Spectral Analysis of a Charged Scalar Field Model with Cutoffs

(切断の入った複素スカラー場の模型のスペクトル解析)

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Abstract

A quantum system of a massless charged scalar field with a self-interaction is investigated. By introducing a spacial cut-off function, a Hamiltonian of the quantum system is realized as a linear operator on a Boson Fock space. Under certain conditions, it is proven that the Hamiltonian is bounded below, self-adjoint and has a ground state for an arbitrary coupling constant. Moreover the Hamiltonian strongly commutes with the total charge operator. This fact implies that the state space of the charged scalar field is decomposed into the infinite direct sum of fixed total charge spaces. A total charge of an eigenstate is discussed.

Contents

1 Introduction

In this thesis, we consider a quantum system of a charged scalar field $\phi(\tilde{x})$ which interacts with itself on the $1+d$ dimensional space-time $\mathbb{R}^{1+d} := \{\tilde{x} = (x^0, x^1, \ldots, x^d) | x^\nu \in \mathbb{R}, \nu = 0, \ldots, d\}$ with the Minkowski metric $g = (g_{\mu\nu})$, $g_{00} = 1$, $g_{jj} = -1$ $(j = 1, \ldots, d)$, $g_{\mu\nu} = 0$ $(\mu \neq \nu)$. The Lagrangian $\mathcal{L}$ of a complex Klein-Gordon equation with a self-interaction term is given by

$$\mathcal{L} = (\partial_\nu \phi)(\partial^\nu \phi)^* - m^2 \phi\phi^* - \frac{\lambda}{4!}(\phi\phi^*)^2, \quad \Big(\partial_\nu := \frac{\partial}{\partial x^\nu}, \ \ \partial^\nu := g^{\nu\rho}\partial_\rho\Big),$$

where the Einstein convention for the sum on repeated Greek induces is used, A^* denotes the complex conjugate of A, $m \geq 0$ is the mass of a particle and $\lambda > 0$ is a coupling constant. Let us consider the following Lagrangian $\mathcal{L}'$:

$$\mathcal{L}' = (\partial_\nu \phi)(\partial^\nu \phi)^* + \mu^2 \phi\phi^* - \frac{\lambda}{4!}(\phi\phi^*)^2, \tag{1}$$

where $\mu > 0$ is merely a parameter. $\mathcal{L}'$ is the deformation of $\mathcal{L}$ by the replacement $m^2 \to -\mu^2$. As is well known, the formal quantization of ϕ yields particles and anti-particles. We denote by $a_+(k)$ (resp. $a_-(k)$) the formal distribution kernel of the annihilation operator for the particle (resp. anti-particle). The formal adjoint $a_+(k)^*$ (resp. $a_-(k)^*$) represents the formal distribution kernel of the creation operator for the particle (resp. anti-particle). We denote by $\phi(x)$ $(x \in \mathbb{R}^d)$ the time-zero charged scalar field of ϕ. Then the Hamiltonian derived from (1) is *formally* given by

$$\begin{aligned} H_{\text{formal}} = &\int_{\mathbb{R}^d} |k| \big(a_+(k)^* a_+(k) + a_-(k)^* a_-(k)\big) \mathrm{d}k \\ &+ \int_{\mathbb{R}^d} \big(-\mu^2 \phi(x)\phi(x)^* + \frac{\lambda}{4!}(\phi(x)\phi(x)^*)^2\big) \mathrm{d}x. \end{aligned} \tag{2}$$

The integrand of the second term on the right hand side of (2) is of the form of the so-called *Higgs potential.* The Lagrangian $\mathcal{L}'$ is introduced as an example of *spontaneous symmetry breaking* in quantum field theory (see, e.g., [18,23]). It is interesting to study about it from an operator theoretical point of view. However we can not analyze (2) directly as a linear operator on a Boson Fock space, since the second term on the right hand side of (2) always diverges even if a vector belongs to a nice class. Therefore we need modifications.

Let ω be a multiplication operator of a non-negative function on $\mathbb{R}^d$ denoting a one-boson Hamiltonian. Then the free Hamiltonian H_0 of a charged scalar field is defined by the second quantization of $\omega \oplus \omega$:

$$H_0 := \mathrm{d}\Gamma_{\mathrm{b}}(\omega \oplus \omega)$$

on a suitable Boson Fock space (see Section 2). Let χ_{sp} be a non-negative function on $\mathbb{R}^d$ which plays a role as a *spacial cut-off.* For $x \in \mathbb{R}^d$, let $\phi(f_x)$ be a charged field operator smeared by a function f_x on $\mathbb{R}^d$. The Hamiltonian H we consider is of the following form:

$$H = \mathrm{d}\Gamma_{\mathrm{b}}(\omega \oplus \omega) + \mu \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\phi(f_x)^*\phi(f_x)\mathrm{d}x + \lambda \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)(\phi(f_x)^*\phi(f_x))^2 \mathrm{d}x, \tag{3}$$

where $\mu \in \mathbb{R}$ and $\lambda > 0$ are coupling constants. A rigorous definition of H is given in Section 2. The integrals on the right hand side of (3) are taken in the sense of strong

Bochner integral. By introducing a spacial cut-off, the quantum system we want to study loses translation invariance. Thus H does not have relativistic covariance. However we are able to analyze H as an approximation version of H_{formal} by virtue of a spacial cut-off. We expect that the study of H will be also a first step towards understanding spontaneous symmetry breaking. In this thesis we study properties of H via operator theoretical methods. In related works, it is assumed the space dimension to be one. However we assume the space dimension to be $d \in \mathbb{N}$ for a mathematical generalization. If $\mu < 0$, H describes a cut-off Hamiltonian of a charged scalar field with a Higgs type potential. If $\mu = 0$, H becomes a complex-$\lambda\phi^4$ model with cut-offs. Hence H unifies two important models. Properties of H_0 are already known. In particular, it is self-adjoint and has a ground state. It is not trivial, however, whether H still holds these properties even if $\mu \geq 0$ and $\lambda = 0$. In view of perturbation theory of linear operators, it is interesting to find the condition that H is still self-adjoint and has a ground state. Since the third term on the right hand side of (3) is not "small" with respect to H_0, we need careful treatment to analyze H. Moreover, we certainly meet a perturbation problem for an embedded eigenvalue since the mass of boson to be zero. Therefore it makes spectral analysis more difficult. As is seen below, the quantum system holds the charge conservation. It means that the Hamiltonian H and the total charge operator strongly commute. In the physical context, this property corresponds to the *global U(1)-gauge symmetry.* Note that this structure is not seen in a real scalar Bose field model.

There are several models similar to (3), which have been studied so far. Glimm-Jaffe [13] considered the real ϕ_2^4 model which describes a real scalar Bose field with quartic interaction in the 2-dimensional space-time. Dereziński-Gérard [8] considered the scattering theory for the real $P(\varphi)_2$ model. Gérard-Panati [12] considered the spectral and scattering theory for an abstract Hamiltonian which include the real $P(\phi)_2$ model. Gérard [11] considered the charged $P(\phi)_2$ model which describes the charged scalar Bose field with a self-interaction in the 2-dimensional space-time. In these studies, the infimum of ω is assumed to be strictly positive but ultraviolet cut off is not imposed. On the other hand, we consider the case of the infimum of ω is 0 and ultraviolet cut off is imposed. An interaction model between quantum mechanical particles and a real scalar Bose field is also established. Recently, some singular perturbed models are studied. Miyao and Sasaki [20] considered the generalized spin-boson model (GSB model) with quadratic interaction. They gave a criteria for the existence of the ground state. Teranishi [27] also considered the same model in terms of the self-adjointness. Takaesu [26] considered the GSB model with ϕ^4-perturbation. He showed the existence of a ground state and the existence of asymptotic fields for a sufficiently small coupling constants. Hidaka [16] considered the Nelson model with perturbation of a form $\sum_{j=1}^{4} c_j \phi^j$ with $c_4 > 0$. He showed the existence of a ground state for arbitrary coupling constants. The study about the total charge operator is already done by Takaesu [25], who treats a model of the quantum electrodynamics. To our best knowledge, there are few results about the charged scalar field with the infimum of ω being zero.

We give our strategy comparing with some related works.

Self-adjointness: To show the self-adjointness of H, we apply the method in [16] and [26]. A key lemma is that the interaction term is H-bounded. To prove this lemma, we need the fact that the second term on the right hand side of (3) is infinitesimally small with respect to the third term of it. We need some technical treatments because of strong Bochner integral.

Existence of ground states: First of all, we show the existence of a ground state of a massive Hamiltonian. After that, we consider the mass zero limit of massive ground states. In the massive case, we apply methods used in [7,8,16] and references therein. In these methods, the so-called *Number-Energy Estimate* (NEE) is important to show the existence of ground states for the massive case. However, it is difficult to prove this lemma in our Hamiltonian since the interaction term is singular. As is seen below, we study the massive case without using a NEE (see Lemma 5.1 and 5.2). To show that the mass zero limit of massive ground states is not zero, we apply methods in [14, 24] and references therein.

Total charge of eigenstates: First of all, we show the strong commutativity of H and the total charge operator (Proposition 6.1). After that we show that the total charge of eigenstates are zero under certain conditions (Theorem 6.1). To prove Theorem 6.1, symmetry between particles and anti-particles plays important roles.

The contents of this thesis are as follows. In Section 2, we recall several notations and symbols about the abstract Boson Fock space. After that, we introduce the Hamiltonian H rigorously. The self-adjointness of H is discussed in Section 3. In Section 4, the spectrum of H is specified. The existence of a ground state is proved in Section 5. The total charge in eigenstates is discussed in Section 6. In Appendix A, some results of the abstract Boson Fock space which are used in this paper are collected. In Appendix B and Appendix C, we summarize the results of [2,5] which we use in Section 3 and Section 4.

Finally, all contents of this thesis are based on [28].

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2 Mathematical Preliminaries

For a Hilbert space $\mathscr{H}$, we denote its inner product and norm by $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ (linear in the right vector) and $\| \cdot \|_{\mathscr{H}}$ respectively. But, if there is no danger of confusion, then we often omit the subscript $\mathscr{H}$ of them.

2.1 Abstract Boson Fock space

Let us recall some notations and symbols about the abstract Boson Fock space.

Let $\mathcal{K}$ be a separable Hilbert space over $\mathbb{C}$. Then the Boson Fock space over $\mathcal{K}$ is given by

$$\mathcal{F}_{\mathrm{b}}(\mathcal{K}) := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \bigotimes_{\mathrm{s}}^{n} \mathcal{K},$$

where $\otimes_{\mathrm{s}}^{n}$ denotes the n-fold symmetric tensor product. The *Fock vacuum* in $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ is denoted by Ω and

$$\Omega := \{1, 0, 0, \cdots\} \in \mathcal{F}_{\mathrm{b}}(\mathcal{K}).$$

Let us introduce the finite particle subspace $\mathcal{F}_{\mathrm{b},0}(\mathcal{K})$ as follows:

$$\mathcal{F}_{\mathrm{b},0}(\mathcal{K}) := \Big\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}}(\mathcal{K}) \Big| \exists N \in \mathbb{N} \text{ s.t. } \Psi^{(n)} = 0 \text{ for all } n \geq N+1 \Big\}.$$

Note that $\mathcal{F}_{\mathrm{b},0}(\mathcal{K})$ is dense in $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$. For each $u \in \mathcal{K}$, the creation operator $A(u)^{\dagger}$ is defined as follows:

$$D(A(u)^{\dagger}) := \Big\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}}(\mathcal{K}) \Big| \sum_{n=1}^{\infty} n \Big\| S_n(u \otimes \Psi^{(n-1)}) \Big\|_{\otimes_{\mathrm{s}}^{n}\mathcal{K}}^{2} < \infty \Big\},$$

$$(A(u)^{\dagger}\Psi)^{(n)} := \sqrt{n} S_n(u \otimes \Psi^{(n-1)}), \quad \Psi \in D(A(u)^{\dagger}), \quad (n \geq 1),$$

and $(A(u)^{\dagger}\Psi)^{(0)} := 0$. Here $D(T)$ denotes the domain of a linear operator T, and S_n denotes the symmetrization operator on $\otimes^{n}\mathcal{K}$. It is known that $A(u)^{\dagger}$ is a closed operator. The annihilation operator with $u \in \mathcal{K}$ is given by the adjoint of $A(u)^{\dagger}$:

$$A(u) := (A(u)^{\dagger})^{*}.$$

For all $u, v \in \mathcal{K}$, the annihilation and creation operators satisfy the following canonical commutation relations on $\mathcal{F}_{\mathrm{b},0}(\mathcal{K})$:

$$[A(u), A(v)] = [A(u)^{\dagger}, A(v)^{\dagger}] = 0, \qquad [A(u), A(v)^{\dagger}] = \langle u, v \rangle_{\mathcal{K}}, \tag{4}$$

where $[X, Y] := XY - YX$. For a subspace D of $\mathcal{K}$, the subspace $\mathcal{F}_{\mathrm{b,fin}}(D)$ is introduced as follows,

$$\mathcal{F}_{\mathrm{b,fin}}(D) := \mathrm{L.H.} \Big\{ \Omega, A(u_1)^{\dagger} \cdots A(u_n)^{\dagger}\Omega \Big| n \in \mathbb{N}, u_j \in D, j = 1, \ldots, n \Big\},$$

where L.H.$\{\cdots\}$ denotes the linear hull of a set $\{\cdots\}$ over $\mathbb{C}$. Note that, if D is dense in $\mathcal{K}$, then $\mathcal{F}_{\mathrm{b,fin}}(D)$ is dense in $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$.

Let T be a densely defined closable operator on $\mathcal{K}$. We denote the closure of T by $\overline{T}$. Then the second quantization of T is given by

$$\mathrm{d}\Gamma_{\mathrm{b}}(T) := 0 \oplus \bigoplus_{n=1}^{\infty} \overline{\sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes \overset{j-th}{T} \otimes I \cdots \otimes I \restriction \hat{\otimes}_{s}^{n} D(T)},$$

where I is the identity on $\mathcal{K}$, $S \restriction \mathcal{D}$ is the restriction of a linear operator S to $\mathcal{D}$ and $\hat{\otimes}_{\mathrm{s}}^n$ denotes the n-fold algebraic symmetric tensor product. It is seen that $\mathrm{d}\Gamma_{\mathrm{b}}(T)$ is closed. If T is self-adjoint, so is $\mathrm{d}\Gamma_{\mathrm{b}}(T)$.

Associated with T, another operator $\Gamma_{\mathrm{b}}(T)$ is also defined. Let T be a contraction operator on $\mathcal{K}$ to another complex Hilbert space $\mathcal{K}'$. Then $\Gamma_{\mathrm{b}}(T)$ is defined as follows:

$$\Gamma_{\mathrm{b}}(T) := 1 \oplus \bigoplus_{n=1}^{\infty} \underbrace{T \otimes \cdots \otimes T}_{n-\text{times}}.$$

It is seen that $\Gamma_{\mathrm{b}}(T)$ is a contraction operator from $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ to $\mathcal{F}_{\mathrm{b}}(\mathcal{K}')$.

2.2 Charged scalar field

For a subspace $\mathcal{D}$ of a Hilbert space $\mathcal{K}$, we use a following notation:

$$[\mathcal{D}] := \mathcal{D} \oplus \mathcal{D} \subset \mathcal{K} \oplus \mathcal{K}.$$

Let $d \in \mathbb{N}$. The state space of a charged scalar field is given by

$$\mathcal{F} := \mathcal{F}_{\mathrm{b}}([L^2(\mathbb{R}^d)]),$$

the Boson Fock space over $[L^2(\mathbb{R}^d)]$. In the physical context under consideration, $[L^2(\mathbb{R}^d)]$ describes the state space of pairs of a particle and an anti-particle. For $u \in L^2(\mathbb{R}^d)$, the operators $a_\pm(u)$ and $a_\pm(u)^\dagger$ on $\mathcal{F}$ are defined as follows:

$$\begin{aligned} a_+(u) &:= A((u,0)), \quad & a_+(u)^\dagger &:= A((u,0))^\dagger, \\ a_-(u) &:= A((0,u)), \quad & a_-(u)^\dagger &:= A((0,u))^\dagger. \end{aligned}$$

The operators $a_+(u)$ and $a_-(u)$ are called the annihilation operator of a particle and an anti-particle with a state function u, respectively. On the other hand, $a_+(u)^\dagger$ and $a_-(u)^\dagger$ are called the creation operator of a particle and an anti-particle, respectively. By (4), these operators satisfy the canonical commutation relations on $\mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])$:

$$\begin{aligned} &[a_\pm(u), a_\pm(v)] = [a_\pm(u)^\dagger, a_\pm(v)^\dagger] = 0, \\ &[a_\pm(u)^\dagger, a_\mp(v)^\dagger] = [a_\pm(u), a_\mp(v)^\dagger] = [a_\pm(u), a_\mp(v)] = 0, \\ &[a_\pm(u), a_\pm(v)^\dagger] = \langle u, v \rangle_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{5}$$

We denote the field operator with a state function $u \in L^2(\mathbb{R}^d)$ by

$$\phi(u) := \frac{1}{\sqrt{2}} \big(a_+(u) + a_-(u)^\dagger \big). \tag{6}$$

It is easy to see that $\phi(u)$ is densely defined and closable. We denote the closure of $\phi(u)$ by the same symbol. By von Neumann's theorem, $\phi(u)^*\phi(u)$ is non-negative self-adjoint operator on $\mathcal{F}$. Note that a concrete action of $\phi(u)^*$ is as follows:

$$\phi(u)^* = \frac{1}{\sqrt{2}} \big(a_+(u)^\dagger + a_-(u) \big), \quad \big(\text{on } \mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])\big).$$

By (5), the field operators satisfy the following commutation relations on $\mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])$:

$$[\phi(u), \phi(v)] = [\phi(u)^*, \phi(v)^*] = 0, \qquad [\phi(u), \phi(v)^*] = i\mathrm{Im}\, \langle u, v \rangle_{L^2(\mathbb{R}^d)},$$

where $\mathrm{Im}\, z$ denotes the imaginary part of $z \in \mathbb{C}$.

2.3 Definition of a model

Let ω be the multiplication operator on $L^2(\mathbb{R}^d)$ by the function

$$\omega(k) := |k| \quad (k \in \mathbb{R}^d).$$

For a linear operator T on $L^2(\mathbb{R}^d)$, we use following notation:

$$[T] := T \oplus T.$$

Then the free Hamiltonian of the charged scalar field H_0 is defined by the second quantization of $[\omega]$:

$$H_0 := \mathrm{d}\Gamma_{\mathrm{b}}([\omega]).$$

The number operator N_{b} is introduced as

$$N_{\mathrm{b}} := \mathrm{d}\Gamma_{\mathrm{b}}([1]).$$

Note that H_0 and N_{b} are non-negative self-adjoint on $\mathcal{F}$. For $q \in \mathbb{R} \setminus \{0\}$, the total charge operator Q is defined as follows:

$$Q := \mathrm{d}\Gamma_{\mathrm{b}}((q \oplus -q)).$$

Note that Q is self-adjoint on $\mathcal{F}$.

Let $\varphi \in D(\omega^{-1/2})$. For $x \in \mathbb{R}^d$, a function f_x is defined as follows:

$$f_x(k) := \frac{\varphi(k)}{\sqrt{\omega(k)}} e^{-ikx} \quad (\text{a.e. } k \in \mathbb{R}^d)$$

with $kx := k_1x_1 + \cdots + k_dx_d$ for $k = (k_1, \ldots, k_d) \in \mathbb{R}^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. φ plays a role as an ultraviolet cut-off and an infrared cut-off.

To describe the interaction of charged scalar field, we pick a function χ_{sp} which satisfies following condition:

Assumption 2.1. *$\chi_{\mathrm{sp}} : \mathbb{R}^d \to \mathbb{R}$ is measurable, non-negative and $\chi_{\mathrm{sp}} \in L^1(\mathbb{R}^d)$.*

Remark 2.1. *Throughout in this thesis, we always assume $\varphi \in D(\omega^{-1/2})$ and Assumption 2.1.*

Now, we introduce two linear operators H_1 and H_2 as follows:

$$H_1 := \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\phi(f_x)^*\phi(f_x)\mathrm{d}x, \quad H_2 := \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\big(\phi(f_x)^*\phi(f_x)\big)^2\mathrm{d}x. \tag{7}$$

The integrals on the right hand sides of (7) are taken in the sense of $\mathcal{F}$-valued strong Bochner integral. Namely, the domain and the action of H_1 and H_2 are defined as follows:

$$D(H_i) := \Big\{\Psi \in \mathcal{F} \Big| \Psi \in \bigcap_{x \in \mathrm{supp}\chi_{\mathrm{sp}}} D((\phi(f_x)^*\phi(f_x))^i),$$

$$\big(\phi(f_x)^*\phi(f_x)\big)^i\Psi \text{ is measurable}, \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\big\|\big(\phi(f_x)^*\phi(f_x)\big)^i\Psi\big\|_{\mathcal{F}}\mathrm{d}x < \infty\Big\},$$

$$H_i\Psi := \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\big(\phi(f_x)^*\phi(f_x)\big)^i\Psi\mathrm{d}x, \quad (i = 1, 2).$$

By using $\chi_{\mathrm{sp}} \in L^1(\mathbb{R}^d)$ and Proposition A.1, it follows that $\mathcal{F}_{\mathrm{b,fin}}([L^2(\mathbb{R}^d)]) \subset D(H_1) \cap D(H_2)$. Thus H_1 and H_2 are densely defined and symmetric.

The Hamiltonian we study in this thesis is as follows:

$$H := H_0 + \overline{\mu H_1 + \lambda H_2}. \tag{8}$$

Here, $\mu \in \mathbb{R}$ and $\lambda > 0$ are coupling constants.

Remark 2.2. *In [8], [11], [12] and [13], there are used "Wick ordering" $: \cdot :$, which is defined in a product of annihilation and creation operators by moving the creation operators to the left and the annihilation operators to the right without canonical commutation relations. For $u \in L^2(\mathbb{R}^d)$, by using the commutation relations for creation and annihilation operators, it follows that*

$$: \big(\phi(u)^*\phi(u)\big)^2 := \big(\phi(u)^*\phi(u)\big)^2 - \frac{1}{2}\|u\|^2\phi(u)^*\phi(u) + \frac{1}{2}\|u\|^2 \quad on \quad \mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)]).$$

Since we impose an ultraviolet and an infrared cut-offs, Wick ordering is not needed. Moreover if we want to know the property of Hamiltonian with Wick ordering, it suffices to study (8).

3 Self-adjointness of H

In this section, we prove the (essential) self-adjointness of H. We set the following assumption:

Assumption 3.1. *It follows that*

(1) $\varphi \in D(\omega^{1/2})$,

(2) $|\varphi(k)| = |\varphi(-k)|$ (a.e. $k \in \mathbb{R}^d$).

Remark 3.1. *As is seen in Lemma 3.1, we can show that H is essentially self-adjoint without Assumption 3.1. To show the self-adjointness of H, Assumption 3.1 is needed. From $|\varphi(k)| = |\varphi(-k)|$, we have $[\phi(f_x), \phi(f_y)^*] = 0$ on $\mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])$. This commutativity is important in our analysis.*

Let us denote the set of infinitely differentiable functions on $\mathbb{R}^d$ with compact support by $C_0^\infty(\mathbb{R}^d)$. The main result of this section is as follows:

Theorem 3.1. *Under Assumption 3.1, H is bounded from below, self-adjoint with $D(H) = D(H_0) \cap D(\overline{H_2})$ and essentially self-adjoint on $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ for arbitrary $\mu \in \mathbb{R}$ and $\lambda > 0$.*

First, we show the essential self-adjointness of H.

Lemma 3.1. *H is bounded from below essentially self-adjoint on $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$.*

Proof. First of all, we check that H satisfies a criterion of essential self-adjointness on $D(H_0) \cap \mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])$ (see Proposition B.1). Since $\mu H_1 + \lambda H_2$ maps $\otimes_{\mathrm{s}}^n([L^2(\mathbb{R}^d)])$ to $\oplus_{j=-4}^4 \otimes_{\mathrm{s}}^{n+j}([L^2(\mathbb{R}^d)])$, we see that

$$\langle \Psi^{(n)}, (\mu H_1 + \lambda H_2)\Psi^{(m)} \rangle = 0, \qquad (\text{whenever} \quad |n-m| \geq 5).$$

If $\mu \geq 0$, then it is obvious that H is bounded from below on $D(H_0) \cap \mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])$.

The case where $\mu < 0$, for any $\Psi \in D(H_0) \cap \mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])$, we see that

$$\begin{aligned}
\langle \Psi, H\Psi \rangle &= \langle \Psi, H_0\Psi \rangle + \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x) \langle \Psi, \{\mu\phi(f_x)^*\phi(f_x) + \lambda(\phi(f_x)^*\phi(f_x))^2\}\Psi \rangle \mathrm{d}x \\
&\geq \int_{\mathbb{R}^d} \int_{t\geq 0} \chi_{\mathrm{sp}}(x)(\mu t + \lambda t^2) \mathrm{d}\|E_x(t)\Psi\|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \int_{t\geq 0} \chi_{\mathrm{sp}}(x) \Big\{ \lambda \Big(t + \frac{\mu}{2\lambda} \Big)^2 - \frac{\mu^2}{4\lambda} \Big\} \mathrm{d}\|E_x(t)\Psi\|^2 \mathrm{d}x \\
&\geq -\frac{\mu^2}{4\lambda} \|\Psi\|^2 \|\chi_{\mathrm{sp}}\|_{L^1} > -\infty,
\end{aligned}$$

where $E_x(\cdot)$ is the spectral measure of $\phi(f_x)^*\phi(f_x)$. The relative boundedness of $\mu H_1 + \lambda H_2$ with respect to $(N_{\mathrm{b}} + 1)^2$ is seen by using Proposition A.1. Therefore H is essentially self-adjoint on $D(H_0) \cap \mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])$.

Since $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ is a core of H_0, for any $\Psi \in D(H_0) \cap \mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])$, there exist an $N \in \mathbb{N}$ and a sequence $\{\Psi_j\}_{j=1}^\infty \subset \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ such that

$$\Psi_j \to \Psi, \quad H_0\Psi_j \to H_0\Psi, \quad (j \to \infty),$$

and $\Psi^{(n)} = 0$, whenever $n > N$. Since $(\mu H_1 + \lambda H_2) \restriction (\oplus_{l=0}^N \otimes_{\mathrm{s}}^l [L^2(\mathbb{R}^d)])$ is bounded , we see that

$$\Psi_j \to \Psi, \quad H\Psi_j \to H\Psi, \quad (j \to \infty).$$

Thus the desired result follows. □

Let ϵ and η be arbitrary positive constants with $\lambda^2 - 2\epsilon - \lambda^2\mu^2\eta/\epsilon > 0$. Then we define a constant $C(\mu, \lambda, \epsilon, \eta)$ as follows:

$$C(\mu, \lambda, \epsilon, \eta) := (\lambda^2 - 2\epsilon - \lambda^2\mu^2\eta/\epsilon)^{-1/2} \Big(\frac{\lambda^2\mu^2}{4\epsilon\eta} \|\chi_{\mathrm{sp}}\|_{L^1}^2 + \frac{\|\chi_{\mathrm{sp}}\|_{L^1}^2}{4\epsilon} + \lambda^2 \|\varphi\|_{L^2}^4 + 1 \Big)^{1/2}.$$

A following lemma is important to show the self-adjointness of H.

Lemma 3.2. *Suppose that Assumption 3.1 is satisfied. Then for all $\Psi \in D(\overline{H})$,*

$$\|\overline{H}_1\Psi\| \leq \theta C(\mu, \lambda, \epsilon, \eta) \|\overline{H}\Psi\| + \big(\theta C(\mu, \lambda, \epsilon, \eta) + \frac{1}{2\theta} \|\chi_{\mathrm{sp}}\|_{L^1} \big) \|\Psi\|, \tag{9}$$

$$\|\overline{H}_2\Psi\| \leq C(\mu, \lambda, \epsilon, \eta) \big(\|\overline{H}\Psi\| + \|\Psi\| \big), \tag{10}$$

where θ is an arbitrary positive constant.

Proof. We recall that $|\varphi(k)| = |\varphi(-k)|$ implies $[\phi(f_x), \phi(f_y)^*] = 0$ on $\mathcal{F}_{\mathrm{b},0}([L^2(\mathbb{R}^d)])$ for all $x, y \in \mathbb{R}^d$. For any $\Psi \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$, it follows that

$$\begin{aligned}
\|H_1\Psi\|^2 &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y) \langle \phi(f_x)^*\phi(f_x)\Psi, \phi(f_y)^*\phi(f_y)\Psi \rangle \mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y) \langle \Psi, \phi(f_x)^*\phi(f_x)\phi(f_y)^*\phi(f_y)\Psi \rangle \mathrm{d}x\mathrm{d}y \\
&\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y) \|\Psi\| \|\phi(f_x)^*\phi(f_x)\phi(f_y)^*\phi(f_y)\Psi\| \mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$\le \epsilon \iint_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\rm sp}(x)\chi_{\rm sp}(y)\big\|\phi(f_x)^*\phi(f_x)\phi(f_y)^*\phi(f_y)\Psi\big\|^2 {\rm d}x{\rm d}y + \frac{1}{4\epsilon}\big\|\chi_{\rm sp}\big\|_{L^1}^2\big\|\Psi\big\|^2 \tag{11}$$

$$= \epsilon \iint_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\rm sp}(x)\chi_{\rm sp}(y)\langle(\phi(f_x)^*\phi(f_x))^2\Psi, (\phi(f_y)^*\phi(f_y))^2\Psi\rangle {\rm d}x{\rm d}y + \frac{1}{4\epsilon}\big\|\chi_{\rm sp}\big\|_{L^1}^2\big\|\Psi\big\|^2$$

$$= \epsilon\big\|H_2\Psi\big\|^2 + \frac{1}{4\epsilon}\big\|\chi_{\rm sp}\big\|_{L^1}^2\big\|\Psi\big\|^2. \tag{12}$$

Here, to get (11), we used the following elementary inequality:

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2 \quad (\text{for } a, b \ge 0,\ \ \epsilon > 0). \tag{13}$$

Thus, H_1 is infinitesimally small with respect to H_2. Next we show that H_2 is H-bounded. For all $\Psi \in \mathcal{F}_{\rm b,fin}([C_0^\infty(\mathbb{R}^d)])$,

$$\begin{aligned}
&\big\|\lambda H_2\Psi\big\|^2 \\
&= \big\|(H - H_0 - \mu H_1)\Psi\big\|^2 \\
&= \big\|H\Psi\big\|^2 - \langle H\Psi, (H_0+\mu H_1)\Psi\rangle - \langle (H_0+\mu H_1)\Psi, H\Psi\rangle + \big\|(H_0+\mu H_1)\Psi\big\|^2 \\
&= \big\|H\Psi\big\|^2 - \lambda\langle H_2\Psi, H_0\Psi\rangle - \lambda\langle H_0\Psi, H_2\Psi\rangle \\
&\qquad\qquad - 2\lambda\mu{\rm Re}\langle H_1\Psi, H_2\Psi\rangle - \big\|(H_0+\mu H_1)\Psi\big\|^2 \\
&\le \big\|H\Psi\big\|^2 - \lambda\langle H_2\Psi, H_0\Psi\rangle - \lambda\langle H_0\Psi, H_2\Psi\rangle + 2\lambda|\mu||{\rm Re}\langle H_1\Psi, H_2\Psi\rangle|,
\end{aligned}$$

where Re z denotes the real part of $z \in \mathbb{C}$. By using (12) and (13), it follows that

$$\begin{aligned}
2\lambda|\mu||{\rm Re}\langle H_1\Psi, H_2\Psi\rangle| &\le 2\lambda|\mu|\big\|H_1\Psi\big\|\big\|H_2\Psi\big\| \\
&\le \epsilon\big\|H_2\Psi\big\|^2 + \frac{\lambda^2\mu^2}{\epsilon}\big\|H_1\Psi\big\|^2 \\
&\le \epsilon\big\|H_2\Psi\big\|^2 + \frac{\lambda^2\mu^2}{\epsilon}\Big(\eta\big\|H_2\Psi\big\|^2 + \frac{1}{4\eta}\big\|\chi_{\rm sp}\big\|_{L^1}^2\big\|\Psi\big\|^2\Big),
\end{aligned}$$

where ϵ and η are arbitrary positive constants. Therefore we have

$$\begin{aligned}
\big\|\lambda H_2\Psi\big\|^2 \le \big\|H\Psi\big\|^2 + \Big(\epsilon + \frac{\lambda^2\mu^2\eta}{\epsilon}\Big)\big\|H_2\Psi\big\|^2 + \frac{\lambda^2\mu^2}{4\epsilon\eta}\big\|\chi_{\rm sp}\big\|_{L^1}^2\big\|\Psi\big\|^2 \\
- \lambda\int_{\mathbb{R}^d}\chi_{\rm sp}(x)\big\langle\Psi, \big\{\big(\phi(f_x)^*\phi(f_x)\big)^2, H_0\big\}\Psi\big\rangle{\rm d}x,
\end{aligned}$$

where $\{X, Y\} := XY + YX$. By using the identity $X^2Y + YX^2 = 2XYX + [X, [X, Y]]$, we have

$$\begin{aligned}
\big\{\big(\phi(f_x)^*\phi(f_x)\big)^2, H_0\big\} = 2\phi(f_x)^*\phi(f_x)H_0\phi(f_x)^*\phi(f_x) \\
+ \Big[\phi(f_x)^*\phi(f_x), \big[\phi(f_x)^*\phi(f_x), H_0\big]\Big].
\end{aligned}$$

Since $\phi(f_x)^*\phi(f_x)H_0\phi(f_x)^*\phi(f_x)$ is non-negative, we see that

$$\begin{aligned}
\big\|\lambda H_2\Psi\big\|^2 \le \big\|H\Psi\big\|^2 + \Big(\epsilon + \frac{\lambda^2\mu^2\eta}{\epsilon}\Big)\big\|H_2\Psi\big\|^2 + \frac{\lambda^2\mu^2}{4\epsilon\eta}\big\|\chi_{\rm sp}\big\|_{L^1}^2\big\|\Psi\big\|^2 \\
- \lambda\int_{\mathbb{R}^d}\chi_{\rm sp}(x)\langle\Psi, [\phi(f_x)^*\phi(f_x), [\phi(f_x)^*\phi(f_x), H_0]]\Psi\rangle{\rm d}x.
\end{aligned}$$

By applying Proposition A.2, we have

$$\Big[\phi(f_x)^*\phi(f_x), \big[\phi(f_x)^*\phi(f_x), H_0\big]\Big] = -2\big\|\varphi\big\|_{L^2}^2\phi(f_x)^*\phi(f_x). \tag{14}$$

Here, to derive (14), $\varphi \in D(\omega^{1/2})$ is needed. Hence it follows that

$$\begin{aligned}\big\|\lambda H_2\Psi\big\|^2 \leq \big\|H\Psi\big\|^2 &+ 2\lambda\big\|\varphi\big\|_{L^2}^2\langle\Psi, H_1\Psi\rangle \\ &+ \Big(\epsilon + \frac{\lambda^2\mu^2\eta}{\epsilon}\Big)\big\|H_2\Psi\big\|^2 + \frac{\lambda^2\mu^2}{4\epsilon\eta}\big\|\chi_{\rm sp}\big\|_{L^1}^2\big\|\Psi\big\|^2.\end{aligned} \tag{15}$$

By using (10) and (13), we have

$$\begin{aligned}2\lambda\big\|\varphi\big\|_{L^2}^2\langle\Psi, H_1\Psi\rangle &\leq 2\lambda\|\varphi\|_{L^2}^2\|\Psi\|\|H_1\Psi\| \\ &\leq \big\|H_1\Psi\big\|^2 + \lambda^2\|\varphi\|_{L^2}^4\|\Psi\|^2 \\ &\leq \epsilon\big\|H_2\Psi\big\|^2 + \Big(\frac{\|\chi_{\rm sp}\|_{L^1}^2}{4\epsilon} + \lambda^2\|\varphi\|_{L^2}^4\Big)\|\Psi\|^2.\end{aligned} \tag{16}$$

From (15) and (12), it is seen that

$$\begin{aligned}\big\|\lambda H_2\Psi\big\|^2 \leq \big\|H\Psi\big\|^2 &+ \Big(2\epsilon + \frac{\lambda^2\mu^2\eta}{\epsilon}\Big)\big\|H_2\Psi\big\| \\ &+ \Big(\frac{\lambda^2\mu^2}{4\epsilon\eta}\big\|\chi_{\rm sp}\big\|_{L^1}^2 + \frac{\|\chi_{\rm sp}\|_{L^1}^2}{4\epsilon} + \lambda^2\|\varphi\|_{L^2}^4\Big)\big\|\Psi\big\|^2.\end{aligned}$$

By choosing constants ϵ and η such that $2\epsilon + \lambda^2\mu^2\eta/\epsilon < \lambda^2$, we have the following inequality:

$$(\lambda^2 - 2\epsilon - \lambda^2\mu^2\eta/\epsilon)\big\|H_2\Psi\big\|^2 \leq \big\|H\Psi\big\|^2 + \Big(\frac{\lambda^2\mu^2}{4\epsilon\eta}\big\|\chi_{\rm sp}\big\|_{L^1}^2 + \frac{\|\chi_{\rm sp}\|_{L^1}^2}{4\epsilon} + \lambda^2\|\varphi\|_{L^2}^4\Big)\big\|\Psi\big\|^2.$$

Thus (10) holds for all $\Psi \in \mathcal{F}_{\rm b,fin}([C_0^\infty(\mathbb{R}^d)])$. Since $\mathcal{F}_{\rm b,fin}([C_0^\infty(\mathbb{R}^d)])$ is a core of $\overline{H}$, (10) follows for all $\Psi \in D(\overline{H})$ from a limiting argument. (9) immediately follows from (10), (12) and (13). □

Proof of Theorem 3.1. We show $\overline{H} = H$ as an operator equality. Then we can conclude that H is self-adjoint since $\overline{H}$ is self-adjoint by Lemma 3.1. $\overline{H} \supset H$ is trivial. To show the inverse, it suffices to show that $D(\overline{H}) \subset D(H)$. For any $\Psi \in D(\overline{H})$, there exists a sequence $\{\Psi_n\}_{n=1}^\infty \subset \mathcal{F}_{\rm b,fin}([C_0^\infty(\mathbb{R}^d)])$ such that

$$\Psi_n \to \Psi, \quad H\Psi_n \to \overline{H}\Psi, \quad (n \to \infty).$$

By Lemma 3.2, H_0 is H-bounded on $\mathcal{F}_{\rm b,fin}([C_0^\infty(\mathbb{R}^d)])$. Indeed we note that the following inequality holds:

$$\begin{aligned}\big\|H_0\Theta\big\| = \big\|(H - \mu H_1 - \lambda H_2)\Theta\big\| \leq \big\|H\Theta\big\| + |\mu|\big\|H_1\Theta\big\| + \lambda\Theta\big\|H_2\Theta\big\|, \\ \big(\Theta \in \mathcal{F}_{\rm b,fin}([C_0^\infty(\mathbb{R}^d)])\big).\end{aligned}$$

Therefore, $\{H_0\Psi_n\}_{n=1}^\infty$ and $\{H_2\Psi_n\}_{n=1}^\infty$ are Cauchy sequences. By the closedness of H_0 and the closability of H_2, it follows that $\Psi \in D(H_0) \cap D(\overline{H}_2) = D(H)$. Reminder assertions follow from Lemma 3.1. □

4 Identification of the spectrum of H

Throughout in this section, we assume Assumption 3.1.

We recall that for a linear operator T, $\sigma(T)$ denotes the spectrum of T. We denote the *essential* spectrum of T by $\sigma_{\mathrm{ess}}(T)$ for a self-adjoint operator T. If T is bounded from below and self-adjoint, then we define

$$E_0(T) := \inf \sigma(T) > -\infty.$$

Our goal of this section is to determine the spectrum of H. A main result of this section is as follows:

Theorem 4.1. *It follows that*

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = [E_0(H), \infty).$$

Let us calculate $[\mu H_1 + \lambda H_2, A((u,v))^\dagger]$ with $u, v \in L^2(\mathbb{R}^d)$. For all $\Psi \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$, we see that

$$\left[\mu H_1 + \lambda H_2, A((u,v))^\dagger\right]\Psi = \frac{1}{\sqrt{2}}(\mu T_1 + \mu T_2 + 2\lambda T_3 + 2\lambda T_4)\Psi,$$

where,

$$\begin{aligned}
T_1 &:= \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\langle f_x, v\rangle \phi(f_x)\mathrm{d}x,\\
T_2 &:= \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\langle f_x, u\rangle \phi(f_x)^*\mathrm{d}x,\\
T_3 &:= \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\langle f_x, v\rangle \phi(f_x)\phi(f_x)^*\phi(f_x)\mathrm{d}x,\\
T_4 &:= \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\langle f_x, u\rangle \phi(f_x)^*\phi(f_x)\phi(f_x)^*\mathrm{d}x.
\end{aligned}$$

Note that these integrals on the right hand side are taken in the sense of $\mathcal{F}$-valued strong Bochner integral.

Lemma 4.1. *T_j $(j = 1,2,3,4)$ are H-bounded on $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$.*

Proof. Let $\Psi \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$. Then

$$\begin{aligned}
&\left\|T_1\Psi\right\|^2\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\langle f_x, v\rangle\langle v, f_y\rangle\langle \phi(f_y)\Psi, \phi(f_x)\Psi\rangle \mathrm{d}x\mathrm{d}y\\
&\le \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\Big|\langle f_x, v\rangle\langle v, f_y\rangle\langle \Psi, \phi(f_y)^*\phi(f_x)\Psi\rangle\Big| \mathrm{d}x\mathrm{d}y\\
&\le \left\|\omega^{-1/2}\omega\varphi\right\|^2\left\|v\right\|^2 \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\Big\{\frac{1}{2}\left\|\Psi\right\|^2 + \frac{1}{2}\left\|\phi(f_y)^*\phi(f_x)\Psi\right\|^2\Big\}\mathrm{d}x\mathrm{d}y\\
&= \frac{1}{2}\left\|\omega^{-1/2}\varphi\right\|^2\left\|v\right\|^2 \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\langle \phi(f_y)^*\phi(f_y)\Psi, \phi(f_x)^*\phi(f_x)\Psi\rangle \mathrm{d}x\mathrm{d}y\\
&\qquad\qquad + \frac{1}{2}\left\|\omega^{-1/2}\varphi\right\|^2\left\|v\right\|^2\left\|\chi_{\mathrm{sp}}\right\|_{L^1}^2\left\|\Psi\right\|^2\\
&= \frac{1}{2}\left\|\omega^{-1/2}\varphi\right\|^2\left\|v\right\|^2\left(\left\|H_1\Psi\right\|^2 + \left\|\chi_{\mathrm{sp}}\right\|_{L^1}^2\left\|\Psi\right\|^2\right).
\end{aligned}$$

By applying Lemma 3.2, T_1 is H-bounded. It is shown that T_2 is also H-bounded.

Next, we show the H-boundedness of T_3. It follows that

$$\begin{aligned}
&\|T_3\Psi\|^2\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\langle f_x, v\rangle\langle v, f_y\rangle\langle \phi(f_y)\phi(f_y)^*\phi(f_y)\Psi, \phi(f_x)\phi(f_x)^*\phi(f_x)\Psi\rangle \mathrm{d}x\mathrm{d}y\\
&\le \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\Big|\langle f_x, v\rangle\langle v, f_y\rangle\langle \phi(f_y)^*\phi(f_x)\Psi, \phi(f_x)^*\phi(f_x)\phi(f_y)^*\phi(f_y)\Psi\rangle\Big| \mathrm{d}x\mathrm{d}y\\
&\le \frac{1}{2}\|\omega^{-1/2}\varphi\|^2\|v\|^2\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\|\phi(f_y)^*\phi(f_x)\Psi\|^2\mathrm{d}x\mathrm{d}y\\
&\qquad + \frac{1}{2}\|\omega^{-1/2}\varphi\|^2\|v\|^2\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\|\phi(f_x)^*\phi(f_x)\phi(f_y)^*\phi(f_y)\Psi\|^2\mathrm{d}x\mathrm{d}y\\
&= \frac{1}{2}\|\omega^{-1/2}\varphi\|^2\|v\|^2\left(\|H_1\Psi\|^2 + \|H_2\Psi\|^2\right).
\end{aligned}$$

Thus T_3 is H-bounded by Lemma 3.2. The case of T_4 is also estimated similarly. Thus the desired results follow. □

Let $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty \subset D(\omega)\cap D(\omega^{-1/2})$ be arbitrary sequences such that

$$\operatorname*{w-lim}_{n\to\infty} u_n = 0,\quad \operatorname*{w-lim}_{n\to\infty} v_n = 0,\quad \|u_n\|^2 + \|v_n\|^2 = 1,\quad (n\in\mathbb{N}),$$

where w-lim denotes weak limit. It is seen that

$$\begin{aligned}
\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)]) \subset D\big((\mu H_1 + \lambda H_2)A((u_n,v_n))^\dagger\big) \bigcap D\big(A((u_n,v_n))^\dagger(\mu H_1 + \lambda H_2)\big)\\
\bigcap D\big((\mu H_1 + \lambda H_2)^* A((u_n,v_n))\big) \bigcap D\big(A((u_n,v_n))(\mu H_1 + \lambda H_2)^*\big).
\end{aligned}$$

By applying Proposition C.1 as $A = \mu H_1 + \lambda H_2$, $B = A((u_n,v_n))^\dagger$, $C = H$ and $\mathcal{D} = \mathcal{E}_C = \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$, we see that the weak commutator $[\mu H_1+\lambda H_2, A((u_n,v_n))^\dagger]_{\mathrm{w},D(H)}$ exists and

$$[\mu H_1+\lambda H_2, A((u_n,v_n))^\dagger]_{\mathrm{w},D(H)} = \frac{1}{\sqrt{2}}\Big(\overline{\mu T_{1.n}} + \overline{\mu T_{2.n}} + \overline{2\lambda T_{3.n}} + \overline{2\lambda T_{4.n}}\Big) \restriction D(H), \quad (17)$$

where

$$\begin{aligned}
T_{1.n} &:= \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\langle f_x, v_n\rangle\phi(f_x)\mathrm{d}x,\\
T_{2.n} &:= \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\langle f_x, u_n\rangle\phi(f_x)^*\mathrm{d}x,\\
T_{3.n} &:= \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\langle f_x, v_n\rangle\phi(f_x)\phi(f_x)^*\phi(f_x)\mathrm{d}x,\\
T_{4.n} &:= \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\langle f_x, u_n\rangle\phi(f_x)^*\phi(f_x)\phi(f_x)^*\mathrm{d}x.
\end{aligned}$$

Proof of Theorem 4.1. We apply Proposition C.2. Hence we need only to show that for all $\Psi \in D(H)$,

$$\lim_{n\to\infty}[\mu H_1 + \lambda H_2, A((u_n,v_n))^\dagger]_{\mathrm{w},D(H)}\Psi = 0.$$

By (17), we have

$$\lim_{n\to\infty}[\mu H_1 + \lambda H_2, A((u_n,v_n))^\dagger]_{\mathrm{w},D(H)}\Psi = \lim_{n\to\infty}\frac{1}{\sqrt{2}}(\mu\overline{T_{1.n}} + \mu\overline{T_{2.n}} + 2\lambda\overline{T_{3.n}} + 2\lambda\overline{T_{4.n}})\Psi.$$

Thus it suffices to show that $\lim_{n\to\infty} \|\overline{T_{j.n}}\Psi\| = 0$ $(j = 1,2,3,4)$. First, we consider $T_{1.n}$. Since $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ is a core of H, there exists a sequence $\{\Psi_k\}_k \subset \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ such that

$$\Psi_k \to \Psi, \quad H\Psi_k \to H\Psi, \ \ (k \to \infty).$$

Then $T_{1.n}\Psi_k \to \overline{T_{1.n}}\Psi$ $(k \to \infty)$ by Lemma 4.1. For any $k \in \mathbb{N}$, we have

$$\begin{aligned}\big\|\overline{T_{1.n}}\Psi\big\| &\le \big\|\overline{T_{1.n}}\Psi - T_{1.n}\Psi_k\big\| + \big\|T_{1.n}\Psi_k\big\| \\ &\le C\big\|H(\Psi - \Psi_k)\big\| + D\big\|\Psi - \Psi_k\big\| \\ &\qquad\qquad + E\big\|(N_{\mathrm{b}}+1)^{1/2}\Psi_k\big\| \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)|\langle f_x, v_n\rangle| \mathrm{d}x,\end{aligned}$$

where C, D and E are positive constants independent of n and k. By the property of v_n, it follows that

$$\lim_{n\to\infty} |\langle f_x, v_n\rangle| = 0, \quad (\text{for } x \in \mathbb{R}^d),$$

and

$$\chi_{\mathrm{sp}}(x)|\langle f_x, v_n\rangle| \le \chi_{\mathrm{sp}}(x)\big\|\omega^{-1/2}\varphi\big\|_{L^2}$$

is integrable on $\mathbb{R}^d$. Hence, by applying the Lebesgue dominated convergence theorem, we have

$$\limsup_{n\to\infty} \big\|\overline{T_{1.n}}\Psi\big\| \le C\big\|H(\Psi - \Psi_k)\big\| + D\big\|\Psi - \Psi_k\big\|.$$

Since $k \in \mathbb{N}$ is arbitrary, we have $\lim_{n\to\infty} \|\overline{T_{1.n}}\Psi\| = 0$ by taking $k \to \infty$. In the same manner, we can show that $\lim_{n\to\infty} \|\overline{T_{j.n}}\Psi\| = 0$ $(j = 2,3,4)$. □

5 Existence of ground states

Throughput in this section, we always assume Assumption 3.1.

5.1 Massive case

For a positive constant $m > 0$, we define the function ω_m by

$$\omega_m(k) := \sqrt{k^2 + m^2} \quad (k \in \mathbb{R}^d).$$

The constant $m > 0$ is regarded as the mass of a Boson. Let us introduce a *massive* Hamiltonian H_m as follows:

$$H_m := \mathrm{d}\Gamma_{\mathrm{b}}([\omega_m]) + \overline{\mu H_1 + \lambda H_2}.$$

In the same way as in the proof of Theorem 3.1, one can show that H_m is self-adjoint, bounded from below and essentially self-adjoint on $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$.

Remark 5.1. *H_1 and H_2 are H_m-bounded with*

$$\big\|\overline{H}_1\Psi\big\| \le \theta C_m(\mu,\lambda,\epsilon,\eta)\big\|H_m\Psi\big\| + \big(\theta C_m(\mu,\lambda,\epsilon,\eta) + \frac{1}{2\theta}\|\chi_{\mathrm{sp}}\|_{L^1}\big)\big\|\Psi\big\|,$$

$$\big\|\overline{H}_2\Psi\big\| \le C_m(\mu,\lambda,\epsilon,\eta)\big(\big\|H_m\Psi\big\| + \big\|\Psi\big\|\big), \quad \Psi \in D(H_m),$$

where θ is arbitrary positive constant and

$$C_m(\mu,\lambda,\epsilon,\eta) := (\lambda^2 - 2\epsilon - \lambda^2\mu^2\eta/\epsilon)^{-1/2} \times \Big(\frac{\lambda^2\mu^2}{4\epsilon\eta}\|\chi_{\rm sp}\|_{L^1}^2 + \frac{\|\chi_{\rm sp}\|_{L^1}^2}{4\epsilon} + \lambda^2\|\omega_m^{1/2}\omega^{-1/2}\varphi\|_{L^2}^4 + 1\Big)^{1/2},$$

with $\epsilon > 0$ and $\eta > 0$ being arbitrary such that $\lambda^2 > 2\epsilon + \lambda^2\mu^2\eta/\epsilon$. Note that $\mathrm{d}\Gamma_{\rm b}([\omega_m])$ is also H_m-bounded.

Let T be a bounded from below self-adjoint operator. In general, we say that T has ground states if $E_0(T)$ is an eigenvalue of T. A goal of this subsection is as follows:

Theorem 5.1. *H_m has ground states for arbitrary $\mu \in \mathbb{R}$ and $\lambda > 0$.*

Let us consider the following extended Hilbert space:

$$\mathcal{F} \otimes \mathcal{F}.$$

Then the *extended Hamiltonian $H_m^{\rm e}$* is defined as follows:

$$H_m^{\rm e} := H_m \otimes 1_{\mathcal{F}} + 1_{\mathcal{F}} \otimes \Gamma_{\rm b}([\omega_m]),$$

$$H_{0,m}^{\rm e} := \mathrm{d}\Gamma_{\rm b}([\omega_m]) \otimes 1_{\mathcal{F}} + 1_{\mathcal{F}} \otimes \mathrm{d}\Gamma_{\rm b}([\omega_m]).$$

Let us introduce a partition of unity. Let j_0 and j_∞ be $\mathbb{R}$-valued functions such that $j_0,\, j_\infty \in C^\infty(\mathbb{R}^d)$, $j_0^2 + j_\infty^2 = 1$, $0 \le j_0, j_\infty \le 1$ and

$$j_0(x) = \begin{cases} 1 & |x| \le 1, \\ 0 & |x| \ge 2, \end{cases}$$

where $C^\infty(\mathbb{R}^d)$ denotes the set of infinitely differentiable functions on $\mathbb{R}^d$. We set for $R > 0$,

$$j_{0,R} := j_0(\cdot/R), \ \ j_{\infty,R} := j_\infty(\cdot/R), \ \ \tilde{j}_{0,R} := j_{0,R}(-i\nabla_k), \ \ \tilde{j}_{\infty,R} := j_{\infty,R}(-i\nabla_k),$$

where $\nabla_k := (\partial/\partial k_1, \ldots, \partial/\partial k_d)$. We introduce an operator $\tilde{j}_R$ which maps $\oplus^2 L^2(\mathbb{R}^d)$ into $\oplus^4 L^2(\mathbb{R}^d)$ as follows:

$$\tilde{j}_R(u,v) := \big(\tilde{j}_{0,R}u,\ \tilde{j}_{0,R}v,\ \tilde{j}_{\infty,R}u,\ \tilde{j}_{\infty,R}v\big), \qquad (u,v) \in [L^2(\mathbb{R}^d)].$$

Note that $\tilde{j}_R$ is isometry. Let us denote the unitary operator which maps $\mathcal{F}_{\rm b}(\oplus^2[L^2(\mathbb{R}^d)])$ to $\mathcal{F}\otimes\mathcal{F}$ by $U_{[L^2(\mathbb{R}^d)],[L^2(\mathbb{R}^d)]}$ (see Proposition A.3). We define an operator $\check{\Gamma}(\tilde{j}_R) : \mathcal{F} \to \mathcal{F}\otimes\mathcal{F}$ by

$$\check{\Gamma}(\tilde{j}_R) := U_{[L^2(\mathbb{R}^d)],[L^2(\mathbb{R}^d)]}\Gamma_{\rm b}(\tilde{j}_R). \tag{18}$$

The following lemma is important to avoid making use of Number-Energy estimate. We set $N_0 := N_{\rm b} \otimes 1_{\mathcal{F}}$ and $N_\infty := 1_{\mathcal{F}} \otimes N_{\rm b}$.

Lemma 5.1. *There exists a constant $C > 0$ independent of R such that the following inequality holds.*

$$\big\|(N_0 + N_\infty + 1)^{-1}\big(H_j \otimes 1_{\mathcal{F}}\check{\Gamma}(\tilde{j}_R) - \check{\Gamma}(\tilde{j}_R)H_j\big)(N_{\rm b}+1)^{-1}\big\| \le C\int_{\mathbb{R}^d}\chi_{\rm sp}(x)\Big(\big\|(1-\tilde{j}_{0,R})f_x\big\| + \big\|\tilde{j}_{\infty,R}f_x\big\|\Big)\mathrm{d}x, \quad (j = 1,2).$$

Proof. We only show the case of $j=2$. The case of $j=1$ is proven similarly and we omit the proof. For any $\Psi \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$, we have

$$\begin{aligned}
&(N_0+N_\infty+1)^{-1}\big(H_2\otimes 1_{\mathcal{F}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_2\big)(N_{\mathrm{b}}+1)^{-1}\Psi \\
&=\int_{\mathbb{R}^d}\chi_{\mathrm{sp}}(x)(N_0+N_\infty+1)^{-1}\Big(\big(\phi(f_x)^*\phi(f_x)\big)^2\otimes 1_{\mathcal{F}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)\big(\phi(f_x)^*\phi(f_x)\big)^2\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times(N_{\mathrm{b}}+1)^{-1}\Psi\mathrm{d}x.
\end{aligned} \tag{19}$$

The integrand on the right hand side of (19) is decomposed as follows:

$$\begin{aligned}
&(N_0+N_\infty+1)^{-1}\Big(\big(\phi(f_x)^*\phi(f_x)\big)^2\otimes 1_{\mathcal{F}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)\big(\phi(f_x)^*\phi(f_x)\big)^2\Big)(N_{\mathrm{b}}+1)^{-1}\Psi \\
&=\sum_{k=0}^{3}(N_0+N_\infty+1)^{-1}\tilde{R}_k\Big(\phi\big((1-\tilde{j}_{0,R})f_x\big)^{\sharp_k}\otimes 1_{\mathcal{F}}-1_{\mathcal{F}}\otimes\phi\big(\tilde{j}_{0,\infty}f_x\big)^{\sharp_k}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\check{\Gamma}(\tilde{j}_R)R_{3-k}(N_{\mathrm{b}}+1)^{-1}\Psi,
\end{aligned} \tag{20}$$

where $R_0 := 1_{\mathcal{F}}$, $R_1 := \phi(f_x)$, $R_2 := \phi(f_x)^*\phi(f_x)$, $R_3 := \phi(f_x)\phi(f_x)^*\phi(f_x)$, $\tilde{R}_k := R_k^*\otimes 1_{\mathcal{F}}$ $(k=0,1,2,3)$, $\phi(u)^{\sharp_k} := \phi(u)^*$ if $k=0,2$, and $\phi(u)^{\sharp_k} := \phi(u)$ if $k=1,3$. To get (20), we used the following property:

$$\check{\Gamma}(\tilde{j}_R)\phi(u)^\sharp=\Big(\phi(\tilde{j}_{0,R}u)^\sharp\otimes 1_{\mathcal{F}}+1_{\mathcal{F}}\otimes\phi(\tilde{j}_{\infty,R}u)^\sharp\Big)\check{\Gamma}(\tilde{j}_R), \tag{21}$$

where $\phi(u)^\sharp$ denotes $\phi(u)$ or $\phi(u)^*$. To estimate (20), we divide the quartic products of field operators into two quadratic products of field operators. After that we apply the later assertion of Proposition A.1 as $T=1_{\mathcal{F}}$ or $T=1_{\mathcal{F}\otimes\mathcal{F}}$. To explain more precisely, we consider $k=0$ term of (20). For simplicity, we introduce following operators

$$\begin{aligned}
G_0 :=&(N_0+N_\infty+1)^{-1}\Big(\phi\big((1-\tilde{j}_{0,R})f_x\big)^*\otimes 1_{\mathcal{F}}-1_{\mathcal{F}}\otimes\phi\big(\tilde{j}_{0,\infty}f_x\big)^*\Big) \\
&\qquad\qquad\qquad\qquad\times\Big(\phi(\tilde{j}_{0,R}f_x)\otimes 1_{\mathcal{F}}+1_{\mathcal{F}}\otimes\phi(\tilde{j}_{\infty,R}f_x)\Big), \\
G_1 :=&\Big(\phi(\tilde{j}_{0,R}f_x)^*\otimes 1_{\mathcal{F}}+1_{\mathcal{F}}\otimes\phi(\tilde{j}_{\infty,R}f_x)^*\Big) \\
&\qquad\times\Big(\phi\big((1-\tilde{j}_{0,R})f_x\big)\otimes 1_{\mathcal{F}}-1_{\mathcal{F}}\otimes\phi\big(\tilde{j}_{0,\infty}f_x\big)\Big)(N_0+N_\infty+1)^{-1}, \\
G_2 :=&\phi(f_x)^*\phi(f_x)(N_{\mathrm{b}}+1)^{-1}.
\end{aligned}$$

By applying Proposition A.1, G_1 and G_2 are bounded. In particular, $\|G_2\|$ is dominated by a constant independent of x. Moreover, there exists a constant $C_0>0$ independent of x and R such that

$$\big\|G_1\big\|\leq C_0\big(\|(1-\tilde{j}_{0,R})f_x\|+\|\tilde{j}_{0,\infty}f_x\|\big). \tag{22}$$

By the general theory of adjoint operators, G_1^* is also bounded and $\|G_1^*\|=\|G_1\|$. Since $G_1^*\supset G_0$ and $\check{\Gamma}(\tilde{j}_R)G_2\Psi\in D(G_0)$, we have

$$\begin{aligned}
\big\|(k=0\ \text{ term of (20)})\big\| &\overset{(21)}{=}\big\|G_0\check{\Gamma}(\tilde{j}_R)G_2\Psi\big\| \\
&=\big\|G_1^*\check{\Gamma}(\tilde{j}_R)G_2\Psi\big\| \\
&\leq\big\|G_1^*\big\|\big\|G_2\big\|\|\Psi\| \\
&\leq C_0\big(\|(1-\tilde{j}_{0,R})f_x\|+\|\tilde{j}_{\infty,R}f_x\|\big)\big\|G_2\big\|\|\Psi\|.
\end{aligned}$$

Similarly, the other terms in (20) are also estimated. By combining these results, there exists a constant $C > 0$ independent of R such that

$$\begin{aligned}\big\|(N_0+N_\infty+1)^{-1}\big(H_2\otimes 1_{\mathcal{F}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_2\big)(N_{\mathrm{b}}+1)^{-1}\Psi\big\| \\ \le C\big\|\Psi\big\|\int_{\mathbb{R}^d}\chi_{\mathrm{sp}}(x)\big(\|(1-\tilde{j}_{0,R})f_x\|+\|\tilde{j}_{0,\infty}f_x\|\big)\mathrm{d}x.\end{aligned}$$

Since $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ is dense in $\mathcal{F}$, the desired result follows by making use of extension theorem of bounded operators. □

Lemma 5.2. *For any* $\chi \in C_0^\infty(\mathbb{R})$,

$$\lim_{R\to\infty}\big\|\chi(H_m^{\mathrm{e}})\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)\chi(H_m)\big\|=0.$$

Proof. By the Helffer-Sjöstrand formula [9,15], it is seen that

$$\begin{aligned}\chi(H_m^{\mathrm{e}})\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)\chi(H_m)=\frac{-i}{2\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\chi}(z)(z-H_m^{\mathrm{e}})^{-1} \\ \times\big(H_m^{\mathrm{e}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_m\big)(z-H_m)^{-1}\mathrm{d}z\mathrm{d}\overline{z},\end{aligned} \tag{23}$$

where $\tilde{\chi}$ is an *almost analytic extension* of χ and $\partial_{\overline{z}}=\frac{1}{2}(\partial_x+i\partial_y)$, $(z=x+iy)$. Let us estimate the integrand on the left hand side of (23). It follows that

$$\begin{aligned}&(z-H_m^{\mathrm{e}})^{-1}\big(H_m^{\mathrm{e}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_m\big)(z-H_m)^{-1} \\ &\quad=(z-H_m^{\mathrm{e}})^{-1}(N_0+N_\infty+1) \\ &\qquad\times(N_0+N_\infty+1)^{-1}\big(H_m^{\mathrm{e}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_m\big)(N_{\mathrm{b}}+1)^{-1} \\ &\qquad\qquad\times(N_{\mathrm{b}}+1)(z-H_m)^{-1}.\end{aligned}$$

It is easy to see that $(z-H_m^{\mathrm{e}})^{-1}(N_0+N_\infty+1)$ is bounded on $D(N_0+N_\infty)$ with the operator norm

$$\big\|(z-H_m^{\mathrm{e}})^{-1}(N_0+N_\infty+1)\big\|\le C\Big(1+(1+|z|)|\mathrm{Im}\,z|^{-1}\Big),$$

where $C>0$ is a constant independent of z and we used the fact that N_{b} is $\mathrm{d}\Gamma_{\mathrm{b}}([\omega_m])$ -bounded and the fact that if a linear operator S is bounded, so is S^*. Similarly one can show that $(N_{\mathrm{b}}+1)(z-H_m)^{-1}$ is also bounded with the operator norm

$$\big\|(N_{\mathrm{b}}+1)(z-H_m)^{-1}\big\|\le D\Big(1+(1+|z|)|\mathrm{Im}\,z|^{-1}\Big),$$

where $D>0$ is a constant independent of z. Thus we have

$$\begin{aligned}&\big\|(z-H_m^{\mathrm{e}})^{-1}\big(H_m^{\mathrm{e}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_m\big)(z-H_m)^{-1}\big\| \\ &\qquad\le CD\Big(1+(1+|z|)|\mathrm{Im}\,z|^{-1}\Big)^2 \\ &\qquad\qquad\times\big\|(N_0+N_\infty+1)^{-1}\big(H_m^{\mathrm{e}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_m\big)(N_{\mathrm{b}}+1)^{-1}\big\|.\end{aligned}$$

By the property of $\tilde{\chi}$, it suffices to show that

$$\lim_{R\to\infty}\big\|(N_0+N_\infty+1)^{-1}\big(H_m^{\mathrm{e}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_m\big)(N_{\mathrm{b}}+1)^{-1}\big\|=0. \tag{24}$$

We have following decomposition.

$$\begin{aligned}H_m^{\mathrm{e}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_m=\Big\{H_{0,m}^{\mathrm{e}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)\mathrm{d}\Gamma_{\mathrm{b}}([\omega_m])\Big\} \\ +\Big\{(\mu H_1+\lambda H_2)\otimes 1_{\mathcal{F}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)(\mu H_1+\lambda H_2)\Big\}.\end{aligned} \tag{25}$$

By the similar argument as in [7, Proof of Lemma 3.4.], [16, Lemma IV.4.], one can show that

$$\lim_{R\to\infty} \left\|(N_0+N_\infty+1)^{-1}\big(H^{\mathrm{e}}_{0,m}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)\mathrm{d}\Gamma_{\mathrm{b}}([\omega_m])\big)(N_{\mathrm{b}}+1)^{-1}\right\| = 0.$$

Next we estimate $(N_0+N_\infty+1)^{-1}(H_j\otimes 1_{\mathcal{F}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R))H_j)(N_{\mathrm{b}}+1)^{-1}$ $(j=1,2)$. By Lemma 5.1, there exists $C>0$ such that

$$\begin{aligned}&\left\|(N_0+N_\infty+1)^{-1}\big(H_j\otimes 1_{\mathcal{F}}\check{\Gamma}(\tilde{j}_R)-\check{\Gamma}(\tilde{j}_R)H_j\big)(N_{\mathrm{b}}+1)^{-1}\right\| \\ &\qquad\le C\int_{\mathbb{R}^d}\chi_{\mathrm{sp}}(x)\Big(\big\|(1-\tilde{j}_{0,R})f_x\big\|+\big\|\tilde{j}_{\infty,R}f_x\big\|\Big)\mathrm{d}x,\ (j=1,2).\end{aligned}$$

By definitions of $\tilde{j}_{0,R}$ and $\tilde{j}_{\infty,R}$, we have

$$\lim_{R\to\infty}\big\|(1-\tilde{j}_{0,R})f_x\big\| = \lim_{R\to\infty}\big\|\tilde{j}_{\infty,R}f_x\big\| = 0,$$
$$\big\|(1-\tilde{j}_{0,R})f_x\big\| \le \big\|\omega^{-1/2}\varphi\big\|,$$
$$\big\|\tilde{j}_{\infty,R}f_x\big\| \le \big\|\omega^{-1/2}\varphi\big\|.$$

By $\chi_{\mathrm{sp}}\in L^1(\mathbb{R}^d)$ and an application of Lebesgue dominated convergence theorem, it is seen that

$$\lim_{R\to\infty}\int_{\mathbb{R}^d}\chi_{\mathrm{sp}}(x)\Big(\big\|(1-\tilde{j}_{0,R})f_x\big\|+\big\|\tilde{j}_{\infty,R}f_x\big\|\Big)\mathrm{d}x = 0.$$

Therefore the desired result follows. □

Proof of Theorem 5.1. We show that for any $\chi\in C_0^\infty(\mathbb{R})$ such that supp $\chi\subset(-\infty,E_0(H_m)+m)$, $\chi(H_m)$ is a compact operator. By applying a general theorem [22, Theorem XIII-77], one sees that $\sigma(H_m)\cap(-\infty,E_0(H_m)+m)$ is purely discrete. In particular, H_m has a ground state.

Let $E_{N_{\mathrm{b}}}$ be the spectral measure of N_{b}. For any $n\in\mathbb{N}$, it follows that

$$\begin{aligned}E_{N_{\mathrm{b}}}(\{n\})\Gamma_{\mathrm{b}}([\tilde{j}^2_{0,R}])\chi(H_m) &= E_{N_{\mathrm{b}}}(\{n\})\Gamma_{\mathrm{b}}([\tilde{j}^2_{0,R}])(\mathrm{d}\Gamma_{\mathrm{b}}([\omega_m])+1)^{-1}\\ &\qquad\times(\mathrm{d}\Gamma_{\mathrm{b}}([\omega_m])+1)\chi(H_m) = J_1J_2,\end{aligned}$$

where

$$J_1 := E_{N_{\mathrm{b}}}(\{n\})\Gamma_{\mathrm{b}}([\tilde{j}^2_{0,R}])(\mathrm{d}\Gamma_{\mathrm{b}}([\omega_m])+1)^{-1},$$
$$J_2 := (\mathrm{d}\Gamma_{\mathrm{b}}([\omega_m])+1)\chi(H_m).$$

Since J_1 is compact (see [7, Lemma 4.2]) and J_2 is bounded, $E_{N_{\mathrm{b}}}(\{n\})\Gamma_{\mathrm{b}}([\tilde{j}^2_{0,R}])\chi(H_m)$ is compact. Note that

$$\begin{aligned}&\left\|\Gamma_{\mathrm{b}}([\tilde{j}^2_{0,R}])\chi(H_m)-\sum_{n=0}^{N}E_{N_{\mathrm{b}}}(\{n\})\Gamma_{\mathrm{b}}([\tilde{j}^2_{0,R}])\chi(H_m)\right\|\\ &\qquad\le\frac{1}{N+1}\big\|\Gamma_{\mathrm{b}}([\tilde{j}^2_{0,R}])(N_{\mathrm{b}}+1)\chi(H_m)\big\|\to 0,\ \ (N\to\infty).\end{aligned}$$

Thus $\Gamma_{\mathrm{b}}([\tilde{j}^2_{0,R}])\chi(H_m)$ is compact. Next we show that $\chi(H_m)$ is compact. Since supp $\chi\subset(-\infty,E_0(H_m)+m)$, it follows that

$$\chi(H^{\mathrm{e}}_m) = (1_{\mathcal{F}}\otimes P_0)\chi(H^{\mathrm{e}}_m), \tag{26}$$

where P_0 is the orthogonal projection onto the subspace $\{z\Omega : z\in\mathbb{C}\}$. Furthermore, the following properties also hold:

$$\check{\Gamma}(\tilde{j}_R)^*\check{\Gamma}(\tilde{j}_R) = 1_{\mathcal{F}},\qquad \check{\Gamma}(\tilde{j}_R)^*(1_{\mathcal{F}}\otimes P_0)\check{\Gamma}(\tilde{j}_R) = \Gamma_{\mathrm{b}}([\tilde{j}^2_{0,R}]).$$

By applying Lemma 5.2, we have

$$\begin{aligned}\chi(H_m) &= \check{\Gamma}(\tilde{j}_R)^*\check{\Gamma}(\tilde{j}_R)\chi(H_m) \\ &= \check{\Gamma}(\tilde{j}_R)^*\chi(H_m^{\mathrm{e}})\check{\Gamma}(\tilde{j}_R) + o(R^0) \\ &= \check{\Gamma}(\tilde{j}_R)^*(1_{\mathcal{F}} \otimes P_0)\chi(H_m^{\mathrm{e}})\check{\Gamma}(\tilde{j}_R) + o(R^0) \\ &= \check{\Gamma}(\tilde{j}_R)^*(1_{\mathcal{F}} \otimes P_0)\check{\Gamma}(\tilde{j}_R)\chi(H_m) + o(R^0) \\ &= \Gamma_{\mathrm{b}}([\tilde{j}_{0,R}^2])\chi(H_m) + o(R^0),\end{aligned}$$

where $o(R^0)$ denotes a bounded operator tending to 0 as $R \to \infty$ in the operator norm topology. Thus $\chi(H_m)$ is compact. □

5.2 Massless case

To show the existence of ground states of H, we need more assumptions:

Assumption 5.1. *It follows that*

(1) φ is a rotation-invariant function and has a compact support.

(2) There exists an open set $V \subset \mathbb{R}^d$ such that $\overline{V}$ =supp φ and φ is continuously differentiable on V. Here for $A \subset \mathbb{R}^d$, $\overline{A}$ denotes the closure of A.

(3) $\varphi \in D(\omega^{-3/2})$.

(4) $\omega^{-5/2}\varphi \in L^p(\mathbb{R}^d)$ and $\omega^{-3/2}\frac{\partial\varphi}{\partial k_j} \in L^p(\mathbb{R}^d)$ $(j = 1, \ldots, d)$ for all $1 \leq p < 2$.

(5) $\displaystyle\int_{\mathbb{R}^d}(1 + |x|^2)\chi_{\mathrm{sp}}(x)dx < \infty$.

Remark 5.2. *We give some comments about (1), (2), (3) and (4) of Assumption 5.1. A rotation invariance of φ implies that V has a cone property. This property and compactness of supp φ are needed to employ "Rellich-Kondrachov theorem". (3) is required to get a Boson number bound (see Lemma 5.5). (4) is important to show that a sequence of ground states belongs to suitable Sobolev spaces and its norm are uniformly bounded (see Lemma 5.9). We note that Assumption 5.1 implies Assumption 2.1 and Assumption 3.1.*

Remark 5.3. *We remark on the assumption of spacial cut-off function χ_{sp}. To show the existence of ground states of massive Hamiltonian H_m, we only use the condition $\chi_{\mathrm{sp}} \in L^1(\mathbb{R}^d)$. On the other hand, in the case of H, more faster decay of χ_{sp} is required to control a behavior of derivatives in ground states (see Lemma 5.6). Therefore as a sufficient condition, Assumption 5.1-(5) is needed.*

A main result of this subsection is as follows:

Theorem 5.2. *Suppose that Assumption 5.1 is satisfied. Then H has ground states for arbitrary $\lambda > 0$ and $\mu \in \mathbb{R}$.*

For $m > 0$, let Φ_m be a ground state of H_m with $\|\Phi_m\| = 1$.

Lemma 5.3. *$H_m \to H$ $(m \to 0)$ in the strong resolvent sense. In particular, $E_0(H_m) \to E_0(H)$ $(m \to 0)$.*

Proof. For any $\Psi \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$, we have $H_m\Psi \to H\Psi$ $(m \to 0)$ by a direct calculation. This fact implies the strong resolvent convergence [21, Theorem VIII 25 (a)]. The strong resolvent convergence implies that

$$\limsup_{m\to 0} E_0(H_m) \le E_0(H).$$

For any $m > 0$, we have

$$E_0(H_m) = \langle \Phi_m, H_m\Phi_m\rangle \ge \langle \Phi_m, H\Phi_m\rangle \ge E_0(H). \tag{27}$$

By taking $\liminf_{m\to 0}$ on the both sides of (27), we obtain the desired result. □

We can identify $\mathcal{F}$ as follows:

$$\mathcal{F} = \bigoplus_{n,n'=0}^{\infty} L^2_{\mathrm{sym}}(\mathbb{R}^{dn} \times \mathbb{R}^{dn'}),$$

where

$$L^2_{\mathrm{sym}}(\mathbb{R}^{dn}) := \Big\{ f \in L^2(\mathbb{R}^{dn}) \Big| f(k_{\pi(1)}, \cdots, k_{\pi(n)}) = f(k_1, \cdots, k_n) \\ \text{for a.e. } k_1, \cdots, k_n \in \mathbb{R}^d \text{ and } \pi \in \mathcal{S}_n \Big\},$$

$$L^2_{\mathrm{sym}}(\mathbb{R}^{dn} \times \mathbb{R}^{dn'}) \\ := \Big\{ f \in L^2(\mathbb{R}^{d(n+n')}) \Big| \text{for a.e. } k_1, \ldots k_n, l_1, \ldots l_{n'} \in \mathbb{R}^d, \sigma \in \mathcal{S}_n, \tau \in \mathcal{S}_{n'}, \\ f(k_{\sigma(1)}, \ldots, k_{\sigma(n)} : l_{\tau(1)}, \ldots, l_{\tau(n')}) = f(k_1, \ldots, k_n : l_1, \ldots, l_{n'}) \Big\},$$

$$L^2_{\mathrm{sym}}(\mathbb{R}^{dn} \times \mathbb{R}^0) := L^2_{\mathrm{sym}}(\mathbb{R}^{dn}),\ L^2_{\mathrm{sym}}(\mathbb{R}^0 \times \mathbb{R}^{dn'}) := L^2_{\mathrm{sym}}(\mathbb{R}^{dn'}),\ L^2_{\mathrm{sym}}(\mathbb{R}^0 \times \mathbb{R}^0) := \mathbb{C}.$$

For $k \in \mathbb{R}^d$, let us introduce linear operators $a_+(k)$ and $a_-(k)$ act on $\mathscr{H}$ are defined as follows:

$$(a_+(k)\Psi)^{(n,n')}(k_1, \ldots, k_n : l_1, \ldots, l_{n'}) \\ := \sqrt{n+1}\,\Psi^{(n+1,n')}(k, k_1, \ldots, k_n : l_1, \ldots, l_{n'}), \quad a.e.,$$

$$(a_-(k)\Psi)^{(n,n')}(k_1, \ldots, k_n : l_1, \ldots, l_{n'}) \\ := \sqrt{n'+1}\,\Psi^{(n,n'+1)}(k_1, \ldots, k_n : k, l_1, \ldots, l_{n'}), \quad a.e..$$

$a_+(\cdot)$ and $a_-(\cdot)$ are called the annihilation kernel of particle and anti-particle, respectively. For each $u \in L^2(\mathbb{R}^d)$, $a_+(u)$ and $a_-(u)$ are represented by using the annihilation kernel as follows:

$$a_\pm(u) = \int_{\mathbb{R}^d} u(k)^* a_\pm(k) \mathrm{d}k, \tag{28}$$

where the integrals on the right hand side of (28) are taken in the sense of $\mathcal{F}$-valued strong Bochner integral.

For $k \in \mathbb{R}^d$, let us introduce the following operators:

$$S_1(k) := \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x) e^{-ikx} \phi(f_x) \mathrm{d}x, \quad S_2(k) := \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x) e^{-ikx} \phi(f_x)\phi(f_x)^*\phi(f_x) \mathrm{d}x,$$

$$L_1(k) := \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x) e^{-ikx} \phi(f_x)^* \mathrm{d}x, \quad L_2(k) := \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x) e^{-ikx} \phi(f_x)^*\phi(f_x)\phi(f_x)^* \mathrm{d}x.$$

Note that these operators are also taken in the sense of $\mathcal{F}$-valued strong Bochner integral.

Lemma 5.4. *For almost every $k \in \mathbb{R}^d$, we have*

$$\begin{aligned} a_+(k)\Phi_m &= \frac{\varphi(k)}{\sqrt{2\omega(k)}}\big(E_0(H_m) - H_m - \omega_m(k)\big)^{-1}\big(\mu S_1(k) + 2\lambda\overline{S_2(k)}\big)\Phi_m, \\ a_-(k)\Phi_m &= \frac{\varphi(k)}{\sqrt{2\omega(k)}}\big(E_0(H_m) - H_m - \omega_m(k)\big)^{-1}\big(\mu L_1(k) + 2\lambda\overline{L_2(k)}\big)\Phi_m. \end{aligned} \tag{29}$$

Proof. Here, we prove only the first equation . The second one is proven similarly, and we omit the proof. Let $\Theta \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ and $g \in C_0^\infty(\mathbb{R}^d)$. Since $\Phi_m \in \mathrm{Ker}(H_m - E_0(H_m))$, we have

$$\begin{aligned} &\langle (H_m - E_0(H_m))\Theta, a_+(g)\Phi_m\rangle \\ &= \langle [a_+(g)^\dagger, H_m - E_0(H_m)]\Theta, \Phi_m\rangle \\ &= -\langle a_+(\omega_m g)^\dagger\Theta, \Phi_m\rangle \\ &\qquad - \frac{1}{\sqrt{2}}\int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)\langle g, f_x\rangle\langle\big(\mu\phi(f_x)^* + 2\lambda\phi(f_x)^*\phi(f_x)\phi(f_x)^*\big)\Theta, \Phi_m\rangle \mathrm{d}x \\ &= -\langle a_+(\omega_m g)^\dagger\Theta, \Phi_m\rangle \\ &\quad - \int_{\mathbb{R}^d\times\mathbb{R}^d} g(k)^* \frac{\varphi(k)}{\sqrt{2\omega(k)}}\chi_{\mathrm{sp}}(x)e^{-ikx}\big\langle\big(\mu\phi(f_x)^* + 2\lambda\phi(f_x)^*\phi(f_x)\phi(f_x)^*\big)\Theta, \Phi_m\big\rangle \mathrm{d}x\mathrm{d}k. \end{aligned} \tag{30}$$

Here to get the last equality of (30), we used Fubini's theorem. By using (28), we have

$$\begin{aligned} &\int_{\mathbb{R}^d} g(k)^*\langle (E_0(H_m) - H_m - \omega_m(k))\Theta, a_+(k)\Phi_m\rangle \mathrm{d}k \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(k)^*\frac{\varphi(k)}{\sqrt{2\omega(k)}}\chi_{\mathrm{sp}}(x)e^{-ikx}\big\langle\big(\mu\phi(f_x)^* + 2\lambda\phi(f_x)^*\phi(f_x)\phi(f_x)^*\big)\Theta, \Phi_m\big\rangle \mathrm{d}x\mathrm{d}k, \end{aligned}$$

Since $g \in C_0^\infty(\mathbb{R}^d)$ is arbitrary, we obtain

$$\begin{aligned} &\langle (E_0(H_m) - H_m - \omega_m(k))\Theta, a_+(k)\Phi_m\rangle \\ &\qquad = \frac{\varphi(k)}{\sqrt{2\omega(k)}}\int_{\mathbb{R}^d}\chi_{\mathrm{sp}}(x)e^{-ikx}\big\langle\big(\mu\phi(f_x)^* + 2\lambda\phi(f_x)^*\phi(f_x)\phi(f_x)^*\big)\Theta, \Phi_m\big\rangle \mathrm{d}x. \end{aligned}$$

Since $\Phi_m \in D(H_m)$, there exists a sequence $\{\Phi_m^j\}_{j=1}^\infty \subset \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ such that $\Phi_m^j \to \Phi_m$, $H_m\Phi_m^j \to H_m\Phi_m$ $(j \to \infty)$. Therefore we have

$$\begin{aligned} &\big\langle\big(E_0(H_m) - H_m - \omega_m(k)\big)\Theta, a_+(k)\Phi_m\big\rangle \\ &\qquad\qquad = \frac{\varphi(k)}{\sqrt{2\omega(k)}}\langle\Theta, \mu S_1(k)\Phi_m\rangle + \frac{\varphi(k)}{\sqrt{2\omega(k)}}\lim_{j\to\infty}\big\langle\Theta, 2\lambda S_2(k)\Phi_m^j\big\rangle, \end{aligned}$$

where we used the H_m-boundedness of $S_1(k)$. We show that for any $k \in \mathbb{R}^d$, $S_2(k)$ is H_m-bounded on $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$. For $\Psi \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$, It follows that

$$\begin{aligned} \big\|S_2(k)\Psi\big\|^2 &\le \int_{\mathbb{R}^d\times\mathbb{R}^d}\chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\big|\langle\phi(f_x)^*\phi(f_x)\phi(f_y)^*\phi(f_y)\Psi, \phi(f_y)\phi(f_x)^*\Psi\rangle\big|\mathrm{d}x\mathrm{d}y \\ &\le \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\langle\big(\phi(f_y)^*\phi(f_y)\big)^2\Psi, \big(\phi(f_x)^*\phi(f_x)\big)^2\Psi\rangle\mathrm{d}x\mathrm{d}y \\ &\qquad\qquad + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi_{\mathrm{sp}}(x)\chi_{\mathrm{sp}}(y)\langle\phi(f_y)^*\phi(f_y)\Psi, \phi(f_x)^*\phi(f_x)\Psi\rangle\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2}\big(\big\|H_2\Psi\big\|^2 + \big\|H_1\Psi\big\|^2\big). \end{aligned}$$

Therefore $S_2(k)$ is H_m-bounded by Remark 5.1. This fact implies that $\{S_2(k)\Phi_m^j\}_{j=1}^\infty$ is a Cauchy sequence. By the closability of $S_2(k)$, we have

$$\langle (E_0(H_m)-H_m-\omega_m(k))\Theta, a_+(k)\Phi_m\rangle = \frac{\varphi(k)}{\sqrt{2\omega(k)}}\Big\{\langle\Theta, \mu S_1(k)\Phi_m\rangle + \langle\Theta, 2\lambda\overline{S_2(k)}\Phi_m\rangle\Big\}.$$

Thus we see that $a_+(k)\Phi_m \in D(E_0(H_m) - H_m - \omega_m(k))$ and

$$(E_0(H_m) - H_m - \omega_m(k))a_+(k)\Phi_m = \frac{\varphi(k)}{\sqrt{2\omega(k)}}\big(\mu S_1(k) + 2\lambda\overline{S_2(k)}\big)\Phi_m.$$

Since $E_0(H_m) - H_m - \omega_m(k)$ has a bounded inverse, the first equation of (29) follows. □

Lemma 5.5. *Suppose that $\varphi \in D(\omega^{-3/2})$. Then,*

$$\limsup_{m\to 0} \|N_{\mathrm{b}}^{1/2}\Phi_m\| < \infty.$$

Proof. By Proposition A.3 and Proposition A.5, we see that

$$\|N_{\mathrm{b}}^{1/2}\Phi_m\|^2 = \int_{\mathbb{R}^d} \|a_+(k)\Phi_m\|^2 \mathrm{d}k + \int_{\mathbb{R}^d} \|a_-(k)\Phi_m\|^2 \mathrm{d}k.$$

Note that $S_1(k), S_2(k), L_1(k)$ and $L_2(k)$ are H_m-bounded uniformly in k. By Remark 5.1, Lemma 5.4 and $\|(E_0(H_m) - H_m - \omega_m(k))^{-1}\| \le \omega(k)^{-1}$, we have

$$\begin{aligned}
&\|N_{\mathrm{b}}^{1/2}\Phi_m\|^2 \\
&\le \int_{\mathbb{R}^d} \frac{|\varphi(k)|^2}{2\omega(k)} \big\|\big(E_0(H_m) - H_m - \omega_m(k)\big)^{-1}\big(\mu S_1(k) + 2\lambda\overline{S_2(k)}\big)\Phi_m\big\|^2 \mathrm{d}k \\
&\quad + \int_{\mathbb{R}^d} \frac{|\varphi(k)|^2}{2\omega(k)} \big\|\big(E_0(H_m) - H_m - \omega_m(k)\big)^{-1}\big(\mu L_1(k) + 2\lambda\overline{L_2(k)}\big)\Phi_m\big\|^2 \mathrm{d}k \\
&\le 2(|\mu|^2 + 4\lambda^2)\big(\|H_1\Phi_m\|^2 + \|H_2\Phi_m\|^2\| + \|\chi_{\mathrm{sp}}\|_{L^1}^2\|\Phi_m\|^2\big)\|\omega^{-3/2}\varphi\|^2 \\
&= D_m(\mu,\lambda,\epsilon,\eta,\theta)(|\mu|^2 + 4\lambda^2)\big(E_0(H_m)^2 + 1\big)\|\omega^{-3/2}\varphi\|_{L^2}^2,
\end{aligned}$$

where for any $\theta > 0$, $D_m(\mu,\lambda,\epsilon,\eta,\theta)$ is defined by

$$D_m(\mu,\lambda,\epsilon,\eta,\theta) := 2C_m(\mu,\lambda,\epsilon,\eta)^2(\theta^2 + 2) + 2\big(\theta C_m(\mu,\lambda,\epsilon,\eta) + \frac{1}{2\theta}\|\chi_{\mathrm{sp}}\|_{L^1}\big)^2.$$

Thus the desired result follows by Remark 5.1, Lemma 5.3 and taking the limit superior. □

Remark 5.4. *To show the existence of ground states for sufficiently small coupling constants under $\varphi \in D(\omega^{-3/2})$, it is important that $\|N_{\mathrm{b}}^{1/2}\Phi_m\|$ tends to zero when μ and λ tend to 0. Now, we fix $\mu = 0$ and consider the behavior of $\|N_{\mathrm{b}}^{1/2}\Phi_m\|$ as $\lambda \to 0$. Then we cannot conclude that* $\lim_{\lambda\to 0}\lambda^2 D_m(0,\lambda,\epsilon,\eta,\theta) = 0$. *Since $\lambda^2(\lambda^2 - 2\epsilon)^{-1} > 1$, it is not expected that $\lambda^2 C_m(0,\lambda,\epsilon,\eta)^2 \to 0$ (as $\lambda \to 0$). As a result, we cannot apply the idea of proof in [6]. Therefore it is interesting to show the existence of ground state for sufficiently small coupling constants under conditions weaker than Assumption 5.1.*

Lemma 5.6. *Suppose that φ is differentiable and $\int_{\mathbb{R}^d}(1+|x|^2)\chi_{\rm sp}(x)dx<\infty$. Then, $\mathbb{R}^d\setminus\{0\}\ni k\mapsto a_\pm(k)\Phi_m\in\mathcal{F}$ is strongly differentiable. Moreover, for $k\neq 0$,*

$$\begin{aligned}
&(D_j a_+)(k)\Phi_m\\
&\quad=\Big\{\frac{1}{\sqrt{2}\omega(k)^{1/2}}\frac{\partial\varphi}{\partial k_j}(k)-\frac{\varphi(k)k_j}{2\sqrt{2}\omega(k)^{5/2}}\Big\}\\
&\qquad\qquad\times\Big(E_0(H_m)-H_m-\omega_m(k)\Big)^{-1}(\mu S_1(k)+2\lambda\overline{S_2(k)})\Phi_m\\
&\qquad+\frac{k_j\varphi(k)}{\omega_m(k)\sqrt{2\omega(k)}}\big(E_0(H_m)-H_m-\omega_m(k)\big)^{-2}(\mu S_1(k)+2\lambda\overline{S_2(k)})\Phi_m\\
&\qquad-\frac{i\varphi(k)}{\sqrt{2\omega(k)}}\big(E_0(H_m)-H_m-\omega_m(k)\big)^{-1}\big(\mu S_{1.j}(k)+2\lambda\overline{S_{2.j}(k)}\big)\Phi_m,
\end{aligned}$$

$$\begin{aligned}
&(D_j a_-)(k)\Phi_m\\
&\quad=\Big\{\frac{1}{\sqrt{2}\omega(k)^{1/2}}\frac{\partial\varphi}{\partial k_j}(k)-\frac{\varphi(k)k_j}{2\sqrt{2}\omega(k)^{5/2}}\Big\}\\
&\qquad\qquad\times\Big(E_0(H_m)-H_m-\omega_m(k)\Big)^{-1}(\mu L_1(k)+2\lambda\overline{L_2(k)})\Phi_m\\
&\qquad+\frac{k_j\varphi(k)}{\omega_m(k)\sqrt{2\omega(k)}}\big(E_0(H_m)-H_m-\omega_m(k)\big)^{-2}(\mu L_1(k)+2\lambda\overline{L_2(k)})\Phi_m\\
&\qquad-\frac{i\varphi(k)}{\sqrt{2\omega(k)}}\big(E_0(H_m)-H_m-\omega_m(k)\big)^{-1}\big(\mu L_{1.j}(k)+2\lambda\overline{L_{2.j}(k)}\big)\Phi_m,
\end{aligned}$$

where

$$\begin{aligned}
S_{1.j}(k)&:=\int_{\mathbb{R}^d}x_j\chi_{\rm sp}(x)e^{-ikx}\phi(f_x)\mathrm{d}x,\\
S_{2.j}(k)&:=\int_{\mathbb{R}^d}x_j\chi_{\rm sp}(x)e^{-ikx}\phi(f_x)\phi(f_x)^*\phi(f_x)\mathrm{d}x,\\
L_{1.j}(k)&:=\int_{\mathbb{R}^d}x_j\chi_{\rm sp}(x)e^{-ikx}\phi(f_x)^*\mathrm{d}x,\\
L_{2.j}(k)&:=\int_{\mathbb{R}^d}x_j\chi_{\rm sp}(x)e^{-ikx}\phi(f_x)^*\phi(f_x)\phi(f_x)^*\mathrm{d}x,
\end{aligned}$$

and D_j is the strong differential operator in the j-th variable k_j.

Proof. Since $(E_0(H_m)-H_m-\omega_m(\cdot))^{-1}$ is differentiable in the operator norm topology and $\varphi/\sqrt{\omega}$ is differentiable for any $k\neq 0$, it suffices to show the strong differentiability of S_1, S_2, L_1 and L_2. Here we only show the case of S_2. Since $\Phi_m\in D(H_m)$, we can take a sequence $\{\Phi_m^j\}_{j=1}^\infty\subset\mathcal{F}_{\rm b,fin}([C_0^\infty(\mathbb{R}^d)])$ such that $\Phi_m^j\to\Phi_m$ and $H_m\Phi_m^j\to H_m\Phi_m$ $(j\to\infty)$. Since $S_2(k)$ and $S_{2.l}(k)$ are H_m-bounded, we have $S_2(k)\Phi_m^j\to\overline{S_2(k)}\Phi_m$ and $S_{2.l}(k)\Phi_m^j\to\overline{S_{2.l}(k)}\Phi_m$ $(j\to\infty)$. Let $\{e_l\}_{l=1}^d$ be the standard orthogonal basis of $\mathbb{R}^d$ and $h\in\mathbb{R}\setminus\{0\}$. It is seen that

$$\begin{aligned}
&\Big\|\frac{\overline{S_2(k+he_l)}-\overline{S_2(k)}}{h}\Phi_m+i\overline{S_{2,l}(k)}\Phi_m\Big\|^2\\
&\quad=\lim_{j\to\infty}\Big\|\frac{S_2(k+he_l)-S_2(k)}{h}\Phi_m^j+iS_{2,l}(k)\Phi_m^j\Big\|^2\\
&\quad\le\lim_{j\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi_{\rm sp}(x)\chi_{\rm sp}(y)\Big|\frac{e^{ihx_l}-1}{h}-ix_l\Big|\Big|\frac{e^{-ihy_l}-1}{h}+iy_l\Big|\\
&\qquad\qquad\times\big|\big\langle\phi(f_x)\phi(f_x)^*\phi(f_x)\Phi_m^j,\phi(f_y)\phi(f_y)^*\phi(f_y)\Phi_m^j\big\rangle\big|\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$\leq \lim_{j\to\infty} \int_{\mathbb{R}^d} \chi_{\rm sp}(x) \Big| \frac{e^{ihx_l}-1}{h} - ix_l \Big|^2 {\rm d}x$$
$$\times \Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\rm sp}(x)\chi_{\rm sp}(y) \big| \langle \phi(f_x)\phi(f_y)^*\Phi_m^j, \phi(f_x)^*\phi(f_x)\phi(f_y)^*\phi(f_y)\Phi_m^j \rangle \big|^2 {\rm d}x{\rm d}y \Big)^{\frac{1}{2}} \tag{31}$$

$$\leq \lim_{j\to\infty} C\|(d\Gamma_{\rm b}([\omega_m])+1)\Phi_m^j\| \int_{\mathbb{R}^d} \chi_{\rm sp}(x) \Big| \frac{e^{ihx_l}-1}{h} - ix_l \Big|^2 {\rm d}x$$
$$\times \Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\rm sp}(x)\chi_{\rm sp}(y) \langle \big(\phi(f_x)^*\phi(f_x)\big)^2\Phi_m^j, \big(\phi(f_y)^*\phi(f_y)\big)^2\Phi_m^j \rangle {\rm d}x{\rm d}y \Big)^{\frac{1}{2}}$$
$$\leq \lim_{j\to\infty} C\big\|(d\Gamma_{\rm b}([\omega_m])+1)\Phi_m^j\big\| \big\|H_2\Phi_m^j\big\| \int_{\mathbb{R}^d} \chi_{\rm sp}(x) \Big| \frac{e^{ihx_l}-1}{h} - ix_l \Big|^2 {\rm d}x. \tag{32}$$

Here to get (31), we used the Schwartz inequality. Since $\mathrm{d}\Gamma_{\rm b}([\omega_m])$ and H_2 are H_m-bounded, the limit in (32) exists. Note that $|(e^{ihx_l}-1)/h - ix_l|^2 \leq 4x_l^2$ and $\int_{\mathbb{R}^d} \chi_{\rm sp}(x)x_l^2 < \infty$. From the Lebesgue dominated convergence theorem, we see that $\overline{S_2(k)}\Phi_m$ is strongly differentiable and its strong derivative is $-i\overline{S_{2.l}(k)}\Phi_m$. By using the Leibniz rule for (29), we obtain the desired results. □

Lemma 5.7. *Suppose that the same assumption as in Lemma 5.6 holds. Then there exist positive constants C_1, C_2 and C_3 independent of m and k such that*

$$\big\|D_j a_\pm(k)\Phi_m\big\|_{\mathcal{F}} \leq C_1 \frac{|\varphi(k)|}{\omega(k)^{3/2}} + C_2 \frac{|\varphi(k)|}{\omega(k)^{5/2}} + C_3 \frac{1}{\omega(k)^{3/2}} \Big| \frac{\partial\varphi}{\partial k_j}(k) \Big| \quad (\textit{for } k \neq 0). \tag{33}$$

Moreover, we suppose that Assumption 5.1 holds. For any $p \in [1,2)$, it follows that

$$\sup_{0<m\leq 1} \sum_{j=1}^d \int_{\mathbb{R}^d} \big\|D_j a_\pm(k)\Phi_m\big\|_{\mathcal{F}}^p dk < \infty. \tag{34}$$

Proof. For $k \neq 0$, it is seen that $\big\|(E_0(H_m) - H_m - \omega_m(k))^{-1}\big\| \leq \omega(k)^{-1}$. Since $S_1(k)$, $S_2(k)$, $L_1(k)$, $L_2(k)$, $S_{1.j}(k)$, $S_{2.j}(k)$, $L_{1.j}(k)$ and $L_{2.j}(k)$ are uniformly H_m-bounded in k, we have (33). (34) is immediately follows from (33). □

We set $\Phi_m = \{\Phi_m^{(n,n')}\}_{n,n'=0}^\infty$. Note that $\Phi_m^{(n,n')}$ is a $d(n+n')$-variable function. We denote $k_j = (k_{j.1}, \ldots, k_{j.d})$ and $l_j = (l_{j.1}, \ldots, l_{j.d})$.

Lemma 5.8. *For $1 \leq i \leq n$ and $1 \leq j \leq d$, let $\partial_{i.j}$ be the distributional derivative in $k_{i.j}$ in V (see Assumption 5.1) and for $n+1 \leq i \leq n+n'$ and $1 \leq j \leq d$, $\partial_{i.j}$ be the distributional derivative with respect to $l_{i.j}$ in V. Suppose that Assumption 5.1 holds. Then,*

$$(\partial_{i.j}\Phi_m^{(n,n')})(k_1,\ldots,k_n : l_1,\ldots,l_{n'})$$
$$= \begin{cases} \frac{1}{\sqrt{n}} D_j a_+(k_i)\Phi_m^{(n-1,n')}(k_1,\ldots,\hat{k_i},\ldots,k_n : l_1,\ldots,l_{n'}), \\ \qquad\qquad\qquad\qquad 1 \leq i \leq n, \\ \frac{1}{\sqrt{n'}} D_j a_-(l_{i-n})\Phi_m^{(n,n'-1)}(k_1,\ldots,k_n : l_1,\ldots,\hat{l_i},\ldots,l_{n'}), \\ \qquad\qquad\qquad\qquad n+1 \leq i \leq n+n', \end{cases}$$

where $\hat{k}$ denotes omitting of k.

Proof. Here, we consider only the case of $1 \leq i \leq n$ and $1 \leq j \leq d$. The other case is proven in a similar manner. Let $f \in C_0^\infty(V^{n+n'})$. Then it suffices to show that

$$\begin{aligned}\int_{\mathbb{R}^{d(n+n')}} &\Phi_m^{(n,n')}(k_1,\dots,k_n : l_1,\dots,l_n)(\partial_{i.j} f)(k_1,\dots,k_n,l_1,\dots,l_{n'})\mathrm{d}^n k \mathrm{d}^{n'} l \\ &+ \frac{1}{\sqrt{n}} \int_{\mathbb{R}^{d(n+n')}} D_j a_+(k_i)\Phi_m^{(n-1,n')}(k_1,\dots,\hat{k_i},\dots,k_n : l_1,\dots,l_{n'}) \\ &\qquad\qquad \times f(k_1,\dots,k_n,l_1,\dots,l_{n'})\mathrm{d}^n k \mathrm{d}^{n'} l \\ &= 0,\end{aligned} \tag{35}$$

where $\mathrm{d}^n k := \mathrm{d}k_1 \cdots \mathrm{d}k_n$, $\mathrm{d}^{n'} l := \mathrm{d}l_1 \cdots \mathrm{d}l_{n'}$. We denote the standard orthogonal basis of $\mathbb{R}^d$ by $\{e_j\}_{j=1}^d$. Also, we introduce the following notations:

$$\begin{aligned}\big(\Delta_h^{i,j} f\big)&(k_1,\cdots,k_n : l_1,\cdots,l_m) \\ &:= f(k_1,\cdots,k_i + he_j,\cdots,k_n : l_1,\cdots,l_m) \\ &\qquad - f(k_1,\cdots,k_i,\cdots,k_n : l_1,\cdots,l_m)\end{aligned}$$

By the definition of classical derivative, the absolute value of the left hand side of (35) is calculated as follows:

$$\begin{aligned}\lim_{h\to 0}\Big|\int_{\mathbb{R}^{d(n+n')}} &\frac{\big(\Delta_h^{i,j}\Phi_m^{(n,n')}\big)(k_1,\dots,k_n : l_1,\dots,l_{n'})}{h} f(k_1,\dots,k_n,l_1,\dots,l_{n'})\mathrm{d}^n k \mathrm{d}^{n'} l \\ &- \frac{1}{\sqrt{n}}\int_{\mathbb{R}^{d(n+n')}} D_j a_+(k_i)\Phi_m^{(n-1,n')}(k_1,\dots,\hat{k_i},\dots,k_n : l_1,\dots,l_{n'}) \\ &\qquad\qquad \times f(k_1,\dots,k_n,l_1,\dots,l_{n'})\mathrm{d}^n k \mathrm{d}^{n'} l\Big|.\end{aligned}$$

Since $\Phi_m^{(n,n')} \in L^2_{\mathrm{sym}}(\mathbb{R}^{dn} \times \mathbb{R}^{dn'})$, we have

$$\begin{aligned}\lim_{h\to 0}\Big|\int_{\mathbb{R}^{d(n+n')}} &\frac{1}{\sqrt{n}}\Big(\frac{1}{h}\big(\delta_h^j a_+\big)(k_i) - D_j a_+(k_i)\Big)\Phi_m^{(n-1,n')}(k_1,\dots,\hat{k_i},\dots,k_n : l_1,\dots,l_{n'}) \\ &\qquad\qquad \times f(k_1,\dots,k_n,l_1,\dots,l_{n'})\mathrm{d}^n k \mathrm{d}^{n'} l\Big|,\end{aligned}$$

where

$$\big(\delta_h^j a_+\big)(k) := a_+(k + he_j) - a_+(k).$$

By applying the Schwarz inequality with respect to $\mathrm{d}k_1 \cdots \mathrm{d}k_{i-1}\mathrm{d}k_{i+1}\cdots \mathrm{d}k_n \mathrm{d}^{n'} l$, we see that it is dominated by

$$\begin{aligned}\lim_{h\to 0}\int_{\mathbb{R}^d} \frac{1}{\sqrt{n}}\Big\|\frac{1}{h}\big(\delta_h^j a_+\big)(k_i)\Phi_m^{(n-1,n')} - D_j a_+(k_i)\Phi_m^{(n-1,n')}\Big\|_{L^2(\mathbb{R}^{d(n+n'-1)})} \\ \times \big\|f(\cdot,k_i,\cdot)\big\|_{L^2(\mathbb{R}^{d(n+n'-1)})}\mathrm{d}k_i.\end{aligned} \tag{36}$$

Since the function $k \mapsto a_+(k)\Phi_m^{(n-1,n')}$ is strongly continuously differentiable in V, the first factor of the integrand of (36) is bounded on V uniformly in h. Therefore, we can apply the Lebesgue dominated convergence theorem and the desired result follows. □

Let us denote the Sobolev space of order 1 and index p on an open set U in $\mathbb{R}^{d(n+n')}$ by $W^{1,p}(U)$.

Lemma 5.9. *Suppose that Assumption 5.1 holds. Then for any $n + n' \geq 1$, $m > 0$ and $1 \leq p < 2$, $\Phi_m^{(n,n')} \in W^{1,p}(V^{n+n'})$ and*

$$\sup_{0<m\leq 1}\big\|\Phi_m^{(n,n')}\big\|_{W^{1,p}(V^{n+n'})} < \infty.$$

Proof. Similar to the proof of [14, Proof of Theorem 2.1, Step 2]. To prove this, we need Lemma 5.7 and Lemma 5.8 □

Proof of Theorem 5.2. Since $\{\Phi_m\}_{0<m\leq 1}$ is a bounded set on $\mathcal{F}$, there exists a sequence $\{\Phi_{m_j}\}_{j=1}^{\infty}$ and a vector $\Phi \in \mathcal{F}$ such that $m_j \to 0$, $(j \to \infty)$ and

$$\operatorname*{w-lim}_{j\to\infty} \Phi_{m_j} = \Phi.$$

Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $\Psi \in \mathcal{F}$ be arbitrary. Then

$$\langle \Psi, (H_{m_j} - z)^{-1}\Phi_{m_j}\rangle = \langle \Psi, (E_0(H_{m_j}) - z)^{-1}\Phi_{m_j}\rangle. \tag{37}$$

By taking $\lim_{j\to\infty}$ on the both sides of (37) and Lemma 5.3, we have,

$$\langle \Psi, (H - z)^{-1}\Phi\rangle = \langle \Psi, (E_0(H) - z)^{-1}\Phi\rangle.$$

This fact implies that $\Phi \in D(H)$ and

$$H\Phi = E_0(H)\Phi.$$

Hence Φ is a ground state of H if $\Phi \neq 0$. Now we assume that $\Phi = 0$. Then we have

$$\begin{aligned} \|\Phi_{m_j}\|^2 &= \sum_{n+n'\leq N} \|\Phi_{m_j}^{(n,n')}\|^2 + \sum_{n+n'>N} \|\Phi_{m_j}^{(n,n')}\|^2 \\ &\leq \sum_{n+n'\leq N} \|\Phi_{m_j}^{(n,n')}\|^2 + \frac{1}{N}\|N_{\mathrm{b}}^{1/2}\Phi_{m_j}\|^2, \end{aligned} \tag{38}$$

where $N \in \mathbb{N}$ is arbitrary. Now we show that for any n and n', $\Phi_{m_j}^{(n,n')}$ converges to $\Phi^{(n,n')} = 0$ strongly in $L^2(\mathbb{R}^{d(n+n')})$ sense. By applying Lemma 5.5 and the definition of the annihilation kernel, we have

$$\text{supp}\ \Phi_{m_j}^{(n,n')} = \overline{V^{n+n'}},$$

since $\Phi_{m_j}^{(n,n')} \in L^2_{\text{sym}}(\mathbb{R}^{dn} \times \mathbb{R}^{dn'})$ (see, e.g., [14, Proof of Theorem 2.1, Step2]). Since the Lebesgue measure of $V^{n+n'}$ is finite, we have $L^s(V^{n+n'}) \subset L^2(V^{n+n'})$ for all $s \geq 2$. Thus, $\Phi_{m_j}^{(n,n')}$ weakly converges to $\Phi^{(n,n')} = 0$ in the $L^p(V^{n+n'})$ sense. By Lemma 5.9, a subsequence of $\{\Phi_{m_j}^{(n,n')}\}_{j=1}^{\infty}$ converges to a vector $\hat{\Phi}^{(n,n')} \in W^{1,p}(V^{n+n'})$ in the $W^{1,p}(V^{n+n'})^*$ sense. It means that for any f_0, f_1, $\ldots$, $f_{d(n+n')} \in L^p(V^{n+n'})^* = L^r(V^{n+n'})$ with $1/r + 1/p = 1$,

$$\begin{aligned} &\int_{V^{n+n'}} f_0\big(\Phi_{m_j}^{(n,n')} - \hat{\Phi}^{(n,n')}\big)\mathrm{d}^n k\mathrm{d}^{n'} l \\ &\qquad + \sum_{i=1}^{d(n+n')} \int_{V^{n+n'}} f_i\partial_i\big(\Phi_{m_j}^{(n,n')} - \hat{\Phi}^{(n,n')}\big)\mathrm{d}^n k\mathrm{d}^{n'} l \to 0, \qquad (j \to \infty). \end{aligned}$$

Hence we have

$$0 = \Phi^{(n,n')}(k_1, \ldots, k_n : l_1, \ldots, l_{n'}) = \hat{\Phi}^{(n,n')}(k_1, \ldots, k_n : l_1, \ldots, l_{n'}), \quad \text{a.e..}$$

Thus we have for all $1 \leq p < 2$, $\Phi_{m_j}^{(n,n')} \to 0$, $(j \to \infty)$ in the weak sense of $W^{1,p}(V^{n+n'})$. By applying the Rellich-Kondrachov theorem (see, e.g.,[1, Theorem 6.3][19, Theorem 8.9]), we have

$$\lim_{j\to\infty} \|\Phi_{m_j}^{(n,n')}\|_{L^q(V^{n+n'})} = 0,$$

for all $q < \frac{d(n+n')p}{d(n+n')-p}$, since V has a cone property. To get $q = 2$, we choose p as

$$\begin{cases} \frac{2d(n+n')}{d(n+n')+2} < p < 2, & \text{if } 2 \leq d(n+n'), \\ p = 1, & \text{if } d(n+n') = 1. \end{cases}$$

Thus, by taking $\limsup_{j\to\infty}$ in (38), we have

$$1 = \limsup_{j\to\infty} \left\|\Phi_{m_j}\right\|^2 \leq \frac{1}{N} \limsup_{j\to\infty} \left\|N_{\mathrm{b}}^{1/2}\Phi_{m_j}\right\|^2.$$

By Lemma 5.5, this is a contradiction since N is arbitrary. Hence $\Phi \neq 0$. □

6 Total charge of eigenstates

Throughout this section, we assume Assumption 3.1. In this section, we discus the total charge of eigenstates. First, we show the following proposition:

Proposition 6.1. *H and Q strongly commute.*

Proof. It is trivial that H_0 and e^{-itQ} commute (see Proposition A.4). By Proposition A.2-(2) and Proposition A.4-(2), the following relations hold:

$$e^{-itQ}a_+(u)e^{itQ} = a_+(e^{-itq}u), \quad e^{-itQ}a_-(u)e^{itQ} = a_-(e^{itq}u),$$
$$e^{-itQ}a_+(u)^\dagger e^{itQ} = a_+(e^{-itq}u)^\dagger, \quad e^{-itQ}a_-(u)^\dagger e^{itQ} = a_-(e^{itq}u)^\dagger, \quad \big(u \in L^2(\mathbb{R}^d)\big).$$

Let $\Psi \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$. Then, $e^{itQ}\Psi \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ and we have

$$e^{-itQ}H_1e^{itQ}\Psi = \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)e^{-itQ}(\phi(f_x)^*\phi(f_x))e^{itQ}\Psi\mathrm{d}x,$$
$$e^{-itQ}H_2e^{itQ}\Psi = \int_{\mathbb{R}^d} \chi_{\mathrm{sp}}(x)e^{-itQ}(\phi(f_x)^*\phi(f_x))^2e^{itQ}\Psi\mathrm{d}x.$$

It follows that on $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$:

$$\begin{aligned} e^{-itQ}\phi(f_x)^*\phi(f_x)e^{itQ} &= \frac{1}{2}(a_+(e^{-itq}f_x)^\dagger + a_-(e^{itq}f_x))(a_+(e^{-itq}f_x) + a_-(e^{itq}f_x)^\dagger) \\ &= e^{-itq}\phi(f_x)^*e^{itq}\phi(f_x) = \phi(f_x)^*\phi(f_x). \end{aligned}$$

Therefore for any $\Psi \in \mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$, we see that

$$e^{-itQ}He^{itQ}\Psi = H\Psi.$$

Since e^{-itQ} is unitary and $\mathcal{F}_{\mathrm{b,fin}}([C_0^\infty(\mathbb{R}^d)])$ is a core of H, the above equality can be extended to the operator equality. By the functional calculus, we have

$$e^{-itQ}e^{-isH}e^{itQ} = e^{-isH}, \quad (s, t \in \mathbb{R}).$$

Hence the desired result follows. □

Remark 6.1. *Also the massive Hamiltonian H_m strongly commutes with Q. The proof is quite similar.*

By Proposition 6.1, $\mathcal{F}$ is decomposed with respect to the spectrum of the total charge operator Q as

$$\mathcal{F} = \bigoplus_{z\in\mathbb{Z}} \mathcal{F}_q(z),$$

where $\mathcal{F}_q(z) :=\mathrm{Ker}(Q - qz)$.

To explain Theorem 6.1, we introduce a linear transform τ on $[L^2(\mathbb{R}^d)]$ by

$$\tau(f, g) := (g, f) \quad \text{for} \quad (f, g) \in [L^2(\mathbb{R}^d)].$$

In the physical context under consideration, $\Gamma_{\mathrm{b}}(\tau)$ changes particles for anti-particles and anti-particles for particles simultaneously. If a self-adjoint operator A and $\Gamma_{\mathrm{b}}(\tau)$ are strongly commute (*i,e,* $\Gamma_{\mathrm{b}}(\tau)A\Gamma_{\mathrm{b}}(\tau) = A$), we say that A has a symmetry with respect to $\Gamma_{\mathrm{b}}(\tau)$. Following theorem says that a total charge of a non-degenerate eigenstate is automatically zero if a self-adjoint operator has symmetry with respect to $\Gamma_{\mathrm{b}}(\tau)$. A main result of this section is as follows:

Theorem 6.1. *Let A be a self-adjoint operator on $\mathcal{F}$ which satisfies following conditions:*

(1) A and Q strongly commute.

(2) There exists an eigenvalue λ such that $\mathrm{dimKer}(A - \lambda) = 1$.

(3) A and $\Gamma_{\mathrm{b}}(\tau)$ strongly commute.

Then for any $\Psi \in \mathrm{Ker}(A - \lambda) \setminus \{0\}$, $\Psi \in \mathcal{F}_q(0)$.

Remark 6.2. *Theorem 6.1 is not only applicable to H but also other models which are similar to H. H_m(see Section 5) and an operator whose form is* $\mathrm{d}\Gamma_{\mathrm{b}}([T]) + P(\phi(f)^*\phi(f))$ *are these examples. Here, $f \in L^2(\mathbb{R}^d)$, T is a non-negative self-adjoint operator on $L^2(\mathbb{R}^d)$ and $P(\cdot)$ is a bounded from below real polynomial. If the ground state is unique in these models, then we can conclude that the total charge of the ground state is zero.*

It is easy to see that H and $\Gamma_{\mathrm{b}}(\tau)$ strongly commute. From this fact, Theorem 5.2 and Theorem 6.1, we have the following consequence.

Corollary 6.1. *Suppose that the ground state of H is unique and Assumption 5.1 is satisfied. Then for any $\Psi \in\mathrm{ker}(H - E_0(H)) \setminus \{0\}$, $\Psi \in \mathcal{F}_q(0)$.*

Remark 6.3. *We expect that if ground states of H exists, it is unique (i.e.* $\mathrm{dimker}(H-E_0(H)) = 1$*). However we have not been solved yet. We left this problem for a future study.*

The following result is a slight generalization of [25, Theorem1.7].

Proposition 6.2. *Let A be a self-adjoint operator on $\mathcal{F}$ which satisfies following conditions:*

(1) A and Q strongly commute.

(2) There exists an eigenvalue λ such that $\mathrm{dimKer}(A - \lambda)$*=1.*

(3) $\Psi \in\mathrm{Ker}(A-\lambda)\setminus\{0\}$ satisfies $\|\Psi\| = 1$, $\Psi \in D(N_{\mathrm{b}}^{1/2})$ and $\|N_{\mathrm{b}}^{1/2}\Psi\|^2 < n_0$ for some $n_0 \in \mathbb{N}$.

Then $\Psi \notin \mathcal{F}_q(z)$ for all $|z| \geq n_0$. In particular, if we can choose n_0 as 1, then $\Psi \in \mathcal{F}_q(0)$.

Proof. An idea is quite similar to [25, Proof of Theorem 1.7] thus we omit the proof. □

Remark 6.4. *The later assertion of Proposition 6.2 is originally established in the case of ground states in [25]. Proposition 6.2 says that the total charge of a non-degenerate eigenvector is dominated by the expectation of number operator. This is natural in the following sense. If there is a state which is constructed by N-bosons, then it is impossible that the absolute value of a total charge of this state is more than N. By virtue of this proposition, we can reduce where the total charge of eigenstate are localized.*

To prove Theorem 6.1, we prepare some results. First, we discus properties of $\Gamma_{\rm b}(\tau)$. For $n, m \in \mathbb{N}$, we define

$$\begin{aligned} W(n,m) &\equiv \text{L.H.}\big\{a_+(f_1)^* \cdots a_+(f_n)^* a_-(g_1)^* \cdots a_-(g_m)^*\Omega \big| \\ &\qquad\qquad\qquad\qquad f_1, \cdots, f_n, g_1, \cdots g_m \in L^2(\mathbb{R}^d)\big\}, \\ W(0,m) &\equiv \text{L.H.}\big\{a_-(g_1)^* \cdots a_-(g_m)^*\Omega \big| g_1, \cdots g_m \in L^2(\mathbb{R}^d)\big\}, \\ W(n,0) &\equiv \text{L.H.}\big\{a_+(f_1)^* \cdots a_+(f_n)^*\Omega \big| f_1, \cdots, f_n \in L^2(\mathbb{R}^d)\big\}, \\ W(0,0) &\equiv \{c\Omega : c \in \mathbb{C}\}. \end{aligned}$$

Proposition 6.3. *Following assertions hold:*

(1) $\Gamma_{\rm b}(\tau)$ is unitary, self-adjoint and $\Gamma_{\rm b}(\tau)^2 = 1_{\mathcal{F}}$.

(2) For any $n, m \in \mathbb{N} \cup \{0\}$, $\Gamma_{\rm b}(\tau)\overline{W(n,m)} = \overline{W(m,n)}$.

(3) For any $z \in \mathbb{Z}$, $\Gamma_{\rm b}(\tau)\mathcal{F}_q(z) = \mathcal{F}_q(-z)$.

(4) As an operator equality, $\Gamma_{\rm b}(\tau)Q\Gamma_{\rm b}(\tau) = -Q$ holds.

Proof. (1) It is seen that τ is unitary, self-adjoint and $\tau^2 = 1$ on $[L^2(\mathbb{R}^d)]$. By the property of $\Gamma_{\rm b}(\cdot)$, $\Gamma_{\rm b}(\tau)$ is unitary, self-adjoint and $\Gamma_{\rm b}(\tau)^2 = 1_{\mathcal{F}}$.

(2) By the definition of $a_\pm$, canonical commutation relations and Proposition A.2 (2), it follows that

$$\begin{aligned} &\Gamma_{\rm b}(\tau)a_+(f_1)^* \cdots a_+(f_n)^* a_-(g_1)^* \cdots a_-(g_m)^*\Omega \\ &\qquad = a_-(f_1)^* \cdots a_-(f_n)^* a_+(g_1)^* \cdots a_+(g_m)^*\Omega \\ &\qquad = a_+(g_1)^* \cdots a_+(g_m)^* a_-(f_1)^* \cdots a_-(f_n)^*\Omega \in W(m,n). \end{aligned}$$

By the limiting argument, we have $\Gamma_{\rm b}(\tau)\overline{W(n,m)} = \overline{W(m,n)}$.

(3) For $z \in \mathbb{Z}$, we can identify $\mathcal{F}_q(z)$ as follows:

$$\mathcal{F}_q(z) = \bigoplus_{n-m=z} \overline{W(n,m)}.$$

By Proposition 6.3-(2), we have $\Gamma_{\rm b}(\tau)\mathcal{F}_q(z) = \mathcal{F}_q(-z)$.

(4) For any $\Psi \in \mathcal{F}_{\rm b,\,fin}([L^2(\mathbb{R}^d)])$, Ψ is decomposed as $\{\Psi^{(z)}\}_{z\in\mathbb{Z}}$ with $\Psi^{(z)} \in \mathcal{F}_q(z)$. By Proposition 6.3-(1) and (3), we have $\Gamma_{\rm b}(\tau)Q\Gamma_{\rm b}(\tau)\Psi^{(z)} = -z\Psi^{(z)}$. Thus we have $\Gamma_{\rm b}(\tau)Q\Gamma_{\rm b}(\tau)\Psi = -Q\Psi$. Since $\Gamma_{\rm b}(\tau)$ is unitary and $\mathcal{F}_{\rm b,fin}([L^2(\mathbb{R}^d)])$ is a core of Q, the desired result follows. □

Lemma 6.1. *Let A be a self-adjoint operator strongly commute with Q. Suppose that A has an eigenvalue λ and $N_\lambda :=$dimKer$(A - \lambda) < \infty$. We denote the orthogonal projection onto $\mathcal{F}_q(z)$ by P_z. Then for any $\Psi \in$Ker$(A - \lambda) \setminus \{0\}$,*

$$1 \le \sharp\{z \in \mathbb{Z} : P_z\Psi \neq 0\} \le N_\lambda,$$

where $\sharp A$ denotes the number of elements of a set A.

Proof. Since $\Psi \neq 0$, $1 \leq \sharp\{z \in \mathbb{Z} : P_z\Psi \neq 0\}$ is trivial. Suppose that there are $z_1, \cdots, z_N \in \mathbb{Z}$ such that $N > N_\lambda$, $z_i \neq z_j$ if $i \neq j$ and $P_{z_i}\Psi \neq 0$. Since A and Q strongly commute, $P_{z_i}\Psi \in \mathrm{Ker}(A-\lambda) \setminus \{0\}$. On the other hand, we have $\langle P_{z_i}\Psi, P_{z_j}\Psi\rangle_{\mathcal{F}} = 0$ if $(i \neq j)$. Thus $P_{z_1}\Psi, \cdots, P_{z_N}\Psi$ are linearly independent. As a result, we have $N_\lambda \geq N$. But it is a contradiction. □

Proof of Theorem 6.1. Let $\Psi \in \mathrm{Ker}(A-\lambda) \setminus \{0\}$. Since $\dim\mathrm{Ker}(A-\lambda) = 1$, there is a unique $z_0 \in \mathbb{Z}$ such that $P_{z_0}\Psi \neq 0$ by Lemma 6.1. Now we set $z_0 \neq 0$. By assumptions of Theorem 6.1, it follows that $\Gamma_{\mathrm{b}}(\tau)P_{z_0}\Psi \in \mathrm{Ker}(A-\lambda) \setminus \{0\}$. Since $P_{z_0}\Psi \in \mathcal{F}_q(z_0)$ and $\Gamma_{\mathrm{b}}(\tau)P_{z_0}\Psi \in \mathcal{F}_q(-z_0)$, we have

$$\langle P_{z_0}\Psi, \Gamma_{\mathrm{b}}(\tau)P_{z_0}\Psi\rangle = 0.$$

In particular $P_{z_0}\Psi$ and $\Gamma_{\mathrm{b}}(\tau)P_{z_0}\Psi$ are linearly independent. Thus we have $\dim\mathrm{Ker}(A-\lambda) \geq 2$. But it is a contradiction. Therefore we have $\Psi \in \mathcal{F}_q(0)$. □

A Some facts on the abstract Boson Fock space

In this appendix, we summarize some facts which are often used in this paper and are well known. We use the same notations and symbols as in Section 2. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces.

Proposition A.1. *[3, Proposition 4.24][4, Lemma 6.32] Let T be a non-negative self-adjoint operator on $\mathcal{X}$ with ker $T = \{0\}$. If $u \in D(T^{-1/2})$, then $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}) \subset D(A(u)) \cap D(A(u)^\dagger)$ and*

$$\begin{aligned}\|A(u)\Psi\| &\leq \|T^{-1/2}u\|\|\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}\Psi\|,\\ \|A(u)^\dagger\Psi\| &\leq \|T^{-1/2}u\|\|\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2}\Psi\| + \|u\|\|\Psi\|,\end{aligned}$$

for all $\Psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T)^{1/2})$. Moreover if $u, v \in D(T) \cap D(T^{-1/2})$, then $D(\mathrm{d}\Gamma_{\mathrm{b}}(T)) \subset D(A(u)^\sharp A(v)^\sharp)$ and

$$\begin{aligned}\|A(u)^\sharp A(v)^\sharp\Psi\| \leq C&\|(\mathrm{d}\Gamma_{\mathrm{b}}(T)+1)\Psi\|\\ &\times \left(\|T^{-1/2}u)\| + \|u\|\right)\left(\|T^{-1/2}v\| + \|v\| + \|Tv\| + \|T^{1/2}v\|\right),\end{aligned}$$

for all $\Psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(T))$. Here $A(\cdot)^\sharp$ denotes $A(\cdot)$ or $A(\cdot)^\dagger$ and $C > 0$ is a constant independent of u, v, T and Ψ.

Proposition A.2. *[3, Proposition 4.26][7, Lemma 2.7 and Lemma 2.8] Let T be a densely defined closable operator on $\mathcal{X}$ and $u \in D(T) \cap D(T^*)$. Then,*

(1)

$$[\mathrm{d}\Gamma_{\mathrm{b}}(T), A(u)] = -A(T^*u), \qquad [\mathrm{d}\Gamma_{\mathrm{b}}(T), A(u)^\dagger] = A(Tu)^\dagger, \quad \text{on } \mathcal{F}_{\mathrm{b,fin}}(D(T)).$$

(2) If $u \in D(T)$, then

$$\Gamma_{\mathrm{b}}(T)A(u)^{\dagger} = A(Tu)^{\dagger}\Gamma_{\mathrm{b}}(T), \quad \text{on } \mathcal{F}_{\mathrm{b,fin}}(D(T)).$$

Moreover, if T is isometry, then

$$\Gamma_{\mathrm{b}}(T)A(u) = A(Tu)\Gamma_{\mathrm{b}}(T).$$

Proposition A.3. *[3, Theorem 4-55 and Theorem 4-56] Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. Then there exists a unique unitary operator $U_{\mathcal{X},\mathcal{Y}}$: $\mathcal{F}_{\mathrm{b}}(\mathcal{X} \oplus \mathcal{Y}) \to \mathcal{F}_{\mathrm{b}}(\mathcal{X}) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{Y})$ satisfying the following (1) and (2):*

(1)

$$U_{\mathcal{X},\mathcal{Y}}\Omega_{\mathcal{X}\oplus\mathcal{Y}} = \Omega_{\mathcal{X}} \otimes \Omega_{\mathcal{Y}},$$

where $\Omega_{\mathcal{X}}$ is the Fock vacuum in $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$.

(2)

$$U_{\mathcal{X},\mathcal{Y}}A(u,v)^{\sharp}U_{\mathcal{X},\mathcal{Y}}^{-1} = \overline{A(u)^{\sharp} \otimes I + I \otimes A(v)^{\sharp}}, \quad \big(u \in \mathcal{X}, v \in \mathcal{Y}\big),$$

and

$$U_{\mathcal{X},\mathcal{Y}}\mathcal{F}_{\mathrm{b,fin}}(\mathcal{X} \oplus \mathcal{Y}) = \mathcal{F}_{\mathrm{b,fin}}(\mathcal{X})\hat{\otimes}\mathcal{F}_{\mathrm{b,fin}}(\mathcal{Y}).$$

Moreover, for all self-adjoint operators T on $\mathcal{X}$ and S on $\mathcal{Y}$,

$$U_{\mathcal{X},\mathcal{Y}}\mathrm{d}\Gamma_{\mathrm{b}}(T \oplus S)U_{\mathcal{X},\mathcal{Y}}^{-1} = \overline{\mathrm{d}\Gamma_{\mathrm{b}}(T) \otimes I + I \otimes \mathrm{d}\Gamma_{\mathrm{b}}(S)}.$$

Remark A.1. *If T and S are non-negative self-adjoint in the above, then*

$$\overline{d\Gamma_{\mathrm{b}}(T) \otimes I + I \otimes \mathrm{d}\Gamma_{\mathrm{b}}(S)} = \mathrm{d}\Gamma_{\mathrm{b}}(T) \otimes I + I \otimes \mathrm{d}\Gamma_{\mathrm{b}}(S).$$

Proposition A.4. *[3, Theorem 4-17 and Theorem 4-20] Let A and B be self-adjoint operators on $\mathcal{K}$.*

(1)A and B strongly commute if and only if $\mathrm{d}\Gamma_{\mathrm{b}}(A)$ and $\mathrm{d}\Gamma_{\mathrm{b}}(B)$ strongly commute.

(2)

$$\Gamma_{\mathrm{b}}(e^{-itA}) = e^{-it\mathrm{d}\Gamma_{\mathrm{b}}(A)}.$$

Let $\mathcal{K} = L^2(\mathbb{R}^d)$. Then $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$ is rewritten as follows:

$$\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d)) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}(\mathbb{R}^{dn}),$$

where

$$L^2_{\mathrm{sym}}(\mathbb{R}^{dn}) := \big\{f \in L^2(\mathbb{R}^{dn}) | f(k_{\pi(1)}, \cdots, k_{\pi(n)}) = f(k_1, \cdots, k_n) \\ \text{for a.e. } k_1, \cdots, k_n \in \mathbb{R}^d \text{ and } \pi \in \mathcal{S}_n\big\}.$$

For a.e. $k \in \mathbb{R}^d$, an annihilation kernel $a(k)$ act on $\mathscr{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$ is defined as follows.

$$(a(k)\Psi)^{(n)}(k_1, \cdots, k_n) := \sqrt{n+1}\Psi^{(n+1)}(k, k_1, \cdots, k_n) \quad \text{a.e. } (k_1, \cdots, k_n) \in \mathbb{R}^{dn}.$$

Proposition A.5. *[3, Proposition 8.6] Let f be a measurable function such that $0 \leq f(k) < \infty$ for a.e. $k \in \mathbb{R}^d$. Then $\Psi \in D(\mathrm{d}\Gamma_{\mathrm{b}}(f)^{1/2})$ if and only if*

$$\int_{\mathbb{R}^d} f(k)\big\|a(k)\Psi\big\|^2 dk < \infty.$$

In that case

$$\big\|\mathrm{d}\Gamma_{\mathrm{b}}(f)^{1/2}\Psi\big\|^2 = \int_{\mathbb{R}^d} f(k)\big\|a(k)\Psi\big\|^2 dk.$$

B A criteria of essential self-adjointness

In this appendix, we summarize the result of [2] For $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{X}_n$ be a Hilbert space. We set $\mathcal{X} \equiv \oplus_{n=0}^{\infty} \mathcal{X}_n$. We introduce the following subspace:

$$\mathcal{X}_{\text{fin}} \equiv \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{X} | \exists N \text{ such that } \Psi^{(n)} = 0 \quad \text{for all } n \geq N+1 \right\}.$$

The number operator $N_{\mathcal{X}}$ is defined by

$$D(N_{\mathcal{X}}) := \left\{ \Psi \in \mathcal{X} \middle| \sum_{n=0}^{\infty} n^2 \|\Psi^{(n)}\|_{\mathcal{X}_n}^2 < \infty \right\},$$

$$(N_{\mathcal{X}}\Psi)^{(n)} := n\Psi^{(n)}, \qquad \left(\Psi \in D(N_{\mathcal{X}}),\ \ n \in \mathbb{N} \cup \{0\}\right).$$

For $n \in \mathbb{N} \cup \{0\}$, let A_n be a self-adjoint operator on $\mathcal{X}_n$, and set $A := \oplus_{n=0}^{\infty} A_n$. Let B be a symmetric operator on $\mathcal{X}$. We identify $\Psi^{(n)} \in \mathcal{X}_n$ as

$$\mathcal{X}_n \ni \Psi^{(n)} = \{0, \cdots, 0, \Psi^{(n)}, 0, \cdots\} \in \mathcal{X}.$$

Proposition B.1. *Suppose that the following (1)-(3) hold:*

(1) $\mathcal{X}_{\text{fin}} \subset D(B)$ *and* $A+B$ *is bounded from below on* $D(A) \cap \mathcal{X}_{\text{fin}}$.

(2) There exists a $p \in \mathbb{N}$ *such that*

$$\langle \Psi^{(n)}, B\Psi^{(m)} \rangle_{\mathcal{X}} = 0, \qquad \left(\Psi \in \mathcal{X}_{\text{fin}},\ \text{ whenever } \ |n-m| \geq p\right).$$

(3) There exist a constant $c > 0$ *and a linear operator* L *on* $\mathcal{X}$ *such that* $D(((A+B) \restriction D(A) \cap \mathcal{X}_{\text{fin}})^*) \subset D(L)$, $\text{Ran}(L \restriction D(L) \cap \mathcal{X}_n) \subset \mathcal{X}_n$ *and*

$$|\langle \Phi, B\Psi \rangle| \leq c \|L\Phi\| \|(N_{\mathcal{X}}+1)^2 \Psi\|, \quad \left(\Psi \in \mathcal{X}_{\text{fin}},\ \Phi \in D(L)\right).$$

Then $A+B$ *is essentially self-adjoint on* $D(A) \cap \mathcal{X}_{\text{fin}}$.

C Location of the essential spectrum

In this appendix, we summarize the result of [5]. Let $\mathcal{K}$ and $\mathcal{X}$ be two Hilbert spaces. In this chapter we consider the following Hilbert space:

$$\mathcal{K} \otimes \mathcal{F}_{\text{b}}(\mathcal{X})$$

Let A be a bounded from below self-adjoint operator on $\mathcal{K}$ and S be a non-negative self-adjoint operator on $\mathcal{X}$ with Ker $S = \{0\}$. Then

$$K_0 := A \otimes 1 + 1 \otimes \text{d}\Gamma_{\text{b}}(S)$$

is self-adjoint on $D(A \otimes 1) \cap D(1 \otimes \text{d}\Gamma_{\text{b}}(S))$. Let K_I be a symmetric operator on $\mathcal{K} \otimes \mathcal{F}_{\text{b}}(\mathcal{X})$ and

$$K := K_0 + K_I.$$

Let us recall the notion of a weak commutator.

Definition C.1. *Let $\mathscr{X}$ be a Hilbert space. Let A and B be densely defined linear operators on $\mathscr{X}$. If there exists a dense subspace $\mathscr{Y}$ and a linear operator L such that $\mathscr{Y} \subset D(L) \cap D(A) \cap D(A^*) \cap D(B) \cap D(B^*)$ and*

$$\langle A^*\psi, B\phi\rangle - \langle B^*\psi, A\phi\rangle = \langle \psi, L\phi\rangle, \qquad \big(\psi, \phi \in \mathscr{Y}\big),$$

then we say that the couple $\langle A, B\rangle$ has the weak commutator on $\mathscr{Y}$ defined by

$$[A, B]_{w,\mathscr{Y}} := L \restriction \mathscr{Y}.$$

The next proposition gives a sufficient condition for $\langle A, B\rangle$ to have a weak commutator.

Proposition C.1. *Let $\mathscr{X}$ be a Hilbert space and $\mathcal{D}$ be a dense subspace of $\mathscr{X}$. Let A and B be densely defined linear operators on $\mathcal{X}$ such that $\mathcal{D} \subset D(A) \cap D(B) \cap D(A^*) \cap D(B^*)$. Assume that there exist a densely defined closed linear operator C on $\mathscr{X}$ and a core $\mathcal{E}_C$ of C with the following properties:*

(1) $\mathcal{E}_C \subset \mathcal{D} \subset D(C)$.

(2) A and B are C-bounded on $\mathcal{E}_C$.

(3) $\mathcal{E}_C \subset D(AB) \cap D(BA)$ and $M := [A, B] \restriction \mathcal{E}_C$ is C-bounded on $\mathcal{E}_C$.

(4) $D(A^*B^*) \cap D(B^*A^*)$ is dense in $\mathscr{X}$.

Then M is closable with $D(C) \subset D(\overline{M})$ and $\langle A, B\rangle$ has a weak commutator on $\mathcal{D}$ which is given by

$$[A, B]_{w,\mathcal{D}} = \overline{M} \restriction \mathcal{D}.$$

Proposition C.2. *Suppose that following (1) and (2) hold.*

(1) K is self-adjoint and bounded from below.

(2) For any $u \in D(S) \cap D(S^{-1/2})$, the couple $\langle K_I, I \otimes A(u)^\dagger\rangle$ has the weak commutator $[K_I, I \otimes A(u)^\dagger]_{\mathrm{w},D(K)}$ on $D(K)$. Furthermore, for any $\Psi \in D(K)$, and any sequences $\{u_n\}_{n=1}^\infty \subset D(S) \cap D(S^{-1/2})$ such that $\|u_n\| = 1$, $\mathrm{w\text{-}lim}_{n\to\infty} u_n = 0$, and

$$\lim_{n\to\infty} [K_I, I \otimes A(u_n)^\dagger]_{w,D(K)}\Psi = 0.$$

If $\sigma(S) = [0, \infty)$, then

$$\sigma(K) = \sigma_{\mathrm{ess}}(K) = [E_0(K), \infty).$$

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