Motion of a graph by R-curvature

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1. Introduction.

In this talk we introduce our recent result:

H. Ishii and T. Mikami, Motion of a graph by R-vurvature, Hokkaido mathematical preprint series, No. 340.

Let us first introduce two definitions.

Definition 1 (R-curvature) *Let $R \in L^1(\mathbf{R}^d : [0,\infty), dx)$. For $u \in C(\mathbf{R}^d : \mathbf{R})$, we define the R-curvature of u as the finite Borel measure $w(R,u,dx)$ on $\mathbf{R}^d$ given by*

$$w(R,u,A) \equiv \int_{\cup_{x\in A}\partial u(x)} R(y)dy \quad \text{for all Borel } A \subset \mathbf{R}^d. \tag{0.1}$$

Definition 2 (Motion by R-curvature) *The graph of $u \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$ is called the motion by R-curvature if the following holds: for any $\varphi \in C_o(\mathbf{R}^d : \mathbf{R})$ and any $t \geq 0$,*

$$\begin{aligned}&\int_{\mathbf{R}^d}\varphi(x)u(t,x)dx-\int_{\mathbf{R}^d}\varphi(x)u(0,x)dx \\ = \ &\int_0^t ds\int_{\mathbf{R}^d}\varphi(x)w(R,u(s,\cdot),dx).\end{aligned} \tag{0.2}$$

By the continuum limit of a class of infinite particle systems, we first show the existence of the motion by R-curvature, and then the uniqueness by the comparison theorem. We also show that the motion by R-curvature is a viscosity solution to

$$(PDE) \qquad \partial u(t,x)/\partial t=\chi(u,Du(t,x),t,x)\mathrm{Det}_+(D^2u(t,x))R(Du(t,x)),$$

where $Du(t,x)\equiv(\partial u(t,x)/\partial x_i)_{i=1}^d$, $D^2u(t,x)\equiv(\partial^2u(t,x)/\partial x_i\partial x_j)_{i,j=1}^d$,

$$\chi(u,p,t,x)\equiv\begin{cases}1 & \text{if } p\in\partial u(t,x),\\ 0 & \text{otherwise,}\end{cases}$$

$\partial u(t,x)$ denotes the subdifferential of the function $x\mapsto u(t,x)$, and for a real $d\times d$-symmetric matrix X,

$$\mathrm{Det}_+X\equiv\begin{cases}\mathrm{Det}X & \text{if } X \text{ is nonnegative definite,}\\ 0 & \text{otherwise.}\end{cases}$$

We introduce the definition of the viscosity solution to (PDE).

Definition 3 (Viscosity solution) *(Viscosity subsolution)* $u\in C((0,\infty)\times\mathbf{R}^d:\mathbf{R})$ *is a viscosity subsolution of (PDE) if whenever* $\varphi\in C^2((0,\infty)\times\mathbf{R}^d:\mathbf{R})$ *and* $u-\varphi\le(u-\varphi)(t_o,x_o)$,

$$\partial\varphi(t_o,x_o)/\partial t \le \chi(u, D\varphi(t_o,x_o), t_o, x_o) Det_+(D^2\varphi(t_o,x_o)) R(D\varphi(t_o,x_o)).$$

(Viscosity supersolution) $u \in C((0,\infty)\times \mathbf{R}^d : \mathbf{R})$ *is a viscosity supersolution of (PDE) if whenever* $\varphi \in C^2((0,\infty)\times \mathbf{R}^d : \mathbf{R})$ *and* $u-\varphi \ge (u-\varphi)(t_o,x_o)$,

$$\partial\varphi(t_o,x_o)/\partial t \ge \chi^-(u, D\varphi(t_o,x_o), t_o, x_o) Det_+(D^2\varphi(t_o,x_o)) R(D\varphi(t_o,x_o)).$$

Here $\chi^-(v,p,t,x) = 1$ *if*

$$v(t,y) > v(t,x) + < p, y-x > \qquad (y \ne x)$$

and if there exists $\varepsilon > 0$ *such that for all* $(s,y) \in (0,\infty)\times \mathbf{R}^d$ *satisfying* $|y| > \varepsilon^{-1}$ *and* $|s-t| < \varepsilon$,

$$v(s,y) > p \cdot y + \varepsilon|y|,$$

and $\chi^-(v,p,t,x) = 0$, *otherwise.*

Remark 1 *If* $\chi^-(v,p,t,x) = 1$ *and* s *is close to* t, *then* $p \in \partial v(s,y)$ *for some* y.

Finally we discuss under what condition the viscosity solution to (PDE) is the motion by R-curvature.

2. Infinite particle systems and the motion by R-curvature.

In this section we construct the motion by R-curvature by the continuum limit of infinite particle systems.

Fix $\varepsilon_n \downarrow 0$ as $n \to \infty$, and put

(A.1). $||R||_{L^1} \equiv \int_{\mathbf{R}^d} R(y)dy > 0$, $R \geq 0$, $h \in C(\mathbf{R}^d : \mathbf{R})$,

(A.2). $|\partial h(\mathbf{R}^d)(\equiv \cup_{x \in \mathbf{R}^d} \partial h(x))| > 0$,

$$\begin{aligned} S_n \equiv & \{v : \mathbf{Z}^d/n \mapsto \mathbf{R} | \sum_{z \in \mathbf{Z}^d/n} (v(z) - h(z)) < \infty, \\ & (v(z) - h(z))/\varepsilon_n \in \mathbf{N} \cup \{0\} \text{ for all } z \in \mathbf{Z}^d/n\}. \end{aligned}$$

Let $\{Y_n(k, \cdot)\}_{0 \leq k}$ be a Markov chain on S_n such that $Y_n(0, \cdot) = h(\cdot)$, and that

$$P(Y_n(k+1, \cdot) = v_{n,z} | Y_n(k, \cdot) = v) = w(R, \hat{v}, \{z\})/w(R, \hat{Y}_n(0, \cdot), \mathbf{R}^d),$$

where

$$v_{n,z}(x) \equiv \begin{cases} v(x) + \varepsilon_n & \text{if } x = z, \\ v(x) & \text{if } x \in (\mathbf{Z}^d/n) \backslash \{z\}. \end{cases}$$

Let $p_n(t)$ be a Poisson process, with parameter $n^d \varepsilon_n^{-1} w(R, \hat{Y}_n(0, \cdot), \mathbf{R}^d)$, which is independent of Y_n. Put

$$Z_n(t, z) \equiv Y_n(p_n(t), z),$$

$$X_n(t, x) \equiv \max(\hat{Z}_n(t, x), h(x)).$$

For f and $g \in C(\mathbf{R}^d : \mathbf{R})$, we put

$d_{C(\mathbf{R}^d : \mathbf{R})}(f, g) \equiv \sum_{m \geq 1} 2^{-m} \min(\sup_{|x| \leq m} |f(x) - g(x)|, 1)$.

Then we show that $X_n(t, x)$ converges to the motion by R-curvature under the following additional conditions.

(A.3). The closure of the set $\{x \in \mathbf{R}^d : \hat{h}(x) < h(x)\}$ does not contain any line which is unbounded in two different directions.

(A.4). For any $p \notin \partial h(\mathbf{R}^d)$ and $C \in \mathbf{R}$,

$$\int_{\mathbf{R}^d} \max(< p, x > +C - h(x), 0) dx = \infty.$$

Theorem 1 *Suppose that (A.1) and (A.3)-(A.4) hold. Then there exists a unique continuous solution u to (1.2) with $u(0, \cdot) = h$. Suppose in addition that (A.2) holds. Then the following holds: for any $\gamma > 0$ and $T > 0$,*

$$\lim_{n \to \infty} P(\sup_{0 \le t \le T} d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(t, \cdot), u(t, \cdot)) \ge \gamma) = 0.$$

Remark 2 *(A.3) holds when $d = 1$. If h is convex, then (A.4) holds.*

We give the properties of the motion by R-curvature.

Theorem 2 *Suppose that (A.1) holds. Let $u \in C([0, \infty) \times \mathbf{R}^d : \mathbf{R})$ be the solution to (1.2) with $u(0, \cdot) = h$. Then:*

(a) $t \mapsto u(t, x)$ is nondecreasing.

(b) $u = \max(\hat{u}, h)$

(c) $u(t, x) - \hat{u}(t, x) \le h(x) - \hat{h}(x)$. In particular, if $h(x) = \hat{h}(x)$, then $u(t, x) = \hat{u}(t, x)$.

Suppose in addition that (A.4) holds. Then:

(d) For any $t > 0$, $\partial u(t, \mathbf{R}^d) = \partial h(\mathbf{R}^d)$.

$$\int_{\mathbf{R}^d} (u(t, x) - h(x)) dx = t \cdot w(R, h, \mathbf{R}^d).$$

(e) Let $\overline{u} \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$ *be the solution to (1.2) with* $u(0,\cdot) = \hat{h}$*. If* $u(s,\cdot) - \hat{u}(s,\cdot) \neq h - \hat{h}$ *for some* $s \in (0,\infty)$*, then* $\overline{u}(t,\cdot) - \hat{u}(t,\cdot) \neq 0$ *for all* $t \geq s$*.*

According to the above theorem, (a) any graph moves upward by R-curvature, (b) those points on any graph moving by R-curvature do not move as far as they stay in its cavities, (c) the height between any graph moving by R-curvature and its convex envelope is nonincreasing as it evolves, (d) any graph moving by R-curvature sweeps in time $t > 0$ a region with volume given by $t \cdot w(R, h, \mathbf{R}^d)$, and (e) for the motion of a graph by R-curvature, taking its convex envelope at time $t > 0$ and evolving up to time t starting with the convex envelope of the initial graph give different graphs in general, if the initial graph is not convex.

3. Motion by R-curvature and the viscosity solution.

In this section we discuss the relation between the motion by R-curvature and the viscosity solution to (PDE).

(A.5). $R \in C(\mathbf{R}^d : [0,\infty))$.

Theorem 3 *Suppose that (A.1) and (A.5) hold. Then a continuous solution* u *to (1.2) with* $u(0,\cdot) = h$ *is a viscosity solution to (PDE).*

Theorem 3 means that the motion by R-curvature is the viscosity solution to (PDE). To discuss under what condition the reverse is true, we discuss the uniqueness of the viscosity solution to (PDE).

(A.6). $R(x) \geq R(rx)$ for any $r \geq 1$ and $x \in \mathbf{R}^d$.

(A.7). $\inf_{x \neq o} h(x)/|x| > 0$.

(A.8). There exists a constant $C > 0$ such that $h(x+y)+h(x-y)-2h(x) \leq C$ for all $(x, y) \in \mathbf{R}^d \times U_1(o)$, where $U_1(o) \equiv \{y \in \mathbf{R}^d : |y| < 1\}$.

Theorem 4 *Suppose that (A.1) and (A.3)-(A.8) hold. Then there exists a unique continuous viscosity solution* u *to (PDE) with* $u(0, \cdot) = h$ *in the space of continuous functions* v *for which*

$$\sup\{|v(t,x) - h(x)| : (t,x) \in [0,T] \times \mathbf{R}^d\} < \infty \textit{ for all } T > 0.$$

u *is also a unique continuous solution to (1.2) with* $u(0, \cdot) = h$.

We restrict our attention to Gauss curvature flow and give a finer result. For $A \subset \mathbf{R}^d$ and $v : A \mapsto \mathbf{R}$, put

$$\mathrm{epi}(v) = \{(x, y) : x \in A,\ y \geq v(x)\}.$$

For $r > 0$, put

$$h^r(x) = \inf\{y \in \mathbf{R} \mid U_r((x,y)) \subset \mathrm{epi}(h)\} \quad (x \in \mathbf{R}^d).$$

Under the following condition, we give the comparison theorem for the continuous viscosity solution to (PDE).

(A.1)'. $R(y) = (1 + |y|^2)^{-(d+1)/2}$ and $h \in C(\mathbf{R}^d : \mathbf{R})$.

(A.2)'.

$$\liminf_{\theta \downarrow 1}\{\liminf_{r \to \infty}[\liminf_{|x| \to \infty}(h(\theta x) - h^r(x))]\} > 0,$$

$$\lim_{\theta\downarrow 1}\{\sup_{x\in\mathbf{R}^d}(h(x)-h(\theta x))\}=0.$$

Theorem 5 *Suppose that (A.1)'-(A.2)' hold. Then for any viscosity subsolution u and supersolution v, of (PDE) in the space $C([0,\infty)\times\mathbf{R}^d:\mathbf{R})$, such that $u(0,\cdot)\le h\le v(0,\cdot)$, $u\le v$.*

Remark 3 *(A.2)' holds if there exists a convex function $h_0:\mathbf{R}^d\mapsto\mathbf{R}$ such that $h_0(x)\to\infty$ as $|x|\to\infty$ and that*

$$\lim_{|x|\to\infty}[h(x)-h_0(x)]=0.$$

In fact, the following holds:

$$\lim_{|x|\to\infty}[h(\theta x)-h^r(x)]=\infty \quad \textit{for all } \theta>1, r>0,$$

$$\lim_{\theta\downarrow 1}\{\sup_{x\in\mathbf{R}^d}[h(x)-h(\theta x)]\}=0.$$

The following corollary is better than Theorem 4 in that we can consider the viscosity solution in the entire space $C(\mathbf{R}^d:\mathbf{R})$.

Corollary 1 *Suppose that (A.1)'-(A.2)' and (A.3)-(A.4) hold. Then there exists a unique continuous viscosity solution u to (PDE) with $u(0,\cdot)=h$. u is also a unique continuous solution to (1.2) with $u(0,\cdot)=h$.*

Acknowledgement: We woule like to thank Prof. K. Ishii for informing us the following paper:

G. Barles, S. Biton and O. Ley, Quelque résultats d'unicité pour l'equation du mouvement par courbure moyenne dans $\mathbf{R}^N$, preprint, Theorem 4.1,

where they studied a similar result to Theorem 5 for the mean curvature flow with a convex coercive initial function.