Acoustic phonons in multiconstituent superlattices

S. Tamura
Department of Physics and Materials Research Laboratory, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801
and Department of Engineering Science, Hokkaido University, Sapporo 060, Japan

J. P. Wolfe
Department of Physics and Materials Research Laboratory, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

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Acoustic phonons propagating parallel to the growth direction of a periodic superlattice composed of more than two different constituent layers are studied theoretically. We give general formulas for the phonon dispersion relation and the width $\Delta \nu$ of frequency gaps existing at the center and boundary of the folded Brillouin zone. We also calculate the structure factor for phonons and find that $\delta$-function peaks corresponding to Bragg reflection have prefactors proportional to $\Delta \nu$. This relation can be used to explain the relative magnitude of transmission dips found at frequencies satisfying the Bragg condition.

I. INTRODUCTION

Acoustic-phonon propagation in periodic and quasi-periodic (Fibonacci) superlattices has been studied extensively both experimentally and theoretically. Though superlattices consist typically of a periodic array of alternating layers, there has been a growing interest in those having more than two layers in a unit cell. Such multiconstituent superlattices have been proposed and fabricated.

Phonons in an ideal, periodic superlattice are Bragg reflected when their wave number $q$ parallel to the growth direction satisfies $q = q_m = m \pi / D$, where $m$ is an integer and $D$ is the period of the superlattice. These wave numbers correspond to the center and boundary of the folded Brillouin zone defined by $-\pi / D \leq q \leq \pi / D$. At $q = q_m$ frequency gaps (phonon stop bands) in the dispersion relation are produced. In a real superlattice with a finite number of periods, phonons in these frequency gaps have a finite transmission probability through the superlattice but large dips in transmission are predicted in general. This phonon filtering effect of a superlattice has been observed experimentally both by phonon spectroscopy and by phonon imaging.

It is well known that the condition for Bragg reflection is determined by the relative magnitudes between the phonon wavelength and the length of a period. However, related effects on phonons due to Bragg reflection, such as the width of frequency gaps and magnitude of transmission dips, depend sensitively on the structure of the layers comprising the unit cell of a superlattice.

In the present work we shall study theoretically the acoustic phonons in superlattices with $N$ layers ($N \geq 2$) per period. This study was motivated by the transmission rate shown in Fig. 1, which is calculated for transverse acoustic (TA) phonons propagating normal to interfaces of a (001)-GaAs/AlAs superlattice. The superlattice assumed is a periodic array $ABA \ldots$ consisting of seven $A$ [(=20-Å GaAs) (17-Å AlAs)] blocks, which is a periodic version of the Fibonacci superlattice fabricated by Merlin et al., i.e., $ABAABA \ldots$. A simple question is why some of transmission dips (especially that corresponding to the fourth-order Bragg reflection) are small and some are large. In order to answer this kind of question we shall calculate the phonon dispersion relation and the structure factor for phonons in the general multilayer superlattice depicted in Fig. 2, where the unit cell composed of $N$ layers repeats periodically. (Note that in the above example, $N = 4$ and $D = 96$ Å.)

![Graphical representation of the transmission rate](image-url)
A. COUCTION PHONONS IN MULTICONSTITUENT SUPERLATTICES

II. DISPERSION RELATION

We consider phonons with wave vectors perpendicular to interfaces. By assuming the decoupling among different phonon modes, we denote by $U_n$ and $S_n$ the nonvanishing components of phonon displacement and stress in $n$th layer with thickness $d_n$ $(1 \leq n \leq N)$. Explicitly, they are written as

$$U_n(z) = A_n \exp(ik_n z) + B_n \exp(-ik_n z),$$

$$S_n(z) = i \omega Z_n \left[ A_n \exp(ik_n z) - B_n \exp(-ik_n z) \right],$$

where $A_n$ and $B_n$ are amplitudes, $k_n$ is the wave number, $\omega$ is the angular frequency, and the acoustic impedance in the $n$th layer $Z_n = \rho_n v_n$, with $\rho_n$ the mass density and $v_n$ the phonon phase velocity. $U_n$ and $S_n$ should satisfy (a) the continuity condition at each interface and (b) an appropriate periodicity condition. Specifically, these conditions are given by

$$W_n = W_{n+1} \left[ \sum_{i=1}^{n} d_i \right] (1 \leq n \leq N - 1),$$

where we have defined $W_n(z) = (U_n(z), S_n(z))$, $D = \sum_{n=1}^{N} d_n$ is the period, and $q$ is the superlattice wave number of phonons. These conditions provide $2N$ equations with respect to $A_1 - A_N$ and $B_1 - B_N$. In order that they have a nontrivial solution, the determinant of $2N \times 2N$ matrix consisting of the coefficients of $A_n$ and $B_n$ $(1 \leq n \leq N)$ should vanish. The phonon dispersion relation $\omega$ versus $q$ is obtained directly from this condition. The calculation is somewhat tedious but straightforward. The result can be written explicitly by introducing operators $Q^{(0)}, Q^{(1)}, Q^{(2)}, \ldots$, which operate on an arbitrary function $g(111, 112, \ldots, 11N)$ of $111, 112, \ldots, 11N$. With the use of these operators, the dispersion relation is

$$\cos(qD) = \frac{1}{2} \prod_{n=1}^{N} Z_n \cos(qD) = \left[ Q^{(0)} - \sum_{i=1}^{N} Q^{(1)}_{i} \right] \left[ + \sum_{i=1}^{N} \sum_{j=i+1}^{N} Q^{(2)}_{ij} - \cdots - (-1)^{[N/2]} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \sum_{p=1}^{N} Q^{(N/2)}_{ijp} \right]$$

$$\times \prod_{n=1}^{N} \left( \eta_n Z_n + \eta_{n+1} Z_{n+1} \right) \cos \left[ \sum_{i=1}^{N} \eta_i \Omega_i \right],$$

where $Z_{N+n} = Z_n$, $\Omega_n = k_n d_n = \omega d_n / v_n$, $\gamma = 1 \left( \frac{1}{2} \right)$ if $N$ is odd (even), and $[n]$ means the maximum integer which does not exceed $n$. For $N = 2$, Eq. (4) is reduced to the well-known form of the dispersion relation given by Rytov. For $N = 3$, Eq. (4) is reduced to that given by Nakayama et al. or by Santos and Ley. However, Eq. (4) is the nontrivial extension of these previous works.

For the four-layer GaAs/AlAs superlattice discussed above, Eq. (4) becomes (by putting $n = 1 = 3 = \text{GaAs}$ and $n = 2 = 4 = \text{AlAs}$)

$$\cos(qD) = \cos(\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4)$$

$$- \frac{(Z_1 - Z_2)^2}{2Z_1 Z_2} \sin(\Omega_1 + \Omega_3) \sin(\Omega_2 + \Omega_4)$$

$$+ \frac{1}{2} \left[ \frac{Z_1^2 - Z_2^2}{Z_1 Z_2} \right] \sin \Omega_1 \sin \Omega_2 \sin \Omega_3 \sin \Omega_4.$$
ted for the TA mode. Frequency gaps are created at \( qD = m \pi \) because for \( Z_1 \neq Z_2 \) the modulus of the right-hand side of Eq. (5) exceeds unity, whereas \( |\cos(qD)| \leq 1 \). It is important to note that the central frequency and frequency width of the gaps are well correlated with the central frequency and depth of transmission dips of Fig. 1, respectively.

### III. WIDTH OF FREQUENCY GAPS

Provided that all layers in a unit cell have acoustic impedances of the same order of magnitude, we can also derive a formula for the width of frequency gap by a perturbation calculation. By keeping terms up to second order in the difference of acoustic impedance we have from Eq. (4)

\[
2^N \left( \prod_{n=1}^{N} Z_n \right) \cos(qD) = \left( \prod_{n=1}^{N} (Z_n + Z_{n+1}) \right) \cos \left( \sum_{l=1}^{N} \Omega_l \right) + \sum_{n=1}^{N} \left( P_n^{(m)} \right) \cos \left( \sum_{l=1}^{N} \eta_l \Omega_l \right),
\]

where

\[
P_n^{(m)} = (Z_n - Z_{n+1}) [(Z_{n+1} - Z_{n+2}) Q_{n+1}^{(1)} + (Z_{n+2} - Z_{n+3}) Q_{n,n+1}^{(2)} + \cdots + \gamma (Z_{n+(N/2)} - Z_{n+1+(N/2)}) Q_{n+1,n+(N/2)}^{(N/2)}].
\]

Now we assume that \( \omega = \omega_m + \delta \omega_m \) at \( q = q_m \), where \( \omega_m \) is the zeroth order in \( \epsilon_{ij} = (Z_i - Z_j)/Z \) (\( Z \equiv \sum_{n=1}^{N} Z_n/N \)) and given by

\[
\omega_m = \frac{m \pi}{\sum_{n=1}^{N} (d_n/\nu_n)}.
\]

This is the phonon frequency at the center of the \( m \)th band gap. Next, we assume that the magnitude of \( (\delta \omega_m/\omega_m)^2 \) is the same order as \( \epsilon_{ij}^2 \). Equating terms of the order of \( \epsilon_{ij}^2 \) in Eq. (6), we have

\[
\frac{\delta \omega_m}{\omega_m} = \frac{1}{(2m \pi \bar{Z})^2} \sum_{n=1}^{N} (Z_n - Z_{n+1})^2 + 2(-1)^m P_n^{(m)} \cos \left( \omega_m \sum_{l=1}^{N} \eta_l \Omega_l \right).
\]

Applying this formula again to the four-layer GaAs/AlAs superlattice exemplified above, we obtain

\[
\frac{\delta \omega_m}{\omega_m} = (-1)^m \left( \frac{\epsilon_{12}}{m \pi} \right)^2 \left[ -\sin \omega_m \left( \frac{d_1 + d_3}{\nu_1} \right) \sin \omega_m \left( \frac{d_2 + d_4}{\nu_2} \right) \right] + 4 \sin \omega_m \left( \frac{d_1}{\nu_1} \right) \sin \omega_m \left( \frac{d_2}{\nu_2} \right) \sin \omega_m \left( \frac{d_3}{\nu_1} \right) \sin \omega_m \left( \frac{d_4}{\nu_2} \right).
\]

The width of the \( m \)th frequency gap is \( \Delta \nu_m = \delta \omega_m / \pi \). In Table I we have tabulated up to \( m = 10 \) the gap width \( \Delta \nu_m^{\text{exact}} \) given exactly from the dispersion relation (5) and \( \Delta \nu_m \) obtained by the above formula (10). The agreement of \( \Delta \nu_m \) with \( \Delta \nu_m^{\text{exact}} \) is excellent, validating the approximation we have employed. Incidentally, we note \( \epsilon_{12} = 0.033 \) for TA mode in GaAs/AlAs heterostructures.

### IV. STRUCTURE FACTOR

The effects on Bragg reflection of a multilayer structure in a unit cell are also understood by examining the structure factor of superlattices for phonons. For phonons with superlattice wave number \( q \) propagating normal to interfaces spaced by \( D \), the interference of reflected phonons is described by the structure factor

\[
S(q) = \sum_n e^{2imqD} = \frac{\pi}{D} \sum_m \delta(q - q_m).
\]

The \( \delta \)-function peaks of \( S(q) \) mean that Bragg reflection occurs for phonons with \( q = q_m \). Here, we note that \( q \) is the wave number of a modulated plane wave propagating in a whole superlattice, independent of the detailed structure in a unit cell. The existence of multilayers in the unit cell causes extra reflections at their boundaries and gives additional phases to the structure factor. This amounts to the modification \( S(q) \rightarrow S_N(q) = f_N(q) S(q) \), where

\[
f_N(q) = \sum_{n=1}^{N} r_{n,n+1} \exp \left[ i \sum_{l=1}^{N} \Omega_l \right],
\]

and \( r_{n,n+1} \) is the amplitude-reflection coefficient at the interface between \( n \)th and \((n+1)\)th layers (for phonons impinging on the interface from the \( n \)th layer). Explicit expression for \( r_{n,n+1} \) is derived from Eqs. (1) and (2) as

\[
r_{n,n+1} = \frac{Z_n - Z_{n+1}}{Z_n + Z_{n+1}}.
\]

Here, we remark that Eq. (12) is derived by neglecting multiple phonon reflections [which add terms of the or-
TABLE I. Frequency $v$ and gap width $\Delta v$ associated with Bragg reflection of TA phonons in 42-Å GaAs-17-Å AlAs-20-Å GaAs-17-Å AlAs superlattice.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$v_m$ (GHz)</th>
<th>$\Delta v_m$ (GHz)</th>
<th>$\Delta v_m$ (GHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>184</td>
<td>7.5</td>
<td>7.5</td>
</tr>
<tr>
<td>2</td>
<td>367</td>
<td>25.7</td>
<td>25.6</td>
</tr>
<tr>
<td>3</td>
<td>551</td>
<td>38.4</td>
<td>38.4</td>
</tr>
<tr>
<td>4</td>
<td>735</td>
<td>1.8</td>
<td>1.7</td>
</tr>
<tr>
<td>5</td>
<td>919</td>
<td>24.7</td>
<td>24.6</td>
</tr>
<tr>
<td>6</td>
<td>1102</td>
<td>4.7</td>
<td>4.6</td>
</tr>
<tr>
<td>7</td>
<td>1286</td>
<td>6.2</td>
<td>6.2</td>
</tr>
<tr>
<td>8</td>
<td>1470</td>
<td>31.1</td>
<td>31.0</td>
</tr>
<tr>
<td>9</td>
<td>1645</td>
<td>11.8</td>
<td>11.8</td>
</tr>
<tr>
<td>10</td>
<td>1837</td>
<td>32.1</td>
<td>32.1</td>
</tr>
</tbody>
</table>

order of $r_{n,n+1}$ to Eq. (12) and assuming perfect phonon transmission at interfaces except at the one from which phonons are reflected. This approximation is valid for most superlattices because $|r_{n,n+1}|^2 < 1$ and $t_{n,n+1} = 1 - |r_{n,n+1}|^2 = 1$, where $t_{n,n+1} = 1 + r_{n,n+1}$ is the amplitude transmission coefficient. For example, at a (001)-GaAs/AlAs interface $|r_{GaAs,AlAs}|^2 = 0.0082$ for the TA mode and 0.0074 for the longitudinal acoustic (LA) mode. Thus the layered structure in a unit cell affects the relative strength of Bragg reflection but not the basic Bragg condition.

Now, we calculate $f_N$ at $q = q_m$ for which the structure factor is nonvanishing. In the lowest order in the difference of acoustic impedance we can put $r_{n,n+1} = e_{n,n+1}/2$ and $\omega = \omega_m$. Hence, after a bit of algebra as described in the Appendix we derive a simple equation

$$S_N(q) = \frac{D}{m} \sum_{n} f_N(q_m) \delta(q - q_m),$$

(14)

where $f_N$ is related to $\delta \omega$ defined by Eq. (9) as

$$|f_N(q_m)|^2 = \left( m \pi(v_n/v_m) \right)^2.$$

(15)

Thus, the modulation factor $f_N(q_m)$ which determines the intensity of the $m$th Bragg reflection is proportional to the width of the corresponding gap in phonon dispersion relation. This establishes the close correlation between the magnitude of transmission dips in Fig. 1 and the width of frequency gaps shown in Fig. 3, which was previously noted.

V. CONCLUSION

To conclude, we have derived a formula for the dispersion relation of acoustic phonons propagating normal to the interfaces of multiconstituent superlattices. The formulation is exact in the context of continuum elasticity theory. The formula for the width of the frequency gap $\Delta v$ in the dispersion relation is also derived from a perturbation calculation, and gives very accurate numerical values for the majority of superlattices. Our calculations also establish a close relationship between $\Delta v$ and the interference effects of phonons reflected at different interfaces of constituent layers. We believe that these general results will be useful for comparison to future experiments because in most spectroscopic experiments phonons propagating normal to the interfaces of a superlattice are detected.

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APPENDIX

In this appendix we shall derive Eq. (15). Multiplying $r_{n,n+1} \exp(-2i\sum_{l=1}^{n} \Omega_{l})$ by Eq. (12) and summing over $n'$, we obtain

$$|f_N(q)|^2 = \sum_{n=1}^{N} r_{n,n+1}^2 + 2 \sum_{j=1}^{N-1} \sum_{n=1}^{j} r_{n,n+1} t_{n,n+j} + t_{n+1,n+j+1} \cos \left( \frac{2 \pi + \Omega_j}{2} \right),$$

(1A)

where $r_{N,N+1} \equiv r_{N,1}$. At $q = q_m$, $\Omega_n = \omega_m d_n/v_n$ and $\sum_{n=1}^{N} \Omega_n = m \pi$ hold in the zeroth order of $\epsilon_{ij}$. Hence, from (1A) we can derive
\[ |f_N(q_m)|^2 = \sum_{n=1}^{N} r_{n,n+1}^2 + 2(-1)^m \sum_{j=1}^{[N/2]} \sum_{n=1}^{N} \gamma_j r_{n,n+j} r_{n+n+j} 2^{(m+1)} + 1 \cos \left( \sum_{j=1}^{N} \eta_j \Omega_j^{(m)} \right), \]  
(A2)

where \( \gamma_j = 1 \) (1 \( \leq j \leq [N/2] - 1 \)) and \( \gamma_{[N/2]} = \gamma \). Using Eq. (13) and keeping the terms up to \( \epsilon_j \), (A2) is reduced to

\[ |f_N(q_m)|^2 = \frac{1}{4Z^2} \sum_{n=1}^{N} (Z_n - Z_{n+1})^2 + \frac{(-1)^m [N/2]}{2Z^2} \sum_{j=1}^{[N/2]} \sum_{n=1}^{N} \gamma_j (Z_n - Z_{n+1})(Z_{n+j} - Z_{n+j+1}) \Omega_j^{(m)} + 1 \cos \left( \sum_{j=1}^{N} \eta_j \Omega_j^{(m)} \right), \]  
(A3)

where \( Z_{n+N} = Z_n \) (1 \( \leq n \leq N - 1 \)). Comparing (A3) with Eqs. (7) and (9), we establish Eq. (15), i.e.,

\[ |f_N(q_m)|^2 = \frac{1}{2Z^2} \delta \varepsilon \omega_m \omega_m. \]