博士学位論文

Low Dimensional Homology of Artin Groups

(アルティン群の低次ホモロジー)

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1 Introduction

The study of *braid groups* is an active topic in diverse areas of mathematics and theoretical physics. In 1925, E. Artin [Art25] introduced the notion of braids in a geometric picture. Fox and Neuwirth [FN62] showed that the configuration space of unordered n points in $\mathbb{C}$ is a classifying space of the braid group $Br(n)$. This led to extensive investigations of the cohomology of braid groups by Arnol'd [Arn69], Fuks [Fuk70], Vaĭnšteĭn [Vaĭ78] and many others. It is remarkable to note that the braid group is closely related to the hyperplane arrangement associated to the symmetric group, known as the braid arrangement. The above mentioned configuration space is nothing but the orbit space of the complement to the complexified braid arrangement with respect to the action of the symmetric group by permutation of coordinates.

A generalization of the relation between braid groups and symmetric groups is that of *Artin groups* and *Coxeter groups*. For a Coxeter graph Γ and the associated Coxeter system $(W(\Gamma), S)$, we associate an Artin group $A(\Gamma)$ obtained by, informally speaking, dropping the relations that each generator has order 2 from the standard presentation of $W(\Gamma)$. The braid group $Br(n)$ is the Artin group associated to the Coxeter graph of type A_{n-1} and the symmetric group $\mathfrak{S}_n$ is the Coxeter group associated to the Coxeter graph of type A_{n-1}. When $W(\Gamma)$ is a finite Coxeter group, we say that $A(\Gamma)$ is of finite type (or spherical type). Recall that a finite Coxeter group $W(\Gamma)$ can be geometrically realized as an orthogonal reflection group acting on $\mathbb{R}^n$ where $n = \#S$ is the rank of W. Let $\mathcal{A}$ be the collection of the reflection hyperplanes (in $\mathbb{R}^n$) determined by W, known as the Coxeter arrangement associated to W. Topologically, it is more interesting to consider the complexified Coxeter arrangement $\mathcal{A}_{\mathbb{C}} = \{H \otimes \mathbb{C} \mid H \in \mathcal{A}\}$ and its complement $M(\Gamma) = \mathbb{C}^n \setminus \cup_{H\in\mathcal{A}} H \otimes \mathbb{C}$. The Coxeter group $W(\Gamma)$ acts freely on $M(\Gamma)$. Set $N(\Gamma) = M(\Gamma)/W(\Gamma)$ to be the quotient space. Brieskorn [Bri71] proved that the fundamental group of $N(\Gamma)$ is isomorphic to the Artin group $A(\Gamma)$. Furthermore, as a consequence of a theorem of Deligne [Del72], $N(\Gamma)$ is a $K(A(\Gamma), 1)$ space. Hence the (co)homology of $N(\Gamma)$ is the (co)homology of the Artin group $A(\Gamma)$ of finite type. There are many computations of (co)homology of Artin groups of finite type in the literature. Besides the above mentioned references for braid groups (type A_n), see [Gor78] for types

C_n and D_n, and [Sal94] for exceptional types. Cohomology ring structure is computed in [Lan00].

When $W(\Gamma)$ is an infinite Coxeter group, we say that the associated Artin group $A(\Gamma)$ is of infinite type (or non-spherical type). In this case, the Coxeter group $W(\Gamma)$ can be realized as a (non-orthogonal) reflection group acting on a convex cone U (called Tits cone, see Subsection 2.1) in $\mathbb{R}^n$ with $n = \#S$ the rank of W. Let $\mathcal{A}$ be the collection of reflection hyperplanes. Consider the complement $M(\Gamma) = (\mathrm{int}(U) + \sqrt{-1}\mathbb{R}) \setminus \cup_{H \in \mathcal{A}} H \otimes \mathbb{C}$ and the $W(\Gamma)$-action on $M(\Gamma)$, the resulting quotient space $N(\Gamma) = M(\Gamma)/W(\Gamma)$ has fundamental group isomorphic to $A(\Gamma)$ ([vdL83]). The celebrated $K(\pi, 1)$ conjecture asks whether $N(\Gamma)$ is a $K(A(\Gamma), 1)$ space. See Subsection 2.3 for a list of Γ for which the $K(\pi, 1)$ conjecture is proved to hold.

The most effective tool in the computation of cohomology of Artin group is the so-called *Salvetti complex* introduced by Salvetti in [Sal87]. In that paper, Salvetti associated a CW-complex (known as Salvetti complex) to each real hyperplane arrangement which has the homotopy type of the complement to the complexified arrangement. Later, Salvetti [Sal94] and De Concini-Salvetti [DCS96] applied the construction of Salvetti complex to reflection arrangements associated to (possibly infinite) Coxeter groups and obtained a very useful algebraic complex that computes the (co)homology of the quotient space $N(\Gamma)$ of the complement $M(\Gamma)$ with respect to the Coxeter group $W(\Gamma)$. Whenever $N(\Gamma)$ is known to be a $K(\pi, 1)$ space, this provides a standard method to compute the (co)homology of the Artin group $A(\Gamma) \cong \pi_1(N(\Gamma))$ over both trivial and twisted coefficients. See Subsection 3.3 for a list of results using this method.

Existing results about (co)homology of Artin groups all rely on the affirmative solution of the $K(\pi, 1)$ conjecture, since the computations are actually the (co)homology of the quotient space $N(\Gamma)$. In Section 4, we compute the second mod 2 homology of arbitrary Artin groups, without assuming an affirmative solution of the $K(\pi, 1)$ conjecture. Our main result is a formula for the second mod 2 homology of arbitrary Artin groups. Our main tool is the classical Hopf's formula on the second homology (or Schur multiplier) of groups, together with Howlett's theorem (Theorem 4.3) on the second homology of Coxeter groups. We are primarily inspired by [Pit99] and [KS03], where the authors computed the second integral homology of the mapping class groups of oriented surfaces using Hopf's formula. Section 4 is based on joint work with Professor Toshiyuki Akita.

In fact, we shall prove that for any Artin group $A(\Gamma)$, the second integral homology fits into a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^{p(\Gamma)+q(\Gamma)} \arrow[r] \arrow[dr] & H_2(A(\Gamma);\mathbb{Z}) \arrow[d] \\
& H_2(W(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}
\end{tikzcd}$$

where all maps are surjective, $p(\Gamma)$ and $q(\Gamma)$ are nonnegative integers associated to the Coxeter graph Γ defined in Theorem 3.8. By taking tensor product with $\mathbb{Z}_2$ for this diagram, we derive that

$$H_2(A(\Gamma);\mathbb{Z}_2) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}.$$

As a corollary, we obtain a sufficient condition of the triviality of the Hurewicz homomorphism

$$h_2 : \pi_2(N(\Gamma)) \to H_2(N(\Gamma);\mathbb{Z}).$$

Furthermore, we conclude that the induced Hurewicz homomorphism

$$h_2 \otimes \mathrm{id}_{\mathbb{Z}_2} : \pi_2(N(\Gamma)) \otimes \mathbb{Z}_2 \to H_2(N(\Gamma);\mathbb{Z}) \otimes \mathbb{Z}_2$$

is always trivial. This provides affirmative evidence for the $K(\pi,1)$ conjecture.

In the last section, we present a computation of the cohomology ring structure of 2-dimensional Artin groups. Our computation relies on a suitable Δ-complex structure of the classifying space.

2 Basic definitions

We collect relevant definitions and properties of Coxeter groups and Artin groups. We refer to [Bou68], [Hum90] and [Par14] for details.

2.1 Coxeter groups

Let S be a finite set. A *Coxeter matrix* over S is a symmetric matrix $M = (m(s,t))_{s,t\in S}$ such that $m(s,s) = 1$ for all $s \in S$ and $m(s,t) = m(t,s) \in \{2, 3, \cdots\} \cup \{\infty\}$ for distinct $s, t \in S$. It is convenient to represent M by a labeled graph Γ, called the *Coxeter graph* of M defined as follows:

- The vertex set $V(\Gamma) = S$;
- The edge set $E(\Gamma) = \{\{s,t\} \subset S \mid m(s,t) \geq 3\}$;
- The edge $\{s,t\}$ is labeled by $m(s,t)$ if $m(s,t) \geq 4$.

Let Γ_{odd} be the subgraph of Γ with $V(\Gamma_{odd}) = V(\Gamma) = S$ and $E(\Gamma_{odd}) = \{\{s,t\} \in E(\Gamma) \mid m(s,t) \text{ is odd}\}$ inheriting labels from Γ. By abuse of notations, we frequently regard Γ (hence also Γ_{odd}) as its underlying 1-dimensional CW-complex with the set of 0-cells S and the set of 1-cells $\{\langle s,t\rangle \mid \{s,t\} \in E(\Gamma)\}$.

For two letters s, t and an integer $m \geq 2$, we shall use the following notation of the word of length m consisting of s and t in an alternating order.

$$(st)_m := \overbrace{sts\cdots}^{m}.$$

Definition 2.1. Let Γ be a Coxeter graph and S its vertex set. The *Coxeter system* associated to Γ is by definition the pair $(W(\Gamma), S)$, where $W(\Gamma)$ is the group presented by

$$W(\Gamma) = \langle S \mid \overline{R_W} \cup Q_W\rangle.$$

The sets of relations are $\overline{R_W} = \{R(s,t) \mid m(s,t) < \infty\}$ and $Q_W = \{Q(s) \mid s \in S\}$, where $R(s,t) := (st)_{m(s,t)}(ts)^{-1}_{m(s,t)}$ and $Q(s) := s^2$.

Note that since $R(s,t) = R(t,s)^{-1}$, we may reduce the relation set $\overline{R_W}$ by introducing a total order on S and put $R_W := \{R(s,t) \mid m(s,t) < \infty, s < t\}$. We have the following presentation with fewer relations

$$W(\Gamma) = \langle S \mid R_W \cup Q_W \rangle.$$

The group $W = W(\Gamma)$ is called the *Coxeter group* of type Γ. We shall omit the reference to Γ if there is no ambiguity. The rank of W is defined to be $\#S$.

Remark 2.2. There is also a standard presentation of $W(\Gamma)$ used in the literature

$$W(\Gamma) = \langle S \mid (st)^{m(s,t)} = 1, \forall s,t \in S \text{ with } m(s,t) \neq \infty \rangle.$$

Each generator $s \in S$ of W has order 2. For distinct $s, t \in S$, the order of st is precisely $m(s,t)$ if $m(s,t) \neq \infty$. In case $m(s,t) = \infty$, the element st has infinite order. Therefore, given a pair (W, S) with the above presentation, it uniquely determines a Coxeter matrix, hence a Coxeter graph.

2.1.1 Parabolic subgroups and Poincaré series

Let (W, S) be a Coxeter system. For a subset $T \subset S$, let W_T denote the subgroup of W generated by T, called a *parabolic subgroup* of W. In particular, $W_S = W$ and $W_\varnothing = \{1\}$. It is known that (W_T, T) is the Coxeter system (cf. Théorème 2 in Chapter IV of [Bou68]) associated to the Coxeter graph Γ_T (the full subgraph of Γ spanned by T inheriting labels). If $T \subset S$ generates a finite parabolic subgroup, we say that T is a spherical subset. Denote by $\mathcal{S}^f$ the collection of all spherical subsets.

Recall that the length $\ell(w)$ of an element $w \in W$ is the defined as the minimal number k such that w can be written as a word $w = s_1 s_2 \cdots s_k$ with $s_i \in S$, and such a word is called a reduced expression of w. We set $\ell(w) = 0$ if $w = 1$. The restriction of the length function to any parabolic subgroup W_T agrees with the length function of the Coxeter system (W_T, T).

Define $W^T := \{w \in W \mid \ell(wt) > \ell(w) \text{ for all } t \in T\}$. Then for $w \in W$, there is a unique element $u \in W^T$ and a unique element $v \in W_T$ such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$. Moreover, u is the unique element of minimal length in the coset wW_T. Hence the set W^T is called the set of minimal coset representatives.

For a subset $X \subset W$, we define the Poincaré series of X

$$X(q) = \sum_{x \in X} q^{\ell(x)}.$$

A consequence of the last paragraph is that for a subset $T \subset S$,

$$W(q) = W_T(q) W^T(q).$$

2.1.2 Geometric representation

As mentioned in the introduction, each Coxeter group can be geometrically realized as a reflection group acting on a Tits cone. In this subsection, we describe this construction.

Let (W, S) be a Coxeter system. Denote by V the real vector space with basis $\{\alpha_s \mid s \in S\}$. We define a symmetric bilinear form B on V by setting

$$B(\alpha_s, \alpha_t) := -\cos \frac{\pi}{m(s,t)}.$$

We agree that $\cos \frac{\pi}{\infty} = 1$. For each $s \in S$, we define the reflection ρ_s on V by

$$\rho_s(x) = x - \frac{2B(\alpha_s, x)}{B(\alpha_s, \alpha_s)} \alpha_s,$$

for $x \in V$. Remark that $\rho_s(\alpha_s) = -\alpha_s$ and ρ_s fixes pointwise the hyperplane H_s orthogonal to α_s with respect to B. The assignment $s \mapsto \rho_s$ defines a faithful representation of W,

$$\rho : W \to \mathrm{GL}(V).$$

Let us consider the contragradient representation

$$\rho^* : W \to \mathrm{GL}(V^*),$$

where V^* is the dual space of V. It is defined by the following

$$\langle \rho^*(w)(f), \alpha \rangle = \langle f, \rho(w^{-1})(\alpha) \rangle,$$

where $f \in V^*, \alpha \in V$ and $\langle \bullet, \bullet \rangle$ is the natural pairing of V^* and V. Let C_0 be the fundamental chamber defined as $C_0 := \{f \in V^* \mid \langle f, \alpha_s \rangle > 0 \text{ for all } s \in S\}$. Define the Tits cone $U := W\overline{C_0}$ as the W-orbit of the closure of C_0. It is a W-stable subset of V^*. Recall some important properties of U.

Proposition 2.3 ([Vin71]). *Let U be defined as above. Then*

(i) U is a convex cone in V^* with vertex 0.

(ii) $\mathrm{int}(U)$ *is open in V^*.*

(ii) The followings are equivalent.

(a) $U = V^*$, (b) W is finite, (c) B is positive definite.

2.1.3 Finite and affine Coxeter groups

A Coxeter system (W, S) is said to be *irreducible* if the Coxeter graph Γ is connected. The following proposition allows us to reduce the study of Coxeter systems to that of irreducible ones.

Proposition 2.4. *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ be the connected components of Γ, and $S_i \subset S$ be the vertex set of Γ_i. Then the Coxeter group W is the direct product of its parabolic subgroups $W_{S_1}, W_{S_2}, \ldots, W_{S_m}$ and the Coxeter system (W_{S_i}, S_i) is irreducible.*

The classification of finite irreducible Coxeter groups is well known.

Theorem 2.5. *Let (W, S) be a finite irreducible Coxeter system, then the Coxeter graph Γ must be one of the listed graphs in Figure 1, where the subscript denotes the rank of the corresponding Coxeter group.*

Among infinite Coxeter groups, there is an important class called *affine* Coxeter groups. They arise as affine reflection groups in the Euclidean space. Recall that a finite Coxeter group W is called *crystallographic* if W stabilizes a lattice in V where the action is given by the geometric representation $\rho : W \to \mathrm{GL}(V)$. Such a group is also known as a Weyl group. They are characterized by the following proposition.

Proposition 2.6. *W is crystallographic if and only if $m(s,t) = 2, 3, 4$ or 6 for each pair of distinct $s, t \in S$.*

We simply rule out H_3, H_4 and $I_2(m)$ for $m = 5, 7, 8, 9, \ldots$ from the list in Figure 1 to get a list of irreducible Weyl groups.

Let $\mathrm{Aff}(V)$ be the group of affine transformations of V, which is the semidirect product of $\mathrm{GL}(V)$ with the group of translations in V. We define the affine Weyl group W_a as the

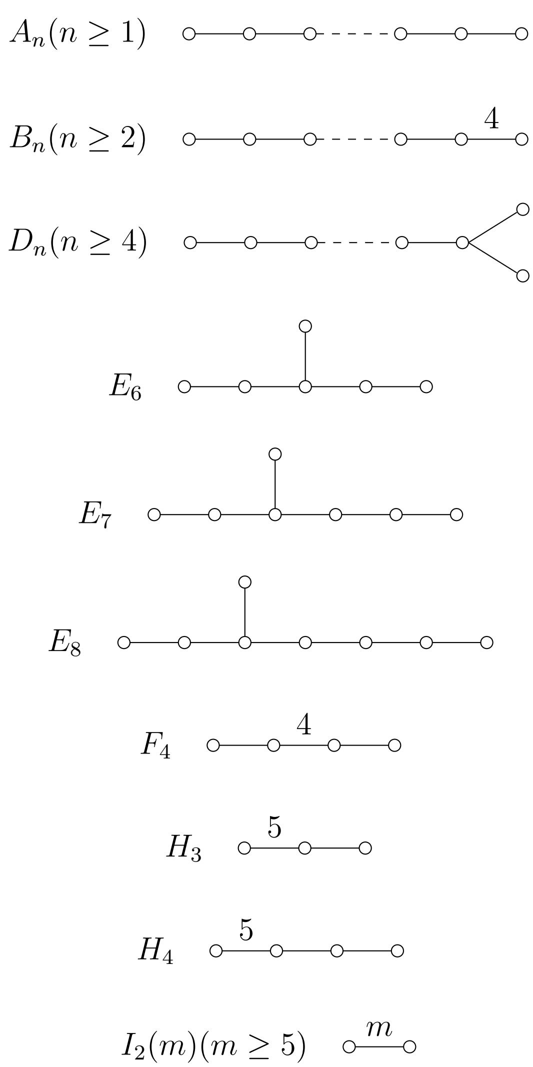

Figure 1: Classification of finite irreducible Coxeter groups

subgroup of $\mathrm{Aff}(V)$ generated by affine reflections along affine hyperplanes $H_{\alpha,k} = \{v \in V \mid (v, \alpha) = k\}$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, where $\Phi = \{w(\alpha_s) \mid s \in S, w \in W\}$ and $(\bullet, \bullet)$ is the scalar product on V induced by the positive definite form B.

We list the Coxeter graphs of irreducible affine Weyl groups in Figure 2. In each graph, the number of vertices is equal to the subscript plus 1.

2.2 Artin groups

The Artin group $A(\Gamma)$ associated to a Coxeter graph Γ is obtained from the presentation of $W(\Gamma)$ by dropping the relation set Q_W.

Definition 2.7. Given a Coxeter graph Γ (hence a Coxeter system (W, S)), we introduce a set $\Sigma = \{a_s \mid s \in S\}$ in one-to-one correspondence with S. Then the Artin system associated to Γ is the pair $(A(\Gamma), \Sigma)$, where $A(\Gamma)$ is the *Artin group* of type Γ defined by the following presentation:

$$A(\Gamma) = \langle \Sigma \mid \overline{R_A} \rangle,$$

where $\overline{R_A} = \{R(a_s, a_t) \mid m(s,t) < \infty\}$ and $R(a_s, a_t) = (a_s a_t)_{m(s,t)} (a_t a_s)^{-1}_{m(s,t)}$.

Note that since $R(a_s, a_t) = R(a_t, a_s)^{-1}$, we may reduce the relation set R_A by introducing a total order on S and put $R_A := \{R(a_s, a_t) \mid m(s,t) < \infty, s < t\}$. We have the following presentation with fewer relations

$$A(\Gamma) = \langle \Sigma \mid R_A \rangle.$$

There is a canonical projection $p : A(\Gamma) \to W(\Gamma)$, $a_s \mapsto s (s \in S)$, the kernel is called the *pure Artin group* of type Γ, denoted by $PA(\Gamma)$. The three groups fit into the exact sequence

$$1 \to PA(\Gamma) \to A(\Gamma) \xrightarrow{p} W(\Gamma) \to 1.$$

The projection p has a canonical set-theoretic section $\psi : W(\Gamma) \to A(\Gamma)$ given by

$$\psi(w) = \psi(s_1 s_2 \cdots s_k) = a_{s_1} a_{s_2} \cdots a_{s_k},$$

where $s_1 s_2 \cdots s_k$ $(s_i \in S)$ is a reduced expression of $w \in W(\Gamma)$, which is well-defined due to a theorem of Matsumoto [Mat64].

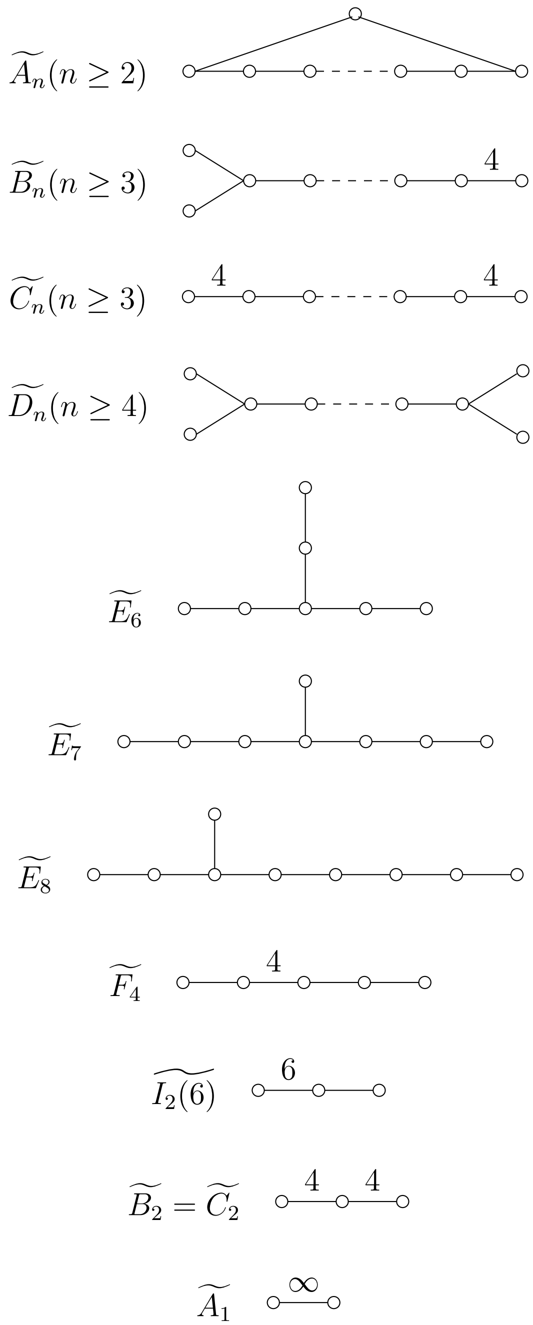

Figure 2: Coxeter graphs for affine Coxeter groups

We say that an Artin group $A(\Gamma)$ is of finite type (or spherical type) if the associated Coxeter group $W(\Gamma)$ is finite. $A(\Gamma)$ is of infinite type (or non-spherical type) if $W(\Gamma)$ is infinite.

2.3 $K(\pi, 1)$ conjecture

Consider a Coxeter graph Γ and the associated Coxeter system (W, S) with W finite and rank $\#S = n$. Recall that W can be realized as a reflection group acting on $\mathbb{R}^n$. Let $\mathcal{A}$ be the collection of the reflection hyperplanes, known as the Coxeter arrangement associated to W. Let

$$M(\Gamma) := \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{C}$$

be the complement to the complexified arrangement $\mathcal{A}_{\mathbb{C}} = \{H \otimes \mathbb{C} \mid H \in \mathcal{A}\}$ in $\mathbb{C}^n$. The Coxeter group W acts freely and properly discontinuously on $M(\Gamma)$. Denote the orbit space by $N(\Gamma) = M(\Gamma)/W$.

Theorem 2.8 ([Bri71])**.** *For an Artin group $A(\Gamma)$ of finite type, the fundamental group of $N(\Gamma)$ is isomorphic to $A(\Gamma)$.*

The classical case $\Gamma = A_n$ is previously proved in [FN62].

The following theorem of Deligne shows that $N(\Gamma)$ is a $K(\pi, 1)$ space.

Theorem 2.9 ([Del72])**.** *Let $\mathcal{A}$ be a finite real central* simplicial *arrangement, then the complement $M(\mathcal{A})$ to the complexification of $\mathcal{A}$ is a $K(\pi, 1)$ space.*

Here a real arrangement is called simplicial if any chamber (a connected component of the complement) is a simplicial cone. Since Coxeter arrangements are simplicial ([Bou68]), we have

Corollary 2.10. *The complement $M(\Gamma)$ is a $K(\pi, 1)$ space and so is $N(\Gamma)$.*

An Artin group $A(\Gamma)$ of finite type thus has a classifying space $N(\Gamma)$. As for Artin groups of infinite type, the above construction can be mildly modified. Suppose now the Coxeter group $W(\Gamma)$ is infinite of rank n, realized as a reflection group acting on a Tits cone $U \subset \mathbb{R}^n$. Let $\mathcal{A}$ be the collection of reflection hyperplanes. Set

$$M(\Gamma) := \left(\mathrm{int}(U) + \sqrt{-1}\mathbb{R}^n\right) \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{C},$$

the complement to the union of complexified reflection hyperplanes in the complexified space with real part the interior of the Tits cone. Then W acts on $M(\Gamma)$ freely and properly discontinuously. Denote the orbit space by

$$N(\Gamma) := M(\Gamma)/W.$$

It is known that

Theorem 2.11 ([vdL83])**.** *The fundamental group of $N(\Gamma)$ is isomorphic to the Artin group $A(\Gamma)$.*

In general, $N(\Gamma)$ is conjectured to be a classifying space of $A(\Gamma)$.

Conjecture 2.12 ($K(\pi, 1)$ conjecture)**.** *Let Γ be an arbitrary Coxeter graph, then the orbit space $N(\Gamma)$ is a $K(\pi, 1)$ space, hence is a classifying space of the Artin group $A(\Gamma)$.*

This conjecture is proved for a few classes of Artin groups besides the finite types. Here is a list of such classes.

- Artin groups of large type ([Hen85]).
- 2-dimensional Artin groups ([CD95]).
- Artin groups of FC type ([CD95]).
- Artin groups of affine types $\widetilde{A_n}, \widetilde{C_n}$ ([Oko79]).
- Artin groups of affine type $\widetilde{B_n}$. [CMS10]

3 Existing results about (co)homology of Artin groups

3.1 Salvetti complex

We briefly recall the central construction, the so-called Salvetti complex $Sal(\Gamma)$ associated to a Coxeter graph Γ. This complex will have the homotopy type of the complement $M(\Gamma)$. The Salvetti complex was first introduce by M. Salvetti in [Sal87] for real hyperplane arrangements. However, we shall follow the construction used in [Par14].

Let Γ be a Coxeter graph and (W, S) the associated Coxeter system. Recall that $\mathcal{S}^f = \{T \subset S \mid W_T \text{ is finite}\}$ is the set of spherical subsets. Define a partial order $\preceq$ on the set $W \times \mathcal{S}^f$ by declaring $(w, T) \preceq (w', T')$ if

$$T \subset T', (w')^{-1}w \in W_{T'}, (w')^{-1}w \text{ is } (\varnothing, T)\text{-minimal},$$

where an element $u \in W$ is called $(\varnothing, T)$-minimal if u is of minimal length in the coset uW_T (such an element of minimal length is unique, see [Bou68] or [Hum90]). This $\preceq$ is indeed a partial order (Lemma 3.2 of [Par14]).

Definition 3.1. For a Coxeter graph Γ, the associated Salvetti complex $Sal(\Gamma)$ is defined as the geometric realization of the derived complex of the poset $(W \times \mathcal{S}^f, \preceq)$.

We shall not distinguish a complex with its geometric realization. Note that W acts on $Sal(\Gamma)$ by $u(w, T) := (uw, T)$ for $u, w \in W, T \in \mathcal{S}^f$. We denote the orbit space by $Z(\Gamma) := Sal(\Gamma)/W$.

Theorem 3.2. *There is a homotopy equivalence $f : Sal(\Gamma) \to M(\Gamma)$, which is W-equivariant. Thus f induces a homotopy equivalence $\bar{f} : Z(\Gamma) \to N(\Gamma)$.*

3.2 Salvetti's algebraic complex for Artin groups

In this subsection, we recall an algebraic complex introduced by Salvetti that is useful in our computation. The contents here could be found in [MSV12], see also [Sal94, DCS96].

Consider a Coxeter system (W, S) with rank $\#S = n$ and its Coxeter graph Γ. The cellular structure of $Z(\Gamma) = Sal(\Gamma)/W$ can be described combinatorially (cf. [Sal94]). Each k-cell of $Z(\Gamma)$ is dual to a unique k-codimensional facet of the fundamental chamber of the arrangement $\mathcal{A}$ and such a facet corresponds to a unique intersection of k hyperplanes of the fundamental chamber. Hence each k-cell of $Z(\Gamma)$ is indexed by a unique spherical subset of S of cardinality k. A total ordering of S then determines an orientation of each cell. The cellular complex of $Z(\Gamma)$ can be identified with an algebraic complex $(\bar{C}_*, \bar{\partial}_*)$ obtained from the Salvetti's algebraic complex which we now describe.

Salvetti introduced an algebraic complex (C_*, ∂_*) for $A(\Gamma)$ whose (co)homology groups coincide with those of $Z(\Gamma)$. Consider a representation of $A(\Gamma)$

$$\lambda : A(\Gamma) \to \mathrm{Aut}(M)$$

where M is a $\mathbb{Z}$-module. Let $\mathcal{L}_\lambda$ be the local system on $Z(\Gamma)$ defined by λ. Define the complex C_* as follows

$$C_k = \bigoplus_{\substack{J \in \mathcal{S}^f \\ \# J = k}} M \cdot e_J,$$

and the boundary map could be written as

$$\partial(a \cdot e_J) = \sum_{s \in J} \left((-1)^{\#\{t \in J | t \leq s\}} \sum_{\beta \in W_J^{J-\{s\}}} (-1)^{\ell(\beta)} \lambda(\psi(\beta))(a) \right) e_{J-\{s\}},$$

where $W_J^{J-\{s\}} = \{\beta \in W_J \mid \ell(\beta t) > \ell(\beta), \forall t \in J - \{s\}\}$ is the set of *minimal coset representatives* of $W_J/W_{J-\{s\}}$, i.e. the collection of the unique element of minimal length in each coset of $W_J/W_{J-\{s\}}$ (see Subsection 2.1.1) and $\psi : W \to A(\Gamma)$ is the canonical section (Subsection 2.2).

Theorem 3.3 ([Sal94])**.** *In the above situation,*

$$H_*(C_*) \cong H_*(Z(\Gamma); \mathcal{L}_\lambda).$$

Remark 3.4. The complex (C^*, δ^*) for cohomology is similar. Precisely, $C^* = C_*$ and

$$\delta(a.e_J) = \sum_{\substack{s \in S \setminus J \\ |W_{J \cup \{s\}}| < \infty}} \left((-1)^{\#\{t \in J | t \leq s\}} \sum_{\beta \in W_J^{J-\{s\}}} (-1)^{\ell(\beta)} \lambda(\psi(\beta))(a) \right) e_{J \cup \{s\}}.$$

Also, $H^*(C^*) \cong H^*(Z(\Gamma); \mathcal{L}_\lambda)$.

The complex will become much simpler if we restrict ourselves to the case $M = \mathbb{Z}$ and λ is trivial. We shall denote this specific complex by $(\bar{C}_*, \bar{\partial}_*)$

$$\bar{C}_k = \bigoplus_{\substack{J \in \mathcal{S}^f \\ \#J = k}} \mathbb{Z}e(J),$$

where we write the basis as $e(J)$ in this specific case. Let $X(q)$ be the Poincaré series of a subset $X \subset W$ (Subsection 2.1.1), then the boundary can be written as

$$\bar{\partial} e(J) = \sum_{s \in J} \left((-1)^{\#\{t \in J | t \leq s\}} W_J^{J-\{s\}}(-1) \right) e(J - \{s\}), \tag{3.1}$$

where $W_J^{J-\{s\}}(-1)$ is the value

$$W_J^{J-\{s\}}(q)|_{q=-1} = \left. \frac{W_J(q)}{W_{J-\{s\}}(q)} \right|_{q=-1}.$$

From now on, we identify the cell of $Z(\Gamma)$ indexed by $J \in \mathcal{S}^f$ with $e(J)$ with orientation compatible with the formula of $\bar{\partial}$ given above.

3.3 Summary of existing results

The Salvetti's algebraic complex provides a relatively simple model for computations. Many computations using this complex for specific types Arin groups exist in the literature. Beside those listed in the introduction, see [DCPSS99, DCPS01] for the cohomology of Artin groups of finite type with coefficient the Laurent polynomial ring $\mathbb{Q}[q^{\pm}]$ on which each standard generator acts as multiplication by $-q$, [CMS08] for type B_n with coefficient the 2-parameter Laurent polynomial ring $\mathbb{Q}[q^{\pm}, t^{\pm}]$ on which the first $n-1$ standard generators act as multiplication by $-q$ and the last acts as multiplication by $-t$, as well as type $\widetilde{A_n}$ with trivial $\mathbb{Q}$ coefficient, [CMS10] for type $\widetilde{B_n}$ with coefficient $\mathbb{Q}[q^{\pm}, t^{\pm}]$ on which the first n standard generators act as multiplication by $-q$ and the last acts as multiplication by $-t$. See also [SV13] for twisted cohomology of Artin groups of exceptional affine types using discrete Morse theory.

3.4 First and second homology of the quotient Salvetti complex

In this subsection, we perform computations of the first and the second homology of the quotient Salvetti complex $Z(\Gamma) = Sal(\Gamma)/W(\Gamma)$ associated to a Coxeter graph Γ using the algebraic Salvetti complex $(\bar{C}_*, \bar{\partial}_*)$ described in Subsection 3.2. We shall remark that the first homology of $Z(\Gamma)$ is exactly the first homology of the Artin group $A(\Gamma) \cong \pi_1(Z(\Gamma))$. The computation of the second homology of $Z(\Gamma)$ was obtained by Clancy-Ellis ([CE10]).

Let Γ be a Coxeter graph with vertex set $S = \{s_1, s_2, \dots, s_n\}$ totally ordered by $s_1 < s_2 < \dots < s_n$. When we want to emphasize the order in a subset of S, we write, for example, $\{s_i < s_j\} \subset S$. For simplicity, we denote the cells $e_i := e(\{s_i\}), 1 \le i \le n$ and $e_{i,j} := e(\{s_i, s_j\}), e_{j,i} := -e(\{s_i, s_j\}), 1 \le i < j \le n$. Also $m_{ij} := m(s_i, s_j)$.

If $\{s_i < s_j\} \in \mathcal{S}^f$, then $W_{\{s_i, s_j\}} \cong D_{2m_{ij}}$ (the dihedral group of order $2m_{ij}$). Using the boundary formula in Subsection 3.2, one computes

$$\begin{aligned}\bar{\partial}_2 e_{i,j} &= (-1)^2 W^{\{s_i\}}_{\{s_i,s_j\}}(-1) e_i + (-1)^1 W^{\{s_j\}}_{\{s_i,s_j\}}(-1) e_j \\ &= (1 + q + q^2 + \dots + q^{m_{ij}-1})|_{q=-1}(e_i - e_j) \\ &= \begin{cases} 0, & m_{ij} : \text{even}; \\ e_i - e_j, & m_{ij} : \text{odd}. \end{cases}\end{aligned}$$

and $\bar{\partial}_1 = 0$.

To describe the homology, recall the definition of Γ_{odd}. Let $\{\Gamma^1_{odd}, \dots, \Gamma^c_{odd}\}$ be the set of connected components of Γ_{odd}. For $1 \le k \le c$, put I_k as the index set of vertices of Γ^k_{odd}, or equivalently $\{s_j \mid j \in I_k\} = V(\Gamma^k_{odd})$. Now set $\Lambda = \{\min I_1, \dots, \min I_c\}$. Hence $c = \#\Lambda$ is the number of connected components of Γ_{odd}. We denote by α_i the homology class represented by e_i. By the above computation, if s_i and s_j are in the same connected component of Γ_{odd}, then $\alpha_i = \alpha_j$. Therefore we have the following result.

Proposition 3.5. *The first homology group $H_1(Z(\Gamma); \mathbb{Z})$ is free abelian of rank $\#\Lambda$ with basis $\{\alpha_i \mid i \in \Lambda\}$, i.e.*

$$H_1(Z(\Gamma); \mathbb{Z}) = \bigoplus_{i \in \Lambda} \mathbb{Z}\alpha_i.$$

Remark 3.6. For an arbitrary Artin group $A(\Gamma)$ with the presentation in Definition 2.7, its abelianization is

$$A(\Gamma)^{ab} = \langle \Sigma \mid a_s = a_t, \text{ if } m(s,t) \text{ is odd} \rangle.$$

Clearly, this coincides with the result in the proposition.

Now let us continue to compute the second homology of $Z(\Gamma)$ following Clancy-Ellis ([CE10]). Set $Q(\Gamma) = \{\{s_i, s_j\} \subset S \mid m_{ij} \text{ is even}\}$, then for $\{s_i < s_j\} \in Q(\Gamma)$, $e_{i,j}$ is a

2-cycle. There may be another kind of 2-cycles. In order to present them explicitly, note that the first homology $H_1(\Gamma_{odd};\mathbb{Z})$ is precisely the finitely generated free abelian group of 1-cycles of Γ_{odd}. Choose a basis $\Omega(\Gamma_{odd})$ for $H_1(\Gamma_{odd};\mathbb{Z})$.

We would like to introduce a 1-dimensional CW-complex (directed graph) $\widetilde{\Gamma_{odd}}$ with 0-cells $\{e_i \mid 1 \leq i \leq n\}$ and 1-cells $\{e_{i,j} \mid \{s_i < s_j\} \in \mathcal{S}^f - Q(\Gamma)\}$ with incidence $\partial e_{i,j} = e_i - e_j$. It is immediate to observe that the graph homomorphism $\iota : \Gamma_{odd} \to \widetilde{\Gamma_{odd}}, s_i \mapsto e_i$ induces a cellular isomorphism. Note that ω is a 1-cycle of Γ_{odd} if and only if $\iota(\omega)$ is a 1-cycle of $\widetilde{\Gamma_{odd}}$. Moreover, a 1-cycle $\iota(\omega)$ of $\widetilde{\Gamma_{odd}}$ is also a 2-cycle of $Z(\Gamma)$. We have the following proposition.

Proposition 3.7. *The kernel* $\mathrm{Ker}\bar{\partial}_2$ *is free abelian of rank* $\#Q(\Gamma) + \#\Omega(\Gamma_{odd})$ *with basis*

$$\{e_{i,j} \mid \{s_i < s_j\} \in Q(\Gamma)\} \cup \{\iota(\omega) \mid \omega \in \Omega(\Gamma_{odd})\},$$

i.e.

$$\mathrm{Ker}\bar{\partial}_2 = \left(\bigoplus_{\{s_i<s_j\}\in Q(\Gamma)} \mathbb{Z}e_{i,j} \right) \oplus \left(\bigoplus_{\omega\in\Omega(\Gamma_{odd})} \mathbb{Z}\iota(\omega) \right).$$

Proof. Let σ be an arbitrary 2-cycle of $Z(\Gamma)$, then σ has the form $\sigma = \sum_{\{s_i<s_j\}\in\mathcal{S}^f} x_{i,j}e_{i,j}$, where $x_{i,j} \in \mathbb{Z}$, such that $\bar{\partial}_2\sigma = 0$. The latter condition is equivalent to

$$\bar{\partial}_2\sigma = \bar{\partial}_2 \left(\sum_{\{s_i<s_j\}\in\mathcal{S}^f - Q(\Gamma)} x_{i,j}e_{i,j} \right) = 0.$$

This is to say that $\tilde{\sigma} = \sum_{\{s_i<s_j\}\in\mathcal{S}^f - Q(\Gamma)} x_{i,j}e_{i,j}$ is a 1-cycle of $\widetilde{\Gamma_{odd}}$, thus uniquely expressed by basis $\{\iota(\omega) \mid \omega \in \Omega(\Gamma_{odd})\}$ of $H_1(\widetilde{\Gamma_{odd}};\mathbb{Z})$. □

Next we state Clancy-Ellis' theorem. Let us first fix some notations associated to a Coxeter graph. Let Γ be a Coxeter graph with vertex set S. Denote by $P(\Gamma)$ the set of pairs of non-adjacent vertices of Γ, namely $P(\Gamma) = \{\{s,t\} \subset S \mid m(s,t) = 2\}$. Write $\{s,t\} \equiv \{s',t'\}$ if two such pairs in $P(\Gamma)$ satisfy $s = s'$ and $m(t,t')$ is odd. This generates an equivalence relation on $P(\Gamma)$, denoted by $\sim$. Denote by $P(\Gamma)/\sim$ the set of equivalence classes. An equivalence class is called a *torsion* class if it is represented by a pair $\{s,t\} \in P(\Gamma)$ such that there exists a vertex $v \in S$ with $m(s,v) = m(t,v) = 3$. In the above situation, we have the following theorem.

Theorem 3.8 ([CE10]). *The second integral homology of the quotient Salvetti complex $Z(\Gamma) = Sal(\Gamma)/W(\Gamma)$ associated to a Coxeter graph Γ is*

$$H_2(Z(\Gamma);\mathbb{Z}) = \mathbb{Z}_2^{p(\Gamma)} \oplus \mathbb{Z}^{q(\Gamma)},$$

where

$$\begin{aligned}
p(\Gamma) &:= \text{ number of torsion classes in } P(\Gamma)/\sim, \\
q_1(\Gamma) &:= \text{ number of non-torsion classes in } P(\Gamma)/\sim, \\
q_2(\Gamma) &:= \#(Q(\Gamma) - P(\Gamma)) = \#\{\{s,t\} \subset S \mid m(s,t) \geq 4 \text{ is even}\}, \\
q_3(\Gamma) &:= \mathrm{rank} H_1(\Gamma_{odd};\mathbb{Z}), \\
q(\Gamma) &:= q_1(\Gamma) + q_2(\Gamma) + q_3(\Gamma).
\end{aligned}$$

Proof. It suffices to continue the argument in the proof of Proposition 3.7. There we have computed the kernel of the differential $\bar{\partial}_2 : \bar{C}_2 \to \bar{C}_1$ in the cellular chain complex of $Z(\Gamma)$.

$$\mathrm{Ker}\bar{\partial}_2 = \bigoplus_{\{s,t\}\in Q(\Gamma)} \mathbb{Z}e(\{s,t\}) \oplus \bigoplus_{\omega\in\Omega(\Gamma_{odd})} \mathbb{Z}\iota(\omega).$$

It remains to compute the image of $\bar{\partial}_3$. To do so, we first classify finite Coxeter groups with rank 3 by their Coxeter graphs. There are 5 types of them: $A_3, B_3, H_3, A_1 \times A_1 \times A_1$ and $A_1 \times I_2(m)$ for $m \geq 3$. Therefore the chain group $\bar{C}_3$ of $Z(\Gamma)$ for any Γ has basis $\{e(T)\}$ where $T \subset S$ spans a Coxeter subgraph Γ_T of one of the above types. Using the boundary formula 3.1, we derive $\bar{\partial}_3 e(T) = 0$ if Γ_T is $B_3, H_3, A_1 \times A_1 \times A_1$ or $I_2(m)$ for even m.

As for $\Gamma_T = A_1 \times I_2(m)$ with m odd, suppose $T = \{r,s,t\}$ with $r < s < t$ and $m(r,s) = m(r,t) = 2, m(s,t) = m$, formula 3.1 shows that

$$\bar{\partial}_3 e(T) = e(\{r,t\}) - e(\{s,t\}).$$

This means that $e(\{r,t\})$ and $e(\{s,t\})$ are homologous if $\{r,t\} \equiv \{s,t\}$ in $P(\Gamma)$. Hence taking quotient modulo $\bar{\partial}_3 e(T)$ for type $A_1 \times I_2(m)$ decreases the number of direct summands $\mathbb{Z}$ in $\mathrm{Ker}\bar{\partial}_2$ by $\#P(\Gamma) - \#P(\Gamma)/\sim$.

Now suppose $\Gamma_T = A_3$, that is $T = \{r,s,t\}$ with $r < s < t$ and $m(r,s) = m(s,t) = 3, m(r,t) = 2$, then formula 3.1 shows that

$$\bar{\partial}_3 e(T) = 2e(\{r,t\}).$$

In other words, $2e(\{r,t\})$ is a boundary if $\{r,t\}$ is a torsion class in $P(\Gamma)/\sim$. Taking quotient modulo $\bar{\partial}_3 e(T)$ for type A_3 then decreases the number of direct summands $\mathbb{Z}$ by $p(\Gamma)$ as well as generates $\mathbb{Z}_2^{p(\Gamma)}$. $\square$

Remark 3.9. Whenever $K(\pi,1)$ conjecture holds for the Artin group $A(\Gamma)$, Theorem 3.8 gives the second integral homology of $A(\Gamma)$.

4 Second mod 2 homology of Artin groups

As we see in the previous section, almost all existing results about (co)homology of Artin groups are based on the affirmative solution of the $K(\pi, 1)$ conjecture. In this section, nevertheless, we shall work on low dimensional homology of *arbitrary* Artin groups, without assuming that the $K(\pi, 1)$ conjecture holds. The first homology is easily derived from the standard presentation given in Definition 2.7, see Proposition 3.5 and Remark 3.6. Our main results are the following theorems.

Theorem 4.1. *The second integral homology of an arbitrary Artin group $A(\Gamma)$ fits into the following commutative diagram*

$$\begin{tikzcd} \mathbb{Z}^{p(\Gamma)+q(\Gamma)} \arrow[r] \arrow[dr] & H_2(A(\Gamma); \mathbb{Z}) \arrow[d] \\ & H_2(W(\Gamma); \mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)} \end{tikzcd} \tag{4.1}$$

where all maps are surjective and the numbers $p(\Gamma)$ and $q(\Gamma)$ are given in Theorem 3.8.

Theorem 4.2. *Let Γ be an arbitrary Coxeter graph and $A(\Gamma)$ the associated Artin group. Then the second mod 2 homology of $A(\Gamma)$ is*

$$H_2(A(\Gamma); \mathbb{Z}_2) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}.$$

The outline of our proof is as follows. In Subsection 4.1, we recall Howlett's theorem on the second integral homology group $H_2(W(\Gamma); \mathbb{Z})$ of the Coxeter group $W(\Gamma)$ associated to Γ. Next in Subsection 4.2, we recall the classical Hopf's formula of the second homology of a group. The key of the proof is that by virtue of Hopf's formula, we are able to find explicitly a minimal set of generators of $H_2(W(\Gamma); \mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}$ (Section 4.3), as well as a set of generators of $H_2(A(\Gamma); \mathbb{Z})$ (Section 4.4). Both sets of generators are images of the basis of a common free abelian group $\mathbb{Z}^{p(\Gamma)+q(\Gamma)}$ under the diagonal and horizontal maps respectively. The commutativity of the diagram will follow from the constructions.

This section is based on joint work with Professor Toshiyuki Akita.

4.1 Howlett's theorem

The group epimorphism $p : A(\Gamma) \to W(\Gamma)$ induces $H_2(A(\Gamma);\mathbb{Z}) \to H_2(W(\Gamma);\mathbb{Z})$. The target $H_2(W(\Gamma);\mathbb{Z})$ turns out to be a finite group. We recall Howlett's theorem ([How88]) on the second integral homology of Coxeter groups.

Theorem 4.3 ([How88]). *The second integral homology of the Coxeter group $W(\Gamma)$ associated to a Coxeter graph Γ is*

$$H_2(W(\Gamma);\mathbb{Z}) = \mathbb{Z}_2^{-n_1(\Gamma)+n_2(\Gamma)+n_3(\Gamma)+n_4(\Gamma)},$$

where

$$\begin{aligned} n_1(\Gamma) &:= \#S, \\ n_2(\Gamma) &:= \#\{\{s,t\} \in E(\Gamma) \mid m(s,t) < \infty\}, \\ n_3(\Gamma) &:= \#P(\Gamma)/\sim, \\ n_4(\Gamma) &:= \mathrm{rank} H_0(\Gamma_{odd};\mathbb{Z}). \end{aligned}$$

Remark 4.4. For a Coxeter graph Γ, the above numbers are related to those used by Clancy-Ellis as follows

$$-n_1(\Gamma) + n_2(\Gamma) + n_3(\Gamma) + n_4(\Gamma) = p(\Gamma) + q(\Gamma).$$

In fact, $n_1(\Gamma) = \#V(\Gamma_{odd})$, $n_2(\Gamma) = q_2(\Gamma) + \#E(\Gamma_{odd})$ and $n_3(\Gamma) = p(\Gamma) + q_1(\Gamma)$. The above equation follows from the Euler-Poincaré theorem applied to Γ_{odd},

$$\#V(\Gamma_{odd}) - \#E(\Gamma_{odd}) = \mathrm{rank} H_0(\Gamma_{odd};\mathbb{Z}) - \mathrm{rank} H_1(\Gamma_{odd};\mathbb{Z}).$$

Example 4.5. Let $\Gamma = I_2(m)$. Thus $W(\Gamma) = D_{2m}$ is the dihedral group of order $2m$. Theorem 4.3 shows that

$$H_2(W(\Gamma);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2, & m \text{ is even;} \\ 0, & m \text{ is odd.} \end{cases}$$

See also Corollary 10.1.27 of [Kar93] for an alternative proof of the same result.

4.2 Hopf's formula

The classical Hopf's formula gives a description of the second integral homology of a group. We first recall some notations. For a group G, the commutator of $x, y \in G$ is the element $[x, y] = xyx^{-1}y^{-1}$. The commutator subgroup $[G, G]$ of G is the subgroup of G generated by all commutators. It is the smallest normal subgroup of G such that the quotient group is abelian. In general, for subgroups H and K of G, we define $[H, K]$ as the subgroup of G generated by $[h, k], h \in H, k \in K$.

Theorem 4.6 (Hopf's formula)**.** *If a group G has a finite presentation $\langle S \mid R \rangle$ and $G = F/N$, then*

$$H_2(G; \mathbb{Z}) \cong \frac{N \cap [F, F]}{[F, N]},$$

where $F = F(S)$ is the free group generated by S and $N = N(R)$ is the normal closure of R (subgroup of F normally generated by the relation set R).

See Section II.5 of [Bro82] for a topological proof. For simplicity we denote by $\langle x \rangle_G = x[F, N] \in F/[F, N]$ the coset of $[F, N]$ represented by $x \in F$ and $\langle x, y \rangle_G = [x, y][F, N] \in [F, F]/[N, F]$ for $x, y \in F$. Thanks to Hopf's formula, second homology classes of G can be considered as $\langle x \rangle_G$ for $x \in N \cap [F, F]$.

To see how the representatives look like, we make the following simple observations.

Lemma 4.7. *The group $N/[F, N]$ is abelian.*

Proof. Note that $N/[F, N]$ is a quotient group of $N/[N, N]$ and the latter is the abelianization of N. □

Thus we write the group $N/[F, N]$ additively. It is clear $\langle n \rangle_G = -\langle n^{-1} \rangle_G$ for $n \in N$. The next two lemmas are useful, their proofs can be found, for example, in [KS03].

Lemma 4.8. *In the abelian group $N/[F, N]$, we have*

$$\langle n \rangle_G = \langle fnf^{-1} \rangle_G$$

for $n \in N$ and $f \in F$.

Proof. Since $[f, n] \in [F, N]$, $\langle f, n \rangle_G = \langle fnf^{-1}n^{-1} \rangle_G = \langle fnf^{-1} \rangle_G - \langle n \rangle_G = 0$. □

Therefore a coset in $N/[F,N]$ is represented by an element of the form $\prod_{r\in R} r^{n_r}(n_r \in \mathbb{Z})$. Hopf's formula implies that a second homology class of G can be represented by an element $\prod_{r\in R} r^{n_r} \in [F,F]$.

Lemma 4.9. *Let $G = F/N$ as in Theorem 4.6. If $x, y, z \in F$ such that $[x,y],[x,z] \in N \cap [F,F]$, then*

$$\langle x, yz\rangle_G = \langle x, y\rangle_G + \langle x, z\rangle_G, \quad \langle x, y^{-1}\rangle_G = -\langle x, y\rangle_G,$$

Proof. Note that $[x,yz] = [x,y]y[x,z]y^{-1}$. Then in the abelian group $N/[F,N]$,

$$\langle x, yz\rangle_G = \langle x, y\rangle_G + \langle y[x,z]y^{-1}\rangle_G.$$

The term $\langle y[x,z]y^{-1}\rangle_G = \langle x, z\rangle_G$ since

$$[x,z]^{-1}y[x,z]y^{-1} = [[x,z]^{-1}, y] \in [N,F].$$

Hence the first equality holds. The second follows immediately from the first. □

4.3 Hopf's formula applied to Coxeter groups

The aim of this subsection is to construct the diagonal map $\mathbb{Z}^{p(\Gamma)+q(\Gamma)} \to H_2(W(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}$ in the diagram in Theorem 4.1. Let us first state our strategy used in the rest of this section. Theorem 4.3 provides a relatively small group receiving a map $H_2(A(\Gamma);\mathbb{Z}) \to H_2(W(\Gamma);\mathbb{Z})$. By applying Hopf's formula, we can write down the map explicitly. More important is that Hopf's formula enables us to find a minimal set of generators of $H_2(W(\Gamma);\mathbb{Z})$. In fact, we construct a surjective group homomorphism Φ_W of the free abelian group with basis the relation set of W onto the abelian group of cosets $\langle x\rangle_W$ with x expressed by a product of relations. By analyzing when x belongs to the commutator subgroup of the free group $F(S)$ on S, we derive a surjective group homomorphism of $\mathbb{Z}^{p(\Gamma)+q(\Gamma)}$ onto $H_2(W(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}$, which takes the basis of the domain to a minimal set of generators of $H_2(W(\Gamma);\mathbb{Z})$. Similar construction applied to Artin group $A(\Gamma)$ yields the horizontal map $Z^{p(\Gamma)+q(\Gamma)} \to H_2(A(\Gamma);\mathbb{Z})$, which takes the basis of the domain to a set of generators of $H_2(A(\Gamma);\mathbb{Z})$. Note that the set of generators of $H_2(A(\Gamma);\mathbb{Z})$ is taken to that of $H_2(W(\Gamma);\mathbb{Z})$ by the vertical map.

Let Γ be a Coxeter graph and (W,S) the associated Coxeter system with S totally ordered. Then $W = \langle S \mid R_W \cup Q_W \rangle$ is as in Definition 2.1. Let $F_W = F(S)$ be the free group on S and $N_W = N(R_W \cup Q_W)$ be the normal closure of $R_W \cup Q_W$. Therefore $W = F_W/N_W$. Using Hopf's formula we obtain $H_2(W;\mathbb{Z}) \cong (N_W \cap [F_W,F_W])/[F_W,N_W]$. Hence a second homology class of W is of the form $\langle x \rangle_W$ with x expressed by $\prod_{R(s,t)\in R_W} R(s,t)^{n_{s,t}} \prod_{Q(s)\in Q_W} Q(s)^{n_s} \in [F_W,F_W]$.

Recall that the relation set R_W decomposes into $R_W = R_W^{even} \sqcup R_W^{odd}$, where $R_W^{even} = \{R(s,t) \mid m(s,t) \text{ is even}, s<t\}$ and $R_W^{odd} = \{R(s,t) \mid m(s,t) \text{ is odd}, s<t\}$ with cardinality $\#R_W^{even} = \#P(\Gamma) + q_2(\Gamma)$ (see Theorem 3.8 for notations) and $\#R_W^{odd} = \#E(\Gamma_{odd})$. Now consider the surjective group homomorphism

$$\Phi_W : \mathbb{Z}^{R_W \cup Q_W} \longrightarrow \frac{N_W}{[F_W,N_W]}$$

$$\sum_{R(s,t)\in R_W} n_{s,t} R(s,t) + \sum_{Q(s)\in Q_W} n_s Q(s) \longmapsto \left\langle \prod_{R(s,t)\in R_W} R(s,t)^{n_{s,t}} \prod_{Q(s)\in Q_W} Q(s)^{n_s} \right\rangle_W$$

where $\mathbb{Z}^{R_W \cup Q_W}$ is the free abelian group with basis $R_W \cup Q_W$. If $R(s,t) \in R_W^{even}$, then $R(s,t)$ is in $[F_W,F_W]$. As a result, Φ_W maps the subgroup $\mathbb{Z}^{R_W^{even}}$ of $\mathbb{Z}^{R_W\cup Q_W} = \mathbb{Z}^{R_W^{even}} \oplus \mathbb{Z}^{R_W^{odd}} \oplus \mathbb{Z}^{Q_W}$ into $(N_W \cap [F_W,F_W])/[F_W,N_W]$.

Define a subgroup of $\mathbb{Z}^{R_W^{odd}} \oplus \mathbb{Z}^{Q_W}$ as follows

$$\mathcal{C}_W = \left\{ \sum_{R(s,t)\in R_W^{odd}} n_{s,t} R(s,t) + \sum_{Q(s)\in Q_W} n_s Q(s) \,\middle|\, \prod_{R(s,t)\in R_W^{odd}} R(s,t)^{n_{s,t}} \prod_{Q(s)\in Q_W} Q(s)^{n_s} \in [F_W,F_W] \right\}.$$

Then Φ_W maps $\mathcal{C}_W$ into $(N_W \cap [F_W,F_W])/[F_W,N_W]$. In other words, Φ_W restricts to a surjective group homomorphism

$$\Phi_W| : \mathbb{Z}^{R_W^{even}} \oplus \mathcal{C}_W \to \frac{N_W \cap [F_W,F_W]}{[F_W,N_W]}.$$

Note that the element

$$2R(s,t) - Q(s) + Q(t)$$

lies in $\mathcal{C}_W$ if $R(s,t) \in R_W^{odd}$ since the word $\left((st)_m (ts)_m^{-1}\right)^2 (s^2)^{-1} t^2$ is mapped to the identity under the abelianization map $F_W \to F_W/[F_W,F_W]$ when m is odd. Let $\mathcal{D}_W$ be the subgroup of $\mathcal{C}_W$ generated by elements $2R(s,t) - Q(s) + Q(t)$ for $R(s,t) \in R_W^{odd}$. The following is a consequence of Example 4.5.

Proposition 4.10. *$\mathcal{D}_W$ lies in the kernel of Φ_W.*

Proof. It suffices to show that the generator $2R(s,t) - Q(s) + Q(t)$ of $\mathcal{D}_W$ is taken to the identity of $N_W/[F_W, N_W]$ by Φ_W, or equivalently, the word $\left((st)_m(ts)_m^{-1}\right)^2 (s^2)^{-1}t^2$ lies in $[F_W, N_W]$ when m is odd. Let $s, t \in S$ with $m := m(s,t)$ odd, consider the parabolic subgroup $W' := W_{\{s,t\}}$ of W, which is the dihedral group D_{2m} of order $2m$. From Example 4.5, we know $H_2(W';\mathbb{Z}) = 0$. On the other hand, Hopf's formula applied to W' shows that $H_2(W';\mathbb{Z}) \cong (N_{W'} \cap [F_{W'}, F_{W'}])/[F_{W'}, N_{W'}]$. Therefore the word $\left((st)_m(ts)_m^{-1}\right)^2 (s^2)^{-1}t^2 \in N_{W'} \cap [F_{W'}, F_{W'}]$ represents the trivial homology class. That is to say

$$\left((st)_m(ts)_m^{-1}\right)^2 (s^2)^{-1}t^2 \in [F_{W'}, N_{W'}] \subset [F_W, N_W].$$

This proves the proposition. □

The restriction $\Phi_W|$ then factors through $\mathbb{Z}^{R_W^{even}} \oplus \mathcal{C}_W/\mathcal{D}_W \to (N_W \cap [F_W, F_W])/[F_W, N_W]$. In the next proposition, we show that the group $\mathcal{C}_W/\mathcal{D}_W$ is isomorphic to $\mathbb{Z}_2^{q_3(\Gamma)}$.

Proposition 4.11. *There is an isomorphism $\mathcal{C}_W/\mathcal{D}_W \to H_1(\Gamma_{odd};\mathbb{Z}_2)$.*

Proof. First we identify $\mathbb{Z}^{R_W^{odd}}$ with the chain group $C_1(\Gamma_{odd})$ by $R(s,t) \mapsto \langle s,t\rangle$, and $\mathbb{Z}^{Q_W}$ with $2C_0(\Gamma_{odd})$, that is the image of the injective homomorphism $\mathbb{Z}^{Q_W} \to C_0(\Gamma_{odd})$ defined by $Q(s) \mapsto 2s$. Under these identifications, the definition of $\mathcal{C}_W$ translates into

$$\mathcal{C}_W = \{(\alpha, \beta) \in C_1(\Gamma_{odd}) \oplus 2C_0(\Gamma_{odd}) \mid \partial\alpha = \beta\}$$

and $\mathcal{D}_W$ is the subgroup of $\mathcal{C}_W$ generated by elements $(2\langle s,t\rangle, -2s + 2t)$ for 1-cells $\langle s,t\rangle$ of Γ_{odd}. On the other hand, $H_1(\Gamma_{odd};\mathbb{Z}_2) = Z_1(\Gamma_{odd};\mathbb{Z}_2) \cong \mathbb{Z}_2^{q_3(\Gamma)}$ is the vector space of 1-cycles of Γ_{odd} over $\mathbb{Z}_2$.

Now define a homomorphism $\mathcal{C}_W \to H_1(\Gamma_{odd};\mathbb{Z}_2)$ by $(\alpha, \partial\alpha) \mapsto \overline{\alpha}$, where $\alpha \in C_1(\Gamma_{odd})$ is a 1-chain of Γ_{odd} such that $\partial\alpha \in 2C_0(\Gamma_{odd})$ and $\overline{\alpha} \in C_1(\Gamma_{odd};\mathbb{Z}_2)$ is the 1-chain obtained from α by reducing each integral coefficient modulo 2. The condition $\partial\alpha \in 2C_0(\Gamma_{odd})$ asserts that $\overline{\alpha}$ is indeed a 1-cycle of Γ_{odd} with coefficients in $\mathbb{Z}_2$. The proposition follows from the obvious observation that this homomorphism is surjective with kernel exactly $\mathcal{D}_W$. □

Let $R_W^2 := \{R(s,t) \in R_W^{even} \mid m(s,t) = 2\}$ and $R_W^{even\geq 4} := \{R(s,t) \in R_W^{even} \mid m(s,t) \geq 4\}$, hence $R_W^{even} = R_W^2 \sqcup R_W^{even\geq 4}$ and $\#R_W^2 = \#P(\Gamma), \#R_W^{even\geq 4} = q_2(\Gamma)$. We introduce an equivalence relation in R_W^2 by setting $R(s,t) \sim R(s',t')$ if $\{s,t\} \sim \{s',t'\}$ in $P(\Gamma)$. Denote the set of equivalence classes by $R_W^2/\sim$. Our last observation is the following.

Proposition 4.12. *If $s < t$ and $s < u$ with $\{s,t\} \equiv \{s,u\}$ in $P(\Gamma)$, that is $m(s,t) = m(s,u) = 2$ and $m(t,u)$ is odd, then $\Phi_W(R(s,t)) = \Phi_W(R(s,u))$.*

Proof. Suppose $s < t$ and $s < u$ with $\{s,t\} \equiv \{s,u\}$ in $P(\Gamma)$. In this case, $R(s,t) = [s,t]$ and $R(s,u) = [s,u]$. In $N_W/[F_W, N_W]$, we have

$$\begin{aligned}\Phi_W(R(s,t)) - \Phi_W(R(s,u)) &= \langle s,t\rangle_W - \langle s,u\rangle_W \\ &= \langle s, R(t,u)\rangle_W \\ &= \langle sR(t,u)s^{-1}R(t,u)^{-1}\rangle_W \\ &= \langle sR(t,u)s^{-1}\rangle_W + \langle R(t,u)^{-1}\rangle_W \\ &= \langle R(t,u)\rangle_W - \langle R(t,u)\rangle_W = 0,\end{aligned}$$

where the second and the fourth equalities follow from Lemma 4.9, the fifth from Lemma 4.8. $\square$

Finally the restriction $\Phi_W|$ factors through a surjective group homomorphism which we denote by $\widetilde{\Phi_W}$:

$$\mathbb{Z}^{R_W^2/\sim} \oplus \mathbb{Z}^{R_W^{even\geq 4}} \oplus H_1(\Gamma_{odd}; \mathbb{Z}_2) \to \frac{N_W \cap [F_W, F_W]}{[F_W, N_W]},$$

where we have identified $\mathcal{C}_W/\mathcal{D}_W$ with $H_1(\Gamma_{odd}; \mathbb{Z}_2) \cong \mathbb{Z}_2^{q_3(\Gamma)}$ in view of Proposition 4.11. Composing with the natural surjective group homomorphism, we obtain the desired map

$$\begin{array}{ccc} \mathbb{Z}^{R_W^2/\sim} \oplus \mathbb{Z}^{R_W^{even\geq 4}} \oplus H_1(\Gamma_{odd}; \mathbb{Z}) \cong \mathbb{Z}^{p(\Gamma)+q(\Gamma)} & & \\ \downarrow & & \\ \mathbb{Z}^{R_W^2/\sim} \oplus \mathbb{Z}^{R_W^{even\geq 4}} \oplus H_1(\Gamma_{odd}; \mathbb{Z}_2) & \xrightarrow[\widetilde{\Phi_W}]{} & \frac{N_W \cap [F_W, F_W]}{[F_W, N_W]} \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)} \end{array} \tag{4.2}$$

which takes a basis of the free abelian group on the top left corner to a minimal set of generators of $\frac{N_W \cap [F_W, F_W]}{[F_W, N_W]} \cong H_2(W(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}$. More precisely, let us choose a basis $R_W^2/\sim \cup R_W^{even\geq 4} \cup \Omega(\Gamma_{odd})$ for $\mathbb{Z}^{R_W^2/\sim} \oplus \mathbb{Z}^{R_W^{even\geq 4}} \oplus H_1(\Gamma_{odd};\mathbb{Z})$, where $\Omega(\Gamma_{odd})$ is a basis for the free abelian group $H_1(\Gamma_{odd};\mathbb{Z}) \cong \mathbb{Z}^{q_3(\Gamma)}$. An equivalence class in $R_W^2/\sim$ represented by $R(s,t)$ is mapped to $\langle R(s,t)\rangle_W = \langle s,t\rangle_W \in \frac{N_W \cap [F_W, F_W]}{[F_W, N_W]}$. An element $R(s,t)$ in $R_W^{even\geq 4}$ is mapped to $\langle R(s,t)\rangle_W \in \frac{N_W \cap [F_W, F_W]}{[F_W, N_W]}$. An element $\omega \in \Omega(\Gamma_{odd})$ is a 1-cycle of Γ_{odd}, it is first mapped to $\overline{\omega} \in H_1(\Gamma_{odd};\mathbb{Z}_2)$ by reducing each coefficient modulo 2. We may assume that $\overline{\omega} = \sum\langle s,t\rangle$ is the sum of some 1-cells $\langle s,t\rangle$ of Γ_{odd}, each $\langle s,t\rangle$ corresponds to a relation $R(s,t) \in R_W^{odd}$. Denote by $I(\overline{\omega}) := \{R(s,t) \in R_W^{odd} \mid \langle s,t\rangle \text{ appears in } \overline{\omega}\}$. Then $\widetilde{\Phi_W}(\overline{\omega}) = \langle\prod_{R(s,t)\in I(\overline{\omega})} R(s,t)s^{-1}t\rangle_W \in \frac{N_W \cap [F_W, F_W]}{[F_W, N_W]}$. Note that the fact $\prod_{R(s,t)\in I(\overline{\omega})} R(s,t)s^{-1}t$ is indeed in $N_W \cap [F_W, F_W]$ is guaranteed by $\overline{\omega} \in H_1(\Gamma_{odd};\mathbb{Z}_2)$.

4.4 Hopf's formula applied to Artin groups

Now we construct the horizontal map $\mathbb{Z}^{p(\Gamma)+q(\Gamma)} \to H_2(A(\Gamma);\mathbb{Z})$ in the diagram in Theorem 4.1. The arguments here are parallel to those in the Coxeter case.

Let Γ be a Coxeter graph with vertex set S totally ordered, $A = A(\Gamma)$ be the Artin group of type Γ with the presentation $A = \langle \Sigma \mid R_A\rangle$ given in Definition 2.7. Let $F_A = F(\Sigma)$ be the free group on Σ and N_A be the normal closure of R_A. Hopf's formula yields $H_2(A;\mathbb{Z}) \cong (N_A \cap [F_A, F_A])/[F_A, N_A]$. For the same reason as before, a second homology class of A is a coset $\langle x\rangle_A$ with x of the form $\prod_{R(a_s,a_t)\in R_A} R(a_s,a_t)^{n_{s,t}} \in [F_A, F_A]$.

Parallel arguments as in the previous subsection then follow. Similarly, $R_A = R_A^{odd} \sqcup R_A^{even}$. We have the following surjective group homomorphism

$$\begin{aligned} \Phi_A : \mathbb{Z}^{R_A} &\longrightarrow \frac{N_A}{[F_A, N_A]} \\ \sum_{R(a_s,a_t)\in R_A} n_{s,t} R(a_s,a_t) &\longmapsto \left\langle \prod_{R(a_s,a_t)\in R_A} R(a_s,a_t)^{n_{s,t}} \right\rangle_A \end{aligned}$$

where $\mathbb{Z}^{R_A}$ is the free abelian group generated by R_A. For the same reason, Φ_A maps the subgroup $\mathbb{Z}^{R_A^{even}}$ of $\mathbb{Z}^{R_A} = \mathbb{Z}^{R_A^{even}} \oplus \mathbb{Z}^{R_A^{odd}}$ into $(N_A \cap [F_A, F_A])/[F_A, N_A]$.

Define a subgroup of $\mathbb{Z}^{R_A^{odd}}$ as follows

$$\mathcal{C}_A = \left\{ \sum_{R(a_s,a_t)\in R_A^{odd}} n_{s,t} R(a_s,a_t) \,\middle|\, \prod_{R(a_s,a_t)\in R_A^{odd}} R(a_s,a_t)^{n_{s,t}} \in [F_A, F_A] \right\}.$$

Then Φ_A maps $\mathcal{C}_A$ into $(N_A \cap [F_A, F_A])/[F_A, N_A]$. Now Φ_A restricts to a surjective group homomorphism $\Phi_A| : \mathbb{Z}^{R_A^{even}} \oplus \mathcal{C}_A \to (N_A \cap [F_A, F_A])/[F_A, N_A]$.

Furthermore, we have the following

Proposition 4.13. *There is an isomorphism $\mathcal{C}_A \to H_1(\Gamma_{odd}; \mathbb{Z})$.*

Proof. The assignment $R(a_s, a_t) \mapsto \langle s, t\rangle$ defines an isomorphism $\mathbb{Z}^{R_A^{odd}} \to C_1(\Gamma_{odd})$. The condition

$$\prod_{R(a_s,a_t)\in R_A^{odd}} R(a_s,a_t)^{n_{s,t}} \in [F_A, F_A]$$

is equivalent to that the image of $\sum_{R(a_s,a_t)\in R_A^{odd}} n_{s,t} R(a_s, a_t)$ is a 1-cycle. Then the above isomorphism maps $\mathcal{C}_A$ isomorphically onto $Z_1(\Gamma_{odd}) \cong H_1(\Gamma_{odd}; \mathbb{Z})$. □

By the exactly same proof as in Proposition 4.12, $\Phi_A|$ factors through a surjective group homomorphism which we denote by $\widetilde{\Phi_A}$:

$$\mathbb{Z}^{R_A^2/\sim} \oplus \mathbb{Z}^{R_A^{even\geq 4}} \oplus H_1(\Gamma_{odd}; \mathbb{Z}) \to \frac{N_A \cap [F_A, F_A]}{[F_A, N_A]},$$

where $R_A^2 := \{R(a_s, a_t) \in R_A^{even} \mid m(s,t) = 2\}$ and $R_A^{even\geq 4} := \{R(a_s, a_t) \in R_A^{even} \mid m(s,t) \geq 4\}$. Note that we have identified $\mathcal{C}_A$ with $H_1(\Gamma_{odd}; \mathbb{Z})$. The set $R_A^2/\sim$ is defined similarly as $R_W^2/\sim$. Hence we obtain the desired surjective group homomorphism

$$\mathbb{Z}^{p(\Gamma)+q(\Gamma)} \cong \mathbb{Z}^{R_A^2/\sim} \oplus \mathbb{Z}^{R_A^{even\geq 4}} \oplus H_1(\Gamma_{odd}; \mathbb{Z}) \to \frac{N_A \cap [F_A, F_A]}{[F_A, N_A]} \cong H_2(A(\Gamma); \mathbb{Z}),$$

which takes a basis of the free abelian group on the left to a set of generators of $H_2(A(\Gamma); \mathbb{Z})$.

4.5 Proof of main results

Let us summarize the results obtained so far by the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^{R_A^2/\sim} \oplus \mathbb{Z}^{R_A^{even\geq 4}} \oplus H_1(\Gamma_{odd}; \mathbb{Z}) \arrow[r, "\widetilde{\Phi_A}"'] \arrow[d, "\Delta"'] & \frac{N_A\cap[F_A,F_A]}{[F_A,N_A]} \arrow[d, "\Xi"] \\
\mathbb{Z}^{R_W^2/\sim} \oplus \mathbb{Z}^{R_W^{even\geq 4}} \oplus H_1(\Gamma_{odd}; \mathbb{Z}_2) \arrow[r, "\widetilde{\Phi_W}"'] & \frac{N_W\cap[F_W,F_W]}{[F_W,N_W]}
\end{tikzcd} \tag{4.3}$$

where we have identified $\mathbb{Z}^{R_A^2/\sim}\oplus\mathbb{Z}^{R_A^{even\geq 4}}\oplus H_1(\Gamma_{odd};\mathbb{Z})$ with $\mathbb{Z}^{R_W^2/\sim}\oplus\mathbb{Z}^{R_W^{even\geq 4}}\oplus H_1(\Gamma_{odd};\mathbb{Z})$ via the obvious bijection between their bases. Therefore Δ restricted to $\mathbb{Z}^{R_A^2/\sim}\oplus\mathbb{Z}^{R_A^{even\geq 4}}$ is simply induced by $R(a_s,a_t)\mapsto R(s,t)$ and Δ restricted to $H_1(\Gamma_{odd};\mathbb{Z})$ is simply reduction of coefficients modulo 2, the map Ξ is induced by $a_s\mapsto s$. The commutativity follows from the constructions of these maps. Furthermore, Ξ is surjective since the other three maps are surjective by definition. Under identifications given by Hopf's formula and Howlett's theorem, we derive the desired commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^{p(\Gamma)+q(\Gamma)} \arrow[r] \arrow[dr] & H_2(A(\Gamma);\mathbb{Z}) \arrow[d] \\
 & H_2(W(\Gamma);\mathbb{Z})\cong\mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}
\end{tikzcd}$$

and finish the proof of Theorem 4.1.

Taking tensor product with $\mathbb{Z}_2$ for terms in the diagram 4.3, we see that $\widetilde{\Phi_W}\circ\Delta$ becomes an isomorphism $\mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}\to\mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}$ since tensoring with $\mathbb{Z}_2$ preserves surjectivity. This forces $H_2(A(\Gamma);\mathbb{Z})\otimes\mathbb{Z}_2\cong\mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}$. By universal coefficient theorem, we have the exact sequence

$$0\to H_2(A(\Gamma);\mathbb{Z})\otimes\mathbb{Z}_2\to H_2(A(\Gamma);\mathbb{Z}_2)\to \mathrm{Tor}(H_1(A(\Gamma);\mathbb{Z}),\mathbb{Z}_2)\to 0,$$

where $\mathrm{Tor}(H_1(A(\Gamma);\mathbb{Z}),\mathbb{Z}_2)=0$ since $H_1(A(\Gamma);\mathbb{Z})$ is torsion free (Proposition 3.5 and Remark 3.6). Now we conclude $H_2(A(\Gamma);\mathbb{Z}_2)\cong\mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}$ and finish the proof of Theorem 4.2.

As a byproduct of the proof, we have the following corollaries.

Corollary 4.14. *The projection $A(\Gamma)\to W(\Gamma)$ induces a surjective homomorphism between the second integral homology*

$$H_2(A(\Gamma);\mathbb{Z})\twoheadrightarrow H_2(W(\Gamma);\mathbb{Z}),$$

and an isomorphism between the second mod 2 homology

$$H_2(A(\Gamma);\mathbb{Z}_2)\xrightarrow{\sim} H_2(W(\Gamma);\mathbb{Z}_2).$$

Corollary 4.15. *Let $M(\Gamma)$ be the complement of the complexified arrangement of reflection hyperplanes associated to the Coxeter group $W(\Gamma)$ and $N(\Gamma) = M(\Gamma)/W(\Gamma)$. If Γ satisfies the following conditions*

- $P(\Gamma)/\sim$ *consists of torsion classes.*
- $\Gamma = \Gamma_{odd}$.
- Γ *is a tree.*

Then $H_2(A(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)}$. Hence the Hurewicz homomorphism $h_2 : \pi_2(N(\Gamma)) \to H_2(N(\Gamma);\mathbb{Z})$ is trivial. Furthermore, for any Coxeter graph, the Hurewicz homomorphism becomes trivial after taking tensor product with $\mathbb{Z}_2$.

Proof. Since $N(\Gamma)$ is path-connected and has fundamental group $\pi_1(N(\Gamma)) = A(\Gamma)$, there is an exact sequence

$$\pi_2(N(\Gamma)) \xrightarrow{h_2} H_2(N(\Gamma);\mathbb{Z}) \xrightarrow{f} H_2(A(\Gamma);\mathbb{Z}) \to 0. \tag{4.4}$$

Suppose that Γ satisfies the three conditions, then $q_1(\Gamma) = q_2(\Gamma) = q_3(\Gamma) = 0$. Theorem 3.8 implies that $H_2(N(\Gamma);\mathbb{Z}) = \mathbb{Z}_2^{p(\Gamma)}$. Then by Theorem 4.1, $H_2(A(\Gamma);\mathbb{Z})$ sits in the following sequence

$$\mathbb{Z}_2^{p(\Gamma)} \twoheadrightarrow H_2(A(\Gamma);\mathbb{Z}) \twoheadrightarrow \mathbb{Z}_2^{p(\Gamma)}$$

whose composition must be an isomorphism, hence $H_2(A(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)}$. As a result, f must be an isomorphism and h_2 must be trivial.

Now suppose Γ is arbitrary. By right-exactness of tensor functor, taking tensor product with $\mathbb{Z}_2$ for 4.4 yields an exact sequence

$$\pi_2(N(\Gamma)) \otimes \mathbb{Z}_2 \xrightarrow{h_2 \otimes \mathrm{id}_{\mathbb{Z}_2}} H_2(N(\Gamma);\mathbb{Z}) \otimes \mathbb{Z}_2 \xrightarrow{f \otimes \mathrm{id}_{\mathbb{Z}_2}} H_2(A(\Gamma);\mathbb{Z}) \otimes \mathbb{Z}_2 \to 0.$$

Note that $f \otimes \mathrm{id}_{\mathbb{Z}_2}$ is an isomorphism as a consequence of Theorem 4.2 and Clancy-Ellis' Theorem 3.8. Hence $h_2 \otimes \mathrm{id}_{\mathbb{Z}_2}$ must be trivial. $\square$

5 Cohomology ring of 2-dimensional Artin groups

In this last section, we restrict ourselves to 2-dimensional Artin groups.

Definition 5.1. Let Γ be a Coxeter graph and (W, S) the associated Coxeter system. The Artin group $A(\Gamma)$ is 2-dimensional if spherical subsets have at most cardinality 2.

It is known that the $K(\pi, 1)$ conjecture holds for 2-dimensional Artin groups ([CD95]). Hence $Z(\Gamma) = Sal(\Gamma)/W$ defined in Subsection 3.1 is a classifying space of $A(\Gamma)$ if $A(\Gamma)$ is 2-dimensional. We shall not distinguish $H_*(A(\Gamma))$ and $H_*(Z(\Gamma))$. The integral homology of $Z(\Gamma)$ computed in Subsection 3.4 then gives the integral homology of a 2-dimensional Artin group $A(\Gamma)$. We repeat the result for convenience.

Theorem 5.2. *Let $A(\Gamma)$ be a 2-dimensional Artin group, then the first homology*

$$H_1(A(\Gamma); \mathbb{Z}) = \bigoplus_{i \in \Lambda} \mathbb{Z}\alpha_i,$$

where α_i is the homology class represented by the 1-cycle e_i. The second homology

$$H_2(A(\Gamma); \mathbb{Z}) = \left(\bigoplus_{\{s_i < s_j\} \in Q(\Gamma)} \mathbb{Z}\beta_{i,j} \right) \oplus \left(\bigoplus_{\omega \in \Omega(\Gamma_{odd})} \mathbb{Z}\gamma_\omega \right),$$

where $\beta_{i,j}$ is the homology class represented by the 2-cycle $e_{i,j}$ and γ_ω is the homology class represented by the 2-cycle $\iota(\omega)$.

Our results of homology groups of 2-dimensional Artin group $A(\Gamma)$ pass to cohomology groups by universal coefficient theorem:

$$H^k(A(\Gamma); \mathbb{Z}) \cong \mathrm{Hom}(H_k(A(\Gamma); \mathbb{Z}), \mathbb{Z}),$$

for $k = 1, 2$. To determine the ring structure, we shall compute the cup product of first cohomology classes.

In what follows, we switch our settings to simplicial cohomology of Δ-complexes (cf. [Hat02]). Our computation of cup products runs over a well-chosen Δ-complex structure of $Z(\Gamma)$. For $i < j$, the cell $e_{i,j}$ (a $2m_{ij}$-gon with edges identified) is now endowed with the Δ-complex structure as shown in Figure 3 for $m := m_{ij}$ even (resp. odd), where each

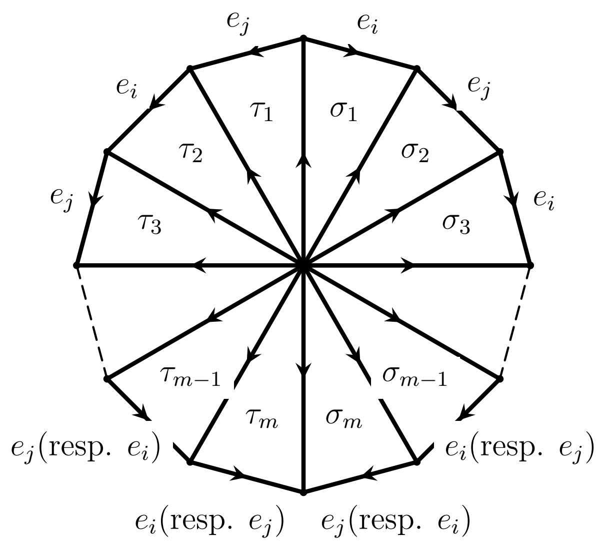

Figure 3: Δ-complex structure

triangle has orientation given by its directed edges, hence $e_{i,j}$ can be viewed as a simplicial 2-chain:

$$e_{i,j} = \sum_{k=1}^{m} \sigma_k - \sum_{k=1}^{m} \tau_k.$$

In this fashion, $Z(\Gamma)$ has been given a Δ-complex structure. We preserve our notations α_i, $\beta_{i,j}$ and γ_ω keeping in mind that they have been translated to simplicial sense. Denote by $\alpha_i^*, \beta_{i,j}^*, \gamma_\omega^*$ the corresponding cohomology classes.

5.1 Dihedral type

The only interesting coholomogy ring of Artin groups with $\#S \leq 2$ is that of $A(I_2(2p))$ (in this case, the Coxeter group is D_{4p}). Although this result is well-known (cf. [Lan00]), we present an alternative computation here as an illustration of our method for general case.

Set the Coxeter graph $\Gamma = I_2(2p)$ for a positive integer p, that is, a graph of two vertices connected by an edge labelled by $2p$, unless in the case $p = 1$, Γ consists of two vertices and no edges. We know the cohomology groups

$$H^1(A(\Gamma); \mathbb{Z}) = \mathbb{Z}\alpha_1^* \oplus \mathbb{Z}\alpha_2^*, \quad H^2(A(\Gamma); \mathbb{Z}) = \mathbb{Z}\beta_{1,2}^*.$$

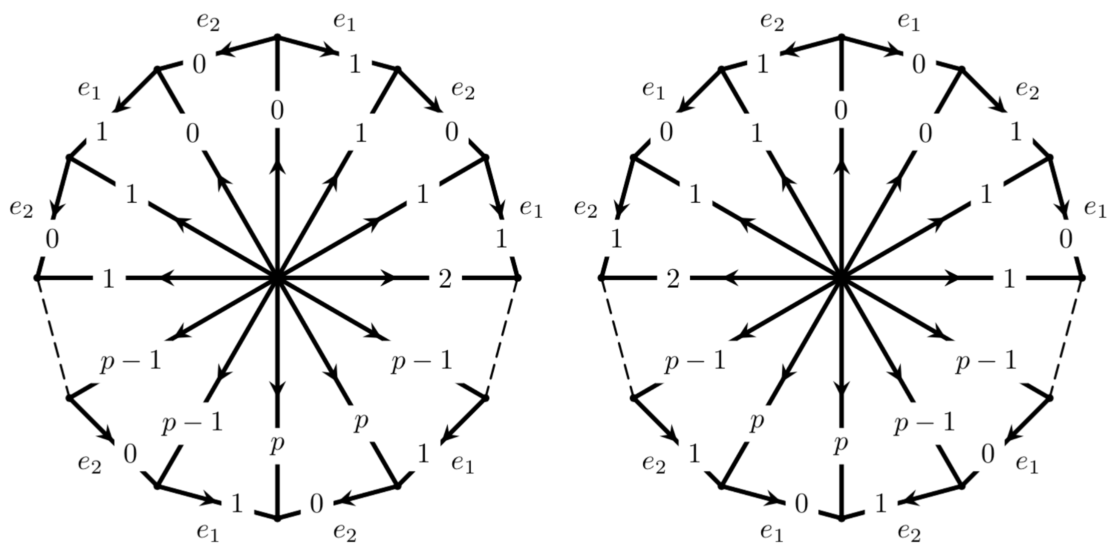

Figure 4: Cocycles φ_1 (left) and φ_2 (right)

In order to compute cup products, we need to find 1-cocyles φ_1 and φ_2 representing α_1^* and α_2^* respectively. They can be chosen as shown in Figure 4.

We explain the choice of φ_1 in details. Since φ_1 should represent α_1^*, as a $\mathbb{Z}$-valued function of 1-simplices (edges), φ_1 must take values 1 on e_1 and 0 on e_2. As for those radius edges, if we assign the "north pointing" edge an arbitrary integer, say 0, then all the other radius edges are assigned values according to the cocycle condition, that is, the sum of (signed) values on edges of each triangle must be 0. Remark that different choices of the assignment of values to radius edges differ by coboundaries hence will certainly not change our result in cohomology level.

It is easy to compute the values that the resulting 2-cocycles take on the 2-cycle $e_{1,2}$

$$(\varphi_1 \smile \varphi_2)(e_{1,2}) = -(\varphi_2 \smile \varphi_1)(e_{1,2}) = p.$$

In cohomology class level, this yields

$$\alpha_1^* \smile \alpha_2^* = -(\alpha_2^* \smile \alpha_1^*) = p\beta_{1,2}^*.$$

Also $\alpha_1^* \smile \alpha_1^* = \alpha_2^* \smile \alpha_2^* = 0$ is trivially understood.

5.2 General case

Now we prove our theorem on the cohomology rings of 2-dimensional Artin groups.

Let us first recall our notations. Let $A(\Gamma)$ be a 2-dimensional Artin group with associated Coxeter system (W, S) and Coxeter graph Γ. The generating set $S = \{s_1, \dots, s_n\}$, as well as the vertex set of Γ, has been given a total order $s_1 < \cdots < s_n$. The subgraph Γ_{odd} of Γ consists of edges labelled by odd numbers. Fix a set of representatives

$$\Lambda = \{i \mid s_i \text{ is the minimum vertex in the connected components of } \Gamma_{odd} \text{ containing it}\}.$$

For each $i \in \Lambda$, it is convenient to denote

$$\Delta_i = \{j \mid s_j \text{ and } s_i \text{ are in the same connected component of } \Gamma_{odd}\}.$$

Hence $\{\Delta_i \mid i \in \Lambda\}$ is a partition of $\{1, \dots, n\}$. Recall also $Q(\Gamma) = \{\{s_i, s_j\} \subset S \mid m_{ij} \text{ is even}\}$ and $\Omega(\Gamma_{odd})$ is a fixed basis of the finitely generated free abelian group $H_1(\Gamma_{odd}; \mathbb{Z})$.

Theorem 5.3. *With the above notations, $A(\Gamma)$ has cohomology groups:*

$$H^1(A(\Gamma); \mathbb{Z}) = \bigoplus_{i \in \Lambda} \mathbb{Z}\alpha_i^*,$$

$$H^2(A(\Gamma); \mathbb{Z}) = \left(\bigoplus_{\{s_i < s_j\} \in Q(\Gamma)} \mathbb{Z}\beta_{i,j}^* \right) \oplus \left(\bigoplus_{\omega \in \Omega(\Gamma_{odd})} \mathbb{Z}\gamma_\omega^* \right).$$

The cohomology ring structure is given by the following relations:

$$\alpha_i^* \smile \alpha_i^* = 0,$$

for $i \in \Lambda$. As for distinct $i, j \in \Lambda$,

$$\alpha_i^* \smile \alpha_j^* = \sum_{\ell \in \Delta_i, k \in \Delta_j} \frac{m_{\ell k}}{2} \beta_{\ell,k}^*.$$

Proof. The cohomology group structure follows from the results of Section **??** and universal coefficient theorem. The relation $\alpha_i^* \smile \alpha_i^* = 0$ is trivial. It remains to compute $\alpha_i^* \smile \alpha_j^*$.

For distinct $i, j \in \Lambda$, we choose 1-cocycles φ_i and φ_j representing α_i^* and α_j^* respectively. They take values

$$\varphi_i(e_{i'}) = \begin{cases} 1, & i' \in \Delta_i; \\ 0, & i' \notin \Delta_i. \end{cases} \qquad \varphi_j(e_{j'}) = \begin{cases} 1, & j' \in \Delta_j; \\ 0, & j' \notin \Delta_j. \end{cases}$$

As for the values they take on the radius edges in the Δ decomposition of each 2 cell, we assign value 0 to the "north pointing" edges as we did in the previous subsection, then all the other edges are assigned values accordingly. Now we are able to compute for $1 \leq \ell < k \leq n$ the value

$$(\varphi_i \smile \varphi_j)(e_{\ell,k}) = \begin{cases} \dfrac{m_{\ell k}}{2}, & \ell \in \Delta_i, k \in \Delta_j; \\ -\dfrac{m_{\ell k}}{2}, & \ell \in \Delta_j, k \in \Delta_i; \\ 0, & \text{otherwise}. \end{cases}$$

A few words should be mentioned about this result. The cases $\ell \in \Delta_i, k \in \Delta_j$ and $\ell \in \Delta_j, k \in \Delta_i$ reduce to the computation in Subsection 4.1. One should notice that $\beta^*_{\ell,k} = -\beta^*_{k,\ell}$. Next suppose $\ell \notin \Delta_i \cup \Delta_j$, then no matter where k lies, at least one of φ_i and φ_j takes value 0 on all edges of $e_{\ell,k}$. This yields $(\varphi_i \smile \varphi_j)(e_{\ell,k}) = 0$. The same result occurs for the case $k \notin \Delta_i \cup \Delta_j$. These exhaust all possible cases. By passing to cohomology class level, we derive the desired result. $\square$

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