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INSTABILITY OF MULTI-SPOT PATTERNS IN SHADOW SYSTEMS OF REACTION-DIFFUSION EQUATIONS

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Abstract. Our aim in this paper is to prove the instability of multi-spot patterns in a shadow system, which is obtained as a limiting system of a reaction-diffusion model as one of the diffusion coefficients goes to infinity. Instead of investigating each eigenfunction for a linearized operator, we characterize the eigenspace spanned by unstable eigenfunctions.

1. **Introduction.** Various reaction-diffusion systems with two or more components have been proposed as mathematical models for pattern formation. Some systems with appropriate reaction terms and diffusion coefficients exhibit steady states with spot patterns. Suppose that a system can be decomposed into two subsystems; one has small diffusion and the other has large one. In this case many numerical results suggest that multiple spots must be unstable and only a single spot can be observed (see Figure 1). In fact, some mathematical analysis has been done for some specific models to prove the instability of multi-spot solutions [19, 20, 21]. One of our aims in this paper is to prove, in a quite general setting, the instability of any multi-spot pattern in the shadow system,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u + f(u,v), & x \in \Omega,\ t > 0, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = g_1(v) + \dfrac{1}{\varepsilon^d} \displaystyle\int_\Omega g_2(u,v)\mathrm{d}x, & t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with C^2 boundary $\partial\Omega$, ν is the outer normal vector on $\partial\Omega$, ε is a positive parameter, f, g_1, g_2 are sufficiently smooth

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functions, $d \in [0, n]$ is a scaling factor. In this paper, we are interested in stationary solutions with multi-spot pattern, which are formally expressed as

$$u(x) \sim \sum_{i=1}^{m} S\left(\frac{x - h_i}{\varepsilon}\right), \quad v \sim \zeta \tag{2}$$

with small ε and $h_i \in \overline{\Omega}$, $i = 1, \ldots, m$. Here the function $S = S(y)$ is a positive radially symmetric solution of

$$\begin{cases} \Delta S + f(S, \zeta) = 0, & y \in \mathbb{R}^n, \\ \qquad\qquad S \to 0, & |y| \to \infty, \end{cases} \tag{3}$$

that decays exponentially as $|y| \to \infty$. If (u, v) satisfies (2), we call it *m-spots*. Spots at $h_i \in \partial\Omega$ are called *boundary spots*, and others *interior spots*. Our main result in this paper is briefly summarized as follows:

Main result. If $\varepsilon > 0$ is sufficiently small, then any multi-spot pattern in (1) is unstable.

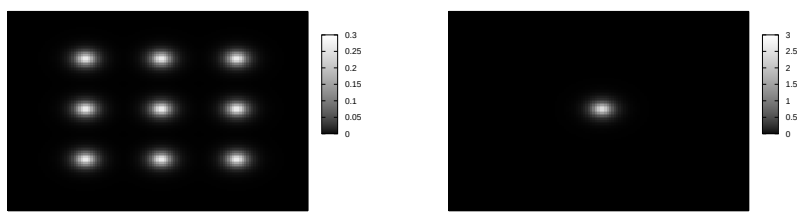

FIGURE 1. Instability of multi-spot patterns in the Gierer-Meinhardt system (7). The white regions denote large values of A. A solution initially has nine spots (the left figure), but converges to a single-spot pattern (the right figure) as $t \to \infty$. Parameter values are $\varepsilon = 0.02$, $\tau = 0.25$, $(p, q, r, s) = (2, 1, 2, 0)$.

Later, we shall give precise conditions on the profile $S(y)$ of the multi-spot solution. These conditions are quite general and technical and will be given in Section 2, see (A1)-(A11), and then the precise statement of our results will be described. On the other hand, we shall not specify the reaction terms, so our results apply to any shadow systems of the form (1). In order to show the instability, we consider the linearized eigenvalue problem associated with (1). As our system consists of semi-linear equations, the linear instability implies the nonlinear instability.

Our motivation to formulate the problem as (1) is as follows. Let us consider the following two-component reaction-diffusion system:

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A + F(A, H), & x \in \Omega,\ t > 0, \\ \dfrac{\partial H}{\partial t} = D\Delta H + G(A, H), & x \in \Omega,\ t > 0, \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial H}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{4}$$

Here we assume that the diffusion constant D of the second component is much larger than the diffusion constant ε^2 of the first component. Then it is believed that the reaction-diffusion system (4) with a suitable nonlinear functions F, G exhibits the Turing instability and generates stationary solutions with some pattern. However to prove this in a general setting is an extremely difficult problem. Therefore, we consider the limit as $D \to \infty$ in (4) and reduce our system to

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A + F(A, H), & x \in \Omega,\ t > 0, \\ \dfrac{\mathrm{d}H}{\mathrm{d}t} = \displaystyle\int_\Omega G(A, H)\mathrm{d}x, & t > 0, \\ \dfrac{\partial A}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{5}$$

where the measure of Ω is supposed to be 1 without loss of generality. The system (5) is called the *shadow system* of (4) (see [15]). Note that the second component H in (5) depends on the time variable only, which makes analysis of the shadow systems easier than the original reaction diffusion systems.

Before dealing with general shadow systems, we discuss two typical systems. One is the FitzHugh-Nagumo model

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A + A(A-a)(1-A) - H, & x \in \Omega,\ t > 0, \\ \tau \dfrac{\mathrm{d}H}{\mathrm{d}t} = \displaystyle\int_\Omega A\mathrm{d}x - \gamma H, & t > 0, \end{cases} \tag{6}$$

where $0 < a < 1/2$ and $\tau, \gamma > 0$ are constants, and the other is the Gierer-Meinhardt model

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A - A + \dfrac{A^p}{H^q}, & x \in \Omega,\ t > 0, \\ \tau \dfrac{\mathrm{d}H}{\mathrm{d}t} = -H + \displaystyle\int_\Omega \frac{A^r}{H^s}\mathrm{d}x, & t > 0, \end{cases} \tag{7}$$

where τ is a positive constant and the exponents p, q, r, s satisfy

$$p > 1,\ q > 0,\ r > 1,\ s \geq 0,\ \frac{p-1}{q} < \frac{r}{s+1}.$$

We suppose that the function A in these systems satisfies the homogeneous Neumann boundary condition on $\partial\Omega$.

For (6), by the same argument as in [8], we can show the existence of stationary solutions with multi-spot pattern in one dimension that are uniformly bounded in $\varepsilon > 0$. On the other hand, it is easy to show that (7) has stationary solutions with multi-spot pattern in one dimension (see [16]). Additionally, as seen in the next proposition, it is known that there exists a stationary solution (A, H) in (7) with m spots for $m \geq 1$ and $n \geq 2$.

Proposition 1 ([7]). *Assume that $1 < p < \infty$ for $n = 2$, and $1 < p < (n+2)/(n-2)$ for $n \geq 3$. Let $w = w(y)$ be a positive radially symmetric solution of*

$$\begin{cases} \Delta w - w + w^p = 0, & y \in \mathbb{R}^n, \\ \qquad\qquad w \to 0, & |y| \to \infty. \end{cases}$$

If $\varepsilon > 0$ is sufficiently small, there is a stationary solution (A, H) in (7) such that (A, H) has an asymptotic behavior

$$A(x) \sim \varepsilon^{-\frac{qn}{qr-(p-1)(s+1)}} \sum_{i=1}^{m} S\left(\frac{x-h_i}{\varepsilon}\right), \quad v \sim \varepsilon^{-\frac{(p-1)n}{qr-(p-1)(s+1)}} \zeta$$

as $\varepsilon \to 0$, where $h_i \in \partial\Omega$ for $i = 1, \dots, m_1$, $h_i \in \Omega$ for $i = m_1 + 1, \dots, m_1 + m_2$, m_1, m_2 are nonnegative integers with $m_1 + m_2 = m \geq 1$, and the function S and the constant ζ are given by

$$S(y) \equiv \zeta^{\frac{q}{p-1}} w(y), \quad \zeta \equiv \left(\frac{m_1 + 2m_2}{2} \int_{\mathbb{R}^n} w^r \mathrm{d}y\right)^{-\frac{p-1}{qr-(p-1)(s+1)}}.$$

This proposition states that the function A is localized at $x = h_i$, and (A, H) goes to ∞ as $\varepsilon \to 0$. To avoid the singularities of A and H, we rescale the variables with respect to ε and replace (A, H) into new variables (u, v) by $u = \varepsilon^{qn/(qr-(p-1)(s+1))} A$ and $v = \varepsilon^{(p-1)n/(qr-(p-1)(s+1))} H$. Then (u, v) satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u - u + \dfrac{u^p}{v^q}, & x \in \Omega,\ t > 0, \\ \tau \dfrac{\mathrm{d}v}{\mathrm{d}t} = -v + \dfrac{1}{\varepsilon^n} \displaystyle\int_\Omega \dfrac{u^r}{v^s} \mathrm{d}x, & t > 0, \end{cases} \tag{8}$$

for which Proposition 1 gives a bounded stationary solution.

Here we note that the scaling factor $1/\varepsilon^n$ appears naturally in the second equation of the system above due to the change of the variables from (A, H) to (u, v). On the other hand, when we consider the instability of a bounded stationary solution with multiple spots in (6), we do not need to rewrite (6) into a new system like (8), that is, we may treat (1) with $d = 0$. Thus, the two examples motivate us to consider the system (1) with the factor $1/\varepsilon^d$ for $0 \leq d \leq n$ instead of (5).

Let us compare our results and known facts. It is shown in many papers that the shadow system can generate stable spatial patterns. On the other hand, since the shadow system is a simplified system, it is revealed in some papers that stable solutions must be structurally simple in certain sense. In fact, Ni-Poláčik-Yanagida [13] treated general shadow systems in a 1-dimensional bounded interval, and proved that any non-monotone steady state must be linearly unstable. Since their analysis was based on a symmetric property of steady states and the Strum-Liouville theorem, the results applies only to the 1-dimensional case. Later, Miyamoto [12] considered the case where the domain is a two-dimensional ball, and showed that any stable stationary solution must have at most two critical points at the boundary of the domain. In these two papers, the diffusion constant ε^2 is not necessarily small, and the asymptotic behavior of the stationary solutions as $\varepsilon \to 0$ is not assumed. Thus, these results are rather general, but the domain is strictly restricted. In our paper, although we must assume that ε is sufficiently small and the stationary solution satisfies some extra conditions, we do not impose any extra conditions on the domain Ω to show the instability of multi-spot solutions.

The instability of multi-spot solutions in some specific reaction-diffusion systems was shown by Wei-Winter [20, 21, 19], Doelman-Gardner-Kaper [4], and others. In these papers, the authors mainly focused on the stability analysis, and did not pay much attention to the number of eigenvalues or the properties of the corresponding eigenfunctions. As described in the previous paragraph, one of our goals is to find *unstable eigenvalues*. (If an eigenvalue has a positive real part, we call it an unstable

eigenvalue.) However the existence of unstable eigenvalues is not sufficient to predict the behavior in time of solutions of (5) under small perturbation to the multi-spot pattern. Our second aim is to study the asymptotic behavior of unstable eigenpairs as $\varepsilon \to 0$, by which we mean a pair of an unstable eigenvalue and its associated eigenfunction.

Stationary solutions with multiple spots may have two or more unstable eigenvalues, and these eigenvalues are close to each other if ε is sufficiently small because of the similarity of each spots. In this case, since it is difficult to obtain the profile of each eigenfunction, we construct an invariant subspace which consists of eigenfunctions associated with unstable eigenvalues. In the proof, we first define a distance between two subspaces of $L^2_\varepsilon(\Omega)\times\mathbb{C}$ with a finite dimension N (The functional space $L^2_\varepsilon(\Omega)$ will be defined in Section 2.). Using this distance, we construct a sequence of subspaces that converges to an invariant space spanned by eigenfunctions.

Recently, some interesting results on the profile of eigenfunctions were obtained by Wakasa-Yotsutani [17]. They considered an eigenvalue problem associated with a scalar reaction-diffusion equation in a one-dimensional bounded interval with a specific nonlinearity. If a diffusion coefficient ε is small, (4) has stationary solutions with m layers. Those stationary solutions are expected to have m eigenvalues close to 0. In [17], it was shown that the eigenfunctions are described by some *cosine rule*, which characterizes the amplitude of their peaks. Although the profiles in our results (see (11) or (12)) are less explicit than those in [17], our results hold true for a wider class of systems.

In Section 5, we apply our theorems to the two specific reaction-diffusion systems (6) and (7). We emphasize here that our result can be applied to the Gray-Scott model [18], the Gierer-Meinhardt system with saturation effect [10], and so on, that are more or less related to multi-spot patterns. If the readers are interested in these models, see these papers and references cited therein.

This paper is organized as follows. In Section 2, we state our results (Theorems 2.1, 2.2, and 2.3) in a mathematically rigorous manner. In addition we describe several assumptions and notation used throughout this paper. In Section 3, we construct a sequence of subspaces in $L^2_\varepsilon(\Omega)\times\mathbb{C}$ and prove that this sequence converges to an invariant subspace of a linearized operator. Finally we complete the proofs of our theorems by using a key lemma (Lemma 3.2) given in Section 4. In Section 5, we treat two typical examples that satisfy all assumptions.

2. **Assumptions and theorems.**

Now we describe several assumptions imposed on a stationary solution and the nonlinear functions f, g_1, g_2. Roughly speaking, we assume that there is a stationary solution (u, v) of (1) with m spots such that (u, v) satisfies (2). More precisely, the asymptotic behavior (2) means that (u, v) satisfies the following assumptions (A1)–(A6):

- (A1): There exist a constant ζ and a positive radially symmetric solution $S = S(y)$ of (3) which decays exponentially as $|y| \to \infty$.
- (A2): $(u, v) = (u(x), v)$ is uniformly bounded in $x \in \overline{\Omega}$ and $\varepsilon > 0$.
- (A3): For nonnegative integers $m_1, m_2 \geq 0$ with $m = m_1 + m_2 \geq 1$, there exist boundary spots $h_i \in \partial\Omega$ for $i = 1, \ldots, m_1$, and interior spots $h_i \in \Omega$ for $i = m_1 + 1, \ldots, m_1 + m_2$ such that $\lim_{\varepsilon\to 0} h_i$ exists $(i = 1, \ldots, m)$, $h_i \in \partial\Omega$ $(i = 1, \ldots, m_1)$, and $\lim_{\varepsilon\to 0} h_i \in \Omega$ $(i = m_1 + 1, \ldots, m)$.
- (A4): $|h_i - h_j|/\varepsilon \to \infty$ as $\varepsilon \to 0$ for any $i \neq j$, where $|\cdot|$ represents the usual Euclidean norm.

(A5): In the case of $d = 0$, for any $R > 0$ independent of ε, $\lim_{\varepsilon\to 0} u(x) = 0$ uniformly in $x \in \overline{\Omega} \setminus \cup_{i=1}^m B_{\varepsilon R}(h_i)$, where $B_{\varepsilon R}(h_i)$ denotes an open ball centered at h_i with a radius εR. In the case of $0 < d \le n$, for any $R > 0$ independent of ε, $|u(x)| \le C\exp(-\alpha|x - h_i|/\varepsilon)$ in $x \in \overline{\Omega} \setminus \cup_{i=1}^m B_{\varepsilon R}(h_i)$ for some positive constants C, α independent of ε, R.

(A6): $u(\varepsilon y + h_i) \to S(y)$ uniformly in $y \in K$, $v \to \zeta$ as $\varepsilon \to 0$ for any compact subset $K \subset \Omega_{i,\varepsilon}$ independent of ε and any $i = 1, \dots, m$, where $\Omega_{i,\varepsilon}$ is defined by $\Omega_{i,\varepsilon} = \{y \in \mathbb{R}^n \mid \varepsilon y + h_i \in \overline{\Omega}\}$.

The assumption (A1) holds true under a general setting. For example, the authors in [1] and [2] give a sufficient condition for the existence of S with (A1). If the function $f(u, \zeta)$ satisfies the following conditions for some ζ, (A1) holds true for $n \ge 3$ (see Theorem 1 in [1]):

(A1-1): $f(\cdot, \zeta) : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0, \zeta) = 0$.

(A1-2): $-\infty < \liminf_{u\to+0} \dfrac{f(u,\zeta)}{u} \le \limsup_{u\to+0} \dfrac{f(u,\zeta)}{u} < 0$.

(A1-3): $-\infty \le \limsup_{u\to+\infty} \dfrac{f(u,\zeta)}{u^l} \le 0$, where $l = \dfrac{n+2}{n-2}$.

(A1-4): There exists $u^* > 0$ such that $\displaystyle\int_0^{u^*} f(u, \zeta)\mathrm{d}u > 0$.

Assumptions (A3), (A4) determine the asymptotic behavior of the positions of spots as $\varepsilon \to 0$. The interior and boundary spots are definitely distinguished by (A3). Each spot is supposed to be separated from each other in the sense of (A4). However, some spots may be located relatively closely in some cases such as $h_i - h_j = O(\varepsilon|\log\varepsilon|)$. Assumptions (A5), (A6) imply that the profile of the solution exhibits a localized pattern, and the stationary solution u is small in the outside of each spot. In addition, the function u can be characterized by S in a neighborhood of each spot. In (A5), we need the stronger condition in the case of $0 < d \le n$ than $d = 0$, because of the singularity with respect to ε in the second equation of (9).

In order to study the stability of the stationary solution (u, v), we consider the linearized eigenvalue problem of (1) represented by

$$\lambda \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \mathcal{L} \begin{pmatrix} \phi \\ \eta \end{pmatrix} \equiv \begin{pmatrix} L_\varepsilon \phi + f_v(u,v)\eta \\ g_1'(v)\eta + \dfrac{1}{\varepsilon^d}\displaystyle\int_\Omega g_{2u}(u,v)\phi \mathrm{d}x + \dfrac{\eta}{\varepsilon^d}\displaystyle\int_\Omega g_{2v}(u,v)\mathrm{d}x \end{pmatrix}, \tag{9}$$

where $L_\varepsilon := \varepsilon^2\Delta + f_u(u, v)$.

The nonlinear functions f, g_1, g_2 are supposed to satisfy the following assumptions in a neighborhood of ζ denoted by $N \subset \mathbb{R}$.

(A7): f, g_2 are defined in $\mathbb{R} \times N$, $g_1$is defined in N. In addition, f, g_1, g_2 are continuously differentiable, and the derivatives $f_u, f_v, g_1', g_{2u}, g_{2v}$ are Hölder continuous.

(A8): $f_u(0, \zeta) < 0$, $g_{2v}(0, v) \equiv 0$ for $v \in N$.

Since the stationary solution (u, v) converges to (S, ξ), $L_\varepsilon \to L_0 \equiv \Delta + f_u(S, \zeta)$ as $\varepsilon \to 0$ in some sense by the stretched coordinate $y = x/\varepsilon$. We hypothesize that some eigenpairs in the eigenvalue problem given by

$$\lambda\psi = L_0\psi \quad \text{in } \mathbb{R}^n \tag{10}$$

satisfy the following assumption (A9).

(A9): $\lambda = \mu > 0$ and $\lambda = 0$ are eigenvalues of (10). For small $\gamma > 0$, there is no essential spectrum of (10) in $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \geq -\gamma\}$. The eigenfunction ψ corresponding to μ is uniquely defined, positive and radially symmetric, while the eigenspace associated with the zero eigenvalue consists of $S_{y_j} = \partial S/\partial y_j$ for $j = 1, \ldots, n$, where the spatial variable y is represented by $y = (y_1, \ldots, y_n)$.

We normalize ψ by $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$ and $\psi_j = S_{y_j}/\|S_{y_j}\|_{L^2(\mathbb{R}^n)}$ for $j = 1, \ldots, n$. In some case, L_0 satisfies (A9) (see [14] or Proposition 2 in Section 5). Similarly, we can easily construct exactly $n-1$ eigenfunctions of $L_0\psi = 0$ in a half space $\mathbb{R}^n_{\nu_i} = \{y \in \mathbb{R}^n \mid y \cdot \nu_i < 0\}$ by linear combinations of S_{y_j} which satisfy the homogeneous Neumann boundary condition on $\mathbb{R}_{\nu_i}$. Here ν_i is the outer normal vector on $\partial\Omega$ at h_i. We denote the $n-1$ eigenfunctions by ψ_{ij} normalized as $\|\psi_{ij}\|_{L^2(\mathbb{R}^n_{\nu_i})} = 1$ for $i = 1, \ldots, m_1$ and $j = 1, \ldots, n-1$. Note that since L_0 is self-adjoint, the assumption (A9) implies

$$\|(\kappa + \mu - L_0)^{-1}\| \leq \frac{C}{\kappa}, \quad \|(\kappa - L_0)^{-1}\| \leq \frac{C}{\kappa},$$

for arbitrarily small $\kappa > 0$ and a constant $C > 0$ independent of κ, where a usual operator norm is denoted by $\|\cdot\|$. The invertible operator for L_0 is supposed to be acting on $L^2(\mathbb{R}^n)$.

Finally we need the following critical conditions for $\mathcal{L}$. The invertible operator for $\mathcal{L}$ is assumed to be acting on $L^2_\varepsilon(\Omega) \times \mathbb{C}$.

(A10): In the case of $d = 0$, $(\mu - f_u(0,\zeta))(\mu - g_1'(\zeta)) - f_v(0,\zeta)g_{2u}(0,\zeta) \neq 0$ and $f_u(0,\zeta)g_1'(\zeta) - f_v(0,\zeta)g_{2u}(0,\zeta) \neq 0$. In the case of $0 < d \leq n$, $f_v(0,v) \equiv 0$, $g_{2u}(0,v) \equiv 0$ for $v \in N$, $\int_{\mathbb{R}^n} f_v(S,\zeta)\psi dy \neq 0$ and $\int_{\mathbb{R}^n} g_{2u}(S,\zeta)\psi dy \neq 0$.

(A11): There is a constant $C > 0$ independent of ε such that for arbitrarily small $\kappa > 0$,

$$\|(\kappa + \mu - \mathcal{L})^{-1}\| \leq \frac{C}{\kappa}, \quad \|(\kappa - \mathcal{L})^{-1}\| \leq \frac{C}{\kappa}.$$

The assumptions (A10) is nothing but a nondegeneracy and technical condition. We need this condition in Section 4 to prove several lemmas. The assumption (A11) is also a nondegeneracy condition associated with the algebraic multiplicity of the eigenvalues close to μ and 0. For stationary solutions with multiple spots, there are many eigenvalues close to μ and 0. Hence the algebraic multiplicity of each eigenvalues of $\mathcal{L}$ may be larger than 1. (see [9, p.181] for the definition of algebraic multiplicity of eigenvalues.). In that case, the analysis becomes more difficult, which we do not treat in this paper. The condition (A11) may hold true if the stationary solution (u, v) has a single-spot pattern. It is easy to verify that the FitzHugh-Nagumo system and the Gierer-Meinhardt system satisfy these conditions. Therefore we believe that the linearized operator $\mathcal{L}$ generically satisfies them.

Our first result is focused on the case of $0 \leq d < n$. In this case, we shall construct m eigenpairs (λ, ϕ, η) which satisfy

$$\left\|\phi - \sum_{i=1}^{m} c_i\psi_i\right\|_{L^2_\varepsilon} \to 0, \quad \eta \to 0, \quad \lambda \to \mu \quad \text{as } \varepsilon \to 0, \tag{11}$$

where $c_1, \dots, c_m$ are complex-valued constants and the functions ψ_i are defined by

$$\psi_i(x) = \begin{cases} \dfrac{1}{\varepsilon^{(n-d)/2}} 2\psi\left(\dfrac{x-h_i}{\varepsilon}\right), & i = 1, \dots, m_1, \\ \dfrac{1}{\varepsilon^{(n-d)/2}} \psi\left(\dfrac{x-h_i}{\varepsilon}\right), & i = m_1+1, \dots, m_1+m_2. \end{cases}$$

We note that $\psi_i(x)$ is normalized in a scaled Lebesgue space

$$L^2_\varepsilon(\Omega) = \{\varphi \in L^2(\Omega) \mid \|\varphi\|^2_{L^2_\varepsilon} \equiv \frac{1}{\varepsilon^d} \int_\Omega |\varphi|^2 \mathrm{d}x < \infty\}.$$

Theorem 2.1. *Fix $m \geq 1$ arbitrarily and assume $0 \leq d < n$. Under assumptions* (A1)–(A11)*, if ε is sufficiently small, there are exactly m eigenpairs of* (9) *satisfying* (11)*.*

In the case of $d = n$, we also find eigenpairs satisfying (11). However the total number of eigenpairs is different. The constants c_i should be constrained by the condition $\sum_{i=1}^m c_i = 0$ so that we obtain only $m-1$ eigenpairs. Note that the coefficient vector $(c_1, \dots, c_m)$ with $\sum_{i=1}^m c_i = 0$ generates an $(m-1)$-dimensional subspace in $\mathbb{C}^m$.

Theorem 2.2. *Fix $m \geq 2$ arbitrarily and assume $d = n$. Under assumptions* (A1)–(A11)*, if ε is sufficiently small, there are exactly $m-1$ eigenpairs of* (9) *satisfying* (11)*.*

Theorems 2.1, 2.2 state that any stationary solution (u, v) with multiple spots has at least one eigenvalue with a positive real part. Then the stationary solution (u, v) must be unstable in (1). Therefore these theorems show the instability of any multi-spot solution and clarify the behavior of the associated eigenpairs as $\varepsilon \to 0$ in shadow systems in a general setting.

As pointed out before the statement of Theorem 2.2, the total number of the eigenvalues around μ is different between the cases of $0 \leq d < n$ and $d = n$. This difference arises from the singularities with respect to ε in the integral terms of (9). In fact, for $d = n$, the singularities crucially influence the eigenvalue problem and decrease the total number of eigenpairs around $\lambda = \mu$.

Since L_0 has a zero eigenvalue, it is easily seen that $\mathcal{L}$ has an eigenvalue close to zero. We shall construct eigenpairs (λ, ϕ, η) which satisfies as $\varepsilon \to 0$,

$$\left\| \phi - \sum_{i=1}^{m_1} \sum_{j=1}^{n-1} c_{ij} \hat{\psi}_{ij} - \sum_{i=m_1+1}^{m} \sum_{j=1}^{n} c_{ij} \hat{\psi}_{ij} \right\|_{L^2_\varepsilon} \to 0, \quad \eta \to 0, \quad \lambda \to 0, \tag{12}$$

where c_{ij} is a complex-valued constant, and

$$\hat{\psi}_{ij}(x) := \begin{cases} \dfrac{1}{\varepsilon^{(n-d)/2}} \psi_{ij}\left(\dfrac{x-h_i}{\varepsilon}\right), & i = 1, \dots, m_1, \quad j = 1, \dots, n-1, \\ \dfrac{1}{\varepsilon^{(n-d)/2}} \psi_{ij}\left(\dfrac{x-h_i}{\varepsilon}\right), & i = m_1+1, \dots, m_1+m_2, \quad j = 1, \dots, n. \end{cases}$$

As seen in (12), the eigenvalues are close to 0 but may not be equal to 0. Hence some of them may be unstable eigenvalues for $\varepsilon > 0$. Actually, it was shown in [3] that if spot pattern exists inside the domain, it cannot be stable in the shadow system of (7) and moves towards a point of the boundary. This behavior of the solution arises from the existence of unstable eigenvalues close to 0. Therefore it is important to consider eigenvalues near 0 in (9).

Theorem 2.3. *Fix $m \geq 1$. Under assumptions* (A1)–(A11)*, if ε is sufficiently small, there are exactly $m_1(n-1)+m_2 n$ eigenpairs of* (9) *satisfying* (12)*.*

Theorem 2.3 can be derived from the same argument as in Theorems 2.1 and 2.2 so that we omit the details of the proof.

Next, we introduce some notation. We treat the eigenvalue problem (9) in the Hilbert space $H^2_{N,\varepsilon}(\Omega)\times\mathbb{C}$ defined by

$$H^2_N(\Omega) = \{\varphi \in H^2(\Omega) \mid \partial\phi/\partial\nu = 0 \text{ on } \partial\Omega\},$$
$$H^2_{N,\varepsilon}(\Omega) = \{\varphi \in H^2_N(\Omega) \mid \|\varphi\|^2_{H^2_\varepsilon} \equiv \|\varphi\|^2_{L^2_\varepsilon} + \varepsilon^2\|\nabla\varphi\|^2_{L^2_\varepsilon} + \varepsilon^4\|\nabla^2\varphi\|^2_{L^2_\varepsilon} < \infty\},$$

where $\nabla\varphi$ and $\nabla^2\varphi$ represent the gradient and the Hessian matrix of φ, respectively, as usual. Throughout this paper, all functions and eigenvalues are mainly supposed to be complex-valued. We define an inner product on $L^2_\varepsilon(\Omega)$ by

$$\langle\varphi_1,\varphi_2\rangle_{L^2_\varepsilon} = \frac{1}{\varepsilon^d}\int_\Omega \varphi_1\overline{\varphi}_2 \mathrm{d}x$$

for $\varphi_1,\varphi_2 \in L^2_\varepsilon(\Omega)$, where $\overline{z}$ represents the complex conjugate of $z\in\mathbb{C}$. This inner product obviously induces the norm on $L^2_\varepsilon(\Omega)$ defined as above. A Hilbert space $L^2_\varepsilon(\Omega)\times\mathbb{C}$ is equipped with the inner product and norms naturally induced by $L^2_\varepsilon(\Omega)$ and $\mathbb{C}$. We denote them by $\langle\cdot,\cdot\rangle_{L^2_\varepsilon\times\mathbb{C}}$ and $\|\cdot\|_{L^2_\varepsilon\times\mathbb{C}}$, respectively. We also define the inner product and norm of $H^2_{N,\varepsilon}(\Omega)\times\mathbb{C}$, represented by similar notation. Note that if $d=0$, $L^2_\varepsilon(\Omega)$ is identical to the usual Lebesgue space $L^2(\Omega)$.

3. Proofs of Theorems 2.1 and 2.2.

Stationary solutions with multiple spots may have one or more unstable eigenvalues, and these eigenvalues are close to each other if ε is sufficiently small because of the similarity of each spots. In this case, since it is difficult to obtain the profile of each eigenfunction, we construct an invariant subspace which consists of eigenfunctions associated with unstable eigenvalues. To obtain an invariant subspace, we construct a sequence of subspaces that converges to an invariant space spanned by eigenfunctions. To this end, we introduce a distance between two subspaces of $L^2_\varepsilon(\Omega)\times\mathbb{C}$ with a finite dimension N as follows. Let Λ be a set of subspaces given by

$$\Lambda = \{K \mid K \text{ is a subspace in } L^2_\varepsilon(\Omega)\times\mathbb{C} \text{ with } \dim K = N\},$$

and define a distance on Λ by

$$\mathrm{dist}(K^1,K^2) := \|P_{K^1} - P_{K^2}\| = \sup_{\|\varphi\|_{L^2_\varepsilon\times\mathbb{C}}=1} \|P_{K^1}\varphi - P_{K^2}\varphi\|_{L^2_\varepsilon\times\mathbb{C}}$$

for $K^1,K^2\in\Lambda$, where P_{K^j} is the usual projection map from $L^2_\varepsilon(\Omega)\times\mathbb{C}$ to K^j, $\|\cdot\|$ is the usual norm of operators. It is easy to see that Λ is a complete metric space induced by the distance d. Moreover, if $d(K^1,K^2)\leq\delta$, then $\phi_1 - P_{K^2}\phi_1$ is orthogonal to K^2 and

$$\|\phi_1 - P_{K_2}\phi_1\|_{L^2_\varepsilon\times\mathbb{C}} \leq \delta\|\phi_1\|_{L^2_\varepsilon\times\mathbb{C}}$$

for any $\phi_1\in K^1$.

We construct a convergent sequence of subspaces as follows. As stated in the next lemma, the convergence of subspaces is equivalent to that of their orthonormal bases.

Lemma 3.1. *Let K^l and K^∞ be subspaces in $L^2_\varepsilon(\Omega)\times\mathbb{C}$ with a finite dimension N, and denote orthonormal bases of K^l and K^∞ by $\{\phi_i^l\}_{i=1}^N$ and $\{\phi_i^\infty\}_{i=1}^N$, respectively. If ϕ_i^l converges to ϕ_i^∞ strongly in $L^2_\varepsilon(\Omega)\times\mathbb{C}$ as $l\to\infty$ for each $i=1,\dots,N$, then dist$(K^l,K^\infty)\to 0$ as $l\to\infty$. Conversely, if dist$(K^l,K^\infty)\to 0$ as $l\to\infty$, then there are orthonormal bases $\{\phi_i^l\}_{i=1}^N$ and $\{\phi_i^\infty\}_{i=1}^N$ of K^l and K^∞ such that ϕ_i^l converges to ϕ_i^∞ strongly in $L^2_\varepsilon(\Omega)\times\mathbb{C}$ as $l\to\infty$ for each $i=1,\dots,N$.*

Let K be an N-dimensional subspace and $\{\phi_i\}_{i=1}^N$ an orthonormal set. We explicitly represent P_K by this orthonormal set $\{\phi_i\}_{i=1}^N$ as

$$P_K\varphi=\sum_{i=1}^N\langle\varphi,\phi_i\rangle_{L^2_\varepsilon\times\mathbb{C}}\,\phi_i.$$

Using this expression, we easily prove Lemma 3.1 and thereby omit details.

Put $T\equiv(\mu+\kappa-\mathcal{L})^{-1}$. Due to the boundedness of the domain Ω, it is easy to see the compactness of T on $L^2_\varepsilon(\Omega)\times\mathbb{C}$. In order to construct suitable eigenfunctions in Theorems 2.1, 2.2, we obtain an invariant space for T close to a subspace $K^0_\varepsilon\subset L^2_\varepsilon(\Omega)\times\mathbb{C}$, where K^0_ε is given by

$$\left\{\sum_{i=1}^m c_i\begin{pmatrix}\psi_i\\0\end{pmatrix}\;\middle|\;c_i\in\mathbb{C}\right\}\qquad\text{if } 0\le d<n,$$
$$\left\{\sum_{i=1}^m c_i\begin{pmatrix}\psi_i\\0\end{pmatrix}\;\middle|\;\sum_{i=1}^m c_i=0,\ c_i\in\mathbb{C}\right\}\quad\text{if } d=n.$$

By setting $N=m$ if $0\le d<n$ and $N=m-1$ if $d=n$, the dimension of K^0_ε is N in both cases. For a set of linearly independent vectors $\{V_i\}_{i=1}^N$ and an N-dimensional subspace $K=\{\sum_{i=1}^N c_iV_i\mid c_i\in\mathbb{C}\}$, we define TK by

$$TK=\left\{\sum_{i=1}^N c_iTV_i\;\middle|\;c_i\in\mathbb{C}\right\}$$

if $\{TV_i\}_{i=1}^N$ is a set of linearly independent vectors. By this notation, we define a sequence of N-dimensional subspaces by $K^l=T^lK^0_\varepsilon$ for $l=1,2,\dots$. We claim that K^l stays near K^0_ε and converges to an N-dimensional subspace K^∞ with respect to the distance d.

Lemma 3.2. *Let $\delta,\kappa>0$ be arbitrarily small constants independent of $\varepsilon>0$ with $\delta<\kappa^2<1$. Suppose that $V\in L^2_\varepsilon(\Omega)\times\mathbb{C}$ satisfies $\|V\|_{L^2_\varepsilon\times\mathbb{C}}=1$, and there is $V_0\in K^0_\varepsilon$ such that $V-V_0$ is orthogonal to K^0_ε and $\|V-V_0\|_{L^2_\varepsilon\times\mathbb{C}}\le\delta$. Then there is a positive constant C independent of $\varepsilon,\kappa,\delta,V$ such that $\|TV-V_0/\kappa\|_{L^2_\varepsilon\times\mathbb{C}}\le C\kappa^2$.*

Remark 1. Let K be an N-dimensional subspace spanned by an orthonormal set $\{V_i\}_{i=1}^N$. Under the assumption of Lemma 3.2, if dist$(K,K^0_\varepsilon)\le\delta$, then there are positive constants C_1,C_2 independent of $\delta,\kappa,\varepsilon$ and K such that

$$\frac{C_1}{\kappa}\le\|TV_i\|_{L^2_\varepsilon\times\mathbb{C}}\le\frac{C_2}{\kappa}.$$

Moreover, if $\kappa^5<\delta^2<\kappa^4$, TK is an N-dimensional subspace of $L^2_\varepsilon(\Omega)\times\mathbb{C}$ such that dist$(TK,K^0_\varepsilon)\le\delta$.

Lemma 3.2 is crucial for a proof of Theorems 2.1 and 2.2. A proof of Lemma 3.2 shall be given in the next section. Using Lemma 3.2 and the compactness of T,

we are able to construct an invariant subspace K^∞ of T, from which we can easily obtain N eigenpairs of $\mathcal{L}$.

Proof of Theorems 2.1 and 2.2. Let κ, δ be small positive constants independent of ε such as $\kappa^5 < \delta^2 < \kappa^4$. In this proof, the constant C represents the generic constant independent of $\varepsilon, \kappa, \delta$ if there is no description. First we construct an invariant subspace K^∞ of T close to K_ε^0. From Remark 1, K^l is an N-dimensional subspace and $\mathrm{dist}(K^l, K_0^\varepsilon) \le \delta$ for any l. Let $\{V_i^l\}_{k=1}^N$ be an orthonormal set of K^l. Then there is $V_i \in L_\varepsilon^2(\Omega) \times \mathbb{C}$ such that $V_i^l \to V_i$ weakly in $L_\varepsilon^2(\Omega) \times \mathbb{C}$ as $l \to \infty$ for each $i = 1, \dots, N$ because of $\|V_i^l\|_{L_\varepsilon^2 \times \mathbb{C}} = 1$. Here we may take a subsequence of l if needed, and use the same notation. From Lemma 3.2, $\{V_i\}_{i=1}^N$ spans an N-dimensional subspace, denoted by K^∞. Since T is compact, TV_i^l converges to TV_i strongly in $L_\varepsilon^2(\Omega) \times \mathbb{C}$ as $l \to \infty$. Since V_i^l can be represented as a linear combination of $\{TV_i^{l-1}\}_{i=1}^N$, V_i^l converges to V_i strongly in $L_\varepsilon^2(\Omega) \times \mathbb{C}$. By Lemma 3.1, $K^l \to K^\infty$ as $l \to \infty$. It is clear that $TK^\infty = K^\infty$ and $\mathrm{dist}(K^\infty, K_0^\varepsilon) \le \delta$.

Secondly we see that K^∞ consists of N eigenpairs with (11). Suppose that $\{V_i\}_{i=1}^N$ is an orthonormal pair of K^∞. Thanks to $\mathrm{dist}(K^\infty, K_\varepsilon^0) \le \delta$, there is a pair of constants $\{c_{ij}\}_{j=1}^m$ for each $i = 1, \dots, N$ such that $\|V_i - \sum_{j=1}^m c_{ij}{}^t(\psi_j, 0)\|_{L_\varepsilon^2 \times \mathbb{C}} \le \delta$. Then,

$$\|TV_i - \frac{1}{\kappa}\sum_{j=1}^m c_{ij}{}^t(\psi_j, 0)\|_{L_\varepsilon^2 \times \mathbb{C}} \le C\kappa^2 \tag{13}$$

due to Lemma 3.2. On the other hand, since K^∞ is an invariant subspace of T, there are constants a_{ij} for $i, j = 1, \dots, N$ such that

$$TV_i = \sum_{j=1}^N a_{ji} V_j. \tag{14}$$

We define an $N \times N$ matrix by $\tilde{T} = (a_{ij})_{i,j=1,\dots,N}$. From (13), (14), we see that $|a_{ij} - \delta_{ij}/\kappa| \le C\kappa$, where δ_{ij} is the Kronecker's delta. It is clear that there is an eigenpair of $\tilde{T}$ at least, denoted by $(\lambda, \mathbf{a})$, and any eigenvalue of $\tilde{T}$ must be close to $1/\kappa$. Put $\mathbf{a} = {}^t(\alpha_1, \dots, \alpha_N)$. Setting $V = \sum_{j=1}^N \alpha_i V_i$, we have

$$TV = \sum_{i=1}^N \alpha_i TV_i = \sum_{j=1}^N \sum_{i=1}^N \alpha_i a_{ji} V_j = \lambda \sum_{i=1}^N \alpha_i V_i = \lambda V.$$

Hence V is an eigenfunction of T. Since each eigenfunction can be represented by a linear combination of $\{V_i\}_{i=1}^N$ and the parameter $\kappa > 0$ is arbitrary, we obtain (11).

Thirdly we show that the matrix T has exactly N eigenpairs, or equivalently $\tilde{T}$ has exactly N eigenpairs. Suppose that there is $(\lambda, \mathbf{a}, \mathbf{b})$ such that $\lambda \in \mathbb{C}$ is an eigenvalue of $\tilde{T}$ which converges to $1/\kappa$ as $\varepsilon \to 0$, $\tilde{T}\mathbf{a} = \lambda\mathbf{a}$, and $(\lambda I - \tilde{T})\mathbf{b} = \mathbf{a}$, where I is the identity matrix on $\mathbb{C}^N$. Set $\mathbf{a} = {}^t(\alpha_1, \dots, \alpha_N)$ and $\mathbf{b} = {}^t(\beta_1, \dots, \beta_N)$. By putting $W = \sum_{i=1}^N \beta_i V_i$ and $V = \sum_{i=1}^N \alpha_i V_i$, it follows that $(\lambda I - T)W = V$. Multiplying $(\kappa + \mu - \mathcal{L})$ to both sides and using $(\kappa + \mu - \mathcal{L})V = V/\lambda$, we see that

$$\left(\kappa + \mu - \frac{1}{\lambda} - \mathcal{L}\right) W = \frac{1}{\lambda^2} V.$$

However it contradicts (A11). Therefore $\tilde{T}$, or equivalently $\mathcal{L}$ does have exactly N eigenpairs. □

4. **Proof of Lemma 3.2.** In this section, we prove Lemma 3.2. We study the asymptotic behavior of TV as $\varepsilon \to 0$ by dividing the domain into two regions, called *outer and inner regions*.

Set $V = {}^t(\sum_{i=1}^m c_i\psi_i + \psi^\perp, \Xi)$, where ${}^t(\sum_{i=1}^m c_i\psi_i, 0) \in K_\varepsilon^0$ and ${}^t(\psi^\perp, \Xi) \in L_\varepsilon^2(\Omega) \times \mathbb{C}$ is orthogonal to K_ε^0. The constants Ξ, c_i, and the function $\psi^\perp$ may depend on ε although we do not write it explicitly. From the assumption of Lemma 3.2, it holds that $\|\psi^\perp\|_{L_\varepsilon^2} \le \delta$ and $|\Xi| \le \delta$. Then $1 - 2\delta \le \|\sum_{i=1}^m c_i\psi_i\|_{L_\varepsilon^2} \le 1 + 2\delta$ so that

$$C_1 \le \sum_{i=1}^m c_i^2 \le C_2$$

for positive constants C_1, C_2 independent of δ, ε.

Put

$$\begin{pmatrix} \phi_\varepsilon \\ \eta_\varepsilon \end{pmatrix} = TV - \frac{1}{\kappa}\sum_{i=1}^m c_i \begin{pmatrix} \psi_i \\ 0 \end{pmatrix}.$$

Then ${}^t(\phi_\varepsilon, \eta_\varepsilon)$ satisfies

$$\begin{pmatrix} (\kappa + \mu - L_\varepsilon)\phi_\varepsilon - f_v(u,v)\eta_\varepsilon \\ (\kappa + \mu - g_1'(v))\eta_\varepsilon - \dfrac{1}{\varepsilon^d}\displaystyle\int_\Omega g_{2u}(u,v)\phi_\varepsilon \mathrm{d}x - \dfrac{\eta_\varepsilon}{\varepsilon^d}\displaystyle\int_\Omega g_{2v}(u,v)\mathrm{d}x \end{pmatrix} = \begin{pmatrix} \psi^\perp \\ \Xi \end{pmatrix} + o(1) \tag{15}$$

as $\varepsilon \to 0$. From the assumption (A11) and $\delta < \kappa^2$,

$$\|\phi_\varepsilon\|_{H_\varepsilon^2} \le C_1\kappa, \quad |\eta_\varepsilon| \le C_1\kappa \tag{16}$$

for a constant $C_1 > 0$ independent of $\varepsilon, \delta, \kappa$. Let χ be a smooth cut-off function satisfying $0 \le \chi \le 1$ and

$$\chi(r) = \begin{cases} 1, & 0 \le r \le \dfrac{1}{2}, \\ 0, & r \ge 1. \end{cases}$$

Set

$$\phi_{\varepsilon,i}(x) = \chi_{\varepsilon,i}(x + h_i)\phi_\varepsilon(x + h_i), \quad \psi_{\varepsilon,i}^\perp(x) = \chi_{\varepsilon,i}(x + h_i)\psi^\perp(x + h_i)$$

for $i = 1, \dots, m$, and

$$\phi_{\varepsilon,0}(x) = \Big(1 - \sum_{i=1}^m \chi_{\varepsilon,i}(x)\Big)\phi_\varepsilon(x), \quad \psi_{\varepsilon,0}^\perp(x) = \Big(1 - \sum_{i=1}^m \chi_{\varepsilon,i}(x)\Big)\psi^\perp(x),$$

where $\chi_{\varepsilon,i}(x) = \chi(|x - h_i|/R\varepsilon)$ for $i = 1, \dots, m$ for R sufficiently large and given independently of ε. Note that $\phi_\varepsilon(x) = \phi_{\varepsilon,0}(x) + \sum_{i=1}^m \phi_{\varepsilon,i}(x - h_i)$ and $\psi^\perp(x) = \psi_{\varepsilon,0}^\perp(x) + \sum_{i=1}^m \psi_{\varepsilon,i}^\perp(x - h_i)$. It follows that

$$|\nabla\chi_{\varepsilon,i}| \le \frac{C}{\varepsilon R}, \quad |\nabla^2\chi_{\varepsilon,i}| \le \frac{C}{\varepsilon^2 R^2},$$

for $i = 1, \dots, m$. (Throughout this section, the constant C represents the generic constant independent of $\varepsilon, R, \kappa, \delta$ if there is no description.)

We estimate $\phi_{\varepsilon,i}$ and study the asymptotic behavior of $\phi_{\varepsilon,i}$ and η_ε as $\varepsilon \to 0$ for each $i = 0, \dots, m$. In the following, we only consider the case of $h_1 \in \partial\Omega$ and $h_i \in \Omega$ for $i = 2, \dots, m$. In addition, without loss of generality, we suppose that $h_1 = 0$ and the outer normal vector of $\partial\Omega$ at 0 is given by $(0, \dots, 0, -1)$. The proof for other cases can be obtained by the same argument as in this case. We first estimate $\phi_{\varepsilon,0}$.

Lemma 4.1. *Let δ, κ be small and fixed, and R be large and fixed. If ε is sufficiently small, then there exists a constant $C > 0$ independent of $\varepsilon, R, \delta, \kappa$ such that*

$$\varepsilon^2 \|\nabla \phi_{\varepsilon,0}\|^2_{L^2_\varepsilon} + (\kappa + \mu - f_u(0,\zeta))\|\phi_{\varepsilon,0}\|^2_{L^2_\varepsilon} \le C\delta^2.$$

Proof. Since u and ψ are small in the outside of a neighborhood of spots, it follows from simple calculations that

$$\begin{aligned} \varepsilon^2 \Delta \phi_{\varepsilon,0} - (\kappa + \mu - f_u(u,v))\phi_{\varepsilon,0} = &- \varepsilon^2 \sum_{i=1}^m (2\nabla \chi_{\varepsilon,i} \cdot \nabla \phi_\varepsilon + \Delta \chi_{\varepsilon,i} \phi_\varepsilon) \\ &- (1 - \sum_{i=1}^m \chi_{\varepsilon,i}) f_v(u,v)\eta_\varepsilon - \psi^\perp_{\varepsilon,0} + o(1). \end{aligned} \tag{17}$$

We multiply $\phi_{\varepsilon,0}$ to both sides of this equality and integrate it by parts. Then the integral over $\partial\Omega \cap B_{\varepsilon R}(0)$ naturally appears because $\phi_{\varepsilon,0}$ may not satisfy the homogeneous Neumann boundary condition on $\partial\Omega \cap B_{\varepsilon R}(0)$. By the similar argument to the proof of the Trace Theorem (see [6]), we readily see that

$$\|\varphi\|^2_{L^2(\partial\Omega)} \le C(\|\varphi\|^2_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}) \tag{18}$$

for $\varphi \in H^1(\Omega)$. Since $h_i \in \Omega$ for $i \ge 2$, we have

$$-\int_\Omega \Delta\phi_{\varepsilon,0}\phi_{\varepsilon,0}dx = \int_\Omega |\nabla\phi_{\varepsilon,0}|^2 dx + \int_{\partial\Omega\cap B_{\varepsilon R}(0)} \frac{\partial \chi_{\varepsilon,1}}{\partial \nu}(1-\chi_{\varepsilon,1})\phi_\varepsilon^2 dS_\sigma$$

and

$$\begin{aligned} \varepsilon^2 \left| \int_{\partial\Omega\cap B_{\varepsilon R}(0)} \frac{\partial \chi_{\varepsilon,1}}{\partial \nu}(1-\chi_{\varepsilon,1})\phi_\varepsilon^2 dS_\sigma \right| &\le \varepsilon^2 \int_{\partial\Omega\cap B_{\varepsilon R}(0)} |\nabla\chi_{\varepsilon,1}||\phi_\varepsilon|^2 dS_\sigma \\ &\le \frac{C\varepsilon}{R}(\|\phi_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla\phi_\varepsilon\|_{L^2(\Omega)}\|\phi_\varepsilon\|_{L^2(\Omega)}), \end{aligned}$$

where S_σ represents the $(n-1)$-dimensional surface measure. From (16), (17) and this estimate, we have

$$\varepsilon^2 \|\nabla\phi_{\varepsilon,0}\|^2_{L^2_\varepsilon} + (\mu + \kappa - f_u(0,\zeta))\|\phi_{\varepsilon,0}\|^2_{L^2_\varepsilon} \le C_1 \left(\frac{1}{R} + \delta \|\phi_{\varepsilon,0}\|_{L^2_\varepsilon} \right)$$

for a constant C_1 independent of ε, R. In the case of $d = 0$, we used $|\eta_\varepsilon| \le C\delta$, which will be shown in the next lemma. Since R is sufficiently large and ε is sufficiently small, we complete the proof. □

We only consider the case of $d = 0$ in the next lemma, which will be necessary in the proof of Lemma 4.3.

Lemma 4.2. *Let $d = 0$. For $(\psi, \xi) \in L^2(\Omega) \times \mathbb{C}$, there exists a constant $C > 0$ independent of $\varepsilon, \delta, \kappa$ such that the solution (ϕ, η) of $(\kappa + \mu - \mathcal{L})^t(\phi, \eta) = {}^t(\psi, \xi)$ in $H^2_N(\Omega) \times \mathbb{C}$ satisfies as $\varepsilon \to 0$,*

$$\left| \int_\Omega \phi dx \right| + |\eta| \le C \left(\left| \int_\Omega \psi dx \right| + |\xi| \right) + o(1). \tag{19}$$

Proof. The pair (ϕ, η) satisfies as $\varepsilon \to 0$

$$\begin{cases} (\kappa + \mu - f_u(0,\zeta)) \displaystyle\int_\Omega \phi dx - f_v(0,\zeta)\eta = \int_\Omega \psi dx + o(1), \\ -g_{2u}(0,\zeta) \displaystyle\int_\Omega \phi dx + (\kappa + \mu - g_1'(\zeta))\eta = \xi + o(1) \end{cases}$$

by the integration over Ω and Gauss's theorem. By the assumption (A10), we have (19) as $\varepsilon \to 0$. $\square$

Next we estimate $\phi_{\varepsilon,i}$. In the case of $0 \le d < n$, $\phi_{\varepsilon,i}(x - h_i)$ obviously tends to 0 as $\varepsilon \to 0$ in $L^2_\varepsilon(\Omega)$ because the domain of integration shrinks to a point. Hence we derive the next result.

Lemma 4.3. *Let $0 \le d < n$. If ε is small, then there is a constant $C > 0$ independent of $\varepsilon, R, \delta, \kappa$ such that $|\eta_0| \le C\delta$ and $\|\phi_\varepsilon\|_{L^2} \le C\delta$.*

Proof. By Lemma 4.1 and the argument above, we see $\|\phi_\varepsilon\|_{L^2_\varepsilon} \le C\delta$. Also, we have already proved $|\eta_\varepsilon| \le C\delta$ for $d = 0$ in Lemma 4.2. If $0 < d < n$, it is easy to show $|\eta_\varepsilon| \le C\delta$ by (A8), (A10), and the second equation of (15). $\square$

Next we consider the case of $d = n$. Let Θ be a neighborhood of the origin independent of ε and R, and let $\pi = \pi(x)$ be a diffeomorphism such that

$$\pi(\Omega \cap \Theta) \subset \mathbb{R}^n_+, \quad \pi(\partial\Omega \cap \Theta) \subset \partial\mathbb{R}^n_+, \quad \pi(0) = 0, \quad D\pi(0) = I,$$

where $\mathbb{R}^n_+ \equiv \{z \in \mathbb{R}^n \mid z_n > 0\}$, I is the identity matrix on $\mathbb{R}^n$ and $D\pi$ represents the Jacobian matrix of π. By the regularity of $\partial\Omega$, π belongs to C^2-class. Roughly speaking, the mapping π straightens out $\partial\Omega$ around the origin. We set $\tilde{\phi}_{\varepsilon,i}(y) = \phi_{\varepsilon,i}(\pi^{-1}(\varepsilon y))$ for $i = 1, \dots, m$ and study the asymptotic behavior of $\tilde{\phi}_{\varepsilon,i}$ as $\varepsilon \to 0$. In particular, it is important to obtain the asymptotic behavior of $\langle \phi_{\varepsilon,i}, \psi_i \rangle_{L^2_\varepsilon}$ as $\varepsilon \to 0$.

The function $\tilde{\phi}_{\varepsilon,1}$ satisfies

$$\begin{aligned}&(\kappa + \mu - \Delta - f_u(\tilde{u}, v))\tilde{\phi}_{\varepsilon,1} - \tilde{\chi}_{\varepsilon,1} f_v(\tilde{u}, v)\eta_\varepsilon - \tilde{\psi}^\perp_{\varepsilon,1} \\ &= 2\nabla\tilde{\chi}_{\varepsilon,1} \cdot \nabla\tilde{\phi}_\varepsilon + \Delta\tilde{\chi}_{\varepsilon,1}\tilde{\phi}_\varepsilon + o(1),\end{aligned}$$

where we denote a stretched function for $\varphi(x)$ by $\tilde{\varphi}(y) = \varphi(\pi^{-1}(\varepsilon y))$ in the same way as $\tilde{\phi}_{\varepsilon,i}$ and $\tilde{\psi}^\perp_{\varepsilon,i}$. Since $\tilde{\phi}_{\varepsilon,1}$ and $\tilde{\psi}^\perp_{\varepsilon,1}$ have bounded supports in $\overline{\mathbb{R}^n_+}$, we can extend them to the half space by a natural way, i.e., set $\tilde{\phi}_{\varepsilon,1} \equiv 0$ and $\tilde{\psi}^\perp_{\varepsilon,1} \equiv 0$ outside their supports. From (A11), we obtain $\|\tilde{\phi}_{\varepsilon,1}\|_{H^1(\mathbb{R}^n_+)} \le C$ for a constant $C > 0$ independent of ε, R. Clearly, $\|\tilde{\psi}^\perp_{\varepsilon,1}\|_{L^2(\mathbb{R}^n_+)} \le C\delta$. Then there are $\phi_{0,1} \in H^1(\mathbb{R}^n_+)$ and $\psi^\perp_{0,1} \in L^2(\mathbb{R}^n_+)$ such that as $\varepsilon \to 0$,

$$\tilde{\phi}_{\varepsilon,1} \to \phi_{0,1} \text{ weakly in } H^1(\mathbb{R}^n_+), \quad \tilde{\psi}^\perp_{\varepsilon,1} \to \psi^\perp_{0,1} \text{ weakly in } L^2(\mathbb{R}^n_+).$$

Similarly, we extend $\tilde{\phi}_{\varepsilon,i}$ and $\tilde{\psi}^\perp_{\varepsilon,i}$ to the whole space for $i = 2, \dots, m$. Then for $\tilde{\phi}_{\varepsilon,i}$ and $\tilde{\psi}^\perp_{\varepsilon,i}$, there are $\phi_{0,i} \in H^1(\mathbb{R}^n)$ and $\psi^\perp_{0,i} \in L^2(\mathbb{R}^n)$ such that as $\varepsilon \to 0$,

$$\tilde{\phi}_{\varepsilon,i} \to \phi_{0,i} \text{ weakly in } H^1(\mathbb{R}^n), \quad \tilde{\psi}^\perp_{\varepsilon,i} \to \psi^\perp_{0,i} \text{ weakly in } L^2(\mathbb{R}^n),$$

respectively. Since η_ε is bounded uniformly in ε, there is a constant η_0 such that $\eta_\varepsilon \to \eta_0$ as $\varepsilon \to 0$. Here we may take a subsequence of ε if necessary, and do not distinguish the subsequence from the original one. Since $\phi_{0,1}$ has a bounded support in $\overline{\mathbb{R}^n_+}$, $\tilde{\phi}_{\varepsilon,1}$ tends to $\phi_{0,1}$ strongly in $L^2(\mathbb{R}^n_+)$ as $\varepsilon \to 0$ because of Rellich's theorem. By the same argument, $\tilde{\phi}_{\varepsilon,i}$ also tends to $\phi_{0,i}$ strongly in $L^2(\mathbb{R}^n)$ as $\varepsilon \to 0$. Hence it is sufficient to estimate $\phi_{0,i}$ and η_0 in order to achieve our goal.

Lemma 4.4. *Let δ, κ be small constants independent of R satisfying $0 < \delta < \kappa^2$. If R is large, then there is a constant $C > 0$ independent of δ, κ, R such that for $i = 2, \dots, m$,*

$$\|\phi_{0,1}\|_{L^2(\mathbb{R}^n_+)} \le C\kappa^2, \quad \|\phi_{0,i}\|_{L^2(\mathbb{R}^n)} \le C\kappa^2, \quad |\eta_0| \le C\kappa^2.$$

Proof. By the standard regularity theory, we have $\phi_{0,1} \in H^2(\mathbb{R}^n_+)$ and $\phi_{0,i} \in H^2(\mathbb{R}^n)$ for $i = 2, \dots, m$. In addition, $\phi_{0,i}$ satisfies

$$(\kappa + \mu - L_0)\phi_{0,i} - f_v(S, \zeta)\eta_0 - \psi_{0,i}^{\perp} = O(1/R). \tag{20}$$

This equation should be considered in the half space $\mathbb{R}^n_+$ for $i = 1$, while it should be in the whole space for $i = 2, \dots, m$.

We shall show $\partial\phi_{0,1}/\partial y_n = 0$ on $\partial\mathbb{R}^n_+$. For an arbitrary smooth function $\varphi = \varphi(y)$ with a bounded support in $\overline{\mathbb{R}^n_+}$, it is clear that

$$\begin{aligned} &-\int_\Omega \Delta_x \phi_{\varepsilon,1} \varphi(\frac{1}{\varepsilon}\pi(x))\mathrm{d}x \\ &= \int_\Omega \nabla_x \phi_{\varepsilon,1} \cdot \nabla_x \left(\varphi(\frac{1}{\varepsilon}\pi(x))\right)\mathrm{d}x - \int_{\partial\Omega} \frac{\partial \phi_{\varepsilon,1}}{\partial \nu}\varphi(\frac{1}{\varepsilon}\pi(x))\mathrm{d}S_\sigma. \end{aligned}$$

Setting $y = \pi(x)/\varepsilon$, we rewrite this equality into

$$-\int_{\mathbb{R}^n_+} \Delta_y \tilde{\phi}_{\varepsilon,1}\varphi \mathrm{d}y = \int_{\mathbb{R}^n_+} \nabla_y \tilde{\phi}_{\varepsilon,1} \cdot \nabla_y \varphi \mathrm{d}y - \frac{1}{\varepsilon^{n-2}} \int_{\partial\Omega} \frac{\partial \phi_{\varepsilon,1}}{\partial \nu}\varphi(\frac{1}{\varepsilon}\pi(x))\mathrm{d}S_\sigma + o(1).$$

Here we show that the second term in the right-hand side tends to 0 as $\varepsilon \to 0$. Thanks to $\partial\phi_\varepsilon/\partial\nu = 0$ on $\partial\Omega$, we have

$$\frac{\partial \phi_{\varepsilon,1}}{\partial \nu}(x) = \nu(x) \cdot \frac{x}{|x|}\frac{1}{\varepsilon R}\chi'(\frac{|x|}{\varepsilon R})\phi_\varepsilon(x).$$

We claim that $\nu(x) \cdot x/|x|$ tends to 0 uniformly in $x \in \partial\Omega \cap B_{\varepsilon R}(0)$ as $\varepsilon \to 0$ because $\nu(x)$ and $x/|x|$ approaches a normal vector and a tangent vector of $\partial\Omega$ at 0, respectively. In addition, it follows from (18) and $\|\phi_\varepsilon\|_{H^1_\varepsilon} \le C_1$ for a constant C_1 independent of ε, R that

$$\frac{1}{\varepsilon^{n-1}} \int_{\partial\Omega} |\phi_\varepsilon|^2 \mathrm{d}S_\sigma \le C.$$

Hence we have

$$\begin{aligned} &\left|\frac{1}{\varepsilon^{n-2}} \int_{\partial\Omega} \frac{\partial \phi_{\varepsilon,1}}{\partial \nu}\varphi(\frac{1}{\varepsilon}\pi(x))\mathrm{d}S_\sigma\right| \\ &\le \frac{C}{R}\|\varphi\|_{L^\infty} \left\{\sup_{x \in \partial\Omega \cap B_{\varepsilon R}(0)} \frac{|\nu(x) \cdot x|}{|x|}\right\} \frac{1}{\varepsilon^{n-1}} \int_{\partial\Omega \cap B_{\varepsilon R}(0)} |\phi_\varepsilon| \mathrm{d}S_\sigma \\ &\le CR^{(n-3)/2}\|\varphi\|_{L^\infty} \left\{\sup_{x \in \partial\Omega \cap B_{\varepsilon R}(0)} \frac{|\nu(x) \cdot x|}{|x|}\right\} \left(\frac{1}{\varepsilon^{n-1}} \int_{\partial\Omega} |\phi_\varepsilon|^2 \mathrm{d}S_\sigma\right)^{1/2} \to 0. \end{aligned}$$

In the latter inequality, we used the Hölder inequality and the fact that the volume $\partial\Omega \cap B_{\varepsilon R}(0)$ can be estimated above by $C(\varepsilon R)^{n-1}$. Therefore we have

$$-\int_{\mathbb{R}^n_+} \Delta_y \phi_{0,1}\varphi \mathrm{d}y = \int_{\mathbb{R}^n_+} \nabla_y \tilde{\phi}_{0,1} \cdot \nabla_y \varphi \mathrm{d}y$$

for any φ so that $\phi_{0,1}$ satisfies the homogeneous Neumann boundary condition on $\partial\mathbb{R}^n_+$. Then we extend $\phi_{0,1}$ to the whole space by setting

$$\hat{\phi}_{0,1}(y_1,\dots,y_{n-1},y_n)=\begin{cases}\phi_{0,1}(y_1,\dots,y_{n-1},y_n), & y_n>0,\\ \phi_{0,1}(y_1,\dots,y_{n-1},-y_n), & y_n<0.\end{cases}$$

Clearly, $\hat{\phi}_{0,1}$ belongs to $H^2(\mathbb{R}^n)$. In the following, we do not distinguish the original function $\phi_{0,1}$ from the extended one $\hat{\phi}_{0,1}$, and use the same notation $\phi_{0,1}$. We also extend $\psi^\perp_{0,1}$ to the whole space by the same way as $\phi_{0,1}$. Then (20) holds in the whole space.

We set

$$\begin{aligned}\varphi_+&=\phi_{0,1}+2\sum_{i=2}^m\phi_{0,i},\quad \varphi_-=\sum_{i=1}^m\alpha_i\phi_{0,i},\\ \psi^\perp_+&=\psi^\perp_{0,1}+2\sum_{i=2}^m\psi^\perp_{0,i},\quad \psi^\perp_-=\sum_{i=1}^m\alpha_i\psi^\perp_{0,i},\\ c_+&=\int_{\mathbb{R}^n}\varphi_+\psi\mathrm{d}y,\quad c_-=\int_{\mathbb{R}^n}\varphi_-\psi\mathrm{d}y,\end{aligned}$$

where $\{\alpha_i\}_{i=1}^m$ is a set of constants with $\sum_{i=1}^m\alpha_i=0$. Then, φ_+ and φ_- satisfy

$$\begin{aligned}&(\kappa+\mu-L_0)\varphi_+-(2m-1)f_v(S,\zeta)\eta_0-\psi^\perp_+=O(1/R),\\ &(\kappa+\mu-L_0)\varphi_--\psi^\perp_-=O(1/R).\end{aligned}\tag{21}$$

Since $(\psi^\perp,\Xi)$ is orthogonal to K^0_ε, we have

$$\int_\Omega\psi^\perp\sum_{i=1}^m\alpha_i\psi_i\mathrm{d}x=0,$$

from which

$$\int_{\mathbb{R}^n}\psi^\perp_-\psi\mathrm{d}y=O(1/R).$$

Hence we have $c_-=O(1/R)$, and for any $\delta>0,\kappa>0$, there is a constant $C>0$ independent of δ,κ,R such that

$$\|\varphi_-\|_{L^2(\mathbb{R}^n)}\le C\delta.$$

Multiplying ψ to both sides of the first equation of (21) and integrating by parts, we have

$$\kappa c_+-(2m-1)\eta_0\int_{\mathbb{R}^n}f_v(S,\zeta)\psi\mathrm{d}y=\int_{\mathbb{R}^n}\psi\psi^\perp_+\mathrm{d}y+O(1/R).\tag{22}$$

In addition, solving the second equation of (15) with respect to η_ε and taking the limit of $\varepsilon\to0$, we have

$$\begin{aligned}&-\frac{1}{2}\int_{\mathbb{R}^n}g_{2u}(S,\zeta)\varphi_+\mathrm{d}y+\left\{\mu+\kappa-g_1'(\zeta)-\frac{2m-1}{2}\int_{\mathbb{R}^n}g_{2v}(S,\zeta)\mathrm{d}y\right\}\eta_0\\ &=\Xi_0+O(1/R),\end{aligned}\tag{23}$$

where Ξ_0 is a limit of Ξ as $\varepsilon\to0$. It follows from (16) that $\|\varphi_+\|_{L^2(\mathbb{R}^n)}\le C\kappa$ and $|\eta_0|\le C\kappa$. Using (22), we have $|\eta_0|\le C\kappa^2$. Since $\varphi_+-c_+\psi$ is orthogonal to ψ, $\|\varphi_+-c_+\psi\|_{L^2(\mathbb{R}^n)}\le C\kappa^2$ from the first equation of (21). By combining these inequalities, (A10), and (23), we obtain $|c_+|\le C\kappa^2$. Thus we complete the proof. □

From Lemmas 4.1, 4.3, and 4.4, Lemma 3.2 follows immediately. We omit details of the proof.

5. **Applications.** In this section we treat the two typical reaction-diffusion equations (6) and (7) and show that there is a stationary solution in these equations such that it satisfies all the assumptions (A1)–(A11). As described in Section 2, these systems have a stationary solution satisfying (A1)–(A6) under a suitable condition. Other conditions (A7), (A8), and (A10) can be easily verified because the nonlinear terms are given by

$$\begin{aligned} f_u(u,v) &= -a + 2(a+1)u - 3u^2 \quad f_v(u,v) = -1, \\ g_1'(v) &= -\frac{\gamma}{\tau}, \quad g_{2u}(u,v) = \frac{1}{\tau}, \quad g_{2v}(u,v) = 0, \end{aligned} \qquad \text{in (6)},$$

and

$$\begin{aligned} f_u(u,v) &= -1 + p\frac{u^{p-1}}{v^q}, \quad f_v(u,v) = -q\frac{u^p}{v^{q+1}}, \\ g_1'(v) &= -\frac{1}{\tau}, \quad g_{2u}(u,v) = \frac{r}{\tau}\frac{u^{r-1}}{v^s}, \quad g_{2v}(u,v) = -\frac{s}{\tau}\frac{u^r}{v^{s+1}}, \end{aligned} \qquad \text{in (7)}.$$

So it remains only to check the properties of $\mathcal{L}$ and L_0. In particular, it is important to verify the condition (A11).

First, we introduce results which show that L_0 given in (10) satisfies (A9).

Proposition 2 ([11], [14])**.** *There exists $\delta > 0$ such that all spectra of L_0 in $\{\lambda \in \mathbb{C} \,|Re\lambda \geq -\delta\}$ are only μ and 0, where Re denotes the real part of complex numbers. The eigenfunction corresponding to μ is uniquely given by ψ while the eigenspace corresponding to 0 is spanned by S_{y_i} for $i = 1, \ldots, n$.*

Next we show that $\mathcal{L}$ satisfies (A11).

Lemma 5.1. *Let κ be small and fixed. Then $(\kappa+\mu-\mathcal{L}) : H^2_{N,\varepsilon}(\Omega)\times\mathbb{C} \to L^2_\varepsilon(\Omega)\times\mathbb{C}$ and $(\kappa-\mathcal{L}) : H^2_{N,\varepsilon}(\Omega)\times\mathbb{C} \to L^2_\varepsilon(\Omega)\times\mathbb{C}$ are invertible, and the invertible operators $(\kappa+\mu-\mathcal{L})^{-1}$ and $(\kappa-\mathcal{L})^{-1}$ are compact on $L^2_\varepsilon(\Omega)\times\mathbb{C}$. Moreover, there are $\varepsilon_0 > 0$ and a constant $C > 0$ independent of ε, κ such that for any $\varepsilon < \varepsilon_0$,*

$$\|(\kappa+\mu-\mathcal{L})^{-1}\| \leq \frac{C}{\kappa}, \quad \|(\kappa-\mathcal{L})^{-1}\| \leq \frac{C}{\kappa}. \tag{24}$$

The proof is divided into two steps. First we have a resolvent estimate with the parameter κ for $L_\varepsilon = \varepsilon^2\Delta + f_u(u,v)$.

Lemma 5.2. *Let κ be small and fixed. Then $(\kappa+\mu-L_\varepsilon) : H^2_{N,\varepsilon}(\Omega) \to L^2_\varepsilon(\Omega)$ and $(\kappa-L_\varepsilon) : H^2_{N,\varepsilon}(\Omega) \to L^2_\varepsilon(\Omega)$ are invertible. Moreover, there are $\varepsilon_0 > 0$ and a constant $C > 0$ independent of ε, κ such that for any $\varepsilon < \varepsilon_0$,*

$$\|(\kappa+\mu-L_\varepsilon)^{-1}\| \leq \frac{C}{\kappa}, \quad \|(\kappa-L_\varepsilon)^{-1}\| \leq \frac{C}{\kappa}. \tag{25}$$

Proof. It is clear from Proposition 2 that $(\kappa+\mu-L_\varepsilon)$ and $(\kappa-L_\varepsilon)$ are invertible, and their invertible operators are compact on $L^2_\varepsilon(\Omega)$ because the domain Ω is bounded. To show (25), the same argument as in the proofs of Lemmas 4.1, 4.3, and 4.4 can be applied to this case. We omit details of the proof. □

Using Lemma 5.2, we obtain the estimates of $(\mu+\kappa-\mathcal{L})^{-1}$ and $(\kappa-\mathcal{L})^{-1}$. We prove Lemma 5.1 for (6) and (7) separately. To derive estimates (24) for (7), we shall use *Sherman-Morrison's formula*, which is useful to study the eigenvalue problem with a nonlocal term like (9) (see [5]).

Proposition 3. *Suppose that $A : D(A) \subset H \to H$ is an invertible operator and $B\phi = \langle \Psi_1, \phi \rangle \Psi_2$, where $\langle \cdot, \cdot \rangle$ represents an inner product in a Hilbert space H. Then $A + B$ is invertible if and only if*

$$M \equiv 1 + \left\langle \Psi_1, A^{-1}\Psi_2 \right\rangle \neq 0.$$

Moreover, the invertible operator of $A + B$ is written as

$$(A+B)^{-1} = \left(1 - \frac{A^{-1}B}{M}\right) A^{-1}.$$

This proposition can be proved easily by direct calculation. Now we are in position to prove Lemma 5.1.

Proof. First we consider (6) and prove the invertibility of $(\mu + \kappa - \mathcal{L})$ and (24) directly. Suppose that for $(\psi, \xi) \in L^2(\Omega) \times \mathbb{C}$ with $\|(\psi, \xi)\|_{L^2 \times \mathbb{C}} = 1$, there is a solution $(\phi, \eta) \in H^2_N(\Omega) \times \mathbb{C}$ of

$$\begin{cases} (\kappa + \mu - L_\varepsilon)\phi + \eta = \psi, \\ (\kappa + \mu + \gamma)\eta - \displaystyle\int_\Omega \phi \mathrm{d}x = \xi. \end{cases} \tag{26}$$

By (19) in Lemma 4.2, we have $|\eta| \leq C$. Thanks to the first equation of (26) and (25), we have $\|\phi\|_{L^2} \leq C/\kappa$. Therefore $(\kappa + \mu - \mathcal{L})$ is invertible, and the first inequality of (24) holds true. By the same argument, we also see that $(\kappa - \mathcal{L})$ is invertible, and the second inequality of (24) holds true.

Next we consider (7). Put $A\phi \equiv (\kappa + \mu - L_\varepsilon)\phi$ and

$$B\phi \equiv -\frac{f_v(u,v)}{\kappa + \mu - g_1'(v) - \frac{1}{\varepsilon^n}\int_\Omega g_{2v}(u,v)\mathrm{d}x} \frac{1}{\varepsilon^n} \int_\Omega g_{2u}(u,v)\phi \mathrm{d}x.$$

Simple computations imply that $(\kappa + \mu - \mathcal{L})$ is invertible if and only if $A + B$ is invertible. From Proposition 3, it suffices to check

$$M \equiv 1 + \frac{1}{\kappa + \mu + \frac{1}{\tau} + \frac{s}{\tau \varepsilon^n v^{s+1}} \int_\Omega u^r \mathrm{d}x} \left\langle \frac{r}{\tau}\frac{u^{r-1}}{v^s}, (\kappa + \mu - L_\varepsilon)^{-1} q \frac{u^p}{v^{q+1}} \right\rangle_{L^2_\varepsilon} \neq 0.$$

Straightforward calculations yield

$$M \to 1 + \frac{qr}{\tau(\kappa + \mu) + 1 + s} \frac{m_1 + 2m_2}{2\zeta^{s+q+1}} \int_{\mathbb{R}^n} S^{r-1}(\kappa + \mu - L_0)^{-1} S^p \mathrm{d}y$$

as $\varepsilon \to 0$. Remember m_1 and m_2 denote the numbers of boundary and interior spots, respectively. Due to the positivity of the function S, the second term in the right-hand side is estimated below by C/κ for a constant $C > 0$ independent of κ. Hence we have $M \geq C/\kappa$ so that $(\mu + \kappa - \mathcal{L})$ is invertible. Using Lemma 5.2 and the Sherman-Morrison's formula again, we easily have

$$\|(\mu + \kappa - \mathcal{L})^{-1}\| \leq \left(1 + C\kappa \|(\mu + \kappa - L_\varepsilon)^{-1}\|\right) \|(\mu + \kappa - L_\varepsilon)^{-1}\| \leq \frac{C}{\kappa}.$$

Finally we estimate the resolvent operator $(\kappa - \mathcal{L})^{-1}$. By the same argument as above, we show

$$M \equiv 1 + \frac{1}{\kappa + \frac{1}{\tau} + \frac{s}{\tau \varepsilon^n v^{s+1}} \int_\Omega u^r \mathrm{d}x} \left\langle \frac{r}{\tau}\frac{u^{r-1}}{v^s}, (\kappa - L_\varepsilon)^{-1} q \frac{u^p}{v^{q+1}} \right\rangle_{L^2_\varepsilon} \neq 0.$$

Taking the limit of $\varepsilon \to 0$, we have

$$M \to 1 + \frac{qr}{\tau\kappa + 1 + s} \frac{m_1 + 2m_2}{2\zeta^{s+q+1}} \int_{\mathbb{R}^n} S^{r-1}(\kappa - L_0)^{-1} S^p \mathrm{d}y. \tag{27}$$

From direct calculations, we see

$$L_0 S = \Delta S - S + p\frac{S^p}{\zeta^q} = (p-1)\frac{S^p}{\zeta^q}.$$

Since S^p is orthogonal to S_{y_i} for any $i = 1, \dots, n$, $L_0^{-1} S^p$ is uniquely determined by $L_0^{-1} S^p = S\zeta^q/(p-1)$. Substituting this into the last term of (27), we have

$$\lim_{\kappa\to 0}\lim_{\varepsilon\to 0} M = 1 - \frac{m_1 + 2m_2}{2}\frac{qr}{(p-1)(1+s)} < 0$$

due to the assumption for the exponents and $m \geq 2$. Thus we see that $(\kappa - \mathcal{L})$ is invertible and

$$\|(\kappa - \mathcal{L})^{-1}\| \leq C\|(\kappa - L_\varepsilon)^{-1}\| \leq \frac{C}{\kappa}.$$

□

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