Convergence of Multigrid Method for Edge-Based Finite-Element Method

K. Watanabe, H. Igarashi, and T. Honma, *Member, IEEE*

***Abstract*—This paper discusses robustness of the multigrid (MG) method against distortion of finite elements. The convergence of MG method becomes considerably worse as the finite elements become flat. It is shown that the smoother used in the MG method cannot effectively eliminate the high-frequency component of the residue for flat elements, and this gives rise to deterioration in the convergence. Moreover, the multigrid method with conjugate gradient (CG) smoother is shown to be more robust against mesh distortion than that with Gauss–Seidel smoother.**

***Index Terms*—Convergence, eigenvalue, multigrid (MG).**

I. Introduction

THE MULTIGRID (MG) method has been applied to electromagnetic field problems so far [1] to show that it can significantly reduce computational time in comparison with conventional linear solvers such as the incomplete Cholesky conjugate gradient (ICCG) solver. However, it has been pointed out [2] that the convergence of the MG method becomes considerably worse as finite elements become flat. It is important to develop the robust MG method for practical use. In this paper, we pay attention to the property of the smoother that plays a crucial role in the MG method. We investigate the robustness of the MG method with different smoothers. Moreover, the residue of a linear system is decomposed into the Fourier components, and each convergence is numerically investigated for different flatness of elements to clarify the reason for such an effect.

II. Formulation

A. Magnetostatic Problem

Let us consider magnetostatic field governed by

$$\nabla \times \nu \nabla \times \boldsymbol{A} = \boldsymbol{J}_0 \tag{1}$$

$$\nabla \cdot \boldsymbol{J}_0 = 0 \tag{2}$$

where ν is the magnetic reluctivity, $\boldsymbol{A}$ is the vector potential, and $\boldsymbol{J}_0$ is the current density. The current vector potential

$$\nabla \times \boldsymbol{T} = \boldsymbol{J}_0 \tag{3}$$

is introduced for satisfaction of (2). Equation (1) now leads to

$$\nabla \times \nu \nabla \times \boldsymbol{A} = \nabla \times \boldsymbol{T}. \tag{4}$$

Finite-element discretization of (4) results in the system of linear equations

$$[K]\{x\} = \{b\} \tag{5}$$

where $[K]$ is a positively semidefinite matrix which is the discrete counterpart of the operator in the left-hand side of (4), $\{x\}$ and $\{b\}$ denote column vectors corresponding to $\boldsymbol{A}$ and $\boldsymbol{T}$, respectively.

B. Multigrid

It is known that the linear solvers such as Gauss–Seidel and conjugate gradient (CG) methods tend to eliminate the high-frequency components of the residue in (5) more rapidly than the low-frequency components. The MG method is based on this property, that is, the high-frequency residual components are eliminated on a fine mesh by small numbers of iterations of the linear solver (smoother). The remaining residual components are then projected onto a more coarse mesh, in which they now have high frequency that can again be eliminated by small numbers of iterations. The MG method solves (5) successively performing these processes. This procedure is usually called the coarse grid correction. Although there are many variations in the MG method, all these variations are based on the coarse grid correction. The procedure of the two-grid V-cycle method that is the simplest MG method is described later.

Step 1 (Smoothing): The smoothing operation is applied to the system equation

$$[K_f]\{x\} = \{b\} \tag{6}$$

for the fine mesh to obtain approximate solution $\{\tilde{x}\}$, where $[K_f]$ denotes the system matrix defined on the fine mesh. In this step, the high-frequency components in the solution error are eliminated.

Step 2: The residual vector $\{r_f\}$ corresponding to the approximate solution $\{\tilde{x}\}$ is calculated

$$\{r_f\} = \{b\} - [K_f]\{\tilde{x}\}. \tag{7}$$

Step 3 (Restriction): The residual vector is projected onto a coarser mesh using the restriction matrix $[R]$

$$\{r_c\} = [R]\{r_f\} \tag{8}$$

where the component of the matrix $[R]$ is obtained by the following integration:

$$R_{ij} = \int_{E_j^f} \boldsymbol{w}_i^c \cdot d\boldsymbol{l} \tag{9}$$

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The authors are with the Division of Systems and Information Engineering, Graduate School of Engineering, Hokkaido University, Kita-ku Sapporo 060-8628, Japan (e-mail: wata@em-si.eng.hokudai.ac.jp).

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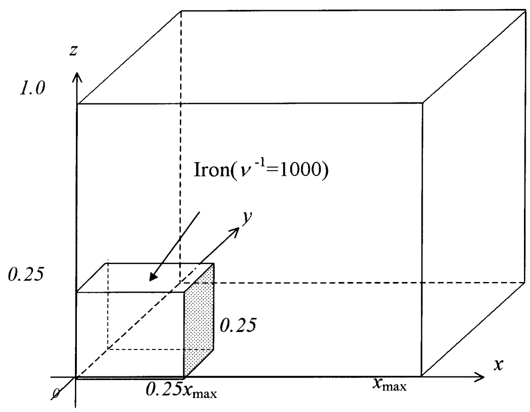


Fig. 1. Simple analysis model (1/8).

and E_j^f denotes jth edge in fine mesh, and $\boldsymbol{w}_i^c$ denotes the interpolation function corresponding to ith edge in the coarse mesh.

Step 4: The residual equation in coarse mesh is solved to obtain the error vector $\{e_c\}$ corresponding to the residual vector $\{r_c\}$

$$[K_c]\{e_c\} = \{r_c\} \tag{10}$$

where $[K_c]$ is the system matrix defined in the coarse mesh. It takes a short amount of time to solve (10) because there are a small number of unknowns in (10). Equation (10) cannot be solved by the direct solver such as Gauss-elimination method, because $[K_c]$ is singular. For this reason, the CG or ICCG method is used in this paper.

Step 5 (Prolongation): The error vector is projected onto the fine mesh using the prolongation matrix $[P]$

$$\{e_f\} = [P]\{e_c\} \tag{11}$$

where $[P]$ is usually chosen as the transpose of $[R]$.

Step 6: The solution $\{\tilde{x}\}$ obtained in Step 1 is corrected using error vector $\{e_f\}$

$$\{x^{\text{new}}\} = \{\tilde{x}\} + \{e_f\}. \tag{12}$$

Step 7 (Post-Smoothing): The smoothing operation is applied to the system equation again. This procedure is called post-smoothing. After post-smoothing, the convergence of the solution is tested. If the convergence condition is not satisfied, we go back to Step 2.

III. Comparison of the Different Smoothers

To investigate the robustness of the MG method against mesh distortion, we analyze a simple magnetostatic problem shown in Fig. 1. Only 1/8 of the model is considered due to the symmetry. The whole region ($0 \le x \le x_{\max}$, $0 \le y$, $z \le 1$) is divided into 768 tetrahedral elements for the coarse mesh. The fine mesh is automatically created from the coarse mesh as follows. First, the coarsest mesh is prepared by a mesh generator. The finer meshes are then obtained by dividing each coarse element into eight finer elements as shown in Fig. 2 [3].

To evaluate the flatness of the elements, two types of aspect ratio are defined: $a_1 = l_{\max}/l_{\min}$ and $a_2 = l_{\max}/d_{\min}$, where $l_{\max}$ is the length of the largest edge, $l_{\min}$ is the length of the smallest edge, and $d_{\min}$ is the smaller distance between the

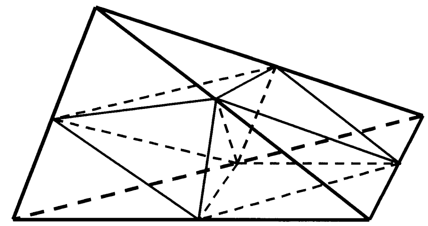


Fig. 2. Division to make the fine mesh.

TABLE I
Aspect Ratio of Elements

$x_{\max}$	Mesh	Aspect ratio (max./mean) a_1	Aspect ratio (max./mean) a_2
1	coarse	1.63/1.63	2.82/2.82
	fine	1.91/1.71	4.06/3.20
10	coarse	10.0/8.38	20.1/15.8
	fine	15.0/8.27	33.6/17.5
20	coarse	20.0/16.7	40.1/31.4
	fine	30.0/16.3	67.1/34.8
50	coarse	50.0/41.6	100/78.4
	fine	75.0/40.3	168/87.0

TABLE II
Comparison of Calculation Time

$x_{\max}$	Method	Time [s]
1	ICCG	0.8
	MG with GS smoother	0.3
	MG with CG smoother	3.3
10	ICCG	22.0
	MG with GS smoother	6.8
	MG with CG smoother	27.4
20	ICCG	172
	MG with GS smoother	59.4
	MG with CG smoother	49.5
50	ICCG	>300
	MG with GS smoother	182
	MG with CG smoother	102

vertex and that diagonal face in an element [4]. Table I shows the aspect ratio of these meshes. Finite elements become flatter as $x_{\max}$ grows.

Table II shows the calculation time of the MG method with two different smoothers (Gauss–Seidel and CG smoother) as well as of the conventional ICCG method. The calculations are performed on a personal computer with Pentium III-1.26 GHz.

It is shown that the convergence of the MG method is strongly influenced by the mesh quality. Moreover, the MG method with CG smoother is shown to be more robust against mesh distortion than that with Gauss–Seidel smoother.

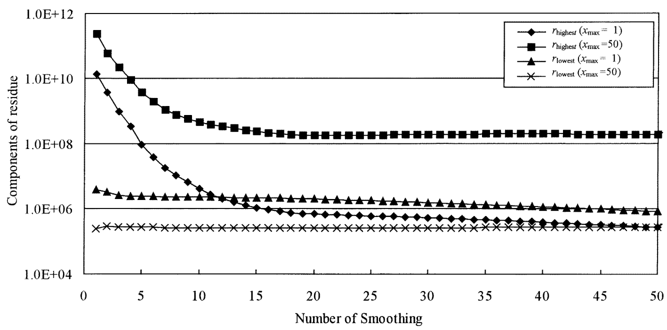
IV. Decomposition of Residual Vector

Here, we consider the property of the eigenvalues in a system matrix. Finite elements become flatter as $x_{\max}$ grows. It is known that convergence of the CG method becomes better (worse) when the condition number

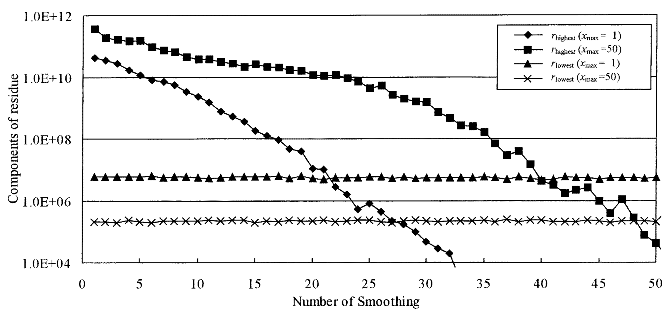
$$k = \lambda_{\max}/\lambda_{\min} \tag{13}$$

TABLE III
CONDITION NUMBER AND EIGENVALUES IN SYSTEM MATRIX

x_{max}	Condition number	Eigenvalue λ_{max}	λ_{min}
1	1.85×10^{4}	100.7	5.438×10^{-3}
10	1.41×10^{5}	365.8	2.140×10^{-3}
20	6.68×10^{5}	723.4	1.083×10^{-3}
50	4.15×10^{6}	1803	4.345×10^{-4}



(a)



(b)

Fig. 3. Reduction of residual components. (a) Gauss–Seidel smoother. (b) CG smoother.

becomes smaller (larger) [5], [6] where $\lambda_{\min}$ is the nonzero smallest eigenvalue and $\lambda_{\max}$ is the largest eigenvalue of the system matrix. Table III shows k for different values of $x_{\max}$. We can see that the condition number becomes larger as $x_{\max}$ increases. The condition number is expected to characterize the convergence of not only the CG method but also the MG method. However, the condition number is not always available because it takes a long time to calculate the eigenvalues.

Next, to consider the cause of the poor convergence in more detail, the convergence of each Fourier component of the residue in the smoothing process of the MG method is plotted in Fig. 3. The residual components r_{lowest} and r_{highest} with the lowest and highest spatial frequency, respectively, are defined by

$$r_{\text{lowest}} = \{v_{\min}\}^T\{r\} \tag{14}$$

$$r_{\text{highest}} = \{v_{\max}\}^T\{r\} \tag{15}$$

where $\{v_{\min}\}$ and $\{v_{\max}\}$ are the eigenvectors corresponding to $\lambda_{\min}$ and $\lambda_{\max}$, and $\{r\}$ is the residual vector after smoothing process.

First, we consider the results of the Gauss–Seidel smoother. When the whole region is nearly a cube ($x_{\max} = 1$), the r_{highest} rapidly reduces within small numbers of iterations as expected. When the element becomes flat ($x_{\max} = 50$), r_{highest} rapidly reduces within small numbers of iterations again. However, there remains the relatively large residue and it hardly decreases any longer. This means that the smoother cannot effectively eliminate the residue for flat elements so that the MG method with Gauss–Seidel smoother requires a number of iterations.

Next, we consider the CG smoother. Although the convergence of r_{highest} is affected by the flatness of element, r_{highest} decreases almost linearly with the iteration. This means that the CG smoother with enough iteration can eliminate the residue even for flat elements. They are consistent with the results shown in Table II.

The residue r_{lowest} in both Gauss–Seidel and CG smoother seemingly unchange because they reduce very slowly. Therefore, r_{lowest} should be reduced by the coarse grid correction.

V. CONCLUSION

This paper discusses dependence of the MG method on the shape of finite elements. The convergence of the MG method becomes considerably worse as the finite elements become flat. It is shown that the smoother used in the MG method cannot effectively eliminate the high-frequency component of the residue for flat elements, and this gives rise to deterioration in the convergence. Moreover, the MG method with CG smoother is shown to be more robust against mesh distortion than that with Gauss–Seidel smoother.

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