Classification of *R*-operators[a]

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We classified the *R*-operators which satisfy the quantum Yang–Baxter equation on a function space. In this study, we gave all the meromorphic solutions of the system of the functional equations which is a necessary and sufficient condition for the *R*-operator to satisfy the Yang–Baxter equation. Most of the solutions were expressed in terms of the elliptic, trigonometric and rational functions. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1367326]

I. INTRODUCTION

During the last 8 years, significant advances have been made in our understanding of the solutions of the (quantum) Yang–Baxter equation on a function space, which we call the *R*-operators.[1–3]

Definition 1 (R-operator[1]): For $x_1,x_2,\ldots,x_n\in\mathbb{C}$ and $r>0$, define the sets $C(x_1,r)$ and $C((x_1,x_2,\ldots,x_n),r)$ by $C(x_1,r)=\{x\in\mathbb{C};|x-x_1|<r\}$ and $C((x_1,x_2,\ldots,x_n),r)=C(x_1,r)\times C(x_2,r)\times\ldots\times C(x_n,r)$. Let functions $A(x)$ and $B(u,x)$ be meromorphic on $C(0,r)$ and $C((0,0),r)$, respectively. For a function f meromorphic on $C((0,0),r/2)$, we define the function $(R(u)f)(z_1,z_2)$ meromorphic on $C(0,r)\times C((0,0),r/2)(\ni(u,z_1,z_2))$ as

$$(R(u)f)(z_1,z_2)=A(z_1-z_2)f(z_1,z_2)-B(u,z_1-z_2)f(z_2,z_1).$$

We call this operator $R(u)$ the *R*-operator.

There are three kinds of the *R*-operators expressed in terms of the elliptic, trigonometric, and rational functions, respectively. The elliptic *R*-operator has been investigated in particular. We found it by taking the limit $n\to\infty$ of Belavin's *R*-matrix.[1] Belavin's *R*-matrix is conversely obtained through restricting the domain of a modified version of the elliptic *R*-operator to a suitable finite-dimensional subspace.[4] This suggests that the properties of Belavin's *R*-matrix are generalized to those of the elliptic *R*-operator. Actually the author constructed the incoming and outgoing intertwining vectors for the elliptic *R*-operator, and proved the vertex-IRF correspondence.[5] The boundary *K*-operators,[6,7] which satisfy the boundary Yang–Baxter equation for the elliptic *R*-operator, are also obtained. We essentially use the elliptic *R*-operator and boundary *K*-operators to construct the (generalized) Ruijsenaars operators,[8–10] the commuting difference operators. Therefore, it is very important to find out new solutions of the Yang–Baxter equation in order to investigate the integrable models. What remains a question is the classification of the *R*-operators.

The aim of this article is to classify the *R*-operators.

Proposition I.1: For any function f meromorphic on $C((0,0,0),r/2)$, a necessary and sufficient condition for the functions $R_{12}(u)R_{13}(u+v)R_{23}(v)f$ and $R_{23}(v)R_{13}(u+v)R_{12}(u)f$ meromorphic on $C((0,0,0,0,0),r/2)$ to satisfy the Yang–Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v)f=R_{23}(v)R_{13}(u+v)R_{12}(u)f$$

is that the meromorphic functions A and B satisfy the following equations on $C((0,0,0,0),r/2)$:

[a]Dedicated to Professor Yoshiyuki Shimizu on the occasion of his sixtieth birthday.

$$B(u+v,x+y)(A(x)A(-x)-A(y)A(-y))$$
$$=B(v,y)B(u+v,x)B(u,y)-B(u,x)B(u+v,y)B(v,x), \tag{1}$$

$$A(y)B(u,x)B(v,x+y)=A(y)(B(u+v,x+y)B(u,-y)+B(v,y)B(u+v,x)). \tag{2}$$

Therefore, in order to classify the R-operators, we gave the complete classification of the meromorphic solutions A and B of the functional equations (1) and (2).

Theorem I.2: *The meromorphic solutions $A(x)$ and $B(u,x)$ of Eqs. (1) and (2) defined on the polydiscs $C(0,r)$ and $C((0,0),r)$, respectively, are one of the following*:
0. *Trivial case:*

$$A(x) \ \ \textit{is arbitrary}, \quad B(u,x)\equiv 0.$$

$$A(x)\equiv 0,$$

$$B(u,x)=\exp(F(x)u)G(u) \ \ \textit{on} \ \ C(0,r)\times C(0,r_1)$$

$$(0<r_1\leqslant r).$$

1. *Generic case:*
1-1. *Elliptic:*

$$A(x)=c\cdot h(x)\frac{\sigma(x+s;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)\sigma(s;\tau_1,\tau_2)},$$

$$B(u,x)=c\exp(\rho ux)\frac{\sigma(x+au;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)\sigma(au;\tau_1,\tau_2)}$$

$$(a,c,\tau_1,\tau_2\in\mathbb{C}\backslash\{0\},\mathrm{Im}\,\tau_2/\tau_1>0,s\in\mathbb{C}\backslash(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2),\rho\in\mathbb{C}).$$

1-2. *Trigonometric:*

$$A(x)=\begin{cases} c\cdot h(x)\dfrac{\sinh(x+s)/\lambda}{\sinh(x/\lambda)\sinh(s/\lambda)}, \\ c\cdot h(x)\dfrac{1}{\sinh(x/\lambda)}, \end{cases}$$

$$B(u,x)=\begin{cases} c\exp(\rho ux)\dfrac{\sinh(x+au)/\lambda}{\sinh(x/\lambda)\sinh(au/\lambda)}, \\ c\exp(\rho ux)\dfrac{\exp(\pm x/\lambda)}{\sinh x/\lambda} \end{cases}$$

$$(a,c,\lambda\in\mathbb{C}\backslash\{0\},s\in\mathbb{C}\backslash\mathbb{Z}\pi\sqrt{-1}\lambda,\rho\in\mathbb{C}).$$

1-3. *Rational:*

$$A(x)=\begin{cases} c\cdot h(x)\dfrac{x+s}{xs}, \\ c\cdot h(x)\dfrac{1}{x}, \end{cases} \qquad B(u,x)=\begin{cases} c\exp(\rho ux)\dfrac{x+au}{axu}, \\ c\exp(\rho ux)\dfrac{1}{x} \end{cases}$$

$$(a,c,s\in\mathbb{C}\backslash\{0\},\rho\in\mathbb{C}).$$

2. *Singular case:*

$$A(x)=c_1h(x),\quad B(u,x)=c_2\exp(\rho ux)\frac{1}{u}$$

$$(c_1,\rho\in\mathbb{C},c_2\in\mathbb{C}\setminus\{0\}).$$

Here the function F is holomorphic on $C(0,r_1)$, *the function* $G(\not\equiv 0)$ *is meromorphic on* $C(0,r)$, *the function h is meromorphic on* $C(0,r)$ *satisfying the relation* $h(x)h(-x)=1$ *and the function* $\sigma(x)=\sigma(x;\tau_1,\tau_2)$ *is the Weierstrass sigma function,*

$$\sigma(x;\tau_1,\tau_2)=x\prod_{\omega=m_1\tau_1+m_2\tau_2}\left\{\left(1-\frac{x}{\omega}\right)\exp\left(\frac{x}{\omega}+\frac{1}{2}\left(\frac{x}{\omega}\right)^2\right)\right\},$$

where (m_1,m_2) *in the product above runs over all the elements in* $\mathbb{Z}^2$ *except* (0, 0).

We can show the following theorem easily.

Theorem I.3: *The functions A and B in Theorem I.2 satisfy Eqs. (1) and (2).*

Our strategy to solve the functional equations (1) and (2) is as follows. We reduced Eqs. (1) and (2) to the functional equation introduced by Braden and Buchstaber:[11]

$$\phi_1(x+y)(\phi_4(x)\phi_5(y)-\phi_4(y)\phi_5(x))=\phi_2(x)\phi_3(y)-\phi_2(y)\phi_3(x). \tag{3}$$

They have proved that the solutions of this functional equation above were characterized by those of the functional equation discussed by Bruschi and Calogero:[12,13]

$$\alpha(x+y)-\alpha(x)\alpha(y)=\varphi(x)\varphi(y)\psi(x+y). \tag{4}$$

Since Kawazumi and the author[14] have given the complete classification of the meromorphic solutions near the origin of Eq. (4), we obtained all the meromorphic solutions of Eqs. (1) and (2) near the origin.

Let us now explain how this article is organized. Section II gives a brief summary of the functional equations above. In Sec. III, we solve the functional equations (1) and (2) on the assumptions that $B\not\equiv 0$ and that $A(x)A(-x)$ is not identically constant. There are three kinds of meromorphic solutions of Eqs. (1) and (2) expressed in terms of the elliptic, trigonometric and rational functions. We discuss the elliptic case in Sec. IV, the trigonometric case in Sec. V and the rational case in Sec. VI, respectively. Section VII presents the classification of the meromorphic solutions of the functional equations (1) and (2) on the assumptions that $B\not\equiv 0$ and that $A(x)A(-x)(\not\equiv 0)$ is identically constant. In the final section, Sec. VIII, we classify the meromorphic solutions of the functional equations (1) and (2) with $A\equiv 0$ or $B\equiv 0$.

After finishing this article, the author found the thesis[15] in which Komori investigated the *R*-operators associated with root algebras. We note that the definition of the *R*-operators in his thesis was slightly different from that in this article.

II. REVIEWS OF CERTAIN FUNCTIONAL EQUATIONS OF ADDITION TYPE

In this section, we review the solutions of the functional equations (3) and (4) of addition type.

A. Solutions of Eq. (4)

Bruschi and Calogero have investigated the general analytic solution of Eq. (4).[12,13] They have obtained the elliptic solution in the most general case and some trigonometric and rational solutions by degenerating the periods of the elliptic functions.

Kawazumi and the author classified the meromorphic solutions near the origin of Eq. (4).

Theorem II.1 (Kawazumi-Shibukawa[14]): *Let* α, φ *and* ψ *be holomorphic functions defined on a punctured disk* $\{x\in\mathbb{C};0<|x|<r'\}$ *for some* $r'>0$. *If they satisfy the functional equation (4), then they are equal to one of the following functions.*

$$\text{(0-i)}\quad \alpha(x)=0 \ \ \textit{or}\ \ \exp(\rho x)\ \ (\rho\in\mathbb{C}),$$

$$\varphi\equiv 0 \ \ \textit{and}\ \ \psi\text{: arbitrary, or}\ \ \varphi\text{: arbitrary and}\ \ \psi\equiv 0.$$

$$\text{(0-ii)}\quad \alpha(x)=C\exp(\rho x),\quad \varphi(x)=C_1\exp(C_2x),$$

$$\psi(x)=C(1-C)C_1^{-2}\exp((\rho-C_2)x)$$

$$(C,\rho,C_1,C_2\in\mathbb{C},C\neq 0,1,C_1\neq 0).$$

$$\text{(I)}\quad \alpha(x)=\exp(\rho x)\frac{\sigma(\nu;\tau_1,\tau_2)\sigma(x+\mu;\tau_1,\tau_2)}{\sigma(\mu;\tau_1,\tau_2)\sigma(x+\nu;\tau_1,\tau_2)},$$

$$\varphi(x)=\exp(C_1x+C_2)\frac{\sigma(x)}{\sigma(x+\nu)},$$

$$\psi(x)=\exp((\rho-C_1)x-2C_2)\frac{\sigma(\nu)\sigma(\mu-\nu)\sigma(x+\mu+\nu)}{\sigma^2(\mu)\sigma(x+\nu)},$$

$$(\rho,\mu,\nu,C_1,C_2\in\mathbb{C},\ \ \tau_1,\tau_2\in\mathbb{C}\backslash\{0\},\ \ \mathrm{Im}\,\tau_2/\tau_1>0\quad \mu,\nu\notin\mathbb{Z}\tau_1+\mathbb{Z}\tau_2,\quad \mu-\nu\notin\mathbb{Z}\tau_1+\mathbb{Z}\tau_2).$$

$$\text{(II)}\quad \alpha(x)=\exp(\rho x)\frac{a(\exp(2x/\lambda)-1)+b}{c(\exp(2x/\lambda)-1)+b},$$

$$\varphi(x)=\exp(C_1x+C_2)\frac{\exp(2x/\lambda)-1}{c(\exp(2x/\lambda)-1)+b},$$

$$\psi(x)=\exp(-C_1x-2C_2)\frac{(a-c)\{-ac(\exp(2x/\lambda)-1)+b^2-b(a+c)\}}{c(\exp(2x/\lambda)-1)+b}$$

$$(\lambda,\rho,a,b,c,C_1,C_2\in\mathbb{C},\ \ \lambda\neq 0,\ \ b(a-c)\neq 0).$$

$$\text{(III)}\quad \alpha(x)=\exp(\rho x)\frac{ax+b}{cx+b},\ \ \varphi(x)=\exp(C_1x+C_2)\frac{x}{cx+b},$$

$$\psi(x)=\exp((\rho-C_1)x-2C_2)\frac{(c-a)\{acx+b(a+c)\}}{cx+b}$$

$$(\rho,a,b,c,C_1,C_2\in\mathbb{C},\ \ b(a-c)\neq 0).$$

All the solutions except for the case (0-i) *extend themselves to meromorphic functions defined on the whole plane* $\mathbb{C}$.

Remark: In Theorem II.1 (I), we use τ_1, τ_2, μ and ν instead of τ_1/λ, τ_2/λ, μ/λ and ν/λ in Ref. 14. Moreover, we note that the condition $\mu-\nu\notin\mathbb{Z}\tau_1+\mathbb{Z}\tau_2$ in Theorem II.1 (I) was dropped in Ref. 14.

B. Solutions of Eq. (3)

Braden and Buchstaber[11] have investigated Eq. (3). They have shown that the solutions of Eq. (3) were characterized by the solutions of Eq. (4). We review their results briefly.

Let ϕ_1 be a holomorphic function on $C(2x_0,2r_0)$ and ϕ_2, ϕ_3, ϕ_4 and ϕ_5 be holomorphic functions on $C(x_0,r_0)$ for some $x_0\in\mathbb{C}$ and $r_0>0$. We assume that they satisfy the following conditions:

(a) Eq. (3) for all $x,y \in C(x_0,r_0)$,
(b) $\phi_2(x_0)\phi_3'(x_0)-\phi_2'(x_0)\phi_3(x_0)\neq 0$,
(c) $\phi_4(x_0)\phi_5'(x_0)-\phi_4'(x_0)\phi_5(x_0)\neq 0$.

Lemma II.2: We define the function $\tilde{\phi}_1$ *holomorphic on* $C(0,2r_0)$ *and the functions* $\tilde{\phi}_2,\ldots,\tilde{\phi}_5$ *holomorphic on* $C(0,r_0)$ *as follows*:

$$\tilde{\phi}_1(x)=c\,\phi_1(x+2x_0),$$

$$\begin{pmatrix}\tilde{\phi}_{2k}(x)\\ \tilde{\phi}_{2k+1}(x)\end{pmatrix}=\begin{pmatrix}\phi_{2k}'(x_0) & \phi_{2k}(x_0)\\ \phi_{2k+1}'(x_0) & \phi_{2k+1}(x_0)\end{pmatrix}^{-1}\begin{pmatrix}\phi_{2k}(x+x_0)\\ \phi_{2k+1}(x+x_0)\end{pmatrix}\quad (k=1,2),$$

where

$$c=\det\begin{pmatrix}\phi_4'(x_0) & \phi_4(x_0)\\ \phi_5'(x_0) & \phi_5(x_0)\end{pmatrix}\Big/\det\begin{pmatrix}\phi_2'(x_0) & \phi_2(x_0)\\ \phi_3'(x_0) & \phi_3(x_0)\end{pmatrix}.$$

Then they satisfy

$$\tilde{\phi}_1(x+y)(\tilde{\phi}_4(x)\tilde{\phi}_5(y)-\tilde{\phi}_4(y)\tilde{\phi}_5(x))=\tilde{\phi}_2(x)\tilde{\phi}_3(y)-\tilde{\phi}_2(y)\tilde{\phi}_3(x)$$

for all $x,\ y\in C(0,r_0)$.

By straightforward computation, we deduce $\tilde{\phi}_{2k}(0)=\tilde{\phi}_{2k+1}'(0)=0$ and $\tilde{\phi}_{2k}'(0)=\tilde{\phi}_{2k+1}(0)=1$ for $k=1,2$.

Lemma II.3: There exist $(0<)r_2\leqslant r_1$, *the functions* γ_k *and* ξ_k $(k=1,2)$ *holomorphic on* $C(0,r_2)$ *such that* $\gamma_k(x)\neq 0$ *for all* $x\in C(0,r_2)$,

$$\begin{pmatrix}\tilde{\phi}_{2k}(x)\\ \tilde{\phi}_{2k+1}(x)\end{pmatrix}=\frac{1}{\gamma_k(x)}\begin{pmatrix}\xi_k(x)\\ \xi_k'(x)\end{pmatrix}$$

for all $x\in C(0,r_2)$, $\xi_k(0)=0$, *and* $\xi_k'(0)=\gamma_k(0)=1$.

For $k=1,2$, define $\tilde{\xi}_k(x)=\exp(-\lambda_k x)\xi_k(x)$, where $\lambda_k=-\tilde{\phi}_{2k}''(0)/2$. Then the functions $\tilde{\xi}_k(x)$ are holomorphic on $C(0,r_2)$ and satisfy $\tilde{\xi}_k(0)=\tilde{\xi}_k''(0)=0$ and $\tilde{\xi}_k'(0)=1$. We define the functions $\tilde{\xi}_0$ on $x\in C(0,2r_2)$ and γ on $x\in C(0,r_2)$ by $\tilde{\xi}_0(x)=\exp((\lambda_1-\lambda_2)x)\tilde{\phi}_1(x)$ and $\gamma(x)=\exp(2(\lambda_1-\lambda_2)x)\gamma_2(x)/\gamma_1(x)$.

Lemma II.4: (1) *The function* $\tilde{\xi}_1(x)/\tilde{\xi}_2(x)$ *is holomorphic on* $C(0,r_2)$.
(2) *For all* $x,y\in C(0,r_2)$

$$\tilde{\xi}_0(x+y)(\tilde{\xi}_2(x)\tilde{\xi}_2'(y)-\tilde{\xi}_2(y)\tilde{\xi}_2'(x))=\gamma(x)\gamma(y)(\tilde{\xi}_1(x)\tilde{\xi}_1'(y)-\tilde{\xi}_1(y)\tilde{\xi}_1'(x)).$$

Since there exists $(0<)r_3\leqslant r_2$ such that $\tilde{\xi}_1(x)\neq 0$ and $\tilde{\xi}_2(x)\neq 0$ for all $x\in C(0,r_3)\backslash\{0\}$, we are led to the following.

Theorem II.5 (Braden–Buchstaber[11]):

(1) $\gamma(x)=(\tilde{\xi}_2(x)/\tilde{\xi}_1(x))^2$ *and* $\tilde{\xi}_0(x)=\tilde{\xi}_2(x)/\tilde{\xi}_1(x)$ *for all* $x\in C(0,r_3)$.
(2) *Define the functions* α *and* φ *holomorphic on* $C(0,r_3)$ *by* $\alpha(x)=\tilde{\xi}_2(x)/\tilde{\xi}_1(x)$ *and* $\varphi(x)=\tilde{\xi}_2(x)$. *Then they satisfy Eq. (4) for all* $x,\ y\in C(0,r_3/2)\backslash\{0\}$.

It is to be mentioned that the function ψ is determined by the functions α and φ. We can reconstruct the solutions $\phi_1,\ldots,\phi_5$ of Eq. (3) from the functions α and φ in the theorem above.

III. GENERIC CASE

In this section, we solve Eqs. (1) and (2) on the assumption below.

Assumption 1: (1) The meromorphic function $A(x)A(-x)$ is not identically constant on $C(0,r)$.

(2) The meromorphic function B is not identically zero on $C((0,0),r)$.

The purpose of this section is to prove the following theorem.

Theorem III.1: (1) *The function $A(x)A(-x)$ meromorphic on the disk $C(0,r)$ is one of the following:*

$$\textit{elliptic:}\quad A(x)A(-x)=\frac{a_1\wp(x;\tau_1,\tau_2)+a_2}{a_3\wp(x;\tau_1,\tau_2)+a_4}, \tag{5}$$

$$\textit{trigonometric:}\quad A(x)A(-x)=\frac{a_1\sinh^{-2}(x/\lambda)+a_2}{a_3\sinh^{-2}(x/\lambda)+a_4},$$

$$\textit{rational:}\quad A(x)A(-x)=\frac{a_1x^{-2}+a_2}{a_3x^{-2}+a_4},$$

where $\wp(x)=\wp(x;\tau_1,\tau_2)$ is the Weierstrass pe function

$$\wp(x;\tau_1,\tau_2)=-\frac{d}{dx}\left(\frac{\sigma'(x;\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)}\right),$$

and the constants $\tau_1,\ \tau_2,\ \lambda\in\mathbb{C}\backslash\{0\}$ and $a_1,\ a_2,\ a_3,\ a_4\in\mathbb{C}$ satisfy the relations $\operatorname{Im}\tau_2/\tau_1>0$ *and* $a_1a_4-a_2a_3\neq 0$.

(2) *There exists $C(u_1,r_1)\subset C(0,r/4)$ such that the function $B(u,x)$ is one of the following:*

$$\textit{elliptic:}\quad B(u,x)=\exp(\rho(u)x)b(u)\frac{\sigma(x+a(u);\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)}, \tag{6}$$

$$\forall(u,x)\in D_1\cap D^e\cap(C(u_1,r_1)\times C(0,r)),$$

$$\textit{trigonometric:}\quad B(u,x)=\exp(\tilde{\rho}(u)x)\tilde{b}(u)\times\frac{c(u)(\exp((x+2\tilde{a}(u))/\lambda)-\exp(-x/\lambda))+\exp(-x/\lambda)}{\sinh(x/\lambda)}, \tag{7}$$

$$\forall(u,x)\in D_1\cap D^t\cap(C(u_1,r_1)\times C(0,r)),$$

$$\textit{rational:}\quad B(u,x)=\exp(\rho(u)x)\frac{b(u)+a(u)x}{x},$$

$$\forall(u,x)\in D_1\cap D^r\cap(C(u_1,r_1)\times C(0,r)),$$

where $\rho(u),\ a(u),\ b(u)\in\mathbb{C}$ for all $u\in C(u_1,r_1)$. Here $D_1(\subset C((0,0),r))$ is the domain of the meromorphic function $B(u,x)$ and

$$D^e=C(0,r)\times(C(0,r)\backslash(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)),$$

$$D^t=C(0,r)\times(C(0,r)\backslash\mathbb{Z}\pi\sqrt{-1}\lambda),$$

$$D^r=C(0,r)\times(C(0,r)\backslash\{0\}).$$

Assumption 1 implies the following lemma. (For part (2), see Lemma 5 in Ref. 11.)

Lemma III.2: (1) *Equations (1) and (2) on* $C((0,0,0,0),r/2)$ *are equivalent to the following equations:*

$$A(x)A(-x)-A(y)A(-y)=B(u,x)B(u,-x)-B(u,y)B(u,-y), \tag{8}$$

$$B(u,x)B(v,x+y)=B(u+v,x+y)B(u,-y)+B(v,y)B(u+v,x) \tag{9}$$

on $C((0,0,0),r)$ *and* $C((0,0,0,0),r/2)$, *respectively.*

(2) *The meromorphic solutions* $A(x)$ *and* $B(u,x)$ *of the previous equations satisfy the equation*

$$\begin{aligned} &B(v,x+y)(A(x)A(-x)-A(y)A(-y)) \\ &\quad =B(u,-x)B(u+v,x)B(v,y)-B(u,-y)B(u+v,y)B(v,x) \end{aligned} \tag{10}$$

as meromorphic functions on $C((0,0,0,0),r/2)$.

Now we intend to apply Sec. II B to Eq. (10).

Lemma III.3: For any $C((u_0',x_0'),r_0')\subset C((0,0),r/2)$, *there exist* $(u_1,x_1)\in C((u_0',x_0'),r_0')$ *and* $r_1(>0)$ *such that*

(0) $C((u_1,x_1),r_1)\subset C((u_0',x_0'),r_0')$,

(1) $B(u,x)$ *is holomorphic on* $C((u_1,x_1),r_1)$, $C(u_1,r_1)\times C(2x_1,2r_1)$ *and* $C(2u_1,2r_1)\times C(x_1,r_1)$,

(2) $B(u,-x)$ *is holomorphic on* $C((u_1,x_1),r_1)$ *and* $C(u_1,r_1)\times C(2x_1,2r_1)$,

(3) $A(x)$ *is holomorphic on* $C(x_1,r_1)$ *and* $C(2x_1,2r_1)$,

(4) $A(-x)$ *is holomorphic on* $C(x_1,r_1)$ *and* $C(2x_1,2r_1)$,

(5) $B(u,x)\neq 0$ *for all* $(u,x)\in C((u_1,x_1),r_1)$.

By $C((u_0',x_0'),r_0')\coloneqq C((0,0),r/4)$ in Lemma III.3, there exist $(u_1,x_1)\in C((0,0),r/4)$ and r_1 (>0) satisfying the conditions in Lemma III.3.

Lemma III.4: (1) $(d/dx)(A(x)A(-x))\not\equiv 0$ *on* $C(x_1,r_1)$.

(2) *For all* u, $v\in C(u_1,r_1)$,

$$\begin{vmatrix} B(u,-x)B(u+v,x) & \dfrac{\partial}{\partial x}(B(u,-x)B(u+v,x)) \\ B(v,x) & \dfrac{\partial B}{\partial x}(v,x) \end{vmatrix} \not\equiv 0$$

on $C(x_1,r_1)$.

Proof: We prove part (2) only. The proof is by contradiction. Assume the assertion were false. Then there would exist u_0, $v_0\in C(u_1,r_1)$ such that

$$\begin{vmatrix} B(u_0,-x)B(u_0+v_0,x) & \dfrac{d}{dx}(B(u_0,-x)B(u_0+v_0,x)) \\ B(v_0,x) & \dfrac{\partial B}{\partial x}(v_0,x) \end{vmatrix} \equiv 0$$

on $C(x_1,r_1)$. Thus there exists $c\in\mathbb{C}$ such that

$$\frac{B(u_0,-x)B(u_0+v_0,x)}{B(v_0,x)}\equiv c \quad \text{on} \quad C(x_1,r_1).$$

By Eq. (10) and Assumption 1 (1), we have $B(v_0,x)\equiv 0$ on $C(2x_1,2r_1)$. From Lemma III.5, we get $B(v_0,x)\equiv 0$ on $C(0,r)$, which implies a contradiction of Assumption 1 (1) because of Eq. (8). □

Lemma III.5: Let $r_1,r_2>0$ and $F(u,x)$ be a function meromorphic on the polydisc $C(0,r_1)\times C(0,r_2)$. For any $v\in C(0,r_1)$ such that the function F is holomorphic at $(v,y)(y\in C(0,r_2))$, the function $F(v,x)$ is meromorphic on $C(0,r_2)$.

Proof: Because the polydisc $C(0,r_1)\times C(0,r_2)$ is Stein and $H^2(C(0,r_1)\times C(0,r_2),\mathbb{Z})=0$, the sharp form of the Poincaré theorem is valid on $C(0,r_1)\times C(0,r_2)$. (See, for example, Chap. V, Sec. 2 in Ref. 16 and Secs. I and K in Ref. 17.) Then there exist two functions g and h holomorphic on $C(0,r_1)\times C(0,r_2)$ such that h is not identically zero, $F(u,x)=g(u,x)/h(u,x)$, and the functions g and h are coprime locally. Since the function F is holomorphic at (v,y), we have $h(v,y)\neq 0$, which implies $h(v,x)\not\equiv 0$ on $C(0,r_2)$. Thus the function $g(v,x)/h(v,x)$ is meromorphic on $C(0,r_2)$. For any $x\in C(0,r_2)$ such that $h(v,x)\neq 0$, $F(v,x)=g(v,x)/h(v,x)$. This completes the proof of the lemma. □

Let u_0, $v_0\in C(u_1,r_1)$. Because of Lemmas III.3 (2) and III.4, we can apply the method introduced in Sec. II B to Eq. (10) for $u:=u_0$ and $v:=v_0$. That is to say, there exists $C(x_0,r_0)\subset C(x_1,r_1)$ such that $\phi_1(x):=B(v_0,x)$ defined on $C(2x_0,2r_0)$, $\phi_2(x):=B(u_0,-x)B(u_0+v_0,x)$, $\phi_3(x):=B(v_0,x)$, $\phi_4(x):=A(x)A(-x)$, and $\phi_5(x)\equiv 1$ defined on $C(x_0,r_0)$ satisfy the conditions (a)–(c) in Sec. II B.

From Theorems II.1 and II.5, the function $\alpha(x)=\tilde{\xi}_2(x)/\tilde{\xi}_1(x)$ defined near the origin is one of the following.

$$(0)\quad \alpha(x)=C\exp(\rho x),$$

$$(\mathrm{I})\quad \alpha(x)=\exp(\rho x)\frac{\sigma(\mu;\tau_1,\tau_2)\sigma(x+\nu;\tau_1,\tau_2)}{\sigma(\nu;\tau_1,\tau_2)\sigma(x+\mu;\tau_1,\tau_2)},$$

$$(\mathrm{II})\quad \alpha(x)=\exp(\rho x)\frac{a(e^{2x/\lambda}-1)+b}{c(e^{2x/\lambda}-1)+b},$$

$$(\mathrm{III})\quad \alpha(x)=\exp(\rho x)\frac{ax+b}{cx+b}.$$

Lemma III.6: $\alpha(x)\neq C\exp(\rho x)$.

Proof: The proof is by contradiction. Assume the assertion were false. With the aid of Theorem II.5, we get $\alpha(0)=1$, and, consequently, $B(v_0,x)=c^{-1}\exp((\rho-\lambda_1+\lambda_2)(x-2x_0))$ near $2x_0$. Here the constant c was defined in Lemma II.2. From Lemma III.5 and the identity theorem for the meromorphic functions, the above equation is also valid on $C(0,r)$, which implies a contradiction of Assumption 1 (1) by virtue of Eq. (8). □

From this lemma, the function φ is uniquely determined by the function α and so are the functions $\tilde{\xi}_1$ and $\tilde{\xi}_2$.

Proof of Theorem III.1 (1): We first note that Assumption 1 implies the condition $a_1a_4-a_2a_3\neq 0$.

By means of $\phi_5(x)\equiv 1$, we have $\tilde{\phi}_5(x)\equiv 1$ and $\gamma_2(x)=\xi_2'(x)$, and $\tilde{\phi}_4(x)=\xi_2(x)/\xi_2'(x)$ as a result. By the definition in Lemma II.2,

$$A(x)A(-x)$$
$$=\begin{cases} \phi_4(x_0)+\dfrac{\phi_4'(x_0)}{\zeta(x-x_0)-\zeta(x-x_0+\nu)+\lambda_2+\zeta(\nu)}, & \text{(I)} \\ \phi_4(x_0) & \\ +\dfrac{\phi_4'(x_0)\left(\exp\left(\dfrac{2(x-x_0)}{\lambda}\right)-1\right)\left\{c\left(\exp\left(\dfrac{2(x-x_0)}{\lambda}\right)-1\right)+1\right\}}{\left(\lambda_2+\dfrac{2c-1}{\lambda}\right)\left(\exp\left(\dfrac{2(x-x_0)}{\lambda}\right)-1\right)\left\{c\left(\exp\left(\dfrac{2(x-x_0)}{\lambda}\right)-1\right)+1\right\}+\dfrac{2}{\lambda}\exp\left(\dfrac{2(x-x_0)}{\lambda}\right)}, & \text{(II)} \\ \phi_4(x_0)+\dfrac{\phi_4'(x_0)(x-x_0)\{c(x-x_0)+1\}}{(\lambda_2+c)(x-x_0)\{c(x-x_0)+1\}+1}, & \text{(III)} \end{cases}$$

near x_0, where $\zeta(x)=\zeta(x;\tau_1,\tau_2)$ is the Weierstrass zeta function $\zeta(x;\tau_1,\tau_2)=\sigma'(x;\tau_1,\tau_2)/\sigma(x;\tau_1,\tau_2)$. With the aid of the identity theorem for the meromorphic functions, the equation above is valid on $C(0,r)$. Because $A(x)A(-x)$ is an even function on $C(0,r)$, we obtain the desired result. □

Now we prove Theorem III.1 (2).

Proposition III.7: Let $u_0\in C(u_1,r_1)$. For any $v_0\in C(u_1,r_1)$, there exist $x_0(v_0)\in C(x_1,r_1)$ and $r_2(v_0)(>0)$ such that the function $B(v_0,x)$ is one of the following: For all $x\in C(2x_0(v_0),r_2(v_0))$,

$$\textit{elliptic:}\quad B(v_0,x)=\exp(\rho(v_0)x)b(v_0)\frac{\sigma(x+a(v_0);\tau_1,\tau_2)}{\sigma(x;\tau_1,\tau_2)},$$

$$\textit{trigonometric:}\quad B(u,x)=\exp(\rho(u)x)\tilde{b}(u)\times\frac{c(u)(\exp((x+2\tilde{a}(u))/\lambda)-\exp(-x/\lambda))+\exp(-x/\lambda)}{\sinh(x/\lambda)},$$

$$\textit{rational:}\quad B(v_0,x)=\exp(\rho(v_0)x)\frac{b(v_0)+a(v_0)x}{x},$$

where $\rho(v_0)$, $a(v_0)$, $b(v_0)\in\mathbb{C}$.

Proof: For the sake of brevity, we only show the elliptic case. For any $v_0\in C(u_1,r_1)$, there exists $C(x_0(v_0),r_0(v_0))\subset C(x_1,r_1)\backslash(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)$ such that $\phi_1(x):=B(v_0,x)$ defined on $C(2x_0(v_0),\ 2r_0(v_0))$, $\phi_2(x):=B(u_0,-x)B(u_0+v_0,x)$, $\phi_3(x):=B(v_0,x)$, $\phi_4(x):=A(x)A(-x)$, and $\phi_5(x)\equiv 1$ defined on $C(x_0(v_0),r_0(v_0))$ satisfy the conditions (a)–(c) in Sec. II B by means of Lemma III.4. Thus we deduce

$$\tilde{\xi}_2(x)=\exp(\zeta(2x_0(v_0);\tau_1,\tau_2)x)\frac{\sigma(2x_0(v_0);\tau_1,\tau_2)\sigma(x;\tau_1,\tau_2)}{\sigma(x+2x_0(v_0);\tau_1,\tau_2)},\tag{11}$$

where τ_1 and τ_2 are in Theorem III.1 (1).

Lemma III.8: $\alpha(x)\neq 0$, $\exp(\rho x)$.

The proof is quite similar to that of Lemma III.6, so we omit it. Equation (11) tells us that the zeroes of the function $\tilde{\xi}_2$ are $\mathbb{Z}\tau_1+\mathbb{Z}\tau_2$. Hence the function α is an elliptic solution of Eq. (4) by means of Lemma III.8, and the periods of the Weierstrass sigma function σ in the function α are τ_1 and τ_2 as a consequence. (See Secs. 3 and 4 in Ref. 14.) Thus there exists $(0<)r_2<2r_1$ such that

$$\tilde{\xi}_1(x)=\exp(\zeta(\mu;\tau_1,\tau_2)x)\,\frac{\sigma(\mu;\tau_1,\tau_2)\sigma(x;\tau_1,\tau_2)}{\sigma(x+\mu;\tau_1,\tau_2)}$$

on $C(0,r_2)$. From Lemma II.2, Theorem II.5, Eq. (11) and $\phi_1(x)=B(v_0,x)$, we have proved the proposition. □

Proof of Theorem III.1 (2): For the sake of brevity, we only prove the elliptic case. For any $v_0\in C(u_1,r_1)$, the function $B(v_0,x)$ is meromorphic on $C(0,r)$ by Lemma III.5. On the other hand, by Proposition III.7,

$$B(v_0,x)=\exp(\rho(v_0)x)b(v_0)\,\frac{\sigma(x+a(v_0))}{\sigma(x)}$$

on some small disk in $C(0,r)$. Because the right hand side of the equation above is meromorphic on $C(0,r)$, we have proved the theorem. □

IV. ELLIPTIC CASE

This section presents the solutions A and B of Eqs. (8) and (9) in the elliptic case of Theorem III.1.

Lemma IV.1: For all $u\in C(u_1,r_1)$, $\sigma(a(u))\neq 0$ and $b(u)\neq 0$.

Proof: We only show that $\sigma(a(u))\neq 0$ for all $u\in C(u_1,r_1)$. The proof is by contradiction. Assume the assertion were false. Then there would exist $u\in C(u_1,r_1)$ such that $\sigma(a(u))=0$. By means of Eq. (6), there exist functions $\tilde{\rho}$ and $\tilde{b}$ such that $B(u,x)=\exp(\tilde{\rho}(u)x)\tilde{b}(u)$ for all $x\in C(x_1,r_1)\backslash(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)$, and this equation is also valid on $C(0,r)$ from Lemma III.5. By Eq. (8), this contradicts Assumption 1 (1). □

By virtue of Eqs. (5), (6), (8) and the lemma above, we conclude Lemma IV.2.

Lemma IV.2: We have $a_3=0$, that is to say, $A(x)A(-x)=\tilde{a}_1\wp(x)+\tilde{a}_2$ on $C(0,r)$, where $\tilde{a}_1=a_1/a_4$ and $\tilde{a}_2=a_2/a_4$.

We note that the relation $a_1a_4-a_2a_3\neq 0$ implies $\tilde{a}_1\neq 0$. It follows from the Lemma IV.2 and Eq. (8) that $b(u)^2\sigma^2(a(u))=-\tilde{a}_1$ for all $u\in C(u_1,r_1)$.

Lemma IV.3: There exist $c\in\mathbb{C}\backslash\{0\}$ and $C(u_2,r_2)\subset C(u_1,r_1)$ such that $b(u)\sigma(a(u))=c$ for all $u\in C(u_2,r_2)$, and

$$B(u,x)=c\,\exp(\rho(u)x)\,\frac{\sigma(x+a(u))}{\sigma(a(u))\sigma(x)}$$

for all $(u,x)\in D_1\cap D^e\cap(C(u_2,r_2)\times C(0,r))$ as a consequence.

For the proof, it suffices to show the following lemma.

Lemma IV.4: There exists $C((u_2,0),r_2)\subset C(u_1,r_1)\times C(0,r)$ such that the function $B(u,x)\sigma(x)$ is holomorphic on $C((u_2,0),r_2)$.

Proof: Since the sharp form of the Poincaré theorem is valid on $C((0,0),r)$, there exist two functions g and h holomorphic on $C((0,0),r)$ such that h is not identically zero, $B(u,x)\sigma(x)=g(u,x)/h(u,x)$, and the functions g and h are coprime locally. By Eq. (6), $g(u,x)=\exp(\rho(u)x)b(u)\sigma(x+a(u))h(u,x)$ for all $(u,x)\in D_1\cap D^e\cap(C(u_1,r_1)\times C(0,r))$.

We fix any $u\in C(u_1,r_1)$. Because the function $\exp(\rho(u)x)b(u)\sigma(x+a(u))$ is holomorphic on $C(0,r)$ and $g(u,x)=\exp(\rho(u)x)b(u)\sigma(x+a(u))h(u,x)$ for all $x\in C(x_1,r_1)\backslash(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)$, we have $g(u,x)=\exp(\rho(u)x)b(u)\sigma(x+a(u))h(u,x)$ for all $x\in C(0,r)$. Thus $g(u,0)=b(u)\sigma(a(u))h(u,0)$, which tells us that $(u,0)$ is not a pole of the function $B(u,x)\sigma(x)$ for all $u\in C(u_1',r_1')$. Since the set of points of indeterminacy of the meromorphic function of two variables is isolated, there exists a regular point $(u_2,0)\in C(u_1',r_1')\times C(0,r)$ of the function $B(u,x)\sigma(x)$. We have thus proved the lemma. □

Using Eq. (8), we are led to the following theorem.

Theorem IV.5: *The elliptic solution $A(x)$ defined on $C(0,r)$ is*

$$A(x)=c\cdot h(x)\frac{\sigma(x+s)}{\sigma(x)\sigma(s)},$$

where $h(x)$ *is a meromorphic function defined on* $C(0,r)$ *satisfying the relation* $h(x)h(-x)=1$ *and* s *is a complex constant such that* $\wp(s)=-\tilde{a}_2/\tilde{a}_1$.

In the sequel, we determine the function B.

Proposition IV.6: There exists $C(u_3',\delta')\subset C(u_2,r_2)$ *such that*

$$B(u,x)=\exp(\tilde{\rho}(u)x)\frac{\sigma(x+a_1(u))}{\sigma(a_1(u))\sigma(x)}$$

for all $(u,x)\in D_1\cap D^e\cap(C(u_3',\delta')\times C(0,r))$, *where the function* a_1 *is holomorphic on* $C(u_3',\delta')$.

We only give the proof in the case that there exists $v_1\in C(u_2,r_2)$ such that $\wp'(a(v_1))\neq 0$ because the proof is rather simple in the case that $\wp'(a(u))\equiv 0$ on $C(u_2,r_2)$. The function $\wp(x)$ is holomorphic at $x=a(v_1)$ by using Lemma IV.1, and the function $\wp$ has a holomorphic inverse g near $a(v_1)$ as a result. (See, for example, p. 215 of Ref. 18.) Then there exists $C(v_1,\delta)\subset C(u_2,r_2)$ such that $\wp(a(u))$ is in the domain of the function g for all $u\in C(v_1,\delta)$. Define a function $\tilde{a}$ holomorphic on $C(v_1,\delta)$ as $\tilde{a}(u)=g(\wp(a(u)))$. There exists a function $\epsilon(u)\in\{0,1\}$ such that $a(u)\equiv(-1)^{\epsilon(u)}\tilde{a}(u)$ $(\mathrm{mod}\,\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)$ for all $u\in C(v_1,\delta)$, and consequently

$$B(u,x)=\exp(\tilde{\rho}(u)x)(-1)^{\epsilon(u)}\frac{\sigma(x+(-1)^{\epsilon(u)}\tilde{a}(u))}{\sigma(\tilde{a}(u))\sigma(x)}$$

for all $(u,x)\in D_1\cap D^e\cap(C(v_1,\delta)\times C(0,r))$, where $\tilde{\rho}(u)\in\mathbb{C}$.

Proposition IV.6 now follows from the following lemma.

Lemma IV.7: There exist $C(v_1',\delta')\subset C(v_1,\delta)$ *such that* $a(u)\equiv\tilde{a}(u)(\mathrm{mod}\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)$ *for all* $u\in C(v_1',\delta')$ *or* $a(u)\equiv-\tilde{a}(u)(\mathrm{mod}\,\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)$ *for all* $u\in C(v_1',\delta')$.

To prove this lemma, it suffices to give the proof in the case that, for all $C(u,\delta_1)\subset C(v_1,\delta)$, there exist $v,w\in C(u,\delta_1)$ such that $\epsilon(v)\neq\epsilon(w)$. By the sharp form of the Poincaré theorem, there exist two functions g and h holomorphic on $C((0,0),r)$ such that h is not identically zero, $B(u,x)\sigma(x)=g(u,x)/h(u,x)$, and the functions g and h are coprime locally. We omit the proof of the lemma below because it is similar to that of Lemma IV.4.

Lemma IV.8: There exists $C(u_3',\delta')\subset C(v_1,\delta)$ *satisfying the following conditions.*

(1) *The function* $B(u,x)\sigma(x)$ *is holomorphic on* $C((u_3',0),\delta')$.
(2) *For all* $(u,x)\in C((u_3',0),\delta')$,

$$c\exp(\tilde{\rho}(u)x)(-1)^{\epsilon(u)}\frac{\sigma(x+(-1)^{\epsilon(u)}\tilde{a}(u))}{\sigma(\tilde{a}(u))}h(u,x)=g(u,x).$$

By means of Lemma IV.8,

$$\exp(\tilde{\rho}(u)x)(-1)^{\epsilon(u)}\sigma(x+(-1)^{\epsilon(u)}\tilde{a}(u))=c^{-1}B(u,x)\sigma(x)\sigma(\tilde{a}(u))$$

for all $(u,x)\in C((u_3',0),\delta')$. Since the function $\tilde{a}(u)$ is holomorphic on $C(u_3',\delta')$, the function $f(u,x):=c^{-1}B(u,x)\sigma(x)\sigma(\tilde{a}(u))$ is holomorphic on $C((u_3',0),\delta')$. The function $(\partial f/\partial x)(u,0)$ is consequently holomorphic on $C(u_3',\delta')$, and

$$\frac{\partial f}{\partial x}(u,0)=\tilde{\rho}(u)\sigma(\tilde{a}(u))+(-1)^{\epsilon(u)}\sigma'(\tilde{a}(u)) \tag{12}$$

for all $u\in C(u_3',\delta')$.

By $B\not\equiv 0$ and Lemma IV.1, we conclude the following.

Lemma IV.9: There exists $C((u_3'',x_3''),\delta'')\subset C((u_3',0),\delta')$ *satisfying the following conditions.*

(1) $C(x_3'',\delta'')\not\ni 0$.
(2) $f(u,x)\neq 0$ *for all* $(u,x)\in C((u_3'',x_3''),\delta'')$.
(3) $\sigma(x-\tilde{a}(u))\neq 0$ *for all* $(u,x)\in C((u_3'',x_3''),\delta'')$.
(4) $\sigma(x+\tilde{a}(u))\neq 0$ *for all* $(u,x)\in C((u_3'',x_3''),\delta'')$.

Lemma IV.7 follows from Lemma IV.10 immediately.

Lemma IV.10: For all $u\in C(u_3'',\delta'')$, $2\tilde{a}(u)\in\mathbb{Z}\tau_1+\mathbb{Z}\tau_2$.

Proof: To prove this lemma, we show

$$-\frac{\sigma(x_3+\tilde{a}(u_3))}{\sigma(x_3-\tilde{a}(u_3))}=\exp(2x_3\zeta(\tilde{a}(u_3)))$$

for all $(u_3,x_3)\in C((u_3'',x_3''),\delta'')$. From Lemma IV.9, the functions $-f(u,x)/\sigma(x-\tilde{a}(u))$ and $f(u,x)/\sigma(x+\tilde{a}(u))$ are holomorphic on $C((u_3'',x_3''),\delta'')$ and satisfy $-f(u,x)/\sigma(x-\tilde{a}(u))\neq 0$ and $f(u,x)/\sigma(x+\tilde{a}(u))\neq 0$ for all $(u,x)\in C((u_3'',x_3''),\delta'')$. For any $(u_3,x_3)\in C((u_3'',x_3''),\delta'')$, let $\mathrm{Log}^{(1)}(x)$ and $\mathrm{Log}^{(2)}(x)$ be branches of the logarithm defined on open connected sets $V_1,V_2\subset\mathbb{C}$ such that $\exp(\tilde{\rho}(u_3)x_3)\in V_1$ and $(-1)^{\epsilon(u_3)+1}f(u_3,x_3)/\sigma(x_3+(-1)^{\epsilon(u_3)+1}\tilde{a}(u_3))\in V_2$, respectively. Because the function $(-1)^{\epsilon(u_3)+1}f(u,x)/\sigma(x+(-1)^{\epsilon(u_3)+1}\tilde{a}(u))$ is continuous at $(u,x)=(u_3,x_3)$, there exist $\tilde{\epsilon}>0$ and $\tilde{\delta}>0$ satisfying the following conditions.

(1) $C((-1)^{\epsilon(u_3)+1}f(u_3,x_3)/\sigma(x_3+(-1)^{\epsilon(u_3)+1}\tilde{a}(u_3)),\tilde{\epsilon})\subset V_2$.
(2) $C((u_3,x_3),\tilde{\delta})\subset C((u_3'',x_3''),\delta'')$.
(3) For all $(u,x)\in C((u_3,x_3),\tilde{\delta})$

$$\frac{(-1)^{\epsilon(u_3)+1}f(u,x)}{\sigma(x+(-1)^{\epsilon(u_3)+1}\tilde{a}(u))}\in C\left(\frac{(-1)^{\epsilon(u_3)+1}f(u_3,x_3)}{\sigma(x_3+(-1)^{\epsilon(u_3)+1}\tilde{a}(u_3))},\tilde{\epsilon}\right).$$

Let $N\in\mathbb{N}$ such that $1/N<\tilde{\delta}$. For all $n\geqslant N$, there exists $\tilde{u}_n\in C(u_3,1/n)$ such that $\epsilon(\tilde{u}_n)\neq\epsilon(u_3)$. (This is the case which we now consider.) Then we have $\epsilon(\tilde{u}_n)\equiv\epsilon(u_3)+1\ (\mathrm{mod}\ 2)$, $\lim_{n\to\infty}\tilde{u}_n=u_3$, and $(-1)^{\epsilon(u_3)+1}f(\tilde{u}_n,x_3)/\sigma(x_3+(-1)^{\epsilon(u_3)+1}\tilde{a}(\tilde{u}_n))\in V_2$ for all $n\geqslant N$. By the conditions (1) and (2) above, $\exp(\tilde{\rho}(\tilde{u}_n)x_3)\in V_2$ for all $n\geqslant N$, and, consequently,

$$\tilde{\rho}(\tilde{u}_n)x_3=\mathrm{Log}^{(2)}\left(\frac{(-1)^{\epsilon(u_3)+1}f(\tilde{u}_n,x_3)}{\sigma(x_3+(-1)^{\epsilon(u_3)+1}\tilde{a}(\tilde{u}_n))}\right)$$

for all $n\geqslant N$. On account of Eq. (12),

$$\begin{aligned}\frac{\partial f}{\partial x}(u_3,0)&=\lim_{n\to\infty}\frac{\partial f}{\partial x}(\tilde{u}_n,0)\\&=\frac{1}{x_3}\mathrm{Log}^{(2)}\left(\frac{(-1)^{\epsilon(u_3)+1}f(u_3,x_3)}{\sigma(x_3+(-1)^{\epsilon(u_3)+1}\tilde{a}(u_3))}\right)\sigma(\tilde{a}(u_3))+(-1)^{\epsilon(u_3)+1}\sigma'(\tilde{a}(u_3)).\end{aligned}$$

Because of $\exp(\tilde{\rho}(u_3)x_3)\in V_1$,

$$\frac{\partial f}{\partial x}(u_3,0)=\frac{1}{x_3}\mathrm{Log}^{(1)}\left(\frac{(-1)^{\epsilon(u_3)}f(u_3,x_3)}{\sigma(x_3+(-1)^{\epsilon(u_3)}\tilde{a}(u_3))}\right)\sigma(\tilde{a}(u_3))+(-1)^{\epsilon(u_3)}\sigma'(\tilde{a}(u_3)).$$

By the straightforward calculation, we obtain the desired result, thereby completing the proof of Proposition IV.6. □

Proposition IV.11: There exists $C(u_3,r_3)\subset C(u_1,r_1)$ *such that*

$$B(u,x)=\exp(\rho_1(u)x)\frac{\sigma(x+a_1(u))}{\sigma(a_1(u))\sigma(x)} \tag{13}$$

as meromorphic functions on $C(u_3,r_3)\times C(0,r)$, *where the functions* ρ_1 and a_1 *are holomorphic on* $C(u_3,r_3)$.

Proof: It is enough to show that the function $\tilde{\rho}$ in Proposition IV.6 is holomorphic locally. Define $f_1(u,x):=e^{\tilde{\rho}(u)}\sigma(x+a_1(u))$. Since the function f_1 is expressed as $f_1(u,x)=c^{-1}B(u,x)\sigma(x)\sigma(a_1(u))$, there exists $C((u_3'',0),\delta'')\subset C((u_3',0),\delta')$ such that the function f_1 is holomorphic on $C((u_3'',0),\delta'')$ (see Lemma IV.4), and $(\partial f/\partial x)(u,0)=\tilde{\rho}(u)\sigma(a_1(u))+\sigma'(a_1(u))$ on $C(u_3'',\delta'')$ as a result. By Lemma IV.1, we are led to $\sigma(a_1(u))\neq 0$ on $C(u_3'',\delta'')$, thereby completing the proof. □

Proposition IV.12: We have $\rho_1(u)=\rho u+\rho_3$ *and* $a_1(u)=au+a_3$, *where* ρ, ρ_3, a, $a_3\in\mathbb{C}$.

For the proof, we need the following.

Lemma IV.13: There exist $C(u_4,r_4)\subset C(u_3,r_3)$ *and a function* a_4 *holomorphic on* $C(u_4,r_4)$ *such that* $\sigma(a_4(u)+a_1(v))\neq 0$ *for all* $u,v\in C(u_4,r_4)$ *and* $\sigma(a_4(u))\neq 0$ *for all* $u\in C(u_4,r_4)$.

Proof: If $\sigma(2a_1(u))\not\equiv 0$ on $C(u_3,r_3)$, put $a_4:=a_1$. The proof in the case that $\sigma(2a_1(u))\equiv 0$ on $C(u_3,r_3)$ is simple, so we omit it. □

We take $C(\tilde{x}_1,\tilde{r}_1)\subset(C(x_1,r_1)\backslash(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2))$ such that $C(2\tilde{x}_1,2\tilde{r}_1)\cap(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)=\emptyset$. From Theorem IV.5 and the three term identity of σ (see, for example, p. 377 of Ref. 19 and p. 461 of Ref. 20),

$$\begin{aligned}&\frac{B(v,x+y)(A(x)A(-x)-A(y)A(-y))}{B(v,x)B(v,y)}\\&\quad=\frac{c\sigma(a_1(v))}{\sigma(a_4(u))\sigma(a_4(u)+a_1(v))}\\&\qquad\times\left(\frac{\sigma(x+a_4(u)+a_1(v))\sigma(x-a_4(u))}{\sigma(x)\sigma(x+a_1(v))}-\frac{\sigma(y+a_4(u)+a_1(v))\sigma(y-a_4(u))}{\sigma(y)\sigma(y+a_1(v))}\right)\end{aligned}$$

for all $u,v\in C(u_4,r_4)$ and $x,y\in C(\tilde{x}_1,\tilde{r}_1)$. By virtue of Eq. (10), for all $u,v\in C(u_4,r_4)$, there exists a constant $\gamma(u,v)\in\mathbb{C}$ such that

$$\frac{B(u,-x)B(u+v,x)}{B(v,x)}=\frac{c\sigma(a_1(v))\sigma(x+a_4(u)+a_1(v))\sigma(x-a_4(u))}{\sigma(x)\sigma(x+a_1(v))\sigma(a_4(u))\sigma(a_4(u)+a_1(v))}+\gamma(u,v) \tag{14}$$

for all $x,y\in C(\tilde{x}_1,\tilde{r}_1)$.

From Lemma III.3 and $C((2u_4,2x_1),2r_4)\subset C((0,0),r/2)$, there exist $(u_1',x_1')\in C((2u_4,2x_1),2r_4)$ and $r_1'(>0)$ such that the conditions in Lemma III.3 hold. The proof of Lemma IV.14 is similar to that of Proposition IV.11, so we omit it.

Lemma IV.14: There exists $C(u_3',r_3')\subset C(u_1',r_1')$ *such that*

$$B(u,x)=\pm c\exp(\rho_2(u)x)\frac{\sigma(x+a_2(u))}{\sigma(x)\sigma(a_2(u))} \tag{15}$$

as meromorphic functions on $C(u_3',r_3')\times C(0,r)$ *with the functions* ρ_2 *and* a_2 *holomorphic on* $C(u_3',r_3')$.

Proof of Proposition IV.12: By Eqs. (13)–(15),

$$\pm\exp((\rho_2(u+v)-\rho_1(u)-\rho_1(v))x)\sigma(x-a_1(u))\sigma(x+a_2(u+v))$$
$$\times\sigma(a_1(v))\sigma(a_4(u))\sigma(a_4(u)+a_1(v))$$
$$=\sigma(a_1(v))\sigma(x+a_4(u)+a_1(v))\sigma(x-a_4(u))\sigma(a_1(u))\sigma(a_2(u+v))$$
$$+\gamma(u,v)\sigma(a_4(u))\sigma(x)\sigma(a_2(u+v))\sigma(x+a_1(v))\sigma(a_4(u)+a_1(v))$$

for any $x\in C(\tilde{x}_1,\tilde{r}_1)$ and $u,v\in C(u_3'/2,r_3'/2)$. We note that the equation above is valid on $\mathbb{C}$ $(\ni x)$ also by means of the identity theorem for the holomorphic functions. Since the both sides of the equation above are quasi-periodic with the periods τ_1 and $\tau_2, a_2(u+v)-a_1(u)-a_1(v)\in\mathbb{Z}\tau_1+\mathbb{Z}\tau_2$. Because the functions a_1 and a_2 are holomorphic and the set $\mathbb{Z}\tau_1+\mathbb{Z}\tau_2$ is discrete, the function $a_2(u+v)-a_1(u)-a_1(v)$ is constant on $C((u_3'/2,u_3'/2),r_3'/2)$ and so is the function $\rho_2(u+v)-\rho_1(u)-\rho_1(v)$. Hence we get the desired result. □

It is to be noted that $\rho_2(u)=\rho u+\rho_4$ and that $a_2(u)=au+a_4$, where $\rho_4,a_4\in\mathbb{C}$. By the straightforward computation, we deduce the following.

Theorem IV.15: *The elliptic solution $B(u,x)$ of Eqs. (8) and (9) defined on the polydisc $C((0,0),r)$ is*

$$B(u,x)=c\exp(\rho ux)\frac{\sigma(x+au)}{\sigma(x)\sigma(au)},$$

where $a\in\mathbb{C}\backslash\{0\}$ and $\rho\in\mathbb{C}$.

V. TRIGONOMETRIC CASE

In this section, we solve Eqs. (8) and (9) in the trigonometric case of Theorem III.1.

The proof of Lemma V.1 is the same as that of Lemmas IV.1 and IV.2, so we omit the proof.

Lemma V.1: (1) *For all $u\in C(u_1,r_1)$, $c(u)(\exp(2\tilde{a}(u)/\lambda)-1)+1\neq 0$ and $\tilde{b}(u)\neq 0$.*

(2) *We have $a_3=0$, that is to say, $A(x)A(-x)=\tilde{a}_1\sinh^{-2}(x/\lambda)+\tilde{a}_2$ on $C(0,r)$, where $\tilde{a}_1=a_1/a_4$ and $\tilde{a}_2=a_2/a_4$.*

From Eqs. (7), (8) and Lemma V.1 (2),

$$A(x)A(-x)-B(u,x)B(u,-x)=\tilde{a}_2+4\tilde{b}(u)^2\exp\left(\frac{2\tilde{a}(u)}{\lambda}\right)c(u)(1-c(u)) \tag{16}$$

for all $u\in C(u_1,r_1), x\in C(x_1,r_1)\backslash\mathbb{Z}\pi\sqrt{-1}\lambda$.

Lemma V.2: There exists $C(u_1',r_1')\subset C(u_1,r_1)$ such that

$$B(u,x)=\exp(\rho(u)x)b(u)\frac{\sinh(x+a(u))/\lambda}{\sinh(x/\lambda)},\quad or\quad B(u,x)=\exp(\rho(u)x)b(u)\frac{1}{\sinh(x/\lambda)}$$

for any $(u,x)\in D_1\cap D^t\cap(C(u_1',r_1')\times C(0,r))$.

If there exists $C(u_1',r_1')\subset C(u_1,r_1)$ such that $c(u)-c(u)^2\neq 0$ for all $u\in C(u_1',r_1')$, then there exists $a(u)\in\mathbb{C}$ such that $c(u)=\exp(a(u)/\lambda)/(2\sinh(a(u)/\lambda))$ for all $u\in C(u_1',r_1')$. For the proof of Lemma V.2, it suffices to show the following lemma.

Lemma V.3: If, for all $C(u,\tilde{r})\subset C(u_1,r_1)$, there exists $u_0\in C(u,\tilde{r})$ such that $c(u_0)-c(u_0)^2=0$, then $c(u)=0$ or 1 for all $u\in C(u_1,r_1)$.

Proof: The proof is by contradiction. Assume the assertion were false. Then there would exist $u_0'\in C(u_1,r_1)$ such that $c(u_0')\neq 0$, 1. We take $N\in\mathbb{N}$ such that $C(u_0',1/N)\subset C(u_1,r_1)$. For all $n\geqslant N$, there exists $u_n\in C(u_0',1/n)$ such that $c(u_n)-c(u_n)^2=0$ and, for all $n\geqslant N$,

$$\tilde{a}_2+4\tilde{b}(u_n)^2\exp\left(\frac{2\tilde{a}(u_n)}{\lambda}\right)c(u_n)(1-c(u_n))=\tilde{a}_2$$

as a result. The left hand side of Eq. (16) is holomorphic on $C(u_1,r_1)$ for a fixed $x \in C(x_1,r_1)\backslash\mathbb{Z}\pi\sqrt{-1}\lambda$, so it is continuous. Since $\lim_{n\to\infty} u_n = u_0'$,

$$\tilde{a}_2 + 4\tilde{b}(u_0')^2 \exp\left(\frac{2\tilde{a}(u_0')}{\lambda}\right) c(u_0')(1-c(u_0')) = \tilde{a}_2,$$

which is a contradiction of the choice of u_0' and Lemma V.1 (1). □

The proof of the theorem below is the same as that in Sec. IV, so we omit it.

Theorem V.4: (1) *The trigonometric solution $A(x)$ of Eqs. (8) and (9) defined on the polydisc $C(0,r)$ is*

$$A(x) = c\cdot h(x)\frac{\sinh(x+s)/\lambda}{\sinh(x/\lambda)\sinh(s/\lambda)} \quad or \quad c\cdot h(x)\frac{1}{\sinh(x/\lambda)},$$

where $c \in \mathbb{C}\backslash\{0\}$, $s \in \mathbb{C}\backslash\mathbb{Z}\pi\sqrt{-1}\lambda$ and $h(x)$ is a meromorphic function defined on $C(0,r)$ satisfying the relation $h(x)h(-x)=1$.

(2) *There exists $C(u_3,r_3)\subset C(u_1,r_1)$ such that the trigonometric solution $B(u,x)$ of Eqs. (8) and (9) is expressed as*

$$B(u,x) = c\exp(\rho_1(u)x)\frac{\sinh(x+a_1(u))/\lambda}{\sinh(a_1(u)/\lambda)\sinh(s/\lambda)}, \quad or \quad c\exp(\rho_1(u)x)\frac{1}{\sinh(x/\lambda)}$$

on $C(u_3,r_3)\times C(0,r)$. Here the functions ρ_1 and a_1 are holomorphic on $C(u_3,r_3)$.

(3) *There exist $C(u_4,r_4)\subset C(u_3,r_3)$ and a function a_4 holomorphic on $C(u_4,r_4)$ such that $\sinh((a_4(u)+a_1(v))/\lambda)\neq 0$ and $\sinh(a_4(u)/\lambda)\neq 0$ for all $u\in C(u_4,r_4)$.*

(4) *There exists $C(u_3',r_3')\subset C(2u_4,2r_4)$ such that the trigonometric solution $B(u,x)$ of Eqs. (8) and (9) is expressed as follows*:

$$B(u,x) = \pm c\exp(\rho_2(u)x)\frac{\sinh(x+a_2(u))/\lambda}{\sinh(a_2(u)/\lambda)\sinh(x/\lambda)} \quad or \quad \pm c\exp(\rho_2(u)x)\frac{1}{\sinh(x/\lambda)}$$

on $C(u_3',r_3')\times C(0,r)$. Here the functions ρ_2 and a_2 are holomorphic on $C(u_3',r_3')$.

We take $C(\tilde{x}_1,\tilde{r}_1)\subset(C(x_1,r_1)\backslash\mathbb{Z}\pi\sqrt{-1}\lambda)$ as $C(2\tilde{x}_1,2\tilde{r}_1)\cap\mathbb{Z}\pi\sqrt{-1}\lambda=\emptyset$, and fix any $u,v \in C(u_3'/2,r_3'/2)$. From Eq. (10) there exists $\gamma(u,v)\in\mathbb{C}$ such that

$$\frac{B(u,-x)B(u+v,x)}{B(v,x)} = \begin{cases} \dfrac{c\sinh(a_1(v)/\lambda)\sinh((x+a_4(u)+a_1(v))/\lambda)\sinh(x-a_4(u))/\lambda}{\sinh((a_4(u)+a_1(v))/\lambda)\sinh((a_4(u)/\lambda)\sinh(x/\lambda)\sinh((x+a_1(v))/\lambda)} + \gamma(u,v), \\ \dfrac{-c\exp(-x/\lambda)}{\sinh(x/\lambda)} + \gamma(u,v), \end{cases} \tag{17}$$

for all $x\in C(\tilde{x}_1,\tilde{r}_1)$, and, as a result, we are led to the four cases below:

$$\mp \exp((\rho_2(u+v)-\rho_1(u)-\rho_1(v))x)$$
$$\times \frac{\sinh((x+a_2(u+v))/\lambda)\sinh((-x+a_1(u))/\lambda)\sinh(a_1(v)/\lambda)}{\sinh(a_2(u+v)/\lambda)\sinh(a_1(u)/\lambda)}$$
$$= \frac{c\sinh(a_1(v)/\lambda)\sinh((x+a_4(u)+a_1(v))/\lambda)\sinh((x-a_4(u))/\lambda)}{\sinh((a_4(u)+a_1(v))/\lambda)\sinh(a_4(u)/\lambda)}$$
$$+\gamma(u,v)\sinh\frac{x}{\lambda}\sinh\frac{x+a_1(v)}{\lambda}, \tag{18}$$

$$\mp \exp((\rho_2(u+v)-\rho_1(u)-\rho_1(v))x)\frac{\sinh((-x+a_1(u))/\lambda)\sinh(a_1(v)/\lambda)}{\sinh(a_1(u)/\lambda)}$$
$$= \frac{c\sinh(a_1(v)/\lambda)\sinh((x+a_4(u)+a_1(v))/\lambda)\sinh((x-a_4(u))/\lambda)}{\sinh((a_4(u)+a_1(v))/\lambda)\sinh(a_4(u)/\lambda)}$$
$$+\gamma(u,v)\sinh\frac{x}{\lambda}\sinh\frac{x+a_1(v)}{\lambda}, \tag{19}$$

$$\mp c\exp((\rho_2(u+v)-\rho_1(u)-\rho_1(v))x)\frac{\sinh((x+a_2(u+v))/\lambda)}{\sinh(a_2(u+v)/\lambda)} = -c\exp\left(-\frac{x}{\lambda}\right)+\gamma(u,v)\sinh\frac{x}{\lambda}, \tag{20}$$

$$\mp c\exp((\rho_2(u+v)-\rho_1(u)-\rho_1(v))x) = -c\exp\left(-\frac{x}{\lambda}\right)+\gamma(u,v)\sinh\frac{x}{\lambda}, \tag{21}$$

for any $x\in C(\tilde{x}_1,\tilde{r}_1)$. We note that the equations above are valid on $\mathbb{C}$. Substitution of 0 in x yields that all the signatures of Eqs. (18)–(21) are -1. From the periodicity of Eqs. (18)–(21),

$$\mp \exp((\rho_2(u+v)-\rho_1(u)-\rho_1(v))\pi\sqrt{-1}\lambda)=1, \tag{22}$$

and consequently, we have the following.

Lemma V.5: There exist $\rho,\rho_3,\rho_4\in\mathbb{C}$ such that $\rho_1(u)=\rho u+\rho_3$ for all $u\in C(u_3,r_3)$ and $\rho_2(u)=\rho u+\rho_4$ for all $u\in C(u_3',r_3')$.

In the case of (20), we can express the function B in two ways

$$B(u,x)=c\exp((\rho u+\rho_3)x)\frac{1}{\sinh(x/\lambda)}, c\exp((\rho u+\rho_4)x)\frac{\sinh((x+a_2(u))/\lambda)}{\sinh(a_2(u)/\lambda)\sinh(x/\lambda)}.$$

This is a contradiction. In the case of (19), we deduce a contradiction in a similar fashion.

From Eq. (22) there exists $n\in\mathbb{Z}$ such that $\rho_4-2\rho_3=n/\lambda$, and one can regard Eqs. (18) and (21) as the polynomials of the variable $\exp(x/\lambda)$. Thus we deduce the following.

Proposition V.6: On $C((0,0),r)$

$$B(u,x)=\begin{cases} c\exp((\rho u+\rho_3)x)\dfrac{\sinh((x+au+a_3)/\lambda)}{\sinh(x/\lambda)\sinh((au+a_3)/\lambda)} & \text{for (18)},\\ c\exp((\rho u+\rho_4)x)\dfrac{1}{\sinh(x/\lambda)} & \text{for (21)},\end{cases}$$

where $a,a_3\in\mathbb{C}$.

From Eq. (17) we get $\rho_3=0$, $\rho_4=\pm 1/\lambda$ and $a_3\in\mathbb{Z}\pi\sqrt{-1}\lambda$, that is to say,

Theorem V.7: *The trigonometric solution $B(u,x)$ of Eqs. (8) and (9) defined on the polydisc $C((0,0),r)$ is*

$$B(u,x)=c\exp(\rho ux)\frac{\sinh(x+au)/\lambda}{\sinh(x/\lambda)\sinh(au/\lambda)},\quad or\ \ c\exp(\rho ux)\frac{\exp(\pm x/\lambda)}{\sinh(x/\lambda)},$$

where c is in Theorem V.4, $a\in\mathbb{C}\backslash\{0\}$ *and* $\rho\in\mathbb{C}$.

VI. RATIONAL CASE

In this section, we continue solving Eqs. (8) and (9) in the rational case of Theorem III.1.

The proof of Theorem VI.1 is the same as that in Sec. IV, so we omit it.

Theorem VI.1: (1) *The rational solution $A(x)$ of Eqs. (8) and (9) defined on the polydisc $C(0,r)$ is*

$$A(x)=c\cdot h(x)\frac{x+s}{xs},\ \ or\ \ c\cdot h(x)\frac{1}{x},$$

where $c,s\in\mathbb{C}\backslash\{0\}$ and $h(x)$ is a meromorphic function defined on $C(0,r)$ satisfying the relation $h(x)h(-x)=1$.

(2) *There exist $C(u_3,r_3)\subset C(u_1,r_1)$ and $C(u_3',r_3')\subset C(2u_3,2r_3)$ such that the rational solution $B(u,x)$ of Eqs. (8) and (9) is expressed as follows:*

$$B(u,x)=\begin{cases}\exp(\rho_1(u)x)\dfrac{a_1(u)x+c}{x}, & on\ \ C(u_3,r_3)\times C(0,r),\\[2ex] \exp(\rho_2(u)x)\dfrac{a_2(u)x\pm c}{x}, & on\ \ C(u_3',r_3')\times C(0,r).\end{cases}$$

Here the functions ρ_1 and a_1 are holomorphic on $C(u_3,r_3)$ and the functions ρ_2 and a_2 are holomorphic on $C(u_3',r_3')$.

We fix any $u,v\in C(u_3'/2,r_3'/2)$. From Eq. (10), there exists $\gamma(u,v)\in\mathbb{C}$ such that

$$\frac{B(u,-x)B(u+v,x)}{B(v,x)}=-\frac{c^2}{x(a_1(v)x+c)}+\gamma(u,v)$$

for all $x\in C(x_1,r_1)\backslash\{0\}$, and consequently

$$\exp((\rho_2(u+v)-\rho_1(u)-\rho_1(v))x)=\frac{c^2}{(-a_1(u)x+c)(a_2(u+v)x\pm c)}-\frac{\gamma(u,v)x(a_1(v)x+c)}{(-a_1(u)x+c)(a_2(u+v)x\pm c)}\tag{23}$$

for all $x\in C(x_1,r_1)\backslash\{0\}$. Since the equation above is valid on $\mathbb{C}$, we obtain the following.

Lemma VI.2: There exist $\rho,\rho_3\in\mathbb{C}$ such that $\rho_1(u)=\rho u+\rho_3$ for all $u\in C(u_3,r_3)$ and $\rho_2(u)=\rho u+2\rho_3$ for all $u\in C(u_3',r_3')$.

From Eq. (23), $a_1(u)a_2(u+v)=(a_1(u)-a_2(u+v))a_1(v)$ for all $u,v\in C(u_3'/2,r_3'/2)$, which implies the following.

Lemma VI.3: The function $a_1(u)$ is identically zero on $C(u_3,r_3)$, or there exists $a,a_3\in\mathbb{C}$ such that

$$\frac{1}{a_1(u)}=\frac{au}{c}+\frac{a_3}{c},\quad \forall u\in C(u_3,r_3),$$

$$\frac{1}{a_2(u)}=\frac{au}{c}+\frac{2a_3}{c},\quad \forall u\in C(u_3',r_3').$$

By the straightforward computation, we deduce the following.

Theorem VI.4: *The rational solution $B(u,x)$ of Eqs. (8) and (9) defined on the polydisc $C((0,0),r)$ is*

$$B(u,x)=c\exp(\rho ux)\frac{x+au}{aux},\quad or\quad c\exp(\rho ux)\frac{1}{x}.$$

Here c is in Theorem VI.1, $a\in\mathbb{C}\backslash\{0\}$ *and* $\rho\in\mathbb{C}$.

VII. SINGULAR CASE

This section describes the solutions A and B of Eqs. (1) and (2) on the assumption that $B\not\equiv 0$ and that $A(x)A(-x)(\not\equiv 0)$ is identically constant. It is to be mentioned that the assumption above and Eqs. (1) and (2) imply Eqs. (9) and

$$B(u,x)B(u,-x)=B(u,y)B(u,-y) \tag{24}$$

on $C((0,0),r)$. Let $D_1,D_2\subset C((0,0),r)$ be the domains of the meromorphic function $B(u,x)$ and $B(u,-x)$, respectively. From Eq. (24), for all $u\in C(0,r)$ such that $(u,x)\in D_1\cap D_2$, there exists $a(u)\in\mathbb{C}$ such that

$$B(u,x)B(u,-x)=a(u)\quad \forall x\in C(0,r)\text{s.t.}(u,x)\in D_1\cap D_2. \tag{25}$$

It follows immediately that $a(u)$ is holomorphic at $u=u_0$ if $(u_0,y_0)\in D_1\cap D_2$.

Lemma VII.1: If $(u_0,y_0)\in D_1\cap D_2$, then $(u_0,0)$ is not a pole of the function $B(u,x)$.

Proof: The proof is by contradiction. Assume the assertion were false. For all $n\in\mathbb{N}$, there would exist $(u_n',x_n')\in C((0,0),r)$ such that $(u_n',x_n')\in C((u_0,0),1/n)\cap D_1$. Then there exists $r_n'>0$ such that $C((u_n',x_n'),r_n')\subset C((u_0,0),1/n)\cap D_1$. Hence there exists $(u_n,x_n)\in D_2$ such that $(u_n,x_n)\in C((u_n',x_n'),r_n')$. Because $(u_n,x_n)\in C((u_0,0),1/n)\cap D_1\cap D_2$, $\lim_{n\to\infty}u_n=u_0$ and $\lim_{n\to\infty}x_n=0$. Since $(u_0,0)$ is a pole of $B(u,x)$, $\lim_{n\to\infty}|B(u_n,x_n)|=\lim_{n\to\infty}|B(u_n,-x_n)|=\infty$, and $\lim_{n\to\infty}|B(u_n,x_n)B(u_n,-x_n)|=\infty$ as a consequence. As we mentioned earlier, we are led to $\lim_{n\to\infty}a(u_n)=a(u_0)$, which is a contradiction of Eq. (25). □

Thus the point $(u_0,0)$ in Lemma VII.1 is a regular point or a point of indeterminacy of $B(u,x)$.

Lemma VII.2. For any $(0<)r'\leqslant r$, there exists $u_0\in C(0,r')$ such that $(u_0,0)$ is a regular point of $B(u,x)$.

Proof: It suffices to consider the case that $(u_0,0)\in C((0,0),r')$ in Lemma VII.1 is a point of indeterminacy of the function $B(u,x)$.

Because the set of the points of the indeterminacy of the meromorphic function with two variables is isolated, there exists $r_0>0$ such that $B(u,x)$ has no points of indeterminacy in $C((u_0,0),r_0)\backslash\{(u_0,0)\}$ and $C((u_0,0),r_0)\subset C((0,0),r')$. That is to say, for any $u_1\in C(u_0,r_0)\backslash\{u_0\}$, $(u_1,0)$ is not a point of indeterminacy of $B(u,x)$, and there exists $s>0$ such that $C((u_1,0),s)\subset C((u_0,0),r_0)\backslash\{(u_0,0)\}$ as a result.

For $(u_3,y_3)\in D_1\cap D_2\cap(C(u_1,s)\times C(0,r'))$, $(u_3,0)$ is not a pole of $B(u,x)$ by means of Lemma VII.1. From $(u_3,0)\in C((u_1,0),s)$, $(u_3,0)$ is not a point of indeterminacy of $B(u,x)$. This point u_3 is the desired one. □

Proposition VII.3: There exist $r_0(>0)$ and $u_0\in C(0,r)$ satisfying the following conditions.

(1) $C((4u_0,0),4r_0)\subset C((0,0),r)$.
(2) $B(u,x)$ *is holomorphic on* $C((u_0,0),r_0)\cup C((2u_0,0),2r_0)\cup C((4u_0,0),4r_0)$.
(3) $B(u,x)\neq 0$ *for all* $(u,x)\in C((u_0,0),r_0)\cup C((2u_0,0),2r_0)\cup C((4u_0,0),4r_0)$.

Proposition VII.3 follows from Lemma VII.4 immediately.

Lemma VII.4: (1) *If there exists* $C((u_0,0),r_0)\subset C((0,0),r)$ *such that* $B(u,x)$ *is holomorphic on* $C((u_0,0),r_0)$ *and* $C((2u_0,0),2r_0)\subset C((0,0),r)$, *then there exists* $C((u_1,0),r_1)\subset C((u_0,0),r_0)$ *such that* $B(u,x)$ *is holomorphic on* $C((2u_1,0),2r_1)$.

(2) *If there exists* $C((u_0,0),r_0)\subset C((0,0),r)$ *such that* $B(u,x)$ *is holomorphic on* $C((u_0,0),r_0)$, *then there exists* $C((u_1,0),r_1)\subset C((u_0,0),r_0)$ *such that* $B(u,x)\neq 0$ *for all* $(u,x)\in C((u_1,0),r_1)$.

Proof: We prove (1) only. We take $C((u_2,y_2),r_2)\subset(C(2u_0,2r_0)\times C(0,r))\cap D_1\cap D_2$, and, for all $u\in C(u_2,r_2)$, there exists $y\in C(y_2,r_2)$ such that $(u,y)\in D_1\cap D_2$ as a result. By Lemma VII.1, $(u,\ 0)$ is not a pole of $B(u,x)$ for all $u\in C(u_2,r_2)$. Because the set of the points of indeterminacy of the meromorphic function of two variables is isolated and $u/2\in C(u_0,r_0)$ for all $u\in C(u_2,r_2)$, there exists $u_1\in C(u_0,r_0)$ such that $(2u_1,0)\in D_1$. Thus there exists $r_1>0$ such that $C((u_1,0),r_1)\subset C((u_0,0),r_0)$ and $C((2u_1,0),2r_1)\subset D_1$. This completes the proof. □

By Eq. (9), there exists $\gamma(u,v)\in\mathbb{C}$ such that

$$\frac{B(u+v,x)}{B(u,x)B(v,x)}=\gamma(u,v) \tag{26}$$

for all $x\in C(0,r_0/2),u,v\in C(u_0,r_0/2)$, and, consequently, we have the following.

Proposition VII.5: We fix any $u,v\in C(u_0,r_0/2)$ *and put*

$$\alpha(x)=\frac{B(u+v,x)}{B(u,x)},\quad \varphi(x)=\frac{1}{B(u,x)},\quad \psi(x)=a(u)\gamma(u,v)B(u+v,x).$$

Then they satisfy Eq. (4) for all $x,y\in C(0,r_0/4)$.

With the aid of Proposition VII.3, the functions α, φ and ψ are all holomorphic on $C(0,r_0/2)$. Moreover, $\varphi(x)\neq 0$ and $\psi(x)\neq 0$ for all $x\in C(0,r_0/2)$. This tells us that the functions α, φ and ψ are the solutions of Eq. (4) with the conditions $\varphi(0)\neq 0$ and $\alpha(x+y)-\alpha(x)\alpha(y)\neq 0$ for all $x,y\in C(0,r_0/4)$. By virtue of Theorem II.1, we conclude the following.

Proposition VII.6: For $u\in C(u_0,r_0/2)$ *and* $x\in C(0,r_0/4)$,

$$B(u,x)=c_1(u)\exp(\rho_1(u)x),$$

where c_1 *and* ρ_1 *are holomorphic on* $C(u_0,r_0/2)$. *The function* c_1 *satisfies* $c_1(u)\neq 0$ *for all* $u\in C(u_0,r_0/2)$.

We obtain Proposition VII.7 in a similar fashion.

Proposition VII.7: For $u\in C(2u_0,r_0)$ *and* $x\in C(0,r_0/2)$,

$$B(u,x)=c_2(u)\exp(\rho_2(u)x),$$

where c_2 *and* p_2 *are holomorphic on* $C(2u_0,r_0)$. *The function* c_2 *satisfies* $c_2(u)\neq 0$ *for all* $u\in C(2u_0,r_0)$.

By virtue of Eqs. (9) and (26), we deduce Theorem VII.8.

Theorem VII.8: *The singular solutions* $A(x)$ *and* $B(u,x)$ *of Eqs. (1) and (2) defined on the polydiscs* $C(0,r)$ *and* $C((0,0),r)$, *respectively, are as follows*:

$$A(x)=c_1h(x),\quad B(u,x)=c_2\exp(\rho ux)\frac{1}{u}.$$

Here c_1, $c_2 \in \mathbb{C}\backslash\{0\}$, $\rho \in \mathbb{C}$ *and* $h(x)$ *is a meromorphic function defined on* $C(0,r)$ *satisfying the relation* $h(x)h(-x)=1$.

VIII. TRIVIAL CASE

In this section, we solve Eqs. (1) and (2) with $A \equiv 0$ or $B \equiv 0$.

Lemma VIII.1: If the function B is identically zero on $C((0,0),r)$, *then, for any function A meromorphic on* $C(0,r)$, *the functions A and* $B(\equiv 0)$ *satisfy Eqs. (1) and (2).*

In the sequel, we assume that $A \equiv 0$ and $B \not\equiv 0$. From the previous assumption, Eqs. (1) and (2) are equivalent to

$$\frac{B(u+v,x)}{B(u,x)B(v,x)} = \frac{B(u+v,y)}{B(u,y)B(v,y)}$$

on $C((0,0,0,0),r/2)$. By differentiating the equation above in the variable x, we get

$$\frac{(\partial B/\partial x)(u+v,x)}{B(u+v,x)} - \frac{(\partial B/\partial x)(u,x)}{B(u,x)} - \frac{(\partial B/\partial x)(v,x)}{B(v,x)} = 0$$

on $C((0,0,0),r/2)$ and, as a result, $(\partial^2 \tilde{B}/\partial u^2)(u,x)=0$ on $C((0,0),r)$, where $\tilde{B}(u,x)=(\partial B/\partial x)(u,x)/B(u,x)$.

Lemma VIII.2: There exists a function f meromorphic on $C(0,r)$ *such that* $\tilde{B}(u,x)=f(x)u$ *as meromorphic functions on* $C((0,0),r)$ *and the function f is holomorphic at* $x=0$.

Proof: We only show that the function f is holomorphic at $x=0$. Let D_1 be the domain of the meromorphic function B. By means of $B \not\equiv 0$, there exists $C((u_1,x_1),r_1) \subset D_1 \backslash (\{0\} \times C(0,r))$ such that $B(u,x) \neq 0$ for all $(u,x) \in C((u_1,x_1),r_1)$. Hence, for all $u \in C(u_1,r_1)$, $f(x)=(\partial B/\partial x)(u,x)/(uB(u,x))$ is meromorphic on $C(0,r)$. (See Lemma III.5) Laurent's expansions near $x=0$ of the functions f and $(\partial B/\partial x)(u,x)/(uB(u,x))$ are

$$f(x)=\sum_{k_1=l}^{\infty} a_{k_1} x^{k_1}, \quad \frac{(\partial B/\partial x)(u,x)}{uB(u,x)} = \frac{1}{u}\sum_{k_2=-1}^{\infty} b_{k_2}(u) x^{k_2} \quad (l, b_{-1}(u) \in \mathbb{Z}),$$

and we get $l=-1$ and $a_{-1}=b_{-1}(u)/u$ for all $u \in C(u_1,r_1)$ as a result. If $b_{-1}(u_1) \neq 0$, then, for all $u \in C(u_1,r_1)$, $u=(b_{-1}(u)/b_{-1}(u_1))u_1 \in \mathbb{Q}u_1$, which is a contradiction. Thus $b_{-1}(u_1)=0$, and consequently $a_{-1}=0$. We have completed the proof. □

Therefore we deduce the following theorem.

Theorem VIII.3: *There exist* $(0<)r_1(\leq r)$, *a function F holomorphic on* $C(0,r_1)$ *and a function G meromorphic on* $C(0,r)$ *such that the function G is not identically zero and* $B(u,x)=\exp(F(x)u)G(u)$ *as meromorphic functions on* $C(0,r)\times C(0,r_1)$.

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[1] Y. Shibukawa and K. Ueno, Lett. Math. Phys. **25**, 239 (1992).
[2] Y. Shibukawa and K. Ueno, Int. J. Mod. Phys. A **3A**, 309 (1993).
[3] Y. Shibukawa and K. Ueno, in *Quantum Groups, Integrable Statistical Models and Knot Theory* (World Scientific, Singapore, 1993), p. 302.
[4] G. Felder and V. Pasquier, Lett. Math. Phys. **32**, 167 (1994).
[5] Y. Shibukawa, Commun. Math. Phys. **172**, 661 (1995).
[6] K. Hikami, Phys. Lett. A **205**, 167 (1995).
[7] H. Fan, B. Hou, and K. Shi, Phys. Lett. A **230**, 19 (1997).

[8] S. N. M. Ruijsenaars, Commun. Math. Phys. **110**, 191 (1987).
[9] Y. Komori and K. Hikami, J. Phys. A **30**, 4341 (1997).
[10] Y. Komori and K. Hikami, Lett. Math. Phys. **43**, 335 (1998).
[11] H. W. Braden and V. M. Buchstaber, SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **28**, 903 (1997).
[12] M. Bruschi and F. Calogero, Commun. Math. Phys. **109**, 481 (1987).
[13] M. Bruschi and F. Calogero, SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **21**, 1019 (1990).
[14] N. Kawazumi and Y. Shibukawa, Publ. RIMS, Kyoto Univ. **36**, 85 (2000).
[15] Y. Komori, thesis, University of Tokyo, 1999.
[16] H. Grauert and R. Remmert, *Theory of Stein Spaces* (Springer-Verlag, Berlin, 1979).
[17] R. C. Gunning, *Introduction to Holomorphic Functions of Several Variables*, Volume III (Wadsworth, Belmont, 1990).
[18] W. Rudin, *Real and Complex Analysis*, 3rd ed. (McGraw-Hill, New York, 1987).
[19] C. Jordan, *Fonctions Elliptiques* (Springer-Verlag, Berlin, 1981).
[20] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed. (Cambridge University Press, Cambridge, 1927).