Proposition 3.3.1 is now completed. □

End of the proof of Theorem 3.1.1. Let $u\in C(\mathbf{R}\times[0,T])$ be the solution of the integral equation, (3.3.1). Setting $S=\lim_{j\to\infty}S_j$, we see from (3.3.6) that $S_j\le S$ for all $j\in\mathbf{N}$. Therefore, (3.3.2) yields

$$\begin{aligned} C_{a,j} &\ge \exp\{p^{j-1}\{\log(C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}-\log E_{p,a}^{1/(p-1)}\} \\ &= E_{p,a}^{-1/(p-1)}\exp\{p^{j-1}\{\log(C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}\}. \end{aligned} \tag{3.3.19}$$

(i) The lifespan in the case of $-1\le a<0$.
We take $\varepsilon_0=\varepsilon_0(g,a,p)>0$ so small that

$$B_1\varepsilon_0^{-(p-1)/(1-a)}\ge 4,$$

where we set

$$B_1=(c_0^p2^{-(a+4)+p(a-3)/(p-1)}p^{-2S}E_{p,a}^{1/(p-1)})^{-(p-1)/p(1-a)}>0.$$

Next, for a fixed $\varepsilon\in(0,\varepsilon_0]$, we suppose that T satisfies

$$T>B_1\varepsilon^{-(p-1)/(1-a)}\ (\ge 4). \tag{3.3.20}$$

Combining (3.3.19) with (3.3.9), we have

$$\begin{aligned} u(x,t)\ge{}& E_{p,a}^{-1/(p-1)}\exp\{p^{j-1}\{\log(C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}\} \\ &\times\left\{\frac{(t-x-1)^2}{(t-x)^{(1+a)}}\right\}^{(p^j-1)/(p-1)} \end{aligned}$$

in Γ_2. Note that $t-x-1 \geq (t-x)/2$ is equivalent to $t-x \geq 2$. Furthermore, we have $(t/2,t) \in \Gamma_2$ for $t \in [4,T]$. Hence we get

$$\begin{aligned} u(t/2,t) &\geq (2^{a-3}E_{p,a})^{-1/(p-1)} \exp\{p^{j-1}\{\log(2^{p(a-3)/(p-1)}C_{a,1}F_{p,a}^{-S}E_{p,a}^{1/(p-1)})\}\} \\ &\quad \times t^{(1-a)(p^j-1)/(p-1)} \\ &= (2^{a-3}E_{p,a})^{-1/(p-1)} \exp\{p^{j-1}K_1(t)\}t^{-(1-a)/(p-1)} \end{aligned}$$

for $t \in [4,T]$, where we set

$$K_1(t) = \log\left(\varepsilon^p c_0^p 2^{-(a+4)+p(a-3)/(p-1)} p^{-2S} E_{p,a}^{1/(p-1)} t^{p(1-a)/(p-1)}\right)$$

(recall (3.3.4) and (3.3.5)).

By (3.3.20) and the definition of B_1, we have $K_1(T) > 0$. Therefore we get $u(T/2,T) \to \infty$ as $j \to \infty$. Hence, (3.3.20) implies that $T_\varepsilon \leq B_1\varepsilon^{-(p-1)/(1-a)}$ for $0 < \varepsilon \leq \varepsilon_0$.

(ii) The lifespan in the case of $a = 0$.

We take $\varepsilon_1 = \varepsilon_1(g,p) > 0$ so small that

$$\phi^{-1}(B_2\varepsilon_1^{-(p-1)}) \geq 4,$$

where ϕ is the one in Theorem 3.1.1 and

$$B_2 = (c_0^p 2^{-1-3p/(p-1)} p^{-2S} E_{p,0}^{1/(p-1)})^{-(p-1)/p} > 0.$$

Next, for a fixed $\varepsilon \in (0,\varepsilon_1]$, we suppose that T satisfies

$$T > \phi^{-1}(B_2\varepsilon^{-(p-1)}) \ (\geq 4). \tag{3.3.21}$$

Combining the estimates (3.3.19) and (3.3.10), we have

$$\begin{aligned} u(t/2,t) &\geq (2^{-2}E_{p,0})^{-1/(p-1)} \exp\{p^{j-1}\{\log(\varepsilon^p c_0^p 2^{-1-2p/(p-1)} p^{-2S} E_{p,0}^{1/(p-1)})\}\} \\ &\quad \times \{t\log(1+t/2)\}^{(p^j-1)/(p-1)} \end{aligned}$$

for $4 \leq t \leq T$. Noticing that

$$\log\left(1+\frac{t}{2}\right) = \log(2+t) - \log 2 \geq \frac{\log(2+t)}{2} \quad \text{for } t \geq 2,$$

we get

$$u(t/2,t) \geq (2^{-3}E_{p,0})^{-1/(p-1)} \exp\{p^{j-1}K_2(t)\}\phi(t)^{-1/(p-1)}$$

for $4 \leq t \leq T$, where we set

$$K_2(t) = \log\left(\varepsilon^p c_0^p 2^{-1-3p/(p-1)} p^{-2S} E_{p,0}^{1/(p-1)} \{\phi(t)\}^{p/(p-1)}\right).$$

Analogously to the case of $-1 \le a < 0$, we have $K_2(T) > 0$ by (3.3.21) and the definition of B_2. Therefore we get $u(T/2, T) \to \infty$ as $j \to \infty$. Hence, (3.3.21) implies that $T_\varepsilon \le \phi^{-1}(B_2 \varepsilon^{-(p-1)})$ for $0 < \varepsilon \le \varepsilon_1$.

(iii) The lifespan in the case of $a > 0$.

We take $\varepsilon_2 = \varepsilon_2(g, a, p) > 0$ so small that

$$B_3 \varepsilon_2^{-(p-1)} \ge 20,$$

where we set

$$B_3 = (c_0^p 2^{-(a+2)-2p/(p-1)} (2p)^{-S} E_{p,a}^{1/(p-1)})^{-(p-1)/p} > 0.$$

Next, for a fixed $\varepsilon \in (0, \varepsilon_2]$, we suppose that T satisfies

$$T > B_3 \varepsilon^{-(p-1)} \ (\ge 20). \tag{3.3.22}$$

Combining the estimates (3.3.19) with (3.3.11), we have

$$u(t/2, t) \ge (2^{-2} E_{p,a})^{-1/(p-1)} \exp\{p^{j-1} K_3(t)\} t^{-1/(p-1)}$$

for $20 \le t \le T$, where we set

$$K_3(t) = \log\left(\varepsilon^p c_0^p 2^{-(a+2)-2p/(p-1)} (2p)^{-S} E_{p,a}^{1/(p-1)} t^{p/(p-1)}\right).$$

Since $K_3(T) > 0$, by (3.3.22) and the definition of B_3, we get $u(T/2, T) \to \infty$ as $j \to \infty$. Hence, (3.3.22) implies that $T_\varepsilon \le B_3 \varepsilon^{-(p-1)}$ for $0 < \varepsilon \le \varepsilon_2$. Therefore, the proof of Theorem 3.1.1 is now completed. □

3.4 Lower bound of the lifespan

In this section, we prove Theorem 3.1.2. First of all, we introduce a Banach space

$$X = \{u \in C(\mathbf{R} \times [0, T]) : \|u\|_{L^\infty(\mathbf{R} \times [0,T])} < \infty\}, \tag{3.4.1}$$

which is equipped with a norm

$$\|u\|_{L^\infty(\mathbf{R} \times [0,T])} = \sup_{(x,t) \in \mathbf{R} \times [0,T]} |u(x, t)|. \tag{3.4.2}$$

We shall construct a solution of the integral equation (3.2.3) in X under suitable assumption on T such as (3.4.7) below. Define a sequence of functions $\{u_n\}_{n \in \mathbf{N}}$ by

$$u_n = u_0 + L(H(\cdot, u_{n-1})), \quad u_0 = \varepsilon u^0, \tag{3.4.3}$$

where L, H and u^0 are given by (3.2.2), (3.1.2) and (3.2.1), respectively. Since $\|u_0\|_{L^\infty(\mathbf{R}\times[0,T])} \le M\varepsilon$, where $M = \|f\|_{L^\infty(\mathbf{R})} + \|g\|_{L^1(\mathbf{R})}$ by (3.2.1), we have $u_0 \in X$.

The following *a priori* estimate plays a key role in the proof of Theorem 3.1.2.

Lemma 3.4.1 *Let $V \in X$, $a \ge -1$, and let $D = D(\tau)$ is a function defined by*

$$D(\tau) = \begin{cases} (1+\tau)^{1-a} & \text{if } -1 \le a < 0, \\ \phi(\tau) & \text{if } a = 0, \\ 1+\tau & \text{if } a > 0, \end{cases} \tag{3.4.4}$$

for $\tau \ge 0$, where ϕ is the one in Theorem 3.1.1. Then, there exists a positive constant C_a such that

$$\left\| L\left(\frac{V}{(1+|\cdot|^2)^{(1+a)/2}} \right) \right\|_{L^\infty(\mathbf{R}\times[0,T])} \le C_a D(T) \|V\|_{L^\infty(\mathbf{R}\times[0,T])}. \tag{3.4.5}$$

Proof. Noticing that $(1+y^2) \ge (1+|y|)^2/2$, the left-hand side in (3.4.5) is dominated by

$$C_a \|V\|_{L^\infty(\mathbf{R}\times[0,T])} \iint_{D(x,t)} \frac{dyds}{\langle y\rangle^{1+a}},$$

where we set $\langle y\rangle = 1+|y|$. Thus, it is enough to show the inequality,

$$I(x,t) \le C_a D(T) \quad \text{for } (x,t) \in \mathbf{R}\times[0,T], \tag{3.4.6}$$

where we set

$$I(x,t) = \iint_{D(x,t)} \frac{dyds}{\langle y\rangle^{1+a}}.$$

We may assume $x \ge 0$. Because $I(x,t)$ is an even function with respect to x. When $t \ge x \ge 0$, we divide the integral domain $D(x,t)$ into two parts $D_j(x,t)$ $(j = 1,2)$, where

$$D_1(x,t) = \{(y,s) \in \mathbf{R}\times[0,\infty) : 0 \le s \le t-x, x-t+s \le y \le t-x-s\},$$
$$D_2(x,t) = \{(y,s) \in [0,\infty)^2 : 0 \le s \le t, |x-t+s| \le y \le x+t-s\}.$$

Namely, we set

$$I_j(x,t) = \iint_{D_j(x,t)} \frac{1}{\langle y\rangle^{1+a}} dyds \quad (j = 1,2),$$

so that $I(x,t) = I_1(x,t) + I_2(x,t)$. We shall estimate I_1. Since $\langle y\rangle$ is an even function, we obtain

$$I_1(x,t) = 2\int_0^{t-x} ds \int_0^{t-x-s} \frac{dy}{(1+y)^{1+a}} \quad \text{for } t \ge x \ge 0.$$

Then, the y-integral is dominated by

$$\begin{cases} -a^{-1}(1+t-x)^{-a} & \text{if } a<0, \\ \log(1+t-x) & \text{if } a=0, \\ a^{-1} & \text{if } a>0. \end{cases}$$

Hence, we get

$$I_1(x,t) \le C_a D(t-x) \le C_a D(T) \quad \text{for } 0 \le x \le t \le T.$$

Next, we shall estimate I_2. It follows that

$$I_2(x,t) = \int_0^t ds \int_{|x-t+s|}^{t-x-s} \frac{dy}{(1+y)^{1+a}} \le \int_0^t ds \int_0^{t+x-s} \frac{dy}{(1+y)^{1+a}}$$

for $t \ge x \ge 0$, and that the y-integral is dominated by

$$\begin{cases} -a^{-1}(1+t+x)^{-a} & \text{if } a<0, \\ \log(1+t+x) & \text{if } a=0, \\ a^{-1} & \text{if } a>0. \end{cases}$$

Noticing that

$$\log(1+2t) \le \log 2 + \log(2+t) \le 2\log(2+t) \quad \text{for } t \ge 0,$$

we get

$$I_2(x,t) \le C_a D(t+x) \le C_a D(T) \quad \text{for } 0 \le x \le t \le T.$$

When $x \ge t$, we have

$$I(x,t) \le \int_0^t \frac{ds}{(1+s)^{1+a}} \int_{x-t+s}^{x+t-s} dy \le 2t \int_0^t \frac{ds}{(1+s)^{1+a}} \le C_a D(T).$$

Therefore, the proof of Lemma 3.4.1 is ended. □

Now, we move on to the proof of Theorem 3.1.2. First of all, we take $T>0$ such that

$$2^{p+1} p C_a D(T) M^{p-1} \varepsilon^{p-1} \le 1, \tag{3.4.7}$$

where C_a is the one in Lemma 3.4.1. We shall show

$$\|u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \le 2M\varepsilon \quad (n \in \mathbf{N}), \tag{3.4.8}$$

by induction. Assume that $\|u_{n-1}\|_{L^\infty(\mathbf{R}\times[0,T])} \le 2M\varepsilon$ $(n \ge 2)$. It follows from (3.4.3) and Lemma 3.4.1 that

$$\begin{aligned} \|u_n\|_{L^\infty(\mathbf{R}\times[0,T])} &\le \|u_0\|_{L^\infty(\mathbf{R}\times[0,T])} + \|L(H(\cdot,u_{n-1}))\|_{L^\infty(\mathbf{R}\times[0,T])} \\ &\le M\varepsilon + C_a D(T)\|u_{n-1}\|^p_{L^\infty(\mathbf{R}\times[0,T])}. \end{aligned}$$

The assumption of the induction yields that

$$\|u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \le M\varepsilon + C_a(2M\varepsilon)^p D(T).$$

This inequality shows (3.4.8), provided (3.4.7) holds.

Next we shall estimate the differences of $\{u_n\}_{n\in\mathbf{N}}$. Since

$$\begin{aligned}|H(y,u_n) - H(y,u_{n-1})| &\le \left|\frac{p}{(1+y^2)^{(1+a)/2}}\int_{u_{n-1}(y,s)}^{u_n(y,s)} |\eta|^{p-1}d\eta\right| \\ &\le \frac{p}{(1+y^2)^{(1+a)/2}}(|u_{n-1}(y,s)|^{p-1} + |u_n(y,s)|^{p-1}) \\ &\quad \times|u_n(y,s) - u_{n-1}(y,s)|\end{aligned}$$

for $(y,s) \in \mathbf{R}\times[0,\infty)$, we see from Lemma 3.4.1 that

$$\begin{aligned}&\|u_{n+1} - u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \\ &\le pC_aD(T)(\|u_n\|^{p-1}_{L^\infty(\mathbf{R}\times[0,T])} + \|u_{n-1}\|^{p-1}_{L^\infty(\mathbf{R}\times[0,T])})\|u_n - u_{n-1}\|_{L^\infty(\mathbf{R}\times[0,T])}.\end{aligned}$$

Making use of (3.4.8), we have

$$\|u_{n+1} - u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \le \frac{1}{2}\|u_n - u_{n-1}\|_{L^\infty(\mathbf{R}\times[0,T])} \quad \text{for } n \in \mathbf{N}$$

provided (3.4.7) holds. Hence, we obtain

$$\|u_{n+1} - u_n\|_{L^\infty(\mathbf{R}\times[0,T])} \le \frac{1}{2^n}\|u_1 - u_0\|_{L^\infty(\mathbf{R}\times[0,T])} \quad \text{for } n \in \mathbf{N}.$$

Therefore, $\{u_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in X provided (3.4.7) holds. Since X is complete, there exists $u \in X$ such that u_n converges uniformly to u in X. Therefore, by taking limits under the integral sign, u satisfies the integral equation (3.2.3), so that u is the C^2-solution of (3.1.1). Hence, the proof of Theorem 3.1.2 is completed. □

Appendix

The aim of this appendix is to derive a representation formula for the solution to the homogeneous wave equation. The proof refers to Courant and Hilbert [4] and Yajima [35].

From now on, we assume that $n \geq 2$ and write $n = 2m,\ 2m+1$ $(m = 1, 2, 3, \cdots)$. Because, the representation formula in one space dimension is written in (3.2.1). We consider the following initial value problem for the wave equation:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = 0,\ u_t(x, 0) = g(x), & x \in \mathbf{R}^n. \end{cases} \tag{A.1}$$

Then, we have the following representation formula for the solutions to (A.1).

Theorem A.1 *Let $g \in C_0^\infty(\mathbf{R}^n)$. Then, the solution of (A.1) is represented as*

$$u(x,t) = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (t^2 - r^2)^{(n-3)/2} r Q_n(x, r) dr, \tag{A.2}$$

where

$$Q_n(x, r) = \frac{1}{\omega_n} \int_{|\omega|=1} g(x + r\omega) dS_\omega, \tag{A.3}$$

and ω_n is the area of the unit sphere in $\mathbf{R}^n$, i.e.

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \begin{cases} \dfrac{2(2\pi)^m}{(2m-1)!!} & \textit{for } n = 2m+1, \\ \dfrac{2\pi^m}{(m-1)!} & \textit{for } n = 2m. \end{cases}$$

Moreover, if $n = 2m+1$ we have

$$u(x,t) = \frac{1}{(n-2)!!} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-3)/2} \left\{ t^{n-2} Q_n(x,t) \right\}, \tag{A.4}$$

and

$$u(x,t) = \frac{1}{(n-2)!!} \sum_{k=0}^{(n-3)/2} \alpha_{(n-3)/2,k} t^{k+1} \left(\frac{\partial}{\partial t} \right)^k Q_n(x,t), \tag{A.5}$$

where $\alpha_{(n-3)/2,k}$ *is a suitable constant satisfying* $\alpha_{(n-3)/2,0} = (n-2)!!$.

If $n = 2m$, *we have*

$$u(x,t) = \frac{1}{(n-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\left\{t^{n-1}H_n(x,t)\right\}, \tag{A.6}$$

where

$$H_n(x,t) = \frac{2}{\omega_{n+1}}\int_{|\xi|\le 1}\frac{g(x+t\xi)d\xi}{\sqrt{1-|\xi|^2}}, \tag{A.7}$$

and

$$u(x,t) = \frac{1}{(n-1)!!}\sum_{k=0}^{(n-2)/2}\alpha_{(n-2)/2,k}t^{k+1}\left(\frac{\partial}{\partial t}\right)^k H_n(x,t), \tag{A.8}$$

where $\alpha_{(n-2)/2,k}$ *is a suitable constant satisfying* $\alpha_{(n-2)/2,0} = (n-1)!!$.

Remark A.1 *It is sufficient to consider the case of* $u(x,0) \equiv 0$ *by the following observation. Let* u *be the solution of (A.1). Set* $v(x,t) = \partial_t u(x,t)$. *Noticing that* $u_{tt}(x,0) = \Delta u(x,0) \equiv 0$, v *satisfies*

$$\begin{cases} v_{tt} - \Delta v = 0, & \text{in } \mathbf{R}^n \times [0,\infty), \\ v(x,0) = g(x),\ v_t(x,0) = 0, & x \in \mathbf{R}^n. \end{cases}$$

Therefore, the solution of

$$\begin{cases} w_{tt} - \Delta w = 0, & \text{in } \mathbf{R}^n \times [0,\infty), \\ w(x,0) = f(x),\ w_t(x,0) = g(x), & x \in \mathbf{R}^n \end{cases}$$

is represented by $w(x,t) = \widetilde{v}(x,t) + u(x,t)$, *where* u *is the solution of (A.1) and* $\widetilde{v}(x,t)$ *is a solution of*

$$\begin{cases} \widetilde{v}_{tt} - \Delta \widetilde{v} = 0, & \text{in } \mathbf{R}^n \times [0,\infty), \\ \widetilde{v}(x,0) = f(x),\ \widetilde{v}_t(x,0) = 0, & x \in \mathbf{R}^n. \end{cases}$$

Before proving the theorem, we state the following lemma:

Lemma A.1 (F. John [12]) *Let* $\phi \in C(\mathbf{R})$. *Then we have*

$$\int_{|\omega|=1}\phi(y\cdot\omega)dS_\omega = \omega_{n-1}\int_{-1}^{1}(1-p^2)^{(n-3)/2}\phi(p|y|)dp \tag{A.9}$$

for $y \in \mathbf{R}^n$.

See [12] for the proof of this lemma.

Proof of (A.2). We introduce the Fourier transform for $\psi \in \mathcal{S}(\mathbf{R}^n)$ by

$$\hat{\psi}(\xi) = \int_{\mathbf{R}^n} \psi(x) e^{-ix\cdot\xi} dx. \tag{A.10}$$

Applying the Fourier transform to (A.1) with respect to the variable x, we have

$$\begin{cases} \partial_t^2 \hat{u} + |\xi|^2 \hat{u} = 0, \\ \hat{u}(\xi, 0) = 0, \partial_t \hat{u}(\xi, 0) = \hat{g}(\xi). \end{cases} \tag{A.11}$$

This is the initial value problem for an ordinary differential equation with parameter ξ. Thus, we get

$$\hat{u}(\xi, t) = C_1(\xi) \cos|\xi|t + C_2(\xi) \sin|\xi|t,$$

where $C_1(\xi)$ and $C_2(\xi)$ are constants independent of t. By the initial condition in (A.11), we see that $C_1(\xi) = \hat{u}(\xi, 0) = 0$ and $C_2(\xi) = \partial_t \hat{u}(\xi, 0)/|\xi| = \hat{g}(\xi)/|\xi|$. Hence we get

$$\hat{u}(\xi, t) = \frac{\sin|\xi|t}{|\xi|} \hat{g}(\xi).$$

By using the inverse Fourier transform, we have

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \frac{\sin|\xi|t}{|\xi|} \hat{g}(\xi) d\xi. \tag{A.12}$$

Since $g \in \mathcal{S}(\mathbf{R}^n)$, we have

$$u(x, t) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi - \varepsilon|\xi|} \frac{\sin|\xi|t}{|\xi|} \hat{g}(\xi) d\xi.$$

First we consider the case of $n = 2m + 1$. For $\varepsilon > 0$, we define

$$E_\varepsilon(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi - \varepsilon|\xi|} \frac{\sin|\xi|t}{|\xi|} d\xi. \tag{A.13}$$

Note that the integrand is an integrable function. Then, we get

$$u(x, t) = \lim_{\varepsilon \to 0} \int_{\mathbf{R}^n} E_\varepsilon(x - y, t) g(y) dy.$$

Hence the proof of (A.2) reduce to the following proposition:

Proposition A.1 *Let $n = 2m+1$ $(m = 1, 2, 3, \cdots)$, and let E_ε is the function defined in (A.13). Then we have*

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbf{R}^n} & E_\varepsilon(x-y,t)g(y)dy \\ &= \frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}}\int_0^t (t^2-r^2)^{(n-3)/2} r Q_n(x,r)dr,\end{aligned} \tag{A.14}$$

where Q_n is defined in (A.3).

Proof. Introducing the polor coordinates $\xi = \rho\omega$ $(|\omega| = 1)$ in the integral of (A.13), we get

$$E_\varepsilon(x,t) = \frac{1}{(2\pi)^n}\int_{|\omega|=1}\left(\int_0^\infty e^{ix\cdot\rho\omega-\varepsilon\rho}\sin(t\rho)\rho^{n-2}d\rho\right)dS_\omega. \tag{A.15}$$

It follows from

$$\sin(t\rho)\rho^{n-2} = (-1)^{(n-1)/2}\left(\frac{\partial}{\partial t}\right)^{n-2}\cos(t\rho)$$

that

$$E_\varepsilon(x,t) = \frac{(-1)^{(n-1)/2}}{(2\pi)^n}\left(\frac{\partial}{\partial t}\right)^{n-2}\left\{\int_{|\omega|=1}\left(\int_0^\infty e^{ix\cdot\rho\omega-\varepsilon\rho}\cos(t\rho)d\rho\right)dS_\omega\right\}.$$

Noticing that

$$e^{ix\cdot\rho\omega-\varepsilon\rho}\cos(t\rho) = \frac{e^{i(x\cdot\omega+t+i\varepsilon)\rho}+e^{i(x\cdot\omega-t+i\varepsilon)\rho}}{2},$$

we get

$$\begin{aligned}&E_\varepsilon(x,t)\\ &= \frac{(-1)^{(n-1)/2}}{2(2\pi)^n}\left(\frac{\partial}{\partial t}\right)^{n-2}\left\{\int_{|\omega|=1}\left(\int_0^\infty\left(e^{i(x\cdot\omega+t+i\varepsilon)\rho}+e^{i(x\cdot\omega-t+i\varepsilon)\rho}\right)d\rho\right)dS_\omega\right\}.\end{aligned}$$

Here the ρ-integral is equals to

$$i\left(\frac{1}{x\cdot\omega+t+i\varepsilon}+\frac{1}{x\cdot\omega-t+i\varepsilon}\right).$$

Making use of the rotational invariance $(\omega\to-\omega)$ of the surface integral, we have

$$\begin{aligned}&E_\varepsilon(x,t)\\ &= \frac{(-1)^{(n-1)/2}}{2(2\pi)^n}\left(\frac{\partial}{\partial t}\right)^{n-2}\int_{|\omega|=1} i\left(\frac{1}{x\cdot(-\omega)+t+i\varepsilon}+\frac{1}{x\cdot\omega-t+i\varepsilon}\right)dS_\omega\\ &= \frac{(-1)^{(n-1)/2}}{(2\pi)^n}\left(\frac{\partial}{\partial t}\right)^{n-2}\int_{|\omega|=1}\frac{\varepsilon}{(x\cdot\omega-t)^2+\varepsilon^2}dS_\omega.\end{aligned} \tag{A.16}$$

Here we apply Lemma A.1 as $\phi(s) = \varepsilon/\{(s-t)^2+\varepsilon^2\}$ for $s \in \mathbf{R}$ and $t \geq 0$. Then we get

$$\int_{|\omega|=1} \phi(x\cdot\omega)dS_\omega = \omega_{n-1}\int_{-1}^{1}(1-p^2)^{(n-3)/2}\frac{\varepsilon}{(|x|p-t)^2+\varepsilon^2}dp.$$

Changing the variables by $s = |x|p$, we have

$$\begin{aligned}\int_{|\omega|=1} \phi(x\cdot\omega)dS_\omega &= \omega_{n-1}\int_{-|x|}^{|x|}\left(1-\frac{s^2}{|x|^2}\right)^{(n-3)/2} P_\varepsilon(s-t)|x|^{-1}ds \\ &= \frac{\omega_{n-1}}{|x|^{n-2}}\int_{-|x|}^{|x|}(|x|^2-s^2)^{(n-3)/2}P_\varepsilon(s-t)ds,\end{aligned}$$

where we set

$$P_\varepsilon(s) = \frac{\varepsilon}{s^2+\varepsilon^2} \quad \text{for } s \in \mathbf{R}.$$

Then (A.16) yileds

$$E_\varepsilon(x,t) = \frac{(-1)^{(n-1)/2}\omega_{n-1}}{(2\pi)^n|x|^{n-2}}\int_{-|x|}^{|x|}(|x|^2-s^2)^{(n-3)/2}P_\varepsilon(s-t)ds,$$

and that

$$\begin{aligned}&\int_{\mathbf{R}^n} E_\varepsilon(x-y,t)g(y)dy = \int_{\mathbf{R}^n} E_\varepsilon(y,t)g(x-y)dy \\ &= \frac{(-1)^{(n-1)/2}\omega_{n-1}}{(2\pi)^n}\int_{\mathbf{R}^n}\left(\frac{\partial}{\partial t}\right)^{n-2}\left(|y|^{2-n}\int_{-|y|}^{|y|}(|y|^2-s^2)^{(n-3)/2}P_\varepsilon(s-t)ds\right) \\ &\quad \times g(x-y)dy.\end{aligned}$$

Noticing that

$$\left|\left(\frac{\partial}{\partial t}\right)^{n-2} P_\varepsilon(s-t)\right| \leq C_{n,\varepsilon} \quad \text{for } s \in \mathbf{R},\ n \geq 3,$$

where $C_{n,\varepsilon}$ is a positive constant depending on n and ε, we have

$$\left|\left(\frac{\partial}{\partial t}\right)^{n-2}\left(|y|^{2-n}\int_{-|y|}^{|y|}(|y|^2-s^2)^{(n-3)/2}P_\varepsilon(s-t)ds\right)g(x-y)\right| \leq C_{n,\varepsilon}|g(x-y)|$$

for $y \in \mathbf{R}^n$. We note that the right hand side of this inequality is an integrable function on $\mathbf{R}^n$, because of $g \in C_0^\infty(\mathbf{R}^n)$. Hence, making use of differentiation under the integral sign, we get

$$\int_{\mathbf{R}^n} E_\varepsilon(x-y,t)g(y)dy = \frac{(-1)^{(n-1)/2}}{(2\pi)^n}\left(\frac{\partial}{\partial t}\right)^{n-2} K_\varepsilon(x,t), \tag{A.17}$$

where we set

$$K_\varepsilon(x,t) = \omega_{n-1}\int_{\mathbf{R}^n} |y|^{2-n}\int_{-|y|}^{|y|}(|y|^2-s^2)^{(n-3)/2}P_\varepsilon(s-t)g(x-y)dsdy.$$

Introducing the polor coordinate $y = r\omega$, with $r = |y|$, $|\omega| = 1$, and using the rotational invariance of the surface integral, we have

$$\begin{aligned}&K_\varepsilon(x,t)\\ &= \omega_{n-1}\int_0^\infty r^{2-n}\left(\int_{-r}^{r}(r^2-s^2)^{(n-3)/2}P_\varepsilon(s-t)\int_{|\omega|=1}g(x+r\omega)dS_\omega ds\right)r^{n-1}dr.\end{aligned}$$

Making use of the definition of Q_n in (A.3), we get

$$K_\varepsilon(x,t) = \omega_{n-1}\omega_n\int_0^\infty r\left(\int_{-r}^{r}(r^2-s^2)^{(n-3)/2}P_\varepsilon(s-t)Q_n(x,r)ds\right)dr.$$

Noticing that

$$\begin{aligned}&\omega_n\int_0^\infty r\left(\int_{-r}^{r}(r^2-s^2)^{(n-3)/2}P_\varepsilon(s-t)|Q_n(x,r)|ds\right)dr\\ &\le 2\varepsilon^{-1}\int_0^\infty r^{n-1}\left(\int_{|\omega|=1}|g(x+r\omega)|dS_\omega\right)dr = 2\varepsilon^{-1}\|g\|_{L^1(\mathbf{R}^n)} < \infty,\end{aligned}$$

we can invert the order of (s,r)-integral, and hence

$$K_\varepsilon(x,t) = \omega_{n-1}\omega_n\int_{-\infty}^{\infty}P_\varepsilon(s-t)\left(\int_{|s|\le r}(r^2-s^2)^{(n-3)/2}rQ_n(x,r)dr\right)ds.$$

It follows from

$$\Gamma\left(\frac{n-1}{2}\right) = \frac{(n-3)!!}{2^{(n-3)/2}} \quad \text{and} \quad \Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!}{2^{(n-1)/2}}\sqrt{\pi},$$

that

$$\begin{aligned}\omega_n\omega_{n-1} &= \frac{2\pi^{(n-1)/2}2\pi^{n/2}}{\Gamma\left((n-1)/2\right)\Gamma\left(n/2\right)}\\ &= \frac{\pi^{n-1/2}2^n}{(n-3)!!(n-2)!!\pi^{1/2}} = \frac{(2\pi)^n}{\pi(n-2)!}.\end{aligned}$$

Hence (A.17) yields

$$\begin{aligned}&\int_{\mathbf{R}^n}E_\varepsilon(x-y,t)g(y)dy\\ &= \frac{(-1)^{(n-1)/2}}{\pi(n-2)!}\left(\frac{\partial}{\partial t}\right)^{n-2}\int_{-\infty}^{\infty}P_\varepsilon(s-t)\left(\int_{|s|\le r}(r^2-s^2)^{(n-3)/2}rQ_n(x,r)dr\right)ds.\end{aligned}$$

Since $(-1)^{(n-1)/2+(n-3)/2} = -1$, we have

$$\int_{\mathbf{R}^n} E_\varepsilon(x-y,t)g(y)dy$$
$$= \frac{-1}{\pi(n-2)!}\left(\frac{\partial}{\partial t}\right)^{n-2}\int_{-\infty}^{\infty} P_\varepsilon(s-t)\left(\int_{|s|\le r}(s^2-r^2)^{(n-3)/2} rQ_n(x,r)dr\right)ds.$$

We set

$$\Psi(s) = \frac{-1}{(n-2)!}\int_{|s|\le r}(s^2-r^2)^{(n-3)/2} rQ_n(x,r)dr$$

for $s \in \mathbf{R}$. We note that $\Psi \in C_0^\infty(\mathbf{R})$ because $g \in C_0^\infty(\mathbf{R}^n)$. Then we get

$$\begin{aligned}\int_{\mathbf{R}^n} E_\varepsilon(x-y,t)g(y)dy &= \frac{1}{\pi}\left(\frac{\partial}{\partial t}\right)^{n-2}\int_{-\infty}^{\infty} P_\varepsilon(s-t)\Psi(s)ds \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} P_\varepsilon(s-t)\frac{d^{n-2}}{ds^{n-2}}\Psi(s)ds.\end{aligned} \tag{A.18}$$

Noticing that P_ε is the Possion kernel, we get

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty} P_\varepsilon(s-t)\frac{d^{n-2}}{ds^{n-2}}\Psi(s)ds = \frac{d^{n-2}}{dt^{n-2}}\Psi(t). \tag{A.19}$$

From the definition, we have

$$\begin{aligned}\frac{d^{n-2}}{dt^{n-2}}\Psi(t) &= \frac{-1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}}\int_{|t|}^{\infty}(t^2-r^2)^{(n-3)/2} rQ_n(x,r)dr \\ &= \frac{-1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}}\left(\int_0^{\infty} - \int_0^{|t|}\right)(t^2-r^2)^{(n-3)/2} rQ_n(x,r)dr.\end{aligned}$$

It follows from that $(t^2-r^2)^{(n-3)/2}$ is a polynomial of degree $n-3$, we have

$$\frac{\partial^{n-2}}{\partial t^{n-2}}\int_0^{\infty}(t^2-r^2)^{(n-3)/2} rQ_n(x,r)dr = 0. \tag{A.20}$$

In fact, we obtain

$$\frac{\partial^{n-3}}{\partial t^{n-3}}(t^2-r^2)^{(n-3)/2} rQ_n(x,r) = C_n rQ_n(x,r)$$

for $r \ge 0$, with a suitable constant C_n. It follows that

$$\begin{aligned}&\omega_n\left|\int_0^{\infty} rQ_n(x,r)dr\right| \le \omega_n\int_0^1 r|Q_n(x,r)|dr + \omega_n\int_1^{\infty} r|Q_n(x,r)|dr \\ &\le \int_0^1\int_{|\omega|=1}|g(x+r\omega)|dS_\omega dr + \int_1^{\infty} r^{n-1}\int_{|\omega|=1}|g(x+r\omega)|dS_\omega dr \\ &\le \|g\|_{L^\infty(\mathbf{R}^n)} + \|g\|_{L^1(\mathbf{R}^n)} < \infty,\end{aligned}$$

because $g \in C_0^\infty(\mathbf{R}^n)$. Thus we get (A.20).
Hence for $t \geq 0$, we obtain

$$\frac{d^{n-2}}{dt^{n-2}}\Psi(t) = \frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}}\int_0^t (t^2-r^2)^{(n-3)/2} r Q_n(x,r)dr.$$

By (A.18) and (A.19), we finally get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathbf{R}^n} E_\varepsilon(x-y,t)g(y)dy\\ &\quad = \frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}}\int_0^t (t^2-r^2)^{(n-3)/2} r Q_n(x,r)dr.\end{aligned}$$

This completes the proof of Proposition A.1, and we get (A.2) in the case of $n = 2m+1$. □

Next we consider the case of $n = 2m$. We use Hadamard's method of descent to get (A.2) in the case $n = 2m$ from the case $n = 2m+1$.

Define $\widetilde{x} = (x, x_{n+1}) \in \mathbf{R}^{n+1}$, where $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$. Suppose that u is a solution of (A.1). Define $\widetilde{u}(\widetilde{x},t) = u(x,t)$ and $\widetilde{g}(\widetilde{x}) = g(x)$. Then $\widetilde{u}$ is a solution of (A.1) in $\mathbf{R}^{n+1}$, where $n+1$ is odd. By (A.2), we have

$$\widetilde{u}(\widetilde{x},t) = \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial t^{n-1}}\int_0^t (t^2-r^2)^{(n-2)/2} r Q_{n+1}(\widetilde{x},r)dr. \tag{A.21}$$

Noticing that

$$\begin{aligned}&\int_{|\widetilde{\omega}|=1,\omega_{n+1}>0} \widetilde{g}(\widetilde{x}+r\widetilde{\omega})dS_{\widetilde{\omega}}\\ &= \int_{|\omega|<1} g(x+r\omega)\sqrt{1+\sum_{j=1}^n\left(\frac{d}{d\omega_j}\omega_{n+1}\right)^2}d\omega\\ &= \int_{|\omega|<1}\frac{g(x+r\omega)}{\sqrt{1-|\omega|^2}}d\omega,\end{aligned}$$

we get

$$\begin{aligned}Q_{n+1}(\widetilde{x},r) &= \frac{1}{\omega_{n+1}}\int_{|\widetilde{\omega}|=1}\widetilde{g}(\widetilde{x}+r\widetilde{\omega})dS_{\widetilde{\omega}}\\ &= \frac{2}{\omega_{n+1}}\int_{|\xi|<1}\frac{g(x+r\xi)}{\sqrt{1-|\xi|^2}}d\xi.\end{aligned} \tag{A.22}$$

Here we introduce the polar coordinate $\xi = \rho\omega$ ($|\omega| = 1$) in the above integral. Then we get

$$Q_{n+1}(\widetilde{x},r) = \frac{2}{\omega_{n+1}}\int_0^1 \frac{\rho^{n-1}}{\sqrt{1-\rho^2}}d\rho\int_{|\omega|=1} g(x+r\rho\omega)dS_\omega.$$

Changing the variables by $r\rho = s$, we have

$$\begin{aligned} Q_{n+1}(\widetilde{x}, r) &= \frac{2}{\omega_{n+1} r^{n-1}} \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} ds \int_{|\omega|=1} g(x + s\omega) dS_\omega \\ &= \frac{2\omega_n}{\omega_{n+1} r^{n-1}} \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds. \end{aligned}$$

Substituting this into (A.21), we get

$$\begin{aligned} u(x,t) = \frac{1}{(n-1)!} \frac{2\omega_n}{\omega_{n+1}} \frac{\partial^{n-1}}{\partial t^{n-1}} \int_0^t \frac{(t^2 - r^2)^{(n-2)/2}}{r^{n-1}} r dr \\ \times \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds. \end{aligned} \tag{A.23}$$

It follows that

$$\frac{\partial}{\partial t} \int_0^t \frac{(t^2 - r^2)^{(n-2)/2}}{r^{n-1}} r dr \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds$$

$$= \begin{cases} \displaystyle\int_0^t \frac{s}{\sqrt{t^2 - s^2}} Q_2(x, s) ds & \text{if } n = 2, \\ \displaystyle(n-2)t \int_0^t \frac{(t^2 - r^2)^{(n-4)/2}}{r^{n-2}} dr \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds & \text{if } n \geq 4. \end{cases}$$

Since $2\omega_2/\omega_3 = 1$, we get (A.2) in the case of $n = 2$ from (A.23). From now on, we assume $n \geq 4$. Then (A.23) yields

$$\begin{aligned} u(x,t) = \frac{n-2}{(n-1)!} \frac{2\omega_n}{\omega_{n+1}} \frac{\partial^{n-2}}{\partial t^{n-2}} t \int_0^t \frac{(t^2 - r^2)^{(n-4)/2}}{r^{n-2}} dr \\ \times \int_0^r \frac{s^{n-1}}{\sqrt{r^2 - s^2}} Q_n(x, s) ds. \end{aligned}$$

Inverting the order of (s, r)-integral, we get

$$\begin{aligned} u(x,t) = \frac{n-2}{(n-1)!} \frac{2\omega_n}{\omega_{n+1}} \frac{\partial^{n-2}}{\partial t^{n-2}} t \int_0^t s^{n-1} Q_n(x, s) ds \\ \times \int_s^t \frac{(t^2 - r^2)^{(n-4)/2}}{r^{n-2}\sqrt{r^2 - s^2}} dr. \end{aligned}$$

Changing the variable by

$$1 - \frac{s^2}{r^2} = \left(1 - \frac{s^2}{t^2}\right) \sigma$$

in the above r-integral and noting that

$$\sigma = \frac{(r^2-s^2)t^2}{(t^2-s^2)r^2},\ 1-\sigma = \frac{(t^2-r^2)s^2}{(t^2-s^2)r^2} \quad \text{and} \quad 2\frac{s^2}{r^3}dr = \left(1-\frac{s^2}{t^2}\right)d\sigma,$$

we see that the r-integral equals to

$$\begin{aligned}
&\int_0^1 \frac{(t^2-s^2)^{(n-4)/2}(1-\sigma)^{(n-4)/2}\left(\frac{r}{s}\right)^{n-4}}{r^{n-2}\frac{r}{t}\sqrt{t^2-s^2}\sqrt{\sigma}} \cdot \frac{r^3(t^2-s^2)}{2s^2t^2}d\sigma \\
&= \frac{(t^2-s^2)^{(n-3)/2}}{2ts^{n-2}} B\left(\frac{1}{2}, \frac{n-2}{2}\right),
\end{aligned}$$

where we set

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx \quad (\alpha,\beta>0).$$

Noticing that

$$\begin{aligned}
&\frac{n-2}{(n-1)!}\frac{2\omega_n}{\omega_{n+1}} B\left(\frac{1}{2}, \frac{n-2}{2}\right) \\
&= \frac{n-2}{(n-1)!} \cdot \frac{2\pi^{n/2}/\Gamma(n/2)}{2\pi^{(n+1)/2}/\Gamma((n+1)/2)} \cdot \frac{\Gamma(1/2)\Gamma((n-2)/2)}{\Gamma((n-1)/2)} \\
&= \frac{n-2}{(n-1)!} \cdot \frac{2\pi^{n/2}\Gamma((n-1)/2)(n-1)/2}{2\pi^{(n+1)/2}\Gamma((n-2)/2)(n-2)/2} \cdot \frac{\pi^{1/2}\Gamma((n-2)/2)}{\Gamma((n-1)/2)} = \frac{1}{(n-2)!},
\end{aligned}$$

we get

$$u(x,t) = \frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (t^2-s^2)^{(n-3)/2} s Q_n(x,s)ds.$$

Therefore we obtain (A.2) in the case of $n = 2m$. □

Proof of (A.4). Let λ be a fixed non-negative integer, and let $\phi \in C^\lambda(\mathbf{R})$. Define

$$U_\lambda(t) = \frac{1}{(2\lambda+1)!}\left(\frac{d}{dt}\right)^{2\lambda+1} \int_0^t (t^2-r^2)^\lambda \phi(r)dr,\ t \geq 0. \tag{A.24}$$

Then, we have the following lemma:

Lemma A.2 *Let U_λ be the function in (A.24). Then we have*

$$U_\lambda(t) = \frac{1}{(2\lambda+1)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^\lambda (t^{2\lambda}\phi(t)) \tag{A.25}$$

for $\lambda = 0, 1, 2, \cdots$.

Proof. Set

$$V_\lambda(t) = \frac{1}{(2\lambda+1)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^\lambda (t^{2\lambda}\phi(t)), \tag{A.26}$$

which is the right hand side of (A.25). When $\lambda = 0$, we have

$$U_0(t) = \frac{d}{dt}\int_0^t \phi(r)dr = \phi(t) = V_0(t).$$

Hence (A.25) holds for $\lambda = 0$.

For $\lambda \geq 1$, (A.24) yields

$$\begin{aligned} U_\lambda(t) &= \frac{1}{(2\lambda+1)!}\left(\frac{d}{dt}\right)^{2\lambda}\left(2\lambda t\int_0^t (t^2-r^2)^{\lambda-1}\phi(r)dr\right) \\ &= \frac{1}{(2\lambda+1)!}\left\{2\lambda t\left(\frac{d}{dt}\right)^{2\lambda}\int_0^t (t^2-r^2)^{\lambda-1}\phi(r)dr \right. \\ &\qquad \left. +(2\lambda)^2\left(\frac{\partial}{\partial t}\right)^{2\lambda-1}\int_0^t (t^2-r^2)^{\lambda-1}\phi(r)dr\right\}. \end{aligned}$$

By (A.24), we have

$$U_\lambda(t) = \frac{1}{(2\lambda+1)!}\left\{2\lambda t(2\lambda-1)!U'_{\lambda-1}(t) + (2\lambda)^2(2\lambda-1)!U_{\lambda-1}(t)\right\}.$$

Hence we get

$$U_\lambda(t) = \frac{1}{2\lambda+1}\left\{tU'_{\lambda-1}(t) + 2\lambda U_{\lambda-1}(t)\right\}. \tag{A.27}$$

On the other hand, (A.26) yields

$$\begin{aligned} V_\lambda(t) &= \frac{1}{(2\lambda+1)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^{\lambda-1}(2\lambda t^{2\lambda-2}\phi(t) + t^{2\lambda-1}\phi'(t)) \\ &= \frac{1}{2\lambda+1}\left\{\frac{2\lambda}{(2\lambda-1)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^{\lambda-1}(t^{2\lambda-2}\phi(t)) \right. \\ &\qquad \left. +\frac{1}{(2\lambda-1)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^{\lambda-1}(t^{2\lambda-1}\phi'(t))\right\}. \end{aligned}$$

By (A.26), we get

$$V_\lambda(t) = \frac{1}{2\lambda+1}\left\{2\lambda V_{\lambda-1}(t) + \frac{1}{(2\lambda-1)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^{\lambda-1}(t^{2\lambda-1}\phi'(t))\right\}.$$

Define $\mathcal{D} = \dfrac{1}{t}\dfrac{d}{dt}$. If one can show

$$\mathcal{D}^{\lambda-1}t^{2\lambda}\mathcal{D}w(t) = t^2\mathcal{D}^{\lambda}t^{2(\lambda-1)}w(t) \tag{A.28}$$

for $w \in C^{\lambda}(\mathbf{R})$, then

$$V_{\lambda}(t) = \frac{1}{2\lambda+1}\left\{2\lambda V_{\lambda-1}(t) + tV'_{\lambda-1}(t)\right\}$$

holds, and we get $U_{\lambda}(t) = V_{\lambda}(t)$ by (A.27). In what follows we show (A.28) by induction.

Obviously (A.28) holds for $\lambda = 1$. Assume that (A.28) holds for $\lambda \geq 1$. Noticing that

$$\mathcal{D}t^2w(t) = 2w(t) + t^2\mathcal{D}w(t), \tag{A.29}$$

we get

$$\begin{aligned}\mathcal{D}^{\lambda}t^{2\lambda+2}\mathcal{D}w(t) &= \mathcal{D}^{\lambda}t^{2\lambda}(\mathcal{D}t^2 - 2)w(t)\\ &= \{\mathcal{D}(\mathcal{D}^{\lambda-1}t^{2\lambda}\mathcal{D})t^2 - 2\mathcal{D}^{\lambda}t^{2\lambda}\}w(t).\end{aligned}$$

Making use of the assumption of induction, we get

$$\begin{aligned}\mathcal{D}^{\lambda}t^{2\lambda+2}\mathcal{D}w(t) &= \{\mathcal{D}(t^2\mathcal{D}^{\lambda}t^{2(\lambda-1)})t^2 - 2\mathcal{D}^{\lambda}t^{2\lambda}\}w(t)\\ &= \{(\mathcal{D}t^2 - 2)\mathcal{D}^{\lambda}t^{2\lambda}\}w(t).\end{aligned}$$

By using (A.29), we get

$$\mathcal{D}^{\lambda}t^{2\lambda+2}\mathcal{D}w(t) = (t^2\mathcal{D})\mathcal{D}^{\lambda}t^{2\lambda}w(t) = t^2\mathcal{D}^{\lambda+1}t^{2\lambda}w(t).$$

Hence (A.28) holds for all $\lambda \geq 1$. Therefore, the proof of Lemma A.2 is completed. □

We turn back to the proof of (A.4). Set $\lambda = (n-3)/2$ and $\phi(r) = rQ_n(x,r)$ in (A.24). Then by (A.2), we have $u(x,t) = U_{\lambda}(t)$. Making use of Lemma A.2, we get

$$u(x,t) = \frac{1}{(n-2)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^{(n-3)/2}(t^{n-2}Q_n(x,t)).$$

Therefore, we get (A.4).

Proof of (A.5). We first prepare the following proposition.

Proposition A.2 *Let l be a fixed non-negative integer, and let $\phi \in C^l(\mathbf{R})$. Then there exist constants $\alpha_{l,k} > 0$ such that*

$$\left(\frac{1}{t}\frac{d}{dt}\right)^{l}(t^{2l+1}\phi(t)) = \sum_{k=0}^{l}\alpha_{l,k}t^{k+1}\frac{d^k}{dt^k}\phi(t). \tag{A.30}$$

Moreover, $\alpha_{l,0} = (2l+1)!!$ holds.

Proof. Let $l = 0$. Then (A.30) holds as $\alpha_{0,0} = 1$.

Suppose that

$$\left(\frac{1}{t}\frac{d}{dt}\right)^{l-1}(t^{2l-1}\phi(t)) = \sum_{k=0}^{l-1}\alpha_{l-1,k}t^{k+1}\frac{d^k}{dt^k}\phi(t).$$

holds for $l \geq 1$. Replacing $\phi(t)$ by $t^2\phi(t)$ in the above, we get

$$(\Phi(t) :=)\left(\frac{1}{t}\frac{d}{dt}\right)^{l}(t^{2l+1}\phi(t)) = \frac{1}{t}\frac{d}{dt}\sum_{k=0}^{l-1}\alpha_{l-1,k}t^{k+1}\frac{d^k}{dt^k}t^2\phi(t).$$

It follows that

$$\begin{aligned}
\Phi(t) &= \frac{1}{t}\frac{d}{dt}\sum_{k=0}^{l-1}\alpha_{l-1,k}t^{k+1}\{t^2\frac{d^k}{dt^k}\phi(t) + 2kt\frac{d^{k-1}}{dt^{k-1}}\phi(t) + k(k-1)\phi(t)\} \\
&= \sum_{k=0}^{l-1}\alpha_{l-1,k}\frac{1}{t}\frac{d}{dt}\left\{t^{k+3}\frac{d^k}{dt^k}\phi(t)\right\} + \sum_{k=1}^{l-1}2k\alpha_{l-1,k}\frac{1}{t}\frac{d}{dt}\left\{t^{k+2}\frac{d^{k-1}}{dt^{k-1}}\phi(t)\right\} \\
&\quad + \sum_{k=2}^{l-1}k(k-1)\alpha_{l-1,k}\frac{1}{t}\frac{d}{dt}\left\{t^{k+1}\frac{d^{k-1}}{dt^{k-1}}\phi(t)\right\} \\
&= \sum_{k=0}^{l-1}(k+3)\alpha_{l-1,k}(k+3)t^{k+1}\frac{d^k}{dt^k}\phi(t) + \sum_{k=1}^{l}\alpha_{l-1,k-1}t^{k+1}\frac{d^k}{dt^k}\phi(t) \\
&\quad + \sum_{k=0}^{l-2}2k(k+1)(k+3)\alpha_{l-1,k+1}t^{k+1}\frac{d^k}{dt^k}\phi(t) \\
&\quad + \sum_{k=1}^{l-1}2k\alpha_{l-1,k}t^{k+1}t^{k+1}\frac{d^k}{dt^k}\phi(t) \\
&\quad + \sum_{k=0}^{l-3}(k+3)(k+2)(k+1)\alpha_{l-1,k+2}t^{k+1}\frac{d^k}{dt^k}\phi(t) \\
&\quad + \sum_{k=1}^{l-2}(k+1)k\alpha_{l-1,k+1}t^{k+1}\frac{d^k}{dt^k}\phi(t) \\
&= \{3\alpha_{l-1,0} + 6\alpha_{l-1,1} + 6\alpha_{l-1,2}\}\, t\phi(t) \\
&\quad + \sum_{k=0}^{l-3}\{3(k+1)\alpha_{l-1,k}(k+3) + \alpha_{l-1,k-1} + 3(k+2)(k+1)\alpha_{l-1,k+1} \\
&\quad + (k+3)(k+2)(k+1)\alpha_{l-1,k+2}\}\, t^{k+1}\frac{d^k}{dt^k}\phi(t) \\
&\quad + \{(3l-1)\alpha_{l-1,l-2} + \alpha_{l-1,l-3} + 3l(l-1)\alpha_{l-1,l-1}\}\, t^{l-1}\frac{d^{l-2}}{dt^{l-2}}\phi(t) \\
&\quad + \{3l\alpha_{l-1,l-1} + \alpha_{l-1,l-2}\}\, t^{l}\frac{d^{l-1}}{dt^{l-1}}\phi(t) + \alpha_{l-1,l-1}t^{l+1}\frac{d^l}{dt^l}\phi(t).
\end{aligned}$$

Therefore, (A.30) holds for all $l \geq 1$. Moreover, if we set $\phi(t) = 1$ in (A.30), then we get

$$\left(\frac{1}{t}\frac{d}{dt}\right)^l t^{2l+1} = \alpha_{l,0} t.$$

Hence we have $\alpha_{l,0} = (2l+1)!!$. This completes the proof. □

We turn back to the proof of (A.5). Setting $l = (n-3)/2$ and $\phi(t) = Q_n(x,t)$ in (A.30), we get (A.5) by (A.4). This completes the proof. □

Proof of (A.6) and (A.8). Again, we shall apply Hadamard's method of descent to get (A.6) and (A.8) from (A.4) and (A.5). In view of the proof of (A.2) in the case of $n = 2m$, we get

$$u(x,t) = \frac{1}{(n-1)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^{(n-2)/2} (t^{n-1} Q_{n+1}(x,t)),$$

where

$$Q_{n+1}(x,r) = \frac{2}{\omega_{n+1}} \int_{|\xi|<1} \frac{g(x+r\xi)}{\sqrt{1-|\xi|^2}} d\xi.$$

Also, we get

$$u(x,t) = \frac{1}{(n-1)!!} \sum_{k=0}^{(n-2)/2} \alpha_{(n-2)/2,k} t^{k+1} \left(\frac{\partial}{\partial t}\right)^k Q_{n+1}(x,t).$$

Since $Q_{n+1}(x,t) = H_n(x,t)$, we get (A.6) and (A.8). Therefore, the proof of Theorem A.1 is completed. □

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