<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Radial Bargmann representation for the Fock space of type B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Asai, Nobuhiro; Bozejko, Marek; Hasebe, Takahiro</td>
</tr>
<tr>
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<tr>
<td><strong>Type</strong></td>
<td>article</td>
</tr>
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</table>

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Radial Bargmann representation for the Fock space of type B

Nobuhiro ASAI†, Marek BOŽEJKO‡ and Takahiro HASEBE†

Abstract

Let \( \nu_{\alpha,q} \) be the probability and orthogonality measure for the \( q \)-Meixner-Pollaczek orthogonal polynomials, which has appeared in BEH15 as the distribution of the \((\alpha,q)\)-Gaussian process (the Gaussian process of type B) over the \((\alpha,q)\)-Fock space (the Fock space of type B). The main purpose of this paper is to find the radial Bargmann representation of \( \nu_{\alpha,q} \). Our main results cover not only the representation of \( q \)-Gaussian distribution by LM95, but also of \( q^2 \)-Gaussian and symmetric free Meixner distributions on \( \mathbb{R} \). In addition, non-trivial commutation relations satisfied by \((\alpha,q)\)-operators are presented.

Keywords: Radial Bargmann representation, deformation, Fock spaces, \( q \)-orthogonal polynomials.

2010 Mathematics Subject Classification: 33D45, 46L53, 60E99.

1 Introduction

Bożejko-Łeśniewski-Hasebe BEH15 considered a deformation of the (algebraic) full Fock space with two parameters \( \alpha \) and \( q \), namely, the \((\alpha,q)\)-Fock space (or the Fock space of type B) \( \mathcal{F}_{\alpha,q}(H) \) over a complex Hilbert space \( H \). The deformation with \( \alpha = 0 \) is equivalent to the \( q \)-deformation by Bożejko-Speicher BS91 and Bożejko-Kümmerer-Speicher BKS97, and the corresponding \( q \)-Bargmann-Fock space has been constructed by van Leeuwen-Maassen LM95.

For the construction of \( \mathcal{F}_{\alpha,q}(H) \), their starting point is to replace the Coxeter group of type A, that is, symmetric group \( S_n \) for the \( q \)-Fock space by the Coxeter group of type B, \( \Sigma_n := \mathbb{Z}_2^n \times S_n \) in (A.3) of the Appendix A. This replacement provides us a more general symmetrization operator on \( H^{\otimes n} \) than that of BS91 to define the \((\alpha,q)\)-inner product \( \langle \cdot, \cdot \rangle_{\alpha,q} \) in (A.3). One can define annihilation \( B^-_{\alpha,q}(f) \) and creation \( B^+_{\alpha,q}(f) \) operators acting on \( \mathcal{F}_{\alpha,q}(H) \), and the \((\alpha,q)\)-Gaussian process (the Gaussian process of type B) \( G_{\alpha,q}(f) \) for \( f \in H \) as the sum of them. \( G_{\alpha,q}(f) := B^-_{\alpha,q}(f) + B^+_{\alpha,q}(f) \). It is one of their main interests to find a probability distribution \( \mu_{\alpha,q,f} \) on \( \mathbb{R} \) of \( G_{\alpha,q}(f) \), \( \|f\|_H = 1 \), with respect to the vacuum state \( \langle \Omega, \Omega \rangle_{\alpha,q} \). \( \mathcal{F}_{\alpha,q}(H) \) equipped with \( \langle \cdot, \cdot \rangle_{\alpha,q} \), \( B^-_{\alpha,q}(f) \), and \( B^+_{\alpha,q}(f) \) is a typical example of interacting Fock spaces in the sense of Accardi-Bożejko AB98. It suggests that the theory of orthogonal polynomials plays intrinsic roles in all previous works mentioned above. In fact, the measure \( \mu_{\alpha,q,f} \) given in BEH15 Theorem 3.3 is derived essentially from the orthogonality measure \( \nu_{\alpha,q} \) associated with the \( q \)-Meixner-Pollaczek orthogonal polynomials \( \{P^{(\alpha,q)}_n(x)\} \) for \( \alpha, q \in (-1,1) \) given by the recurrence relation,

\[
\begin{align*}
P^{(\alpha,q)}_0(x) &= 1, & P^{(\alpha,q)}_1(x) &= x, \\
P^{(\alpha,q)}_n(x) &= P^{(\alpha,q)}_{n-1}(x) + (1 + \alpha q^{n-1}) q^n P^{(\alpha,q)}_{n-1}(x), & n &\geq 1
\end{align*}
\]

where \( |n|_q = 1 + q + \cdots + q^{n-1} \) is the \( q \)-number. However, the Bargmann representation (measure on \( \mathbb{C} \)) of \( \nu_{\alpha,q} \) has not been obtained yet except the case of \( \alpha = 0 \) for \( 0 \leq q < 1 \) LM95, for \( q = 1 \) Barg61 AKBK03, for \( q = 0 \) BR94, and \( t \)-deformed cases of these AKW16 KW11, and for \( q > 1 \) Kr98.

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Therefore, the main purpose of this paper is to find the radial Bargmann representation of the probability measure \( \nu_{\alpha,q} \) on \( \mathbb{R} \). Our results cover the radial Bargmann representations of \( q \)-Gaussian, symmetric free Meixner (Kesten) and \( q^2 \)-Gaussian distributions on \( \mathbb{R} \).

The organization of this paper will be as follows. In Section 2 we shall explain how the \((\alpha,q)\)-Fock space is related to the notion of one-mode interacting Fock spaces and Bargmann representation. In Section 3, the radial Bargmann representation of \( \nu_{\alpha,q} \) is constructed explicitly in Theorem 3.11. In Section 4, commutation relations satisfied by one-mode \((\alpha,q)\)-annihilation and creation operators will be treated. In the Appendix, we shall give a minimum reference on the Coxeter group of type B extracted from [BEH15].

## 2 Key ideas and our purpose

Let us point out some of the keys to calculate the distribution of \( G_{\alpha,q}(f) \) in [BEH15]. It is shown that a linear map, \( \Phi : \text{Span}\{ f_{\otimes n} \mid f \in H, n \geq 0 \} \rightarrow L^2(\mathbb{R}, \mu_{\alpha,q,f}) \) given by \( \Phi(f_{\otimes n}) = P_n^{(\alpha, f, \mathcal{F}, H, q)}(x) \), is an isometry and a relation under \( \|f\|_H = 1 \),

\[
G_{\alpha,q}(f) f_{\otimes n} = (B^+_{\alpha,q}(f) + B^-_{\alpha,q}(f)) f_{\otimes n}
\]

\[
= f_{\otimes (n+1)} + (1 + \alpha(f, H, q)) q^{n-1} [n] q f_{\otimes (n-1)},
\]

is satisfied where \( \mathcal{F} \) denotes a self-adjoint involution of \( f \in H \) in (A.2). This corresponds to the three terms recursion relation satisfied by \( P_n^{(\alpha, f, \mathcal{F}, H, q)}(x) \) through \( \Phi \). Then, it is proved that \( \mu_{\alpha,q,f} = \nu_{\alpha(f, \mathcal{F}, H, q)} \) (see \( \nu_{\alpha,q} \) in [3.3]) in the sense of

\[
\langle \Omega, G_{\alpha,q}(f)^n \Omega \rangle_{\alpha,q} = \int x^n \mu_{\alpha,q,f}(dx)
\]

(2.1)

where \( \Omega \) denotes the vacuum vector. Therefore, in order to get the Bargmann representation of \( \nu_{\alpha(f, \mathcal{F}, H, q)} \), it is enough to consider that of \( \nu_{\alpha,q} \) in the sense of Definition 2.2 given later.

Since the structure mentioned above can be reduced to the one-mode analogue of \((\alpha,q)\)-Fock spaces, let us recall fundamental relationships between one-mode interacting Bargmann-Fock spaces and the theory of orthogonal polynomials of one variable.

**Definition 2.1.** Let \( \{\omega_n\}_{n=0}^{\infty} \) with \( \omega_0 := 1 \) be an infinite sequence of positive real numbers and \( \{\alpha_n\}_{n=0}^{\infty} \) be of real numbers. A one-mode interacting Bargmann-Fock space \( \mathcal{B} \) is defined as \( \bigoplus_{n=0}^{\infty} \mathbb{C} \Phi_n \) equipped with \( \Phi_n := z^n/|\omega_n|! \), \( |\omega_n|! := \prod_{k=0}^{n} \omega_k \), the inner product \( \langle \Phi_m, \Phi_n \rangle_\mathcal{B} = \delta_{m,n} \) for all \( m, n \in \mathbb{N} \cup \{0\} \), operators of creation \( a^+ \), annihilation \( a^- \), and conservation \( a^0 \) defined by

\[
\begin{align*}
    a^+ \Phi_n & := \sqrt{\omega_{n+1}} \Phi_{n+1}, & n \geq 0, \\
    a^- \Phi_n & := \sqrt{\omega_{n-1}} \Phi_{n-1}, & n \geq 1, \\
    a^0 \Phi_n & := \alpha_n \Phi_n, & n \geq 0.
\end{align*}
\]

Let \( \{\omega_n\}_{n=0}^{\infty}, \{\alpha_n\}_{n=0}^{\infty} \) be a pair of sequences in Definition 2.1 and define a sequence of monic polynomials \( \{P_n(x)\} \) recurrently by

\[
\begin{align*}
P_0(x) & = 1, & P_1(x) & = x - \alpha_0, \\
xP_n(x) & = P_{n+1}(x) + \omega_n P_{n-1} + \alpha_n P_n(x), & n \geq 1.
\end{align*}
\]

(2.3)

Then there exists a probability measure \( \mu \) on \( \mathbb{R} \) with finite moments of all orders such that \( \{P_n(x)\} \) is the orthogonal polynomials with \( \langle P_m(x), P_n(x) \rangle_{L^2(\mathbb{R}, \mu)} = \delta_{m,n} |\omega_n|! \) for all \( m, n \in \mathbb{N} \cup \{0\} \). (See [CH78], [HO07], for example.)

It is easy to see that a linear map

\[
U : \mathcal{B} = \bigoplus_{n=0}^{\infty} \mathbb{C} \Phi_n \rightarrow L^2(\mathbb{R}, \mu)
\]
defined by \( U \left( \sqrt{\omega_n} \Phi_n \right) = P_n(x) \) is an isometry and in addition \( a^+ a^- + a^0 = U^* X U \) is satisfied due to (2.2) and (2.3), where \( X \) represents the multiplication operator by \( x \) in \( L^2(\mathbb{R}, \mu) \). This intertwining relation provides a notion of the quantum decomposition of a classical random variable \( X \) and

\[
\langle \Phi_0, (a^+ + a^- + a^0)^n \Phi_0 \rangle_X = \int x^n \mu(dx).
\]

(2.4)

Therefore, if \( \omega_n = (1 + \alpha q^{n-1})[n]_q \), \( \alpha_n = 0 \), the equality in (2.4) is a one-mode analogue of (2.1).

Now it is interesting to consider the following moment problem to realize the inner product by the integral:

**Problem 1.** For a given \( \{\omega_n\} \) of \( \mu \), find a probability measure \( \gamma_\mu \) satisfying the equality,

\[
\int_{\mathbb{C}} z^n \gamma_\mu(d^2z) = \delta_{m,n}[\omega_n]!
\]

for all \( m, n \in \mathbb{N} \cup \{0\} \).

**Definition 2.2.** A measure \( \gamma_\mu \) satisfying the equality (2.5) is called a Bargmann representation (measure on \( \mathbb{C} \)) of a measure \( \mu \) on \( \mathbb{R} \).

It was proved in [Sz07] (see also [AKW16] [KW14]) that if a measure \( \mu \) admits any Bargmann representation, then it also admits a radial (rotation invariant) Bargmann representation

\[
\gamma_\mu(d^2z) = \frac{1}{2\pi} \lambda_{[0,2\pi]}(d\theta) \rho_\mu(dr), \quad z = re^{i\theta}, \quad r \geq 0, \quad \theta \in [0,2\pi),
\]

where \( \lambda_{[0,2\pi]} \) is the Lebesgue measure on \([0,2\pi]\). It says that the angular part takes care of orthogonality of (2.5). Therefore, Problem 1 can be transformed into the following Problem 2:

**Problem 2.** Find a positive radial measure \( \rho_\mu \) satisfying

\[
\int_0^\infty r^{2n} \rho_\mu(dr) = [\omega_n]!
\]

for all \( m, n \in \mathbb{N} \cup \{0\} \).

**Main Purpose:** We shall consider Problem 2 associated with \( \omega_n = (1 + \alpha q^{n-1})[n]_q\), \( \alpha_n = 0 \) of \( \nu_{\alpha,q} \) in Section 3. Furthermore, commutation relations satisfied by \( a^+, a^- \) acting on \( B \) associated with \( \omega_n = (1 + \alpha q^{n-1})[n]_q \) will be presented in Section 4.

**Remark 2.3.** (1) One can notice that \( \gamma_\mu \) and \( \rho_\mu \) are determined only by \( [\omega_n]! \). Therefore, it is enough in general for the Bargmann representation in the above sense to consider the symmetric measure \( \mu \) with \( \alpha_n = 0 \) for all \( n \), which implies that \( a^0 \) is a zero operator.

(2) If \( \mu \) is symmetric, then \( \alpha_n = 0 \) for all \( n \) is implied. The converse statement is true if \( \mu \) is determined by its moments.

3 \((\alpha,q)\)-Bargmann representation

3.1 \(q\)-Meixner-Pollaczek polynomials

Let us recall standard notations from \( q \)-calculus, which can be found in [GR04] [KLS10] for example. The \( q \)-shifted factorials are defined by

\[
(a;q)_0 := 1, \quad (a;q)_k := \prod_{\ell=1}^{k}(1 - aq^{\ell-1}), \quad k = 1, 2, \ldots, \infty,
\]

and the product of \( q \)-shifted factorials is defined by

\[
(a_1, a_2; q)_k := (a_1; q)_k(a_2; q)_k, \quad k = 1, 2, \ldots, \infty.
\]
Remark 3.1. The $q$-shifted factorials are a natural extension of the Pochhammer symbol $(\cdot)_n$ because one can see that $\lim_{q \to 1} [k]_q = k$ implies

$$
\lim_{q \to 1} \frac{(q^k; q)_n}{(1 - q)^n} = (k)_n,
$$

(3.1)

where $(k)_0 := 1$, $(k)_n := k(k + 1) \cdots (k + n - 1)$, $n \geq 1$.

As we have mentioned, $\{P_n^{(\alpha,q)}(x)\}$ for $\alpha, q \in (-1,1)$ is the $q$-Meixner-Pollaczek polynomials satisfying the recurrence relation,

$$
P_0^{(\alpha,q)}(x) = 1, \quad P_1^{(\alpha,q)}(x) = x,
$$

$$
P_n^{(\alpha,q)}(x) = P_n^{(\alpha,q)}(x) + (1 + \alpha q^{n-1})[n]_q P_{n-1}^{(\alpha,q)}(x), \quad n \geq 1.
$$

(3.2)

It is known in [KLS10 14.9.2] and [BEH15, page 1781] that the orthogonality measure $\nu_{\alpha,q}$ for such polynomials has the density of the form,

$$
\frac{(q, \gamma^2; q)_{\infty}}{2\pi} \sqrt{\frac{1 - q}{4 - (1 - q)x^2}} \left( \frac{g(x, 1; q)g(x, -1; q)g(x, \sqrt{q}; q)g(x, -\sqrt{q}; q)}{g(x, i\gamma; q)g(x, -i\gamma; q)} \right),
$$

(3.3)

supported on the interval $(-2/\sqrt{1-q}, 2/\sqrt{1-q})$ where

$$
g(x, b; q) = \prod_{k=0}^{\infty} (1 - 4bx(1 - q)^{-1/2}q^k + b^2q^{2k}),
$$

and

$$
\gamma = \begin{cases} \sqrt{-\alpha}, & \alpha < 0, \\ i\sqrt{-\alpha}, & \alpha \geq 0. \end{cases}
$$

Example 3.2. (1) If $\alpha = 0$, then $q$-Meixner-Pollaczek polynomials get back to the $q$-Hermite polynomials $H_n^{(q)}(x)$ whose orthogonality measure is the standard $q$-Gaussian measure on $(-2/\sqrt{1-q}, 2/\sqrt{1-q})$ given by

$$
\nu_q(dx) := \frac{\sqrt{1-q}}{\pi} \sin \theta \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2i\theta})^2 dx,
$$

where $x\sqrt{1-q} = 2\cos \theta$, $\theta \in [0, \pi]$. Furthermore, one can get the standard Gaussian law as $q \to 1$, the Bernoulli law as $q \to 1$, and the standard Wigner’s semi-circle law if $q = 0$. See [BKS97, BS00].

(2) The measure $\nu_{0,0}$ is the symmetric free Meixner law [An02] [BB06] [SY01].

(3) The measure $\nu_{q,0}$ is the $q^2$-Gaussian law scaled by $\sqrt{1+q}$.

(4) If $\alpha = -q^{2\beta}$ as suggested in Remark [3.1] then the measure $\nu_{-q^{2\beta},q}$ under a certain scaling converges to the classical symmetric Meixner law as $q \uparrow 1$,

$$
\frac{2^{2\beta}}{2\pi I(2\beta)} |I(\beta + ix)|^2 dx, \quad x \in \mathbb{R}.
$$

(3.4)

See also [KLS10 14.9.15].

3.2 Problem

For $\alpha, q \in (-1,1)$, we would like to know when there exists a radial measure $\rho_{\nu_{\alpha,q}}$ satisfying

$$
\int_0^\infty r^{2k} \rho_{\nu_{\alpha,q}}(dr) = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}.
$$

(3.5)

Here $[k]_q!$ denotes the $q$-factorials defined by

$$
[k]_q! := 1, \quad [k]_q! := \prod_{\ell=1}^{k} [\ell]_q = \frac{(q; q)_k}{(1 - q)^k}, \quad k \geq 1.
$$
It is easy to get the inequality for \( \alpha, q \in (-1, 1) \),
\[
|(-\alpha; q)_k[k]_q|! \leq \left( \frac{4}{1 - |q|} \right)^k, \quad k \in \mathbb{N} \cup \{0\}.
\]
(3.6)

Due to Carleman criterion for the moment problem, this inequality implies that a radial measure \( \rho_{\nu \alpha, q} \) is determined uniquely by the sequence \( \{(-\alpha; q)_k[k]_q!\} \).

We shall follow the procedure below to construct \( \rho_{\nu \alpha, q} \) in (3.5).

1. Recall the radial part of the \( q \)-Gaussian measure on \( \mathbb{C} \) (\( q \)-Bargmann measure), \( \rho_{\nu q} = \rho_{\nu 0, q} \), obtained in [LM95],
\[
\int_0^\infty r^{2k} \rho_{\nu q}(dr) = [k]_q!.
\]
(3.7)

2. Find a radial (possibly signed) measure \( \rho_{\alpha, q} \) having the moment \( (-\alpha; q)_k \).

3. Compute the multiplicative (Mellin) convolution \( \rho_{\nu q} \ast \rho_{\alpha, q} \) to get \( \rho_{\nu \alpha, q} \).

Remark 3.3. It is known [LM95] that a radial measure \( \rho_{\nu q} \) in (3.7) does not exist for \( q < 0 \). However, one can see that the positivity assumption on \( q \) can be relaxed for \( \rho_{\nu \alpha, q} \) if \( \alpha = q \). It will be discussed right after the proof of Proposition 3.6 and in Proposition 3.7.

### 3.3 Construction of \( (\alpha, q) \)-radial measures

#### Lemma 3.4.
Suppose that \( \alpha \in (-1, 1) \) and \( q \in [0, 1) \). Let
\[
\rho_{\alpha, q} := (-\alpha; q)_\infty \sum_{n=0}^\infty \frac{(\alpha)^n}{(q; q)_n} \delta_{q^n/2},
\]
which is a signed measure. Then we have
\[
\int_0^\infty r^{2k} \rho_{\alpha, q}(dr) = (-\alpha; q)_k, \quad k \in \mathbb{N} \cup \{0\}.
\]
In particular, if taking \( \alpha = -q \), then one can see \( \rho_{\nu q} = D_{(1-q)^{-1/2}}(\rho_{-q, q}) \), namely,
\[
\int_0^\infty r^{2k} D_{(1-q)^{-1/2}}(\rho_{-q, q})(dr) = \frac{(q; q)_k}{(1 - q)^k} = [k]_q!,
\]
where \( D_t(\lambda) \) is the push-forward of \( \lambda \) by the map \( x \mapsto tx \) for a measure \( \lambda \) on \( \mathbb{R} \).

**Proof.** Firstly, we have
\[
\int_0^\infty r^{2k} \rho_{\alpha, q}(dr) = (-\alpha; q)_\infty \sum_{n=0}^\infty \frac{(-\alpha q^k)^n}{(q; q)_n}.
\]
Since Euler’s formula (see [GR04, 1.3.15]),
\[
\frac{1}{(a; q)_\infty} = \sum_{n=0}^\infty \frac{a^n}{(q; q)_n},
\]
(3.8)
is known, we replace \( a \) by \( -\alpha q^k \) in (3.8) to obtain
\[
\int_0^\infty r^{2k} \rho_{\alpha, q}(dr) = \frac{(-\alpha; q)_\infty}{(-\alpha q^k; q)_\infty} = (-\alpha; q)_k.
\]
The proof is complete.  \( \square \)
Remark 3.5. (1) The last equality in proof is due to the formula

\[(a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty}.
\]

See [GR04, 1.2.30], for example.

(2) Euler’s formula is considered as the $q$-analogue of exponential function $e^a$ due to

\[
\lim_{q \to 1} \frac{1}{(1-q)a; q)_n} = e^a.
\]

Let

\[
\left[ \begin{array}{c} n \\ \ell \end{array} \right]_q := \frac{[n]_q!}{[\ell]_q! [n-\ell]_q!} = \frac{(q; q)_n}{(q; q)_\ell (q; q)_{n-\ell}}
\]

be the $q$-binomial coefficients and $h_n(z \mid q)$ be the Rogers-Szegő polynomials [GR04, S05] defined by

\[
h_n(z \mid q) = \sum_{\ell=0}^{n} \left[ \begin{array}{c} n \\ \ell \end{array} \right]_q z^\ell.
\]

Proposition 3.6. Suppose that $\alpha \in (-1, 1)$ and $q \in [0, 1)$. Let

\[
\rho_{\nu, q} := \left\{ \begin{array}{ll}
(-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} \mid q) \delta_{1-q}^{1/2}, & q > 0, \\
-\alpha \delta_0 + (1 + \alpha) \delta_1, & q = 0,
\end{array} \right.
\]

which is a signed measure in general. Then we have

\[
\int_0^\infty r^{2k} \rho_{\nu, q}(dr) = \frac{(-\alpha, q; q)_k}{(1-q)^k} = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}.
\]  

(3.10)

Proof. First of all, it is easy to show (3.10) for the case $q = 0$. Therefore, let us assume $q > 0$.

One can compute the multiplicative (Mellin) convolution $\hat{\otimes}$ to get $\rho_{\nu, q}$ as follows:

\[
\rho_{\nu, q} = \rho_{\alpha, q} \hat{\otimes} D_{1-q}^{-1/2}(\rho_{\nu, q})
\]

\[
= (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \left( \sum_{\ell=0}^{n} \frac{(-\alpha)^\ell q^{n-\ell}}{(q; q)_\ell (q; q)_{n-\ell}} \right) \delta_{1-q}^{1/2}
\]

\[
= (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} \mid q) \delta_{1-q}^{1/2}.
\]

On the other hand, by Lemma 3.3, we have

\[
\int_0^\infty r^{2k} D_{1-q}^{-1/2}(\rho_{\nu, q})(dr) = \frac{(q; q)_k}{(1-q)^k} = [k]_q!.
\]

Therefore, one can get

\[
\int_0^\infty r^{2k} \rho_{\nu, q}(dr) = \int_0^\infty r^{2k} \rho_{\alpha, q}(dr) \int_0^\infty r^{2k} D_{1-q}^{-1/2}(\rho_{\nu, q})(dr)
\]

\[
= (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}.
\]

In Proposition 3.6, we have obtained $\rho_{\nu, q}$ for $\alpha \in (-1, 1)$ and $q \in (0, 1)$. Due to the term

\[
\delta_{1-q}^{1/2} \text{ in } \rho_{\nu, q},
\]

it seems impossible for $q \in (-1, 0)$ to define $\rho_{\nu, q}$. However, if $-1 < \alpha = q < 0$ then $\nu_{q, q}$ coincides with a scaled $q^2$-Gaussian measure, and hence the Bargmann measure exists.
Proposition 3.7. Suppose $-1 < \alpha = q < 0$. We define

$$\rho_{\nu_{q,\alpha}} := D_{(1+q)^{1/2}}(\rho_{\nu_{q,\alpha}})$$

$$= (q^2; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q)^{-1/2}(-q)^n}.$$  \hspace{1cm} (3.11)

Then one can see

$$\int_{0}^{\infty} r^{2k} \rho_{\nu_{q,\alpha}}(dr) = (1 + q)^k |k| q! = (-q; q)_k |k| q!.$$  

Proof. One can see by direct computations

$$(-q; q)_k |k| q! = \left\{ \prod_{\ell=1}^{k} (1 - (-q)q^{\ell-1}) \right\} \left\{ \prod_{\ell=1}^{k} \frac{1 - q^{2\ell}}{1 - q} \right\}$$

$$= (1 + q)^k \prod_{\ell=1}^{k} \frac{1 - q^{2\ell}}{1 - q}$$

$$= (1 + q)^k |k| q!.$$  

Thus $\rho_{\nu_{q,\alpha}}$ can be defined as the radial measure for $q^2$-Gaussian measure on $\mathbb{C}$ scaled by $(1+q)^{1/2}$, namely, $\rho_{\nu_{q,\alpha}} = D_{(1+q)^{1/2}}(\rho_{\nu_{q,\alpha}}).$ \hfill \qed

Remark 3.8. If we use the fact that $h_n(-1 \mid q) = 0$ for odd $n \geq 1$ (see proof of Lemma 3.9 below), we can extend the definition (3.9) to the case $-1 < \alpha = q < 0$. This will give an alternative way to define $\rho_{\nu_{q,\alpha}}$ for $-1 < q < 0$, but both definitions give the same measure.

We need some properties of the Rogers-Szegő polynomials to know when the measure $\rho_{\nu_{q,\alpha}}$ becomes positive.

Lemma 3.9 (MGH90). Suppose that $q \in (-1, 1)$.

1. If $n \geq 0$ is odd, then $h_n(x \mid q) \geq 0$ if and only if $x \geq -1$. Moreover, the point $x = -1$ is the unique zero of $h_n(x \mid q)$ on $\mathbb{R}$.

2. If $n \geq 0$ is even, then $h_n(x \mid q) > 0$ for all $x \in \mathbb{R}$.

Proof. It is known that all the zeros of $h_n(z \mid q)$ lie on the unit circle $|z| = 1$. See [MGH90] or [S05 Theorem 1.6.11]. Note that the result was obtained for $q \in [0, 1)$, but the proof can be extended to $q \in (-1, 1)$ without any modifications.

By definition, one can see

$$\left[ \begin{array}{c} n \\ \ell \end{array} \right]_q = \frac{(1 - q^{n-\ell+1})(1 - q^{n-\ell+2}) \cdots (1 - q^n)}{(1 - q)(1 - q^2) \cdots (1 - q^\ell)} > 0,$$

and hence $h_n(1 \mid q) > 0$ for all $n \geq 0$. Thus, $h_n(x \mid q) \neq 0$ for $x \in \mathbb{R} \setminus \{-1\}$. It then suffices to show that $h_n(-1 \mid q) > 0$ for all even $n \geq 0$ and $h_n(-1 \mid q) = 0$ for all odd $n \geq 1$. The recurrence relation for the Rogers-Szegő polynomials is known to be

$$h_{n+1}(z \mid q) = (z + 1)h_n(z \mid q) - (1 - q^\alpha)zh_{n-1}(z \mid q), \hspace{1cm} n \geq 1.$$ \hspace{1cm} (3.12)

See [S05 1.6.76] (note that formula (1.6.76) has an error of a sign). It is easy to see that $h_0(-1 \mid q) = 1 > 0, h_1(-1 \mid q) = 0$, so by induction and (3.12) one can show $h_n(-1 \mid q) > 0$ for all even $n \geq 0$ and $h_n(-1 \mid q) = 0$ for all odd $n \geq 1$. \hfill \qed

We need the following lemma in proof of Theorem 3.11 for the non-existence part of a radial Bargmann measure.

Lemma 3.10. Let $\mu$ be a signed measure on $\mathbb{R}$ with compact support and let $\nu$ be a nonnegative measure on $\mathbb{R}$. If $\mu$ and $\nu$ have the same finite moments of all orders, then $\mu = \nu$.\hfill \qed
\textbf{Proof.} We denote by \(m_n\) the moments of \(\mu\) (and \(\nu\) by assumption). Since \(\mu\) is compactly supported, say on \([-R, R]\),
\[
|m_n| = \left| \int_{[-R, R]} x^n \mu(dx) \right| \leq \|\mu\| R^n, \quad n \in \mathbb{N} \cup \{0\},
\]
where \(\|\mu\|\) denotes the total variation of \(\mu\). Therefore, \(\nu\) is also supported on \([-R, R]\). By Weierstrass’ approximation, we have
\[
\int_{[-R, R]} f(x) \mu(dx) = \int_{[-R, R]} f(x) \nu(dx)
\]
for all \(f \in C([-R, R])\). This implies that \(\mu = \nu\) since, if we use the Hahn decomposition \(\mu = \mu_+ - \mu_-\), then (3.13) implies
\[
\int_{[-R, R]} f(x) \mu_+(dx) = \int_{[-R, R]} f(x) (\nu + \mu_-)(dx),
\]
and hence \(\mu_+ = \nu + \mu_-\) as nonnegative finite measures. \(\square\)

In summary, the complete answer to the unique existence of a radial Bargmann representation of \(\nu_{\alpha,q}\) is stated as follows:

\textbf{Theorem 3.11.} Suppose that \(\alpha, q \in (-1, 1)\). The probability measure \(\nu_{\alpha,q}\) has a radial Bargmann representation if and only if either (i) \(q \geq 0\) and \(\alpha \leq q\) or (ii) \(\alpha = q \neq 0\).

In fact, the radial measure is given uniquely by
\[
\rho_{\nu_{\alpha,q}} = \begin{cases} 
-\alpha \delta_0 + (1 + \alpha) \delta_1 & (\alpha \leq q = 0), \\
(\alpha, q; q) \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q) -1/2, q^{n/2}} & (q > 0, \alpha < q), \\
(q^2; q^2) \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q) -1/2, q^n} & (\alpha = q \neq 0).
\end{cases}
\]

\textbf{Proof.} 1. \textbf{Existence and uniqueness.} If \(q \in [0, 1)\) and \(\alpha \leq q\), then by Proposition \(3.6\) and Lemma \(3.7\) the signed measure \(\rho_{\nu_{\alpha,q}}\) is in fact a nonnegative measure and becomes the radial part of a Bargmann measure. The case \(\alpha = q < 0\) was discussed in Proposition \(3.7\). Due to Carleman criterion for the moment problem, the inequality given in (3.10) guarantees the uniqueness of \(\rho_{\nu_{\alpha,q}}\) for these cases.

2. \textbf{Non-existence.} (1) If \(q \in (0, 1)\) and \(\alpha > q\), then \(\rho_{\nu_{\alpha,q}}\) is not a nonnegative measure and is really a signed measure since \(h_n(-\alpha/q | q) < 0\) for odd integers \(n \geq 0\) and \(q > 0\) from Lemma \(3.9\). By Lemma \(3.10\) if a radial Bargmann measure exists, then it must be equal to the signed measure \(\nu_{\alpha,q}\). This is a contradiction. Thus, a radial Bargmann measure does not exist.

(2) If \(q = 0\) and \(\alpha > q = 0\) then by \(3.9\) \(\nu_{0,0}\) is really a signed measure, and hence by the same argument as above, a radial Bargmann measure does not exist.

(3) Let
\[
\beta_k(\alpha, q) := (-\alpha; q)_k |q|, \quad k \geq 0, \alpha, q \in (-1, 1).
\]

Given \(q < 0\) and \(\alpha \neq q\), suppose that there exists a radial part of a Bargmann measure, \(\rho\). Let \(\rho^2\) be the push-forward of \(\rho\) by the map \(x \mapsto x^2\). Then,
\[
\beta_k(\alpha, q) = \int_0^{\infty} x^k \rho^2(dx) = \int_0^{\infty} x^{2k} \rho(dx).
\]

By the way, by Proposition \(3.6\) it holds that \(\beta_k(\alpha, q') = \int_0^{\infty} x^{2k} \rho_{\nu_{\alpha,q'}}(dx)\) for any \(q' > 0\), that is,
\[
\beta_k(\alpha, q') = (-\alpha, q'; q') \sum_{n=0}^{\infty} \frac{(q')^n}{(q'; q')_n} h_n(-\alpha(q')^{-1} | q') \frac{(q')^kn}{(1 - q')^k}, \quad q' > 0,
\]

which is true even for \(q' = q\) by analytic continuation.

Now let us consider the signed measure
\[
\mu := (-\alpha, q; q) \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q) -1/2, q^n}, \quad \alpha \neq q < 0,
\]

8
supported on the points \( \frac{a^n}{t^q} \) for \( n = 0, 1, 2, 3, \ldots \). Then by (3.15) for \( q' = q \) and by (3.14),
\[
\int_{\mathbb{R}} x^k \mu(dx) = \beta_k(\alpha, q) = \int_0^\infty x^k \rho^2(dx), \quad k \in \mathbb{N} \cup \{0\}.
\]

By Lemma 3.10 the signed measure \( \mu \) and the probability measure \( \rho^2 \) should be equal. However, the support of \( \mu \) is not contained in \([0, \infty)\), and hence \( \mu \) cannot be equal to \( \rho^2 \). This is a contradiction. \( \square \)

**Example 3.12.** (1) The radial measure \( \rho_{\nu_0, q} \) for \( q \in [0, 1) \) is of the \( q \)-Bargmann \([LM95]\).
(2) The radial measure \( \rho_{\nu_0, q} \) for \( q \in (-1, 1) \) is of the \( q^2 \)-Bargmann.
(3) \( \lim_{q \uparrow 1} \rho_{\nu_0, q} \) is of the classical Bargmann \([BW98, BW01]\), \([AKK03]\).
(4) Consider \( \alpha = -q^{2\beta}, \beta > 0 \). This choice of \( \alpha \) is suggested by (3.1) in Remark 3.1. In fact, one can see
\[
\lim_{q \uparrow 1} \frac{(1 - q^{2\beta + n - 1})|n|^2}{4(1 - q)} = \frac{1}{4} (n + 2\beta - 1)n.
\]

This limit sequence is the Jacobi sequence of the symmetric Meixner distribution in (3.4), so that \( \rho_{\nu_0, q} \) under suitable scaling converges weakly as \( q \uparrow 1 \) to the radial measure with the density,
\[
\frac{2\pi r}{I(2\beta)} \int_0^\infty h(r, t/4)e^{-t/2\beta}dt
\]
where
\[
h(r, t) = \frac{1}{\pi t} \exp \left( -\frac{r^2}{t} \right), \quad r \in \mathbb{R}, \quad t > 0.
\]

This is an integral representation of the radial density for the Bessel kernel measure, which can be also represented by the modified Bessel function \([As05, As09]\).
(5) \( \rho_{\nu_0, q} \) for \( \alpha \in (-1, 0) \) is the radial measure for the symmetric free Meixner distribution. See Remark 3.13 below.

**Remark 3.13.** Let \( \mu_t \) be a \( t \)-deformed probability measure of a probability measure \( \mu \) on \( \mathbb{R} \) defined through the Cauchy transform \( G_\mu \) of \( \mu \),
\[
\frac{1}{G_{\mu_t}(z)} := \frac{t}{G_{\mu}(z)} + (1 - t)z, \quad t \geq 0,
\]
examined by Bożejko-Wysoczanski \([BW98, BW01]\), Krystek-Wojakowski \([KW14]\) discussed the radial Bargmann representation of a \( t \)-deformed probability measure \( \mu_t \), \( t \)-Bargmann representation for short, and obtained necessary and sufficient condition for the admissibility of the representation. The classical Bargmann representation of the Kesten measure \( \kappa_t \) has the form,
\[
\rho_{\kappa_t} = \left( 1 - \frac{1}{t} \right) \delta_0 + \frac{1}{t} \delta_{\sqrt{t}}, \quad t \geq 1.
\]

In \([AKW16]\), the \( t \)-Bargmann representation of a symmetric free Meixner law \( \varphi_{s,t} \) with two positive parameters \( s, t \) is treated and is admitted if and only if \( t \geq 1 \). In fact, one can see \( \rho_{\varphi_{s,t}} = D_s(\rho_{\kappa_t}) \) and hence
\[
\rho_{\varphi_{(1-t)s,t}} = D_{1/\sqrt{t}}(\rho_{\kappa_t}), \quad t \geq 1.
\]
Therefore, the case \( (5) \) in Example 3.12 can be viewed as a \( t \)-Bargmann representation, too.

Furthermore, let us state the \( t \)-deformed version of Theorem 3.11 for \( q \neq 0 \) without proof:

**Proposition 3.14.** The \( t \)-deformed version of \( \rho_{\nu_0, q} \) for \( q \neq 0 \) is given by
\[
\left( 1 - \frac{1}{t} \right) \delta_0 + \frac{1}{t} \rho_{\nu_0, q}, \quad t \geq 1.
\]
3.15. The $t$-Bargmann representation of $\nu_q$ is treated in [KW14] for $q = 1$ and [AKW16] for $0 \leq q < 1$.

Before closing this section, let us give a short remark about relations with the free infinite divisibility. Many of particular examples have so far suggested that the free infinite divisibility of a probability measure implies the existence of a radial Bargmann representation. The converse is not true in general because the Askey-Wimp-Kerov distribution $\mu_{q/10}$ for instance, discussed in [BRLS11], is not freely infinitely divisible, but it has a Bargmann representation with a gamma distribution as its radial measure. However, not many counterexamples have been found.

Therefore, we conjecture that the free infinite divisibility of our $(\alpha, q)$-Gaussian distribution is equivalent to the existence of its radial Bargmann measure:

**Conjecture.** Suppose that $\alpha, q \in (-1, 1)$. The probability measure $\nu_{\alpha,q}$ is freely infinitely divisible if and only if $\alpha = q$ or $\alpha < q \geq 0$.

This conjecture is guaranteed to be true in the restricted subfamilies $\{\nu_{\alpha,0} \mid \alpha \in (-1,1)\}$ ([SY01, Theorem 3.2]), $\{\nu_{0,q} \mid -1 < q < 1\}$ ([ABBL10] and [AH13, Example 3.11] for the free infinite divisibility), and $\{\nu_{q,q} \mid q \in (-1,1)\}$ (all measures in this family are freely infinitely divisible since they are $q^2$-Gaussians).

4 Commutation relations among one-mode $(\alpha, q)$-operators

**Definition 4.1.** Suppose that $\alpha, q \in (-1, 1)$ and $f$ is analytic on $\mathbb{C}$.

(1) Let $Z$ be the multiplication operator defined by

$$(Zf)(z) := zf(z).$$

(2) Let $D_q$ be the Jackson derivative given by

$$(D_qf)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

(3) The $\alpha$-deformed Jackson derivative is given as

$$D_{\alpha,q} := \begin{cases} D_q + \alpha q^{2N} D_{1/q}, & q \neq 0, \\ D_0 + \alpha \frac{D}{dz}\big|_{z=0}, & q = 0, \end{cases}$$

where $N$ is the number operator. For $q \neq 0$, we can also write

$$D_{\alpha,q} = D_q + \frac{\alpha}{q^2} D_{1/q} q^{2N}.$$  

**Remark 4.2.** It is easy to check that the $\alpha$-deformed Jackson derivative is equivalently defined as

$$(D_{\alpha,q}f)(z) = (D_qf)(z) + \alpha (D_{1/q}f)(q^2 z), \quad q \neq 0.$$  

For example, if $f(z) = z^n$, $(D_{\alpha,q}f)(z) = (1 + \alpha q^{n-1}) [n]_q z^{n-1}$ holds. In fact, the $\alpha$-deformed Jackson derivative is an analogue of the operator in [BEH15, Theorem 2.5].

Then, one can realize one-mode analogue of $(\alpha, q)$-operators on an appropriate domain of the one-mode interacting Bargmann-Fock space $B$ with $\omega_n = (1 + \alpha q^{n-1}) [n]_q z^{n-1}$ and $\alpha_n = 0$ by

$$a^+ := Z, \quad a^- := D_{\alpha,q}, \quad \text{and} \quad \Phi_n := \frac{z^n}{\sqrt{\omega_n!}}.$$

In fact, it is easy to check that

$$a^+ \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1},$$

$$a^- \Phi_n = \sqrt{\omega_n} \Phi_{n-1}.$$
hold and the $q$-commutation relation, one-mode analogue of \( A.1 \),

\[
[a^-, a^+]_q \Phi_n := (a^−a^+ − qa^+a^−)\Phi_n = (I + \alpha q^{2N})\Phi_n,
\]

is satisfied. Let us put $M_{\alpha,q} = I + q^{2N}$ and then one can get the expression,

\[
M_{\alpha,q} = (1 + \alpha)I - \alpha(1 - q^2)ZD_{q^2},
\]

due to $(ZD_{q^2})\Phi_n = [n]_q^q\Phi_n$.

Therefore one can obtain the following

**Theorem 4.3.** Suppose $\alpha \in (-1, 1)$ and $q \in (-1, 1)$. Then the following are satisfied.

1. \([a^-, a^+]_q = M_{\alpha,q}, \quad [a^-, M_{\alpha,q}]_q = (1 - q^2)a^-, \quad [M_{\alpha,q}, a^+]_q = (1 - q^2)a^+.\)
2. $M_{\alpha,q} = (1 + \alpha)I - \alpha(1 - q^2)ZD_{q^2}$. 
3. In particular, if $\alpha = q$, then one can obtain a more refined relation, \([a^-, a^+]_q = (1 + q)I.\)

**Example 4.4.** (1) $\alpha = 0$ implies \([a^-, a^+]_q = I.\) Hence $M_{0,q} = I$ commutes with both $a^+$ and $a^−$,

\[
[a^-, M_{0,q}]_1 = [M_{0,q}, a^+]_1 = 0.
\]

Therefore, the case $\alpha \neq 0$ provides non-trivial commutation relations.

(2) If $\alpha = -q^{2\beta}$ for $\beta > 0$, then the limiting case of the scaled operator is obtained as

\[
\lim_{q \uparrow 1} M_{-q^{2\beta}, q} = \lim_{q \uparrow 1} I - \frac{q^{2\beta}q^{2N}}{1 - q^2} = N + \beta.
\]

Moreover, let us consider the scaled operators,

\[
A^\pm := \lim_{q \uparrow 1} \frac{a^\pm}{\sqrt{1 - q^2}}
\]

Then one can get

\[
[A^-, A^+]_1 = N + \beta
\]

and hence

\[
[A^-, N]_1 = A^−, \quad [N, A^+]_1 = A^+.
\]

It should be noted that these are the commutation relations for the classical Meixner-Pollaczek polynomials with respect to the symmetric Meixner distribution in \( A.4 \). See \[As08\].

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**A Appendix**

Let $\Sigma_n$ be the set of bijections $\sigma$ of the $2n$ points $\{\pm 1, \pm 2, \cdots, \pm n\}$ with $\sigma(−k) = −\sigma(k)$. Equipped with the composition operation as a product, $\Sigma_n$ becomes what is called a Coxeter group of type B. It is generated by $\pi_0 := (1,−1)$ and $\pi_i := (i, i+1), \quad 1 \leq i \leq n−1$, which satisfy the generalized braid relations

\[
\begin{align*}
\pi_i^2 &= e, & 0 \leq i \leq n − 1, \\
(\pi_0\pi_1)^4 &= (\pi_i\pi_{i+1})^3 = e, & 1 \leq i \leq n − 1, \\
(\pi_i\pi_j)^2 &= e, & |i − j| \geq 2, \quad 0 \leq i, j \leq n − 1.
\end{align*}
\]
An element \( \sigma \in \Sigma_n \) expresses an irreducible form,
\[
\sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \leq i_1, \ldots, i_k \leq n - 1,
\]
and in this case
\[
\ell_1(\sigma) := \text{the number of } \pi_0 \text{ in } \sigma,
\]
\[
\ell_2(\sigma) := \text{the number of } \pi_i, \quad 1 \leq i \leq n - 1, \text{ in } \sigma
\]
are well defined. Let \( H \) be a separable Hilbert space. For a given self-adjoint involution \( f \mapsto \overline{f} \) for \( f \in H \), an action of \( \Sigma_n \) on \( H^{\otimes n} \) is defined by
\[
\begin{cases}
\pi_0(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes f_2 \otimes \cdots \otimes f_n, & n \geq 1, \\
\pi_i(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n, & n \geq 2, 1 \leq i \leq n - 1.
\end{cases}
\tag{A.2}
\]

The \((\alpha, q)\)-inner product on the full Fock space \( \mathcal{F}(H) \) is defined by
\[
\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha, q} := \delta_{m,n} \sum_{\sigma \in \Sigma_n} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \prod_{j=1}^{n} \langle f_j, g_{\sigma(j)} \rangle_H, \quad \alpha, q \in (-1,1)
\tag{A.3}
\]
with conventions \( 0^0 = 1 \) and \( g_{-k} = \overline{g_k}, \; k = 1, 2, \ldots, n \). For example, if one may define the involution as \( \overline{f} := -f \), then \( g_{-k} = -g_k \). Equipped with this inner product the full Fock space \( \mathcal{F}(H) \) is denoted by \( \mathcal{F}_{\alpha, q}(H) \) to emphasize on the dependence of the inner product on \( \alpha, q \).

The \((\alpha, q)\)-creation operator \( B_{\alpha, q}^+(f) \) is the usual left creation operator on the full Fock space, and the \((\alpha, q)\)-annihilation operator \( B_{\alpha, q}^-(f) \) is its adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_{\alpha, q} \). They satisfy the commutation relation
\[
B_{\alpha, q}^-(f)B_{\alpha, q}^+(g) - qB_{\alpha, q}^+(g)B_{\alpha, q}^-(f) = \langle f, g \rangle_H I + \alpha(\overline{f}, g)_{H} q^2 N, \quad f, g \in H.
\tag{A.4}
\]
The readers can consult [BEH15] for details.

References


