A Note on the Use of Symmetric and Antisymmetric Conditions in the Finite-Element Analysis of Acoustic Waveguides

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Abstract—The application of the finite-element method to the problem of wave propagation in acoustic waveguides with planes of symmetry is discussed. In particular the discussion is how to use the symmetric and antisymmetric conditions on a plane of symmetry whose normal direction is not coincident with the direction of a coordinate axis.

I. INTRODUCTION

IN THE finite-element analysis of acoustic waveguides with planes of symmetry, symmetric and antisymmetric conditions are often used on each plane of symmetry [1]-[6]. When the symmetric condition is used on a plane with normal unit vector $\mathbf{n}$ (see Fig. 1), the particle displacement component parallel to $\mathbf{n}$ and the two stress components normal to $\mathbf{n}$ are zero on the plane. When the antisymmetric condition is used on a plane with normal unit vector $\mathbf{n}$, the two particle displacement components normal to $\mathbf{n}$ and the stress component parallel to $\mathbf{n}$ are zero on the plane. Application of these conditions reduces the number of elements, and therefore it is possible to use computer memory more economically. In earlier finite-element analyses [1]-[6], however, only the plane of symmetry whose normal direction is coincident with the direction of a coordinate axis is considered. In this report, we indicate how to use the symmetric and antisymmetric conditions on a

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plane of symmetry whose normal direction is not coincident with the direction of a coordinate axis.

II. Development of Equations

The coordinate system employed is shown in Fig. 1, where \( z \) is taken in the direction of propagation and the unit vector \( n \) normal to the plane of symmetry \( \Gamma \) lies at an angle \( \theta \) from the \( x \) axis in the \( xy \) plane. Using the finite-element method for only the region \( \Omega \), we obtain

\[
\begin{bmatrix}
[A_{xx}] & [A_{xy}] & [A_{xz}] & [A_{xy'}] & [A_{xz'}] \\
[A_{yx}] & [A_{yy}] & [A_{yz}] & [A_{yy'}] & [A_{yz'}] \\
[A_{zx}] & [A_{zy}] & [A_{zz}] & [A_{zy'}] & [A_{zz'}] \\
[A_{x'y'}] & [A_{y'y'}] & [A_{x'z'}] & [A_{y'y'}] & [A_{x'z'}] \\
[A_{x'z'}] & [A_{y'z'}] & [A_{z'z'}] & [A_{y'z'}] & [A_{z'z'}]
\end{bmatrix}
\]

\[
\begin{pmatrix}
[u_x] \\
[u_y] \\
[u_z] \\
[f_x'] \\
[f_y'] \\
[f_z']
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
\{u_x\} \\
\{u_y\} \\
\{u_z\} \\
\{f_x'\} \\
\{f_y'\} \\
\{f_z'\}
\end{pmatrix}
\]

\[
[A] = [K(\theta)] - \omega^2 [M]
\]

\[
\{f_t\} = \sum_{e^*} \left\{N\right\} T_{t_{n'|\Gamma}} d\Gamma, \quad i' = x', y', z',
\]

where the components of the \( \{u_t\} \) vector are the values of the particle displacement \( u_t \) at all nodal points in \( \Omega \) except \( \Gamma \), the components of the \( \{u_{t'}\} \) vector are the values of \( u_t \) at all nodal points on \( \Gamma \), \( \omega \) is the angular frequency, \( \beta \) is the phase constant in the \( z \) direction, \([K(\theta)]\) and \([M]\) are the stiffness and mass matrices \([1] - [6]\), respectively, \([A_{xx}], [A_{xy}], \ldots\), and \([A_{z'z'}]\) are the submatrices of \([A]\), \( T_{t_{n'|\Gamma}} \) is the stress on \( \Gamma \), \( \left\{N\right\} \) is the shape function \([1] - [6]\), \( \{0\} \) is the null vector, and \( \Sigma_{e^*} \) extends over the elements related to \( \Gamma \).

Using the symmetric condition

\[
\{u_x\} = -\tan \theta \{u_y\}, \quad \{f_x'\} = \cot \theta \{f_y'\}, \quad \{f_z'\} = \{0\} \tag{2}
\]

on \( \Gamma \), from (1) we obtain

\[
\begin{bmatrix}
[A_{xx}] & [A_{xy}] & [A_{xz}] & [A_{xy'}] & [A_{xz'}] \\
[A_{yx}] & [A_{yy}] & [A_{yz}] & [A_{yy'}] & [A_{yz'}] \\
[A_{zx}] & [A_{zy}] & [A_{zz}] & [A_{zy'}] & [A_{zz'}] \\
[A_{x'y'}] & [A_{y'y'}] & [A_{x'z'}] & [A_{y'y'}] & [A_{x'z'}] \\
[A_{x'z'}] & [A_{y'z'}] & [A_{z'z'}] & [A_{y'z'}] & [A_{z'z'}]
\end{bmatrix}
\]

\[
\begin{pmatrix}
\{u_x\} \\
\{u_y\} \\
\{u_z\} \\
\{f_x'\} \\
\{f_y'\} \\
\{f_z'\}
\end{pmatrix} = \begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\} \\
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

Using the antisymmetric condition

\[
\begin{align*}
[A]_{y'y'} &= [A_{y'y'}] - \tan \theta ([A_{x'y'}] + [A_{y'x'}]) \\
&\quad + \tan^2 \theta [A_{x'x'}]
\end{align*}
\]

\[
[A]_{y'y'} = [A_{y'y'}] - \tan \theta [A_{jx'}], \quad j = x, y, z, z'. \tag{3}
\]

Using the antisymmetric condition

\[
\begin{pmatrix}
\{u_x\} \\
\{u_y\} \\
\{u_z\} \\
\{f_x'\} \\
\{f_y'\} \\
\{f_z'\}
\end{pmatrix} = \begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\} \\
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

on \( \Gamma \), from (1) we obtain

\[
\begin{bmatrix}
[A_{xx}] & [A_{xy}] & [A_{xz}] & [A_{xy'}] & [A_{xz'}] \\
[A_{yx}] & [A_{yy}] & [A_{yz}] & [A_{yy'}] & [A_{yz'}] \\
[A_{zx}] & [A_{zy}] & [A_{zz}] & [A_{zy'}] & [A_{zz'}] \\
[A_{x'y'}] & [A_{y'y'}] & [A_{x'z'}] & [A_{y'y'}] & [A_{x'z'}] \\
[A_{x'z'}] & [A_{y'z'}] & [A_{z'z'}] & [A_{y'z'}] & [A_{z'z'}]
\end{bmatrix}
\]

\[
\begin{pmatrix}
\{u_x\} \\
\{u_y\} \\
\{u_z\} \\
\{f_x'\} \\
\{f_y'\} \\
\{f_z'\}
\end{pmatrix} = \begin{pmatrix}
\{0\} \\
\{0\} \\
\{0\} \\
\{0\} \\
\{0\} \\
\{0\}
\end{pmatrix}
\]

\[
[A]_{y'y'} = [A_{y'y'}] + \cot \theta ([A_{x'y'}] + [A_{y'x'}]) \\
&\quad + \cot^2 \theta [A_{x'x'}]
\]

\[
[A]_{y'y'} = [A_{y'y'}] + \cot \theta [A_{jx'}], \quad j = x, y, z, z'. \tag{5}
\]

When \( \tan \theta \to \infty \) in (3) and \( \cot \theta \to \infty \) in (5), \( \{u_y\}, [A_{y'y'}], [A_{jy'}] \) and \( [A_{y'y'}] \) should be replaced by \( \{u_x\}, [A_{x'x'}], [A_{jx'}] \) and \( [A_{x'y'}] \), respectively.

III. Results

First, let us consider an isotropic square rod with four planes of symmetry \( x = 0, y = 0, \) and \( x = \pm y \), as in Fig. 2. We subdivide one quarter or one eighth of the cross section into second-order triangular elements as shown in Fig. 3. Table I gives the numerical results for the normalized phase velocities \( v/(c_{44}/\rho)^{1/2} \) for the fundamental branches of the longitudinal (\( L \)), first screw (\( S^{(1)} \)), torsional (\( T \)), and second screw (\( S^{(2)} \)) modes in a square rod with Poisson's ratio \( \sigma = 0.3 \), where \( v = \omega/\beta, c_{44} \) is the stiffness constant and \( \rho \) is the mass density. The fields of the \( L \)
Fig. 2. Square rod.

Fig. 3. Finite-element divisions of a square rod.

Table I: Normalized Phase Velocities as a Function of ωt for an Isotropic Square Rod (ε = 0.3)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>Fig.3(a)aj(b)</td>
</tr>
<tr>
<td>L₁</td>
<td>1</td>
<td>1.5511</td>
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<td>1.5512</td>
<td>1.5512</td>
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<tr>
<td></td>
<td>3</td>
<td>1.069</td>
<td>1.0668</td>
<td>1.0688</td>
<td>1.0690</td>
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<td>0.9632</td>
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<tr>
<td></td>
<td>10</td>
<td>0.936</td>
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<tr>
<td></td>
<td>14</td>
<td>0.939</td>
<td>0.931</td>
<td>0.9313</td>
<td>0.9519</td>
</tr>
<tr>
<td>S₁(1)</td>
<td>1</td>
<td>2.165</td>
<td>2.1646</td>
<td>2.1646</td>
<td>2.1650</td>
</tr>
<tr>
<td></td>
<td>3</td>
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<td>0.9759</td>
<td>0.9764</td>
</tr>
<tr>
<td></td>
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<td>0.908</td>
<td>0.9081</td>
<td>0.9108</td>
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<tr>
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<td>10</td>
<td>0.9121</td>
<td>0.903</td>
<td>0.9035</td>
<td>0.9208</td>
</tr>
<tr>
<td></td>
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<td>0.9234</td>
<td></td>
<td>0.9040</td>
<td>0.9368</td>
</tr>
<tr>
<td>T₁</td>
<td>1</td>
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<td>0.9182</td>
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<td>0.9185</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.9134</td>
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<td>0.9163</td>
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<td>5</td>
<td>0.9064</td>
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<td>0.9112</td>
<td>0.9151</td>
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<td>0.9018</td>
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<td>0.9053</td>
<td>0.9245</td>
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<td>0.905</td>
<td>0.9045</td>
<td>0.9386</td>
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<tr>
<td>S₁(2)</td>
<td>1</td>
<td>1.969</td>
<td>1.9680</td>
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<tr>
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<td>3</td>
<td>1.068</td>
<td>1.0677</td>
<td>1.0678</td>
<td>1.0686</td>
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<td>5</td>
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<td>0.9922</td>
<td>0.9922</td>
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<td>0.952</td>
<td>0.942</td>
<td>0.9424</td>
<td>0.9627</td>
</tr>
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</table>

Next, let us consider anisotropic square rods with four planes of symmetry $x = 0, y = 0$, and $x = \pm y$. We subdivide one eighth of the cross section into second-order triangular elements as in Fig. 3(d).

The normalized phase velocities for the fundamental branches of the $L$, $S₁(1)$, $T$, and $S₁(2)$ modes are presented in Tables II, III, and IV for tetragonal (NiSO₄), hexagonal (zinc), and cubic (copper) materials [10], respectively. For NiSO₄ and copper the $L₁$, and $S₁(1)$ branches intersect. The results that Nigro [10] has identified with the first two branches of the $L$ mode
### TABLE II
Normalized Phase Velocities as a Function of $\beta a$ for a Tetragonal Square Rod (NiSO$_4$)

<table>
<thead>
<tr>
<th>$\beta a$</th>
<th>$L_1$</th>
<th>$S_1^{[1]}$</th>
<th>$T_1$</th>
<th>$S_1^{[2]}$</th>
<th>$T_1$</th>
<th>$S_1^{[2]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.5879</td>
<td>13.8540</td>
<td>0.9187</td>
<td>22.1852</td>
<td>1.585</td>
<td>13.84</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5878</td>
<td>2.7767</td>
<td>0.9179</td>
<td>4.3769</td>
<td>1.585</td>
<td>2.77</td>
</tr>
<tr>
<td>1.0</td>
<td>1.5875</td>
<td>1.4290</td>
<td>0.9153</td>
<td>2.4270</td>
<td>1.427</td>
<td>1.58</td>
</tr>
<tr>
<td>2.0</td>
<td>1.5100</td>
<td>0.9022</td>
<td>0.9045</td>
<td>1.3368</td>
<td>0.901</td>
<td>1.25</td>
</tr>
<tr>
<td>3.0</td>
<td>1.1878</td>
<td>0.8324</td>
<td>0.8876</td>
<td>1.1148</td>
<td>0.831</td>
<td>1.07</td>
</tr>
<tr>
<td>4.0</td>
<td>1.0697</td>
<td>0.8265</td>
<td>0.8708</td>
<td>1.0340</td>
<td>0.828</td>
<td>1.00</td>
</tr>
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<td>5.0</td>
<td>1.0178</td>
<td>0.8362</td>
<td>0.8596</td>
<td>0.9983</td>
<td>0.84</td>
<td>0.97</td>
</tr>
<tr>
<td>6.0</td>
<td>0.9916</td>
<td>0.8424</td>
<td>0.8540</td>
<td>0.9904</td>
<td>0.84</td>
<td>0.95</td>
</tr>
<tr>
<td>7.0</td>
<td>0.9677</td>
<td>0.8494</td>
<td>0.8513</td>
<td>0.9648</td>
<td>0.86</td>
<td>0.94</td>
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<td>8.0</td>
<td>0.9612</td>
<td>0.8514</td>
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<td>0.9613</td>
<td>0.86</td>
<td>0.94</td>
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<td>9.0</td>
<td>0.9558</td>
<td>0.8531</td>
<td>0.8530</td>
<td>0.9590</td>
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<td></td>
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<tr>
<td>10.0</td>
<td></td>
<td></td>
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</table>

### TABLE III
Normalized Phase Velocities as a Function of $\beta a$ for a Hexagonal Square Rod (Zinc)

<table>
<thead>
<tr>
<th>$\beta a$</th>
<th>$L_1$</th>
<th>$S_1^{[1]}$</th>
<th>$T_1$</th>
<th>$S_1^{[2]}$</th>
<th>$T_1$</th>
<th>$S_1^{[2]}$</th>
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</thead>
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<tr>
<td>0.1</td>
<td>0.9536</td>
<td>27.8624</td>
<td>0.9187</td>
<td>22.1878</td>
<td>0.954</td>
<td>27.84</td>
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<tr>
<td>0.5</td>
<td>0.9503</td>
<td>5.3884</td>
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<td>4.3262</td>
<td>0.951</td>
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<td>1.0</td>
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<td>2.5278</td>
<td>0.9166</td>
<td>2.1231</td>
<td>0.939</td>
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<tr>
<td>2.0</td>
<td>0.8790</td>
<td>1.1931</td>
<td>0.9095</td>
<td>1.1560</td>
<td>0.879</td>
<td>1.19</td>
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<tr>
<td>3.0</td>
<td>0.7893</td>
<td>0.8679</td>
<td>0.8958</td>
<td>0.9418</td>
<td>0.790</td>
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<tr>
<td>4.0</td>
<td>0.7503</td>
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<td>0.8736</td>
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<td>0.76</td>
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<tr>
<td>5.0</td>
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<td>0.7632</td>
<td>0.8451</td>
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<tr>
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<td>0.7591</td>
<td>0.7674</td>
<td>0.8190</td>
<td>0.8373</td>
<td>0.76</td>
<td>0.77</td>
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<tr>
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### TABLE IV
Normalized Phase Velocities as a Function of $\beta a$ for a Cubic Square Rod (Copper)

<table>
<thead>
<tr>
<th>$\beta a$</th>
<th>$L_1$</th>
<th>$S_1^{[1]}$</th>
<th>$T_1$</th>
<th>$S_1^{[2]}$</th>
<th>$T_1$</th>
<th>$S_1^{[2]}$</th>
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<td>0.9874</td>
<td>2.6441</td>
<td>0.9168</td>
<td>3.7881</td>
<td>0.987</td>
<td>2.64</td>
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<td>1.0</td>
<td>0.9392</td>
<td>1.3039</td>
<td>0.9105</td>
<td>1.8649</td>
<td>0.939</td>
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<td>0.7667</td>
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<td></td>
</tr>
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</table>
are also shown in Tables II-IV (branch 1 and branch 2), and we see that for NiSO₄ and copper the branches 1 and 2 must be reidentified as in Tables II and IV [11]. Nigro's reidentified results are in good agreement with the present results for the $L_1$ and $S_1^{(1)}$ branches. Our results for the $T_1$ and $S_1^{(2)}$ branches agree well with those of Nigro and O'Malley [12], [13].

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REFERENCES


