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Crossed products of totally disconnected spaces by $\mathbb{Z}_2 * \mathbb{Z}_2$

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Abstract. Let Ω be a totally disconnected compact metrizable space, and let α be a minimal homeomorphism of Ω. Let σ be a homeomorphism of order 2 on Ω such that $\alpha\sigma = \sigma\alpha^{-1}$, and assume that σ or $\alpha\sigma$ has a fixed point. We prove (Theorem 3.5) that the crossed product $C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2$ is an AF-algebra.

0. *Introduction*

We prove the result stated in the abstract by an elaboration of Putnam's tower construction in [**Put2**]. He proves, without the assumptions involving σ, that any finite number of elements in $C(\Omega) \times_\alpha \mathbb{Z}$ can be approximated by elements in a unital subalgebra of the form

$$[M_{J_1} \otimes C(\mathbb{T})] \oplus M_{J_2} \oplus M_{J_3} \oplus \ldots \oplus M_{J_K}$$

and as a consequence $C(\Omega) \times_\alpha \mathbb{Z}$ has stable rank one.

In §1 we make a σ-covariant version of Putnam's construction, and the main result is Theorem 1.1.

In §2 we use spectral theory to prove, in a σ-covariant way, that $C(\Omega) \times_\alpha \mathbb{Z}$ contains an increasing sequence of algebras of the above form with dense union—see Theorem 2.1. A similar theorem, without σ-covariance and injectivity follows from Theorem 4.3 of [**Ell**]. As a corollary, $C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2$ contains an increasing sequence of subalgebras of the form

$$\tilde{B}_0 \oplus M_{n_1} \oplus M_{n_2} \oplus \ldots \oplus M_{n_N},$$

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where

$$\tilde{B}_0 = \{x \in C(I, M_{2J_1}) : Ex(-1) = x(-1)E \quad \text{and} \quad Ex(1) = x(1)E\}$$

and $I = [-1, 1]$ is the unit interval, and $E \in M_{2J_1}$ is a projection of dimension J_1—see Corollary 2.4.

In §3 we extend the methods of [**BEEK1**] to prove from Corollary 2.4, together with the fact that $C(\Omega) \times_\alpha \mathbb{Z}$ has real rank zero, that $C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2$ is AF, see Theorem 3.5.

Finally, in §4, we use Kumjian's method from [**Kum2**] to compute the K-theory of $C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2$.

In a subsequent paper, [**BK**], the methods of this paper will be extended to prove that the flip-invariant part of the irrational rotation algebra is AF. The irrational rotation algebra is the universal algebra generated by two unitaries U, V with $VU = e^{2\pi i\theta}UV$, where θ is irrational, and the flip σ is defined by $\sigma(V) = V^{-1}$, $\sigma(U) = U^{-1}$, [**Rie**], [**BEEK2**], [**BEEK3**]. The methods used in [**BK**] are somewhat different from those of [**Put3**]. Instead of cutting up the circle, the projections in [**Kum1**] are used.

1. *The tower construction and Berg's technique for $\mathbb{Z} \times_\sigma \mathbb{Z}_2$*

Let Ω be a totally disconnected compact metrizable space. Let $\alpha : \Omega \to \Omega$ be a minimal action on Ω, i.e. α is a homeomorphism of Ω such that the orbit $\{\alpha^n\omega; n \in \mathbb{Z}\}$ is dense in Ω for each $\omega \in \Omega$. Let $\sigma : \Omega \to \Omega$ be an action of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ on Ω, i.e. σ is a homeomorphism of Ω such that $\sigma^2 = \iota$, where ι is the identity. Assume that

$$\alpha\sigma = \sigma\alpha^{-1}. \tag{1.1}$$

In particular, this entails that each of the homeomorphisms $\alpha^n\sigma$, $n \in \mathbb{Z}$, are of order two. To prove our theorem we shall also need the assumption that there exists some $\omega \in \Omega$ such that

$$\alpha\sigma\omega = \omega, \tag{1.2}$$

and we do not know if the theorem is true without this hypothesis. It should, however, be pointed out that since the relation between $\sigma\alpha$ and α is the same as that between σ and α, given by (1.1), one could replace σ by $\sigma\alpha = \alpha^{-1}\sigma$ in all subsequent arguments, and hence (1.2) could be replaced throughout by

$$\sigma\omega = \omega \tag{1.2$_0$}$$

or, for that sake, by

$$\alpha^n\sigma\omega = \omega \tag{1.2$_n$}$$

for any $n \in \mathbb{Z}$. But since e.g. $(1.2)_n$ implies $\alpha^{n-1}\sigma\alpha^{-1}\omega = \omega$, i.e. $\alpha^{n-2}\sigma(\alpha^{-1}\omega) = \alpha^{-1}\omega$, it follows that the assumption $(1.2)_n$ is the same as (1.2) if n is odd, and the same as $(1.2)_0$ if n is even.

At this point it is instructive to consider the case that Ω is finite, since the proof in the general case is to some extent modelled on this case. Then Ω is necessarily homeomorphic to $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ in such a way that α is homeomorphic to the shift

$$\alpha n = n + 1.$$

A simple computation shows that σ must have the form

$$\sigma n = M - n$$

for $M = \sigma 0 \in \mathbb{Z}_N$, so if N is even σ either has none or two fixed points whilst $\alpha\sigma$ has two or none, and if N is odd σ and $\alpha\sigma$ have exactly one fixed point each. In this case an explicit computation shows that

$$C(\Omega) \times_\alpha \mathbb{Z} \cong M_N \otimes C(\mathbb{T})$$

and

$$C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2 \cong \{A \in M_{2N} \otimes C(I), A(0)E = EA(0), A(1)E = EA(1)\},$$

where E is an N-dimensional projection in M_{2N}, **[BEEK1]**, **[BE1]**. Alternatively

$$C(\Omega) \times_\alpha \mathbb{Z}_N \cong M_N$$

and

$$C(\Omega) \times_\alpha \mathbb{Z}_N \times_\sigma \mathbb{Z}_2 \cong M_N \oplus M_N.$$

Throughout the rest of the paper we shall assume that

$$\Omega \text{ is infinite.} \tag{1.3}$$

We shall also identify α, σ with the corresponding actions on $C(\Omega)$ by

$$\alpha f(\omega) = f(\alpha^{-1}\omega), \quad \sigma f(\omega) = f(\sigma\omega).$$

We shall follow the general notation of **[Put2]**, but change the formalism a little. For example, we keep the convention that a partition of Ω is a finite partition of Ω into closed and open (clopen) subsets, and if $\mathcal{P}$ is a partition, then $C(\mathcal{P})$ is the finite dimensional abelian C^*-algebra of functions on Ω which are constant over each set in $\mathcal{P}$.

The part of the following theorem which does not involve σ is Theorem 2.1 of **[Put2]**, and the new proof is executed by an extension of the techniques of **[Put2]**, which again is based on what is called Berg's technique in **[Ver1]**, **[Ver2]**.

THEOREM 1.1. *Adopt the notation and assumptions above. It follows that for any finite partition $\mathcal{P}$ of Ω (into clopen subsets) and any $\varepsilon > 0$ there is a unital C^*-subalgebra $A \subseteq C(\Omega) \times_\alpha \mathbb{Z}$ of the form*

$$[M_{J_1} \otimes C(\mathbb{T})] \oplus M_{J_2} \oplus \ldots \oplus M_{J_K} \tag{1.4}$$

for some integers $J_1, \ldots, J_K$ such that $C(\mathcal{P}) \subseteq A$, and there is a unitary $u' \in A$ such that $\|u - u'\| < \varepsilon$, where u is the canonical unitary in $C(\Omega) \times_\alpha \mathbb{Z}$. Furthermore $\sigma(A) = A$, and σ acts on the canonical unitary $z \to z$ in $1_{J_1} \otimes C(\mathbb{T})$ by sending it into $z \to \bar{z}$, and on a certain set of matrix units e^1_{ij}, $i, j = 0, \ldots, J_1 - 1$ of $M_{J_1} \otimes 1$ by sending them into $e^1_{J_1-1-i, J_1-1-j}$, respectively. On the remaining part

$$M_{J_2} \oplus \ldots \oplus M_{J_K}$$

of A, σ acts by interchanging summands M_{J_i}, M_{J_j} with $J_i = J_j$ or by globally fixing summands M_{J_k}, sending e^k_{ij} into $e^k_{J_k-1-i, J_k-1-j}$. (In our construction J_1 is even.)

We first establish the following subsidiary result.

PROPOSITION 1.2. *Adopt the notation and assumptions before Theorem 1.1. It follows that for any finite partition $\mathcal{P}$ of Ω into clopen subsets and any $N \in \mathbb{N}$ there exists clopen sets $Y_1, Y_2, \ldots, Y_K$ in Ω and integers $J_1, \ldots, J_K$ such that*
(1.5) $J_k \geq 2N + 2$ *for* $k = 1, \ldots, K$.
(1.6) *The sets* $\alpha^k(Y_i)$, $k = 0, 1, \ldots, J_i - 1$, $i = 1, \ldots, K$ *are mutually disjoint with union* Ω, *and thus constitute a partition* $\mathcal{P}_0$ *of* Ω.
(1.7) $\mathcal{P}_0$ *is a refinement of* $\mathcal{P}$.
(1.8) $\{\sigma(Y_1), \ldots, \sigma(Y_K)\} = \{\alpha^{J_1-1}(Y_1), \ldots, \alpha^{J_K-1}(Y_K)\}$ *(as unordered sets). Define* $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_K$. *Then, furthermore*
(1.9) *For* $k = 0, 1, \ldots, N$ *there exists* $A_k, B_k \in \mathcal{P}$ *such that*

$$\alpha^k(Y) \subseteq A_k \quad \sigma\alpha^k(Y) = \alpha^{-k}\sigma(Y) \subseteq B_k.$$

Remark 1.3. It follows immediately from (1.6) and (1.8) that the tower structure defined by $Y_1, Y_2, \ldots, Y_K$ is σ-invariant, i.e. if

$$\sigma(Y_i) = \alpha^{J_j-1}(Y_j)$$

then $J_j = J_i$ and

$$\sigma(Y_j) = \alpha^{J_i-1}(Y_i). \tag{1.10}$$

(One uses the involutory nature of the homeomorphisms $\alpha^k\sigma = \sigma\alpha^{-k}$ together with an induction argument, starting with the tower of greatest height J_i.)

Other consequences of Proposition 1.2 are

$$\mathcal{P}_0 \text{ is } \sigma\text{-invariant.} \tag{1.11}$$

(In fact it follows from (1.8) and $\sigma\alpha^k = \alpha^{-k}\sigma$ that σ applied to a tower, either reverses the tower or interchanges it with another tower, reversing both.)

$$Y = \bigcup_{k=1}^{K} \alpha^{J_k}(Y_k) \tag{1.12}$$

(follows from (1.6)).

$$\alpha\sigma(Y) = Y \tag{1.13}$$

(follows from (1.12) and (1.8)).

As a preliminary to Proposition 1.2 again we prove:

LEMMA 1.4. *Let Y be a clopen set in Ω with the property (1.13):*

$$\alpha\sigma(Y) = Y.$$

Define

$$\lambda(\omega) = \min\{n > 0; \alpha^n(\omega) \in Y\}$$

for $\omega \in Y$. Then λ is continuous, and thus has a finite range

$$\lambda(Y) = \{J_1, \ldots, J_K\},$$

where $J_1 < J_2 < \ldots < J_K$. Define

$$Y_i = \lambda^{-1}(J_i).$$

Then

$$\{\alpha^k(Y_i); k = 0, \ldots, J_i - 1, i = 1, \ldots, K\}$$

is a partition of Ω into clopen sets, and

$$\sigma(Y_i) = \alpha^{J_i - 1}(Y_i) \tag{1.14}$$

for $i = 1, \ldots, K$.

Proof. This lemma has already been established in §2 of [**Put2**] apart from the property (1.14). (Note that the continuity of λ alternatively follows from the relation $\lambda^{-1}(J) = \alpha^{-J}(Y) \bigcap_{0<j<J}(\alpha^{-j}(\Omega \backslash Y)) \cap Y$ for $J = 1, 2, \ldots$.) But as

$$\alpha^{J_i}(Y_i) \subseteq Y$$

one has

$$\sigma\alpha^{J_i - 1}(Y_i) = \alpha\sigma\alpha^{J_i}(Y_i) \subseteq \alpha\sigma(Y) = Y.$$

Now, assume

$$\rho \in \sigma\alpha^{J_i - 1}(Y_i),$$

i.e. there is a $\omega \in Y_i$ such that

$$\rho = \sigma\alpha^{J_i - 1}(\omega) = \alpha^{-J_i + 1}\sigma(\omega) = \alpha^{-J_i}\alpha\sigma(\omega)$$

and thus

$$\alpha^{J_i}(\rho) = \alpha\sigma(\omega) \in Y.$$

It follows that

$$\lambda(\rho) \leq J_i.$$

Now, if J_i is the smallest of the J's, i.e. $i = 1$, then necessarily $\lambda(\rho) = J_1$ and $\rho \in Y_1$. Thus

$$\sigma\alpha^{J_1 - 1}(Y_1) \subseteq Y_1.$$

But, conversely, as $\sigma\alpha^{J_1 - 1}$ is its own inverse

$$Y_1 \subseteq \sigma\alpha^{J_1 - 1}(Y_1)$$

and hence

$$Y_1 = \sigma\alpha^{J_1 - 1}(Y_1).$$

Repeating this argument for the successively higher values $J_2 < J_3 < \ldots < J_K$ one establishes by induction that

$$Y_k = \sigma\alpha^{J_k - 1}(Y_k)$$

for $k = 2, 3, \ldots, K$.

Proof of Proposition 1.2. We shall prove the proposition by making a careful choice of Y in Lemma 1.3, and then refine the partition. At this point, we must use the existence of a fixed point for $\alpha\sigma$, (1.2). So let ω_0 be a fixed point

$$\alpha\sigma\omega_0 = \omega_0.$$

For N given, there exist sets A_k, B_k in $\mathcal{P}$ for $k = 0, 1, \ldots, N$ such that

$$\alpha^k\omega_0 \in A_k \quad \sigma\alpha^k\omega_0 = \alpha^{-k}\sigma\omega_0 \in B_k.$$

Put

$$Z = \bigcap_{k=0}^{N}(\alpha^{-k}(A_k) \cap \sigma\alpha^k(B_k)).$$

Then $\omega_0 \in Z$, and $\alpha^k(Z) \subseteq A_k$, $\sigma\alpha^k(Z) \subseteq B_k$ for $k = 0, 1, \ldots, N$. Now, as α is minimal and $\alpha\sigma\omega_0 = \omega_0$, it follows that all the points $\alpha^k\omega_0$, $\sigma\alpha^k\omega_0 = \alpha^{-k}\sigma\omega_0$, $k = 0, 1, \ldots$ are distinct. Hence, choosing Z even smaller, but still containing ω_0, we may furthermore assume that the sets

$$\alpha^k(Z), \sigma\alpha^k(Z), \quad k = 0, 1, \ldots, N$$

are disjoint. Put

$$Y = Z \cap \alpha\sigma(Z).$$

Then $Y \neq \emptyset$ since $\omega_0 \in Y$, the sets

$$\alpha^k(Y), \sigma\alpha^k(Y) \quad k = 0, 1, \ldots, N$$

are pairwise disjoint, and

$$\begin{matrix} \alpha^k(Y) \subseteq A_k \\ \sigma\alpha^k(Y) \subseteq B_k \end{matrix} \quad k = 0, 1, \ldots, N. \tag{1.9}$$

Now, constructing $Y_1, \ldots, Y_k$ as in Lemma 1.3, all the conditions of Proposition 1.2 are fulfilled with the possible exception of (1.7), since $\mathcal{P}$ has not entered the construction yet. But by further cutting up the towers $\alpha^i(Y_k)$, $i = 0, \ldots, J_k - 1$ from bottom to top as in **[Put2]**, more precisely, partitioning each Y_k into a $\sigma\alpha^{J_k-1}$-invariant family of subsets, one may also ensure that (1.7) is fulfilled as well as (1.8). This ends the proof of Proposition 1.2.

At this point, equip Ω with a probability measure μ which is both α- and σ-invariant. This is possible as Ω is compact and $\mathbb{Z} \times_\sigma \mathbb{Z}_2$ is amenable. Let $u(\alpha)$, $u(\sigma)$ be the unitaries implementing α and σ on $L^2(\Omega, \mu)$,

$$u(\alpha)\psi(\omega) = \psi(\alpha^{-1}\omega), \quad u(\sigma)\psi(\omega) = \psi(\sigma\omega). \tag{1.15}$$

Represent $C(\Omega)$ on $L^2(\Omega, \mu)$ in the standard way

$$f\psi(\omega) = f(\omega)\psi(\omega). \tag{1.16}$$

If X is a clopen subset of Ω, χ_x denotes the characteristic function of X.

LEMMA 1.5. *Adopt the assumptions of Proposition 1.2, and let A_0 be the C^*-algebra on $L^2(\Omega, \mu)$ generated by $C(\mathcal{P}_0)$ and the operator $u(\alpha)\chi_{\Omega\setminus\sigma(Y)}$. It follows that A_0 is finite-dimensional, and the operators*

$$e_{ij}^k = u(\alpha)^i \chi_{Y_k} u(\alpha)^{*j} = u(\alpha)^{i-j} \chi_{\alpha^j(Y_k)} \tag{1.17}$$

for $i, j = 0, 1, \ldots, J_k - 1$, $k = 1, 2, \ldots, K$ constitute a complete set of matrix units for A_0. Furthermore, A_0 is invariant under $\mathrm{Ad}(u(\sigma))$ and

$$u(\sigma) e_{ij}^k u(\sigma) = e_{J_k-1-i, J_k-1-j}^{\ell}, \tag{1.18}$$

where $k = \ell$, or $k \neq \ell$ with $J_k = J_\ell$.

Proof. It is easily verified from Proposition 1.2 that $\{e_{ij}^k\}$ constitute a complete set of matrix units, and (1.18) follows from (1.10) in Remark 1.3. One has

$$e_{ii}^k = \chi_{\alpha^i(Y_k)}, \quad i = 0, \ldots, J_k - 1, \ k = 1, \ldots, K \tag{1.19}$$

and

$$\sum_{k=1}^{K} \sum_{i=0}^{J_i-2} e_{i+1,i}^k = \sum_{k=1}^{K} \sum_{i=0}^{J_i-2} u(\alpha) \chi_{\alpha^i(Y_k)} = u(\alpha) \chi_{\Omega\setminus\sigma(Y)}, \tag{1.20}$$

where we used that the roof of the tower is $\sigma(Y) = \bigcup_{k=1}^{K} \alpha^{J_k-1}(Y_k)$. These relations imply that $A_0 = C^*(C(\mathcal{P}_0), u(\alpha)\chi_{\Omega-\sigma(Y)})$ is exactly the C^*-algebra defined by the matrix units.

Still following **[Put2]**, we next modify $u(\alpha)$ to a unitary operator v_0 in A_0, i.e.

$$\begin{aligned} v_0 &= \sum_{k=1}^{K} \sum_{i=0}^{J_k-2} e_{i+1,i}^k + \sum_{k=1}^{K} e_{0,J_k-1}^k \\ &= u(\alpha)\chi_{\Omega\setminus\sigma(Y)} + \sum_{k=1}^{K} u(\alpha)^{*J_k-1} \chi_{\alpha^{J_k-1}(Y_k)} \\ &= u(\alpha)\chi_{\Omega\setminus\sigma(Y)} + \sum_{k=1}^{K} \chi_{Y_k} u(\alpha)^{*J_k-1}. \end{aligned} \tag{1.21}$$

Thus v_0 is a sum of cyclic unitaries, one for each tower. The unitary v_0 lifts each floor of each tower one floor up except for the top floor which is mapped onto the bottom one. We also introduce another unitary operator u_0 measuring how far v_0 is from $u(\alpha)$, i.e.

$$u(\alpha) = u_0 v_0. \tag{1.22}$$

Thus

$$u_0 = u(\alpha) v_0^* = \chi_{\Omega\setminus Y} + \sum_{k=1}^{K} u(\alpha)^{J_k} \chi_{Y_k}. \tag{1.23}$$

To proceed, we need even more structure in Proposition 1.2, i.e.

PROPOSITION 1.6. *The clopen subsets $Y_1, Y_2, \ldots, Y_K$ in Proposition 1.2 may be chosen so that they have the following further properties in addition to (1.5)–(1.9):*

$$Y_k \cap \alpha\sigma(Y_k) = \emptyset \quad \textit{for } k = 1, \ldots, K. \tag{1.24}$$

Proof. Note that if the set Y in Lemma 1.4 is replaced by an $\alpha\sigma$-invariant clopen subset, the tower over each point becomes higher. Thus we take the Y used in the proof of Proposition 1.2 and throw away a clopen neighbourhood of the $\alpha\sigma$-fixed points in Y. Since $\alpha\sigma$ anticommutes with α, each α-orbit contains at most one fixed point for $\alpha\sigma$, and since α is minimal, it follows that the set of $\alpha\sigma$-fixed points contains no open set. Hence the complement of the set of $\alpha\sigma$-fixed points is open and dense, and hence we may arrange that Y is still non-empty after throwing away the clopen neighbourhood of the $\alpha\sigma$-fixed points. Since the new set Y contains no $\alpha\sigma$-fixed points, $\alpha\sigma(Y)$ contains no $\alpha\sigma$-fixed point, so replacing Y by $\alpha\sigma(Y) \cup Y$, we may assume that the new Y still satisfies

$$\alpha\sigma(Y) = Y.$$

Since Y does not contain any $\alpha\sigma$-fixed point we can find a partition $\mathcal{P}_1$ of Y such that $\mathcal{P}_1$ is $\alpha\sigma$-invariant, and $\alpha\sigma(A) \cap A = \emptyset$ for any $A \in \mathcal{P}_1$. Now repeat the proof of Proposition 1.2 from Lemma 1.3, but replace the old partition $\mathcal{P}$ by the joint refinement of $\mathcal{P}$ and $\mathcal{P}_1$. This ensures the property (1.24), and since each of the new Y_k's are contained in one of the old ones we do keep property (1.9).

We next explore some consequences of Propositions 1.2 and 1.6.

LEMMA 1.7. *Assume that $Y = Y_1 \cup \ldots \cup Y_K$ satisfy the conclusions of Propositions 1.2 and 1.6. It follows that there exist some Y_k, Y_1 say, such that*

$$\text{There is a } \omega \in Y_1 \text{ such that } \sigma\alpha^{J_1-1}(\omega) = \omega, \tag{1.25}$$

$$\sigma\alpha^{J_1-1}(Y_1) = Y_1, \tag{1.26}$$

$$J_1 \text{ is even.} \tag{1.27}$$

Moreover, Y_1 can be taken to be any Y_k such that an $\alpha\sigma$-fixed point ω_0 lies in the tower over Y_1.

Proof. Since (1.24) implies that Y contains no $\alpha\sigma$-fixed point, it is clear that the $\alpha\sigma$-fixed point must lie in the tower over some Y_k, say Y_1, and not in the bottom floor Y_1 of the tower. Since $\alpha(\sigma\omega_0) = \omega_0$, it follows that $\sigma\omega_0$ also lies in the Y_1 tower in the floor below ω_0. Hence there is some $k < J_1 - 1$ and a $\omega \in Y_1$ such that

$$\alpha^k\omega = \sigma(\omega_0).$$

But then, as $\sigma\alpha^k$ is its own inverse,

$$\alpha^k\omega_0 = \sigma(\omega).$$

Hence

$$\omega_0 = \alpha\sigma\omega_0 = \alpha^{k+1}\omega$$

and then

$$\sigma(\omega) = \alpha^k\omega_0 = \alpha^{2k+1}\omega. \tag{1.28}$$

Since σ reverses the towers by (1.18), and the two points ω_0 and $\sigma\omega_0$ are mapped into each other by σ, it follows that these two points lie in the middle of the tower over Y_1. It follows from (1.28) and (1.8) that $J_1 = 2k+2$, $\sigma(Y_1) = \alpha^{J_1-1}(Y_1)$ and $\sigma\alpha^{J_1-1}\omega = \omega$.

LEMMA 1.8. *Assume that $Y = Y_1 \cup \ldots \cup Y_K$ satisfy the conclusions of Propositions 1.2 and 1.6, and choose Y_1 as in Lemma 1.7. It follows that Y_1 contains three mutually disjoint clopen subsets A, B, C such that*

$$\alpha^{J_1-1}(A) = \sigma(A), \tag{1.29}$$

$$\alpha^{J_1-1}(B) = \sigma(C), \tag{1.30}$$

$$\alpha^{J_1-1}(C) = \sigma(B), \tag{1.31}$$

and if k is the smallest positive integer such that $\alpha^k\sigma(A) \cap Y_1 \neq \emptyset$, then

$$B = \alpha^k\sigma(A), \quad A \cap \alpha^j(A) = \emptyset \quad \text{if } 0 \leq j < k. \tag{1.32}$$

Proof. By Lemma 1.7, $\sigma\alpha^{J_1-1}$ is a homeomorphism of Y_1 of order 2 with a fixed point ω, and hence ω has a neighbourhood basis of clopen sets which is invariant under $\sigma\alpha^{J_1-1}$. Thus, if A is one of the sets in the basis, then

$$\alpha^{J_1-1}(A) = \sigma(A).$$

Since $\alpha^{J_1-1}\omega = \sigma\omega$ and α is free, it follows that $\alpha^k\sigma\omega \neq \omega$ for $k = 1, 2, \ldots$. Hence, choosing A small enough, we may ensure that if k is the smallest positive k such that $\alpha^k\sigma(A) \cap Y_1 \neq \emptyset$, then $\alpha^k\sigma(A) \cap A = \emptyset$, and choosing A even smaller we may ensure that $\alpha^k\sigma(A) \subset Y_1$ for this k. By choosing A even smaller we may also ensure that

$$\sigma\alpha^{J_1-1}\alpha^k\sigma(A) \cap \alpha^k\sigma(A) = \emptyset.$$

This is possible since

$$\alpha^{-J_1+1-k}\omega \neq \alpha^k\sigma\omega$$

for all $k = 1, 2, \ldots$, because $\alpha^{J_1-1}\omega = \sigma\omega$. Now put

$$B = \alpha^k\sigma(A), \quad C = \sigma\alpha^{J_1-1}B,$$

and use $(\sigma\alpha^{J_1-1})^2 = \iota$ to verify (1.30) and (1.31). Finally, choosing A as an even smaller clopen neighbourhood of ω, one may ensure that

$$\alpha^j(A) \cap A = \emptyset$$

for $j = 0, 1, \ldots, k-1$, since α is free.

Next we shall repeat the tower construction with Y replaced by

$$X = A \cup \alpha\sigma(A), \tag{1.33}$$

where A is defined in Lemma 1.8. Define v_1, u_1, A_1 for the new tower construction as v_0, u_0, A_0 were defined for the old, but such that the role of $\mathcal{P}$ is replaced by $\mathcal{P}_0$, i.e. the new tower partition $\mathcal{P}_1$ is a refinement of $\mathcal{P}_0$.

LEMMA 1.9. *One has*

$$\mathrm{Ad}(v_0u(\sigma))(v_1v_0^*) = (v_1v_0^*)^* \tag{1.34}$$

and

$$v_1v_0^* \in A_1. \tag{1.35}$$

Proof. Since $\mathcal{P}_1$ is a refinement of $\mathcal{P}_0$, one sees from (1.17) that $A_0 \subseteq A_1$, and hence $v_1 v_0^* \in A_1$. Next, from (1.18) and (1.21)

$$u(\sigma)v_0u(\sigma) = \sum_{k=1}^{K}\sum_{i=0}^{J_k-2} e^k_{J_k-i-1,J_k-i} + \sum_{k=1}^{K} e^k_{J_k-1,0} = v_0^* \tag{1.36}$$

and similarly

$$u(\sigma)v_1u(\sigma) = v_1^*. \tag{1.37}$$

Thus

$$\mathrm{Ad}(v_0u(\sigma))(v_1v_0^*) = v_0u(\sigma)v_1u(\sigma)u(\sigma)v_0^*u(\sigma)v_0^* = v_0v_1^*v_0v_0^* = v_0v_1^* = (v_1v_0^*)^*.$$

To understand the significance of the next lemma, we have to analyse the action of $v_1v_0^*$ on the towers corresponding to X. Each of the towers are left globally invariant but the floors are shuffled as follows displayed in Figure 1, in a typical tower.

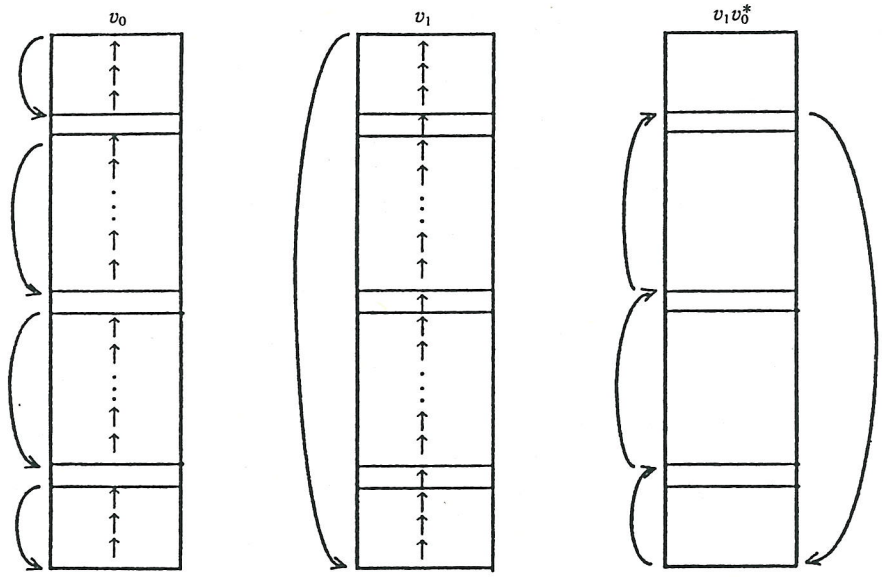

FIGURE 1.

Here the marked subtowers are parts of the Y-towers. Hence inside each minimal projection of the center of A_1, $v_1v_0^*$ is a direct sum of the identity and a cyclic unitary, and the order of the cyclic unitary is equal to the number of floors which intersect Y (and then are contained in Y).

LEMMA 1.10. *If $X = X_1 \cup X_2 \cup \ldots \cup X_{K'}$ is the partition of X defined by the tower construction, then for any k such that σ maps the tower over X_k into itself (i.e. $\alpha^{J'_k-1}X_k = \sigma(X_k)$) the number of floors in this tower contained in Y is odd, and hence the restriction of $v_1v_0^*$ to the corresponding central projection in A_1 has odd order.*

Proof. If σ maps the tower over X_k into itself, then σ reverses the floorplan, by (1.18). Since $\alpha\sigma(Y) = Y$, it follows that if D is a floor in the tower and D is contained in Y,

then $\sigma(D)$ is a floor in the tower and hence, unless D is the ground floor X_k (and thus $\sigma(D)$ is the top floor), $\alpha\sigma(D)$ is another floor in the tower and $\alpha\sigma(D)$ is contained in Y. Furthermore $\alpha\sigma(D)$ is distinct from D since D is contained in some Y_i, and Y_i is disjoint from $\alpha\sigma(Y_i)$ by Proposition 1.6. Thus, excluding the ground floor, the floors contained in Y occur in distinct pairs $D\alpha\sigma(D)$. Therefore, counting also the ground floor, the number of floors in the tower which are contained in Y is odd.

LEMMA 1.11. *There exists a unitary operator $w \in A_1$ such that*

$$w\chi_{\Omega\setminus Y} = \chi_{\Omega\setminus Y} \tag{1.38}$$

$$w^{2N} = v_1 v_0^* \tag{1.39}$$

$$\mathrm{Ad}(v_0 u(\sigma))(w) = w^* \tag{1.40}$$

$$||1 - w|| \leq \pi/2N. \tag{1.41}$$

Proof. Let

$$v_1 v_0^* = \sum_\lambda \lambda e(\lambda)$$

be the spectral decomposition of $v_1 v_0^*$. It follows from (1.34) that

$$\mathrm{Ad}(v_0 u(\sigma))(e(\lambda)) = e(\bar{\lambda}).$$

Thus, if -1 is not in the spectrum of $v_1 v_0^*$, we may define

$$w = \sum_\lambda \lambda^{1/2N} e(\lambda),$$

where $z^{1/2N}$ is the branch of the holomorphic function with $1^{1/2N} = 1$ and cut along the negative real axis. The properties (1.38)–(1.41) are then immediate. However, if $e(-1) \neq 0$, we must find a decomposition

$$e(-1) = e_+ + e_-$$

of $e(-1)$ such that $\mathrm{Ad}(v_0 u(\sigma))(e_+) = e_-$ and $\mathrm{Ad}(v_0 u(\sigma))(e_-) = e_+$, and then define

$$w = \sum_{\lambda \neq -1} \lambda^{1/2N} e(\lambda) + e^{\pi i/2N} e_+ + e^{-\pi i/2N} e_-.$$

The existence of such a decomposition follows from Lemma 1.10. Given the central projection P_{X_k} corresponding to the tower over X_k, there are two possibilities: if this tower is mapped into itself by σ, then $v_1 v_0^* P_{X_k}$ has odd order, hence -1 is not an eigenvalue of $v_1 v_0^* P_{X_k}$ and there is no problem. If on the other hand the tower is interchanged with the tower over X_ℓ by σ, then $\sigma(P_{X_k}) = P_{X_\ell}$ and $\sigma(P_{X_\ell}) = P_{X_k}$. If all such pairs are ordered, and P_+ is the sum of the P_{X_k}'s corresponding to the first member of the pair, and P_- the sum over the second members, then P_+, P_- are central projections in A_1 such that $\sigma(P_+) = P_-$, $\sigma(P_-) = P_+$ and $P_+ P_- = 0$. Now put

$$e_+ = e(-1)P_+, \quad e_- = e(-1)P_-.$$

We have already computed that

$$\mathrm{Ad}(v_0 u(\sigma))(e(-1)) = e(-1)$$

and as v_0 commutes with the central projections P_+ and P_- in A_1, we have

$$\mathrm{Ad}(v_0 u(\sigma))(P_+) = P_-, \quad \mathrm{Ad}(v_0 u(\sigma))(P_-) = P_+.$$

This establishes the desired properties

$$\mathrm{Ad}(v_0 u(\sigma))(e_+) = e_-, \quad \mathrm{Ad}(v_0 u(\sigma))(e_-) = e_+.$$

We also have to construct another unitary operator u:

LEMMA 1.12. *There exists a unitary operator $u \in A_1$ such that*

$$u\chi_{\Omega\setminus Y} = \chi_{\Omega\setminus Y} \tag{1.42}$$

$$u^N \chi_{Y_1} u^{-N} \geq w^{-N} \chi_X w^N \tag{1.43}$$

$$\mathrm{Ad}(v_0(v_0 u(\sigma))(u) = u \tag{1.44}$$

$$||1 - u|| \leq \pi/N. \tag{1.45}$$

Proof. It suffices to construct a unitary operator u^N in the finite-dimensional algebra A_1 with the properties

$$u^N \chi_{\Omega\setminus Y} = \chi_{\Omega\setminus Y} \tag{1.46}$$

$$u^N \chi_{Y_1} u^{-N} \geq w^{-N} \chi_X w^N \tag{1.43}$$

$$\mathrm{Ad}(v_0 u(\sigma))(u^N) = u^N \tag{1.47}$$

and then define u by spectral theory.

First note that as $X = A \cup \alpha\sigma(A)$, one of the towers in the X-tower construction is $A, \alpha(A), \alpha^2(A), \ldots, \alpha^{J_1-1}(A) = \sigma(A)$, and thus

$$v_1 \chi_{\sigma(A)} v_1^* = \chi_A.$$

But this is also part of the tower over Y_1 in the Y-tower construction, and thus

$$v_0^* \chi_A v_0 = \chi_{\sigma(A)}.$$

Hence

$$v_1 v_0^* \chi_A v_0 v_1^* = v_1 \chi_{\sigma(A)} v_1^* = \chi_A,$$

or

$$v_1 v_0^* \chi_A = \chi_A v_1 v_0^*. \tag{1.48}$$

In particular, this means that all the spectral projections $e(\lambda)$ of $v_1 v_0^*$ commute with χ_A, and since $\chi_A \in A_1$ also the central projections P_+ and P_- constructed in the proof of

Lemma 1.11 commutes with χ_A. Hence, inspecting the proof of Lemma 1.11, all the spectral projections of w commute with χ_A, and thus

$$w\chi_A = \chi_A w. \tag{1.49}$$

Let k be the positive integer defined by (1.32) in Lemma 1.8, put

$$\ell = k - 1, \tag{1.50}$$

and define an operator V by

$$V = w^{-N} v_1^{-\ell} \chi_B + v_0 u(\sigma) w^{-N} v_1^{-\ell} \chi_B u(\sigma) v_0^*. \tag{1.51}$$

As $v_0 u(\sigma) w^{-N} = w^N v_0 u(\sigma)$ by (1.40) and

$$\begin{aligned} \chi_B u(\sigma) v_0^* &= u(\sigma) \chi_{\sigma(B)} u(\sigma)^2 v_0^* \\ &= u(\sigma) \chi_{\sigma(B)} v_0^* \\ &= u(\sigma) v_0^* \chi_C, \end{aligned}$$

where the last equality follows from (1.31), we have

$$V = w^{-N} v_1^{-\ell} \chi_B + w^N v_0 u(\sigma) v_1^{-\ell} u(\sigma) v_0^* \chi_C. \tag{1.52}$$

Since B and C are disjoint, we thus obtain, using the expression in (1.51) for the last term,

$$VV^* = w^{-N} v_1^{-\ell} \chi_B v_1^{\ell} w^N + w^N v_0 u(\sigma) v_1^{-\ell} \chi_B v_1^{\ell} u(\sigma) v_0^* w^{-N}. \tag{1.53}$$

But

$$v_1^{-\ell} \chi_B v_1^{\ell} = \chi_{\alpha\sigma(A)}. \tag{1.54}$$

To prove this, we must verify that the iterates $\alpha^j \alpha\sigma(A)$ for $j = 0, \ldots, k$ do not hit $X = A \cup \alpha\sigma(A)$ before hitting B for $j = k$. The iterates do not hit A (or even Y_1) before they hit B by (1.32). But if

$$\alpha^j \alpha\sigma(A) \cap \alpha\sigma(A) \neq \emptyset$$

for some $j = 0, \ldots, k-1$, then, one has

$$\alpha^j(A) \cap A \neq \emptyset,$$

but this is impossible by the last statement of Lemma 1.8. This proves (1.54).

Inserting (1.54) into (1.53), using $\sigma\alpha\sigma = \alpha^{-1}$, we obtain

$$VV^* = w^{-N} \chi_{\alpha\sigma(A)} w^N + w^N v_0 \chi_{\alpha^{-1}(A)} v_0^* w^{-N}. \tag{1.55}$$

Now, as $w^{2N} = v_1 v_0^*$ we have

$$w^N = w^{-N} v_1 v_0^*$$

and inserting this in the last expression of (1.55) we obtain

$$VV^* = w^{-N} \chi_{\alpha\sigma(A)} w^N + w^{-N} v_1 \chi_{\alpha^{-1}(A)} v_1^* w^N. \tag{1.56}$$

By construction, v_1 maps $\sigma(X)$ into X. But

$$\sigma(X) = \sigma(A \cup \alpha\sigma(A)) = \sigma(A) \cup \sigma\alpha\sigma(A) = \sigma(A) \cup \alpha^{-1}(A),$$

where the union is disjoint. But $\sigma(A)$ is part of the roof of the X-towers and is mapped onto A by v_1. Thus $\sigma(X)\backslash\sigma(A) = \alpha^{-1}(A)$ is mapped onto $X\backslash A = \alpha\sigma(A)$ by v_1, i.e.

$$v_1 \chi_{\alpha^{-1}(A)} v_1^* = \chi_{\alpha\sigma(A)}. \tag{1.57}$$

Inserting (1.57) into (1.56) we see that

$$VV^* = 2w^{-N}\chi_{\alpha\sigma(A)}w^N$$

so VV^* is twice a projection. Thus

$$V_1 = V/\sqrt{2}$$

is a partial isometry with

$$V_1V_1^* = w^{-N}\chi_{\alpha\sigma(A)}w^N.$$

On the other hand, by (1.52),

$$V_1^*V_1 \leq \chi_{B\cup C}$$

and by (1.51), as $v_0u(\sigma)$ has order two,

$$\mathrm{Ad}(v_0u(\sigma))(V_1) = V_1.$$

Now, extend V_1 to another partial isometry V_2 in A, by setting

$$V_2 = \chi_A + V_1.$$

Since $V_1 = V_1\chi_{B\cup C}$ and $B \cup C$ is disjoint from A,

$$V_2V_2^* = \chi_A + V_1V_1^* = \chi_A + w^{-N}\chi_{\alpha\sigma(A)}w^N.$$

But χ_A commutes with w by (1.49), and $X = A \cup \alpha\sigma(A)$ where the union is disjoint and hence

$$V_2V_2^* = w^{-N}\chi_{A\cup\alpha\sigma(A)}w^N = w^{-N}\chi_X w^N. \tag{1.58}$$

Thus V_2 is indeed a partial isometry, and

$$V_2^*V_2 \leq \chi_{A\cup B\cup C} \leq \chi_{Y_1}. \tag{1.59}$$

Also, as

$$\mathrm{Ad}(v_0u(\sigma))(\chi_A) = \mathrm{Ad}(v_0)(\chi_{\sigma(A)}) = \chi_A$$

we have

$$\mathrm{Ad}(v_0u(\sigma))(V_2) = V_2. \tag{1.60}$$

Since V_2 is contained in the finite-dimensional fixed-point subalgebra of A_1 under the automorphism $\mathrm{Ad}(v_0u(\sigma))$, it follows that V_2 can be extended to a unitary u^N in this algebra, and then from (1.58)–(1.59)

$$u^N\chi_{Y_1}u^{-N} \geq w^{-N}\chi_X w^N$$

which is (1.43), while (1.47) follows from the construction. Since V_2 lives on $A \cup B \cup C \subseteq Y_1 \subseteq Y$ and $V_2 V_2^* = w^{-N} \chi_X w^N$ where $X \subseteq Y$ and $w \chi_{\Omega \setminus Y} = \chi_{\Omega \setminus Y}$ it is clear that we can construct the extension u^N of V_2 such that

$$u^N \chi_{\Omega \setminus Y} = \chi_{\Omega \setminus Y}.$$

(We use that $\mathrm{Ad}(v_0 u(\sigma))(\chi_Y) = \mathrm{Ad}(v_0)(\chi_{\sigma(Y)}) = \chi_Y$ to first construct u^N inside χ_Y, and then extend it by setting it equal to 1 on the orthogonal complement of χ_Y.)

Next we use w and u to define still another unitary operator z in A_1, with the following properties:

LEMMA 1.13. *There exists a unitary operator z in A_1 with the following properties:*

$$z \chi_{Y_1} z^* \geq \chi_X \tag{1.61}$$

$$z u(\sigma) = u(\sigma) z \tag{1.62}$$

$$z v_0 z^* v_0^* |_{L^2(Y)} = v_1 v_0^* |_{L^2(Y)} \tag{1.63}$$

$$||z v_0 z^* - v_1|| \leq 3\pi/2N. \tag{1.64}$$

Proof. Define

$$\begin{aligned} z \;=\; & \sum_{k=0}^{N} v_0^k w^{N-k} u^{N-k} v_0^{-k} \chi_{\alpha^k(Y)} + \sum_{k=0}^{N} u(\sigma) v_0^k w^{N-k} u^{N-k} v_0^{-k} u(\sigma) \chi_{\alpha^{-k}\sigma(Y)} \\ & + \chi_{\Omega \setminus \{\bigcup_{k=0}^N \alpha^k(Y) \cup \bigcup_{k=0}^N \alpha^{-k}\sigma(Y)\}}. \end{aligned} \tag{1.65}$$

Since v_0^{-k} maps $\alpha^k(Y)$ onto Y, and $v_0^{-k} u(\sigma)$ maps $\alpha^{-k}\sigma(Y)$ via $\alpha^k(Y)$ onto Y for $0 \leq k \leq N$ by (1.21), (1.18) and (1.5), and both u and w restrict to unitary operators on $L^2(Y)$ by (1.38) and (1.42), it is clear that z is unitary and leave each of the subspaces $L^2(\alpha^k(Y))$ and $L^2(\alpha^{-k}\sigma(Y))$ invariant for $k = 0, 1, \ldots, N$. Also as A_1 is σ-invariant, $z \in A_1$. As $u(\sigma)\chi_{\alpha^{-k}\sigma(Y)} = \chi_{\alpha^k(Y)} u(\sigma)$ it is clear that z is the mean of an operator in A_1 and its conjugate under σ, and hence $\sigma(z) = z$, which is (1.62). To prove (1.61) note that when z hits χ_{Y_1} only the first term in the first sum defining z survives, and

$$z \chi_{Y_1} z^* = w^N u^N \chi_{Y_1} u^{-N} w^{-N} \geq w^N w^{-N} \chi_X w^N w^{-N} = \chi_X,$$

where the inequality follows from (1.43). As for (1.63) note that v_0^* maps $L^2(Y)$ onto $L^2(\sigma(Y))$, and on $L^2(\sigma(Y))$ the unitary z^* acts like $u(\sigma) u^{-N} w^{-N} u(\sigma)$. Since

$$v_0 u(\sigma) u^{-N} w^{-N} u(\sigma) v_0^* = u^{-N} w^N$$

by (1.40) and (1.44), $z v_0 z^* v_0$ acts on $L^2(Y)$ as

$$w^N u^N u^{-N} w^N = w^{2N} = v_1 v_0^*,$$

where the last equality is (1.39). This proves (1.63).

To prove (1.64), we first study the restriction of $z v_0 z^* v_0^*$ to each of the subspaces $L^2(\alpha^k(Y))$ and $L^2(\alpha^{-k}\sigma(Y))$ for $k = 0, 1, \ldots, N$. We have, for $k \neq 0$,

$$\begin{aligned} z v_0 z^* v_0^* |_{L^2(\alpha^k(Y))} &= v_0^k w^{N-k} u^{N-k} v_0^{-k} v_0 v_0^{k-1} u^{-(N-(k-1))} w^{-(N-(k-1))} v_0^{-(k-1)} v_0^{-1} |_{L^2(\alpha^k(Y))} \\ &= v_0^k w^{N-k} u^{-1} w^{-N+k-1} v_0^{-k} |_{L^2(\alpha^k(Y))}. \end{aligned}$$

Thus

$$(zv_0z^*v_0^* - 1)|_{L^2(\alpha^k(Y))} = v_0^k w^{N-k}(u^{-1}w^{-1} - 1)w^{-N+k}v_0^{-k}|_{L^2(\alpha^k(Y))}$$

and hence

$$\begin{aligned} ||(zv_0z^*v_0^* - 1)|_{L^2(\alpha^k(Y))}|| &\leq ||u^{-1}w^{-1} - 1|| \\ &\leq ||u - 1|| + ||w - 1|| \leq \frac{\pi}{N} + \frac{\pi}{2N} = \frac{3\pi}{2N} \end{aligned}$$

by (1.45) and (1.41). But as

$$v_1v_0^*|_{L^2(\Omega\setminus Y)} = 1_{L^2(\Omega\setminus Y)}, \tag{1.66}$$

(see, e.g., the figure before the statement of Lemma 1.10), it follows that

$$||(zv_0z^*v_0^* - v_1v_0^*)|_{L^2(\alpha^k(Y))}|| \leq \frac{3\pi}{2N} \tag{1.67}$$

for $k = 1, 2, \ldots, N$. But in the special case that $k = 0$ we have already established that

$$zv_0z^*v_0^*|_{L^2(Y)} = v_1v_0^*|_{L^2(Y)}$$

in (1.63), so (1.67) holds also for $k = 0$ (with the right-hand side replaced by 0).

Similarly, for $0 \leq k \leq N - 1$

$$\begin{aligned} zv_0z^*v_0^*|_{L^2(\alpha^{-k}\sigma(Y))} &= u(\sigma)v_0^k w^{N-k}u^{N-k}v_0^{-k}u(\sigma)v_0u(\sigma)v_0^{k+1}u^{-(N-(k+1))} \\ &\quad \times w^{-(N-(k+1))}v_0^{-k-1}u(\sigma)v_0^{-1}|_{L^2(\alpha^{-k}\sigma(Y))}. \end{aligned}$$

As $u(\sigma)v_0u(\sigma) = v_0^* = v_0^{-1}$ by (1.36), we get further

$$zv_0z^*v_0^*|_{L^2(\alpha^{-k}\sigma(Y))} = u(\sigma)v_0^k w^{N-k}uw^{-N+k+1}v_0^{-k}u(\sigma)|_{L^2(\alpha^{-k}\sigma(Y))}$$

and as before this implies

$$||(zv_0z^*v_0^* - v_1v_0^*|_{L^2(\alpha^{-k}\sigma(Y))}|| \leq \frac{3\pi}{2N} \tag{1.68}$$

for $0 \leq k \leq N - 1$. But if $k = N$ one computes

$$\begin{aligned} zv_0z^*v_0^*|_{L^2(\alpha^{-N}\sigma(Y))} &= zv_0v_0^*|_{L^2(\alpha^{-N}\sigma(Y))} = z|_{L^2(\alpha^{-N}\sigma(Y))} \\ &= u(\sigma)v_0^N v_0^{-N}u(\sigma)|_{L^2(\alpha^{-N}\sigma(Y))} = 1|_{L^2(\alpha^{-N}\sigma(Y))} \end{aligned}$$

and hence (1.68) holds, with right side zero, for $k = N$.

Next, one uses the fact that z acts as the identity outside $\bigcup_{k=0}^N \alpha^k(Y) \cup \bigcup_{k=0}^N \alpha^{-k}\sigma(Y)$ to compute that $zv_0z^*v_0^* = v_0v_0^* = 1$ on the L^2-space on the complement of this set. Since both $zv_0z^*v_0^*$ and $v_1v_0^*$ leave all the spaces $L^2(\alpha^k(Y))$, $L^2(\alpha^{-k}\sigma(Y))$ invariant for $k = 0, 1, \ldots, N$, as well as the orthogonal complement of these spaces, it follows finally from (1.67) and (1.68) that

$$||zv_0z^*v_0^* - v_1v_0^*|| \leq 3\pi/2N$$

which is (1.64).

Proof of Theorem 1.1. Recall that $u_1 = u(\alpha)v_1^*$, and define

$$A = C^*(zA_0z^*, u_1). \tag{1.69}$$

We shall show that A is the subalgebra of the form (1.4) alluded to in Theorem 1.1.

First we show that

$$C(\mathcal{P}) \subseteq zA_0z^* \subseteq A. \tag{1.70}$$

We have already noticed in Lemma 1.5 that

$$C(\mathcal{P}) \subseteq A_0.$$

Further, note that z leaves each of the spaces $L^2(\alpha^k(Y))$, $L^2(\alpha^{-k}\sigma(Y))$, $k = 0, 1, \ldots, N$ invariant and acts as the identity $L^2(\Omega \backslash (\bigcup_{k=0}^N \alpha^k(Y) \cup \bigcup_{k=0}^N \alpha^{-k}\sigma(Y)))$. Since each of the sets $\alpha^k(Y)$, $\alpha^{-k}\sigma(Y)$ is contained in a single element of $\mathcal{P}$ by (1.9), it follows that z commutes with $C(\mathcal{P})$, and hence (1.70) is clear.

Next, as $u_1 \in A$ and $v_0 \in A_0$, we have

$$u' = u_1 z v_0 z^* \in A.$$

As $u(\alpha) = u_1 v_1$ we have

$$||u' - u(\alpha)|| = ||zv_0z^* - v_1|| \leq 3\pi/2N \tag{1.71}$$

by (1.64). Thus, if N is chosen so large that $3\pi/2N < \varepsilon$, the canonical unitary in the crossed product $C(\Omega) \times_\alpha \mathbb{Z}$ is contained within ε in A.

To prove the remaining properties of A we introduce the element

$$V = \sum_{k=0}^{J_1-1} (zv_0^k \chi_{Y_1} z^*) u_1 (z\chi_{Y_1} v_0^{-k} z^*) = \sum_{k=0}^{J_1-1} (ze_{k0}^1 z^*) u_1 (ze_{0k}^1 z^*), \tag{1.72}$$

where we used the matrix units introduced in Lemma 1.5. (V should not be confused with the V used in the proof of Lemma 1.12.) The last expression for V shows that V commutes with zA_0z^*. Furthermore, as

$$ze_{00}^1 z^* = z\chi_{Y_1} z^* \geq \chi_X \tag{1.73}$$

by (1.61), and u_1 acts as the identity on $L^2(\Omega \backslash X)$ by (1.23), it follows that u_1 is contained in the algebra generated by zA_0z^* and V, i.e.

$$A = C^*(zA_0z^*, V). \tag{1.74}$$

Since $ze_{ij}^k z^*$, where e_{ij}^k are defined by Lemma 1.5, constitute a full set of matrix units for zA_0z^*, and V is a unitary on $z(\sum_{i=0}^{J_1-1} e_{ii}^1)z^* L^2(\Omega)$ commuting with zA_0z^*, in order to prove that A has the form (1.4) it suffices to show that V has full spectrum, i.e.

$$Sp(V) = \mathbb{T}. \tag{1.75}$$

But by the K-theoretic reasoning at the end of §2 in **[Put2]**, $[u_1]$ is the generator of $K_1(C(\Omega) \times_\alpha \mathbb{Z})$ which is $\mathbb{Z}$, and hence u_1 has full spectrum. Since, as we already remarked, u_1 'lives' on $ze_{00}^1 z^*$, it follows from (1.72) that V has full spectrum, and hence A has the form (1.4).

Finally, we have to prove the statements of Theorem 1.1 pertaining to σ. As $zu(\sigma) = u(\sigma)z$ by (1.62), the statements concerning the action of σ on zA_0z^* are immediate from (1.18) in Lemma 1.5. It only remains to show that

$$u(\sigma)Vu(\sigma) = V^*. \tag{1.76}$$

But $u(\sigma)zu(\sigma) = z$ and $u(\sigma)e^1_{k0}u(\sigma) = e^1_{J_1-k-1,J_1-1}$, and as $u_1 = u(\alpha)v_1^*$ we have

$$u(\sigma)u_1u(\sigma) = u(\alpha)^*v_1 = v_1^*u_1^*v_1.$$

We conclude that

$$u(\sigma)Vu(\sigma) = \sum_{k=0}^{J_1-1} ze^1_{k,J_1-1}z^*v_1^*u_1^*v_1ze^1_{J_1-1,k}z^*. \tag{1.77}$$

But $e^1_{J_1-1,J_1-1} = \chi_{\sigma(Y_1)} \leq \chi_{\sigma(Y)}$, and z carries $L^2(\sigma(Y))$ into itself by the definition (1.65), and

$$v_1|_{L^2(\sigma(Y))} = zv_0z^*|_{L^2(\sigma(Y))}$$

by (1.63). Hence, from (1.77),

$$u(\sigma)Vu(\sigma) = \sum_{k=0}^{J_1-1} ze^1_{k,J_1-1}z^*zv_0^*z^*u_1^*zv_0z^*ze^1_{J_1-1,k}z^*$$

and as $v_0e^1_{J_1-1,k} = e^1_{0,k}$ by (1.21), we get

$$u(\sigma)Vu(\sigma) = \sum_{k=0}^{J_1-1} ze^1_{k,0}z^*u_1^*ze^1_{0,k}z^* = V^*,$$

which is (1.76). This ends the proof of Theorem 1.1, apart from the last parenthetical remark, which is (1.27).

2. *Inductive limits*

The main result of this section is the following Theorem 2.1, as well as Corollary 2.4.

THEOREM 2.1. *Let A be a unital separable C^*-algebra, and let σ be an automorphism of order 2 of A. Assume that for any $\varepsilon > 0$, and any finite number $x_1, \ldots, x_n$ of elements in A there exist a C^*-subalgebra B of A, with the same unit as A, such that*

$$B \cong [M_{J_1} \otimes C(\mathbb{T})] \oplus M_{J_2} \oplus \ldots \oplus M_{J_K} \tag{2.1}$$

for suitable natural numbers $J_1, J_2, \ldots, J_K$, with the following properties:
(2.2) *There exists elements $y_1, \ldots, y_n$ in B with*

$$||y_k - x_k|| < \varepsilon$$

for $k = 1, \ldots, n$.
(2.3) *$\sigma(B) = B$, and, moreover, σ leaves the two subalgebras corresponding to*

$$[M_{J_1} \otimes 1] \oplus 0 \oplus \ldots \oplus 0$$

and

$$0 \otimes 0 \oplus M_{J_2} \oplus \ldots \oplus M_{J_K}$$

invariant.

(2.4) *σ maps the canonical generator $z \to z$ for $1_{J_1} \otimes C(\mathbb{T})$ into $z \to \bar{z}$, and this generator is in a nontrivial K_1-class, in A.*

It follows that there exists an increasing sequence $A_1 \subseteq A_2 \subseteq \ldots$ of unital C^*-subalgebras of A such that each A_k has the form (2.1), each A_k is σ-invariant and the action $\sigma|_{A_k}$ has the properties (2.3) and (2.4), and, finally,

$$\overline{\bigcup_{k=1}^{\infty} A_k} = A, \tag{2.5}$$

where the bar denotes norm closure.

Before going to the proof we remark that a similar theorem, but without the extra structure given by σ, and without injectivity of the embedding, $A_k \hookrightarrow A$, is Theorem 4.3 in **[Ell]**.

First, for completeness, we state a known lemma.

LEMMA 2.2 (**[Gli, Bra]**). *For any $\varepsilon > 0$ and any natural number n there exists a $\delta(\varepsilon, n) > 0$ with the following property: if A is a C^*-algebra, and B is a finite-dimensional $*$-subalgebra with (linear) dimension not exceeding n, and C is another C^*-subalgebra of A such that any element in the unit sphere of B has distance at most $\delta(\varepsilon, n)$ to C, then there exists an injective morphism*

$$\varphi : B \to C \tag{2.6}$$

such that

$$||\varphi(x) - x|| \leq \varepsilon ||x|| \tag{2.7}$$

for all $x \in B$.

Proof. This is essentially **[Gli**, Lemma 1.10] or **[Bra**, Lemma 2.1].

LEMMA 2.3. *If A is a unital C^*-algebra with an automorphism σ of order 2, and B is a globally σ-invariant C^*-subalgebra of A with the same unit as A such that B has the form (2.1), and the restriction of σ to B has the form (2.3) and (2.4), and $x_1, \ldots, x_m$ are elements in B, then for any $\varepsilon > 0$ there exists a $\delta > 0$ (depending on $x_1, \ldots, x_n$ and B) such that if C is another globally σ-invariant C^*-subalgebra of A such that the generators e_{ij}^k, $i, j = 0, \ldots, J_k - 1$, $k = 1, \ldots, K$ and $z \to z$ of B all can be approximated by elements of C within δ, then there exists an injective morphism*

$$\varphi : B \to C \tag{2.8}$$

such that

$$||\varphi(x_i) - x_i|| \leq \varepsilon ||x_i|| \tag{2.9}$$

for $i = 1, \ldots, m$, and

$$\varphi\sigma x = \sigma\varphi x \tag{2.10}$$

for all $x \in B$.

Proof. If B has the form (2.1), define B_0 as the subalgebra corresponding to

$$[M_{J_1} \otimes 1] \oplus M_{J_2} \oplus \ldots \oplus M_{J_K} \tag{2.11}$$

and u as the unitary operator corresponding to

$$[1 \otimes (z \to z)] \oplus 1 \oplus \ldots \oplus 1. \tag{2.12}$$

Then B_0 is finite-dimensional, $\sigma(B_0) = B_0$, $\sigma(u) = u^*$, u commutes with B_0 and B is generated as C^*-algebra by u and B_0. Moreover, B can be characterized abstractly as the C^*-algebra generated by a finite-dimensional C^*-algebra B_0 of the form (2.11) together with a unitary u with spectrum $\mathbb{T}$ commuting with B_0 such that

$$u(1 - P_{J_1}) = 1 - P_{J_1}, \tag{2.13}$$

where P_{J_1} is the central projection in B_0 corresponding to the first summand in (2.11).

As σ have order two, for a given $k = 1, 2, \ldots, K$ there are two possibilities. Either σ maps M_{J_k} onto itself or σ interchanges M_{J_k} with some M_{J_ℓ} with $J_k = J_\ell$. (Here and later we identify B_0 with (2.11), to save notation.) When $k = 1$ only the first alternative occurs. When the first alternative occurs, the restriction of σ to M_{J_k} is implemented by a self-adjoint unitary since σ has order two, and hence we may choose matrix units e^k_{ij} such that $\sigma(e^k_{ij})$ is either $+e^k_{ij}$ or $-e^k_{ij}$ for each pair (i, j). In particular $\sigma(e^k_{ii}) = e^k_{ii}$ for all i. When the second alternative occurs, we may use the choice

$$e^\ell_{ij} = \sigma(e^k_{ij})$$

for matrix units for M_{J_ℓ} once the matrix units e^k_{ij} for M_{J_k} are chosen, and then

$$e^k_{ij} = \sigma(e^\ell_{ij}).$$

Now, the elements $x_1, \ldots, x_n$ can be approximated arbitrary close by polynomials in e^k_{ij}'s, u and u^*. Thus if we can find an injective morphism $\varphi : B \to C$ such that $||\varphi(e^k_{ij}) - e^k_{ij}||$ and $||\varphi(u) - u||$ all are sufficiently small, then (2.9) will be fulfilled since φ is contractive. We shall argue that we can find such a φ provided e^k_{ij} and u all are sufficiently close to C. First it follows from Lemma 2.2 that we can find a set of matrix units f^k_{ij} in C such that f^k_{ij} is close to e^k_{ij} for each i, j, k. We now use techniques from [**Gli**] and [**Bra**] to modify the f^k_{ij}. In fact we may first apply Lemma 2.2 to the pair B_0^σ, C^σ of fixed-point algebras under σ instead of B_0, C to find a morphism $\varphi_\sigma : B_0^\sigma \to C^\sigma$ such that φ_σ is close to 1. (Note that if $x \in B_0^\sigma$ and $y \in C$ with $||x - y|| \leq \delta$, then $||x - \frac{1}{2}(\sigma(y) + y)|| \leq \delta$ and $\frac{1}{2}(\sigma(y) + y) \in C^\sigma$.) To extend φ_σ to B_0 we operate as follows. If M_{J_k} is a summand invariant under σ, and e^k_{ij} is a matrix element, there are two possibilities: either $\sigma(e^k_{ij}) = e^k_{ij}$, then simply replace f^k_{ij} by $g^k_{ij} \equiv \varphi_\sigma(e^k_{ij})$, or $\sigma(e^k_{ij}) = -e^k_{ij}$. In the latter case, as $\sigma(e^k_{ii}) = e^k_{ii}$ and $\sigma(e^k_{jj}) = e^k_{jj}$, we have

$$\sigma(g^k_{ii} f^k_{ij} g^k_{jj}) \approx -(g^k_{ii} f^k_{ij} g^k_{jj}).$$

If one now introduces

$$y = \tfrac{1}{2}(1 - \sigma)(g^k_{ii} f_{ij} g^k_{jj})$$

then

$$\sigma(y) = -y \quad \text{and} \quad y \approx e_{ij}^k.$$

Since $y^*y \approx e_{jj}^k$ one computes that the spectrum of y^*y is concentrated near 0 and 1. If g_{ij}^k is the partial isometry corresponding to that part of the partial isometry in the polar decomposition of y which lives on the part of y^*y near 1, then

$$\sigma(g_{ij}^k) = -g_{ij}^k$$

and

$$g_{ij}^k g_{ij}^{k*} = g_{ii}^k$$

and

$$g_{ij}^{k*} g_{ij}^k = g_{jj}^k.$$

In this way one constructs g_{ij}^k unless g_{ij}^k can be defined from already constructed g_{ij}^k's by using

$$g_{ij}^k = g_{ji}^{k*}$$

or

$$g_{ij}^k = g_{i\ell}^k g_{\ell j}^k.$$

(The most systematic way is to construct $g_{00}^k, g_{01}^k, \ldots, g_{0J_k-1}^k$ as above, and then define the other g_{ij}^k's by matrix relations.)

The other main case is that M_{J_k} and M_{J_ℓ} are interchanged by σ. Then

$$e_{ij} \equiv e_{ij}^k + e_{ij}^\ell$$

form a complete set of matrix units for $(M_{J_k} + M_{J_\ell})^\sigma$. Put

$$g_{ij} = \varphi_\sigma(e_{ij}).$$

Let $f \in C$ be a self-adjoint approximant to $e_{00}^k - e_{00}^\ell$. We may assume $g_{00} f = f g_{00} = f$ by cutting down with g_{00}. Then

$$\sigma(f) \approx -f$$

and

$$f^2 \approx g_{00}.$$

Replacing f by $\frac{1}{2}(1-\sigma)$, we may assume $\sigma(f) = -f$. Then if h is the partial isometry of the polar decomposition of f, then h is self-adjoint,

$$\sigma(h) = -h, \quad h^2 = g_{00}.$$

Now define

$$g_{00}^k = \tfrac{1}{2}(g_{00} + h) \quad g_{00}^\ell = \tfrac{1}{2}(g_{00} - h)$$

and verify

$$g_{00}^k g_{00}^\ell = 0, \quad g_{00}^k + g_{00}^\ell = g_{00}, \quad \sigma(g_{00}^k) = g_{00}^\ell,$$
$$\sigma(g_{00}^\ell) = g_{00}^k, \quad g_{00}^k \approx e_{00}^k, \quad g_{00}^\ell \approx e_{00}^\ell.$$

Next, define

$$g_{ij}^k = g_{i0}g_{00}^k g_{0j}, \quad g_{ij}^\ell = g_{i0}g_{00}^\ell g_{0j}$$

and verify

$$\sigma(g_{ij}^k) = g_{ij}^\ell$$

etc.

We now extend φ_σ to a morphism $B_0 \to C$ by setting

$$\varphi(e_{ij}^k) = g_{ij}^k$$

and then φ is close to ι on B_0 and $\sigma\varphi = \varphi\sigma$ on B_0.

We next have to extend φ to u, i.e. we have to construct a unitary operator $\tilde{u} \in C$ such that

$$\tilde{u}\varphi(1 - P_{J_1}) = \varphi(1 - P_{J_1}), \tag{2.14}$$

$$\tilde{u} \in \varphi(B_0)', \tag{2.15}$$

$$Sp(\tilde{u}) = \mathbb{T}, \tag{2.16}$$

$$\sigma(\tilde{u}) = \tilde{u}^* \tag{2.17}$$

$$\tilde{u} \approx u. \tag{2.18}$$

So let x be an approximant to $P_{J_1}uP_{J_1}$ in C. We may assume $\varphi(P_{J_1})x\varphi(P_{J_1}) = x$, and by integrating vxv^* over v in the unitary group of $\varphi(B_0)$ we may assume that $x \in \varphi(B_0)' \cap C \equiv C_1$, since it already approximately lies there. But as $\sigma(u)^* \approx u$ we have $\sigma(x)^* \approx x$, so replacing x by $\frac{1}{2}(x + \sigma(x)^*)$ we may assume

$$\sigma(x) = x^*.$$

Now, let v be the partial isometry of the polar decomposition of x inside $\varphi(P_{J_1})C_1\varphi(P_{J_1})$. The partial isometry is actually unitary and contained in $\varphi(P_{J_1})C_1\varphi(P_{J_1})$ since x is approximately unitary there. As $|x|^2 = x^*x$ we have

$$\sigma(|x|^2) = \sigma(x^*)\sigma(x) = xx^* = |x^*|^2$$

and hence

$$\sigma(|x|) = |x^*|.$$

Now, applying σ to both sides of

$$x = v|x|$$

we get

$$x^* = \sigma(v)|x^*|.$$

But as $|x^*| = v|x|v^*$ we obtain

$$|x|v^* = \sigma(v)v|x|v^*$$

and hence

$$\sigma(v)v = \varphi(P_{J_1})$$

and

$$\sigma(v) = v^*.$$

Hence

$$\tilde{u} = v + \varphi(1 - P_{J_1})$$

has the properties (2.14), (2.15), (2.17) and (2.18). But just because $\tilde{u}$ is close to u, it is in the same K_1-class, and as this is nontrivial it follows that $\tilde{u}$ has full spectrum, which is (2.16). This ends the proof of Lemma 2.3.

Proof of Theorem 2.1. Let $x_1, x_2, \ldots$ be a dense sequence in A. We inductively construct a sequence B_n of subalgebras of A of the form (2.1)–(2.4), as well as elements $y_{n,1}, y_{n,2}, \ldots, y_{n,k(n)}$ in B_n and a dense sequence $(z_{n,i})_{i=1}^{\infty}$ in B_n and injective morphisms $\varphi_n : B_n \to B_{n+1}$, as follows. Let $B_1 = \mathbb{C}1$, and when $B_1, \ldots, B_n$ have been constructed, choose B_{n+1} as follows: apply Lemma 2.3 with $\varepsilon = 2^{-n}$ and $\{x_1, \ldots, x_m\} = \{y_{n,1}, \ldots, y_{n,k(n)}\}$ to find a δ with the properties cited there. Then use Theorem 1.1 to find a subalgebra B_{n+1} of the apposite form such that the distances of the generators e_{ij}^k, u of B_n to B_{n+1} are less than δ and the distances of the elements $x_1, \ldots, x_{n+1}$ to B_{n+1} are less than 2^{-n}. Construct $\varphi = \varphi_n$ as in Lemma 2.3, and let the new set of y's be the union of the following three sets:

(1) The images of the previous y's under φ_n.

(2) The images of $z_{m,1}, \ldots, z_{m,n}$ under $\varphi_n \varphi_{n-1} \ldots \varphi_m$ for $m = 1, \ldots, n$.

(3) A set of $n+1$ new y's approximating $x_1, \ldots, x_{n+1}$ to within 2^{-n}.

Then, let $(z_{n+1,i})_{i=1}^{\infty}$ be any countable dense sequence in B_{n+1} containing the new y's and such that the set of elements in the sequence is closed under addition, multiplication, involution and scalar multiplication by rational complex numbers. (If any dense sequence is given, we obtain the latter property by considering all *-polynomials in the sequence with rational complex coefficients.) In particular, we have constructed injective morphisms

$$\varphi_n : B_n \to B_{n+1} \tag{2.19}$$

such that

$$||\varphi_n(y_{n,k}) - y_{n,k}|| < 2^{-n}||y_{n,k}|| \tag{2.20}$$

for $k = 1, \ldots, k(n)$, and

$$\varphi_n \sigma = \sigma \varphi_n. \tag{2.21}$$

Now let B be the inductive limit of the system

$$\ldots \to B_n \xrightarrow{\varphi_n} B_{n+1} \to \ldots \tag{2.22}$$

and let σ' be the automorphism of order 2 of B which is defined by σ. The automorphism σ' is well defined because of (2.21). For each n, let φ be the canonical injection of B_n into B. Then $\varphi(B_n)$ is an increasing sequence of subalgebras of B with dense union in B. Since each $\varphi(B_n)$ has the form (2.1), Theorem 2.1 will be proved once we can show that B is isomorphic to A by an isomorphism intertwining σ and σ'. We shall define such an isomorphism η explicitly as follows:

First we define η on $\varphi(B_n)$, i.e. we define an injection $\eta_n : B_n \to A$ as follows: if $x \in B_n$ and $x = z_{n,k}$ for a suitable k, then for $m \geq \max\{n, k\}$ we have

$$\varphi_m \varphi_{m-1} \ldots \varphi_n(x) \in y\text{-set of } B_{m+1}.$$

It follows that

$$||(\varphi_m - 1)\varphi_{m-1}\varphi_{m-2} \ldots \varphi_n(x)|| \leq 2^{-m}||\varphi_{m-1} \ldots \varphi_n(x)|| = 2^{-m}||x||$$

for $m > \max\{n, k\}$. Thus $m \to \varphi_m \varphi_{m-1} \ldots \varphi_n(x)$ is a Cauchy sequence in A. Let $\eta_n(x)$ be its limit. As $||\varphi_m \ldots \varphi_n(x)|| = ||x||$ for all n, we have that η_n is an isometry of the *-algebra $\{z_{n,i}\}_{i=1}^{\infty}$ over the rational complex numbers, and it is clear by limiting that η_n is a *-morphism. We now extend η_n to B_n by continuity.

It is clear from the definition that

$$\eta_{n+1}\varphi_n = \eta_n \tag{2.23}$$

and hence we may consistently define an isometric *-morphism

$$\eta : \bigcup_n \varphi(B_n) \to A$$

by

$$\eta \circ \varphi|_{B_n} = \eta_n. \tag{2.24}$$

Then η extends by continuity to an injection of B into A, and

$$\eta\sigma' = \sigma\eta.$$

Furthermore, η is surjective by the following reasoning. If $x \in A$, then x lies in the closure of the set $\{x_n\}_{n=1}^{\infty}$. Hence, for any $\varepsilon > 0$ there is a natural number n such that

$$||x - x_n|| < \varepsilon/3.$$

Now choose $m > n$ so that $2^{-m+1} < \varepsilon/3$. There exists a y in the y-set of B_m such that

$$||x_n - y|| < \varepsilon/3.$$

But as

$$||\varphi_m(y) - y|| \leq 2^{-m}||y||$$

$$||\varphi_{m+1}\varphi_m(y) - \varphi_m(y)|| \leq 2^{-m-1}||y||$$

etc, we have

$$||\eta_m(y) - y|| \leq 2^{-m+1}||y|| \leq \tfrac{1}{3}\varepsilon||y||.$$

But $\eta_m(y) = \eta(\varphi(y))$ and hence

$$||x - \eta(\varphi(y))|| < \tfrac{2}{3}\varepsilon + \tfrac{1}{3}\varepsilon||y||.$$

Since $||x - y|| < \frac{2}{3}\varepsilon$, and ε was arbitrary, it follows that x is contained in the closure of the range of η. But this range is closed, so η is surjective.

We have proved that the C^*-dynamical systems (A, σ) and (B, σ') are isomorphic, and this ends the proof of Theorem 2.1.

COROLLARY 2.4. *Let Ω be a totally disconnected compact metrizable space, and let α be a minimal homeomorphism on Ω. Let σ be a homeomorphism of order 2 on Ω such that*

$$\alpha\sigma = \sigma\alpha^{-1} \tag{2.25}$$

and assume that σ or $\alpha\sigma$ has a fixed point. It follows that $C(\Omega) \times_\alpha \mathbb{Z} \times_\alpha \mathbb{Z}_2$ contains an increasing sequence of unital subalgebras B_n with dense union, such that each B_n has the form

$$\tilde{B}_0 \oplus M_{n_1} \oplus M_{n_2} \oplus \ldots \oplus M_{n_N} \tag{2.26}$$

where

$$\tilde{B}_0 = \{x \in C(I, M_{4n_0}) : Ex(-1) = x(-1)E \quad \text{and} \quad Ex(1) = x(1)E\}. \tag{2.27}$$

Here $I = [-1, 1]$ is the unit interval, E is a projection in M_{4n_0} of dimension $2n_0$, and $C(I, M_{4n_0})$ denotes the C^*-algebra of continuous functions from I into M_{4n_0}.

Proof. As mentioned after (1.2) we may for the purposes of this corollary assume that $\alpha\sigma$ has a fixed point, and hence, by Theorems 1.1 and 2.1, it suffices to prove that the crossed product of an algebra of the form (1.4) by an automorphism σ of order 2 satisfying the conditions in Theorem 1.1 has the form (2.26). But if σ is an automorphism of order 2 of any C^*-algebra B then

$$B \times_\alpha \mathbb{Z}_2 \cong \begin{pmatrix} B^\sigma & B^\sigma(-1) \\ B^\sigma(-1) & B^\sigma \end{pmatrix}, \tag{2.28}$$

where

$$B^\sigma = \{x \in B : \sigma(x) = x\} \tag{2.29}$$

and

$$B^\sigma(-1) = \{x \in B : \sigma(x) = -x\} \tag{2.30}$$

see e.g. [**BEEK2**,(4.3)]. From this it is easy to see that if σ flips two summands M_{J_k} and M_{J_ℓ}, with $J_k = J_\ell$, this gives rise to a summand M_{2J_k} in the crossed product, and if σ leaves a summand M_{J_k} invariant, this gives rise to a summand $M_{J_k} \oplus M_{J_k}$ in the crossed product. Finally, the crossed product of $M_{J_1} \otimes C(\mathbb{T})$ by σ has the form (2.27) with $2n_0 = J_1$, see e.g. [**BEEK1**], [**BE1**].

3. *The AF-algebra*

In this section we shall prove that $C(\Omega) \times_\alpha \mathbb{Z} \times_\alpha \mathbb{Z}_2$ is an AF-algebra. We start with:

LEMMA 3.1 ([**Ell**]). *The algebra $C(\Omega) \times_\alpha \mathbb{Z}$ has real rank zero.*

Proof. This is referred to before the statement of Theorem 4.3 in [**Ell**]. By [**Put1**, Corollary 5, p 345] there is a canonical one-one correspondence between tracial states on $C(\Omega) \times_\alpha \mathbb{Z}$ and α-invariant probability measures on Ω. Since Ω is totally disconnected, the projections in $C(\Omega)$ separate all probability measures on Ω, and hence projections in $C(\Omega) \times_\alpha \mathbb{Z}$ separate the trace states on $C(\Omega) \times_\alpha \mathbb{Z}$. Hence, by Theorem 1.3 of [**BBEK**], or Theorem 2 of [**BDR**], together with Theorem 2.1, $C(\Omega) \times_\alpha \mathbb{Z}$ has real rank zero.

Now, let

$$B_k \cong [M_{J_1(k)} \otimes C(\mathbb{T})] \oplus M_{J_2(k)} \oplus \ldots \oplus M_{J_{K_k}(k)} \tag{3.1}$$

be a definite increasing sequence of unital C^*-subalgebras of $C(\Omega) \times_\alpha \mathbb{Z}$ such that $\bigcup_k B_k$ is dense, and such that the restriction of σ to B_k has the form indicated in Theorem 2.1. Let

$$\tilde{B}_k \cong \tilde{B}_{k,0} \oplus M_{n_1(k)} \oplus \ldots \oplus M_{n_{N_k}(k)} \tag{3.2}$$

be the corresponding sequence growing to $C(\Omega) \times_\alpha \mathbb{Z} \times_\alpha \mathbb{Z}_2$, Corollary 2.4.

For $k < \ell$ given, if $z \in \mathbb{T}$, then z defines an irreducible representation of

$$B_\ell \cong [M_{J_1(\ell)} \otimes C(\mathbb{T})] \otimes M_{J_2(k)} \otimes \ldots \otimes M_{J_{K_k}(k)}$$

by evaluation. The restriction of this representation to the first summand $B_{k,0} \cong M_{J_1(k)} \otimes C(\mathbb{T})$ of B_k decomposes into a certain number $[\ell : k]$ of irreducible representations of $B_{k,0}$, given by evaluation at $[\ell : k]$ points $z_1(z), \ldots, z_{[\ell:k]}(z)$, where the number $[\ell : k]$ is independent of z, and the mapping $\hat{\phi}_{k,\ell} : \mathbb{T} \to S^{[\ell:k]}\mathbb{T}$, which to z assigns the image of $(z_1(z), \ldots, z_{[\ell:k]}(z))$ in the $[\ell : k]$-fold symmetric product $S^{[\ell:k]}\mathbb{T}$ of $\mathbb{T}$, is continuous, **[DNNP]**, **[BE2]**. Here $S^{[\ell:k]}\mathbb{T} = \mathbb{T}^{[\ell:k]}/\sum_{[\ell:k]}$ where the symmetric group on $[\ell : k]$ elements acts on $\mathbb{T}^{[\ell:k]}$ permuting the coordinates.

Note that as σ acts on $1 \otimes C(\mathbb{T})$ by flipping the circle, whether in B_k or B_ℓ, and the morphism of $M_{1_1(k)} \otimes C(\mathbb{T})$ into $M_{1_1(\ell)} \otimes C(\mathbb{T})$ intertwines σ, we have

$$\hat{\phi}_{k,\ell}(\bar{z}) = \overline{\hat{\phi}_{k,\ell}(z)} \tag{3.3}$$

where the conjugation in $\mathbb{T}^{[\ell:k]}/\sum_{[\ell:k]}$ is coordinatewise.

Note that as B_k and B_ℓ are finite-dimensional apart from the first summand and the embedding of B_k and B_ℓ is injective, the embedding of $B_{k,0}$ into $B_{\ell,0}$ is non-zero, and hence $[\ell : k] \geq 1$ and the embedding is injective.

Now, by **[BBEK**, Theorem 1.3], the algebra $C(\Omega) \times_\alpha \mathbb{Z}$ has small eigenvalue-variation since it has real rank zero. By the characterization of small eigenvalue variation given in **[BE2]**, this means that $C(\Omega) \times_\alpha \mathbb{Z}$ has small metric variation, i.e.

LEMMA 3.2. *For any k and any ε there exists an L such that if $\ell \geq L$, then the diameter of the range of $\hat{\phi}_{k,\ell}$ in $S^{[\ell:k]}\mathbb{T}$ is less than ε. (The metric on $S^n\mathbb{T}$ is defined by $d(z, y) = \inf_{\tau \in \sum_n} \sup_{1 \leq k \leq n} d(z_k, y_{\tau(k)})$ for $z = (z_1, \ldots, z_n)$, $y = (y_1, \ldots, y_n)$.)*

We now embed each B_k into $\tilde{B}_k$. By (2.88)

$$\tilde{B}_k \cong \begin{pmatrix} B_k^\sigma & B_k^\sigma(-1) \\ B_k^\sigma(-1) & B_k^\sigma \end{pmatrix}$$

and by **[BEEK2**, (4.1)] the concrete embedding of B_k into $\tilde{B}_k$ is given by

$$x \in B_k \to \begin{pmatrix} P_+(x) & P_-(x) \\ P_-(x) & P_+(x) \end{pmatrix} \in \tilde{B}_k, \tag{3.4}$$

where $P_\pm = \frac{1}{2}(1 \pm \sigma)$.

The embeddings θ of $B_{k,0}$ into $\tilde{B}_{k,0}$ can be described analogously. For our purposes, it is convenient to describe the embedding more concretely as follows. An element $f \in B_{k,0}$ is then mapped into the function

$$t \in [-1, 1] \to \begin{pmatrix} f(t + i\sqrt{1-t^2}) & 0 \\ 0 & f(t - i\sqrt{1-t^2}) \end{pmatrix} \tag{3.5}$$

while all of $\tilde{B}_{k,0}$ can be characterised as the set of functions

$$t \in [-1, 1] \to \begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{pmatrix} \tag{3.6}$$

which commute with the self-adjoint unitary $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ at the points -1 and 1. This is consistent with Corollary 2.4 which states that the spectrum $\tilde{I}$ of $\tilde{B}_{k,0}$ consists of the open interval $\langle -1, 1\rangle$ together with two limit points at -1 and two limit points at $+1$. We may define a set-valued map $\tilde{\theta} : \tilde{I} \to \mathbb{T}$ dual to θ (analogously to $\hat{\phi}_{k,\ell}$ dual to $\phi_{k,\ell}$) by requiring that the point $t \in \langle -1, 1\rangle$ is mapped into the two points $t \pm i\sqrt{1-t^2} \in \mathbb{T}$, and the two points at -1 are both mapped into -1, and the two at $+1$ into $+1$. For our purposes it is better to view $\tilde{\theta}$ as a map from I into $\mathbb{T}$ by $\tilde{\theta}(t) = t \pm i\sqrt{1-t^2}$.

If $\psi_{k,\ell}$ denotes the embedding $\tilde{B}_{k,0} \hookrightarrow \tilde{B}_{\ell,0}$, then the dual $\tilde{\psi}_{k,\ell}$ maps $\tilde{I}$ into subsets of $\tilde{I}$. But since the diagram

$$\begin{array}{ccccc} B_{k,0} & \xrightarrow{\sigma} & B_{k,0} & \overset{\theta}{\hookrightarrow} & \tilde{B}_{k,0} \\ \downarrow \phi_{k,\ell} & & \downarrow \phi_{k,\ell} & & \downarrow \psi_{k,\ell} \\ B_{\ell,0} & \xrightarrow{\sigma} & B_{\ell,0} & \overset{\theta}{\hookrightarrow} & \tilde{B}_{\ell,0} \end{array} \tag{3.7}$$

commutes, the diagram

$$\begin{array}{ccccc} \mathbb{T} & \overset{z \to \bar{z}}{\leftarrow} & \mathbb{T} & \overset{\tilde{\theta}}{\leftarrow} & \tilde{I} \\ \uparrow \hat{\phi}_{k,\ell} & & \uparrow \hat{\phi}_{k,\ell} & & \uparrow \hat{\Psi}_{k,\ell} \\ \mathbb{T} & \overset{z \to \bar{z}}{\leftarrow} & \mathbb{T} & \overset{\tilde{\theta}}{\leftarrow} & \tilde{I} \end{array} \tag{3.8}$$

properly interpreted, commutes. A little consideration of the four particular subcases that $\hat{\psi}_{k,\ell}$ maps some endpoint, respectively interior point of $\tilde{I}$ into some end-point, respectively interior point of $\tilde{I}$, show that $\hat{\psi}_{k,\ell}$ can be lifted to a map $I \to I$ by merging the two points at -1, resp.$+1$, whenever they occur, and then $\hat{\psi}_{k,\ell}$ maps any point of I into a set of cardinality $[\ell : k]$ in I. The set $\hat{\psi}_{k,\ell}(x)$ is then nothing but the spectrum of the image of the function $z \to \operatorname{Re} z$ in $B_{k,0}$ over the point x in the spectrum $\tilde{B}_{\ell,0}$ (when the end-points are merged). Hence, by Lemma 3.2, or directly by small eigenvalue variation of this element, we obtain

LEMMA 3.3. *For any k and any ε there exists an L such that if $\ell \geq L$, then the diameter of the range of $\hat{\psi}_{k,\ell}$ in $S^{[\ell:k]}I$ is less than ε.*

If we order the points $\lambda_1(t), \lambda_2(t), \ldots, \lambda_{[\ell:k]}(t)$ in $\hat{\psi}_{k,\ell}(t)$ in increasing order, then the condition in Lemma 3.3 can be expressed as

$$|\lambda_k(t) - \lambda_k(s)| \leq \varepsilon$$

for $k = 1, \ldots, [\ell : k]$ and all pairs $t, s \in [-1, 1]$, see [CE]. The functions $t \to \lambda_k(t)$ are continuous, and if $x \in \tilde{B}_{k,0}$ is arbitrary its image $\psi_{k,\ell}(x)$ in $B_{\ell,0}$, evaluated at $t \in [-1, 1]$, is unitarily equivalent to the matrix

$$\beta(x)(t) = \begin{pmatrix} x(\lambda_1(t)) & & & \\ & x(\lambda_2(t)) & & \\ & & \ddots & \\ & & & x(\lambda_{[\ell:k]}(t)) \end{pmatrix}. \tag{3.9}$$

More precisely, this matrix should also have some more zeros on the diagonal coming from the embedding of the other matrix summands of $\tilde{B}_k$ into $\tilde{B}_{\ell,0}$, but we leave the minor extra complications due to this to the reader.

Note that the unitary $u(t)$ such that

$$(\psi_{k,\ell}x)(t) = u(t)\beta(x)(t)u(t)^* \tag{3.10}$$

can be taken to be independent of x, but it cannot in general be taken to depend continuously on t at points where some of the eigenvalues $\lambda_1(x), \ldots, \lambda_{[\ell:k]}(x)$ coincide. However, if $\tilde{B}_{k,0}$ and $\tilde{B}_{\ell,0}$ had been the full homogeneous algebras $C(I, M_{2J_1(k)})$ and $C(I, M_{2J_1(\ell)})$ it was proved in [**Tho**, Theorem 3.1] that there exists a sequence u_n of continuous unitary-valued maps such that

$$(\psi_{k,\ell}x)(t) = \lim_{n\to\infty} u_n(t)\beta(x)(t)u_n(t)^*, \tag{3.11}$$

uniformly in t for each $x \in C(I, M_{2J_1(k)})$. In our case, we have the extra complication with the two end-points of I. For example, $\psi_{k,\ell}$ is not necessarily extendable to a morphism of $C(I, M_{2J_1(k)})$ into $C(I, M_{2J_1(\ell)})$. For example, if $\tilde{B}_{k,0} = \{x \in C(I, M_2) | x_{12}(-1) = x_{21}(-1) = x_{12}(1) = x_{21}(1) = 0\}$, $\tilde{B}_{\ell,0} = \tilde{B}_{k,0}$ and

$$\psi_{k,\ell}(x)(t) = \begin{pmatrix} x_{11}(t) & \varphi(t)x_{12}(t) \\ \overline{\varphi(t)}x_{21} & x_{22}(t) \end{pmatrix}$$

where φ is a continuous function from $\langle -1, 1\rangle$ into $\mathbb{T}$, then $\psi_{k,\ell}$ is a morphism, but $\psi_{k,\ell}$ is non-extendable if φ is not extendable to a continuous function on $[-1, 1]$. However, $\psi_{k,\ell}$ can be approximated strongly by extendable morphisms by replacing φ by, say, φ_n where

$$\varphi_n(t) = \begin{cases} \varphi(-1+n^{-1}) & \text{if } -1 \le t \le -1+n^{-1} \\ \varphi(t) & \text{if } -1+n^{-1} \le t \le 1-n^{-1} \\ \varphi(1-n^{-1}) & \text{if } 1-n^{-1} \le t \le 1. \end{cases}$$

Employing this device systematically, and using Thomsen's theorem, we can also prove (3.11) in our case. Moreover, since $U(2J_1(\ell))$ is a compact group, we may assume that $u_n(-1)$ and $u_n(+1)$ converge as $n \to \infty$, and then, modifying $u_n(t)$ near ± 1, we may assume that $u_n(-1)$ and $u_n(+1)$ are independent of n. This is a plausibility argument of:

LEMMA 3.4. *There exists a sequence u_n of continuous $U(2J_1(\ell))$-valued maps such that*

$$\psi_{k,\ell}(x)(t) = \lim_{n\to\infty} u_n(t)\beta(x)(t)u_n(t)^* \tag{3.12}$$

for all $x \in \tilde{B}_{k,0}$, where the convergence is uniform in t for each x, and such that

$$u_n(-1) = u(-1), \quad u_n(+1) = u(+1) \tag{3.13}$$

are both independent of n.

Proof. We write $\tilde{B}_{k,0} = M_{J_1(k)} \otimes A_0$ where

$$A_0 = \{f \in C(I) \otimes M_2 : f(\pm 1) \in D_2\} \tag{3.14}$$

and where D_2 are the diagonal matrices of M_2. The relative commutant of the image of $M_{J_1(k)}$ in $C(I) \otimes M_{J_1(\ell)} \otimes M_2$ is again of this form. This allows us to reduce to the case $J_1(k) = 1$; we assume that $\Psi_{k,\ell} = \varphi$ where φ is a unital embedding of A_0 in $C(I) \otimes M_m \otimes M_2$, and that $\dim \varphi(1 \otimes e_{ii})(t) = m$ for any $t \in I$. In other words we assume that $\pi_t \circ \varphi$ has ε^1_{-1} and ε^2_{-1} (respectively ε^1_1 and ε^2_1) with the same multiplicity, where $\pi_t : C(I) \otimes M_m \otimes M_2 \to M_m \otimes M_2$ is the evaluation at $t \in I$, and $\varepsilon^i_{\pm 1} : A_0 \to \mathbb{C}$ is defined by $\varepsilon^i_{\pm 1}(f) = f(\pm 1)_{ii}$.

First of all there is a unitary $u \in C(I) \otimes M_m \otimes M_2$ such that

$$\operatorname{Ad} u \circ \varphi(1 \otimes e_{ii}) = 1 \otimes 1 \otimes e_{ii}. \tag{3.15}$$

There is a maximal Abelian subalgebra C_i of $C(I) \otimes M_m$ such that

$$\operatorname{Ad} u \circ \varphi(C(I) \otimes e_{ii}) \subset C_i \otimes e_{ii}. \tag{3.16}$$

Hence there is a unitary $v \in C(I) \otimes M_m \otimes D_2$ such that

$$\operatorname{Ad} vu \circ (C(I) \otimes e_{ii}) \subset C(I) \otimes D_k \otimes e_{ii}, \tag{3.17}$$

where D_m are the diagonal matrices of M_m. Now take $\operatorname{Ad} vu \circ \varphi$ for φ.

Let $\lambda_i(t)$, $i = 1, \ldots, m$ be continuous functions on I such that $\lambda_i(t) \le \lambda_{i+1}(t)$ and $\pi_t \varphi \simeq \otimes_i \varepsilon_{\lambda_i(t)}$ where ε_λ is the evaluation map of A_0 at λ. Note that $\bigcup_i \{\lambda \in I | \{t \in I : |\lambda_i(t)| = \lambda\}^0 \neq \emptyset\}$ is countable. We choose a sufficiently small $\delta > 0$ such that $\{t \in I : |\lambda_i(t)| = 1 - \delta\}$ has no interior points for any i.

Define

$$\begin{aligned} \hat{u}(t) &= \pi_t \circ \varphi(\chi_{[-1+\delta, 1-\delta]} \otimes e_{12}) = u(t) \otimes e_{12} \\ \hat{E}_{-1}(t) &= \pi_t \circ \varphi(\chi_{[-1,-1+\delta]} \otimes e_{22}) = E_{-1}(t) \otimes e_{22} \\ \hat{E}_0(t) &= \pi_t \circ \varphi(\chi_{[-1+\delta, 1-\delta]} \otimes e_{22}) = E_0(t) \otimes e_{22} \\ \hat{E}_1(t) &= \pi_t \circ \varphi(\chi_{[1-\delta, 1]} \otimes e_{22}) = E_1(t) \otimes e_{22} \end{aligned} \tag{3.18}$$

and similarly define $\hat{F}_{-1}$, $\hat{F}_0$, F_1 like $\hat{E}$, with e_{11} in place of e_{22}. (The definition makes sense by approximating the characteristic functions by continuous functions.) Here $u(t)$ is a partial isometry of M_k such that $u(t)^* u(t) = E_0(t)$, $u(t)u(t)^* = F_0(t)$ and E, F are projections of M_k such that $E_{-1}(t) + E_0(t) + E_1(t) = 1$, etc.

Note that $u(t)$, $E_*(t)F_*(t)$ are continuous on $\bigcap_i \{t \in I | \lambda_i(t)| \neq 1 - \delta\}$, which is a dense open subset of I. For any $t_0 \in I$ such that $u(t)$ is not continuous at t_0, we may choose $0 < \delta' < \delta$ and construct $u' = u$, f' etc, as above such that $u'(t)$ etc, are continuous at t_0. Then $u'(t)$ is an extension of $u(t)$, i.e. $u'(t)E_0(t) = u(t)$. Thus it

easily follows that for each $s \in I$, there is an interval $(s-\delta_s, s+\delta_s)$ such that there is a continuous function u_s on $(s-\delta_s, s+\delta_s) \cap I$ into the unitaries of M_k such that

$$\begin{aligned} u_s(t)E_0(t) &= u(t) \\ u_s(t)E_{-1}(t)u_s(t)^* &= F_{-1}(t) \\ u_s(t)E_1(t)u_s(t)^* &= F_1(t). \end{aligned} \tag{3.19}$$

Since I is compact, there is a finite number of points $s_1 < \ldots < s_n$ such that $\bigcup_i (s_i - \delta_{s_i}, s_i + \delta_{s_i}) \supset I$ and each s_i has a u_{s_i} as above. To find a unitary v in $C(I) \otimes M_k$ such that

$$v(t)E_0(t) = u(t), \quad v(t)E_{-1}(t)v(t)^* = F_{-1}(t), \quad v(t)E_1(t)v(t)^* = F_1(t) \tag{3.20}$$

we have to connect $u^{(1)} = u_{s_i}$ on $(a_1, b_1) = (s_i - \delta_{s_i}, s_i + \delta_{s_i})$ and $u^{(2)} = u_{s_{i+1}}$ on $(a_2, b_2) = (s_{i+1} - \delta_{s_{i+1}}, s_{i+1} - \delta_{s_{i+1}})$ into one v on (a_1, b_2), keeping the condition (3.20). Since u, E, F are continuous on a dense open subset of I, there is an interval $[c, d] \subset (a_2, b_1)$ such that they are continuous on $[c, d]$. Then it is easy to find a continuous w on $[c, d]$ such that

$$\begin{aligned} w(c) = u^{(1)}(c), &\quad w(d) = u^{(2)}(d) \\ w(t)E_0(t) &= u(t) \\ \operatorname{Ad} w(t)(E_{\pm 1})(t) &= F_{\pm 1}(t). \end{aligned} \tag{3.21}$$

Thus we obtain a v combining $u^{(1)}, w, u^{(2)}$ as desired.

By using v satisfying (3.20), we define a map φ_δ of $C(I) \otimes M_2$ into $C(I) \otimes M_k \otimes M_2$ by

$$\begin{aligned} \varphi_\delta(1 \otimes e_{ii}) &= 1 \otimes 1 \otimes e_{ii} \\ \varphi_\delta(1 \otimes e_{12}) &= v \otimes e_{12} \\ \varphi_\delta(f \otimes 1) &= \varphi(f \otimes e_{11}) + v^* \otimes e_{21}\varphi(f \otimes e_{11})v \otimes e_{12}. \end{aligned} \tag{3.22}$$

Since $\varphi_\delta(C(I) \otimes 1)$ commutes with $\varphi_\delta(1 \otimes e_{ij})$, the map φ_δ actually defines a homeomorphism. It is injective since φ is injective. We claim that if $x \in A_0$ is constant on an open neighbourhood of $[-1, -1+\delta] \cup [1-\delta, 1]$, it follows that $\varphi_\delta(x) = \varphi(x)$. First for $x = f \otimes e_{ij}$ with $\operatorname{supp} f \subset [-1+\delta, 1-\delta]$, the equality follows. For example, if $\varphi(f \otimes 1) = a \otimes e_{11} + b \otimes e_{22}$ with $a, b \in C(I) \otimes D_m$, then

$$\begin{aligned} \varphi(f \otimes e_{12}) &= \varphi(f \otimes e_{11})\varphi(\chi_{[-1+\delta, 1-\delta]} \otimes e_{12}) \\ &= au \otimes e_{12} \\ &= ub \otimes e_{12}. \end{aligned} \tag{3.23}$$

Thus $b = u^*au$ and so

$$\begin{aligned} \varphi_\delta(f \otimes e_{22}) &= v^* \otimes e_{21}\varphi(f \otimes e_{11})v \otimes e_{12} \\ &= v^*av \otimes e_{22} = b \otimes e_{22} \\ &= \varphi(f \otimes e_{22}). \end{aligned} \tag{3.24}$$

(u is not in $C(I) \otimes M_m$ but it behaves as a multiplier for a, b.) Next let $x = f \otimes e_{22}$ be such that $f(t) = f(-1)$ for $t \in [-1, -1+\delta']$ with $\delta' > \delta$ and $\operatorname{supp} f \subset [-1, 1-\delta]$. (We should also consider $f \otimes e_{11}$, for this f, and similar elements concentrated at $+1$.)

For each $t_0 \in I$, we find $\delta'' \in (\delta, \delta')$ such that $|\lambda_i(t_0)| \neq 1 - \delta''$, and the u'', E'', F'' constructed for δ'' in place of δ are continuous at t_0. Then it easily follows that

$$\begin{aligned} \pi_{t_0} \circ \varphi(f \otimes e_{22}) &= F''_{-1}(t_0) \otimes e_{22} + \pi_{t_0} \circ \varphi(g \otimes e_{22}) \\ \pi_{t_0} \circ \varphi(f \otimes e_{11}) &= E''_{-1}(t_0) \otimes e_{11} + \pi_{t_0} \circ \varphi(g \otimes e_{11}), \end{aligned} \tag{3.25}$$

where $g \in C(I)$ satisfies $\operatorname{supp} g \subset [-1+\delta, 1-\delta]$. (Express f as the sum of $(f-g)$ and g such that $f - g$ behaves like the characteristic function of $[-1, -1+\delta'']$.) Then

$$\begin{aligned} \pi_{t_0} \circ \varphi_\delta(f \otimes e_{22}) &= (v^* \otimes e_{21} \cdot \varphi(f \otimes e_{11}) v \otimes e_{12})(t_0) \\ &= v(t_0)^* F''_{-1}(t_0) v(t_0) \otimes e_{22} + \pi_{t_0} \circ \varphi(g \otimes e_{22}) \\ &= E''_{-1}(t_0) \otimes e_{22} + \pi_{t_0} \circ \varphi(g \otimes e_{22}) \\ &= \pi_{t_0} \circ \varphi(g \otimes e_{22}), \end{aligned} \tag{3.26}$$

where $v(t_0) F''_{-1}(t_0) v(t_0) = E''_{-1}(t_0)$ since $\delta'' > \delta$.

The other cases can be treated similarly. This implies that $\varphi_\delta(x) \to \varphi(x)$ for $x \in A_0$ as $\delta \downarrow 0$. This ends the proof of Lemma 3.4.

We are now ready to prove

THEOREM 3.5. *Let Ω be a totally disconnected compact metrizable space, and let α be a minimal homeomorphism of Ω. Let σ be a homeomorphism of order 2 on Ω such that*

$$\alpha\sigma = \sigma\alpha^{-1} \tag{3.27}$$

and assume that σ or $\alpha\sigma$ has a fixed point.

It follows that $C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2$ is an AF algebra.

Proof. By [**Bra**, Theorem 2.2] and Corollary 2.4 it suffices to show that if $\varepsilon > 0$ and $x_1, \ldots, x_m \in \tilde{B}_{k,0}$, then there exists a $\ell \geq k$ and a finite-dimensional subalgebra F_0 of $\tilde{B}_{\ell,0}$ such that the distance between each of the elements $\psi_{k,\ell}(x_i)$, $i = 1, \ldots, m$, and F_0 is less than ε.

For this we first use Lemma 3.3 to choose ℓ so large that

$$||\beta(x_i)(t) - \beta(x_i)(s)|| \leq \varepsilon/3 \tag{3.28}$$

for $i = 1, \ldots, m$, $t, s \in [-1, 1]$, where β is defined by (3.9). Next we use Lemma 3.4 to choose n so large that

$$||u_n(t)\beta(x_i)(t)u_n(t)^* - \psi_{k,\ell}(x_i)(t)|| \leq \varepsilon/3 \tag{3.29}$$

for $i = 1, \ldots, m$, $t \in [-1, 1]$. Now let

$$A = u_n^* \tilde{B}_{k,0} u_n \tag{3.30}$$

i.e. the elements in A are continuous functions from I into $M_{2J_1(\ell)}$ of the form

$$t \to u_n(t)^* x(t) u_n(t), \tag{3.31}$$

where $x \in \tilde{B}_{\ell,0}$. Thus

$$\begin{aligned} A &= \{x \in C(I, M_{2J_1(\ell)}); [x(-1), u(-1)^* E u(-1)] = 0 \\ &\quad \text{and} [x(+1), u(+1)^* E u(+1)] = 0\}, \end{aligned} \tag{3.32}$$

where E is the projection defined in Corollary 2.4 (for $n = \ell$). In particular, since $u_n(-1) = u(-1)$ and $u_n(+1) = u(+1)$ are independent of n, we have

$$\beta(\tilde{B}_{k,0}) \subseteq A. \tag{3.33}$$

Let D_{-1}, resp. D_{+1}, be the finite-dimensional (actually 2-dimensional) abelian subalgebra of $M_{2J_1(k)}$ generated by the projection E of Corollary 2.4 for $n = k$. (Of course, $D_{-1} = D_{+1}$, but we view $D_{\pm 1}$ as 'sitting over' the points ± 1 in I, respectively.) Let C_{-1}, respectively C_{+1}, be the Abelian algebra generated by $u(-1)^* E u(-1)$, resp. $u(1)^* E u(1)$, where E now is the E for $n = \ell$. Thus

$$A = \{x \in C(I, M_{2J_1(\ell)}); x(-1) \in C'_{-1}, x(+1) \in C'_{+1}\}. \tag{3.34}$$

Define integers n_-, n_+, m_-, $m_+ \geq 0$ by

$$\begin{aligned} \lambda_1(-1) &= \lambda_2(-1) = \ldots = \lambda_{n_-}(-1) = -1 < \lambda_{n_-+1}(-1) \leq \cdots \\ &\leq \lambda_{[\ell:k]-n_+-1}(-1) < 1 = \lambda_{[\ell:k]-n_+}(-1) = \lambda_{[\ell:k]-n_++1}(-1) = \ldots = \lambda_{[\ell:k]}(-1) \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} \lambda_1(1) &= \lambda_2(1) = \ldots = \lambda_{m_-}(1) = -1 < \lambda_{m_-+1}(1) \leq \cdots \\ &\leq \lambda_{[\ell:k]-m_+-1}(1) < 1 = \lambda_{[\ell:k]-m_+}(1) = \lambda_{[\ell:k]-m_++1}(1) = \ldots = \lambda_{[\ell:k]}(1), \end{aligned} \tag{3.36}$$

and

$$N = \max\{n_-, m_-\}, \quad M = \max\{n_+, m_+\}. \tag{3.37}$$

Let

$$B_{-1} = \beta(\tilde{B}_{k,0})(-1)' \cap M_{2J_1(\ell)}. \tag{3.38}$$

Then, from (3.33) and (3.34),

$$C_{-1} \subseteq B_{-1} \tag{3.39}$$

and from the definition of β on $\tilde{B}_{k,0}$, B_{-1} has the form

$$B_{-1} = \begin{pmatrix} \underbrace{\begin{matrix} D_{-1} & \ldots & D_{-1} \\ \vdots & & \vdots \\ D_{-1} & \ldots & D_{-1} \end{matrix}}_{n_-} & \text{scalars or scalar matrices down diagonal} & \underbrace{\begin{matrix} D_{+1} & \ldots & D_{+1} \\ \vdots & & \vdots \\ D_{+1} & \ldots & D_{+1} \end{matrix}}_{n_+} \end{pmatrix}, \tag{3.40}$$

where the matrix elements are in $M_{2J_1(k)}$. Thus

$$F_{-1} = \begin{pmatrix} D_{-1} & & & & & & \\ & \ddots & & & & & \\ & & D_{-1} & & & & \\ & & & \text{scalars} & & & \\ & & & \text{along} & & & \\ & & & \text{diagonal} & & & \\ & & & & D_{+1} & & \\ & & & & & \ddots & \\ & & & & & & D_{+1} \end{pmatrix} \tag{3.41}$$

is a maximal abelian algebra in B_{-1}, and we can find a unitary $V_{-1} \in B_{-1}$ such that

$$V_{-1} C_{-1} V_{-1}^* \subseteq F_{-1}. \tag{3.42}$$

Now, repeat all this at $+1$ instead of -1, and find a unitary $V_{+1} \in B_{+1} = \beta(\tilde{B}_{k,0})(+1)' \cap M_{2J_1(\ell)}$ such that

$$V_{+1} C_{+1} V_{+1}^* \subseteq F_{+1}. \tag{3.43}$$

Finally, define the finite-dimensional subalgebra F of $M_{2J_1(\ell)}$ by

$$F = \begin{pmatrix} D'_{-1} & & & & & & & \\ & \ddots & & & & & & \\ & & D'_{-1} & & & & & \\ & & & M_{2J_1(k)} & & & & \\ & & & & \ddots & & & \\ & & & & & M_{2J_1(k)} & & \\ & & & & & & D'_{+1} & \\ & & & & & & & \ddots \\ & & & & & & & D'_{+1} \\ \underbrace{\qquad N \qquad} & & & & & & \underbrace{\qquad M \qquad} & \end{pmatrix}. \tag{3.44}$$

Then F commutes with both F_{-1} and F_{+1}, in fact $F = F'_{-1} \cap F'_{+1}$. Let $V(t)$, $-1 \le t \le 1$, be a continuous family of unitaries such that

$$\begin{aligned} V(t) &\in B_{-1} \quad \text{for } -1 \le t \le 0 \\ V(0) &= 1 \\ V(t) &\in B_{+1} \quad \text{for } 0 \le t \le 1 \\ V(-1) &= V_{-1} \\ V(+1) &= V_{+1}. \end{aligned} \tag{3.45}$$

For any of the x_i's, define

$z_i =$

$$\begin{pmatrix} x_i(-1) & & & & & & & & \\ & \ddots & & & & & & & \\ & & x_i(-1) & & & & & & \\ & & & x_i(\lambda_{N+1}(0)) & & & & & \\ & & & & \ddots & & & & \\ & & & & & x_i(\lambda_{[\ell:k]-M-1}(0)) & & & \\ & & & & & & x_i(1) & & \\ & & & & & & & \ddots & \\ & & & & & & & & x_i(1) \\ \underbrace{\qquad\qquad}_{N} & & & & & & & \underbrace{\qquad}_{M} & \end{pmatrix}. \tag{3.46}$$

Then the z_i's all lie in the finite-dimensional C^*-algebra F. Defining

$$(V^* z_i V)(t) = V^*(t) z_i V(t)$$

then $V^* z_i V$ all lie in the finite-dimensional algebra $V^* F V$, and the latter algebra is contained in A by (3.42) and (3.43).

We now argue that

$$||\beta(x_i) - V^* z_i V|| \leq 2\varepsilon/3 \tag{3.47}$$

i.e. that

$$||V(t)\beta(x_i)(t)V(t)^* - z_i|| \leq 2\varepsilon/3 \tag{3.48}$$

for all t. But $\beta(x_i)(-1) \in B'_{-1}$ and $V(t) \in B_{-1}$ for $-1 \leq t < 0$, hence by (3.28),

$$\begin{aligned} ||\beta(x_i)(-1) - V(t)\beta(x_i)(t)V(t)^*|| &= ||V(t)\beta(x_i)(-1)V(t)^* - V(t)\beta(x_i)(t)V(t)^*|| \\ &= ||\beta(x_i)(-1) - \beta(x_i)(t)|| < \varepsilon/3 \end{aligned} \tag{3.49}$$

for $-1 \leq t \leq 0$. Similarly

$$||V(t)\beta(x_i)(t)V(t)^* - \beta(x_i)(+1)|| \leq \varepsilon/3 \tag{3.50}$$

for $0 \leq t \leq 1$. But

$\beta(x_i)(-1) =$

$$\begin{pmatrix} x_i(-1) & & & & & & & \\ & \ddots & & & & & & \\ & & x_i(-1) & & & & & \\ & & & x_i(\lambda_{n_-+1}(-1)) & & & & \\ & & & & \ddots & & & \\ & & & & & x(\lambda_{[\ell:k]-n_+-1}(-1)) & & \\ & & & & & & x_{[\ell:k]}(-1) & \\ & & & & & & & \ddots \\ & & & & & & & & x_{[\ell:k]}(-1) \end{pmatrix}. \tag{3.51}$$

Comparing this with (3.46), and using (3.28) (which implies that the variation of $x_i(\lambda_k(t))$ with t is at most $\varepsilon/3$), it follows that

$$||\beta(x_i)(-1) - z_i|| < \varepsilon/3, \tag{3.52}$$

and similarly

$$||\beta(x_i)(+1) - z_i|| < \varepsilon/3. \tag{3.53}$$

Combining this with (3.49) and (3.50), we obtain (3.48) and hence (3.47).

Finally

$$u_n V^* F V u_n^* \subseteq u_n A u_n^* = \tilde{B}_{\ell,0},$$

where the last equality follows from (3.30). Combining (3.47) with (3.29) we have

$$||\psi_{k,\ell}(x_i) - u_n V^* z_i V u_n^*|| \leq 2\varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus $\psi_{k,\ell}(x_i)$ all lie within ε of the finite-dimensional subalgebra $u_n V^* F V u_n^*$ of $\tilde{B}_{\ell,0}$. Thus, by the first remark of the proof, $C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2$ is AF.

4. *K-theory*

In this section we shall compute $K_0(C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2)$ (as an abelian group) in the case that σ and $\alpha\sigma$ have at most a finite number of fixed points. (The computation is valid even when σ and $\alpha\sigma$ have no fixed points, i.e. outside the range of validity of Theorem 3.5.)

The starting point is that

$$K_0(C(\Omega)) \cong C(\Omega, \mathbb{Z}) \tag{4.1}$$

since Ω is totally disconnected. The actions α_*, σ_* defined by α, σ on $K_0(C(\Omega))$ are given by

$$(\alpha_* f)(\omega) = f(\alpha^{-1}\omega), \quad (\alpha_* f)(\omega) = f(\sigma\omega) \tag{4.2}$$

for $f \in C(\Omega, \mathbb{Z})$, $\omega \in \Omega$. Now, by the Pimsner–Voiculescu exact sequence, **[PV, Bla]**, the following sequence is exact

$$0 \to \mathbb{Z} \xrightarrow{i} C(\Omega, \mathbb{Z}) \xrightarrow{1-\alpha_*} C(\Omega, \mathbb{Z}) \xrightarrow{j_*} K_0(C(\Omega) \times_\alpha \mathbb{Z}) \to 0, \tag{4.3}$$

where i maps $\mathbb{Z}$ into the constant functions and $j : C(\Omega) \to C(\Omega) \times_\alpha \mathbb{Z}$ is the natural embedding. It follows that

$$K_0(C(\Omega) \times_\alpha \mathbb{Z}) \cong C(\Omega, \mathbb{Z})/(1 - \alpha_*)(C(\Omega, \mathbb{Z})) \tag{4.4}$$

and σ_* acts naturally on the latter group.

(Note that $(1 - \alpha_*)(C(\Omega, \mathbb{Z})) = \sigma_*((1 - \alpha_*)(C(\Omega, \mathbb{Z})))$ since

$$\begin{aligned} \sigma_*((1 - \alpha_*)(C(\Omega, \mathbb{Z}))) &= (1 - \alpha_*^{-1})(\sigma_*(C(\Omega, \mathbb{Z}))) \\ &= (\alpha_* - 1)(\alpha_*^{-1}\sigma_*(C(\Omega, \mathbb{Z}))) = (1 - \alpha_*)(C(\Omega, \mathbb{Z})).) \end{aligned}$$

We first state a general theorem, and later, in Corollary 4.4, consider the special case that $\Omega = \mathbb{T}_\theta$, where θ is irrational and $\mathbb{T}_\theta$ is the circle cut up at all the points of the orbit $\mathbb{Z}\theta$.

THEOREM 4.1. *Let Ω be a totally disconnected compact metrizable space and let α be a minimal homeomorphism of Ω. Let σ be a homeomorphism of order 2 on Ω such that $\alpha\sigma = \sigma\alpha^{-1}$, and assume that σ and $\alpha\sigma$ has at most a finite number n_σ and $n_{\alpha\sigma}$ of fixed points.*

It follows that $K_0(C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2)$ is isomorphic to

$$(1+\sigma_*)(C(\Omega,\mathbb{Z})/(1-\alpha_*)(C(\Omega,\mathbb{Z}))) \oplus \mathbb{Z}^{n_\sigma+n_{\alpha\sigma}}. \tag{4.5}$$

Let $\mathbb{Z}_2 * \mathbb{Z}_2$ be the free group product, with generators $1*0$ and $0*1$. Define an action γ of $\mathbb{Z}_2 * \mathbb{Z}_2$ on $C(\Omega)$ by $\gamma_{1*0} = \sigma$ and $\gamma_{0*1} = \alpha\sigma$. Then it follows from **[Kum2]** that $C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2$ is naturally isomorphic to $C(\Omega) \times_\gamma \mathbb{Z}_2 * \mathbb{Z}_2$.

LEMMA 4.2. *The following sequence is exact:*

$$\begin{aligned} 0 \quad &\to \quad K_0(C(\Omega)) \xrightarrow{i_{1*}-i_{2*}} K_0(C(\Omega) \times_\sigma \mathbb{Z}_2) \oplus K_0(C(\Omega) \times_{\alpha\sigma} \mathbb{Z}_2) \\ &\xrightarrow{j_{1*}+j_{2*}} \quad K_0(C(\Omega) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2) \to 0, \end{aligned} \tag{4.6}$$

where $i_1 : C(\Omega) \to C(\Omega)\times_\sigma\mathbb{Z}_2$, $j_1 : C(\Omega)\times_\sigma\mathbb{Z}_2 \to C(\Omega)\times_\gamma\mathbb{Z}_2*\mathbb{Z}_2 \simeq C(\Omega)\times_\alpha\mathbb{Z}\times_\sigma\mathbb{Z}_2$, etc, are natural embeddings.

Proof. See **[Nat]** and **[Kum2]**. The injectivity of $i_{1*} - i_{2*}$ follows from minimality of α.

We now identify $K_0(C(\Omega))$ with $C(\Omega;\mathbb{Z})$. Let $x_1, x_2, \ldots, x_{n_\sigma}$ be the fixed points of σ. Then $C(\Omega) \times_\sigma \mathbb{Z}_2$ is isomorphic to

$$\{f \in C(\Omega; M_2) | f \circ \sigma = \mathrm{Ad}\, u \circ f\} \tag{4.7}$$

where $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus it follows that $K_0(C(\Omega) \times_\sigma \mathbb{Z})$ is identified with

$$G_\sigma = \{(f,a,b) \in C(\Omega,\mathbb{Z}) \oplus \mathbb{Z}^{n_\sigma} \oplus \mathbb{Z}^{n_\sigma} : \sigma(f) = f,\ f(x_i) = a_i + b_i, \quad i = 1,\ldots,n_\sigma\} \tag{4.8}$$

and under this identification, the map $i_{1*} : C(\Omega,\mathbb{Z}) \to G_\sigma$ is given by

$$i_{1*}(h) = (h + \sigma(h)) \oplus (\oplus_{i=1}^{n_\sigma} h(x_i)) \oplus (\oplus_{i=1}^{n_\sigma} h(x_i)). \tag{4.9}$$

Similarly, using the fixed points $y_1, \ldots, y_{n_{\alpha\sigma}}$ of $\alpha\sigma$, we define $G_{\alpha\sigma}$ and describe $i_{2*} : C(\Omega,\mathbb{Z}) \to G_{\alpha\sigma}$.

From now, identify $K_0(C(\Omega) \times_\alpha \mathbb{Z})$ with $C(\Omega,\mathbb{Z})/(1-\alpha_*)(C(\Omega,\mathbb{Z}))$ as in (4.4). We define a map $\beta : G_\sigma \oplus G_{\alpha\sigma} \to K_0(C(\Omega) \times_\alpha \mathbb{Z}) \oplus \mathbb{Z}^{n_\sigma} \oplus \mathbb{Z}^{n_{\alpha\sigma}}$ as follows:

$$\beta((f,a,b) \oplus (g,c,d)) = (j_*(f) + j_*(g)) \oplus (a-b) \oplus (c-d). \tag{4.10}$$

It is obvious that $j_*(f) + j_*(g)$ are σ_*-invariant. We assert:

LEMMA 4.3. *The sequence*

$$C(\Omega,\mathbb{Z}) \xrightarrow{i_{1*}-i_{2*}} G_\sigma \oplus G_{\alpha\sigma} \xrightarrow{\beta} K_0(C(\Omega) \times_\alpha \mathbb{Z}) \oplus \mathbb{Z}^{n_\sigma} \oplus \mathbb{Z}^{n_{\alpha\sigma}} \tag{4.11}$$

is exact at $G_\sigma \oplus G_{\alpha\sigma}$.

Proof. It is obvious that $\beta \circ (i_{1*} - i_{2*}) = 0$. We only have to show that Im $(i_{1*} - i_{2*}) \supset \operatorname{Ker} \beta$. Suppose that $\beta((f, a, b) \oplus (g, c, d)) = 0$. Then $j_*(f+g) = 0$, $a = b$, $c = d$ and $f(x_i) = 2a_i$ and $g(y_j) = 2c_j$ are even. By (4.3) there exists $\varphi \in C(\Omega, \mathbb{Z})$ such that

$$f + g = \varphi - \alpha(\varphi).$$

Note that

$$f + \sigma(g) = \sigma(\varphi) - \sigma\alpha(\varphi), \quad \alpha(f) + g = \alpha\sigma(\varphi) - \sigma(\varphi)$$

and that

$$f - \alpha(f) = \varphi - \alpha(\varphi) - \alpha\sigma(\varphi) + \sigma(\varphi)$$

$$g - \sigma(g) = \varphi - \alpha(\varphi) - \sigma(\varphi) + \alpha\sigma(\varphi).$$

By computation it follows that

$$f - \sigma(\varphi) - \varphi - \alpha(f - \sigma(\varphi) - \varphi) = 0,$$

$$g + \sigma(\varphi) + \alpha(\varphi) - \alpha^{-1}(g + \sigma(\varphi) + \alpha(\varphi)) = g + \sigma(\varphi) + \alpha(\varphi) - \sigma(g + \sigma(\varphi) + \alpha(\varphi)) = 0$$

since $\alpha^{-1} = \sigma^2\alpha^{-1} = \sigma \circ \alpha\sigma$ and $g + \sigma(\varphi) + \alpha(\varphi)$ is $\alpha\sigma$-invariant. Thus

$$f - \sigma(\varphi) - \varphi = \lambda 1, \quad g + \sigma(\varphi) + \alpha(\varphi) = \mu 1$$

for some constants λ, μ. Since

$$f + g - \varphi + \alpha(\varphi) = (\lambda + \mu)1$$

it follows that $\lambda + \mu = 0$. Since $f(x_i)$ is even, $\lambda = f(x_i) - 2\varphi(x_i)$ is even. Let $h = \sigma(\varphi) + \frac{1}{2}\lambda$. Then $h \in C(\Omega, \mathbb{Z})$ and

$$f = h + \sigma(h), \quad -g = h + \alpha\sigma(h).$$

Thus it follows that $(i_{1*} - i_{2*})(h) = (f, a, b) \oplus (g, c, d)$.

To conclude the proof of Theorem 4.1 it suffices to show that

$$\operatorname{Im} \beta \simeq (1 + \sigma_*) K_0(C(\Omega) \times \mathbb{Z}) \oplus \mathbb{Z}^{n_\sigma} \oplus \mathbb{Z}^{n_{\alpha\sigma}}.$$

Let $\{E_i\}$ be a mutually disjoint family of σ-invariant clopen sets of Ω such that $E_i \ni x_i$, and $\{F_j\}$ a mutually disjoint family of $\alpha\sigma$-invariant clopen sets of Ω such that $F_j \ni y_j$. We assert that the image of β is generated by

$$K \equiv (1 + \sigma_*) K_0(C(\Omega) \times_\alpha \mathbb{Z}) \oplus 0 \oplus 0$$

$$j_* \chi_{E_i} \oplus \delta_i \oplus 0, \quad i = 1, \ldots, n_\sigma$$

$$j_* \chi_{F_j} \oplus 0 \oplus \gamma_j, \quad j = 1, \ldots, n_{\alpha\sigma},$$

where $\{\delta_i\}$ (resp. $\{\gamma_j\}$) is a canonical basis for $\mathbb{Z}^{n_\sigma}$ (respectively $\mathbb{Z}^{n_{\alpha\sigma}}$). It is obvious that they are contained in Im β and that the latter $n_\sigma + n_{\alpha\sigma}$ elements generate the subgroup H, which is isomorphic to $\mathbb{Z}^{n_\sigma} \oplus \mathbb{Z}^{n_{\alpha\sigma}}$. It is also obvious that $H \cap K = \{0\}$. We now conclude the proof by showing that $H + K \supset \operatorname{Im} \beta$.

Let $(f, a, b) \in G_\sigma$ and $(g, c, d) \in G_{\alpha\sigma}$. Let

$$x = j_*\left(\sum_{i=1}^{n_\sigma}(b_i - a_i)\chi_{E_i}\right) \oplus (b - a) \oplus 0 \in G_\sigma$$

$$y = j_*\left(\sum_{j=1}^{n_{\alpha\sigma}}(d_j - c_j)\chi_{F_j}\right) \oplus (d - c) \oplus 0 \in G_{\alpha\sigma}.$$

Then $\beta(x \oplus y) \in H$ and

$$\beta((f, a, b) \oplus (g, c, d) + x \oplus y) = j_*(\varphi_1 + \varphi_2) \oplus 0 \oplus 0,$$

where

$$\varphi_1 = f + \sum(b_i - a_i)\chi_{E_i}$$

$$\varphi_2 = f + \sum(d_j - c_j)\chi_{F_j}.$$

Let

$$\psi_1 = \varphi_1 - \sum 2b_i\chi_{E_i}, \quad \psi_2 = \varphi_2 - \sum 2d_j\chi_{F_j}.$$

Since $\psi_1(x_i) = 0$ and $\sigma(\psi_1) = \psi_1$ (respectively $\psi_2(y_j) = 0$ and $\alpha\sigma(\psi_2) = \psi_2$), it easily follows that there is an element ψ_1' (resp. ψ_2') in $C(\Omega, \mathbb{Z})$ such that

$$\psi_1 = \psi_1' + \sigma(\psi_1') \quad (\text{respectively } \psi_2 = \psi_2' + \alpha\sigma(\psi_2')).$$

Letting

$$\varphi_1' = \psi_1 + \sum b_i\chi_{E_i} \quad (\text{respectively } \varphi_2' = \psi_2 + \sum d_j\chi_{F_j})$$

one obtains that $\varphi_1 = \varphi_1' + \sigma(\varphi_1')$, and $\varphi_2 = \varphi_2' + \alpha\sigma(\varphi_2')$ for $\varphi_1', \varphi_2' \in C(\Omega, \mathbb{Z})$. Hence

$$j_*(\varphi_1 + \varphi_2) = j_*(\varphi_1' + \varphi_2') + \sigma_*(j_*(\varphi_1' + \varphi_2'))$$

which imples that $j_*(\varphi_1 + \varphi_2) \oplus 0 \oplus 0 \in K$. This completes the proof of Theorem 4.1.

We next consider a special case of Theorem 4.1. Let θ be an irrational number between 0 and $\frac{1}{2}$ and let $\mathbb{T}_\theta$ be the totally disconnected space obtained from $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0, 1)$ by replacing each $x_n = n\theta + \mathbb{Z}$ by two points x_n^+, x_n^- for $n \in \mathbb{Z}$, with topology induced from the total order on $\mathbb{T}_\theta$, inheriting the order on $\mathbb{T}$, satisfying $x_n^- < x_n^+$, $n \in \mathbb{Z}$. Define a homeomorphism α of $\mathbb{T}_\theta$ by

$$\alpha(x) = x + \theta(\text{mod}\,\mathbb{Z}), \quad x \notin \theta\mathbb{Z} + \mathbb{Z}$$

$$\alpha(x_n^\pm) = x_{n+1}^\pm, \quad n \in \mathbb{Z}$$

and a σ by

$$\sigma(x) = 1 - x(\text{mod}\,\mathbb{Z}), \quad x \notin \theta\mathbb{Z} + \mathbb{Z}$$

$$\alpha(x_n^\pm) = x_{-n}^\mp, \quad n \in \mathbb{Z}.$$

Note that $\alpha\sigma$ has a fixed point, i.e. $\alpha(\frac{1}{2}\theta) = \frac{1}{2}\theta$ and that α is minimal. From the previous sections it follows that $C(\mathbb{T}_\theta) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2$ is AF.

COROLLARY 4.4. *For $\mathbb{T}_\theta$, α, σ as above,*

(i) $K_0(C(\mathbb{T}_\theta) \times_\alpha \mathbb{Z}) \simeq \mathbb{Z}^2$

(ii) $K_0(C(\mathbb{T}_\theta) \times_\alpha \mathbb{Z} \times_\sigma \mathbb{Z}_2) \simeq \mathbb{Z}^5$.

Proof. By Theorem 2.1 of **[Put1]**, (i) follows since α^{-1} defines an interval exchange transformation on $[0, 1)$, exchanging $[0, \theta)$ and $[\theta, 1)$. Since σ has one fixed point and $\alpha\sigma$ has two, (ii) follows since σ_* is the identity on $K_0(C(\Omega) \times_\alpha \mathbb{Z})$.

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