Radial Bargmann representation for the Fock space of type B

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Abstract

Let $\nu_{\alpha,q}$ be the probability and orthogonality measure for the $q$-Meixner-Pollaczek orthogonal polynomials, which has appeared in [BEH15] as the distribution of the $(\alpha,q)$-Gaussian process (the Gaussian process of type B) over the $(\alpha,q)$-Fock space (the Fock space of type B). The main purpose of this paper is to find the radial Bargmann representation of $\nu_{\alpha,q}$. Our main results cover not only the representation of $q$-Gaussian distribution by [LM95], but also of $q^2$-Gaussian and symmetric free Meixner distributions on $\mathbb{R}$. In addition, non-trivial commutation relations satisfied by $(\alpha,q)$-operators are presented.

Keywords: Radial Bargmann representation, deformation, Fock spaces, $q$-orthogonal polynomials.

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1 Introduction

Bożejko-Ejsmont-Hasebe [BEH15] considered a deformation of the (algebraic) full Fock space with two parameters $\alpha$ and $q$, namely, the $(\alpha,q)$-Fock space (or the Fock space of type B) $\mathcal{F}_{\alpha,q}(H)$ over a complex Hilbert space $H$. The deformation with $\alpha = 0$ is equivalent to the $q$-deformation by Bożejko-Speicher [BS91] and Bożejko-Kümmerer-Speicher [BKS97], and the corresponding $q$-Bargmann-Fock space has been constructed by van Leeuwen-Maassen [LM95].

For the construction of $\mathcal{F}_{\alpha,q}(H)$, their starting point is to replace the Coxeter group of type A, that is, symmetric group $S_n$ for the $q$-Fock space by the Coxeter group of type B, $\Sigma_n := Z_2^\alpha \ltimes S_n$ in (A.1) of the Appendix A. This replacement provides us a more general symmetrization operator on $H^{\otimes n}$ than that of [BS91] to define the $(\alpha,q)$-inner product $\langle \cdot \mid \cdot \rangle_{\alpha,q}$ in (A.3). One can define annihilation $B^{-}_{\alpha,q}(f)$ and creation $B^{+}_{\alpha,q}(f)$ operators acting on $\mathcal{F}_{\alpha,q}(H)$, and the $(\alpha,q)$-Gaussian process (the Gaussian process of type B) $G_{\alpha,q}(f)$ for $f \in H$ as the sum of them, $G_{\alpha,q}(f) := B^{-}_{\alpha,q}(f) + B^{+}_{\alpha,q}(f)$. It is one of their main interests to find a probability distribution $\mu_{\alpha,q,f}$ on $\mathbb{R}$ of $G_{\alpha,q}(f)$, $\|f\|_H = 1$, with respect to the vacuum state $(\Omega, \Omega)_{\alpha,q}$. $\mathcal{F}_{\alpha,q}(H)$ equipped with $(\cdot, \cdot)_{\alpha,q}$, $B_{\alpha,q}(f)$, and $B^{+}_{\alpha,q}(f)$ is a typical example of interacting Fock spaces in the sense of Accardi-Bożejko [AB98]. It suggests that the theory of orthogonal polynomials plays intrinsic roles in all previous works mentioned above. In fact, the measure $\mu_{\alpha,q,f}$ given in [BEH15] Theorem 3.3] is derived essentially from the orthogonality measure $\nu_{\alpha,q}$ associated with the $q$-Meixner-Pollaczek orthogonal polynomials $\{P_{n}^{(\alpha,q)}(x)\}$ for $\alpha, q \in (-1, 1)$ given by the recurrence relation,

$$
\begin{align*}
P_{0}^{(\alpha,q)}(x) &= 1,
p_{1}^{(\alpha,q)}(x) = x,
p_{n}^{(\alpha,q)}(x) = p_{n+1}^{(\alpha,q)}(x) + (1 + \alpha q^{n-1})[n]_{q}p_{n-1}^{(\alpha,q)}(x), \quad n \geq 1
\end{align*}
$$

where $[n]_{q} = 1 + q + \cdots + q^{n-1}$ is the $q$ number. However, the Bargmann representation (measure on $\mathbb{C}$) of $\nu_{\alpha,q}$ has not been obtained yet except the case of $\alpha = 0$ for $0 \leq q < 1$ [LM95], for $q = 1$ [Barg61] [AKK93], for $q = 0$ [BR97], and $t$-deformed cases of these [AKW10] [KW14], and for $q > 1$ [Kr98].

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Therefore, the main purpose of this paper is to find the radial Bargmann representation of the probability measure \(\nu_{\alpha,q}\) on \(\mathbb{R}\). Our results cover the radial Bargmann representations of \(q\)-Gaussian, symmetric free Meixner (Kesten) and \(q^2\)-Gaussian distributions on \(\mathbb{R}\).

The organization of this paper will be as follows. In Section 2, we shall explain how the \((\alpha,q)\)-Fock space is related to the notion of one-mode interacting Fock spaces and Bargmann representation. In Section 3, the radial Bargmann representation of \(\nu_{\alpha,q}\) is constructed explicitly in Theorem 3.11. In Section 4, commutation relations satisfied by one-mode \((\alpha,q)\)-annihilation and creation operators will be treated. In the Appendix, we shall give a minimum reference on the Coxeter group of type B extracted from [BEH15].

## 2 Key ideas and our purpose

Let us point out some of the keys to calculate the distribution of \(G_{\alpha,q}(f)\) in [BEH15]. It is shown that a linear map, \(\Phi : \text{Span}\{f^\otimes n \mid f \in H, n \geq 0\} \rightarrow L^2(\mathbb{R}, \mu_{\alpha,q})\) given by \(\Phi(f^\otimes n) = P_n^{(\alpha(f,\overline{T}))}(x)\), is an isometry and a relation under \(\|f\|_H = 1\),

\[
G_{\alpha,q}(f)f^\otimes n = \left(\begin{array}{c} B_{\alpha,q}(f) + B_{\alpha,q}(f) \end{array}\right) (f^\otimes n) \\
= f^\otimes (n+1) + (1 + \alpha(f,\overline{T})_{\mathbb{R}})q^{n-1}[n]_q f^\otimes (n-1),
\]

is satisfied where \(\overline{T}\) denotes a self-adjoint involution of \(f \in H\) in (A.2). This corresponds to the three terms recursion relation satisfied by \(P_n^{(\alpha(f,\overline{T}))}(x)\) through \(\Phi\). Then, it is proved that \(\mu_{\alpha,q} f = \nu_{\alpha(f,\overline{T}),\mathbb{R}}\) (see \(\nu_{\alpha,q}\) in (3.3)) in the sense of

\[
\langle \Omega, G_{\alpha,q}(f)^n \Omega, \alpha,q \rangle = \int x^n \mu_{\alpha,q} f(dx)
\]

where \(\Omega\) denotes the vacuum vector. Therefore, in order to get the Bargmann representation of \(\nu_{\alpha(f,\overline{T}),\mathbb{R}}\), it is enough to consider that of \(\nu_{\alpha,q}\) in the sense of Definition 2.2 given later.

Since the structure mentioned above can be reduced to the one-mode analogue of \((\alpha,q)\)-Fock spaces, let us recall fundamental relationships between one-mode interacting Bargmann-Fock spaces and the theory of orthogonal polynomials of one variable.

### Definition 2.1.

Let \(\{\omega_n\}_{n=0}^\infty\) with \(\omega_0 := 1\) be an infinite sequence of positive real numbers and \(\{\alpha_n\}_{n=0}^\infty\) be of real numbers. A one-mode interacting Bargmann-Fock space \(\mathcal{B}\) is defined as \(\bigoplus_{n=0}^\infty \mathbb{C}\Phi_n\) equipped with \(\Phi_n := z^n/|\omega_n|!\), \([\omega_n]! := \prod_{k=0}^n \omega_k\), the inner product \(\langle \Phi_m, \Phi_n \rangle_{\mathcal{B}} = \delta_{m,n}\) for all \(m,n \in \mathbb{N} \cup \{0\}\), operators of creation \(a^+\), annihilation \(a^-\), and conservation \(a^0\) defined by

\[
\begin{align*}
a^+ \Phi_n &:= \sqrt{\omega_{n+1}} \Phi_{n+1}, & n \geq 0, \\
a^- \Phi_n &:= \sqrt{\omega_{n}} \Phi_{n-1}, & n \geq 1, \\
a^0 \Phi_n &:= \alpha_n \Phi_n, & n \geq 0.
\end{align*}
\]

Let \(\{\omega_n\}_{n=0}^\infty, \{\alpha_n\}_{n=0}^\infty\) be a pair of sequences in Definition 2.1 and define a sequence of monic polynomials \(\{P_n(x)\}\) recurrently by

\[
\begin{align*}
P_0(x) &= 1, & P_1(x) &= x - \alpha_0, \\
xP_n(x) &= P_{n+1}(x) + \omega_n P_{n-1} + \alpha_n P_n(x), & n \geq 1.
\end{align*}
\]

Then there exists a probability measure \(\mu\) on \(\mathbb{R}\) with finite moments of all orders such that \(\{P_n(x)\}\) is the orthogonal polynomials with \(\langle P_m(x), P_n(x) \rangle_{L^2(\mathbb{R}, \mu)} = \delta_{m,n}|\omega_n|!\) for all \(m,n \in \mathbb{N} \cup \{0\}\). (See [Chi78] [HO07], for example.)

It is easy to see that a linear map

\[
U : \mathcal{B} = \bigoplus_{n=0}^\infty \mathbb{C}\Phi_n \rightarrow L^2(\mathbb{R}, \mu)
\]
defined by $U \left( \sqrt{\omega_n} \Phi_n \right) = P_n(x)$ is an isometry and in addition $a^+ + a^- + a^\circ = U^* X U$ is satisfied due to (2.2) and (2.3), where $X$ represents the multiplication operator by $x$ in $L^2(\mathbb{R}, \mu)$. This intertwining relation provides a notion of the quantum decomposition of a classical random variable $X$ and

$$
\langle \Phi_0, (a^+ + a^- + a^\circ)^n \Phi_0 \rangle_B = \int x^n \mu(dx).
$$

(2.4)

Therefore, if $\omega_n = (1 + \alpha q^\alpha-1)^n q^n$, $\alpha_n = 0$, the equality in (2.4) is a one-mode analogue of (2.1).

Now it is interesting to consider the following moment problem to realize the inner product by the integral:

**Problem 1.** For a given $\{\omega_n\}$ of $\mu$, find a probability measure $\gamma_\mu$ satisfying the equality,

$$
\int \mathbb{C} \bar{z}^m z^n \gamma_\mu(d^2z) = \delta_{m,n}[\omega_n]!
$$

(2.5)

for all $m, n \in \mathbb{N} \cup \{0\}$.

**Definition 2.2.** A measure $\gamma_\mu$ satisfying the equality (2.5) is called a Bargmann representation (measure on $\mathbb{C}$) of a measure $\mu$ on $\mathbb{R}$.

It was proved in [Sz07] (see also [AKW16] [KW14]) that if a measure $\mu$ admits any Bargmann representation, then it also admits a radial (rotation invariant) Bargmann representation $\gamma_\mu(d^2z) = \frac{1}{2\pi} \lambda_{[0, 2\pi]}(d\theta) \rho_\mu(dr)$, $z = re^{i\theta}$, $r \geq 0$, $\theta \in [0, 2\pi)$, $\lambda_{[0, 2\pi]}$ is the Lebesgue measure on $[0, 2\pi)$. It says that the angular part takes care of orthogonality of (2.5). Therefore, Problem 1 can be transformed into the following Problem 2:

**Problem 2.** Find a positive radial measure $\rho_\mu$ satisfying

$$
\int_0^\infty r^{2n} \rho_\mu(dr) = [\omega_n]!
$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

**Main Purpose:** We shall consider Problem 2 associated with $\omega_n = (1 + \alpha q^\alpha-1)^n q^n$, $\alpha_n = 0$ of $\nu_{\alpha, q}$ in Section 3. Furthermore, commutation relations satisfied by $a^+$, $a^-$ acting on $\mathcal{B}$ associated with $\omega_n = (1 + \alpha q^\alpha-1)^n q^n$ will be presented in Section 4.

**Remark 2.3.** (1) One can notice that $\gamma_\mu$ and $\rho_\mu$ are determined only by $[\omega_n]!$. Therefore, it is enough in general for the Bargmann representation in the above sense to consider the symmetric measure $\mu$ with $\alpha_n = 0$ for all $n$, which implies that $a^\circ$ is a zero operator.

(2) If $\mu$ is symmetric, then $\alpha_n = 0$ for all $n$ is implied. The converse statement is true if $\mu$ is determined by its moments.

3 (α, q)-Bargmann representation

3.1 q-Meixner-Pollaczek polynomials

Let us recall standard notations from $q$-calculus, which can be found in [GR04] [KLS10] for example. The $q$-shifted factorials are defined by

$$
(a; q)_0 := 1, \quad (a; q)_k := \prod_{\ell=1}^k (1 - aq^{\ell-1}), \quad k = 1, 2, \ldots, \infty,
$$

and the product of $q$-shifted factorials is defined by

$$
(a_1, a_2; q)_k := (a_1; q)_k (a_2; q)_k, \quad k = 1, 2, \ldots, \infty.
$$
Remark 3.1. The \( q \)-shifted factorials are a natural extension of the Pochhammer symbol \((\cdot)_n\) because one can see that \(\lim_{q \to 1}[k]_q = k\) implies

\[
\lim_{q \to 1} \frac{(q^n; q)_n}{(1-q)^n} = (k)_n, \tag{3.1}
\]

where \((k)_0 := 1, (k)_n := k(k+1)\cdots(k+n-1), n \geq 1\).

As we have mentioned, \(\{P_n^{(\alpha, q)}(x)\}\) for \(\alpha, q \in (-1, 1)\) is the \(q\)-Meixner-Pollaczek polynomials satisfying the recurrence relation,

\[
\begin{cases}
P_0^{(\alpha, q)}(x) = 1, & P_1^{(\alpha, q)}(x) = x, \\
P_n^{(\alpha, q)}(x) = P_{n+1}^{(\alpha, q)}(x) + (1 + \alpha q^{n-1})[n]_q P_n^{(\alpha, q)}(x), & n \geq 1.
\end{cases} \tag{3.2}
\]

It is known in [KLS10, 14.9.2] and [BEH15, page 1781] that the orthogonality measure \(\nu_{\alpha, q}\) for such polynomials has the density of the form,

\[
\frac{(q, \gamma^2)_{\infty}}{2\pi} \sqrt{\frac{1-q}{4-(1-q)x^2}} g(x, 1; q)g(x, -1; q)g(x, \sqrt{q}; q)g(x, -\sqrt{q}; q),
\]

supported on the interval \((-2/\sqrt{1-q}, 2/\sqrt{1-q})\) where

\[
g(x, b; q) = \prod_{k=0}^{\infty} (1 - 4bx(1-q)^{-1/2}q^k + b^2q^{2k}),
\]

and

\[
\gamma = \begin{cases} 
\sqrt{-\alpha}, & \alpha < 0, \\
i\sqrt{\alpha}, & \alpha \geq 0.
\end{cases}
\]

Example 3.2. (1) If \(\alpha = 0\), then \(q\)-Meixner-Pollaczek polynomials get back to the \(q\)-Hermite polynomials \(H_n^{(q)}(x)\) whose orthogonality measure is the standard \(q\)-Gaussian measure on \((-2/\sqrt{1-q}, 2/\sqrt{1-q})\) given by

\[
\nu_q(dx) := \frac{\sqrt{1-q}}{\pi} \sin \theta \prod_{n=1}^{\infty} (1-q^n)|1-q^n e^{2i\theta}|^2 dx,
\]

where \(x\sqrt{1-q} = 2\cos \theta, \theta \in [0, \pi]\). Furthermore, one can get the standard Gaussian law as \(q \to 1\), the Bernoulli law as \(q \to -1\), and the standard Wigner’s semi-circle law if \(q = 0\). See [BKS97], [BS91].

(2) The measure \(\nu_{q, 0}\) is the symmetric free Meixner law [An03], [BB06], [SY01].

(3) The measure \(\nu_{\alpha, q}\) is the \(q^2\)-Gaussian law scaled by \(\sqrt{1+q}\).

(4) If \(\alpha = -q^{2\beta}\) as suggested in Remark 3.1 then the measure \(\nu_{-q^{2\beta}, q}\) under a certain scaling converges to the classical symmetric Meixner law as \(q \uparrow 1\),

\[
\frac{2^{2\beta}}{2\pi} \int |\Gamma(\beta + ix)|^2 dx, \quad x \in \mathbb{R}. \tag{3.4}
\]

See also [KLS10, 14.9.15].

3.2 Problem

For \(\alpha, q \in (-1, 1)\), we would like to know when there exists a radial measure \(\nu_{\alpha, q}\) satisfying

\[
\int_0^{\infty} r^{2k} \rho_{\alpha, q}(dr) = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}. \tag{3.5}
\]

Here \([k]_q!\) denotes the \(q\)-factorials defined by

\[
[0]_q! := 1, \quad [k]_q! := \prod_{\ell=1}^{k} [\ell]_q = \frac{(q; q)_k}{(1-q)^k}, \quad k \geq 1.
\]
It is easy to get the inequality for $\alpha, q \in (-1, 1)$,

$$|(-\alpha; q)_k[k]_q|! \leq \left( \frac{4}{1 - |q|} \right)^k, \quad k \in \mathbb{N} \cup \{0\}. \quad (3.6)$$

Due to Carleman criterion for the moment problem, this inequality implies that a radial measure $\rho_{\alpha,q}$ is determined uniquely by the sequence $\{(\alpha; q)_k[k]_q|!\}$.

We shall follow the procedure below to construct $\rho_{\alpha,q}$ in (3.5).

(1) Recall the radial part of the $q$-Gaussian measure on $\mathbb{C}$ ($q$-Bargmann measure), $\rho_{\nu_q} = \rho_{\nu_{0,q}}$, obtained in [LM95],

$$\int_0^\infty r^{2k} \rho_{\nu_q}(dr) = [k]_q^!.$$ \quad (3.7)

(2) Find a radial (possibly signed) measure $\rho_{\alpha,q}$ having the moment $(-\alpha; q)_k$.

(3) Compute the multiplicative (Mellin) convolution $\rho_{\nu_q} \ast \rho_{\alpha,q}$ to get $\rho_{\nu_{\alpha,q}}$.

Remark 3.3. It is known [LM95] that a radial measure $\rho_{\nu_q}$ in (3.7) does not exist for $q < 0$. However, one can see that the positivity assumption on $q$ can be relaxed for $\rho_{\nu_{\alpha,q}}$ if $\alpha = q$. It will be discussed right after the proof of Proposition 3.6 and in Proposition 3.7.

3.3 Construction of $(\alpha, q)$-radial measures

Lemma 3.4. Suppose that $\alpha \in (-1, 1)$ and $q \in [0, 1)$. Let

$$\rho_{\alpha,q} := (-\alpha; q)_\infty \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{(q; q)_n} \delta_{q^n/2},$$

which is a signed measure. Then we have

$$\int_0^\infty r^{2k} \rho_{\alpha,q}(dr) = (-\alpha; q)_k, \quad k \in \mathbb{N} \cup \{0\}.$$ \quad (3.8)

In particular, if taking $\alpha = -q$, then one can see $\rho_{\nu_q} = D_{(1-q)^{-1/2}}(\rho_{-q,q})$, namely,

$$\int_0^\infty r^{2k} D_{(1-q)^{-1/2}}(\rho_{-q,q})(dr) = \frac{(q; q)_k}{(1 - q)^k} = [k]_q^!,$$

where $D_\lambda(\lambda)$ is the push-forward of $\lambda$ by the map $x \mapsto tx$ for a measure $\lambda$ on $\mathbb{R}$.

Proof. Firstly, we have

$$\int_0^\infty r^{2k} \rho_{\alpha,q}(dr) = (-\alpha; q)_\infty \sum_{n=0}^{\infty} \frac{(-\alpha q^k)^n}{(q; q)_n}.$$ \quad (3.8)

Since Euler’s formula (see [GR04, 1.3.15]),

$$\frac{1}{(a; q)_\infty} = \sum_{n=0}^{\infty} \frac{a^n}{(q; q)_n},$$

is known, we replace $a$ by $-\alpha q^k$ in (3.8) to obtain

$$\int_0^\infty r^{2k} \rho_{\alpha,q}(dr) = \frac{(-\alpha; q)_\infty}{(-\alpha q^k; q)_\infty} = (-\alpha; q)_k.$$ \quad (3.8)

The proof is complete. \qed
Remark 3.5. (1) The last equality in proof is due to the formula

\[ (a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty}. \]

See [GR04] 1.2.30, for example.

(2) Euler’s formula is considered as the $q$-analogue of exponential function $e^a$ due to

\[ \lim_{q \to 1} \frac{1}{((1 - q) a; q)_n} = e^a. \]

Let

\[ \binom{n}{\ell}_q := \frac{[n]_q!}{[\ell]_q! [n - \ell]_q!} = \frac{(q; q)_n}{(q; q)_\ell (q; q)_{n - \ell}} \]

be the $q$-binomial coefficients and $h_n(z \mid q)$ be the Rogers-Szegö polynomials [GR04] [S05] defined by

\[ h_n(z \mid q) = \sum_{\ell = 0}^n \binom{n}{\ell}_q z^\ell. \]

Proposition 3.6. Suppose that $\alpha \in (-1, 1)$ and $q \in [0, 1)$. Let

\[ \rho_{\nu_{\alpha,q}} := \begin{cases} (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} \mid q) \delta_{(1-q)^{1/2} q^n / 2}, & q > 0, \\ -\alpha \delta_0 + (1 + \alpha) \delta_1, & q = 0, \end{cases} \]

which is a signed measure in general. Then we have

\[ \int_0^\infty r^{2k} \rho_{\nu_{\alpha,q}}(dr) = \frac{(-\alpha, q; q)_k}{(1 - q)^k} = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}. \] (3.10)

Proof. First of all, it is easy to show (3.10) for the case $q = 0$. Therefore, let us assume $q > 0$.

One can compute the multiplicative (Mellin) convolution $\oplus$ to get $\rho_{\nu_{\alpha,q}}$ as follows:

\[ \rho_{\nu_{\alpha,q}} = \rho_{\alpha,q} \oplus D_{(1-q)^{-1/2} (\rho_{\alpha,q})} \]

\[ = (-\alpha, q; q)_\infty \sum_{\ell=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{(-\alpha)^{\ell} q^{n-\ell}}{(q; q)_\ell (q; q)_{n-\ell}} \right) \delta_{(1-q)^{-1/2} q^n / 2} \]

\[ = (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} \mid q) \delta_{(1-q)^{-1/2} q^n / 2}. \]

On the other hand, by Lemma 3.3 we have

\[ \int_0^\infty r^{2k} D_{(1-q)^{-1/2} (\rho_{\alpha,q})}(dr) = \frac{(q; q)_k}{(1 - q)^k} = [k]_q!. \]

Therefore, one can get

\[ \int_0^\infty r^{2k} \rho_{\nu_{\alpha,q}}(dr) = \int_0^\infty r^{2k} \rho_{\alpha,q}(dr) \int_0^\infty r^{2k} D_{(1-q)^{-1/2} (\rho_{\alpha,q})}(dr) \]

\[ = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}. \]

□

In Proposition 3.6 we have obtained $\rho_{\nu_{\alpha,q}}$ for $\alpha \in (-1, 1)$ and $q \in (0, 1)$. Due to the term

\[ \delta_{(1-q)^{-1/2} q^n / 2} \text{ in } \rho_{\nu_{\alpha,q}}, \]

it seems impossible for $q \in (-1, 0)$ to define $\rho_{\nu_{\alpha,q}}$. However, if $-1 < \alpha = q < 0$ then $\nu_{q,q}$ coincides with a scaled $q^2$-Gaussian measure, and hence the Bargmann measure exists.
Proposition 3.7. Suppose \(-1 < \alpha = q < 0\). We define
\[
\rho_{\nu_{\alpha,q}} := D_{(1+q)^{1/2}}(\rho_{\nu_{\alpha,q}})
= (q^2; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q)^{-1/2}(-q)^n}.
\]
(3.11)
Then one can see
\[
\int_{0}^{\infty} r^{2k} \rho_{\nu_{\alpha,q}}(dr) = (1+q)^k [k]_q^2! = (-q;q)_k [k]_q^2!.
\]
Proof. One can see by direct computations
\[
(-q;q)_k [k]_q^2! = \left\{ \prod_{\ell=1}^{k} (1 - (-q)^{\ell-1}) \right\} \left\{ \prod_{\ell=1}^{k} \frac{1 - q^{2\ell}}{1 - q} \right\}
= (1+q)^k \prod_{\ell=1}^{k} \frac{1 - q^{2\ell}}{1 - q}.
= (1+q)^k [k]_q^2!.
\]
Thus \(\rho_{\nu_{\alpha,q}}\) can be defined as the radial measure for \(q^2\)-Gaussian measure on \(\mathbb{C}\) scaled by \((1+q)^{1/2}\), namely, \(\rho_{\nu_{\alpha,q}} = D_{(1+q)^{1/2}}(\rho_{\nu_{\alpha,q}})\).

Remark 3.8. If we use the fact that \(h_n(-1 | q) = 0\) for odd \(n \geq 1\) (see proof of Lemma 3.9 below), we can extend the definition 3.9 to the case \(-1 < \alpha = q < 0\). This will give an alternative way to define \(\rho_{\nu_{\alpha,q}}\) for \(-1 < q < 0\), but both definitions give the same measure.

We need some properties of the Rogers-Szegö polynomials to know when the measure \(\rho_{\nu_{\alpha,q}}\) becomes positive.

Lemma 3.9 \textbf{[MGH90].} Suppose that \(q \in (-1, 1)\).

1. If \(n \geq 0\) is odd, then \(h_n(x | q) \geq 0\) if and only if \(x \geq -1\). Moreover, the point \(x = -1\) is the unique zero of \(h_n(x | q)\) on \(\mathbb{R}\).

2. If \(n \geq 0\) is even, then \(h_n(x | q) > 0\) for all \(x \in \mathbb{R}\).

Proof. It is known that all the zeros of \(h_n(z | q)\) lie on the unit circle \(|z| = 1\). See \textbf{[MGH90]} or \textbf{[S05]} Theorem 1.6.11]. Note that the result was obtained for \(q \in [0, 1)\), but the proof can be extended to \(q \in (-1, 1)\) without any modifications.

By definition, one can see
\[
\left[ \begin{array}{c} n \\ \ell \end{array} \right]_q = \frac{(1 - q^{n-\ell+1})(1 - q^{n-\ell+2}) \cdots (1 - q^n)}{(1-q)(1-q^2) \cdots (1-q^\ell)} > 0,
\]
and hence \(h_n(1 | q) > 0\) for all \(n \geq 0\). Thus, \(h_n(x | q) \neq 0\) for \(x \in \mathbb{R} \setminus \{-1\}\). It then suffices to show that \(h_n(-1 | q) > 0\) for all even \(n \geq 0\) and \(h_n(-1 | q) = 0\) for all odd \(n \geq 1\). The recurrence relation for the Rogers-Szegö polynomials is known to be
\[
h_{n+1}(z | q) = (z+1)h_n(z | q) - (1-q^n)zh_{n-1}(z | q), \quad n \geq 1.
\]
(3.12)
See \textbf{[S05]} 1.6.76] (note that formula (1.6.76) has an error of a sign). It is easy to see that \(h_0(-1 | q) = 1 > 0, h_1(-1 | q) = 0\), so by induction and 3.12 one can show \(h_n(-1 | q) > 0\) for all even \(n \geq 0\) and \(h_n(-1 | q) = 0\) for all odd \(n \geq 1\).

We need the following lemma in proof of Theorem 3.11 for the non-existence part of a radial Bargmann measure.

Lemma 3.10. Let \(\mu\) be a signed measure on \(\mathbb{R}\) with compact support and let \(\nu\) be a nonnegative measure on \(\mathbb{R}\). If \(\mu\) and \(\nu\) have the same finite moments of all orders, then \(\mu = \nu\).
Proof. We denote by $m_n$ the moments of $\mu$ (and $\nu$ by assumption). Since $\mu$ is compactly supported, say on $[-R, R]$, 
\[
|m_n| = \left| \int_{[-R,R]} x^n \mu(dx) \right| \leq \|\mu\| R^n, \quad n \in \mathbb{N} \cup \{0\},
\]
where $\|\mu\|$ denotes the total variation of $\mu$. Therefore, $\nu$ is also supported on $[-R, R]$. By Weierstrass' approximation, we have 
\[
\int_{[-R,R]} f(x) \mu(dx) = \int_{[-R,R]} f(x) \nu(dx)
\]
for all $f \in C([-R, R])$. This implies that $\mu = \nu$ since, if we use the Hahn decomposition $\mu = \mu_+ - \mu_-$, then (3.13) implies 
\[
\int_{[-R,R]} f(x) \mu_+(dx) = \int_{[-R,R]} f(x) (\nu + \mu_-)(dx),
\]
and hence $\mu_+ = \nu + \mu_-$ as nonnegative finite measures. \qed

In summary, the complete answer to the unique existence of a radial Bargmann representation of $\nu_{\alpha,q}$ is stated as follows:

**Theorem 3.11.** Suppose that $\alpha, q \in (-1, 1)$. The probability measure $\nu_{\alpha,q}$ has a radial Bargmann representation if and only if either (i) $q \geq 0$ and $\alpha \leq q$ or (ii) $\alpha = q \neq 0$.

In fact, the radial measure is given uniquely by

\[
\rho_{\nu_{\alpha,q}} = \begin{cases} 
-\alpha \delta_0 + (1 + \alpha) \delta_1 & (\alpha \leq q = 0), \\
(\alpha, q; q) \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{1/2} q^n} & (q > 0, \alpha < q), \\
(q^2; q^2) \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q)^{1/2} | q^n} & (\alpha = q \neq 0).
\end{cases}
\]

**Proof.**

1. **Existence and uniqueness.** If $q \in [0, 1]$ and $\alpha \leq q$, then by Proposition 3.6 and Lemma 3.9 the signed measure $\rho_{\nu_{\alpha,q}}$ is in fact a nonnegative measure and becomes the radial part of a Bargmann measure. The case $\alpha = q < 0$ was discussed in Proposition 3.7. Due to Carleman criteria for the moment problem, the inequality given in (3.13) guarantees the uniqueness of $\rho_{\nu_{\alpha,q}}$ for these cases.

2. **Non-existence.** (1) If $q \in (0, 1)$ and $\alpha > q$, then $\rho_{\nu_{\alpha,q}}$ is not a nonnegative measure and is really a signed measure since $h_n(-\alpha/q | q) < 0$ for odd integers $n \geq 0$ and $q > 0$ from Lemma 3.9. By Lemma 3.10 if a radial Bargmann measure exists, then it must be equal to the signed measure $\nu_{\alpha,q}$. This is a contradiction. Thus, a radial Bargmann measure does not exist.

(2) If $q = 0$ and $\alpha > q = 0$ then by (3.6), $\nu_{0,0}$ is really a signed measure, and hence by the same argument as above, a radial Bargmann measure does not exist.

(3) Let 
\[
\beta_k(\alpha, q) := (-\alpha; q)_k |q|^k, \quad k \geq 0, \alpha, q \in (-1, 1).
\]

Given $q < 0$ and $\alpha \neq q$, suppose that there exists a radial part of a Bargmann measure, $\rho$. Let $\rho^2$ be the push-forward of $\rho$ by the map $x \mapsto x^2$. Then,

\[
\beta_k(\alpha, q) = \int_0^{\infty} x^k \rho^2(dx) = \int_0^{\infty} x^{2k} \rho(dx).
\]

By the way, by Proposition 3.6 it holds that $\beta_k(\alpha, q') = \int_0^{\infty} x^{2k} \rho_{\nu_{\alpha,q'}}(dx)$ for any $q' > 0$, that is,

\[
\beta_k(\alpha, q') = (-\alpha, q'; q') \sum_{n=0}^{\infty} \frac{(q')^n}{(q'; q')_n} h_n(-\alpha(q')^{-1} | q') \frac{(q')^{kn}}{(1-q')^k}, \quad q' > 0,
\]

which is true even for $q' = q$ by analytic continuation.

Now let us consider the signed measure
\[
\mu := (-\alpha, q; q) \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{-1} q^n}, \quad \alpha \neq q < 0,
\]
supported on the points $\frac{n^2}{t} q$ for $n = 0, 1, 2, 3, \ldots$. Then by (3.14) for $q' = q$ and by (3.13),

$$
\int_R x^k \mu(dx) = \beta_k(\alpha, q) = \int_0^{\infty} x^k \rho^2(dx), \quad k \in \mathbb{N} \cup \{0\}.
$$

By Lemma 3.10, the signed measure $\mu$ and the probability measure $\rho^2$ should be equal. However, the support of $\mu$ is not contained in $[0, \infty)$, and hence $\mu$ cannot be equal to $\rho^2$. This is a contradiction. \( \square \)

**Example 3.12.** (1) The radial measure $\rho_{\nu_0, q}$ for $q \in [0, 1)$ is of the $q$-Bargmann [LM95].

(2) The radial measure $\rho_{\nu_q, q}$ for $q \in (-1, 1)$ is of the $q^2$-Bargmann.

(3) $\lim_{q \uparrow 1} \rho_{\nu_q, q}$ is of the classical Bargmann [Barg61] [AKK03].

(4) Consider $\alpha = -q^{2\beta}, \beta > 0$. This choice of $\alpha$ is suggested by (3.1) in Remark 3.1. In fact, one can see

$$
\lim_{q \uparrow 1} \frac{(1 - q^{2\beta+n-1})[n]_q}{4(1-q)} = \frac{1}{4}(n + 2\beta - 1)n.
$$

This limit sequence is the Jacobi sequence of the symmetric Meixner distribution in (3.14), so that $\rho_{\nu_{-q^2, q}}$ under suitable scaling converges weakly as $q \uparrow 1$ to the radial measure with the density,

$$
\frac{2\pi r}{1(2\beta)} \int_0^{\infty} h(r,t/4)e^{-t^{2\beta-1}}dt
$$

where

$$
h(r,t) = \frac{1}{\pi t} \exp \left( -\frac{r^2}{t} \right), \ r \in \mathbb{R}, \ t > 0.
$$

This is an integral representation of the radial density for the Bessel kernel measure, which can be also represented by the modified Bessel function [As05] [As09].

(5) $\rho_{\nu_{\alpha, q}}$ for $\alpha \in (-1,0]$ is the radial measure for the symmetric free Meixner distribution. See Remark 3.13 below.

**Remark 3.13.** Let $\mu_t$ be a $t$-deformed probability measure of a probability measure $\mu$ on $\mathbb{R}$ defined through the Cauchy transform $G_\mu$ of $\mu$,

$$
G_{\mu_t}(z) := \frac{t}{G_\mu(z)} + (1-t)z, \quad t \geq 0,
$$

examined by Bożejko-Wysoczański [BW98] [BW01]. Krystek-Wojakowski [KW14] discussed the radial Bargmann representation of a $t$-deformed probability measure $\mu_t$, $t$-Bargmann representation for short, and obtained necessary and sufficient condition for the admissibility of the representation. The $t$-Bargmann representation of the Kesten measure $\kappa_t$ has the form,

$$
\rho_{\kappa_t} = \left( 1 - \frac{1}{t} \right) \delta_0 + \frac{1}{t} \delta \sqrt{t}, \quad t \geq 1.
$$

In [AKW16], the $t$-Bargmann representation of a symmetric free Meixner law $\varphi_{s,t}$ with two positive parameters $s, t$ is treated and is admitted if and only if $t \geq 1$. In fact, one can see $\rho_{\varphi_{s,t}} = D_s(\rho_{\kappa_t})$ and hence

$$
\rho_{\nu_{(1-t)/s,0}} = \rho_{\varphi_{1/\sqrt{t}}}, \quad D_1(\rho_{\kappa_t}), \quad t \geq 1.
$$

Therefore, the case (5) in Example 3.12 can be viewed as a $t$-Bargmann representation, too.

Furthermore, let us state the $t$-deformed version of Theorem 3.11 for $q \neq 0$ without proof:

**Proposition 3.14.** The $t$-deformed version of $\rho_{\nu_{\alpha, q}}$ for $q \neq 0$ is given by

$$
\left( 1 - \frac{1}{t} \right) \delta_0 + \frac{1}{t} \rho_{\nu_{\alpha, q}}, \quad t \geq 1.
$$
Remark 3.15. The $t$-Bargmann representation of $\nu_q$ is treated in [KW14] for $q = 1$ and [AKW16] for $0 \leq q < 1$.

Before closing this section, let us give a short remark about relations with the free infinite divisibility. Many of particular examples have so far suggested that the free infinite divisibility of a probability measure implies the existence of a radial Bargmann representation. The converse is not true in general because the Askey-Wimp-Kerov distribution $\mu_{9/10}$ for instance, discussed in [BBLS11], is not freely infinitely divisible, but it has a Bargmann representation with a gamma distribution as its radial measure. However, not many counterexamples have been found.

Therefore, we conjecture that the free infinite divisibility of our $(\alpha, q)$-Gaussian distribution is equivalent to the existence of its radial Bargmann measure:

**Conjecture.** Suppose that $\alpha, q \in (-1, 1)$. The probability measure $\nu_{\alpha, q}$ is freely infinitely divisible if and only if $\alpha = q$ or $\alpha < q \geq 0$.

This conjecture is guaranteed to be true in the restricted subfamilies $\{\nu_{\alpha, 0} \mid \alpha \in (-1, 1)\}$ ([SY01 Theorem 3.2]), $\{\nu_{0, q} \mid -1 < q < 1\}$ ([ABBL10] and [AH13, Example 3.11] for the free infinite divisibility), and $\{\nu_{q, q} \mid q \in (-1, 1)\}$ (all measures in this family are freely infinitely divisible since they are $q^2$-Gaussians).

### 4 Commutation relations among one-mode $(\alpha, q)$-operators

**Definition 4.1.** Suppose that $\alpha, q \in (-1, 1)$ and $f$ is analytic on $\mathbb{C}$.

1. Let $Z$ be the multiplication operator defined by
   \[(Zf)(z) := z f(z).\]

2. Let $D_q$ be the Jackson derivative given by
   \[(D_qf)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1 - q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}\]

3. The $\alpha$-deformed Jackson derivative is given as
   \[D_{\alpha,q} := \begin{cases} D_q + \alpha q^2 N D_{1/q}, & q \neq 0, \\ D_0 + \alpha \frac{d}{dz} \bigg|_0, & q = 0, \end{cases}\]
   where $N$ is the number operator. For $q \neq 0$, we can also write
   \[D_{\alpha,q} = D_q + \frac{\alpha}{q^2} D_{1/q} q^{2N}.\]

**Remark 4.2.** It is easy to check that the $\alpha$-deformed Jackson derivative is equivalently defined as
   \[(D_{\alpha,q}f)(z) = (D_qf)(z) + \alpha (D_{1/q}f)(q^2z), \quad q \neq 0.\]

For example, if $f(z) = z^n$, $(D_{\alpha,q}f)(z) = (1 + \alpha q^{n-1})[n]_q z^{n-1}$ holds. In fact, the $\alpha$-deformed Jackson derivative is an analogue of the operator in [BEH15, Theorem 2.5].

Then, one can realize one-mode analogue of $(\alpha, q)$-operators on an appropriate domain of the one-mode interacting Bargmann-Fock space $B$ with $\omega_n = (1 + \alpha q^{n-1})[n]_q z^{n-1}$ holds for $n = 0$ by
\[a^+ := Z, \quad a^- := D_{\alpha,q}, \quad \text{and} \quad \Phi_n := \frac{z^n}{\sqrt{[\omega_n]!}}.\]

In fact, it is easy to check that
\[
\begin{align*}
a^+ \Phi_n &= \sqrt{\omega_{n+1}} \Phi_{n+1}, \\
(a^- \Phi_n &= \sqrt{\omega_n} \Phi_{n-1},
\end{align*}
\]
hold and the $q$-commutation relation, one-mode analogue of \( \{A, A\} \),

\[
[a^-, a^+]_q \Phi_n := (a^- a^+ - q a^+ a^-) \Phi_n
= (I + \alpha q^{2n}) \Phi_n,
\]

is satisfied. Let us put

\[
M_{\alpha, q} = I + \alpha q^{2N}
\]

and then one can get the expression,

\[
M_{\alpha, q} = (1 + \alpha) I - \alpha(1 - q^2) ZD_{q^2},
\]

de to \((ZD_{q^2}) \Phi_n = [n]_{q^2} \Phi_n\).

Therefore one can obtain the following

**Theorem 4.3.** Suppose \( \alpha \in (-1, 1) \) and \( q \in (-1, 1) \). Then the following are satisfied.

1. \( [a^-, a^+]_q = M_{\alpha, q}, \ [a^-, M_{\alpha, q}]_q = (1 - q^2)a^-, \ [M_{\alpha, q}, a^+]_q = (1 - q^2)a^+ \).

2. \( M_{\alpha, q} = (1 + \alpha) I - \alpha(1 - q^2) ZD_{q^2}. \)

3. In particular, if \( \alpha = q \), then one can obtain a more refined relation, \( [a^-, a^+]_q = (1 + q) I \).

**Example 4.4.** (1) \( \alpha = 0 \) implies \( [a^-, a^+]_q = I \). Hence \( M_{0, q} = I \) commutes with both \( a^+ \) and \( a^- \),

\[
[a^-, M_{0, q}]_1 = [M_{0, q}, a^+]_1 = 0.
\]

Therefore, the case \( \alpha \neq 0 \) provides non-trivial commutation relations.

(2) If \( \alpha = -q^{2\beta} \) for \( \beta > 0 \), then the limiting case of the scaled operator is obtained as

\[
\lim_{q \uparrow 1} M_{-q^{2\beta}, q} = \lim_{q \uparrow 1} \frac{I - q^{2\beta} q^{2N}}{1 - q^2} = N + \beta.
\]

Moreover, let us consider the scaled operators,

\[
A^\pm := \lim_{q \uparrow 1} \frac{a^{\pm}}{\sqrt{1 - q^2}}.
\]

Then one can get

\[
[A^-, A^+]_1 = N + \beta
\]

and hence

\[
[A^-, N]_1 = A^-, \ [N, A^+]_1 = A^+.
\]

It should be noted that these are the commutation relations for the classical Meixner-Pollaczek polynomials with respect to the symmetric Meixner distribution in \( 3.4 \). See As08.

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**A Appendix**

Let \( \Sigma_n \) be the set of bijections \( \sigma \) of the \( 2n \) points \( \{\pm 1, \pm 2, \cdots, \pm n\} \) with \( \sigma(-k) = -\sigma(k) \). Equipped with the composition operation as a product, \( \Sigma_n \) becomes what is called a Coxeter group of type B. It is generated by \( \pi_0 := (1, -1) \) and \( \pi_i := (i, i+1), \ 1 \leq i \leq n-1 \), which satisfy the generalized braid relations

\[
\begin{align*}
\pi_i^2 &= e, & 0 \leq i \leq n-1, \\
(\pi_0 \pi_1)^4 &= (\pi_i \pi_{i+1})^3 = e, & 1 \leq i \leq n-1, \\
(\pi_i \pi_j)^2 &= e, & |i-j| \geq 2, \ 0 \leq i, j \leq n-1. 
\end{align*}
\]
An element $\sigma \in \Sigma_n$ expresses an irreducible form,
\[ \sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \leq i_1, \ldots, i_k \leq n - 1, \]
and in this case
\[ \ell_1(\sigma) := \text{the number of } \pi_0 \text{ in } \sigma, \]
\[ \ell_2(\sigma) := \text{the number of } \pi_i, \quad 1 \leq i \leq n - 1, \quad \text{in } \sigma \]
are well defined. Let $H$ be a separable Hilbert space. For a given self-adjoint involution $f \mapsto \overline{f}$ for $f \in H$, an action of $\Sigma_n$ on $H^{\otimes n}$ is defined by
\[
\begin{aligned}
\pi_0(f_1 \otimes \cdots \otimes f_n) &= \overline{f_1} \otimes f_2 \cdots \otimes f_n, \\
\pi_i(f_1 \otimes \cdots \otimes f_n) &= f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \cdots \otimes f_n, & n \geq 1, \\
\pi_i(f_1 \otimes \cdots \otimes f_n) &= f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \cdots \otimes f_n, & n \geq 2, \quad 1 \leq i \leq n - 1.
\end{aligned}
\]
(A.2)

The $(\alpha, q)$-inner product on the full Fock space $F(H)$ is defined by
\[
\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha, q} := \delta_{m,n} \sum_{\sigma \in \Sigma_n} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \prod_{j=1}^n \langle f_j, g_{\sigma(j)} \rangle_H, \quad \alpha, q \in (-1, 1) \tag{A.3}
\]
with conventions $\delta^0 = 1$ and $g_{-k} = \overline{g_k}$, $k = 1, 2, \ldots, n$. For example, if one may define the involution as $\overline{f} := -f$, then $g_{-k} = -g_k$. Equipped with this inner product the full Fock space $F(H)$ is denoted by $F_{\alpha, q}(H)$ to emphasize on the dependence of the inner product on $\alpha, q$.

The $(\alpha, q)$-creation operator $B_{\alpha, q}^+(f)$ is the usual left creation operator on the full Fock space, and the $(\alpha, q)$-annihilation operator $B_{\alpha, q}^-(f)$ is its adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{\alpha, q}$. They satisfy the commutation relation
\[
B_{\alpha, q}^-(f) B_{\alpha, q}^+(g) - q B_{\alpha, q}^+(g) B_{\alpha, q}^-(f) = \langle f, g \rangle_H I + \alpha \langle \overline{f}, g \rangle_H q 2^N, \quad f, g \in H. \tag{A.4}
\]

The readers can consult [BEH15] for details.

References


