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Radial Bargmann representation for the Fock space of type B

Nobuhiro ASAI‡, Marek BOZEJKO† and Takahiro HASEBE‡

Abstract

Let $\nu_{\alpha,q}$ be the probability and orthogonality measure for the $q$-Meixner-Pollaczek orthogonal polynomials, which has appeared in [BEH15] as the distribution of the $(\alpha,q)$-Gaussian process (the Gaussian process of type B) over the $(\alpha,q)$-Fock space (the Fock space of type B). The main purpose of this paper is to find the radial Bargmann representation of $\nu_{\alpha,q}$. Our main results cover not only the representation of $q$-Gaussian distribution by [LM95], but also of $q^2$-Gaussian and symmetric free Meixner distributions on $\mathbb{R}$. In addition, non-trivial commutation relations satisfied by $(\alpha,q)$-operators are presented.

Keywords: Radial Bargmann representation, deformation, Fock spaces, $q$-orthogonal polynomials.

2010 Mathematics Subject Classification: 33D45, 46L53, 60E99.

1 Introduction

Bożejko-Ejsmont-Hasebe [BEH15] considered a deformation of the (algebraic) full Fock space with two parameters $\alpha$ and $q$, namely, the $(\alpha,q)$-Fock space (or the Fock space of type B) $\mathcal{F}_{\alpha,q}(H)$ over a complex Hilbert space $H$. The deformation with $\alpha = 0$ is equivalent to the $q$-deformation by Bożejko-Speicher [BS91] and Bożejko-Kümmerer-Speicher [BKS97], and the corresponding $q$-Bargmann-Fock space has been constructed by van Leeuwen-Maassen [LM95].

For the construction of $\mathcal{F}_{\alpha,q}(H)$, their starting point is to replace the Coxeter group of type A, that is, symmetric group $S_n$ for the $q$-Fock space by the Coxeter group of type B, $\Sigma_n := Z_2^n \rtimes S_n$ in (A.1) of the Appendix A. This replacement provides us a more general symmetrization operator on $H^{\otimes n}$ than that of [BS91] to define the $(\alpha,q)$-inner product $\langle \cdot, \cdot \rangle_{\alpha,q}$ in (A.3). One can define annihilation $B_{\alpha,q}^-(f)$ and creation $B_{\alpha,q}^+(f)$ operators acting on $\mathcal{F}_{\alpha,q}(H)$, and the $(\alpha,q)$-Gaussian process (the Gaussian process of type B) $G_{\alpha,q}(f)$ for $f \in H$ as the sum of them, $G_{\alpha,q}(f) := B_{\alpha,q}^-(f) + B_{\alpha,q}^+(f)$. It is one of their main interests to find a probability distribution $\mu_{\alpha,q,f}$ on $\mathbb{R}$ of $G_{\alpha,q}(f)$, $\|f\|_1 = 1$, with respect to the vacuum state $(\Omega, \Omega)_{\alpha,q}$. $\mathcal{F}_{\alpha,q}(H)$ equipped with $\langle \cdot, \cdot \rangle_{\alpha,q}$, $B_{\alpha,q}^-(f)$, and $B_{\alpha,q}^+(f)$ is a typical example of interacting Fock spaces in the sense of Accardi-Bożejko [AB98]. It suggests that the theory of orthogonal polynomials plays intrinsic roles in all previous works mentioned above. In fact, the measure $\mu_{\alpha,q,f}$ given in [BEH15] Theorem 3.3] is derived essentially from the orthogonality measure $\nu_{\alpha,q}$ associated with the $q$-Meixner-Pollaczek orthogonal polynomials $\{P^{(\alpha,q)}_n(x)\}$ for $\alpha, q \in (-1,1)$ given by the recurrence relation,

\[
\begin{align*}
P^{(\alpha,q)}_0(x) &= 1, & P^{(\alpha,q)}_1(x) &= x, \\
2P^{(\alpha,q)}_n(x) &= P^{(\alpha,q)}_{n+1}(x) + (1 + \alpha q^{n-1})[q]_q P^{(\alpha,q)}_{n-1}(x), & n &\geq 1
\end{align*}
\]

where $[q]_q = 1 + q + \cdots + q^{n-1}$ is the $q$ number. However, the Bargmann representation (measure on $C$) of $\nu_{\alpha,q}$ has not been obtained yet except the case of $\alpha = 0$ for $0 \leq q < 1$ [LM95], for $q = 1$ [Barg61] [AKK03], for $q = 0$ [B97], and $t$-deformed cases of these [AKW10] [KW11], and for $q > 1$ [Kr98].

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Therefore, the main purpose of this paper is to find the radial Bargmann representation of the probability measure \( \nu_{\alpha,q} \) on \( \mathbb{R} \). Our results cover the radial Bargmann representations of \( q \)-Gaussian, symmetric free Meixner (Kesten) and \( q^2 \)-Gaussian distributions on \( \mathbb{R} \).

The organization of this paper will be as follows. In Section 2 we shall explain how the \((\alpha,q)\)-Fock space is related to the notion of one-mode interacting Fock spaces and Bargmann representation. In Section 3, the radial Bargmann representation of \( \nu_{\alpha,q} \) is constructed explicitly in Theorem 3.1. In Section 4 commutation relations satisfied by one-mode \((\alpha,q)\)-annihilation and creation operators will be treated. In the Appendix, we shall give a minimum reference on the Coxeter group of type B extracted from [BEH15].

2 Key ideas and our purpose

Let us point out some of the keys to calculate the distribution of \( G_{\alpha,q}(f) \) in [BEH15]. It is shown that a linear map, \( \Phi : \text{Span}\{f^{\otimes n} \mid f \in H, \ n \geq 0\} \to L^2(\mathbb{R}, \mu_{\alpha,q,f}) \) given by \( \Phi(f^{\otimes n}) = P_n^{(\alpha,f)}(\mu_{\alpha,q,f}) \), is an isometry and a relation under \( \|f\|_H = 1 \),

\[
G_{\alpha,q}(f)f^{\otimes n} = (B_{\alpha,q}^+(f) + B_{\alpha,q}^-(f))(f^{\otimes n})
\]

\[
= f^{\otimes (n+1)} + (1 + \alpha(f,\overline{f}))q^{n-1}[n]_q f^{\otimes (n-1)},
\]

is satisfied where \( \overline{f} \) denotes a self-adjoint involution of \( f \in H \) in (A.2). This corresponds to the three terms recursion relation satisfied by \( P_n^{(\alpha,f)}(\mu_{\alpha,q,f}) \) through \( \Phi \). Then, it is proved that \( \nu_{\alpha,q,f} = \nu_{(f,\overline{f})} \) (see \( \nu_{\alpha,q} \) in 3.3) in the sense of

\[
\langle \Omega, G_{\alpha,q}(f)^n \Omega \rangle_{\alpha,q} = \int x^n d\mu_{\alpha,q,f}(dx)
\]

(2.1)

where \( \Omega \) denotes the vacuum vector. Therefore, in order to get the Bargmann representation of \( \nu_{(f,\overline{f})} \), it is enough to consider that of \( \nu_{\alpha,q} \) in the sense of Definition 2.2 given later.

Since the structure mentioned above can be reduced to the one-mode analogue of \((\alpha,q)\)-Fock spaces, let us recall fundamental relationships between one-mode interacting Bargmann-Fock spaces and the theory of orthogonal polynomials of one variable.

Definition 2.1. Let \( \{\omega_n\}_{n=0}^{\infty} \) with \( \omega_0 := 1 \) be an infinite sequence of positive real numbers and \( \{\alpha_n\}_{n=0}^{\infty} \) be of real numbers. A one-mode interacting Bargmann-Fock space \( \mathcal{B} \) is defined as \( \bigoplus_{n=0}^{\infty} \mathbb{C} \Phi_n \) equipped with \( \Phi_n := z^n/|\omega_n|!\), \( \{\omega_n|! := \prod_{k=0}^{n} \omega_k \) the inner product \( \langle \Phi_m, \Phi_n \rangle_{\mathcal{B}} = \delta_{m,n} \) for all \( m,n \in \mathbb{N} \cup \{0\} \), operators of creation \( a^+ \), annihilation \( a^- \), and conservation \( a^0 \) defined by

\[
\begin{align*}
& a^+ \Phi_n := \sqrt{\omega_{n+1}} \Phi_{n+1}, & n \geq 0, \\
& a^- \Phi_n := \sqrt{\omega_{n-1}} \Phi_{n-1}, & n \geq 1, \\
& a^0 \Phi_n := \alpha_n \Phi_n, & n \geq 0.
\end{align*}
\]

(2.2)

Let \( \{\{\omega_n\}_{n=0}^{\infty}, \{\alpha_n\}_{n=0}^{\infty}\} \) be a pair of sequences in Definition 2.1 and define a sequence of monic polynomials \( \{P_n(x)\} \) recurrently by

\[
\begin{align*}
P_0(x) &= 1, \ P_1(x) = x - \alpha_0, \\
xP_n(x) &= P_{n+1}(x) + \omega_n P_{n-1} + \alpha_n P_n(x), & n \geq 1.
\end{align*}
\]

(2.3)

Then there exists a probability measure \( \mu \) on \( \mathbb{R} \) with finite moments of all orders such that \( \{P_n(x)\} \) is the orthogonal polynomials with \( \langle P_m(x), P_n(x) \rangle_{L^2(\mathbb{R},\mu)} = \delta_{m,n}|\omega_n|! \) for all \( m,n \in \mathbb{N} \cup \{0\} \). (See [Ch78], [HO07], for example.)

It is easy to see that a linear map

\[
U : \mathcal{B} = \bigoplus_{n=0}^{\infty} \mathbb{C} \Phi_n \to L^2(\mathbb{R}, \mu)
\]
defined by $U \left( \sqrt{[\omega_n]} \Phi_n \right) = P_n(x)$ is an isometry and in addition $a^+ + a^- + a^\circ = U^*XU$ is satisfied due to (2.2) and (2.3), where $X$ represents the multiplication operator by $x$ in $L^2(\mathbb{R}, \mu)$. This intertwining relation provides a notion of the quantum decomposition of a classical random variable $X$ and

$$\langle \Phi_0, (a^+ + a^- + a^\circ)^n \Phi_0 \rangle_B = \int x^n \mu(dx).$$  \hfill (2.4)

Therefore, if $\omega_n = (1 + \alpha q^{n-1})[n]_q$, $\alpha_n = 0$, the equality in (2.4) is a one-mode analogue of (2.1).

Now it is interesting to consider the following moment problem to realize the inner product by the integral:

**Problem 1.** For a given $\{\omega_n\}$ of $\mu$, find a probability measure $\gamma_\mu$ satisfying the equality,

$$\int_{\mathbb{C}} z^n \gamma_\mu(d^2z) = \delta_{m,n} [\omega_n]!$$  \hfill (2.5)

for all $m, n \in \mathbb{N} \cup \{0\}$.

**Definition 2.2.** A measure $\gamma_\mu$ satisfying the equality (2.5) is called a Bargmann representation (measure on $\mathbb{C}$) of a measure $\mu$ on $\mathbb{R}$.

It was proved in [Sz07] (see also [AKW16] [KW14]) that if a measure $\mu$ admits any Bargmann representation, then it also admits a radial (rotation invariant) Bargmann representation

$$\gamma_\mu(d^2z) = \frac{1}{2\pi} \lambda_{[0,2\pi]}(d\theta) \rho_\mu(dr), \quad z = re^{i\theta}, \quad r \geq 0, \quad \theta \in [0, 2\pi),$$

where $\lambda_{[0,2\pi)}$ is the Lebesgue measure on $[0, 2\pi)$. It says that the angular part takes care of orthogonality of (2.5). Therefore, Problem 1 can be transformed into the following Problem 2:

**Problem 2.** Find a positive radial measure $\rho_\mu$ satisfying

$$\int_0^\infty r^{2n} \rho_\mu(dr) = [\omega_n]!$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

**Main Purpose:** We shall consider Problem 2 associated with $\omega_n = (1 + \alpha q^{n-1})[n]_q$, $\alpha_n = 0$ of $\nu_{\alpha,q}$ in Section 3. Furthermore, commutation relations satisfied by $a^+$, $a^-$ acting on $B$ associated with $\omega_n = (1 + \alpha q^{n-1})[n]_q$ will be presented in Section 4.

**Remark 2.3.** (1) One can notice that $\gamma_\mu$ and $\rho_\mu$ are determined only by $[\omega_n]!$. Therefore, it is enough in general for the Bargmann representation in the above sense to consider the symmetric measure $\mu$ with $\alpha_n = 0$ for all $n$, which implies that $a^\circ$ is a zero operator.

(2) If $\mu$ is symmetric, then $\alpha_n = 0$ for all $n$ is implied. The converse statement is true if $\mu$ is determined by its moments.

3  \quad (\alpha, q)$\text{-}$Bargmann representation

3.1  \quad q$\text{-}$Meixner-Pollaczek polynomials

Let us recall standard notations from $q$-calculus, which can be found in [GR04] [KLS10] for example. The $q$-shifted factorials are defined by

$$(a; q)_0 := 1, \quad (a; q)_k := \prod_{\ell=1}^{k} (1 - aq^{\ell-1}), \quad k = 1, 2, \ldots, \infty,$$

and the product of $q$-shifted factorials is defined by

$$(a_1, a_2; q)_k := (a_1; q)_k(a_2; q)_k, \quad k = 1, 2, \ldots, \infty.$$
Remark 3.1. The \(q\)-shifted factorials are a natural extension of the Pochhammer symbol \((\cdot)_n\) because one can see that \(\lim_{q \to 1} [k]_q = k\) implies
\[
\lim_{q \to 1} \frac{(q^n; q)_n}{(1 - q)^n} = (k)_n, \tag{3.1}
\]
where \((k)_0 := 1, (k)_n := (k+1) \cdots (k+n), n \geq 1.\)

As we have mentioned, \(\{P_n^{(\alpha,q)}(x)\}\) for \(\alpha, q \in (-1, 1)\) is the \(q\)-Meixner-Pollaczek polynomials satisfying the recurrence relation,
\[
\begin{cases}
P_0^{(\alpha,q)}(x) = 1, & P_1^{(\alpha,q)}(x) = x, \\
\alpha x P_n^{(\alpha,q)}(x) = P_n^{(\alpha,q)}(x) + (1 + \alpha q^{n-1})[n]_q P_{n-1}^{(\alpha,q)}(x), & n \geq 1. 
\end{cases} \tag{3.2}
\]

It is known in [KLS10, 14.9.2] and [BEH15, page 1781] that the orthogonality measure \(\nu_{\alpha,q}\) for such polynomials has the density of the form,
\[
\frac{q, \gamma^2; q)_\infty}{2\pi \sqrt{4 - (1 - q)x^2}} \frac{g(x, 1; q)g(x, -1; q)g(x, \sqrt{q}; q)g(x, -\sqrt{q}; q)}{g(x, i\gamma; q)g(x, -i\gamma; q)}, \tag{3.3}
\]
supported on the interval \((-2/\sqrt{1-q}, 2/\sqrt{1-q})\) where
\[
g(x, b; q) = \prod_{k=0}^\infty (1 - 4bx(1-q)^{-1/2}q^k + b^2 q^{2k}),
\]
and
\[
\gamma = \left\{ \begin{array}{ll}
\sqrt{-\alpha}, & \alpha < 0, \\
(i\sqrt{\alpha}), & \alpha \geq 0.
\end{array} \right.
\]

Example 3.2. (1) If \(\alpha = 0\), then \(q\)-Meixner-Pollaczek polynomials get back to the \(q\)-Hermite polynomials \(H_n^{(0)}(x)\) whose orthogonality measure is the standard \(q\)-Gaussian measure on \((-2/\sqrt{1-q}, 2/\sqrt{1-q})\) given by
\[
\nu_q(dx) := \frac{\sqrt{1-q}}{\pi} \sin \theta \prod_{n=1}^\infty (1 - q^n)|1 - q^n e^{2i\theta}|^2 dx,
\]
where \(x\sqrt{1-q} = 2\cos \theta, \theta \in [0, \pi]\). Furthermore, one can get the standard Gaussian law as \(q \to 1\), the Bernoulli law as \(q \to -1\), and the standard Wigner’s semi-circle law if \(q = 0\). See [BKS97] [BS01].

(2) The measure \(\nu_{\alpha,0}\) is the symmetric free Meixner law [Am03] [BB06] [SY01].

(3) The measure \(\nu_{0,q}\) is the \(q^2\)-Gaussian law scaled by \(\sqrt{1+q}\).

(4) If \(\alpha = -q^{2\beta}\) as suggested in Remark [3.1] then the measure \(\nu_{-q^{2\beta},q}\) under a certain scaling converges to the classical symmetric Meixner law as \(q \uparrow 1,\)
\[
\frac{2^{2\beta}}{2\pi} |\Gamma(\beta + ix)|^2 dx, \quad x \in \mathbb{R}. \tag{3.4}
\]

See also [KLS10] 14.9.15.

3.2 Problem

For \(\alpha, q \in (-1, 1)\), we would like to know when there exists a radial measure \(\rho_{\nu_{\alpha,q}}\) satisfying
\[
\int_0^\infty r^{2k} \rho_{\nu_{\alpha,q}}(dr) = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}. \tag{3.5}
\]

Here \([k]_q!\) denotes the \(q\)-factorials defined by
\[
[0]_q! := 1, \quad [k]_q! := \prod_{\ell=1}^k [\ell]_q = \frac{(q; q)_k}{(1-q)^k}, \quad k \geq 1.
\]
It is easy to get the inequality for \( \alpha, q \in (-1, 1) \),
\[
|(-\alpha; q)_k[k]_q!| \leq \left(\frac{4}{1 - |q|}\right)^k, \quad k \in \mathbb{N} \cup \{0\}.
\] (3.6)

Due to Carleman criterion for the moment problem, this inequality implies that a radial measure \( \rho_{\alpha,q} \) is determined uniquely by the sequence \( \{(-\alpha; q)_k[k]_q!\} \).

We shall follow the procedure below to construct \( \rho_{\alpha,q} \) in (3.5).

1. Recall the radial part of the \( q \)-Gaussian measure on \( \mathbb{C} \) (\( q \)-Bargmann measure), \( \rho_{\nu_q} = \rho_{\nu_0,q} \), obtained in [LM95],
\[
\int_0^\infty r^{2k} \rho_{\nu_q}(dr) = [k]_q!.
\] (3.7)

2. Find a radial (possibly signed) measure \( \rho_{\alpha,q} \) having the moment \( (-\alpha; q)_k \).

3. Compute the multiplicative (Mellin) convolution \( \rho_{\nu_q} \ast \rho_{\alpha,q} \) to get \( \rho_{\nu_\alpha,q} \).

Remark 3.3. It is known [LM95] that a radial measure \( \rho_{\nu_q} \) in (3.7) does not exist for \( q < 0 \). However, one can see that the positivity assumption on \( q \) can be relaxed for \( \rho_{\nu_\alpha,q} \) if \( \alpha = q \). It will be discussed right after the proof of Proposition 3.6 and in Proposition 3.7.

3.3 Construction of \((\alpha, q)\)-radial measures

Lemma 3.4. Suppose that \( \alpha \in (-1, 1) \) and \( q \in [0, 1) \). Let
\[
\rho_{\alpha,q} := (-\alpha; q)_\infty \sum_{n=0}^\infty \frac{(-\alpha)^n}{(q;q)_n} \delta_{q^n/z},
\]
which is a signed measure. Then we have
\[
\int_0^\infty r^{2k} \rho_{\alpha,q}(dr) = (-\alpha; q)_k, \quad k \in \mathbb{N} \cup \{0\}.
\]

In particular, if taking \( \alpha = -q \), then one can see \( \rho_{\nu_q} = D_{(1-q)^{-1/2}}(\rho_{-q,q}) \), namely,
\[
\int_0^\infty r^{2k} D_{(1-q)^{-1/2}}(\rho_{-q,q})(dr) = \frac{(q;q)_k}{(1-q)^k} = [k]_q!.
\]

where \( D_l(\lambda) \) is the push-forward of \( \lambda \) by the map \( x \mapsto tx \) for a measure \( \lambda \) on \( \mathbb{R} \).

Proof. Firstly, we have
\[
\int_0^\infty r^{2k} \rho_{\alpha,q}(dr) = (-\alpha; q)_\infty \sum_{n=0}^\infty \frac{(-\alpha q^k)_n}{(q;q)_n}.
\]

Since Euler’s formula (see [GR04] 1.3.15]),
\[
\frac{1}{(a; q)_\infty} = \sum_{n=0}^\infty \frac{a^n}{(q; q)_n},
\] (3.8)
is known, we replace \( a \) by \( -\alpha q^k \) in (3.8) to obtain
\[
\int_0^\infty r^{2k} \rho_{\alpha,q}(dr) = \frac{(-\alpha; q)_\infty}{(-\alpha q^k; q)_\infty} = (-\alpha; q)_k.
\]

The proof is complete. \( \square \)
Remark 3.5. (1) The last equality in proof is due to the formula
\[(a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty}.
\]
See [GR04 1.2.30], for example.

(2) Euler’s formula is considered as the \(q\)-analogue of exponential function \(e^a\) due to
\[
\lim_{q \to 1} \frac{1}{(1-q)a; q)_n} = e^a.
\]

Let
\[
[n]_q! := \frac{[n]_q[1]}{[n]_q[1]} = \frac{(q; q)_n}{(q; q)_\ell(q; q)_{n-\ell}}
\]
be the \(q\)-binomial coefficients and \(h_n(z; q)\) be the Rogers-Szegő polynomials [GR04] [S05] defined by
\[
h_n(z; q) = \sum_{\ell=0}^{n} \binom{n}{\ell}_q z^\ell.
\]

Proposition 3.6. Suppose that \(\alpha \in (-1, 1)\) and \(q \in [0, 1)\). Let
\[
(\alpha, q; q)_{n+1}h_n(-\alpha q^{-1}|q) \delta_{(1-q)^{-1/2} q^{n/2}}, \quad q > 0,
\]
\[
-\alpha \delta_0 + (1 + \alpha) \delta_1, \quad q = 0,
\]
which is a signed measure in general. Then we have
\[
\int_0^\infty r^{2k} \rho_{\nu, q}(dr) = \frac{(-\alpha, q; q)_{k}}{(1-q)^k} = (-\alpha; q)_k[k]_q!, \quad k \in \mathbb{N} \cup \{0\}.
\]

Proof. First of all, it is easy to show (3.10) for the case \(q = 0\). Therefore, let us assume \(q > 0\).

One can compute the multiplicative (Mellin) convolution \(\otimes\) to get \(\rho_{\nu, q}\) as follows:
\[
\rho_{\nu, q} = \rho_{\alpha, q} \otimes D_{(1-q)^{-1/2} (\rho - \alpha, q)}
\]
\[
= (-\alpha, q; q)_{\infty} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(-\alpha)^k q^{\ell} q^{n-k}}{(q; q)_{\ell} (q; q)_{n-\ell}} \right) \delta_{(1-q)^{-1/2} q^{n/2}}
\]
\[
= (-\alpha, q; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_{\infty}} h_n(-\alpha q^{-1}|q) \delta_{(1-q)^{-1/2} q^{n/2}}.
\]

On the other hand, by Lemma 3.3, we have
\[
\int_0^\infty r^{2k} D_{(1-q)^{-1/2} (\rho - \alpha, q)}(dr) = \frac{(q; q)_{k}}{(1-q)^k} = [k]_q!.
\]

Therefore, one can get
\[
\int_0^\infty r^{2k} \rho_{\nu, q}(dr) = \int_0^\infty r^{2k} \rho_{\alpha, q}(dr) \int_0^\infty r^{2k} D_{(1-q)^{-1/2} (\rho - \alpha, q)}(dr)
\]
\[
= (-\alpha; q)_k[k]_q!, \quad k \in \mathbb{N} \cup \{0\}.
\]

In Proposition 3.6, we have obtained \(\rho_{\nu, q}\) for \(\alpha \in (-1, 1)\) and \(q \in (0, 1)\). Due to the term
\[
\delta_{(1-q)^{-1/2} q^{n/2}} \text{ in } \rho_{\nu, q},
\]
it seems impossible for \(q \in (-1, 0)\) to define \(\rho_{\nu, q}\). However, if \(-1 < \alpha = q < 0\) then \(\nu_{q, q}\) coincides with a scaled \(q^2\)-Gaussian measure, and hence the Bargmann measure exists.
Proposition 3.7. Suppose $-1 < \alpha = q < 0$. We define
\[ \rho_{\nu,q} := D_{(1+q)^{1/2}}(\rho_{\nu,q}) \]
\[ = (q^2; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} (1-q)^{-(1/2)-(q)n}. \] (3.11)

Then one can see
\[ \int_{0}^{\infty} r^{2k} \rho_{\nu,q}(dr) = (1 + q)^k |k|q^2! = (-q;q)_k |k|q^2!. \]

Proof. One can see by direct computations
\[ (-q;q)_k |k|q^2! = \left\{ \prod_{\ell=1}^{k} (1 - (-q)q^{\ell-1}) \right\} \left\{ \prod_{\ell=1}^{k} \frac{1 - q^{2\ell}}{1 - q^2} \right\} \]
\[ = (1 + q)^k \prod_{\ell=1}^{k} \frac{1 - q^{2\ell}}{1 - q^2} \]
\[ = (1 + q)^k |k|q^2! . \]

Thus $\rho_{\nu,q}$ can be defined as the radial measure for $q^2$-Gaussian measure on $\mathbb{C}$ scaled by $(1+q)^{1/2}$, namely, $\rho_{\nu,q} = D_{(1+q)^{1/2}}(\rho_{\nu,q})$.

Remark 3.8. If we use the fact that $h_n(-1 \mid q) = 0$ for odd $n \geq 1$ (see proof of Lemma 3.9 below), we can extend the definition (3.9) to the case $-1 < \alpha = q < 0$. This will give an alternative way to define $\rho_{\nu,q}$ for $-1 < q < 0$, but both definitions give the same measure.

We need some properties of the Rogers-Szegő polynomials to know when the measure $\rho_{\nu,q}$ becomes positive.

Lemma 3.9 (MGH90). Suppose that $q \in (-1,1)$.

1. If $n \geq 0$ is odd, then $h_n(x \mid q) \geq 0$ if and only if $x \geq -1$. Moreover, the point $x = -1$ is the unique zero of $h_n(x \mid q)$ on $\mathbb{R}$.

2. If $n \geq 0$ is even, then $h_n(x \mid q) > 0$ for all $x \in \mathbb{R}$.

Proof. It is known that all the zeros of $h_n(z \mid q)$ lie on the unit circle $|z| = 1$. See [MGH90] or [S05, Theorem 1.6.11]. Note that the result was obtained for $q \in [0,1)$, but the proof can be extended to $q \in (-1,1)$ without any modifications.

By definition, one can see
\[ \left[ \begin{array}{c} n \\ \ell \end{array} \right]_q = \frac{(1 - q^{n-\ell+1})(1 - q^{n-\ell+2}) \cdots (1 - q^n)}{(1 - q)(1 - q^2) \cdots (1 - q^\ell)} > 0, \]
and hence $h_n(1 \mid q) > 0$ for all $n \geq 0$. Thus, $h_n(x \mid q) \geq 0$ for $x \in \mathbb{R} \setminus \{ -1 \}$. It then suffices to show that $h_n(-1 \mid q) > 0$ for all even $n \geq 0$ and $h_n(-1 \mid q) = 0$ for all odd $n \geq 1$. The recurrence relation for the Rogers-Szegő polynomials is known to be
\[ h_{n+1}(z \mid q) = (z + 1)h_n(z \mid q) - (1 - q^n)zh_{n-1}(z \mid q), \quad n \geq 1. \] (3.12)

See [S05, 1.6.76] (note that formula (1.6.76) has an error of a sign). It is easy to see that $h_0(-1 \mid q) = 1 > 0, h_1(-1 \mid q) = 0$, so by induction and (3.12) one can show $h_n(-1 \mid q) > 0$ for all even $n \geq 0$ and $h_n(-1 \mid q) = 0$ for all odd $n \geq 1$.

We need the following lemma in proof of Theorem 3.11 for the non-existence part of a radial Bargmann measure.

Lemma 3.10. Let $\mu$ be a signed measure on $\mathbb{R}$ with compact support and let $\nu$ be a nonnegative measure on $\mathbb{R}$. If $\mu$ and $\nu$ have the same finite moments of all orders, then $\mu = \nu$. 

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Proof. We denote by $m_n$ the moments of $\mu$ (and $\nu$ by assumption). Since $\mu$ is compactly supported, say on $[-R,R],

|m_n| = \left| \int_{[-R,R]} x^n \mu(dx) \right| \leq \|\mu\| R^n, \quad n \in \mathbb{N} \cup \{0\},

where $\|\mu\|$ denotes the total variation of $\mu$. Therefore, $\nu$ is also supported on $[-R,R]$. By Weierstrass’ approximation, we have

$$\int_{[-R,R]} f(x) \mu(dx) = \int_{[-R,R]} f(x) \nu(dx) \quad (3.13)$$

for all $f \in C([-R,R])$. This implies that $\mu = \nu$ since, if we use the Hahn decomposition $\mu = \mu_+ - \mu_-$, then (3.13) implies

$$\int_{[-R,R]} f(x) \mu_+(dx) = \int_{[-R,R]} f(x) (\nu + \mu_-)(dx),$$

and hence $\mu_+ = \nu + \mu_-$ as nonnegative finite measures.

In summary, the complete answer to the unique existence of a radial Bargmann representation of $\nu_{\alpha,q}$ is stated as follows:

**Theorem 3.11.** Suppose that $\alpha, q \in (-1,1)$. The probability measure $\nu_{\alpha,q}$ has a radial Bargmann representation if and only if either (i) $q \geq 0$ and $\alpha \leq q$ or (ii) $\alpha = q \neq 0$.

In fact, the radial measure is given uniquely by

$$\rho_{\nu_{\alpha,q}} = \begin{cases} -\alpha \delta_0 + (1 + \alpha) \delta_1 & (\alpha \leq q = 0), \\ (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} \mid q) \delta_{(1-q)^{-1/2}|q|^{n}} & (q > 0, \alpha < q), \\ (q^2, q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q)^{-1/2}|q|^{n}} & (\alpha = q \neq 0). \end{cases}$$

**Proof.** 1. Existence and uniqueness. If $q \in [0,1)$ and $\alpha \leq q$, then by Proposition 3.6 and Lemma 3.9 the signed measure $\rho_{\nu_{\alpha,q}}$ is in fact a nonnegative measure and becomes the radial part of a Bargmann measure. The case $\alpha = q < 0$ was discussed in Proposition 3.7. Due to Carleman criterion for the moment problem, the inequality given in (3.14) guarantees the uniqueness of $\rho_{\nu_{\alpha,q}}$ for these cases.

2. Non-existence. (1) If $q \in (0,1)$ and $\alpha > q$, then $\rho_{\nu_{\alpha,q}}$ is not a nonnegative measure and is really a signed measure since $h_n(-\alpha/q \mid q) < 0$ for odd integers $n \geq 0$ and $q > 0$ from Lemma 3.9. By Lemma 3.10 if a radial Bargmann measure exists, then it must be equal to the signed measure $\nu_{\alpha,q}$. This is a contradiction. Thus, a radial Bargmann measure does not exist.

(2) If $q = 0$ and $\alpha > q = 0$ then by (3.14) $\nu_{0,0}$ is really a signed measure, and hence by the same argument as above, a radial Bargmann measure does not exist.

(3) Let

$$\beta_k(\alpha, q) := (-\alpha; q)k[k]_q^k, \quad k \geq 0, \alpha, q \in (-1,1).$$

Given $q < 0$ and $\alpha \neq q$, suppose that there exists a radial part of a Bargmann measure, $\rho$. Let $\rho^2$ be the push-forward of $\rho$ by the map $x \mapsto x^2$. Then,

$$\beta_k(\alpha, q) = \int_{0}^{\infty} x^k \rho^2(dx) = \int_{0}^{\infty} x^{2k} \rho(dx). \quad (3.14)$$

By the way, by Proposition 3.6 it holds that $\beta_k(\alpha, q') = \int_{0}^{\infty} x^{2k} \rho_{\nu_{\alpha,q'}}(dx)$ for any $q' > 0$, that is,

$$\beta_k(\alpha, q') = (-\alpha, q'; q')_\infty \sum_{n=0}^{\infty} \frac{(q')^n}{(q'; q')_n} h_n(-\alpha(q')^{-1} \mid q') \frac{(q')^{kn}}{(1-q')^k}, \quad q' > 0, \quad (3.15)$$

which is true even for $q' = q$ by analytic continuation.

Now let us consider the signed measure

$$\mu := (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} \mid q) \delta_{(1-q)^{-1/2}|q|^{n}}, \quad \alpha \neq q < 0,$$
supported on the points \( \frac{q^n}{t^n} \) for \( n = 0, 1, 2, 3, \ldots \). Then by (3.15) for \( q' = q \) and by (3.14),

\[
\int_{\mathbb{R}} x^k \mu(dx) = \beta_k(\alpha, q) = \int_{0}^{\infty} x^k \rho^2(dx), \quad k \in \mathbb{N} \cup \{0\}.
\]

By Lemma 3.10, the signed measure \( \mu \) and the probability measure \( \rho^2 \) should be equal. However, the support of \( \mu \) is not contained in \([0, \infty)\), and hence \( \mu \) cannot be equal to \( \rho^2 \). This is a contradiction. \( \Box \)

**Example 3.12.** (1) The radial measure \( \rho_{\nu_{0,q}} \) for \( q \in [0, 1) \) is of the \( q \)-Bargmann [LM95].

(2) The radial measure \( \rho_{\nu_{q,q}} \) for \( q \in (-1, 1) \) is of the \( q^2 \)-Bargmann.

(3) \( \lim_{q \uparrow 1} \rho_{\nu_{q,q}} \) is of the classical Bargmann [Barg61] [AKK03].

(4) Consider \( \alpha = -q^{2\beta}, \beta > 0 \). This choice of \( \alpha \) is suggested by (3.1) in Remark 3.1. In fact, one can see

\[
\lim_{q \uparrow 1} \frac{(1 - q^{2\beta + n - 1})|n|^2}{4(1 - q)} = \frac{1}{4}(n + 2\beta - 1)\pi^2.
\]

This limit sequence is the Jacobi sequence of the symmetric Meixner distribution in (3.4), so that \( \rho_{\nu_{q,q}} \) under suitable scaling converges weakly as \( q \uparrow 1 \) to the radial measure with the density,

\[
\frac{2\pi r}{T(2\beta)} \int_{0}^{\infty} h(t, r/4)e^{-t^2 + 1}dt
\]

where

\[
h(t, r) = \frac{1}{\pi t^2} \exp \left(-\frac{r^2}{2t} \right), \quad r \in \mathbb{R}, \quad t > 0.
\]

This is an integral representation of the radial density for the Bessel kernel measure, which can be also represented by the modified Bessel function [As05] [As09].

(5) \( \rho_{\nu_{q,q}} \) for \( \alpha \in (-1, 0) \) is the radial measure for the symmetric free Meixner distribution. See Remark 3.13 below.

**Remark 3.13.** Let \( \mu_t \) be a \( t \)-deformed probability measure of a probability measure \( \mu \) on \( \mathbb{R} \) defined through the Cauchy transform \( G_\mu \) of \( \mu \),

\[
\frac{1}{G_{\mu_t}(z)} := \frac{t}{G_\mu(z)} + (1 - t)z, \quad t \geq 0,
\]

examined by Bożejko-Wysoczanski [BW98] [BW01], Krystek-Wojakowski [KW14] discussed the radial Bargmann representation of a \( t \)-deformed probability measure \( \mu_t \), Bargmann representation for short, and obtained necessary and sufficient condition for the admissibility of the representation. The \( t \)-Bargmann representation of the Kesten measure \( \kappa_t \) has the form,

\[
\rho_{\kappa_t} = \left(1 - \frac{1}{t}\right) \delta_0 + \frac{1}{\sqrt{t}} \mu_{\kappa_t}, \quad t \geq 1.
\]

In [AKW16], the \( t \)-Bargmann representation of a symmetric free Meixner law \( \varphi_{s,t} \) with two positive parameters \( s, t \) is treated and is admitted if and only if \( t \geq 1 \). In fact, one can see \( \rho_{\varphi_{s,t}} = D_s(\rho_{\kappa_t}) \) and hence

\[
\rho_{\varphi_{(1-t)/x, 0}} = \rho_{\varphi_{1, t}} = D_{1/\sqrt{t}}(\rho_{\kappa_t}), \quad t \geq 1.
\]

Therefore, the case (5) in Example 3.12 can be viewed as a \( t \)-Bargmann representation, too.

Furthermore, let us state the \( t \)-deformed version of Theorem 3.11 for \( q \neq 0 \) without proof:

**Proposition 3.14.** The \( t \)-deformed version of \( \rho_{\nu_{q,q}} \) for \( q \neq 0 \) is given by

\[
\left(1 - \frac{1}{t}\right) \delta_0 + \frac{1}{\sqrt{t}} \rho_{\nu_{q,q}}, \quad t \geq 1.
\]
Remark 3.15. The $t$-Bargmann representation of $\nu_q$ is treated in [KW14] for $q = 1$ and [AKW16] for $0 \leq q < 1$.

Before closing this section, let us give a short remark about relations with the free infinite divisibility. Many of particular examples have so far suggested that the free infinite divisibility of a probability measure implies the existence of a radial Bargmann representation. The converse is not true in general because the Askey-Wimp-Kerov distribution $\mu_{9/10}$ for instance, discussed in [BBLS11], is not freely infinitely divisible, but it has a Bargmann representation with a gamma distribution as its radial measure. However, not many counterexamples have been found.

Therefore, we conjecture that the free infinite divisibility of our $(\alpha, q)$-Gaussian distribution is equivalent to the existence of its radial Bargmann measure:

**Conjecture.** Suppose that $\alpha, q \in (-1, 1)$. The probability measure $\nu_{\alpha, q}$ is freely infinitely divisible if and only if $\alpha = q$ or $\alpha < q \geq 0$.

This conjecture is guaranteed to be true in the restricted subfamilies $\{\nu_{\alpha, 0} \mid \alpha \in (-1, 1)\}$ ([SY01, Theorem 3.2]), $\{\nu_{0, q} \mid -1 < q < 1\}$ ([ABBL10] and [AH13, Example 3.11] for the free infinite divisibility), and $\{\nu_{q, q} \mid q \in (-1, 1)\}$ (all measures in this family are freely infinitely divisible since they are $q^2$-Gaussians).

4 Commutation relations among one-mode $(\alpha, q)$-operators

**Definition 4.1.** Suppose that $\alpha, q \in (-1, 1)$ and $f$ is analytic on $\mathbb{C}$.

(1) Let $Z$ be the multiplication operator defined by

$$(Zf)(z) := zf(z).$$

(2) Let $D_q$ be the Jackson derivative given by

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

(3) The $\alpha$-deformed Jackson derivative is given as

$$D_{\alpha, q} := \begin{cases} D_q + \alpha q^{2N}D_{1/q}, & q \neq 0, \\ D_0 + \alpha N |_0, & q = 0, \end{cases}$$

where $N$ is the number operator. For $q \neq 0$, we can also write

$$D_{\alpha, q} = D_q + \frac{\alpha}{q^n}D_{1/q}q^{2N}.$$ 

**Remark 4.2.** It is easy to check that the $\alpha$-deformed Jackson derivative is equivalently defined as

$$(D_{\alpha, q} f)(z) = (D_q f)(z) + \alpha(D_{1/q} f)(q^2z), \quad q \neq 0.$$ For example, if $f(z) = z^n$, $(D_{\alpha, q} f)(z) = (1 + \alpha q^{n-1})[n]_q z^{n-1}$ holds. In fact, the $\alpha$-deformed Jackson derivative is an analogue of the operator in [BEH15, Theorem 2.5].

Then, one can realize one-mode analogue of $(\alpha, q)$-operators on an appropriate domain of the one-mode interacting Bargmann-Fock space $B$ with $\omega_n = (1 + \alpha q^{n-1})|n|_q z^{n-1}$ and $\alpha_n = 0$ by

$$a^+ := Z, \quad a^- := D_{\alpha, q}, \quad \text{and} \quad \Phi_n := \frac{z^n}{\sqrt{\omega_n}}.$$ In fact, it is easy to check that

$$\begin{align*}
\alpha^+ \Phi_n &= \sqrt{\omega_{n+1}} \Phi_{n+1}, \\
\alpha^- \Phi_n &= \sqrt{\omega_n} \Phi_{n-1},
\end{align*}$$
hold and the \( q \)-commutation relation, one-mode analogue of \([A.A]\),
\[
[a^-, a^+]_q \Phi_n := (a^- a^+ - qa^+ a^-) \Phi_n = (I + \alpha q^{2N}) \Phi_n,
\]
is satisfied. Let us put \( M_{\alpha,q} = I + \alpha q^{2N} \) and then one can get the expression,
\[
M_{\alpha,q} = (1 + \alpha) I - \alpha(1 - q^2) ZD_2,
\]
due to \((ZD_2) \Phi_n = [n]_q^2 \Phi_n\).

Therefore one can obtain the following

**Theorem 4.3.** Suppose \( \alpha \in (-1,1) \) and \( q \in (-1,1) \). Then the following are satisfied.

1. \( [a^-, a^+]_q = M_{\alpha,q}, \quad [a^-, M_{\alpha,q}]_q = (1 - q^2) a^-, \quad [M_{\alpha,q}, a^+]_q = (1 - q^2) a^+ \).
2. \( M_{\alpha,q} = (1 + \alpha) I - \alpha(1 - q^2) ZD_2. \)
3. In particular, if \( \alpha = q \), then one can obtain a more refined relation, \( [a^-, a^+]_q = (1 + q) I. \)

**Example 4.4.** (1) \( \alpha = 0 \) implies \( [a^-, a^+]_q = I \). Hence \( M_{0,q} = I \) commutes with both \( a^+ \) and \( a^- \),
\[
[a^-, M_{0,q}]_1 = [M_{0,q}, a^+]_1 = 0.
\]

Therefore, the case \( \alpha \neq 0 \) provides non-trivial commutation relations.

(2) If \( \alpha = -q^{2\beta} \) for \( \beta > 0 \), then the limiting case of the scaled operator is obtained as
\[
\lim_{q \uparrow 1} \frac{M_{-q^{2\beta},q}}{I - q^{2\beta}} = \lim_{q \uparrow 1} \frac{I - q^{2\beta} q^{2N}}{1 - q^{2\beta}} = N + \beta.
\]

Moreover, let us consider the scaled operators,
\[
A^\pm := \lim_{q \uparrow 1} \frac{a^\pm}{\sqrt{1 - q^2}}.
\]

Then one can get
\[
[A^-, A^+]_1 = N + \beta
\]
and hence
\[
[A^-, N]_1 = A^-, \quad [N, A^+]_1 = A^+.
\]

It should be noted that these are the commutation relations for the classical Meixner-Pollaczek polynomials with respect to the symmetric Meixner distribution in \([3.4]\). See \([As08]\).

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**A Appendix**

Let \( \Sigma_n \) be the set of bijections \( \sigma \) of the \( 2n \) points \( \{\pm 1, \pm 2, \cdots, \pm n\} \) with \( \sigma(-k) = -\sigma(k) \). Equipped with the composition operation as a product, \( \Sigma_n \) becomes what is called a Coxeter group of type \( B \). It is generated by \( \pi_0 := (1, -1) \) and \( \pi_i := (i, i+1), 1 \leq i \leq n - 1, \) which satisfy the generalized braid relations

\[
\begin{align*}
\pi_i^2 &= e, \quad 0 \leq i \leq n - 1, \\
(\pi_0 \pi_1)^3 &= (\pi_i \pi_{i+1})^3 = e, \quad 1 \leq i \leq n - 1, \\
(\pi_i \pi_j)^2 &= e, \quad |i - j| \geq 2, 0 \leq i, j \leq n - 1.
\end{align*}
\]
An element $\sigma \in \Sigma_n$ expresses an irreducible form,

$$\sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \leq i_1, \ldots, i_k \leq n - 1,$$

and in this case

$$\ell_1(\sigma) := \text{the number of } \pi_0 \text{ in } \sigma,$$

$$\ell_2(\sigma) := \text{the number of } \pi_i, \quad 1 \leq i \leq n - 1, \text{ in } \sigma$$

are well defined. Let $H$ be a separable Hilbert space. For a given self-adjoint involution $f \mapsto \overline{f}$ for $f \in H$, an action of $\Sigma_n$ on $H^\otimes n$ is defined by

$$\begin{cases}
\pi_0(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes f_2 \otimes \cdots \otimes f_n, & n \geq 1, \\
\pi_i(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_{i-1} \otimes f_i \otimes f_{i+1} \otimes \cdots \otimes f_n, & n \geq 2, \quad 1 \leq i \leq n - 1.
\end{cases} \tag{A.2}$$

The $(\alpha, q)$-inner product on the full Fock space $\mathcal{F}(H)$ is defined by

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha, q} := \delta_{m,n} \sum_{\sigma \in \Sigma_n} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \prod_{j=1}^n \langle f_j, g_{\sigma(j)} \rangle_H, \quad \alpha, q \in (-1,1) \tag{A.3}$$

with conventions $0^0 = 1$ and $g_{-k} = \overline{g_k}$, $k = 1, 2, \ldots, n$. For example, if one may define the involution as $\overline{f} := -f$, then $g_{-k} = -g_k$. Equipped with this inner product the full Fock space $\mathcal{F}(H)$ is denoted by $\mathcal{F}_{\alpha, q}(H)$ to emphasize on the dependence of the inner product on $\alpha, q$.

The $(\alpha, q)$-creation operator $B_{\alpha, q}^+(f)$ is the usual left creation operator on the full Fock space, and the $(\alpha, q)$-annihilation operator $B_{\alpha, q}^-(f)$ is its adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{\alpha, q}$. They satisfy the commutation relation

$$B_{\alpha, q}^-(f)B_{\alpha, q}^+(g) - qB_{\alpha, q}^+(g)B_{\alpha, q}^-(f) = \langle f, g \rangle_H I + \alpha(f, g)_H q^{2N}, \quad f, g \in H. \tag{A.4}$$

The readers can consult [BEH15] for details.

References


