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Study on Kerr/Fluid Duality and Singularity of Solutions to the Fluid Equation

( カー解と流体の双対性および流体方程式における解の特異点に関する研究)

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Abstract

An equation for a viscous incompressible fluid on a spheroidal surface that is dual to the perturbation around the near-near-horizon extreme Kerr (near-NHEK) black hole is derived. It is also shown that an expansion scalar $\theta$ of a congruence of null geodesics on the perturbed horizon of the perturbed near-NHEK spacetime, which is dual to a viscous incompressible fluid, is not in general positive semidefinite, even if initial conditions on the velocity are smooth. Unless initial conditions are appropriately adjusted, caustics of null congruence will occur on the perturbed horizon in the future. A similar result is obtained for a perturbed Schwarzschild black hole spacetime which is dual to a viscous incompressible fluid on $S^2$. An initial condition that $\theta$ be positive semidefinite at any point on $S^2$ is a necessary condition for the existence of smooth solutions to the incompressible Navier-Stokes equation on $S^2$. 
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1 Introduction

The fluid/Gravity correspondence has a long history, starting with [2] and [3]. In the context of anti-de Sitter / conformal field theory (AdS/CFT), this subject has been extensively studied[4] [5]. In the case of asymptotically flat spacetimes, a prescription was developed in [6] and the incompressible Navier-Stokes (NS) equation on a plane, which is dual to Rindler space [7], and that on a sphere, which is dual to Schwarzschild black hole [8] [9], were derived. Furthermore, the Petrov type-I condition on the hypersurface geometry reduces the Einstein equation to the NS equation [10]. To our knowledge, however, an equation of fluid mechanics that is dual to a more realistic black hole, a rotating one in an asymptotically flat spacetime has not been obtained until now.

In this paper, we will derive an extension of an incompressible NS equation to that which is dual to the Kerr black hole solution in four-dimensional asymptotically flat spacetime. For simplicity, we will consider the near-near-horizon extreme Kerr (near-NHEK) limit[11], which is introduced in the context of Kerr/CFT correspondence [12][13]. This problem is interesting from the point of view of symmetry. The Kerr black hole has only axial symmetry and the horizon of the Kerr black hole is not a round sphere, but a distorted, spheroidal surface. This reduces the symmetry of the fluid equation, and it makes it difficult to infer the form of the fluid equation solely from symmetry considerations.

Another subject of this paper is the expansion scalar $\theta$ of the null congruence of the perturbed event horizon of the near-NHEK spacetime. In this paper this perturbed event horizon $H$ is defined by the condition that $H$ coincides with the stationary event horizon in the absence of gravitational perturbation, and $H$ is null. This definition was also adopted in, e.g., [14]. This $H$ is generated by null geodesics. In the case of the spacetime that we study in this paper such a null surface $H$ is uniquely determined by the $\lambda$ expansion which is introduced in the derivation of the generalized incompressible NS equation. We will compute $\theta$ to the leading nontrivial order of the $\lambda$ expansion and find that it is not necessarily positive semidefinite. If $\theta$ is negative at some point the congruence of null geodesics will end with caustics in finite time due to the focusing equation. At the caustics we have $\theta = -\infty$ and the velocity $v_i$ of the fluid becomes singular, since $\theta$ is a polynomial of $v_i$ and its derivatives. This means that the solution to the generalized incompressible NS equation has singularities, even if it was smooth in the past. To avoid such a singularity we need to impose an initial condition that $\theta$ be nonnegative anywhere on $H$ at some time $\tau = \tau_0$. The
calculation of the expansion scalar is carried out for both near-NHEK and Schwarzschild k holes. This result will be relevant to the problem\cite{15} of the existence and smoothness of solutions to the generalized incompressible NS equation in spaces with topologies different from a plane.\footnote{It can be shown by a similar calculation that $\theta$ is positive semidefinite in the case of a two- and three-dimensional planar horizons in Rindler space.}

By the assumption of cosmic censorship\cite{16}, it is considered that, with a smooth initial condition, there are no singularities on and outside the event horizon in the future. It is known that the entropy of a black hole is related to the area of the event horizon. The area of the event horizon and the entropy will not decrease \cite{17}. In the nonstationary case, there are some proposals for definition of the horizons \cite{18}. In the previous paragraph, we defined the event horizon of a perturbed spacetime as a null surface $\mathcal{H}$ which agrees with the event horizon of the unperturbed one when the perturbation is switched off. Then, if $\theta$ were negative at some point on $\mathcal{H}$, $\mathcal{H}$ would not be a true event horizon. The true event horizon will be outside $\mathcal{H}$. However, even if this is the case, the above argument for the singularity of the solution to the fluid equation still applies, since the argument is valid on some null surface $\mathcal{H}$. In contrast, when $\theta$ is nonnegative all over $\mathcal{H}$, $\mathcal{H}$ might as well be regarded as an event horizon.

This paper is organized as follows. There are three parts. In the first part, the basic facts and knowledge to study the fluid/gravity correspondence are gathered. The second part is for the review of the fluid/gravity correspondence and geometry of the Kerr black hole in an extreme limit we use. The third part is the main subject of the thesis, in which we deal with the fluid/gravity correspondence for a Kerr black hole. Moreover, we discuss the singularity of the solutions to the fluid equation. Section 2 is mainly devoted to gravity and geometry of manifolds including Killing vectors, hypersurfaces, geodesic congruences. These subjects are needed for discussing a singularity of a spacetime, a black hole and its horizon. The definition and some nature of black holes are given in section 3. In section 4, we review the fluid mechanics. In section 5 the fluid/gravity correspondence is reviewed. The procedure for constructing fluid mechanics dual to asymptotically flat spacetime\cite{7}\cite{8} is reviewed especially. In section 6 the limit of near-horizon extreme Kerr (NHEK) spacetime and the near-NHEK spacetime is explained. In section 7 a generalized incompressible NS equation on a spheroidal surface that is dual to the near-NHEK is derived. This equation contains several extra terms. In section 8 the expansion scalar $\theta$ for the congruence of the null geodesics on the perturbed event horizon $\mathcal{H}$
is computed. Discussions are given in section 9. In appendix A quantities related to the unperturbed horizon, a spheroidal $S^2$, are collected. In appendix B the perturbation part of the metric that is dual to the fluid is presented. In appendix C we report on some results for a fluid equation that is dual to a perturbed Schwarzschild black hole that was obtained by using the same prescription as the near-NHEK spacetime.

This paper is based on [1].
Part I  
Fundamentals

In this part, we summarize ingredients to understand current researches of gravity and facts about the fluid dynamics. Section 2 is mainly devoted to gravity \cite{19} \cite{20} and geometry of manifolds \cite{21} including Killing vectors, hypersurfaces and congruences. These subjects are needed for discussing a singularity of a spacetime, a black hole and its horizon. Their definitions and some nature of black holes \cite{22} \cite{23} \cite{24} \cite{25} \cite{26} \cite{27} are given in section 3. In section 4, we review the NS equations and dynamical equations describing relativistic fluids \cite{30} \cite{31} briefly.

2  Gravity and Geometry

2.1  General Relativity

The action of general relativity with matter in $d$-dimensional spacetime is called the Einstein-Hilbert action and given by

$$S_{\text{EH}} = \int d^d x \sqrt{-g} \left( \frac{c^4}{16\pi G} (R - 2\Lambda) + \mathcal{L}_{\text{mat}} \right),$$  \hfill (2.1)

where $c$ is the speed of light, $G$ the Newton constant, $\Lambda$ a cosmological constant. $\mathcal{L}_{\text{mat}}$ denotes a Lagrangian for matter coupled to the gravity. Hereafter, we set $c = 1$, and moreover $d = 4$ except in some sections. $R$ is the Ricci scalar obtained by contraction of the Ricci tensor $R_{\mu\nu}$ (or Riemann tensor $R^\lambda{}_{\rho\mu\nu}$)

$$R \equiv g^{\mu\nu} R_{\mu\nu} \equiv g^{\mu\nu} R^\lambda{}_{\mu\lambda\nu}.$$  \hfill (2.2)

The Riemann tensor

$$R^\lambda{}_{\rho\mu\nu} \equiv \partial_\mu \Gamma^\lambda{}_{\rho\nu} + \Gamma^\lambda{}_{\mu\alpha} \Gamma^\alpha{}_{\nu\rho} - (\mu \leftrightarrow \nu)$$  \hfill (2.3)

is defined by the commutation relation of covariant derivatives of a vector $v^\mu$

$$[\nabla_\mu, \nabla_\nu] v^\lambda = R^\lambda{}_{\rho\mu\nu} v^\rho.$$  \hfill (2.4)
More generally for an arbitrary \((n, m)\) rank tensor, the commutation relation is written as

\[
[\nabla_\mu, \nabla_\nu] T^{\lambda_1 \cdots \lambda_n}_{\sigma_1 \cdots \sigma_m} = \sum_{i=1}^{n} R^{\lambda_i}_{\sigma \mu \nu} T^{\lambda_1 \cdots \lambda_n}_{\sigma_1 \cdots \sigma_{i-1} \sigma_i \sigma_{i+1} \cdots \sigma_m} - \sum_{i=1}^{m} R^{\sigma}_{\rho_i \mu \nu} T^{\lambda_1 \cdots \lambda_n}_{\rho_1 \cdots \rho_{i-1} \sigma_i \rho_{i+1} \cdots \rho_m}.
\] (2.5)

The Christoffel symbol appearing in the definition of the Riemann tensor (2.3) is given by

\[
\Gamma^\lambda_{\mu \nu} \equiv \frac{1}{2} g^{\lambda \sigma} (\partial_\mu g_{\nu \sigma} + \partial_\nu g_{\mu \sigma} - \partial_\sigma g_{\mu \nu}).
\] (2.6)

Variation of the action (2.1) gives the equations of motion

\[
R_{\mu \nu} - \frac{1}{2} (R - 2\Lambda) g_{\mu \nu} = 8\pi G T_{\mu \nu}
\] (2.7)

where

\[
T_{\mu \nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_{\text{mat}})}{\delta g^{\mu \nu}}.
\] (2.8)

is called the stress-energy tensor. Taking the divergence on equation (2.7) with the Bianchi identity

\[
\nabla_\nu R^\mu_{\sigma \nu \lambda} + \nabla_\nu R^\sigma_{\mu \lambda \nu} + \nabla_\lambda R^\mu_{\sigma \mu \nu} = 0,
\] (2.9)

we find that

\[
\nabla^\mu T_{\mu \nu} = 0.
\] (2.10)

Equation (2.7) is called the Einstein equation. When no matter exists, \(T_{\mu \nu} = 0\), the equation is called especially by the vacuum Einstein equation.

2.2 Geodesics

2.2.1 Vectors and Parallel Transport

Let us consider a spacetime \((M, g_{\mu \nu})\). A curve \(\gamma(\sigma)\) on \(M\) is defined by a map \(\mathbb{R} \to M\) which is parametrized by a parameter \(\sigma\). \(\gamma(\sigma)\) is represented by \(x^\mu(\sigma)\) in some coordinates \(x^\mu\). If \(\gamma(\sigma)\) is smooth and does not intersect itself, there exists a tangent vector

\[
k^\mu \equiv \frac{dx^\mu(\sigma)}{d\sigma}.
\] (2.11)
By definition, $k^\mu$ is tangent to the curve $\gamma(\sigma)$. The vector $k^\mu$ is introduced by the variation of a scalar field $f(x^\lambda)$ along $\gamma(\sigma)$

$$\frac{df(x^\lambda)}{d\sigma} = \frac{\partial f(x^\lambda)}{\partial x^\mu} \frac{dx^\mu(\sigma)}{d\sigma} = k^\mu \frac{\partial}{\partial x^\mu} f(x^\lambda) = k^\mu \partial_{\mu} f.$$  \hspace{1cm} (2.12)

Conversely, we can find smooth curves corresponding to a given smooth vector field.

The vector $k^\mu$ \footnote{Although we call $v^\mu$ a vector, it is not indeed a vector but a component of a vector. More precisely, $v = v^\mu \partial_\mu$ should be called a vector.} transforms as

$$k'^\mu = \frac{dy^\mu(\sigma)}{d\sigma} = \frac{\partial y^\mu}{\partial x^\nu} \frac{dx^\nu(\sigma)}{d\sigma} = \frac{\partial y^\mu}{\partial x^\nu} k^\nu.$$ \hspace{1cm} (2.13)

under a coordinate transformation and it is indeed the transformation of vectors. There are three types of vectors classified by its norm; a vector is called time-like for $g_{\mu\nu} k^\mu k^\nu < 0$, space-like for $g_{\mu\nu} k^\mu k^\nu > 0$ and null (or light-like) for $g_{\mu\nu} k^\mu k^\nu = 0$.

How does a vector field $v^\mu(x^\nu)$ vary along a curve? The above discussion (2.12) for a scalar field is no longer valid for $v^\mu(x^\nu)$ since

$$\frac{dv^\mu}{d\sigma} = k^\nu \partial_{\nu} v^\mu$$ \hspace{1cm} (2.14)

is not a tensor. A vector (or a tensor) is defined in the tangent space which is a vector space spanned at each point on $M$. In order to compare tensors which belong to tangent spaces at different points, we have to introduce a connection. We adopt the Christoffel symbol (2.6) as the connection. Then we can define the variation of $v^\mu$ along $\gamma(\sigma)$

$$\frac{Dv^\mu}{D\sigma} \equiv k^\nu \nabla_{\nu} v^\mu.$$ \hspace{1cm} (2.15)

Here $\nabla_{\mu}$ is a covariant derivative associated with the metric $g_{\mu\nu}$. This variation is generalized for an arbitrary rank tensor $T_{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_m}$ as

$$\frac{DT_{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_m}}{D\sigma} \equiv k^\lambda \nabla_{\lambda} T_{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_m}.$$ \hspace{1cm} (2.16)

A vector $v^\mu$ is said to be parallel transported along the curve $\gamma(\sigma)$ if it satisfies the equation

$$\frac{D}{D\sigma} v^\mu(\sigma) = k^\nu \nabla_{\nu} v^\mu = 0.$$ \hspace{1cm} (2.17)
2.2.2 Affinely/Non-affinely parameterized Geodesics

A geodesic is defined by a curve whose tangent vector \( k^\mu \) is parallel transported along itself, namely, the geodesic equation is

\[
k^\nu \nabla_\nu k_\mu = 0. \tag{2.18}
\]

The above definition (2.18) is, however, too restrictive. We can loosen the condition in such a way that after some infinitesimal transportation of \( k^\mu \) along the geodesic the transported vector should be proportional to the original one:

\[
k^\nu \nabla_\nu k_\mu = \kappa(\sigma) k_\mu, \tag{2.19}
\]

where \( \kappa \) is a scalar function. Geodesics following equation (2.19) are said to be non-affinely parameterized. Equation (2.19) reduces to equation (2.18) when \( \kappa = 0 \) and it is said to be affinely parameterized. We shall explain these terminologies, affine/non-affine, in the next paragraph.

For the affinely parameterized equation (2.18), introducing a new parameter

\[
\sigma \to \tau = f(\sigma) \tag{2.20}
\]

gives

\[
\frac{d^2 x^\mu(\tau)}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu(\tau)}{d\tau} \frac{dx^\lambda(\tau)}{d\tau} = - \frac{f''}{f'^2} \frac{dx^\mu(\tau)}{d\tau}. \tag{2.21}
\]

where \( f' = df(\sigma)/d\sigma \) and we used the fact that the vector \( k^\mu \) can be written as equation (2.11). If \( f'' = 0 \) is satisfied, equation (2.21) is the same as equation (2.18) except replacing \( \sigma \) with \( \tau \). We find that a reparameterization with constants \( a \) and \( b \)

\[
\tau = f(\sigma) = a \sigma + b \tag{2.22}
\]

satisfies the \( f'' = 0 \). Parameters \( \tau \) and \( \sigma \) related by (2.22) are called affine parameters. Equation (2.18) is invariant under the reparameterization (2.22) This is the reason why equation (2.18) is called affinely parameterized.

In equation (2.21), we see that

\[
\kappa(\tau) = \kappa(f(\sigma)) = - \frac{f''}{f'^2}, \tag{2.23}
\]

or integrating equation (2.23)

\[
\frac{d\sigma}{d\tau} = \exp \int^\tau d\alpha \kappa(\alpha). \tag{2.24}
\]
Equation (2.24) can be obtained as follows

\[
\int \kappa \, d\tau = - \int \frac{d\tau}{f'^2} f'' f' = - \int \frac{d\sigma}{f'} f'' f' = \int d\sigma \frac{f''}{f'} = - \log f' + \text{const.} \quad (2.25)
\]

For a given non-affinely parameterized geodesic equation, if we solve equation (2.24) with respect to \( \tau \), we can obtain the affinely parameterized one. We can bring equation (2.19) to the form of equation (2.17) by such reparameterization without loss of generality.

The norm of the tangent vector of non-affinely parameterized geodesics varies along them as

\[
\frac{d}{d\sigma} (k^\nu k_\nu) = 2 (k^\nu \nabla_\nu k^\mu) k_\mu = 2\kappa k^2. \quad (2.26)
\]

If \( k^\mu \) is null at a point, the norm is zero everywhere on the geodesic as we expected. Even in the case where \( k^\mu \) is non-null, the value of \( k^2 \) does not change along the geodesics.

### 2.2.3 Physical Interpretation

From a physical point of view, a geodesic can be regarded as a worldline of a test particle. The particle moves on the minimal path in a spacetime, in other words, it moves as possible as straight. Let us consider a length \( l \) of a curve whose tangent vector is \( k^\mu \) between two points \( P \) and \( Q \),

\[
l = \int_P^Q d\sigma \sqrt{-g_{\mu\nu} k^\mu k^\nu}. \quad (2.27)
\]
Straightforward calculation of the variation of $l$ gives the geodesic equations. In practice, the variation of $l$ with respect to $x$

$$
\delta l = \int \frac{1}{2\sqrt{g}} \left( -\delta g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} - 2g_{\mu\nu} \frac{d(\delta x^\mu)}{d\sigma} \frac{dx^\nu}{d\sigma} \right)
$$

$$
= \int \frac{1}{2\sqrt{g}} \left( -\partial\gamma g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} + 2\delta x^\lambda \frac{d}{d\sigma} \left( g_{\lambda\nu} \frac{dx^\nu}{d\sigma} \right) \right)
$$

$$
= \int \frac{1}{2\sqrt{g}} \delta x^\lambda \left( -\partial\gamma g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} + 2 \left( \partial\gamma g_{\lambda\nu} \frac{dx^\nu}{d\sigma} + g_{\lambda\nu} \frac{d^2 x^\nu}{d\sigma^2} \right) \right)
$$

(2.28)

gives the geodesic equation

$$
\frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\sigma} \frac{dx^\lambda}{d\sigma} = 0.
$$

(2.29)

Here we used the integral by parts, boundary terms from the operation vanish due to $\delta x = 0$ and a relation

$$
\partial\gamma g_{\mu\nu} = \Gamma_{\lambda\mu\nu} + \Gamma_{\lambda\nu\mu}
$$

(2.30)

following from the definition of the Christoffel symbol (2.6).

Above discussion does not apply to massless particles. In order to discuss massless cases, we need to generalize the above length to a new action

$$
S[x, e] = \frac{1}{2} \int d\sigma \left( \delta x^\alpha \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} e(\sigma)^{-1} - m^2 e(\sigma) \right)
$$

(2.31)

with an auxiliary field $e(\sigma)$ and a mass $m$. Variation with respect to $e(\sigma)$ gives an equation of motion

$$
\delta x^\alpha \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} + m^2 e(\sigma) = 0.
$$

(2.32)

If $m \neq 0$, we can solve equation (2.32). The solution

$$
e(\sigma) = \frac{1}{m} \sqrt{-\delta x^\alpha \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}
$$

(2.33)

reduces the new action (2.31) to the old one (2.27) up to a multiplicative constant $m$. For the massless case $m = 0$, the action (2.31) becomes

$$
S[x, e] = \frac{1}{2} \int d\sigma e(\lambda)^{-1} g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}
$$

(2.34)

It can be show that this action (2.34) with gauge fixing $e(\lambda) = 1$ leads to the null geodesic equation.
2.3 Killing Vectors

The Killing vector is a vector satisfying the Killing equation and describes the symmetry of a spacetime. The Killing equation is defined via the Lie derivative of the metric. Firstly, we give a definition of the Lie derivative. Then, the Killing equation and its nature are treated.

2.3.1 Lie Derivatives

We define the Lie derivative $\mathcal{L}_k v^\mu$ of a vector $v^\mu$ along a curve $\gamma$ with tangent $k^\mu$ by

$$\mathcal{L}_k v^\mu = k^\nu \partial_\nu v^\mu - v^\nu \partial_\nu k^\mu,$$

or more generally for a $(n, m)$ rank tensor $T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m}$

$$\mathcal{L}_k T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m} = k^\lambda \partial_\lambda T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m} - \sum_{i=1}^{n} T^{\mu_1 \ldots \lambda \ldots \mu_i}_{\nu_1 \ldots \nu_m} \partial_\lambda k^{\mu_i} + \sum_{j=1}^{m} T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \lambda \ldots \nu_m} \partial_{\nu_j} k^\lambda.$$

It can be found by following discussion.

Consider a curve $\gamma(\sigma)$ with the tangent vector $k^\mu$ and a vector field $v^\mu$. Let $P$ and $Q$ be two neighbouring points on the curve $\gamma(\sigma)$. In some coordinates $x^\mu(\sigma)$, points $P$ and $Q$ correspond to $x^\mu$ and $x^\mu + \Delta x^\mu$, respectively. We have the relation

$$\Delta x^\mu = \frac{dx^\mu(\sigma)}{d\sigma} \Delta \sigma = k^\mu \Delta \sigma,$$

and introduce new coordinates $x'^\mu$ as

$$x'^\mu = x^\mu + \Delta x^\mu = x^\mu + k^\nu \Delta \sigma.$$

From the transformation of the vector (2.13), we obtain

$$v'^\nu(x') = \frac{\partial x'^\mu}{\partial x^\nu} v^\nu(x)$$

$$= \delta^\mu_\nu (x^\mu + \Delta x^\mu) v^\nu$$

$$= (\delta^\mu_\nu + \partial_\nu k^\mu \Delta \sigma) v^\nu.$$

In other words,

$$v'^\nu(Q) = v^\nu(P) + v^\nu(P) \partial_\nu k^\mu \Delta \sigma.$$
On the other hand, $v^\mu(Q)$ can be expressed as

$$v^\mu(Q) = v^\mu(x + \Delta x)$$

$$= v^\mu(x) + \partial_\nu v^\mu(x) \Delta x^\nu$$

$$= v^\mu(P) + \partial_\nu v^\mu(P) k^\nu \Delta \sigma. \quad (2.41)$$

We define the Lie derivative of a vector by the difference of equations (2.40) and (2.41)

$$\mathcal{L}_k v^\mu(P) \equiv \lim_{\Delta \sigma \to 0} \frac{v^\mu(Q) - v^\mu(P)}{\Delta \sigma}. \quad (2.42)$$

Then we obtain the above result (2.35). Some more discussion reveals that the Lie derivatives of tensors can be defined in equation (2.36).

It is not clear whether the Lie derivative of a vector is a tensor or not. We can easily check this point in a handy way with covariant derivative. Equation (2.35) can be rewritten as

$$\mathcal{L}_k v^\mu = k^\nu \partial_\nu v^\mu - v^\nu \partial_\nu k^\mu = k^\nu \nabla_\nu v^\mu - v^\nu \nabla_\nu k^\mu. \quad (2.43)$$

Equation (2.43) suggests that the Lie derivatives of vectors are manifestly tensors. The same things happen for the Lie derivatives of tensors. Note that the definition of the Lie derivative is independent of the covariant derivative. A more mathematically rigorous way reveals that point [21]. However, here we do not adopt such a treatment.

### 2.3.2 Killing Vectors

We define the Killing equation by vanishing of the Lie derivative of the metric $g_{\mu\nu}$ with a vector $\xi^\mu$

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\sigma} \partial_\mu \xi^\lambda + g_{\mu\lambda} \partial_\nu \xi^\lambda = 0. \quad (2.44)$$

$\mathcal{L}_\xi g_{\mu\nu}$ can be rewritten as

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \nabla_\lambda g_{\mu\nu} + g_{\lambda\sigma} \nabla_\mu \xi^\lambda + g_{\mu\lambda} \nabla_\nu \xi^\lambda$$

$$= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \quad (2.45)$$

The vector $\xi$ is called the Killing vector which satisfies the Killing equation (2.44).
2.4 Hypersurfaces

2.4.1 Basics

A hypersurface $\Sigma$ is defined by a codimension-one submanifold in a manifold $M$ [20] [32].$^3$ Of course, we consider a spacetime $(M, g_{\mu\nu})$ with a metric $g_{\mu\nu}$ as the manifold $M$ and we specifically call $(M, g_{\mu\nu})$ encapsulating $\Sigma$ an ambient spacetime. There are two different ways to describe hypersurfaces. One is to impose a single constraint and the other is to relate the coordinates on $M$ to ones restricted on $\Sigma$. We adopt the first way here, while the second one in section 2.4.3.

In order to specify a hypersurface $\Sigma$, we need a condition with a function $\Phi$

$$\Phi(x^\mu) = 0. \tag{2.46}$$

$\Sigma$ is characterized by the normal vector

$$n_\mu = -\partial_\mu \Phi \tag{2.47}$$

and its tangent vectors. All vectors $v^\mu$ tangent to $\Sigma$ are orthogonal to the normal

$$v^\mu n_\mu = 0. \tag{2.48}$$

If the character of $n^\mu$ (whether it is timelike, spacelike or null) is constant over $\Sigma$,

$$\Sigma \text{ is called } \begin{cases} \text{spacelike} & \text{for } n^\mu n_\mu < 0, \\ \text{timelike} & \text{for } n^\mu n_\mu > 0, \\ \text{null} & \text{for } n^\mu n_\mu = 0. \end{cases}$$

For instance, since all vectors tangent to $\Sigma$ with time-like normal vector $n^\mu$ are space-like vectors, it is said that $\Sigma$ is a space-like surface.

2.4.2 Hypersurface orthogonality and Frobenius’ Theorem

A vector $n^\mu$ is said to be hypersurface orthogonal if $n^\mu$ is expressed by a derivative of a scalar $\Phi(x)$

$$n^\mu \propto g^{\mu\nu} \partial_\nu \Phi. \tag{2.49}$$

Hypersurface orthogonality is a concept for a vector field. At a glance, this definition seems to depend on a hypersurface since equation (2.49) is

$^3$In this section, we mainly consider 4-dimensional manifolds as an ambient spacetime $M$. Most discussion can be generalized to $d$-dimensional manifolds cases straightforwardly.
similar to equation (2.47). Frobenius’ theorem, however, gives an equivalent expression in terms of only $n^\mu$.

Now we review the Frobenius’ theorem. We focus on the necessary and sufficient condition of the hypersurface orthogonality although the theorem can apply to more general codimensional submanifolds. Let $n = n^\mu \partial_\mu$ be a vector. According to the theorem, the following statements are equivalent.

(1) $n = n^\mu \partial_\mu$ is hypersurface orthogonal.

(2) The condition $n_{[\mu} \nabla_{\nu} n_{\lambda]} = 0$ is satisfied.

(3) The rotation tensor $\omega_{\mu\nu} \equiv \nabla_{[\mu} n_{\nu]}$ vanishes.

For brevity, we only give partial proofs; (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3), since the proof of (2) $\Rightarrow$ (1) is harder.

(1) $\Rightarrow$ (2):

(1) implies

$$n^\mu = f g^{\mu\nu} \partial_\nu \Phi$$

with some functions $f$ and $\Phi$. Then straightforward calculation reveals

$$n_{[\mu} \partial_{\nu} n_{\lambda]} = n_{[\mu} \partial_{\nu} \Phi \partial_{\lambda]} (f \partial_{\lambda]} \Phi)$$

$$= f \partial_{[\mu} \Phi \partial_{\nu} f \partial_{\lambda]} \Phi + f^2 \partial_{[\mu} \Phi \partial_{\nu} \partial_{\lambda]} \Phi$$

$$= 0,$$

the first equality comes from equation (2.50), second from the Libnitz rule, and third from the fact that the symmetry of $\partial \Phi$ in the first term and $\partial \partial \Phi$ in the second term.

(3) $\Rightarrow$ (2):

It follows that

$$n_{[\mu} \partial_{\nu} n_{\lambda]} = 0 \Leftrightarrow n_{[\mu} \nabla_{\nu} n_{\lambda]} = 0$$

from the fact that the lower indices symmetry of $\Gamma^\mu_{\nu\lambda}$. Then it leads to

$$n_{[\mu} \nabla_{\nu} n_{\lambda]} = 0 \Leftrightarrow n_{[\mu} \omega_{\nu\lambda]} = 0.$$  

If $\omega_{\mu\nu} = 0$, (2) is satisfied obviously.

(2) $\Rightarrow$ (3):

For the non-null vector $n^\mu$, by contracting RHS of (2.53) with $n^\lambda$, we obtain

$$\omega_{\mu\nu} = 0.$$  

Here we used the fact that $\omega_{\mu\nu}$ is purely transverse. For the null vector $n^\mu$, we can find a normal vector $N^\mu$ such that

$$n^\mu N_\mu = -1, \quad N^\mu N_\mu = 0.$$

Then contracting RHS of (2.53) with $N^\mu$ leads to (2.54).
2.4.3 Induced Metric

Before constructing the induced metric, let us classify terminologies: intrinsic, extrinsic, induced and transverse. These words are almost used as adjectives for the metric. Consider an ambient spacetime \((M, g_{\mu\nu})\) and a hypersurface \((\Sigma, h_{ab})\) as its submanifold. In the notation \((M, g_{\mu\nu})\), the metric \(g_{\mu\nu}\) is intrinsic in \(M\). An intrinsic metric of a manifold is defined in the manifold itself and describes the intrinsic geometry. So \(h_{ab}\) is also said to be the intrinsic metric of \(\Sigma\) itself. \(h_{ab}\) can, however, be induced (or deduced) from \(M\) and called the induced metric.

A hypersurface \(\Sigma\) is always embedded in an ambient spacetime \(M\) by definition. We cannot say anything how \(\Sigma\) is embedded by studying intrinsic geometry of \(\Sigma\). In order to know the extrinsic geometry, how \(\Sigma\) is bent in \(M\), we need to study the relation between \(\Sigma\) and \(M\). To accomplish this task, we need the transverse metric as a projector matrix. The transverse metric is useful in discussion of congruences. We introduce the transverse metric in section 2.4.5 though do not examine the extrinsic geometry in the thesis.

Now we derive the induced metric of \(\Sigma\) from an ambient spacetime \(M\). Introduce two coordinates \(x^\mu\) for \(M\) and \(y^a\) for \(\Sigma\), they are related by

\[
x^\mu = x^\mu(y^a).
\]

The Jacobian \(E^\mu_a\) can be calculated as

\[
E^\mu_a = \frac{\partial x^\mu}{\partial y^a},
\]

which is a \(4 \times 3\) matrix whose rank is 3. The line element of \(M\) is

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu
\]

and one of \(\Sigma\) is

\[
ds^2_\Sigma = h_{ab}dy^a dy^b.
\]

By restricting \(ds^2\) on \(\Sigma\)

\[
ds^2|_{\Sigma} = g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu|_{\Sigma}
\]

\[
= g_{\mu\nu}(x^\lambda(y^a))\frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} dy^a dy^b,
\]

we obtain the induced metric

\[
h_{ab} = g_{\mu\nu}E^\mu_a E^\nu_b.
\]
2.4.4 Properties of Null Surfaces

Consider a null surface $N$ whose normal vector is $k^\mu$. Since the normal $k^\mu$ is null, it is tangent to $N$ simultaneously. It leads to

$$\eta^\mu = f k^\mu$$

(2.62)

with a certain tangent vector $\eta^\mu$ and a function $f(x^\mu)$. Since the normal vector $k^\mu$ is null, we cannot normalize it as for the non-null normal.

If a null surface $N$ is described by

$$\Phi(x^\mu) = 0,$$

(2.63)

the normal can be written by

$$k_\mu = \partial_\mu \Phi.$$  

(2.64)

In this situation, it can be shown that $k^\mu$ satisfies the geodesic equation with a scalar $\kappa$

$$k^\nu \nabla_\nu k_\mu = \kappa k_\mu.$$  

(2.65)

From equation (2.64), it follows that

$$k^\nu \nabla_\nu k_\mu = k^\nu \nabla_\nu (\partial_\mu \Phi) = k^\nu \nabla_\mu k_\nu = \frac{1}{2} \nabla_\mu (k^\nu k_\nu).$$  

(2.66)

Although $k^\mu k_\mu = 0$ on $N$, the variation can be nonzero to directions of the normal $k^\mu$. Then we obtain equation (2.65).

Let us summarize this subsection. If a null vector $k^\mu$ satisfies the hypersurface orthogonality condition, there exist null geodesics related to $k^\mu$. Moreover, since $k^\mu$ is tangent to $N$, a point on $N$ lies on these null geodesics. In this sense, these null geodesics are called the null generators of $N$.

2.4.5 Transverse Metric

We shall define the transverse metric for a null surface $N$. The definitions for null and non-null surfaces differ and the former is given here. In preparation for that, again we introduce the induced metric of $N$.

Consider a null surface $N$ and the null generators $\gamma(\sigma)$. We choose the coordinates $y^a$ on $N$ as

$$y^a = (v = \sigma, y^i),$$

(2.67)
where $y^i$ represents the coordinates of 2-dimensional spatial directions. The geodesics $\gamma(\sigma)$ are labelled by the spatial coordinates $y^i$. In the coordinates $y^a$, we obtain the Jacobian

$$E^\mu_\nu = \frac{\partial x^\mu}{\partial \sigma} = k^\mu, \quad E^\mu_i = \frac{\partial x^\mu}{\partial y^i}, \quad (2.68)$$

and the induced metric defined in (2.61)

$$h_{\nu\nu} = g_{\mu\nu}k^\mu k^\nu = 0, \quad h_{\nu i} = g_{\mu\nu}k^\mu E^\nu_i = 0, \quad h_{ij} = \gamma_{ij} = g_{\mu\nu}E^\mu_i E^\nu_j. \quad (2.69)$$

Here we used the fact that the normal $k^\mu$ is orthogonal to $E^\mu_i$ tangent to $N$. $h_{\nu\nu} = 0$ and $h_{\nu i} = 0$ imply that the induced metric $h_{ij}$ is degenerate, i.e., $h_{ab}$ is a $3 \times 3$ matrix but reduces to a $2 \times 2$ matrix $\gamma_{ij}$ essentially. $\gamma_{ij}$ is the purely spatial 2-dimensional metric. The line element restricted onto $N$ is

$$ds^2|_N = \gamma_{ij}dy^i dy^j = g_{\mu\nu}E^\mu_i E^\nu_j dy^i dy^j. \quad (2.70)$$

In order to project tensors in an ambient spacetime $M$ onto $N$, we use the projector matrix, or the transverse metric. For the null surfaces, due to the nullness of the normal $k^\mu$, we should find another null vector $N^\mu$ to construct the transverse metric. There is a freedom to choose $N^\mu$ but it does not influence how tensors are projected. We adopt $N^\mu$ such that

$$k^\mu N_\mu = -1, \quad E^\mu_i N_\mu = 0, \quad N^\mu N_\mu = 0. \quad (2.71)$$

We define the transverse metric $h_{\mu\nu}$ by

$$h_{\mu\nu} \equiv g_{\mu\nu} + k_\mu N_\nu + k_\nu N_\mu. \quad (2.72)$$

$h_{\mu\nu}$ is a symmetric tensor and satisfies

$$h_{\mu\nu} k^\mu = 0, \quad h_{\mu\nu} N^\mu = 0. \quad (2.73)$$

Its trace

$$g^{\mu\nu}h_{\mu\nu} = g^{\mu\nu}(g_{\mu\nu} + k_\mu N_\nu + k_\nu N_\mu) = 2 \quad (2.74)$$

implies that $h_{\mu\nu}$ behaves like 2-dimensional metric. For raising and lowering its indices the intrinsic metric $g_{\mu\nu}$ should be used like

$$h^{\mu\nu} = g^{\mu\lambda}h_{\lambda\nu}. \quad (2.75)$$

$h^{\mu\nu}$ plays indeed a role of a projector

$$h^{\mu\lambda} h_{\lambda\nu} = h^{\mu\nu}. \quad (2.76)$$
The induced metric can be written by means of the projector as

\[ \gamma_{ij} = g_{\mu \nu} E_i^\mu E_j^\nu \]

\[ = (h_{\mu \nu} - k_\mu N_\nu - k_\nu N_\mu) E_i^\mu E_j^\nu \]

\[ = h_{\mu \nu} E_i^\mu E_j^\nu. \quad (2.77) \]

Conversely, from the fact that \( k^\mu \) and \( N^\mu \) are orthogonal to \( E_i^\mu \), we have the relation dual to (2.77)

\[ h^{\mu \nu} = \gamma^{ij} E_i^\mu E_j^\nu. \quad (2.78) \]

Roughly speaking, the difference between two tensors \( h_{\mu \nu} \) and \( \gamma_{ij} \) is the size of matrices. They have the same nonzero components and so they are almost equivalent physically. The induced metric \( h_{\mu \nu} \) acts, however, as a tensor in \( M \) not \( \Sigma \) while \( \gamma_{ij} \) does in \( \Sigma \) not \( M \).

2.5 Congruences

A congruence is a family of curves which do not intersect each other. A congruence of geodesics is said to be a geodesic congruence. In this subsection, we consider deviation of neighbouring geodesics and how it varies along them. One of the aspect is captured by the Raychaudhuri equation. With an energy condition, the Raychaudhuri equation leads to the focusing theorem. For later purposes, we will focus on the non-affinely parametrized null geodesic congruence.

2.5.1 Deviation Vectors

Let us introduce a deviation vector for a geodesic congruence. Consider a family of neighbouring geodesics \( \gamma_\tau(\sigma) \) whose affine parameter is \( \sigma \) and tangent vector is \( k^\mu \). A parameter \( \tau \) is a label for specifying a geodesic in the congruence. The congruence can be represented by \( x^\mu(\sigma, \tau) \), and \( k^\mu = \partial x^\mu / \partial \sigma \) satisfies equation (2.18). Now we can vary the parameter \( \tau \) with fixed \( \sigma \), and obtain curves across the congruence. Then the tangent vector \( \xi^\mu = \partial x^\mu / \partial \tau \) along the new curves will be defined. This vector \( \xi^\mu \) is a deviation vector. The fact that two partial derivatives commute

\[ \frac{\partial^2 x^\mu}{\partial \sigma \partial \tau} = \frac{\partial^2 x^\mu}{\partial \tau \partial \sigma} \quad (2.79) \]

leads to

\[ k^\nu \nabla_\nu \xi^\mu = \xi^\nu \nabla_\nu k^\mu. \quad (2.80) \]
Equation (2.80) can also be written with the Lie derivative by

$$\mathcal{L}_k \xi^\mu = \mathcal{L}_k k^\mu = 0.$$  

(2.81)

Moreover it follows that

$$\xi^\mu \nabla_\mu k^2 = 0,$$  

(2.82)

since all geodesics in a null geodesic congruence have null tangent vector. Equations (2.19), (2.80) and (2.82) yield

$$\frac{d}{d\sigma} (k^\nu \xi_\nu) = (k^\mu \nabla_\mu k^\nu) \xi_\nu + k^\nu (k^\mu \nabla_\mu \xi_\nu)$$

$$= \kappa k^\mu \xi_\mu + \kappa k^\nu \xi^\mu \nabla_\mu k^\nu$$

$$= \kappa k^\mu \xi_\mu + \frac{1}{2} \xi_{(\mu} \nabla_{\nu)} k^2$$

$$= \kappa k^\mu \xi_\mu.$$  

(2.83)

Hence if $\xi^\mu$ is orthogonal to $k^\mu$,

$$\xi^\mu k_\mu = 0,$$  

(2.84)

at some point on $\gamma_\tau(\sigma)$, equation (2.84) holds on the whole geodesic.

### 2.5.2 Variation of Non-affinely Parameterized Null Congruences

Now we can discuss how the deviation vector varies along the null congruence. Before proceeding to calculation, we show again the equations we have derived above.

$$k^2 = 0, \quad k^\mu \xi_\mu = 0, \quad k^\nu \nabla_\nu \xi_\mu = \xi^\nu \nabla_\nu k_\mu, \quad k^\nu \nabla_\nu k_\mu = \kappa k_\mu.$$  

(2.85)

The variation of $\xi^\mu$ along $\gamma_\tau(\sigma)$ is failure to parallel transportation

$$\frac{D}{D\sigma} \xi_\mu = k^\nu \nabla_\nu \xi_\mu = B_{\mu\nu} \xi^\nu.$$  

(2.86)

Here we defined a tensor

$$B_{\mu\nu} \equiv \nabla_\nu k_\mu.$$  

(2.87)

We have following relations

$$k^\mu B_{\mu\nu} = 0, \quad B_{\mu\nu} k^\nu = \kappa k_\mu,$$  

(2.88)

where the first one follows from $k^2 = 0$. 

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In order to isolate the purely transverse part of the deviation, we use the projector matrix

\[ h_{\mu\nu} \equiv g_{\mu\nu} + k_\mu N_\nu + N_\mu k_\nu, \quad (2.89) \]
\[ h^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta}. \quad (2.90) \]

where \( N^\mu \) is a vector which satisfies

\[ N^\mu N_\mu = 0, \quad N^\mu k_\mu = -1. \quad (2.91) \]

The nature of the projection \( h_{\mu\nu} \) is the same as the transverse metric (2.72)

\[ h_{\mu\nu} k^\nu = h_{\mu\nu} k^\mu = 0, \quad (2.92) \]
\[ h_{\mu\nu} N^\mu = h_{\mu\nu} N^\nu = 0, \quad (2.93) \]
\[ h_{\mu\nu} g^{\mu\nu} = h_{\mu\nu} h^{\mu\nu} = 2. \quad (2.94) \]

With this projector matrix \( h_{\mu\nu} \), we define a tensor \( \tilde{B}_{\mu\nu} \) by

\[ \tilde{B}_{\mu\nu} = h_\mu^\alpha h_\nu^\beta B_{\alpha\beta} \quad (2.95) \]

which is the purely transverse part of \( B_{\mu\nu} \). \( \tilde{B}_{\mu\nu} \) appears in the fully transverse version of equation (2.86)

\[ h_{\mu\nu} \frac{D}{D\sigma} \left( h^\lambda_\chi \xi^\chi \right) = \tilde{B}_{\mu\nu} \xi^\nu. \quad (2.96) \]

\( \tilde{B}_{\mu\nu} \) is orthogonal to \( k^\mu \) and \( N^\mu \)

\[ \tilde{B}_{\mu\nu} k^\mu = \tilde{B}_{\mu\nu} k^\nu = 0, \quad (2.97) \]
\[ \tilde{B}_{\mu\nu} N^\mu = \tilde{B}_{\mu\nu} N^\nu = 0. \quad (2.98) \]

The tensor \( \tilde{B}_{\mu\nu} \) now can be decomposed as

\[ \tilde{B}_{\mu\nu} = \frac{1}{2} \theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}. \quad (2.99) \]

The definitions of tensor appearing RHS in (2.99) as follows. The expansion scalar \( \theta \) is defined by

\[ \theta \equiv g^{\mu\nu} \tilde{B}_{\mu\nu} = h^{\mu\nu} \tilde{B}_{\mu\nu} = h^{\mu\nu} B_{\mu\nu} \quad (2.100) \]

which is the trace part of \( \tilde{B}_{\mu\nu} \), the shear tensor

\[ \sigma_{\mu\nu} = \frac{1}{2} \left( \tilde{B}_{\mu\nu} + \tilde{B}_{\nu\mu} \right) - \frac{1}{2} \theta h_{\mu\nu}. \quad (2.101) \]
which is the traceless symmetric part of $\tilde{B}_{\mu\nu}$, and the rotation (or twist) tensor
\[
\omega_{\mu\nu} = \frac{1}{2} \left( \tilde{B}_{\mu\nu} - \tilde{B}_{\nu\mu} \right) \tag{2.102}
\]
which is the antisymmetric part of $\tilde{B}_{\mu\nu}$. The definition of $\theta$ with equation (2.88) implies that
\[
\theta = h^{\mu\nu} B_{\mu\nu} = (g^{\mu\nu} + k^\mu N^\nu + k^\nu N^\mu) B_{\mu\nu} = \nabla^\mu k_\mu - \kappa. \tag{2.103}
\]
On the other hand, from equation (2.99) we can check that
\[
\tilde{B}_{\mu\nu} \tilde{B}^{\nu\mu} = \frac{1}{2} \theta^2 + \sigma_{\mu\nu} \sigma^{\mu\nu} - \omega_{\mu\nu} \omega^{\mu\nu}. \tag{2.104}
\]
Moreover, it follows that
\[
B_{\mu\nu} B^{\nu\mu} = \tilde{B}_{\mu\nu} \tilde{B}^{\nu\mu} + \kappa^2. \tag{2.105}
\]
Equation (2.105) is derived as follows. From the definition of $\tilde{B}_{\mu\nu}$ (2.95), we obtain
\[
\tilde{B}_{\mu\nu} = (\delta^\alpha_\mu + N_\mu k^\alpha + N^\alpha k_\mu) \left( \delta^\beta_\nu + N_\nu k^\beta + N^\beta k_\nu \right) B_{\alpha\beta} = (\delta^\alpha_\mu + N_\mu k^\alpha + N^\alpha k_\mu) \left( B_{\alpha\nu} + \kappa N_\nu k_\alpha + k_\nu B_{\alpha\beta} N^\beta \right) = B_{\mu\nu} + k_\mu N^\alpha B_{\alpha\nu} + k_\nu B_{\mu\alpha} N^\alpha + k_\mu k_\nu N^\alpha N^\beta B_{\alpha\beta}, \tag{2.106}
\]
where we used equations (2.88). From equations (2.88), (2.97) and (2.106), we can calculate $\tilde{B}_{\mu\nu} \tilde{B}^{\nu\mu}$ as
\[
\tilde{B}_{\mu\nu} \tilde{B}^{\nu\mu} = \tilde{B}_{\mu\nu} \left( B^{\nu\mu} + k^\nu N^\alpha B_{\alpha\mu} + k^\mu B^{\nu\alpha} N^\alpha + k^\nu k^\mu N^\alpha N^\beta B_{\alpha\beta} \right) = \left( B_{\mu\nu} + k_\mu N^\alpha B_{\alpha\nu} + k_\nu B_{\mu\alpha} N^\alpha + k_\mu k_\nu N^\alpha N^\beta B_{\alpha\beta} \right) B^{\nu\mu} = B_{\mu\nu} B^{\nu\mu} - \kappa^2. \tag{2.107}
\]
This is the equation (2.105).
2.5.3 Raychaudhuri Equation

We would like to know an evolution of the expansion $\theta$ which is obtained from

\[
k^\alpha \nabla_\alpha B_{\mu \nu} = k^\alpha \nabla_\alpha \nabla_\nu k_\mu
\]
\[
= k^\alpha \left( [\nabla_\alpha, \nabla_\nu] + \nabla_\nu \nabla_\alpha \right) k_\mu
\]
\[
= -k^\alpha R^\lambda_{\mu \alpha \nu} k_\lambda + \nabla_\nu \left( k^\alpha \nabla_\alpha k_\mu \right) - \nabla_\nu k^\alpha \nabla_\alpha k_\mu. \tag{2.108}
\]

Taking trace with $g^{\mu \nu}$ and using equation (2.103) lead to

\[
\frac{d}{d\sigma} (\theta + \kappa) = -R_{\alpha \beta} k^\alpha k^\beta + \nabla_\mu \left( \kappa k^\mu \right) - B_{\mu \nu} B^{\mu \nu}. \tag{2.109}
\]

The second term in RHS of equation (2.109) is calculated as

\[
\nabla_\mu \left( \kappa k^\mu \right) = k^\mu \nabla_\mu \kappa + \kappa \nabla_\mu k^\mu
\]
\[
= \frac{d}{d\sigma} \kappa + \kappa \theta + \kappa^2, \tag{2.110}
\]

Then, from equations (2.105) and (2.110) we finally obtain the Raychaudhuri equation for non-affinely null geodesics

\[
\frac{d}{d\sigma} \theta = \kappa \theta - \frac{1}{2}\theta^2 - \sigma_{\mu \nu} \sigma^{\mu \nu} + \omega_{\mu \nu} \omega^{\mu \nu} - R_{\alpha \beta} k^\alpha k^\beta. \tag{2.111}
\]

2.5.4 Energy Conditions

In order to proceed to derive the focusing theorem, we require an extra condition. Let us consider the Einstein-Hilbert action. We need a condition on the matter coupled to gravity for the spacetime behaving well physically. This condition is the energy condition. While there are some versions of the condition, one of them is the null energy condition

\[
T_{\mu \nu} k^\mu k^\nu \geq 0. \tag{2.112}
\]

Here $T_{\mu \nu}$ is defined in equation (2.8) and $k^\mu$ is an arbitrary future-directed null vector. This condition combined with the Einstein equation (2.7) implies

\[
R_{\mu \nu} k^\mu k^\nu \geq 0. \tag{2.113}
\]

This is a key assumption to derive the singularity theorem [28] or area increase theorem [17].
2.5.5 Focusing Theorem

Since $\sigma_{\alpha\beta}$ is purely spatial, $\sigma^{\alpha\beta}\sigma_{\alpha\beta}$ must be nonnegative

$$\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0. \quad (2.114)$$

Assuming affinely parameterization ($\kappa = 0$), hypersurface orthogonality $\omega_{\mu\nu} = 0$ and the null energy condition implies that the RHS in equation (2.111) becomes negative semidefinite. Then we obtain

$$\frac{d}{d\sigma} \theta \leq -\frac{1}{2} \theta^2. \quad (2.115)$$

or

$$\frac{d}{d\sigma} \theta^{-1} \geq \frac{1}{2}. \quad (2.116)$$

The statement of the focusing theorem is as follows. Equation (2.116) means that if we can find the form of function $\theta^{-1}(\sigma)$, its gradient must be greater than 1/2, that is, $\theta^{-1}(\sigma)$ is a monotonic increasing function. On the one hand, if an integration constant (or an initial condition) $\theta_0^{-1} = \theta^{-1}(\sigma = 0)$ is negative, it must be zero at a finite time $\sigma_0$. Therefore, it concludes that $\lim_{\sigma \to \sigma_0} \theta(\sigma) = -\infty$. On the other hand, if $\theta_0^{-1}$ is positive, then $\theta(\sigma)$ is positive and should asymptote to zero.

2.5.6 Interpretation of Expansion

We can interpret the expansion $\theta$ as a rate of change the area of a congruence. Let us consider an affinely parameterized null congruence $\gamma_{\tau}(\sigma)$ and foliation of null hypersurfaces $N_\sigma$ whose normal vector field $k^\mu$ generates the congruence. As we saw earlier in section 2.4.5, the metric of $N_\sigma$, $\gamma_{ij}$, is defined in equation (2.70). Calculate the derivative of $\gamma_{ij}$ along the geodesics as

$$\frac{d}{d\sigma} \gamma_{ij} = k^\lambda \nabla_\lambda \left( g_{\mu\nu} E^\mu_i E^\nu_j \right)$$

$$= (k^\lambda \nabla_\lambda g_{\mu\nu}) E^\mu_i E^\nu_j + (k^\lambda \nabla_\lambda E^\mu_i) g_{\mu\nu} E^\nu_j$$

$$= (E^\nu_i \nabla_\lambda k^\mu) g_{\mu\nu} E^\nu_j + (E^\nu_j \nabla_\lambda k^\mu) g_{\mu\nu} E^\nu_i$$

$$= \nabla_\lambda k_{ij} E^\lambda_i E^\nu_j + \nabla_\lambda k_{ij} E^\nu_j E^\lambda_i$$

$$= (B_{\mu\nu} + B_{\nu\mu}) E^\mu_i E^\nu_j. \quad (2.117)$$
where we used equation (2.77) and interpreted $E_i^\mu$ as the deviation vector $\xi^\mu$ in (2.80). Then we find with equation (2.78)

$$\gamma^{ij} \frac{d}{d\sigma} \gamma_{ij} = (B_{\mu\nu} + B_{\nu\mu}) \left( \gamma^{ij} E_i^\mu E_j^\nu \right)$$

$$= (B_{\mu\nu} + B_{\nu\mu}) h^{\mu\nu}$$

$$= 2\theta. \quad (2.118)$$

Finally, we obtain

$$\theta = \frac{1}{2} \gamma^{ij} \frac{d}{d\sigma} \gamma_{ij}$$

$$= \frac{1}{\sqrt{\gamma}} \frac{d}{d\sigma} \sqrt{\gamma}.$$

(2.119)

Here $\gamma = \det \gamma_{ij}$.
3 Black Holes

A black hole is one of exact solutions to the Einstein equation [19] [20]. It captures the nature of dynamics of spacetimes. It is interesting not just from theoretical point of view, e.g., applications to gauge/gravity correspondence, but experimentally, because objects which seem like black holes are observed in our universe. Black holes are, however, highly non-local objects by definition, which is defined in a whole spacetime including the future infinity. This is the distinctive point in comparison with other physical objects since usual objects are defined in local region. There are lots of attempts to generalize non-local black holes to quasi-local ones so far [22] [33].

In this section, we review stationary black holes. After giving the definition of black holes, horizons, singularities and the area theorem are treated. Then, as concrete examples, we review the Schwarzschild and the Kerr solutions [34].

3.1 Horizons

Boundaries of black holes are called horizons. There exist many types of horizons; event horizons, Killing horizons, apparent horizons and so on. Some of them can be defined only for stationary black holes and others are developed for capturing local pictures of black holes beyond the nonlocalness. Here we mainly focus on the horizons for stationary black holes.

3.1.1 Definition and Event Horizons

Roughly speaking, a black hole is a region from which even lights cannot escape. For this reason, we should consider the structure of spacetime at infinity since we should check whether lights can escape from the region to the infinity or not.

Let $M$ be an asymptotically flat (or anti de-Sitter) spacetime. We define $\mathcal{J}^-(U)$ as the causal past of a set of points $U \subset M$ and $\mathcal{J}^+(U)$ as its closure including the edge. From these, the boundary of $\mathcal{J}^-(U)$ is defined by

$$\mathcal{J}^-(U) = \mathcal{J}_-(U) - \mathcal{J}^-(U).$$

Then the region of a black hole is defined by

$$\mathcal{B} = M - \mathcal{J}^-(\mathcal{I}^+)$$

and future event horizon of $M$ is given by

$$\mathcal{H}^+ = \mathcal{J}^+(\mathcal{I}^+),$$
where $\mathcal{I}^+$ denotes the future null infinity. $\mathcal{H}^+$ means the boundary of closure of the causal past of $\mathcal{I}^+$. As we mentioned before, the above definition needs the global structure of the spacetime including infinity. This is the reason why black holes are non-local objects. $\mathcal{H}^+$ is defined via causal past, and so it is a null surface. Moreover, we can show that generators of $\mathcal{H}^+$ have no future endpoint.

### 3.1.2 Killing Horizons

A null hypersurface $\mathcal{N}$ is a Killing horizon $\mathcal{K}$ of a Killing vector $\xi$ if $\xi$ is normal to $\mathcal{N}$ on $\mathcal{N}$. From the knowledge of hypersurfaces in section 2.4, the normal vector $\xi$ is tangent to $\mathcal{N}$ simultaneously and so the vector field $\xi$ can be regarded as null generators of $\mathcal{N}$. It gives a geodesic equation

$$\xi^\nu \nabla_\nu \xi^\mu = \kappa_\xi \xi^\mu \text{ on } \mathcal{K}$$

(3.4)

with a parameter $\kappa_\xi$.

The parameter $\kappa$ in equation (2.65) for a null geodesic has no physical meaning in general since it is changed by a reparameterization of the parameter $\sigma$. This ambiguity comes from the ambiguity of the norm of the normal vector. When we consider, however, a Killing horizon as a null surface, the parameter $\kappa_\xi$ has physical significance and it is called the surface gravity. Now the normalization of $\xi$ is restricted by the Killing equation, and then $\xi \rightarrow a \xi$ with a constant $a$ is only permitted. The constant $a$ can be fixed from such as a condition that for a time-like Killing vector $\xi$. For instance, $\xi$ should be normalized naturally at spatial infinity;

$$\xi^2 \rightarrow -1 \quad (r \rightarrow \infty).$$

(3.5)

Let us calculate the surface gravity of the Killing horizon $\mathcal{K}$ with a normal Killing vector $\xi$. A Killing vector $\xi$ satisfies

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0,$$

(3.6)

and it yields

$$\nabla_\mu \xi_\nu = \nabla_{[\mu} \xi_{\nu]} = \frac{1}{2} (\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu).$$

(3.7)

Moreover from the hypersurface orthogonality or (2) of the Frobenius’ theorem in section 2.4.2, $\xi$ also satisfies on $\mathcal{K}$

$$\xi_{[\mu} \nabla_\nu \xi_{\lambda]}|_{\mathcal{K}} = 0.$$
Equation (3.6) reduces equation (3.8) to
\[
(\xi_\mu \nabla_\nu \xi_\lambda + \xi_\nu \nabla_\lambda \xi_\mu - \xi_\lambda \nabla_\nu \xi_\mu) |_K = 0. \tag{3.9}
\]
Multiplying \(\nabla^\nu \xi^\lambda\) to equation (3.9) gives
\[
\xi_\mu (\nabla_\nu \xi_\lambda) \left( \nabla^\nu \xi^\lambda \right) = -2 \left( \nabla^\nu \xi^\lambda \right) \xi_\nu (\nabla_\lambda \xi_\mu). \tag{3.10}
\]
The RHS of equation (3.10) with equation (3.4) is rewritten as
\[
-2 \left( \nabla^\nu \xi^\lambda \right) \xi_\nu (\nabla_\lambda \xi_\mu) = -2 \left( \xi_\nu \nabla^\nu \xi^\lambda \right) \nabla_\lambda \xi_\mu
= -2 \kappa_\xi \xi^\lambda \nabla_\lambda \xi_\mu
= -2 \kappa_\xi^2 \xi_\mu. \tag{3.11}
\]
Finally, we obtain the surface gravity
\[
\kappa_\xi^2 = -\frac{1}{2} (\nabla^\mu \xi^\nu) (\nabla_\mu \xi_\nu) |_K. \tag{3.12}
\]
We show that the \( \kappa_\xi \) is constant along the geodesic on \( K \). From the Bianchi identity
\[
R^\rho [\lambda_\mu_\nu] = 0, \tag{3.13}
\]
it follows that
\[
\nabla_\lambda \nabla_\mu \xi_\nu = R^\rho [\lambda_\mu_\nu] \xi_\rho. \tag{3.14}
\]
Consider the derivative of \( \kappa_\xi^2 \) along the geodesic
\[
\xi^\lambda \nabla_\lambda \kappa_\xi^2 = \xi^\lambda \nabla_\lambda \left( -\frac{1}{2} (\nabla^\mu \xi^\nu) (\nabla_\mu \xi_\nu) \right)
= -\xi^\lambda (\nabla^\mu \xi^\nu) \nabla_\lambda \nabla_\mu \xi_\nu
= -\xi^\lambda (\nabla^\mu \xi^\nu) R_{\rho\lambda_\mu_\nu} \xi_\rho
= 0, \tag{3.15}
\]
hence \( \kappa_\xi \) is constant along the geodesic. We have shown the surface gravity \( \kappa_\xi \) is constant along the geodesic on \( K \). Further discussion reveals that for a stationary black hole \( \kappa_\xi \) is constant over \( K \) [35].

By definition, the Killing horizon can be defined only if a spacetime has isometries. For static or stationary spacetimes, we can always find a timelike Killing vector and define the Killing horizon. According to [29], the event horizon of a stationary black hole must be the Killing horizon.
As we will see in section 3.4.3, there exist only three parameters of a stationary black hole in four dimensions. Within a classical treatment, these parameters are related with an analogy of the first law of thermodynamics, in which the surface gravity plays the role of temperature. Moreover, quantum level discussions, e.g., the Hawking effect [36] and the Unruh effect [37], also suggest that the surface gravity identified with the temperature of the black hole

$$T = \frac{\kappa}{2\pi}$$

in units with $G = c = \hbar = k = 1$.

### 3.2 Singularities

#### 3.2.1 Singularities

Here we treat the singularities in general relativity [38]. We expect that singularities have some pathological behaviors at some points on a spacetime. It is, however, difficult to define singularities by itself since a point on a spacetime is not singular by definition. At first glance, there might be several ways to define them by such as the divergence of the metric or curvature scalars like $R, R_{\mu\nu}R^{\mu\nu}$. Even though these definitions are natural intuitively, they do not work well as following reasons. First, their behaviors depend on coordinates we chose. A good example is the event horizon of the Schwarzschild black hole. In some coordinates, the metic of the black hole is written as (3.19). This metric diverges at $r = 2M$, but it can be removed by change of coordinates as we shall see in section 3.4.2. Then we find that the surface $r = 2M$ is a pseudo singularity. Second, we know examples that there are singularities in some spacetimes even at which the curvature invariants vanish. For these reasons, we give up to define singularities along the directions.

What we would like to call singularity is a spot at which a particle disappear from the spacetime in a finite time. To describe this situation, we usually define a singularity by an incomplete geodesic. For giving the definition of incomplete, we need to introduce inextensible curves. If a curve $\gamma(\sigma)$ converges around a point $P$, that is, $\gamma(\sigma)$ for $\sigma > \sigma_0$ is in the neighbourhood of $P$, then we say that the curve has future endpoint. An inextensible curve is defined as the curve which has no endpoint. For example, a time-like curve towards the future infinity is future inextensible. Roughly speaking, inextensible curves go to the edge of spacetimes. A geodesic is said to be incomplete if the geodesic is inextensible and has a finite range of affine parameter. If a spacetime includes such an incomplete geodesic, there exists a

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singularity.

3.2.2 Cosmic Censorship

The cosmic censorship is a conjecture about the naked singularity. It claims that singularities should not be observed, hence for a black hole they must lie in the event horizon. The statement is neither derived from some underlying fundamental assumptions nor indicates that a spacetime which has naked singularities is not physical immediately.

3.3 Area Theorem

The Raychaudhuri equation for null geodesics and the cosmic censorship lead to the area theorem. The area theorem insists that the area of the event horizon of any black holes will increase.

In order to understand the theorem, let us consider the event horizon of a black hole. First, we would like to show that the expansion scalar $\theta$ is nonnegative over the horizon. Assume $\theta < 0$ at a point $P$ on the horizon. Then in the neighbourhood of $P$ outward the horizon $\theta$ must be also negative. According to the focusing theorem, $\theta < 0$ implies that the null generators have caustics in a finite range of affine parameter. A caustic of null geodesics leads to an incomplete geodesic and it means the emergence of a singularity. Now the cosmic censorship requires that there exist no singularity outside the event horizon. To avoid the naked singularity, it concludes that $\theta \geq 0$ over the event horizon. As we saw in section 2.5.5, $\theta$ can be interpreted as the rate of change of the area. $\theta \geq 0$ over the event horizon indicates that the area of the event horizon will increase.

3.4 Black Holes and Exact Solutions

3.4.1 Exact Solutions

Here we focus on the vacuum Einstein equation (2.7) with $T_{\mu\nu} = 0$ and $\Lambda = 0$. Even in the simplification, the complexity of equation (2.7), which is a set of non-linear partial differential equations, prevents us from solving them analytically for general spacetimes. If the spacetimes, however, have some specific isometries, equation (2.7) becomes much simpler.

One simplification is that a metric does not depend on time. More precisely, we should give definitions of stationary or static. An asymptotically flat spacetime is stationary if there exists a time-like Killing vector. For a
stationary spacetime, the metric can be written as
\[ ds^2 = g_{tt}(x^i)dt^2 + 2g_{ti}(x^i)dx^t + g_{ij}(x^i)dx^i dx^j. \] (3.17)

The condition of static is stronger than stationary. A stationary spacetime is static if it is invariant under time-reversal \( t \rightarrow -t \). Cross terms \( 2g_{ti}(x^i)dx^t \) should vanish and the static metric can be written as
\[ ds^2 = g_{tt}(x^i)dt^2 + g_{ij}(x^i)dx^i dx^j. \] (3.18)

Thus we have a few exact solutions. In fact, we can find the Schwarzschild solution, which is a static, spherically symmetric black hole. Moreover, we obtain the Kerr solution, Reissner-Nordström solution and Kottler solution as some generalization of the Schwarzschild. The Kerr solution is the rotating version of the Schwarzschild solution, the Reissner-Nordström is the charged version and the Kottler is similar to the Schwarzschild including a nonzero cosmological constant \( \Lambda \). Here we focus on the Schwarzschild and Kerr black holes.

### 3.4.2 Schwarzschild Solution

The Schwarzschild solution, which is one of exact solutions to the vacuum Einstein equation, is a static and spherically symmetric black hole. The metric is given by
\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad f(r) = 1 - \frac{2M}{r}, \] (3.19)

where \( M > 0 \) is the mass of the black hole. In the metric (3.19), there are two singularities at \( r = 0 \) and \( r = 2M \). The former singularity at \( r = 0 \) is a physical singularity but the latter one is due to the choice of coordinates.

In order to remove the pseudo singularity at \( r = 2M \), let us consider transformations
\[ r^* = \int \frac{dr}{f(r)} = r + 2M \log (r - 2M) \] (3.20)

and then
\[ u = t - r^*, \]
\[ v = t + r^*. \] (3.21)

Then we obtain a metric
\[ ds^2 = -f(r)dv^2 + 2dvdr + r^2d\Omega^2 \] (3.22)
in the ingoing Eddington-Finkelstein coordinates and
\[ ds^2 = -f(r)du^2 - 2dudr + r^2d\Omega^2 \]  
(3.23)
in the outgoing Eddington-Finkelstein coordinates. In these coordinates, 
\( r = 2M \) is no longer singular.

Let us calculate the surface gravity \( \kappa_\xi \) of the Schwarzschild black hole. 
From the norm of the Killing vector \( \xi \)

\[ g_{\mu\nu} \xi^\mu \xi^\nu = -\left( 1 - \frac{2M}{r} \right), \]  
(3.24)
we find \( r = 2M \) at which \( \xi \) becomes null. Equation (3.12) leads to

\[ \kappa_\xi = \frac{1}{4M}. \]  
(3.25)
As we saw in equation (3.16), this surface gravity is proportional to the 
temperature of the system.

### 3.4.3 Kerr Solution

In four-dimensional asymptotically flat spacetimes, the most general exact 
solution is the Kerr-Newman solution representing a rotating black hole with 
a charge. Parameters of the black hole are only the mass \( M \), the angular 
momentum \( J \) and the charge \( Q \). In the Boyer-Lindquist coordinates, the 
metric is given by

\[
\begin{align*}
    ds^2 &= -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 - 2a \sin^2 \theta \frac{r^2 + a^2 - \Delta}{\Sigma} dt d\varphi \\
    &\quad + \left( \frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\varphi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,
\end{align*}
\]  
(3.26)
where

\[
\begin{align*}
    \Sigma &= r^2 + a^2 \cos^2 \theta \\
    \Delta &= r^2 - 2Mr + a^2 + Q^2.
\end{align*}
\]  
(3.27)
(3.28)
A new parameter \( a \) is given by

\[ a = \frac{J}{M}. \]  
(3.29)
Taking some limits reduces to special cases. The Kerr-Newman black hole reduces to

$$\begin{align*}
\text{Kerr} & \quad \text{for } Q \to 0, \\
\text{Reissner-Nordström} & \quad \text{for } J \to 0, \\
\text{Schwarzschild} & \quad \text{for } Q, J \to 0, \\
\text{Minkowski} & \quad \text{for } Q, J, M \to 0.
\end{align*}$$

We focus on the Kerr black hole with no charge since the generalization to with nonzero charge is almost straightforward.

The uncharged Kerr black hole is parametrized by $J$ (or $a$) and $M$. The function $\Delta$ (3.28) is a quadratic polynomial in $r$, and the solutions to $\Delta = 0$ are

$$r_\pm = M \pm \sqrt{M^2 - a^2}.$$  \hspace{1cm} (3.30)

The metric (3.26) in $a \to 0$ and the knowledge of the Schwarzschild black hole imply that $r_\pm$ might be related to the event horizon. In fact, we find that $r = r_+$ for $a \leq M$ represents the event horizon. In the case of $a \geq M$, there exist no event horizon but a naked singularity forbidden by the cosmic censorship. For this reason, the parameter space is restricted as $a \leq M$. When $a = M$, the black hole is called the extreme Kerr black hole. Henceforth, we consider only in $a \leq M$.

There exist coordinates analogous to the Eddington-Finkelstein coordinates in the Schwarzschild black hole. We can remove the pseudo singularity in the coordinates and compute the surface gravity as

$$\kappa_\xi = \frac{\sqrt{M^2 - a^2}}{r_+^2 + a^2}.$$  \hspace{1cm} (3.31)

From this quantity, the Hawking temperature is calculated.
4 Fluid Mechanics

In this section, we review the fluid mechanics [30] briefly. Although various fluids have been considered in the literature, we focus on non-relativistic incompressible fluids described by the NS equation and some relativistic fluids.

It is believed that the NS equation, which is a set of nonlinear partial differential equations, describes realistic fluids like flows of water in our real world. Hence the NS equation is not only theoretically interesting but have the value of its engineering application. From a mathematical point of view, however, it is not clear whether the NS equations have the unique solution when a well-behaved initial condition is given. This problem is so-called the existence and smoothness problem of the NS equations, which is one of the millennium problems by the Clay Mathematics Institute [15].

On the other hand, if fluid velocity becomes much large, we should consider relativistic effects. Fluids including the relativistic effects are the relativistic fluids. We will encounter them in the context of fluid/gravity correspondence.

4.1 Navier-Stokes Equation

4.1.1 Basics

The NS equation for an incompressible fluid in $\mathbb{R}^d$ is given by

\begin{align}
\partial_t v_i + v_j \partial_j v_i + \partial_i P - \nu \partial^2 v_i &= f_i, \\
\partial_i v_i &= 0.
\end{align}

(4.1) (4.2)

Here, $v^i$ denotes a velocity of the fluid, $P$ the pressure, $f_i$ a given external force and a positive constant $\nu$ viscosity. All functions $v_i, P, f_i$ depend on time $t$ and space coordinates $x^i$. The indices $i, j$ refer to $d$-dimensional spatial indices. The second equation (4.2) represents the incompressibility condition. The equations (4.1) and (4.2) are a set of nonlinear partial differential equations. It causes difficulty of solving the NS equation analytically.

If we drop a viscosity term $-\nu \partial^2 v_i$ from equation (4.1), the equation reduces to the Euler equation

\begin{equation}
\partial_t v_i + v_j \partial_j v_i + \partial_i P = f_i.
\end{equation}

(4.3)

Equation (4.3) describes perfect fluids while the NS equation does viscous fluids or the so-called Newtonian fluids.
4.1.2 Existence and Smoothness of Solution

One of important problems on the NS equation is existence and smoothness of the solution. This problem is famous and summarized as the millennium problem [15]. The situation is very different between in (2 + 1)- and (3 + 1)-dimensional spacetime. In fact, the problem in (2 + 1)-dimensional spacetime was already solved [39]. We show below the statement of the millennium problem.

Consider the NS equation for incompressible fluids (4.1) and (4.2) with initial conditions

\[ v(x^i, t = 0) = v_{\text{init}}(x^i). \]  

Here, \( v_{\text{init}} \) is a given \( C^\infty \) vector field satisfying (4.2). We assume that

\[ |\partial^n v_{\text{init}}| \leq \frac{C_{\alpha K}}{(1 + |x|)^K} \text{ on } \mathbb{R}^d \]  

and

\[ |\partial^n \partial^m f(x, t)| \leq \frac{C_{\alpha m K}}{(1 + |x| + t)^K} \text{ on } \mathbb{R}^d \times [0, \infty), \]  

where \( C_{\alpha K}, C_{\alpha m K}, \alpha, m \) and \( K > 0 \) are any constants. This conditions mean essentially that the velocity of the fluids and an external force must be close to zero at infinity. Moreover, we would like to require that \( v \) and \( P \) be smooth

\[ p, v \in C^\infty \left( \mathbb{R}^d \times [0, \infty) \right) \]  

and the kinetic energy of the fluid velocity be bounded

\[ \int_{\mathbb{R}^d} |v(x, t)|^2 dx < E \text{ for all } t \geq 0, \]  

in order for the solutions to (4.1) and (4.2) to be physically reasonable.

The statement of the existence and smoothness problem is as follows. Take \( \nu > 0, d = 3 \) and \( f_i(x, t) = 0 \). Let \( v_{\text{init}} \) be any smooth vector field satisfying (4.2) and (4.5). Then there exist smooth functions \( v_i(x, t) \) and \( P(x, t) \) on \( \mathbb{R}^3 \times [0, \infty) \) which satisfy (4.6), (4.7) and (4.8). Conversely, for the breakdown of the NS equation, we should proof that no smooth solutions \( v \) and \( P \) with a nonzero external force \( f_i(x, t) \).

In order to avoid the treatment of the behavior of the velocity at infinity, we may impose periodic boundary conditions

\[ v_{\text{init}}(x + e_j) = v_{\text{init}}, \quad f(x + e_j, t) = f(x, t) \text{ for } 1 \leq j \leq 3 \]  

for 1 \leq j \leq 3
and
\[ v(x, t) = v(x + e_j, t) \quad \text{on } \mathbb{R}^3 \times [0, \infty) \quad \text{for } 1 \leq j \leq 3, \quad (4.10) \]
where \( e_j \) is a unit vector. The topology of the spacetime is changed, and in this case, the NS equation is considered on a torus \( T^3 \).

### 4.1.3 Generalization onto Curved Spacetimes

We have considered so far the NS equation in flat space with some topologies \( \mathbb{R}^3 \) and \( T^3 \). It could be generalized to some curved spacetimes or other topologies such as a \( d \)-dimensional sphere \( S^d \). In that case, the partial derivatives should be replaced by the covariant derivatives. Moreover, because the Riemann or Ricci tensor is nonzero in general, some terms associated with them can appear in the NS equation. Such an equation
\[ \nabla_i p + \partial_t v_i + v_j \nabla_j v_i - \nu (\nabla^2 v_i + R_{ij} v^j) = f_i \quad (4.11) \]
is suggested in [5] and [8].

### 4.2 Relativistic Fluids

The NS equations are only valid for small fluid velocity at low energy scale. In order to describe fluids in high energy regime, we should treat them relativistically. We begin with the divergence of the Einstein equation (2.10)
\[ \nabla^\mu T_{\mu \nu} = 0. \quad (4.12) \]

Determination of the form of \( T_{\mu \nu} \) depends on what kinds of fluids are under consideration. Fluid is not an elementary particle but a set of particles. We wish to describe such a macroscopic object with effective long-range, infrared theories. Dynamical variables or some coefficients in the theory are consequences from the unrevealed complex microscopic properties. Here we need to assume the form of \( T_{\mu \nu} \) phenomenologically.

The simplest candidate for \( T_{\mu \nu} \) is the perfect fluid for which we take \( T_{\mu \nu} \) as
\[ T_{\mu \nu} = e v_\mu v_\nu + P h_{\mu \nu} \]
\[ = (e + P) v_\mu v_\nu + P \eta_{\mu \nu}, \quad (4.13) \]
where \( h_{\mu \nu} = \eta_{\mu \nu} + v_\mu v_\nu, \eta_{\mu \nu} = \text{diag}(-1, 1, 1, 1), v_\mu \) is a velocity of the fluid, \( P \) is a pressure and \( e \) is an energy density.
If we wish to study some viscous fluids, we should add viscosity terms to (4.13) and take $T_{\mu\nu}$ as

$$T_{\mu\nu} = (e + P) v_\mu v_\nu + P \eta_{\mu\nu} - 2\eta \sigma_{\mu\nu} - \xi \theta h_{\mu\nu}$$  

(4.14)

where $\eta$ and $\xi$ are viscosity coefficients,

$$\sigma_{\mu\nu} = h^{\mu\alpha} h^{\nu\beta} \left( \nabla_\alpha v_\beta + \nabla_\beta v_\alpha - \eta_{\alpha\beta} \nabla^\lambda v_\lambda \right)$$  

(4.15)

is called a shear tensor and

$$\theta = \nabla^\alpha v_\alpha$$  

(4.16)

an expansion.

Further corrections can be considered in a similar manner [40]. $T_{\mu\nu}$ is written in the derivative expansion

$$T_{\mu\nu} = T^{(0)}_{\mu\nu}(v, T) + T^{(1)}_{\mu\nu}(\partial v, T) + \cdots ,$$  

(4.17)

where $T$ is the temperature. In equation (4.17), all terms satisfying the tensor structure and keeping the symmetries of the theory should be picked out. For example, for a conformal fluid the condition

$$T^{\mu}_{\mu} = 0$$  

(4.18)

will be satisfied due to the invariance under the scale transformation.
Part II
Review

5 Fluid/Gravity Correspondence

The fluid/gravity correspondence [41] [42] [43] [44] based on the gauge/gravity correspondence [45] gives hydrodynamic descriptions of gauge theories from the dual gravity. According to the gauge/gravity correspondence, $(d + 1)$-dimensional gravity corresponds to the $d$-dimensional gauge theory. This correspondence is the bulk/boundary duality and the fluid/gravity correspondence is also originally. The dual fluid lives on the boundary of the spacetime and its nature is determined by the bulk gravity theory. Further, the fluid/gravity correspondence is expected to be valid in the whole spacetime and the dual fluid would live on the boundary but also on the horizon.

Fluid mechanics is an effective theory which is valid in the long wavelength regime near thermal equilibrium. The fluid variables such as the fluid velocity and temperature should vary slowly compared to the microscopic scale. This picture would be taken over to the dual gravity. On the dual gravitational side, we consider a solution corresponding to the thermal state and add perturbations around the solution. This procedure will be done in the derivative expansion, which is justified if the fluid description is allowed.

The fluid/gravity correspondence was first considered for the Type IIB string theory in the $AdS_5 \times S^5$ spacetime as well as the original work of the gauge/gravity correspondence by J. M. Maldacena [45] and it gives the relativistic conformal fluid describing the $\mathcal{N} = 4$ super Yang-Mills theory. The Brown-York tensor [46] on the boundary of the spacetime is calculated from the solution to the Einstein equation in bulk gravity [47]. The conservation law with the Brown-York tensor on the boundary is regarded as the dynamical equations of the dual relativistic fluid.

The procedure has been generalized to some asymptotically flat spacetimes such as the Rindler spacetime [6] [7] and Schwarzschild black hole [8]. In this case, the dual fluids are non-relativistic and live on the horizon instead of the boundary of spacetimes.
5.1 Fluid/Gravity Correspondence in AdS Spacetime

The metric of the near extremal black brane, which solves to the equation of motion of Type IIB string theory, in the large black hole limit is

$$ds^2 = \frac{r^2}{L^2} \left(-f(br) dt^2 + dx_i dx^i + \frac{L^2}{r^2} f^{-1}(br) dr^2 + L^2 d\Omega_5^2\right), \quad f(r) = 1 - \frac{1}{r^4}. \quad (5.1)$$

By omitting $S^5$ and boosting this metric, we obtain

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu, \quad (5.2)$$

where $b$ is a constant inverse temperature and $u^\mu$ is a constant normalized vector

$$u_\mu u_\nu g^{\mu\nu} = -1, \quad (5.3)$$

which represents the boost. Then the solution is parameterized by four parameters, the inverse temperature $b$ and the fluid velocity $u_\mu$. In order to describe dynamics around the solution, we replace the constant parameters $b$ and $u_\mu$ by slowly varying functions $b(x^\mu)$ and $u_\mu(x^\mu)$ as

$$ds^2 = -2u_\mu(x) dx^\mu dr - r^2 f(b(x)r)u_\mu(x)u_\nu(x) dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu. \quad (5.4)$$

After adding fluctuations to the brane solution in this way, the metric is no longer the solution to the Einstein equation. With a formal parameter $\epsilon$ counting the number of derivatives, the metric is written as

$$g_{MN} = g_{MN}^{(0)} + \epsilon g_{MN}^{(1)} + \epsilon^2 g_{MN}^{(2)} + \cdots, \quad (5.5)$$

and similarly,

$$b(x^\mu) = b^{(0)}(x^\mu) + \epsilon b^{(1)}(x^\mu) + \cdots,$$

$$u_\mu(x^\mu) = u_\mu^{(0)}(x^\mu) + \epsilon u_\mu^{(1)}(x^\mu) + \cdots. \quad (5.6)$$

Then we can solve the Einstein equations order by order in $\epsilon$. The expansion in $\epsilon$ here corresponds to the derivative expansion introduced in section 4.2. After the above procedure done, we obtain the solution up to the second order of $\epsilon$, whose explicit form is given in [43] or [48].

To relate with the physics on the boundary, we need to consider the Einstein-Hilbert action with some boundary terms

$$S = S_{EH} - \frac{1}{8\pi G} \int_{\partial M} d^d x \sqrt{-\gamma} K + \frac{1}{8\pi G} S_{ct} \quad (5.7)$$
and calculate the Brown-York tensor $T_{\mu\nu}$ [46] [47]

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{\mu\nu}}. \quad (5.8)$$

The Brown-York tensor (5.8) is written in terms of extrinsic curvature $K_{\mu\nu}$ of the boundary surface

$$T^{\mu\nu} = \frac{1}{8\pi G} \left( K^{\mu\nu} - K \gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{ct}}}{\delta \gamma_{\mu\nu}} \right). \quad (5.9)$$

So what we need to do is to calculate the induced metric $\gamma_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$. Then we regard the conservation law

$$\nabla_\mu T^{\mu\nu} = 0 \quad (5.10)$$

as the equation of motion on the boundary. Substituting the solution with fluctuations into (5.9), we find that equation (5.10) represents a conformal relativistic fluid

$$T^{\mu\nu} = \frac{1}{b^4} (\eta^{\mu\nu} + 4u^{\mu}u^{\nu}) - \frac{2}{b^3} \omega^{\mu\nu} + \cdots \quad (5.11)$$

expected to describe the $\mathcal{N} = 4$ Super Yang-Mills theory.

### 5.2 Fluid/Gravity Correspondence in Asymptotically Flat Spacetime

This section gives some ingredients for deriving the generalized NS (GNS) equation from Einstein gravity. We review seminal works [6] [7] [8] [10] by A. Strominger et al. which is deeply related to our original work. We will restrict discussions to asymptotically-flat spacetimes and start with some exact solutions.

An idea is to introduce a cutoff surface $\Sigma_c$ at some fixed radius $r = r_c$ in an asymptotically flat spacetime. We then impose some boundary condition for gravitational perturbations on $\Sigma_c$ and look for solutions which are nonsingular at the horizon. Einstein equation has a solution space which is much larger than that of the GNS equation. By choosing suitable boundary conditions on $\Sigma_c$ it is possible to make some degrees of freedom remain which describe fluids on the horizon. In [6], [7] and [10], they derive the NS equation in a Rindler spacetime background. Moreover, the framework is extended to the Schwarzschild black hole in [8].
5.2.1 Fluid/Gravity Correspondence in Rindler Spacetime

We review how the NS equation is connected to the Einstein equation in [7]. The authors consider the metric perturbed around a Rindler spacetime and find a solution to the vacuum Einstein equation which is parametrized by fluid variables following the incompressible NS equation. To obtain the result, we need to expand the metric in the non-relativistic and long wavelength limit.

The non-relativistic limit is given by the scale transformation

\[ v_i \to \epsilon v_i, \quad P \to \epsilon^2 P, \quad \partial_i \to \epsilon \partial_i, \quad \partial_r \to \epsilon^2 \partial_r \quad (5.12) \]

with a small parameter \( \epsilon \), which reduces equation (4.12) describing relativistic fluids to the non-relativistic ones. Furthermore, this scale transformation suppresses the corrections to the NS equation. For instance, consider the NS equation with additional correction terms

\[ \partial_r v_i - \nu \partial^2 v_i + \partial_i P + v^j \partial_j v_i + v^k v^j \partial_j \partial_k v_i + \partial^2_r v_i = 0. \quad (5.13) \]

After scaling (5.12), we obtain

\[ \partial_r v_i - \nu \partial^2 v_i + \partial_i P + v^j \partial_j v_i + \epsilon^2 \left( v^k v^j \partial_j \partial_k v_i + \partial^2_r v_i \right) = 0. \quad (5.14) \]

In the limit \( \epsilon \to 0 \), the last two terms are suppressed in \( \mathcal{O}(\epsilon^2) \). From the above discussion, taking

\[ v_i \sim \mathcal{O}(\epsilon), \quad P \sim \mathcal{O}(\epsilon^2), \quad \partial_i \sim \mathcal{O}(\epsilon), \quad \partial_r \sim \mathcal{O}(\epsilon^2). \quad (5.15) \]

and expanding in \( \epsilon \) is also identical to the long wavelength expansion counting the number of derivatives.

Consider the \((d + 2)\)-dimensional Minkowski spacetime in the ingoing Rindler coordinates

\[ ds^2_{d+2} = -r d\tau^2 + 2d\tau dr + dx_1 dx^i. \quad (5.16) \]

We add perturbations to the background metric (5.16) with a boundary condition that the metric should be flat on a time-like surface \( \Sigma_\epsilon : r = r_\epsilon \) as

\[ ds^2|_{\Sigma_\epsilon} = \gamma_{ab} dx^a dx^b = -r_\epsilon d\tau^2 + dx_1 dx^i. \quad (5.17) \]
Introduce the metric with perturbations
\[
\begin{align*}
&ds^2 = -r d\tau^2 + 2d\tau dr + dx_i dx^i \\
&- 2 \left( 1 - \frac{r}{r_c} \right) v_i dx^i d\tau - 2 \frac{v_i}{r_c} dx^i dr \\
&+ \left( 1 - \frac{r}{r_c} \right) \left[ (v^2 + 2P) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j \right] + \left( \frac{v^2}{r_c} + \frac{2P}{r_c} \right) d\tau dr \\
&- \frac{r_c^2}{r_c} \partial^2 v_i dx^i d\tau + \cdots, 
\end{align*}
\]
where the functions \( v_i = v_i(\tau, x^i) \) and \( P = P(\tau, x^i) \) will be identified as the fluid velocity and the pressure, respectively. The metric (5.18) is certainly expanded in \( \epsilon \).

Now we calculate the vacuum Einstein equations of the metric (5.18)
\[
R_{\mu\nu} = 0, 
\]
order by order with an expansion parameter \( \epsilon \). That is, we single out terms which are same order of \( \epsilon \) by the relation (5.15). This operation enables us to find the incompressibility condition at order \( \epsilon^2 \)
\[
\partial_t v^i = 0, 
\]
and the NS equation at order \( \epsilon^3 \)
\[
\partial_\tau v_i + v^j \partial_j v_i + \partial_t P - \nu \partial^2 v_i = 0. 
\]

We omitted higher-order terms as \( \cdots \) in the metric (5.18). We expect that if the choice of the higher-order terms is correct, then components of the Ricci tensor \( R_{\mu\nu} \) must vanish except for equations (5.20) and (5.21). In fact, we can make the Einstein equations (5.19) valid up to \( \mathcal{O}(\epsilon^4) \).

We have investigated the system in the limit \( \epsilon \to 0 \) so far. On the other hand, there exists an alternative expression with another limit. This is the near-horizon limit. We show that the long wavelength limit and near-horizon one are equivalent mathematically for the Rindler spacetime. For this purpose, consider following transformations from \((r, \tau, x^i)\) to \((\hat{r}, \hat{\tau}, \hat{x}^i)\)
\[
x^i = \frac{r_c \hat{x}^i}{\epsilon}, \quad \tau = \frac{r_c \hat{\tau}}{\epsilon^2}, \quad r = r_c \hat{r}. 
\]
In the new coordinates $(\hat{\tau}, \hat{x}^i)$, the metric (5.18) is written as

$$ds_{d+2}^2 = -\frac{\hat{r}r_c^3}{\epsilon^4} d\hat{\tau}^2 + \frac{2r_c^2}{\epsilon^2} d\hat{\tau} d\hat{r} + \frac{r_c^2}{\epsilon^2} d\hat{x}_i d\hat{x}^i$$

$$- 2r_c^2 \frac{1 - \hat{r}}{\epsilon^2} \hat{v}_i d\hat{x}^i d\hat{\tau} - 2r_c \hat{v}_i d\hat{x}^i d\hat{\tau}$$

$$+ (1 - \hat{r}) \left[ \frac{r_c^2}{\epsilon^2} \hat{v}^2 + \frac{2\hat{P}}{\epsilon} d\hat{\tau}^2 + r_c \hat{v}_i \hat{v}_j d\hat{x}^i d\hat{x}^j \right] + r_c \left( \hat{v}^2 + 2\hat{P} \right) d\hat{\tau} d\hat{r}$$

$$- (\hat{r}^2 - 1) r_c \hat{\dot{v}}^2 \hat{v}_i d\hat{x}^i d\hat{\tau} + \cdots. \quad (5.23)$$

The rescaled metric

$$ds_{d+2}^2 = \left( \frac{\epsilon}{r_c} \right)^2 ds_{d+2}^2 \quad (5.24)$$

does also give the same equation of motion as before rescaling. Introducing a new expansion parameter

$$\lambda = \frac{\epsilon^2}{r_c}, \quad (5.25)$$

we obtain

$$ds_{d+2}^2 = -\frac{\hat{r}}{\lambda} d\hat{\tau}^2$$

$$+ \left( 2d\hat{\tau} d\hat{r} + 2(1 - \hat{r}) \hat{v}_i d\hat{x}^i d\hat{\tau} + (1 - \hat{r})(\hat{v}^2 + 2\hat{P}) d\hat{\tau}^2 \right)$$

$$+ (1 - \hat{r}) \hat{v}_i \hat{v}_j d\hat{x}^i d\hat{x}^j - 2\hat{v}_i d\hat{x}^i d\hat{\tau} + (\hat{v}^2 + 2\hat{P}) d\hat{\tau} d\hat{r}$$

$$+ (1 - \hat{r}^2) \hat{\dot{v}}^2 \hat{v}_i d\hat{x}^i d\hat{\tau} + \cdots. \quad (5.26)$$

On the other hand, consider the near-horizon limit in the metric (5.18). By changing coordinates $(r, \tau, x^i)$ to $(\hat{r}, \hat{\tau}, x^i)$ as

$$r = r_c \hat{r}, \quad \tau = \frac{\hat{\tau}}{r_c} \quad (5.27)$$

with $r_c \to 0$, we obtain the metric of the background in new coordinates

$$ds_{d+2}^2 = -\frac{\hat{r}}{r_c} d\hat{\tau}^2 + 2d\hat{\tau} d\hat{r} + dx_i dx^i. \quad (5.28)$$
By adding perturbations to the metric (5.28) we obtain
\[ ds_{d+2}^2 = -\frac{\dot{r}}{r_c} d\tilde{r}^2 + 2d\tilde{r}d\tilde{r} + dx_i dx^i - 2(1 - \dot{r})v_i dx^i d\tilde{r} + (1 - \dot{r})(v^2 + 2\dot{P})d\tilde{r}^2 + r_c \left( (1 - \dot{r})v_i v_j dx^i dx^j - 2v_i dx^i d\tilde{r} + (v^2 + 2\dot{P})d\tilde{r} d\tilde{r} + (1 - r^2)\partial^2 v_i dx^i d\tilde{r} \right) + \cdots. \] (5.29)

This metric reproduces the Navier-Stokes equation again. In the metric (5.29), if we change the notations as \( v_i \rightarrow \dot{v}_i, \ x^i \rightarrow \dot{x}^i, \ r_c \rightarrow \lambda, \) (5.30) we obtain the metric (5.26). It suggests that we can regard the near-horizon limit as the long wavelength limit.

5.2.2 Fluid/Gravity Correspondence in Schwarzschild Black Hole

Let us summarize the results for the Schwarzschild black hole. Here we specify the dimension of the spacetime to four, though results are obtained for arbitral dimensions in [8]. The Schwarzschild metric in 4-dimensional spacetime is
\[ ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\Omega^2_2, \quad f(r) = 1 - \frac{2m}{r}. \] (5.31)

In the Eddington-Finkelstein coordinates, the metric (5.31) becomes
\[ ds^2 = -f(r) dv^2 + 2dv dr + r^2 d\Omega^2 \] (5.32)

Consider the ansatz metric including perturbations
\[ ds^2 = -\frac{\rho}{\lambda} dt^2 + \frac{\rho(5 + \rho)}{4m^2} dt^2 + 2dt d\rho + 4m^2 d\Omega^2 + (1 - \rho) \left( v^2 dt^2 - 2v_i dtdx^i \right) - 2\rho P dt^2 + \lambda \left( 2(\rho + 4m^2 P) d\Omega^2 + (1 - \rho)v_i v_j dx^i dx^j \right) \]
\[ - (\rho^2 - 1)(\nabla^2 v_i + R_{ij} v^j) dtdx^i - 2v_i dp dx^i + (v^2 + 2P) dtd\rho + 2(1 - \rho)\phi_i dtdx^i \] + \( \mathcal{O}(\lambda^2). \) (5.33)

Here \( R_{ij} \) is Ricci tensor on a round sphere and \( \nabla_i \) is a covariant derivative associated with the metric on a 2-sphere \( S^2. \) \( \lambda \) is an expansion parameter,
and $v_i$ and $P$ is a fluid velocity and pressure, respectively. For this metric, we obtain following Einstein equations

\begin{align}
\mathcal{R}_{\tau\tau} - \nabla^i v_i &= 0 \quad \text{at } \mathcal{O}(\lambda^{-1}), \quad (5.34) \\
\mathcal{R}_{\tau i} &= \partial_i v_i + v^j \nabla_j v_i + \nabla_i P - (\nabla^2 v_i + R_{ij} v^j) = 0 \quad \text{at } \mathcal{O}(\lambda^0). \quad (5.35)
\end{align}

The former one is the incompressibility condition and the latter is the NS equations with unit viscosity on $S^2$. One notable difference from the case of Rindler space (5.16) is that it is not possible to impose a boundary condition such that the metric on $\Sigma_c$ be that of a direct product of time and a round sphere of radius $r_c$, but what can be done at most is that the induced metric on $\Sigma_c$ is conformal to such a fixed metric.

Another difference is the expansion in $\lambda$. In the case of Rindler space, there exist non-relativistic and long wavelength limit. It is shown that the limit is equivalent mathematically to the near-horizon limit. Here, however, we only have the near-horizon limit and we cannot show the equivalence.

### 5.3 Local Event Horizon

In [14], the authors consider the event horizon of the perturbed black brane (5.4). Since non-linear viscous terms (and other higher-order correction terms) included in equation (5.10) cause the dissipative nature, we expect that flow of the fluid will calm down at the time enough later. Hence we assume that $u_\mu(x) = u^{(0)}_\mu$ and $T(x) = T^{(0)}$ in the late time. Under the assumption, we obtain the location of the horizon for the solution dual to the conformal fluid in the derivative expansion.

We assume that the event horizon $H$ is described by the equation

\begin{equation}
S_H(r, x) = r - r_H(x) = 0. \quad (5.36)
\end{equation}

The normal vector $\xi$ of $H$ is also given by

\begin{equation}
\xi^M = g^{MN} \partial_N S_H(r, x). \quad (5.37)
\end{equation}

We require that $H$ be a null surface

\begin{equation}
\xi^2 = G^{MN} \partial_M S_H \partial_N S_H = 0. \quad (5.38)
\end{equation}

From this constraint, we can determine the event horizon order by order in $\epsilon$

\begin{equation}
r_H(x) = \frac{1}{b(x)} + \sum_{n=1}^{\infty} \epsilon^n r^{(n)}(x), \quad (5.39)
\end{equation}

where $r^{(n)}$ denotes the corrections due to the dynamical variables $b(x)$ and $u_\mu(x)$. 

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6 Near Horizon Extreme Kerr

6.1 NHEK

The metric of the Kerr black hole in Boyer-Lindquist coordinates \((\hat{r}, \hat{t}, \hat{\theta}, \hat{\varphi})\) is given by

\[
ds_{BL}^2 = -\frac{\Delta}{\Sigma} \left( d\hat{t} - a \sin^2 \hat{\theta} d\hat{\varphi} \right)^2 + \frac{\sin^2 \hat{\theta}}{\Sigma} \left( (\hat{r}^2 + a^2) d\hat{\varphi} - ad\hat{t} \right)^2 + \frac{\Sigma}{\Delta} d\hat{r}^2 + \Sigma d\hat{\theta}^2
\]

where

\[
\Sigma(\hat{r}, \hat{\theta}) = \hat{r}^2 + a^2 \cos^2 \hat{\theta}, \quad \Delta(\hat{r}) = \hat{r}^2 - 2M\hat{r} + a^2.
\]

This metric has two parameters: the mass of the black hole \(M\) and the angular momentum \(J = aM\). There is an event horizon at \(\hat{r} = r_+\), where \(r_+ = M \pm \sqrt{M^2 - a^2}\) are solutions to \(\Delta(\hat{r}) = 0\). Due to the square root in the above definition for \(r_\pm\), to avoid a naked singularity, \(a\) must satisfy an inequality \(|a| \leq M\). The Hawking temperature, angular velocity of the horizon and entropy are

\[
T_H = \frac{r_+ - r_-}{8\pi Mr_+}, \quad \Omega_H = \frac{a}{2Mr_+}, \quad S_{BH} = 2\pi Mr_+.
\]

We wish to remove the apparent singularity of (6.1) at the horizon, and we change the coordinates to ingoing Kerr coordinates. These are analogous to the Eddington-Finkelstein coordinates for the Schwarzschild black hole, which are appropriate for discussing perturbations around the horizon. The necessary coordinate transformations and the resulting metric are, respectively,

\[
dr = d\hat{r}, \quad dt = d\hat{t} + (\hat{r}^2 + a^2) \frac{d\hat{r}}{\Delta(\hat{r})}, \quad d\theta = d\hat{\theta}, \quad d\varphi = d\hat{\varphi} + a \frac{d\hat{r}}{\Delta(\hat{r})}
\]

and

\[
ds_{EF}^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{2Mr}{\Sigma} a^2 \sin^2 \theta \right) \sin^2 \theta d\varphi^2 + 2dtdr - 2a \sin^2 \theta drd\varphi - \frac{4Mr}{\Sigma} a \sin^2 \theta dt d\varphi.
\]
The extreme Kerr has \( r_+ = r_-, \ T_H = 0 \). In this case the upper bound for \( J \) is saturated: \( J = M^2 \). We can blow up the near-horizon region by the following replacement

\[
    r \rightarrow M + \lambda Mr, \quad t \rightarrow \frac{2M}{\lambda} t, \quad \theta \rightarrow \theta, \quad \phi \rightarrow \phi + \frac{t}{\lambda},
\]

followed by a limit \( \lambda \rightarrow 0 \). The resulting metric is given by

\[
    ds_{\text{NHEK}}^2 = 2J \Gamma(\theta) \left(-r^2 dt^2 + 2dtdr + d\theta^2 + \Lambda(\theta)^2(d\phi + r dt)^2\right),
\]

where

\[
    \Gamma(\theta) = \frac{1 + \cos^2 \theta}{2}, \quad \Lambda(\theta) = \frac{2 \sin \theta}{1 + \cos^2 \theta},
\]

and the associated geometry is the near-horizon extremal Kerr (NHEK) \[11\] \[49\].

\[\text{6.2 near-NHEK}\]

While NHEK has vanishing Hawking temperature, a near-NHEK spacetime was constructed using its generalization with non-vanishing temperature \[11\]. This is achieved by taking the limit \( \lambda \rightarrow 0 \), while keeping the associated temperature

\[
    T_R \equiv \frac{2MT_H}{\lambda}
\]

finite and nonzero. This means that the parameter \( a \) is adjusted according to

\[
    a = \frac{M \sqrt{1 - 4\pi T_R \lambda}}{1 - 2\pi T_R \lambda}
\]

Now we make the following coordinate transformations to the metric (6.7):

\[
    r \rightarrow r_+ + \lambda r_r + r, \quad t \rightarrow \frac{2M}{\lambda} t, \quad \theta \rightarrow \theta, \quad \phi \rightarrow \phi + \frac{t}{\lambda}
\]

instead of the transformation (6.8).

In the limit \( \lambda \rightarrow 0 \), metric (6.7) becomes

\[
    ds_{\text{NHEK}}^2 = \left(\frac{-16M^2 \pi^2 T_R^2 (\Gamma - 1)}{\Gamma} + \frac{M^2 (17 - 19\Gamma + 6\Gamma^2)}{4\Gamma} (4\pi T_R r + r^2)\right) dt^2
\]

\[
    + 4M^2 \Gamma dr dt - \frac{8M^2 (\Gamma - 1)}{\Gamma} (2\pi T_R + r) dtd\phi + 2M^2 \Gamma d\theta^2 - \frac{4M^2 (\Gamma - 1)}{\Gamma} d\phi^2.
\]

This is called a near-NHEK geometry \[11\].
Part III
Original Work

7 GNS Equation Dual to Near-NHEK

In this section, we will derive a generalization of the NS equation by considering perturbation around the near-NHEK background and see how the perturbative solution is found. The parameter of the perturbation is $\lambda$, which is the same as the one introduced in the previous section. Firstly, let us consider the symmetry of the Kerr black hole in section 7.1. In section 7.2, to derive a GNS equation we will introduce an extra scale transformation with $\lambda$. In section 7.3, we will add a perturbation to the metric of the scaled near-NHEK. This perturbed metric is expanded in powers of $\lambda$. Then the Ricci tensor is also expanded in powers of $\lambda$ and calculated from the perturbed metric. We will determine the perturbative part of the metric from the vacuum Einstein equation up to the subleading order in $\lambda$. A perturbative solution can be found and expressed in terms of $v_i$ and $P$. These functions $v_i$ and $P$ obey the incompressibility condition and GNS equation. We will comment on the fact that one of the components of the Einstein equations generally yields an extra constraint on $v_i$ and $P$ in addition to the GNS equation. As we will see in section 7.4, this can, however, be avoided, by a suitable choice of the fluctuation metric. Section 7.5 is for a study of a stationary solution.

7.1 Symmetry of Kerr Black Hole

In the above study symmetry principles are important to understand the final fluid equations. The equation (5.21) is invariant under spacetime translation, rotation, scale transformation and parity. The symmetries which originate from the isometries of the background spacetime restrict the form of the equation. In the background of Schwarzschild black hole, there are also enough symmetries to determine the form of the fluid equation. Actually, Killing vectors of round $S^2$ solve (5.35) and (5.34) [8].

In the case of Kerr black hole, however, the spacetime does not have sufficient symmetries to determine the form of the fluid equation uniquely. Actually, rotation symmetry is reduced to only $\partial_\varphi$. The horizon is not a round sphere, but a deformed, spheroidal surface. This suggests that the equation for the fluid on the spherical horizon also has only a rotation symmetry $\partial_\varphi$. To the best of our knowledge, a form of the NS equation or its
generalization, which is dual to such a curved spacetime without sufficient isometry has not been studied. Hence we cannot predict a priori the exact form of GNS equation solely from the symmetry argument due to the lack of symmetries. In this paper we will show that this fluid equation can be determined uniquely by requiring that the equation can be written in terms of covariant derivatives associated with the metric on the horizon.

Another problem associated with the Kerr spacetime is the fact that it is more complicated than that of Schwarzschild. It is known that the Kerr spacetime can be simplified by considering near-horizon limit of the extremal Kerr or that of near extreme Kerr, and the correspondence with conformal field theories has been studied. In this paper we will treat this limiting case of Kerr spacetime.

7.2 Scale Transformation

We wish to obtain the solution to the Einstein equation around a Kerr black hole, which is dual to the solution to GNS equation. We expect that it will be done in a similar manner to the case of Schwarzschild black hole and then it will be need to consider the near-horizon expansion. More concretely, we assume that the near-horizon limit would be related to the near-NHEK limit.

We will perform an extra rescaling of the coordinates \( r \) and \( t \) (6.13) to (6.7) by

\[
    r = \lambda \rho, \quad t = \frac{\tau}{\lambda}
\]

with the same parameter \( \lambda \) as in the previous section. As \( \lambda \to 0 \), the coordinate frame is infinitely accelerated, and the near-horizon dynamics can be probed. The metric after the transformation (7.1) and expansion in
power series of $\lambda$ is given by
\[
ds^2 = \frac{1}{\lambda^2} 16M^2 \pi^2 T_R^2 (1 - \Gamma) d\tau^2
\]
\[
+ \frac{1}{\lambda} \left( \left( -\frac{32M^2 \pi^2 T_R^2 (1 - 4 \Gamma + 3 \Gamma^2)}{\Gamma^2} - \frac{8M^2 \pi T_R (-2 + 2 \Gamma + \Gamma^2)}{\Gamma} \right) d\tau^2
\]
\[
+ \frac{16M^2 \pi T_R (1 - \Gamma)}{\Gamma} \right) d\tau d\varphi \right)
\]
\[
+ 4M^2 \Gamma d\rho d\tau + 2M^2 \Gamma d\theta^2 + \frac{4M^2 (1 - \Gamma)}{\Gamma} d\varphi^2
\]
\[
+ \left( -\frac{16M^2 \pi^4 T_R^4 (-4 + 24 \Gamma - 49 \Gamma^2 + 29 \Gamma^3)}{\Gamma^3}
\]
\[
- \frac{16M^2 \pi^2 T_R^2 (3 - 10 \Gamma + 6 \Gamma^2 + 2 \Gamma^3)}{\Gamma^2} \rho - \frac{2M^2 (-2 + 2 \Gamma + \Gamma^2)}{\Gamma} \rho^2 \right) d\tau^2
\]
\[
- \left( \frac{16M^2 \pi^2 T_R^2 (2 - 7 \Gamma + 5 \Gamma^2)}{\Gamma^2} + \frac{8M^2 (-1 + \Gamma)}{\Gamma} \right) d\tau d\varphi + O(\lambda^2)
\]
(7.2)

and it turns out that a term with a power $\lambda^{-2}$ appears in front of $d\tau^2$. This $\lambda^{-2}$ order term can be removed by means of a shift in $\varphi$.

\[
\varphi \rightarrow \varphi - \frac{2\pi T_R}{\lambda} \tau
\]
(7.3)

After this shift we obtain
\[
ds^2 = -\frac{8M^2 \pi T_R \Gamma \rho}{\lambda} d\tau^2 + 4M^2 \Gamma d\tau d\rho
\]
\[
+ \left( \frac{8M^2 \pi^4 T_R^4 \sin^2 \theta}{\Gamma^2} - \frac{16M^2 \pi^2 T_R^2 \cos^2 \theta}{\Gamma^2} - \frac{M^2 (3 + 28 \cos 2\theta + \cos 4\theta)}{16\Gamma} \rho^2 \right) d\tau^2
\]
\[
+ \left( \frac{8M^2 \pi^2 T_R^2 \sin^2 \theta}{\Gamma^2} + \frac{4M^2 \sin^2 \theta}{\Gamma} \right) d\tau d\varphi + 2M^2 \Gamma d\theta^2 + \frac{2M^2 \sin^2 \theta}{\Gamma} d\varphi^2
\]
\[
+ \lambda \left( \frac{16M^2 \pi^2 T_R^2 (2 + \cos 2\theta) \sin^2 \theta}{\Gamma^2} - \frac{32M^2 \pi^3 T_R^3 \cos^4 \theta}{\Gamma^2} \rho \right)
\]
\[
+ \frac{M^2 \pi T_R (23 + 12 \cos 2\theta - 3 \cos 4\theta)}{8\Gamma^2} \rho^2 \right) d\tau^2 + 8M^2 \pi T_R d\tau d\rho
\]
\[
+ \left( \frac{16M^2 \pi^3 T_R^3 (5 + 3 \cos 2\theta) \sin^2 \theta}{\Gamma^2} + \frac{M^2 \pi T_R (35 + 28 \cos 2\theta + \cos 4\theta) \sin^2 \theta}{4\Gamma^2} \rho \right) d\tau d\varphi
\]
\[
+ 4M^2 \pi T_R d\theta^2 + \frac{M^2 \pi T_R \sin^2 2\theta}{\Gamma^2} d\varphi^2 \right) + O(\lambda^2).
\]
(7.4)
We also present some components of the inverse metric up to $O(\lambda^1)$.

$$g^{\rho\rho} = \frac{12\pi T_R \rho}{\lambda M^4 \Gamma} + \frac{\rho(-8\pi^2 T_R^2 + \Gamma \rho)}{2M^2 \Gamma^2} + O(\lambda^1), \quad g^{\rho\sigma} = \frac{1}{2M^2 \Gamma} + O(\lambda^1)$$

$$g^{\rho\varphi} = -\frac{2\pi^2 T_R^2 + \rho}{2M^2 \Gamma} + O(\lambda^1), \quad g^{\theta\theta} = \frac{1}{2M^2 \Gamma} + O(\lambda^1), \quad g^{\varphi\varphi} = \frac{-1 + 2 \csc^2 \theta}{4M^2} + O(\lambda^1).$$

The leading order of $g^{\rho\rho}$ is also $\lambda^{-1}$ like that of $g_{\tau\tau}$.

We define $\Sigma_c$ as a hypersurface located at $\rho = 1$. By taking a limit $\lambda \to 0$, $\Sigma_c$ approaches the event horizon $r = r_+$, which is a null surface. This is a distorted 2-sphere $S^2$, whose metric $\gamma_{ij}$ is given in (A.1). We attempted to impose a boundary condition on $\Sigma_c$ such that the induced metric on $\Sigma_c$ is (possibly conformal to) that of a semidirect sum of time and the distorted sphere. It turns out, however, that this is not possible.

### 7.3 Derivation of the GNS Equation

We consider perturbation around the Kerr background, and we wish to find a fluid equation that is an extension of the NS equation with possibly some appropriate extra terms. The metric with perturbations will be denoted as

$$ds^2 = ds^2 + g^{(\text{pert.})}_{\mu\nu} dx^\mu dx^\nu,$$

where the perturbation part of the metric is expanded in a power series of $\lambda$ and $\rho$:

$$g^{(\text{pert.})}_{\mu\nu} = \sum_{n=0}^{\infty} g^{(n)}_{\mu\nu} (\rho, \tau, \theta, \varphi) \lambda^n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g^{(n,m)}_{\mu\nu} (\tau, \theta, \varphi) \lambda^n \rho^m. \quad (7.7)$$

The coefficient functions $g^{(n)}_{\mu\nu} (\rho, \tau, \theta, \varphi)$ and $g^{(n,m)}_{\mu\nu} (\tau, \theta, \varphi)$ must be expressed in terms of the velocity $v^i$ and the pressure $P$ such that if $v^i$ and $P$ satisfy a generalization of the NS equation, the vacuum Einstein equation $\mathcal{R}_{\mu\nu} = 0$ is also satisfied. In other words, the solution to the vacuum Einstein equation is parametrized by $v$ and $P$. Each component of the Ricci tensor is also expanded in a power series of $\lambda$ and $\rho$ in the same fashion as that of the metric

$$\mathcal{R}_{\mu\nu} = \sum_n \mathcal{R}^{(n)}_{\mu\nu} \lambda^n = \sum_n \sum_m \mathcal{R}^{(n,m)}_{\mu\nu} \lambda^n \rho^m. \quad (7.8)$$

\[^4\text{Here and henceforth, Greek indices are reserved for four-dimensional spacetime } (\rho, \tau, \theta, \varphi) \text{ and Latin ones for the distorted 2-sphere } (\theta, \varphi). \text{ } \mathcal{R}_{\mu\nu} \text{ represents the Ricci tensor in four dimensions, and } \mathcal{R}_{ij} \text{ that in a two-dimensional sphere.}\]
$\mathcal{R}^{(n)}_{\mu\nu}$ refers to the Ricci tensor at order $\lambda^n$ and $\mathcal{R}^{(n,m)}_{\mu\nu}$ at order $\lambda^n \rho^m$. We found that the integer $n$ in the summation of (7.8) should start from $-1$. All $\mathcal{R}^{(n,m)}_{\mu\nu}$ should vanish if the metric is to be a solution to the Einstein equation. We analyzed equations both at the leading order in $\lambda$ ($n = -1$) and the subleading ($n = 0$) one. According to [8], we expect that $\mathcal{R}^{(-1,0)}_{\tau\tau} = 0$ will be identified as the incompressibility condition, and $\mathcal{R}^{(0,0)}_{\tau i} = 0$ as a GNS equation.

To relate $g^{(n)}_{\mu\nu}$ to fluid variables $v^i$ and $P$ such that if $v^i$ and $P$ satisfy fluid equations, then $g^{(n)}_{\mu\nu}$ satisfy Einstein equations, we will impose the following requirements.

1. We choose a gauge $g_{\rho\rho} = 0$ not only for the background field but also for the full gravitational field including perturbations.

2. The leading order of the perturbations $g^{(n)}_{\mu\nu}$ is $n = 0$. This is expected because the leading behavior of $g_{\tau\tau}$ is $O(\lambda^{-1})$ after the shift (7.3).

3. The leading and subleading perturbations $g^{(0)}_{\mu\nu}$ and $g^{(1)}_{\mu\nu}$ are cubic polynomials of $\rho$. Actually, it turns out they can even be quadratic polynomials.

4. We require that the metric should be smooth on the boundary $\Sigma_c$, and that the fluid equation be expressed covariantly in terms of covariant derivatives with respect to the metric on the distorted 2-sphere (A.1). It turns out that this requirement is strong enough to restrict the form of the fluid equation. We do not, however, require the induced metric on $\Sigma_c$ to coincide with or be conformal to that of a semidirect sum of time and the distorted sphere as in [8].

5. We do not require that the perturbation part of the metric be written in terms of covariant derivatives associated with the distorted sphere.

Now we will analyze the equations $\mathcal{R}^{(n,m)}_{\mu\nu} = 0$ ($n = -1, 0, m = 0, \ldots, 7$),\footnote{The explicit forms of these equations are too messy to be presented here.} and solve for $g^{(n,m)}_{\mu\nu}$. In most of the equations $\mathcal{R}^{(n,m)}_{\mu\nu} = 0$ are algebraic equations of some components of the metric $g^{(n,m)}_{\mu\nu}$ and can be solved straightforwardly. Among the equations, however, $\mathcal{R}^{(-1,0)}_{\tau\tau} = 0$, $\mathcal{R}^{(0,0)}_{\tau i} = 0$ and $\mathcal{R}^{(0,0)}_{\tau r} = 0$ cannot be solved explicitly. These will be dealt with separately in the following paragraphs and in section 7.4. Our purpose is to express the
metric in terms of the velocity field $v$ and pressure $P$, which will be introduced by analyzing the structures of the equations $R_{\tau\tau}^{(-1,0)} = 0$, $R_{\tau r}^{(0,0)} = 0$ and $R_{\tau r}^{(0,0)} = 0$. The perturbative part of the metric tensor that is obtained and that is dual to the fluid equation is presented in appendix B.

Firstly, we found that by setting

$$g_{\tau i}^{(0,0)} = c_0 v_i(\tau, \theta, \varphi),$$

(7.9)

where $c_0$ is a constant, it is possible to identify $R_{\tau\tau}^{(-1,0)} = 0$ as the incompressibility condition

$$\nabla_i v^i = 0.$$  

(7.10)

Here $\nabla_i$ is covariant derivative associated with the metric on the distorted $S^2$.

Secondly, we also found that $R_{\tau r}^{(0,0)} = 0$ can be identified as a generalization of the NS equation on the distorted $S^2$:

$$R_{\tau r}^{(0,0)} = \frac{2c_2}{c_0} \partial_\tau v_i - 2c_2 v^j \nabla_j v_i + c_1 \nabla_i P - \frac{c_0}{2} (\nabla^2 v_i + R_{ij} v^j) - \frac{4c_2 \pi^2 T_R^2}{c_0} \nabla_\varphi v_i - \frac{8 \cos \theta}{M^2 c_0 (3 + \cos 2\theta)^3} (24c_2 M^2 \pi^2 T_R^2 + 5c_0^2 + (8c_2 M^2 \pi^2 T_R^2 - c_0^2) \cos 2\theta) \epsilon_{ij} v^j = 0,$$

(7.11)

where $P(\tau, \theta, \varphi)$ is a pressure, introduced in $g_{\tau r}^{(1,0)}$ and $g_{\tau r}^{(0,1)}$ as in (7.12), $c_0$, $c_1$, and $c_2$ are constants, and $\epsilon_{ij}$ is a unit antisymmetric tensor defined in (A.3). There are some extra terms in equation (7.11). The term $R_{ij} v^j$ also appeared in the fluid equation (5.35) that is dual to the Schwarzschild black hole [5] [8]. The appearance of $\epsilon_{ij} v^j$ in the last line is permitted since a rotation about an axis breaks parity.\(^7\) The term proportional to $\nabla_\varphi v_i$ in (7.11) is also consistent with the isometry $\partial_\varphi$ of the background. In (7.11) $\cos \theta$ and $\cos 2\theta$ can be re-expressed in terms of scalar curvature $R$ in (A.5). Hence this equation is covariant except for the presence of the term proportional to $\nabla_\varphi v_i$, which is also consistent with the symmetry.

\(^6\)This is a standard prescription adopted in [8], which leads to the GNS equation on $S^2$.

\(^7\)In [50] and [51], a DC thermoelectric conductivity of field theory was considered within the context of AdS/CFT correspondence, and it was shown that this conductivity can be obtained by solving a system of generalized Stokes equations on perturbed black hole horizons. We are informed that the term $v^j d \chi_{ij}^{(0)}$ in the Stokes equation (3.1) of [50] is similar to the last term proportional to $\epsilon_{ij} v^j$ in the middle of (7.11).
The explicit form of the fluid equation (7.11) is obtained by the requirement 4 above. It is found that the equation $\mathcal{R}_{ri}^{(0,0)} = 0$ ($i = \theta, \varphi$) depends only on the metric components $g_{ri}^{(0,0)}$, $g_{ri}^{(0,1)}$, $g_{\mu i}^{(1,0)}$, $g_{\rho r}^{(1,0)}$, and $g_{\tau r}^{(0,1)}$. The equation $\mathcal{R}_{ri}^{(0,0)} = 0$ after substitution of (7.9) into this equation and imposition of the ansatz $^8$ 

\begin{align}
  g_{ri}^{(0,1)} &= \alpha_{ij}^{(1)}(\theta)v^j, \\
  g_{\mu i}^{(1,0)} &= \alpha_{ij}^{(2)}(\theta)v^j, \\
  g_{\rho r}^{(1,0)} &= \frac{M^2\Gamma}{\pi T_R} \left( \beta^{(1)}(\theta)v^2 + \beta^{(2)}(\theta)P \right), \\
  g_{\tau r}^{(0,1)} &= 4M^2\Gamma \left( \beta^{(3)}(\theta)v^2 + \beta^{(4)}(\theta)P \right),
\end{align}

(7.12)

where $\alpha^{(n)}(\theta)$ and $\beta^{(n)}(\theta)$ are unknown functions of $\theta$, turned out to take a complicated structure. In particular, although the GNS equation must be a vector equation, terms such as $v_i v_j$, $v_i \nabla_j v_k$, $v^2$, $P$, $\partial_i P$, $v_i$, $\partial_r v_i$ and $\nabla_i v_j$, which are noncovariant, are present in this equation. To remove extra terms that cannot be combined into a vector quantity, many conditions must be imposed on $\alpha^{(n)}(\theta)$ and $\beta^{(n)}(\theta)$. Some of them are differential equations and solving them yields the integration constants $c_n$ and determines $\alpha^{(n)}(\theta)$, $\beta^{(n)}(\theta)$ except for $\beta^{(2)}(\theta)$ and $\beta^{(3)}(\theta)$. This procedure determines the form (7.11) of the GNS equation. In this way, it turns out the requirement that the GNS equation be covariant (requirement 4.) provides sufficient constraints as strong as the boundary conditions on $\Sigma_c$.\footnote{It is clear from (B.7), the first equation of (7.12), and (7.9) ((B.7)-(B.9)) that we cannot impose a boundary condition that $g_{ri}^{(0)} = 0$ at $\rho = 1$.} We checked that equation (7.11) is the most general form, and no new terms appear.\footnote{If terms that are higher orders in $v_j$ were added to (7.12), then higher-order terms of the fluid equation would appear.}

Finally, we would like to point out that when the coefficients $c_i$ of those terms in (7.11), which are also present in the ordinary NS equation, are fixed,\footnote{This is a generalization of the metric perturbation for the Schwarzschild black hole.}
then all the coefficients of the additional terms in (7.11) are also determined. We cannot freely add or eliminate these additional terms.

7.4 $\mathcal{R}_{\tau\tau}^{(0,0)} = 0$

In the analysis of the Einstein equation up to $\mathcal{O}(\lambda)$ there remains the following equation which must be analysed carefully:

$$\mathcal{R}_{\tau\tau}^{(0,0)} = -2\pi T_R \nabla_i g^{(1,0)}_{\tau\tau} + \mathcal{G} \left( g_{\tau\tau}^{(0,0)}, g_{ij}^{(1,0)}, v_i, P \right) = 0. \quad (7.13)$$

Here a divergence of $g^{(1,0)}_{\tau\tau}$ is defined by

$$\nabla_i g^{(1,0)}_{\tau\tau} = \left( \Gamma - 2 \right) \cot \theta g^{(1,0)}_{\tau\tau} + \frac{1}{2M^2 \Gamma^2} \partial_\theta g^{(1,0)}_{\tau\tau} + \frac{\Gamma}{2M^2 \sin^2 \theta} \partial_\varphi g^{(1,0)}_{\tau\varphi}, \quad (7.14)$$

and $\mathcal{G} \left( g_{\tau\tau}^{(0,0)}, g_{ij}^{(1,0)}, v_i, P \right)$ is a known function, which we, however, avoid writing down explicitly for simplicity. $v_i$ and $P$ appear in arguments of $\mathcal{G}$ since the metric components are already determined as in equations (7.9) and (7.12).

Now, although equation (7.13) might seem to determine the function $g_{\tau\tau}^{(0,0)}$ or $g_{ij}^{(1,0)}$, derivatives of them appear in $\mathcal{G}$ and this fact prevents us from solving the equation for them explicitly. Since we are obtaining a perturbed solution parametrized by $v_i$ and $P$, the components of the metric $g_{\tau\tau}^{(1,0)}, g_{ij}^{(0,0)}$ and $g_{ij}^{(1,0)}$ in equation (7.13) are expressed in terms of $v_i$ and $P$ after solving the Einstein equations. Then equation (7.13) will be interpreted as a constraint between $v_i$ and $P$. Because $v_i$ and $P$ are identified as the velocity and pressure of fluids, respectively, this constraint must be avoided. As we will see, $\mathcal{R}_{\tau\tau}^{(0,0)}$, however, can always be algebraically set to zero by fine-tuning the form of $g_{\tau\tau}^{(1,0)}(v_i, P), g_{\tau\tau}^{(0,0)}(v_i, P)$ and $g_{ij}^{(1,0)}(v_i, P)$. Then the appearance of the constraint is avoided and the equations involving $v_i$ and $P$ turn out to be only (7.10) and (7.11).

First, we will set the following ansatz

$$g_{\tau\tau}^{(0,0)} = 0,$$

$$g_{0\theta}^{(1,0)} = \frac{c_1 M^2 \Gamma}{\pi T_R} P + \frac{c_2}{\pi T_R} v_0^2 - \frac{c_0}{\pi T_R} c_0 \sin^2 \theta - \frac{2c_2 \Gamma^2}{\pi T_R} v_\varphi^2,$$

$$g_{\theta\varphi}^{(1,0)} = 3 \frac{c_2}{\pi T_R} v_0 v_\varphi + \frac{3c_0}{2\pi T_R} \partial_\varphi v_\theta,$$

$$g_{\varphi\varphi}^{(1,0)} = -\frac{c_1 M^2 \sin^2 \theta}{\pi T_R \Gamma} P - \frac{2c_2 \sin^2 \theta}{\pi T_R \Gamma^2} v_0^2 + \frac{c_0}{\pi T_R \Gamma} v_\varphi + \frac{c_2}{\pi T_R} v_\varphi^2. \quad (7.15)$$
Then the incompressibility condition (7.9) and the GNS equation (7.11) force \( G \) to take the form
\[
G \left( \eta^{(0,0)}_{\tau\tau}, \eta^{(1,0)}_{ij}, v_i, P \right) = \nabla^i V_i = \nabla^i V_i = \left( \Gamma - 2 \right) \cot \theta \frac{\partial V}{\partial \theta} + \frac{1}{2M^2 \Gamma} \frac{\partial}{\partial \theta} V + \frac{\Gamma}{2M^2 \sin^2 \theta} \frac{\partial}{\partial \varphi} V = \nabla^i V_i.
\] (7.16)

where
\[
V_\theta = 2c_1 c_0 P \frac{\sin^2 \theta}{8M^2 \Gamma^3} - \frac{c_0^2}{M^2 \Gamma} \frac{2}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \left( \cos \Gamma \right) - \frac{\partial^2}{\partial \varphi^2} + \frac{c_0^2}{4M^2 \sin^2 \theta} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} V - \frac{c_0^2}{4M^2 \sin^2 \theta} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} V \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} V = \nabla^i V_i.
\] (7.17)

\[
V_\varphi = \left( \frac{c_0^2}{4M^2 \Gamma^3} - \frac{32c_2 M^2 \pi^2 T^2 R}{4M^2 \Gamma^3} \right) \sin^2 \theta \frac{\partial^2}{\partial \varphi^2} + \left( \frac{3c_0^2}{2M^2 \Gamma^2} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} + \frac{c_0^2}{M^2 \sin^2 \theta} \frac{\partial}{\partial \varphi} \right) \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi}
\] (7.18)

Now equation (7.13) can be written as
\[
\mathcal{R}^{(0,0)}_{\tau\tau} = \nabla^i \left( V_i - 2\pi T R g^{(1,0)}_{\tau i} \right) = 0.
\] (7.19)

Finally, by choosing
\[
g^{(1,0)}_{\tau i} = \frac{1}{2\pi T R} V_i,
\] (7.20)

\( \mathcal{R}^{(0,0)}_{\tau\tau} = 0 \) is satisfied.

This result on the independence of \( v^i \) and \( P \) is quite different from that in [8] for the study of the dual fluid of the Schwarzschild black hole. In that case, time derivatives of \( v^2 \) and \( P \) appear in the equation that corresponds to \( \mathcal{R}^{(0,0)}_{\tau\tau} = 0 \) and determines \( g^{(1,0)}_{\tau i} = \phi_i \) in [8]), and the authors of [8] had to remove the zero mode of \( \partial_t P + \partial_i v^2 / 2 \) on \( S^2 \) by shifting \( P \) by some integral over \( S^2 \). In (7.19), however, time derivatives of \( v^2 \) and \( P \) do not appear.
This is because we did not require the perturbation part of the metric to be written in terms of covariant derivatives. This leads to a solution that does not require a constraint between $v^i$ and $P$. A similar prescription for the Schwarzschild black hole leads to a new metric that is dual to the GNS fluid: see appendix C.

7.5 Stationary Solution to the GNS Equation on the Distorted 2-sphere

Kerr spacetime has isometry $\varphi \rightarrow \varphi + \text{const}$. Hence we expect a Killing vector $\vec{v}^i = (0, \xi)$, where $\xi$ is a constant, i.e.,

$$
\vec{v}_i = \xi \gamma_{ij} \vec{v}^j = \begin{pmatrix} \xi \gamma_{\theta\theta} \dot{v}^\theta \\ \xi \gamma_{\varphi\varphi} \dot{v}^\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \xi \gamma_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2M^2 \xi \sin^2 \theta}{\Gamma} \end{pmatrix}
$$

(7.21)

to solve the incompressibility condition and the generalized NS equation. Substituting (7.21) into the NS equation (7.11) yields two conditions:

$$
\frac{\xi \left( c_0 (\Gamma - 2) + 4 c_2 M^2 \Gamma \xi \right) \sin 2\theta}{2 \Gamma^3} + c_1 \partial_\varphi P(\tau, \theta, \varphi) = 0,
$$

(7.22)

$$
\partial_\varphi P(\tau, \theta, \varphi) = 0.
$$

(7.23)

The solution for the pressure $P$ is given by

$$
P(\tau, \theta, \varphi) = c_3 + \frac{\xi}{c_1 \Gamma^2} \left( c_0 (1 - \Gamma) - 4 c_2 M^2 \Gamma \xi \right),
$$

(7.24)

where $c_3$ is an integration constant. Choosing $c_3 = \frac{4 c_2 M^2 \xi^2}{c_1}$ reduces (7.24) to

$$
P(\tau, \theta, \varphi) = -\frac{c_2}{c_1} \vec{v}^2 + \frac{c_0}{4 c_1 M^2 \Gamma} \vec{v} \dot{\varphi}.
$$

(7.25)

Hence we have shown the existence of a stationary solution, which is a necessary condition for equation (7.11) to be identified as an equation describing fluids.
8 Singularity of the Solution to the Fluid Equation

In this section, we will consider a fluctuating event horizon $\mathcal{H}$. Its null normal is denoted as $k^\mu$. This normal vector defines a family of null geodesics on $\mathcal{H}$, and, by extending it off $\mathcal{H}$ appropriately, we will obtain an expansion scalar $\theta$ of null geodesics. This represents a logarithmic derivative of local area on $\mathcal{H}$, and, if this is a true event horizon, the second law of black hole thermodynamics\cite{17} asserts that $\theta$ is not negative on the future horizon.

In the case of a nonstationary black hole, there are some proposals for definition of the event horizon\cite{14}. In this paper we define the event horizon of a perturbed black hole around a stationary black hole as a null surface $\mathcal{H}$, which coincides with the event horizon of the unperturbed black hole when the gravitational perturbation is turned off. The focusing theorem implies that if $\theta$ is negative at some point on $\mathcal{H}$, then $\theta$ becomes $-\infty$ within a finite time and the null geodesics meet at caustic points. Hence if $\theta$ is not positive semidefinite, the horizon $\mathcal{H}$ defined above may not be a true event horizon in the light of the area theorem. The purpose of this section, however, is to study the singularity of the solution to the fluid equation, and we will continue to use $\mathcal{H}$ and study $\theta$ associated with $\mathcal{H}$. If $\theta$ is negative at some point, to avoid a singularity of geodesics we need to impose an initial condition on the fluid velocity $v^\nu$ at some moment in the past and make $\theta$ take nonnegative values at that time. Then $\theta$ will be positive or zero at later times and the solution to the NS equation has no singularity in the future.

8.1 Congruence of Null Geodesics

Let the horizon $\mathcal{H}$ be a hypersurface $\Phi(x) = 0$. The normal vector derived from $\Phi$,

$$ k_\mu = \partial_\mu \Phi $$

is null ($k^2 = 0$) on $\mathcal{H}$, but in general not null off $\mathcal{H}$. To compute the expansion scalar $\theta$ associated with a congruence of null geodesics, it is convenient to extend the null normal vector $k^\mu$ off the horizon $\mathcal{H}$\cite{12}. This procedure will be briefly explained. We introduce another normal vector $N_\mu$ which satisfies the following conditions off $\mathcal{H}$:

$$ N^2 = 0, \quad N \cdot k = -1 $$

\footnote{A method different from that below is used in \cite{52} and \cite{53}.}
and define
\[ \hat{k}_\mu = k_\mu + \frac{1}{2} \alpha N_\mu \Phi. \] (8.3)

Here \( \alpha \) is a function defined by
\[ k^2 = \alpha \Phi. \] (8.4)

Now the vector \( \hat{k}_\mu \) is null in the neighborhood of \( \mathcal{H} \) and defines a congruence of null geodesics on and off \( \mathcal{H} \). The vector \( \hat{k}_\mu \) is, however, hypersurface orthogonal only on \( \mathcal{H} \). A projection matrix onto the subspace spanned by \( \hat{k}_\mu \) and \( N_\mu \) is defined by
\[ h^{\mu\nu} = g^{\mu\nu} + \hat{k}^\mu N^\nu + N^\mu \hat{k}^\nu. \] (8.5)

Now the following tensor is introduced:\(^{13}\)
\[ \tilde{B}_{\mu\nu} = h^\lambda_\mu h^\nu_\lambda \nabla_\rho \hat{k}_\rho \] (8.6)
\[ = \frac{1}{2} \theta h_{\mu\nu} + \sigma_{(\mu\nu)} + \omega_{[\mu\nu]} \] (8.7)

It can be shown that although the null vector \( \hat{k}_\mu \) is not hypersurface orthogonal off \( \mathcal{H} \), the rotation tensor \( \omega_{\mu\nu} \) vanishes on \( \mathcal{H} \).

The expansion scalar
\[ \theta = h^{\mu\nu} \tilde{B}_{\mu\nu} \] (8.8)
obeys the Raychaudhuri equation,
\[ \frac{d\theta}{d\mu} = \kappa \theta - \frac{1}{2} \theta^2 - \sigma^{\mu\nu} \sigma_{\mu\nu} + \omega^{\mu\nu} \omega_{\mu\nu} - R_{\mu\nu} \hat{k}^\mu \hat{k}^\nu, \] (8.9)
where \( \mu \) parametrizes the null geodesics, \( \hat{k}^\nu = dx^\nu(\mu)/d\mu \), and \( \kappa(\mu) \) is a function defined by
\[ \hat{k}^\nu \nabla_\nu \hat{k}^\mu = \kappa(\mu) \hat{k}^\mu. \] (8.10)

A special parametrization for which \( \kappa = 0 \) is called an affine parametrization and this is achieved by a change of parametrization \( \mu \to \chi \) (affine parameter) with \( d\chi/d\mu = \exp(\int^\mu \kappa(\mu')d\mu') \). Then \( \theta \) is replaced by \( \bar{\theta} = \theta \exp(-\int^\mu \kappa(\mu')d\mu') \) and \( \bar{\theta} \) obeys (8.9) with \( \kappa = 0 \). If the matter stress-energy tensor satisfies the null energy condition \( (\hat{k}^\mu \hat{k}^\nu T_{\mu\nu} \geq 0) \), then \( R_{\mu\nu} \hat{k}^\mu \hat{k}^\nu \geq \)

\(^{13}\)For review of congruence of geodesics, see [32].
owing to the Einstein equation, and, because \( \omega_{\mu\nu} = 0 \) on \( \mathcal{H} \), \( \tilde{\theta} \) satisfies the focusing theorem on \( \mathcal{H} \):

\[
\frac{d\tilde{\theta}}{d\chi} \leq -\frac{1}{2} \tilde{\theta}^2. \tag{8.11}
\]

Hence, if initially \( \tilde{\theta} = \tilde{\theta}_0 < 0 \) at \( \chi = \chi_0 \), then \( \tilde{\theta} \to -\infty \) within the finite affine parameter \( \chi - \chi_0 \leq 2/|\tilde{\theta}_0| \). This signals the occurrence of a caustic. For the singularity in \( \theta \) to be absent \( \tilde{\theta} \) must be nonnegative. Because \( \theta \) and \( \tilde{\theta} \) are related by a positive multiplicative factor, we will deal with \( \tilde{\theta} \) in what follows.

### 8.2 Sign of expansion scalar \( \theta \)

When the perturbations in near-NHEK are absent, the event horizon is located at \( \rho = 0 \). When the perturbations are switched on, the horizon is deformed [14]:

\[
\Phi(\rho, \tau, \theta, \varphi) \equiv \rho - \lambda F_1(\tau, \theta, \varphi) - \lambda^2 F_2(\tau, \theta, \varphi) - \lambda^3 F_3(\tau, \theta, \varphi) + \cdots = 0, \tag{8.12}
\]

where \( F_a \) are unknown functions to be determined by a null normal condition, whose solution defines a new surface \( \mathcal{H} \). If \( F_a \) are chosen such that \( k \cdot k \) vanishes, then \( \mathcal{H} \) becomes a null surface. Generally, this procedure would yield a set of partial differential equations for \( F_a \) and could not be solved explicitly.

In the present case the condition of a null surface becomes algebraic equations for \( F_1 \) and \( F_2 \), because \( g^{\rho\rho} \) takes a form \( \frac{1}{2} a(\tau, \theta, \varphi) + b(\tau, \theta, \varphi) + \mathcal{O}(\lambda) \), where \( a(\tau, \theta, \varphi) \) and \( b(\tau, \theta, \varphi) \) are known functions. By substituting (8.12) we have \( g^{\rho\rho} = aF_1 + b + \cdots \) and then the equation \( k^2 = g^{\rho\rho} + \mathcal{O}(\lambda^1) = 0 \) is solved to yield

\[
F_1 = -\frac{b}{a} = \frac{1}{16\pi M^4 T_R \Gamma^2} \left[ - \left( g^{(0)}_{\tau\tau} \right)^2 - \Gamma^2 \csc^2 \theta \left( g^{(0)}_{\tau\varphi} \right)^2 
+ 2M^2 \Gamma \left( g^{(0)}_{\tau\tau} - 4\pi^2 T_R^2 g^{(0)}_{\tau\varphi} \right) \right]_{\rho = 0}. \tag{8.13}
\]

We have checked that the induced metric on \( \mathcal{H} \) is degenerate.

---

14 This paper considers only the vacuum solution and the null energy condition is not necessary.

15 This procedure to define an event horizon is in the same spirit as in [14]. This surface, however, may turn out not to be a true event horizon. In the discussion of this section, however, it is not important whether our \( \mathcal{H} \) is a true event horizon or not.

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By using the procedure given in the previous subsection, \( \theta \) is computed to \( \mathcal{O}(\lambda^1) \). After using the Einstein equation, we have \( \theta = \theta^{(0)} + \lambda^1 \theta^{(1)} + \mathcal{O}(\lambda^2) \) with \( \theta^{(0)} = 0 \) owing to the incompressibility condition, and

\[
\theta^{(1)} = \frac{c_0^2}{8 M^6 \pi T_R \Gamma^3} (\partial_\theta v_\theta)^2 + \frac{c_0^2}{16 M^6 \pi T_R \Gamma^4} v_\theta \partial_\theta v_\theta + \frac{c_0^2 \csc^2 \theta}{32 M^6 \pi T_R \Gamma} (\partial_\varphi v_\varphi)^2 \\
+ \frac{c_0^2 \csc^2 \theta}{16 M^6 \pi T_R \Gamma} \partial_\varphi v_\varphi \partial_\theta v_\varphi - \frac{c_0^2 \cot \theta \csc^2 \theta}{8 M^6 \pi T_R \Gamma^2} v_\varphi \partial_\varphi v_\varphi v_\theta + \frac{c_0^2 \csc^2 \theta}{32 M^6 \pi T_R \Gamma} (\partial_\theta v_\varphi)^2 \\
- \frac{c_0^2 \cot \theta \csc^2 \theta}{8 M^6 \pi T_R \Gamma^2} v_\varphi \partial_\theta v_\varphi + \frac{c_0^2 \sin^2 2\theta}{128 M^6 \pi T_R \Gamma^3} (v_\theta)^2 + \frac{c_0^2 \cot^2 \theta \csc^2 \theta}{8 M^6 \pi T_R \Gamma^3} (v_\varphi)^2
\]

(8.14)

Although several terms are positive semidefinite, some are not. It must still be taken into account that in the above result the incompressibility condition (7.9) is not completely solved. In two spatial dimensions, however, the velocity of an incompressible fluid can be expressed in terms of a scalar potential \( f(\tau, \theta, \varphi) \):

\[
v_i = \varepsilon_{ij} \partial_j f.
\]

(8.15)

We checked that even when these equations are substituted into (8.14), the result is still not positive semidefinite. The result (8.14) does not depend on the ansatz (7.12) and the choice of \( c_1 \) and \( c_2 \).

The above result implies that, in order to avoid the singularity of the expansion \( \theta \) in the future, an initial condition on the velocity field \( v_i \) must be imposed at some time \( \tau = \tau_0 \) such that \( \theta^{(1)}(\tau_0, \theta, \varphi) \geq 0 \) for all regions of \( \theta \) and \( \varphi \) as well as the condition of smoothness. Then the velocity field \( v_i \) will remain nonsingular afterwards. If \( \theta^{(1)}(\tau_0, \theta, \varphi) \geq 0 \) is not obeyed, however, caustics will occur on \( \mathcal{H} \) in the future. We also performed similar analysis on the expansion scalar for the Schwarzschild black hole and obtained the same result (see appendix C).

The above result is relevant to the existence and smoothness problem of the incompressible NS equation[15][54][8]. Although this problem is not solved in three spatial dimensions, it was studied in two dimensions (especially for \( \mathbb{R}^2 \) and \( \mathbb{T}^2 \))[55]. Indeed, we can show that the expansion scalar is positive semidefinite in the case of a two-dimensional planar horizon in Rindler space (5.16):\(^{16}\)

\[
\theta(t, x^1, x^2) = 4 (\partial_t \partial_0 f)^2 + (\partial^2_t f - \partial^2_0 f)^2 \geq 0,
\]

\[
v_i(t, x^1, x^2) \equiv \varepsilon_{ij} \partial_j f(t, x^1, x^2) \quad (i, j = 1, 2)
\]

(8.16)

\(^{16}\)In the case of a three-dimensional planar horizon, \( \theta \) is also positive semidefinite. Surely the positive semidefiniteness of \( \theta \) may not be a sufficient condition for existence of a smooth solution to the incompressible NS equation.
However, the solution to this problem in the case of a sphere is not yet known. Study of fluid/gravity correspondence along these lines may shed some light on the solution of this problem for space with topologies other than a plane. The above result is obtained only for $O(\lambda^1)$. It would be interesting to investigate whether higher-order analysis imposes further restrictions on the initial conditions for $v^i$ to avoid caustics.
9 Discussions

In this paper, we derived an equation for a viscous incompressible fluid that is dual to the perturbations of a near-NHEK black hole. This equation contains terms proportional to $\nabla \phi v_i$ and $\epsilon_{ij} v^j$, in addition to $R_{ij} v^j$ which is also present in the fluid equation dual to the perturbation of the Schwarzschild black hole [8]. We called the fluid equation that contains extra terms compared to the ordinary NS equation a generalized NS (GNS) equation. An important point is that such extra terms cannot be introduced arbitrarily. The GNS equation does not depend on the expansion parameter $\lambda$, and we cannot modify the structure of the fluid equation by adding new terms to the metric tensor in a series expansion in $\lambda$. Furthermore, the coefficients of the extra terms are also determined, if those of the ordinary NS equation are fixed.

Although duality between Einstein gravity and fluid theory has been reported for AdS space, Minkowski space in a Rindler patch, and the Schwarzschild black hole, it is nontrivial to identify whether a similar duality holds for other rotating black hole geometries. The structure of this hydrodynamic equation is interesting by itself and deserves further study. In this paper we considered only the near-NHEK geometry. This is simply because in the case of general Kerr geometry, we did not succeed in finding appropriate expressions of the metric perturbation in terms of quantities in fluid dynamics such that, if $v_i$ and $P$ satisfy the viscous NS equations, the Einstein equation is also satisfied. It would be interesting to study whether this near-horizon limit is essential, or whether there exist dual fluids for any angular momentum parameter $a$ ($\leq J$).

In this work we could not make metric perturbations to satisfy some boundary conditions on a cutoff surface $\Sigma_c$ except for smoothness. Nonetheless the condition that the fluid equation can be expressed covariantly in terms of covariant derivatives with respect to the metric of the distorted 2-sphere is sufficient to restrict the form of the fluid equation. It is also possible to derive a GNS equation on the sphere dual to the Schwarzschild black hole with only the requirements of section 7.3 (see appendix C).

In section 8 singularity of the null geodesics on $\mathcal{H}$ (a null surface that agrees with the event horizon when the perturbation of the metric is turned off) in the future is studied. By computing the expansion scalar $\theta$, we found that for some choices of the initial condition on the velocity field, $\theta$ is negative in some regions of the horizon and this causes the null geodesics to focus at caustic points in the future, and in turn, the singularity of the solution to the incompressible NS equations. To avoid such singularities, it
is necessary to set suitable initial conditions on the velocity at some time.

As for future studies, an extension of the present work to the duality of fluids with a higher-dimensional rotating black hole and a study of turbulence based on the new fluid equation may be attempted.
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A Tensors on the distorted 2-sphere

The background metric on the distorted 2-sphere in the near-NHEK limit is

\[
\gamma_{ij} = \begin{pmatrix} 2M^2\Gamma(\theta) & 0 \\ 0 & 2M^2\sin^2\theta \end{pmatrix} \frac{1}{\Gamma(\theta)}
\]  \hspace{1cm} (A.1)

where

\[
\Gamma(\theta) \equiv \frac{1 + \cos^2\theta}{2} = \frac{3 + \cos 2\theta}{4}.
\]  \hspace{1cm} (A.2)

A covariant derivative associated with \(\gamma_{ij}\) are denoted as \(\nabla_i\). The epsilon tensor and the Ricci tensor are

\[
\epsilon_{ij} = \sqrt{\det \gamma_{ij}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2M^2\sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]  \hspace{1cm} (A.3)

\[
R_{ij} = -\frac{8(1 + 3\cos 2\theta)}{M^2(3 + \cos 2\theta)^3} \gamma_{ij} = -\frac{1 + 3\cos 2\theta}{8M^2\Gamma^3} \gamma_{ij}.
\]  \hspace{1cm} (A.4)

The Ricci scalar and the derivative are

\[
R = -\frac{1 + 3\cos 2\theta}{4M^2\Gamma^3}, \hspace{0.5cm} \partial_\theta R = \frac{3\cos \theta \sin^3 \theta}{M^2\Gamma^4}.
\]  \hspace{1cm} (A.5)

The nonzero Christoffel symbols are

\[
\Gamma^\theta_{\theta\theta} = -\frac{\sin 2\theta}{4\Gamma}, \hspace{0.5cm} \Gamma^\theta_{\varphi\varphi} = -\frac{\sin 2\theta}{2\Gamma^3}, \hspace{0.5cm} \Gamma^\varphi_{\theta\varphi} = -\cot \theta \Gamma.
\]  \hspace{1cm} (A.6)
B Fluctuation of the metric tensor

We found the following results on the perturbations of the metric tensor, 

$$g^{(\text{pert.})}_{\mu\nu} = \sum_{n=0}^{\infty} g^{(n)}_{\mu\nu}(\rho, \tau, \theta, \varphi) \lambda^n = \sum_{n=0}^{3} \sum_{m=0}^{3} g^{(n,m)}_{\mu\nu}(\tau, \theta, \varphi) \lambda^n \rho^m,$$  

by an expansion in a power series of \(\lambda\) and \(\rho\):

$$g^{(n,m)}_{\mu\nu} = 0, \quad \text{(for } n, m \text{)}$$  

$$g^{(0,3)}_{\tau\tau} = 0,$$  

$$g^{(0,2)}_{\tau\tau} = \left(-c_0^2 + 8c_2M^2\pi^2T_R^2\Gamma\right) \sin 2\theta \frac{v_\theta}{2\pi^2 T_R^2 c_0} + \left(-c_0^2 + 8c_2M^2\pi^2T_R^2\Gamma + c_0^2\Gamma\right) \frac{v_\varphi}{2\pi^2 T_R^2 c_0}$$  

$$+ \left(-16c_2M^2\pi^2T_R^2 c_0^2 + c_0^4 + 128c_2^2M^4\pi^4T_R^4\Gamma\right) \frac{v^2}{2\pi^2 T_R^2} - 2\pi T_R g^{(1,1)}_{\rho\rho},$$  

$$g^{(0,1)}_{\tau\tau} = 4M^2\Gamma \left(\beta^{(3)}(\theta)v^2 + \left(c_1 - \beta^{(2)}(\theta)\right) P(\tau, \theta, \varphi)\right),$$  

$$g^{(0,0)}_{\tau\tau} = 0,$$  

$$g^{(0,m)}_{\tau\tau} = 0, \quad \text{(for } m \geq 2\text{)}$$  

$$g^{(0,1)}_{\tau i} = \frac{1}{\rho} v^i$$  

$$= -\frac{8c_2M^2\Gamma}{c_0} \gamma_{ij} v^j$$  

$$= \begin{pmatrix} -\frac{8c_2M^2\Gamma}{c_0} & 0 \\ 0 & -\frac{8c_2M^2\Gamma}{c_0} \end{pmatrix} \begin{pmatrix} v_\theta \\ v_\varphi \end{pmatrix},$$  

$$g^{(0,0)}_{\tau i} = c_0 v_i,$$  

$$g^{(1,3)}_{\rho\rho} = 0,$$  

$$g^{(1,2)}_{\rho\rho} = 0,$$  

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\[ g^{(1,0)}_{\rho \tau} = \frac{M^2 \Gamma}{\pi T_R} \left( c_2 - \frac{2c_0^2}{M^2 \pi^2 T_R^2 \Gamma} \right) v^2 + \beta^{(3)}(\theta) v^2 + \beta^{(2)}(\theta) P(\tau, \theta, \varphi), \]  
(B.12)

\[ g^{(1,m)}_{\rho i} = 0, \quad (\text{for } m \geq 1) \]  
(B.13)

\[ g^{(1,0)}_{\rho i} = \alpha^{(2)}_{ij}(\theta) v^j \]
\[ = \frac{-c_0^2 + 32c_2 M^2 \pi^2 T_R^2 \Gamma}{8\pi^3 T_R^3 c_0} \gamma_{ij} v^j \]
\[ = \begin{pmatrix} \frac{-c_0^2 + 32c_2 M^2 \pi^2 T_R^2 \Gamma}{8\pi^3 T_R^3 c_0} & 0 \\ 0 & \frac{-c_0^2 + 32c_2 M^2 \pi^2 T_R^2 \Gamma}{8\pi^3 T_R^3 c_0} \end{pmatrix} \begin{pmatrix} v_\varphi \\ v_\varphi \end{pmatrix}, \]  
(B.14)

\[ g^{(1,3)}_{\tau i} = 0, \]  
(B.15)
\[ g^{(1,2)}_{\tau \theta} = \frac{g^{(1,1)}_{\nu \tau}}{\Gamma} + \frac{1}{2} \partial_{\theta} g^{(1,1)}_{\nu \tau} \]
\[ + \frac{c_2 \left( -16c_2M^2\pi^2T_R^2c_0^2 + c_0^4 + 128c_2^2M^4\pi^4T_R^4\Gamma \right) v_\theta}{4\pi^3T_R^3c_0^2 \Gamma} v_\theta^2 + \frac{c_2 \left( c_0^2(-1 + \Gamma) + 8c_2M^2\pi^2T_R^2\Gamma \right) v_\theta v_\varphi}{2\pi^3T_R^3c_0^2} \]
\[ + \frac{-16c_2M^2\pi^2T_R^2c_0^2 + c_0^4 + 128c_2^2M^4\pi^4T_R^4\Gamma}{32M^2\pi^5T_R^2c_0^2} v_\theta \nabla_\theta v_\theta \]
\[ + \frac{c_2 \Gamma \cot \theta}{2\pi^3T_R^3} v_\theta \left( -\nabla_\theta v_\varphi + \nabla_\varphi v_\theta \right) - \frac{2c_2^2M^2\pi^2 \Gamma}{\pi T_R c_0^2} v_\varphi^2 \]
\[ + \frac{\Gamma^2 \left( -32c_2M^2\pi^2T_R^2c_0^2 + c_0^4 + 384c_2^2M^4\pi^4T_R^4\Gamma \right)}{32M^2\pi^5T_R^2c_0^2 \sin^2 \theta} v_\varphi \nabla_\theta v_\varphi \]
\[ - \frac{c_2 \Gamma^2 \left( -c_0^2 + 16c_2M^2\pi^2T_R^2\Gamma \right)}{2\pi^3T_R^3c_0^2 \sin^2 \theta} v_\varphi \nabla_\varphi v_\theta \]
\[ + \frac{-8c_2M^2\pi^2T_R^2(-1 + \Gamma)\Gamma + c_0^2(-1 + 4\Gamma - 2\Gamma^2) - 16c_2\pi^3T_R^3\Gamma^2 g^{(1,1)}_{\nu \tau}}{8\pi^3T_R c_0 \Gamma^2} v_\varphi \]
\[ + \frac{\left( c_0^2(1 + \Gamma) + 8c_2M^2\pi^2T_R^2\Gamma(-7 + 4\Gamma) \right) \cot \theta}{8\pi^3T_R c_0^2 \Gamma} v_\varphi \]
\[ - \frac{\left( c_0^2 - 8c_2M^2\pi^2T_R^2\Gamma \right) \sin \theta}{16\pi^3T_R^3c_0^2 \Gamma} \nabla_\theta v_\theta + \frac{c_0^2(-1 + \Gamma) + 24c_2M^2\pi^2T_R^2\Gamma}{8\pi^3T_R^3c_0^2} \nabla_\theta v_\varphi \]
\[ - \frac{2c_2M^2\Gamma}{\pi T_R c_0} \nabla_\varphi v_\theta + \frac{c_2M^2\Gamma^3}{\pi T_R c_0 \sin^2 \theta} \left( -\nabla_\theta \nabla_\varphi v_\varphi + \nabla_\varphi \nabla_\varphi v_\theta \right), \quad (B.16) \]
\begin{align}
g^{(1,2)}_{\varphi \varphi} &= + \frac{g^{(1,1)}_{\rho \rho}}{2\Gamma^2} + \frac{1}{2} \partial_\varphi g^{(1,1)}_{\rho \rho} + c_2 \left(-16c_2M^2\pi^2R_0^2c_3^0 + c_0^4 + 128c_2^2M^4\pi^4T_R^4\Gamma\right) v_\varphi \left(v_\varphi^2 + \frac{\Gamma^2}{\sin^2\theta} v_\varphi^2\right) \\
&+ \frac{c_2 \left(-16c_2M^2\pi^2\Gamma c_0^3\right) \sin 2\theta}{4\pi^4T_R^4c_0^3} v_\varphi v_\varphi + \frac{c_2 \left(c_0^2 - 16c_2M^2\pi^2\Gamma\right)}{2\pi^3T_R^3c_0^3} v_\varphi \nabla_\varphi v_\varphi \\
&+ \frac{-32c_2M^2\pi^2\Gamma c_0^2 + c_0^4 + 384c_2^2M^4\Gamma}{32M^2\pi^5c_0^5} v_\varphi \nabla_\varphi v_\varphi \\
&+ \frac{-c_0^4 + 128c_2^2M^4\Gamma c_0^2 + 16c_2M^2\pi^2\Gamma c_0^3(-1 + 2\Gamma)}{64M^2\pi^5c_0^5} v_\varphi \\
&+ \frac{c_2 \Gamma \cot \theta}{2\pi^3T_R^3} v_\varphi \left(-\nabla_\theta v_\varphi + \nabla_\varphi v_\theta\right) \\
&+ \frac{\Gamma^2 \left(-16c_2M^2\pi^2\Gamma c_0^3 + c_0^4 + 128c_2^2M^4\Gamma c_0^2\right)}{32M^2\pi^5\Gamma^2 c_0^2} \sin 2\theta v_\varphi \nabla_\varphi v_\varphi \\
&- \frac{\left(c_0^2 - (1 + \Gamma) + 8c_2M^2\pi^2\Gamma(-5 + 2\Gamma)\right)}{16\pi^3T_R^3c_0^3}\sin 2\theta v_\varphi \\
&+ \frac{8c_2^2M^2\pi^2\Gamma c_0^2 + c_0^4(-1 + 4\Gamma - 2\Gamma^2) - 16c_2M^2\pi^2\Gamma c_0^2}{8\pi^3T_R^3c_0^3}\Gamma^2 v_\varphi \\
&- \frac{c_2M^2\sin 2\theta}{\pi T_R c_0} \nabla_\theta v_\varphi + \frac{\left(c_0^2 - 24c_2^2M^2\pi^2\Gamma\right) \sin 2\theta}{16\pi^3T_R^3c_0^3}\nabla_\varphi v_\theta \\
&+ \frac{\frac{c_2^2(1 - \Gamma)}{8\pi^3T_R^3c_0} \nabla_\varphi v_\varphi + \frac{c_2M^2\Gamma}{\pi T_R c_0}(\nabla_\theta \nabla_\varphi v_\varphi - \nabla_\theta \nabla_\varphi v_\theta),}{(B.17)}
\end{align}

\begin{align}
g^{(1,0)}_{\varphi i} &= \frac{1}{2\pi T_R} V_i, \quad (B.18) \\
g^{(1,m)}_{ij} &= 0, \quad (for \ m \geq 2) \quad (B.19)
\end{align}

\begin{align}
g^{(1,1)}_{ij} &= - \frac{4c_2^2M^2\Gamma}{\pi T_R c_0^2} v_i v_j + \left(-\frac{c_0}{8\pi^3T_R^3} + \frac{2c_2M^2\Gamma}{\pi T_R c_0}\right) \nabla_i v_j - \frac{c_0}{8\pi^3T_R^3}\nabla_i v^k \epsilon_{kj} \\
&- \frac{2c_2M^4\Gamma^4}{3\pi T_R c_0^2} v_i \left(\partial_j R + \frac{1}{\cos^2\theta} \epsilon_j^k \partial_k R\right) - \frac{M^2c_0^4\Gamma^4}{24\pi^3 T_R^5 \sin^2 \theta \cos \theta} \epsilon_{ik} v^k \partial_j R \\
&+ (i \leftrightarrow j), \quad (B.20)
\end{align}
\[ g_{\theta\theta}^{(1,0)} = \frac{c_1 M^2 \Gamma}{\pi T_R} P + \frac{c_2}{\pi T_R} v_\theta^2 - \frac{c_0^2}{\pi T_R c_0 \sin^2 \theta} v_\phi + \frac{2 c_2 \Gamma^2}{\pi T_R \sin^2 \theta} v_\phi, \]
\[ g_{\theta\phi}^{(1,0)} = \frac{3 c_2}{\pi T_R} v_\theta v_\phi + \frac{3 c_0}{2 \pi T_R} \partial_\phi v_\theta, \]
\[ g_{\phi\phi}^{(1,0)} = \frac{c_1 M^2 \sin^2 \theta}{\pi T_R \Gamma} P - \frac{2 c_2 \sin^2 \theta}{\pi T_R \Gamma^2} v_\theta^2 + \frac{c_0}{\pi T_R \Gamma^2} v_\phi + \frac{c_2}{\pi T_R} v_\phi^2. \]  

\( \alpha^{(i)}(\theta) \) and \( \beta^{(i)}(\theta) \) are undetermined functions of \( \theta \) and \( V_i \) is defined in (7.17) and (7.18). The terms that do not appear above can be arbitrary functions up to the order that we consider.
C Metric for the perturbation of Schwarzschild black hole and expansion scalar

We also derive a GNS equation for the perturbation to the Schwarzschild background. The boundary condition on $\Sigma_c$ is only that the functions $v_i$ and $P$ are smooth. Here we do not require that the perturbation part of the metric be written covariantly in terms of covariant derivatives associated with the 2-sphere.

The background metric in a power series of $\lambda$ is

$$ds^2_{\text{Sch}} = 2d\tau d\rho + \left(-\frac{\rho}{2M\lambda} + \frac{\rho^2}{4M^2} - \frac{\rho^3\lambda}{8M^3}\right) d\tau^2 + \left(1 + \frac{\rho\lambda}{M}\right) 4M^2 d\Omega^2 + \mathcal{O}(\lambda^2)$$  \hspace{1cm} (C.1)

where $M$ is the mass of the Schwarzschild black hole.

The perturbation of the metric is expressed as

$$g^{(\text{pert.})}_{\mu\nu} = \sum_{n=0}^{\infty} g^{(n)}_{\mu\nu}(\rho, \tau, \theta, \varphi) \lambda^n = \sum_{n=0}^{\infty} \sum_{m=0}^{3} g^{(n,m)}_{\mu\nu}(\tau, \theta, \varphi) \lambda^n \rho^m.$$  \hspace{1cm} (C.2)

The nonvanishing coefficient functions are given as follows:

$$g^{(0)}_{\tau\tau} = -\frac{\rho}{\nu} P,$$  \hspace{1cm} (C.3)

$$g^{(0)}_{\tau\rho} = \left(-1 + \frac{\rho}{\nu}\right) v_i,$$  \hspace{1cm} (C.4)

$$g^{(1)}_{\rho\rho} = \frac{M}{\nu^2} (-3\nu + \rho) v^2,$$  \hspace{1cm} (C.5)

$$g^{(1,0)}_{\varphi\varphi} = \frac{2M}{\nu} P v_\varphi + \frac{\cot \theta}{2M} v^2_\varphi + \frac{\csc^2 \theta}{2M} v_\varphi \nabla_\theta v_\varphi - \frac{\cot \theta \csc^2 \theta}{2M} v^2_\varphi$$
$$- \frac{\csc^2 \theta}{M} v_\varphi \nabla_\theta v_\varphi - \frac{\csc^2 \theta}{2M} v^2_\varphi,$$  \hspace{1cm} (C.6)

$$g^{(1,2)}_{\tau\varphi} = -\frac{1}{4M\nu} v_\varphi + \frac{1}{2M\nu^2} \nabla_\theta v_\varphi + \frac{\csc^2 \theta \nabla_\theta v_\varphi}{4M\nu} + \frac{\csc^2 \theta \nabla_\varphi v_\varphi}{2M\nu^2} - \frac{\csc^2 \theta}{4M\nu} \nabla^2 v_\theta,$$  \hspace{1cm} (C.7)
\[ g^{(1,0)}_{\tau \varphi} = \frac{2M}{\nu} P v_\varphi + \frac{1}{M} v_\varphi \nabla_\vartheta v_\vartheta + \frac{1}{2M} v_\vartheta \nabla_\varphi v_\vartheta + \frac{\csc^2 \theta}{2M} v_\varphi \nabla_\varphi v_\varphi, \quad (C.8) \]

\[ g^{(1,2)}_{\tau \varphi} = -\frac{1}{4M\nu} \nabla_\varphi^2 v_\varphi + \frac{1}{2M\nu^2} v_\vartheta \nabla_\varphi v_\varphi + \frac{1}{4M\nu} \nabla_\vartheta \nabla_\varphi v_\vartheta - \frac{1}{2M\nu} v_\varphi + \frac{\csc^2 \theta}{2M\nu^2} v_\varphi \nabla_\varphi v_\varphi, \quad (C.9) \]

\[ g^{(1)}_{\theta \theta} = \frac{2M}{\nu} - \frac{4M \csc^2 \theta}{\nu} v_\varphi^2 + \rho \left( -\frac{2M}{\nu^2} v_\vartheta^2 + \frac{4M}{\nu} \nabla_\theta v_\vartheta \right), \quad (C.10) \]

\[ g^{(1)}_{\theta \varphi} = \frac{6M}{\nu} v_\vartheta v_\varphi + \frac{2M \rho}{\nu} \left( \nabla_\theta v_\varphi + \nabla_\varphi v_\theta - \frac{1}{\nu} v_\vartheta v_\varphi \right), \quad (C.11) \]

\[ g^{(1)}_{\varphi \varphi} = -\frac{4M \sin^2 \theta}{\nu} v_\theta^2 + \frac{2M}{\nu} v_\varphi^2 + 8M \sin^2 \theta \left( \nabla_\theta v_\theta + \csc^2 \theta \nabla_\varphi v_\varphi \right) + \rho \left( -\frac{2M}{\nu^2} v_\varphi^2 + \frac{4M}{\nu} \nabla_\varphi v_\varphi \right). \quad (C.12) \]

Like (7.20), \( g^{(1,0)}_{\tau i} \) is explicitly solved. This metric satisfies the Einstein equation provided that \( v^i \) and \( P \) satisfy an incompressibility condition,

\[ R^{(-1,0)}_{\tau \tau} = \nabla^i v_i = 0, \quad (C.13) \]

and a GNS equation,

\[ R^{(0,0)}_{\tau \tau} = \partial_\tau v_i + v^j \nabla_j v_i + \nabla_i P - \nu \left( \nabla^2 v_i + R_{ij}v^j \right) = 0, \quad (C.14) \]

where \( R_{ij} \) and \( \nabla_i \) are the Ricci tensor and covariant derivative associated with the distorted 2-sphere. It is checked that all other components of the Ricci tensor \( R^{(-1)}_{\mu \nu} \) and \( R^{(0)}_{\mu \nu} \) vanish except for \( R^{(0,m)}_{\mu \nu} (m = 1, 2, 3, 4) \). In order to make these vanish \( g^{(2,m)} \) must be adjusted.

The expansion scalar \( \theta \) with perturbation in the Schwarzschild black hole is given by

\[ \theta = \lambda \left( \cot^2 \theta \csc^2 \theta \cot \theta \csc^2 \theta v_\varphi^2 - \frac{\cot \theta \csc^2 \theta}{2M^3} v_\varphi \partial_\varphi v_\varphi - \frac{\cot \theta \csc^2 \theta}{2M^3} v_\varphi \partial_\theta v_\varphi + \frac{\csc^2 \theta}{8M^3} \left( \partial_\varphi v_\theta \right)^2 
+ \frac{\csc^2 \theta}{4M^3} \partial_\varphi v_\theta \partial_\theta v_\varphi + \frac{1}{2M^3} \left( \partial_\theta v_\theta \right)^2 + \frac{\csc^2 \theta}{8M^3} \left( \partial_\theta v_\varphi \right)^2 \right) + O(\lambda^2). \quad (C.15) \]

This is not positive semidefinite as in the case of the Kerr black hole. We checked that this conclusion does not change, even if the metric in [8] is used.
References


