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Motion of a graph by R-curvature

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## 1. Introduction.

In this talk we introduce our recent result:

H. Ishii and T. Mikami, Motion of a graph by *R*-vurvature, Hokkaido mathematical preprint series, No. 340.

Let us first introduce two definitions.

**Definition 1 (R-curvature)** Let  $R \in L^1(\mathbf{R}^d : [0, \infty), dx)$ . For  $u \in C(\mathbf{R}^d : \mathbf{R})$ , we define the R-curvature of u as the finite Borel measure w(R, u, dx) on  $\mathbf{R}^d$  given by

$$w(R, u, A) \equiv \int_{\bigcup_{x \in A} \partial u(x)} R(y) dy \quad \text{for all Borel } A \subset \mathbf{R}^d.$$
(0.1)

**Definition 2 (Motion by** *R*-curvature) The graph of  $u \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$  is called the motion by *R*-curvature if the following holds: for any  $\varphi \in C_o(\mathbf{R}^d : \mathbf{R})$  and any  $t \ge 0$ ,

$$\int_{\mathbf{R}^d} \varphi(x) u(t, x) dx - \int_{\mathbf{R}^d} \varphi(x) u(0, x) dx \qquad (0.2)$$
$$= \int_0^t ds \int_{\mathbf{R}^d} \varphi(x) w(R, u(s, \cdot), dx).$$

By the continuum limit of a class of infinite particle systems, we first show the existence of the motion by R-curvature, and then the uniqueness by the comparison theorem. We also show that the motion by R-curvature is a viscosity solution to

$$(PDE) \qquad \partial u(t,x)/\partial t = \chi(u, Du(t,x), t, x) \operatorname{Det}_+(D^2 u(t,x)) R(Du(t,x)),$$

where  $Du(t,x) \equiv (\partial u(t,x)/\partial x_i)_{i=1}^d$ ,  $D^2u(t,x) \equiv (\partial^2 u(t,x)/\partial x_i\partial x_j)_{i,j=1}^d$ ,

$$\chi(u, p, t, x) \equiv \begin{cases} 1 & \text{if } p \in \partial u(t, x), \\ 0 & \text{otherwise,} \end{cases}$$

 $\partial u(t, x)$  denotes the subdifferential of the function  $x \mapsto u(t, x)$ , and for a real  $d \times d$ -symmetric matrix X,

$$\operatorname{Det}_{+} X \equiv \begin{cases} \operatorname{Det} X & \text{if } X \text{ is nonnegative definite,} \\ 0 & \text{otherwise.} \end{cases}$$

We introduce the definition of the viscosity solution to (PDE).

**Definition 3 (Viscosity solution)** (Viscosity subsolution)  $u \in C((0, \infty) \times \mathbf{R}^d : \mathbf{R})$  is a viscosity subsolution of (PDE) if whenever  $\varphi \in C^2((0, \infty) \times \mathbf{R}^d : \mathbf{R})$  and  $u - \varphi \leq (u - \varphi)(t_o, x_o)$ ,

$$\partial \varphi(t_o, x_o) / \partial t \le \chi(u, D\varphi(t_o, x_o), t_o, x_o) Det_+(D^2 \varphi(t_o, x_o)) R(D\varphi(t_o, x_o)).$$

(Viscosity supersolution)  $u \in C((0,\infty) \times \mathbf{R}^d : \mathbf{R})$  is a viscosity supersolution of (PDE) if whenever  $\varphi \in C^2((0,\infty) \times \mathbf{R}^d : \mathbf{R})$  and  $u - \varphi \ge (u - \varphi)(t_o, x_o)$ ,

$$\partial \varphi(t_o, x_o) / \partial t \ge \chi^-(u, D\varphi(t_o, x_o), t_o, x_o) Det_+(D^2 \varphi(t_o, x_o)) R(D\varphi(t_o, x_o)).$$

Here  $\chi^-(v, p, t, x) = 1$  if

$$v(t,y) > v(t,x) + \langle p, y - x \rangle$$
  $(y \neq x)$ 

and if there exists  $\varepsilon > 0$  such that for all  $(s, y) \in (0, \infty) \times \mathbf{R}^d$  satisfying  $|y| > \varepsilon^{-1}$  and  $|s - t| < \varepsilon$ ,

$$v(s,y) > p \cdot y + \varepsilon |y|,$$

and  $\chi^{-}(v, p, t, x) = 0$ , otherwise.

**Remark 1** If  $\chi^{-}(v, p, t, x) = 1$  and s is close to t, then  $p \in \partial v(s, y)$  for some y.

Finally we discuss under what condition the viscosity solution to (PDE) is the motion by R-curvature.

## 2. Infinite particle systems and the motion by *R*-curvature.

In this section we construct the motion by R-curvature by the continuum limit of infinite particle systems.

Fix  $\varepsilon_n \downarrow 0$  as  $n \to \infty$ , and put

(A.1).  $||R||_{L^1} \equiv \int_{\mathbf{R}^d} R(y) dy > 0, R \ge 0, h \in C(\mathbf{R}^d : \mathbf{R}),$ (A.2).  $|\partial h(\mathbf{R}^d) (\equiv \bigcup_{x \in \mathbf{R}^d} \partial h(x))| > 0,$ 

$$S_n \equiv \{ v : \mathbf{Z}^d / n \mapsto \mathbf{R} | \sum_{z \in \mathbf{Z}^d / n} (v(z) - h(z)) < \infty,$$
$$(v(z) - h(z)) / \varepsilon_n \in \mathbf{N} \cup \{0\} \text{ for all } z \in \mathbf{Z}^d / n \}$$

Let  $\{Y_n(k, \cdot)\}_{0 \le k}$  be a Markov chain on  $S_n$  such that  $Y_n(0, \cdot) = h(\cdot)$ , and that

$$P(Y_n(k+1,\cdot) = v_{n,z} | Y_n(k,\cdot) = v) = w(R, \hat{v}, \{z\}) / w(R, Y_n(0,\cdot), \mathbf{R}^d),$$

where

$$v_{n,z}(x) \equiv \begin{cases} v(x) + \varepsilon_n & \text{if } x = z, \\ v(x) & \text{if } x \in (\mathbf{Z}^d/n) \setminus \{z\}. \end{cases}$$

Let  $p_n(t)$  be a Poisson process, with parameter  $n^d \varepsilon_n^{-1} w(R, \hat{Y}_n(0, \cdot), \mathbf{R}^d)$ , which is independent of  $Y_n$ . Put

$$Z_n(t,z) \equiv Y_n(p_n(t),z),$$

$$X_n(t,x) \equiv \max(\hat{Z}_n(t,x), h(x)).$$

For f and  $g \in C(\mathbf{R}^d : \mathbf{R})$ , we put

 $d_{C(\mathbf{R}^d:\mathbf{R})}(f,g) \equiv \sum_{m\geq 1} 2^{-m} \min(\sup_{|x|\leq m} |f(x) - g(x)|, 1).$ 

Then we show that  $X_n(t, x)$  converges to the motion by *R*-curvature under the following additional conditions. (A.3). The closure of the set  $\{x \in \mathbf{R}^d : \hat{h}(x) < h(x)\}$  does not contain any line which is unbounded in two different directions.

(A.4). For any  $p \notin \partial h(\mathbf{R}^d)$  and  $C \in \mathbf{R}$ ,

$$\int_{\mathbf{R}^d} \max(\langle p, x \rangle + C - h(x), 0) dx = \infty.$$

**Theorem 1** Suppose that (A.1) and (A.3)-(A.4) hold. Then there exists a unique continuous solution u to (1.2) with  $u(0, \cdot) = h$ . Suppose in addition that (A.2) holds. Then the following holds: for any  $\gamma > 0$  and T > 0,

$$\lim_{n \to \infty} P(\sup_{0 \le t \le T} d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(t, \cdot), u(t, \cdot)) \ge \gamma) = 0.$$

**Remark 2** (A.3) holds when d = 1. If h is convex, then (A.4) holds.

We give the properties of the motion by R-curvature.

**Theorem 2** Suppose that (A.1) holds. Let  $u \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$  be the solution to (1.2) with  $u(0, \cdot) = h$ . Then: (a)  $t \mapsto u(t, x)$  is nondecreasing. (b)  $u = \max(\hat{u}, h)$ (c)  $u(t, x) - \hat{u}(t, x) \leq h(x) - \hat{h}(x)$ . In particular, if  $h(x) = \hat{h}(x)$ , then  $u(t, x) = \hat{u}(t, x)$ .

Suppose in addition that (A.4) holds. Then: (d) For any t > 0,  $\partial u(t, \mathbf{R}^d) = \partial h(\mathbf{R}^d)$ .

$$\int_{\mathbf{R}^d} (u(t,x) - h(x)) dx = t \cdot w(R,h,\mathbf{R}^d).$$

(e) Let  $\overline{u} \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$  be the solution to (1.2) with  $u(0,\cdot) = \hat{h}$ . If  $u(s,\cdot) - \hat{u}(s,\cdot) \neq h - \hat{h}$  for some  $s \in (0,\infty)$ , then  $\overline{u}(t,\cdot) - \hat{u}(t,\cdot) \neq 0$  for all  $t \geq s$ .

According to the above theorem, (a) any graph moves upward by Rcurvature, (b) those points on any graph moving by R-curvature do not move as far as they stay in its cavities, (c) the height between any graph moving by R-curvature and its convex envelope is nonincreasing as it evolves, (d) any graph moving by R-curvature sweeps in time t > 0 a region with volume given by  $t \cdot w(R, h, \mathbf{R}^d)$ , and (e) for the motion of a graph by R-curvature, taking its convex envelope at time t > 0 and evolving up to time t starting with the convex envelope of the initial graph give different graphs in general, if the initial graph is not convex.

## 3. Motion by *R*-curvature and the viscosity solution.

In this section we discuss the relation between the motion by R-curvature and the viscosity solution to (PDE).

(A.5).  $R \in C(\mathbf{R}^d : [0, \infty)).$ 

**Theorem 3** Suppose that (A.1) and (A.5) hold. Then a continuous solution u to (1.2) with  $u(0, \cdot) = h$  is a viscosity solution to (PDE).

Theorem 3 means that the motion by R-curvature is the viscosity solution to (PDE). To discuss under what condition the reverse is true, we discuss the uniqueness of the viscosity solution to (PDE).

(A.6).  $R(x) \ge R(rx)$  for any  $r \ge 1$  and  $x \in \mathbf{R}^d$ . (A.7).  $\inf_{x \ne o} h(x)/|x| > 0$ . (A.8). There exists a constant C > 0 such that  $h(x+y)+h(x-y)-2h(x) \le C$ for all  $(x,y) \in \mathbf{R}^d \times U_1(o)$ , where  $U_1(o) \equiv \{y \in \mathbf{R}^d : |y| < 1\}$ .

**Theorem 4** Suppose that (A.1) and (A.3)-(A.8) hold. Then there exists a unique continuous viscosity solution u to (PDE) with  $u(0, \cdot) = h$  in the space of continuous functions v for which

$$\sup\{|v(t,x) - h(x)| : (t,x) \in [0,T] \times \mathbf{R}^d\} < \infty \text{ for all } T > 0.$$

u is also a unique continuous solution to (1.2) with  $u(0, \cdot) = h$ .

We restrict our attention to Gauss curvature flow and give a finer result. For  $A \subset \mathbf{R}^d$  and  $v : A \mapsto \mathbf{R}$ , put

$$epi(v) = \{(x, y) : x \in A, y \ge v(x)\}.$$

For r > 0, put

$$h^r(x) = \inf\{y \in \mathbf{R} \mid U_r((x,y)) \subset \operatorname{epi}(h)\} \quad (x \in \mathbf{R}^d).$$

Under the following condition, we give the comparison theorem for the continuous viscosity solution to (PDE).

(A.1)'.  $R(y) = (1 + |y|^2)^{-(d+1)/2}$  and  $h \in C(\mathbf{R}^d : \mathbf{R})$ . (A.2)'.

$$\liminf_{\theta \downarrow 1} \{\liminf_{r \to \infty} [\liminf_{|x| \to \infty} (h(\theta x) - h^r(x))] \} > 0,$$

$$\lim_{\theta \downarrow 1} \{ \sup_{x \in \mathbf{R}^d} (h(x) - h(\theta x)) \} = 0.$$

**Theorem 5** Suppose that (A.1)'-(A.2)' hold. Then for any viscosity subsolution u and supersolution v, of (PDE) in the space  $C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$ , such that  $u(0, \cdot) \leq h \leq v(0, \cdot), u \leq v$ .

**Remark 3** (A.2)' holds if there exists a convex function  $h_0 : \mathbf{R}^d \mapsto \mathbf{R}$  such that  $h_0(x) \to \infty$  as  $|x| \to \infty$  and that

$$\lim_{|x| \to \infty} [h(x) - h_0(x)] = 0.$$

In fact, the following holds:

$$\lim_{|x|\to\infty} [h(\theta x) - h^r(x)] = \infty \quad \text{for all } \theta > 1, r > 0,$$

$$\lim_{\theta \downarrow 1} \{ \sup_{x \in \mathbf{R}^d} [h(x) - h(\theta x)] \} = 0.$$

The following corollary is better than Theorem 4 in that we can consider the viscosity solution in the entire space  $C(\mathbf{R}^d : \mathbf{R})$ .

**Corollary 1** Suppose that (A.1)'-(A.2)' and (A.3)-(A.4) hold. Then there exists a unique continuous viscosity solution u to (PDE) with  $u(0, \cdot) = h$ . u is also a unique continuous solution to (1.2) with  $u(0, \cdot) = h$ .

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where they studied a similar result to Theorem 5 for the mean curvature flow with a convex coercive initial function.