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Mathematical Approach to the Statistical-Mechanical Models in Random Media

（ランダム媒質中の統計力学模型に対する数学的アプローチ）

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Mathematical Approach to the Statistical-Mechanical Models in Random Media

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Preface

So many interesting and amazing phenomena exist around us. A lots of scientists have challenged to understand these phenomena for a long time. In statistical mechanics, we would like to prospect that the macroscopic phenomena which we have enjoyed and sometimes suffered from should be explained by the microscopic interactions among numerous small particles. For instance, thermodynamics has been explained by the kinetic theory of gasses. However, we can not treat these numerous interactions by deterministic equations, so we apply statistics and probability theory for analyzing the connection between microscopic interactions and macroscopic phenomena. It is known that we have the phase transitions and the critical phenomena in most statistical-mechanical models. In recent years, we have introduced random environments for those models. By inducing a random environment, it has been known that some qualitative changes occur in the phase transition and the critical phenomenon of each model. Our main interest in this thesis is the behavior of linear polymers consisting of many monomers which interact with one another and lying in a random environment. The behavior of linear polymers is also affected by the random environment.

In this thesis, we consider self-avoiding walk (SAW) on a random environment and the pining model as the models of linear polymers in random media. The systems of the statistical-mechanical models in random media are called the disordered systems. In the disordered systems we mainly consider how the random environment affects the critical point and the critical behavior of the original statistical-mechanical models. Many kinds of models of the disordered systems have been studied frequently and actively since 1980's. Classical examples in mathematics are Sinai's one-dimensional random walk in a random medium [60] and Smith and Wilkinson's branching processes in random environments [62]. More recent examples are the pinning models [26, 41] and the directed polymer models [25]. We have two cases in the disordered systems. One is the quenched case and another is the annealed case. In the disorder systems, we have two randomness, random phenomenon and random environment. In the quenched case we consider the random environment which is randomized much more slowly than that of the random phenomenon. For instance, the particles in the air collide randomly to the glass consisting of molecules which is put randomly. Therefore, we consider the system, letting the random environment fixed. On the other hand, the annealed case is the case that time scale of the environment is greater than that of the random phenomenon. So that we consider the system in the averaged environment. Under a certain condition, it is known that the disorder is irrelevant. This is one of the most interesting problems of the disordered systems.

SAW is similar to random walk, however, it has self-avoidance constraint, i.e., the walk never visits the same site once it visited. By this property, SAW is no longer a Markov chain. This self-avoidance constraint is a very natural assumption for linear polymers. Many interesting results on SAW have been proven, however, in spite of the simple definition of SAW, many of the most fundamental questions are left open and are difficult to solve mathematically rigorously.

The pinning model is a model of discrete time and one-dimensional space. This model has studied for considering the behavior of linear polymers consisting of numerous monomers. For example, we can treat linear polymers consisting of hydrophilic and hydorophobic monomers. These two kinds of monomers are put randomly in a linear polymer and we consider the interface between oil and water. Then, it is known that there is a phase transition in the sense that the polymer either localizes at the interface or delocalizes, and the random environment affects this critical phenomenon.

This thesis is organized as follows. In Chapter 1 we consider SAW on a random environment. We first review SAW in a homogeneous setting with some facts which are well-known, and then we introduce a random environment which we consider random conductor on each edge of the graph SAW lies in. The main results of this chapter are displayed in Section 1.2, which are about the critical point for the quenched susceptibility. This section has two parts. First we give the qualitative result of the quenched critical point. We show that the quenched critical point is independent of the reference point and is a degenerate random variable. Second we estimate it quantitatively. We give upper and lower bounds of the quenched critical point by using the Paley-Zygmund inequality and also give an application of this inequality for SAW on RC. In the last section of this chapter, we consider the model on a homogeneous degree tree. We apply the fractional moment method to provide the exact value of the quenched critical point. Indeed, it has already been known by Derrida and Spohn [27] and Baffet, Patrick, and Pulé [7], however our approach is simple, heuristic, and is a good example of the fractional moment method which is a strong method for understanding the quenched case. In Chapter 2 we consider the pinning model. In the first two sections, we review what has been known on both the pinning model in a homogeneous setting and in an i.i.d. random environment and see some comparisons with the Markovian model which is defined the pinning model in an environment equipped with the Markov property. Section 2.3 is the main part of this chapter. We introduce the pinning model on renewal set, which belongs to the class of long-range correlations. This is a joint work with Dimitris Cheliotis and Julien Poisat. We prove some propositions and theorems for the annealed case by analyzing the affection between two renewals and conclude with a discussion about the quenched case, which is an ongoing project.

Chapter 1

Self-avoiding walk (SAW) on random conductors

This chapter is based on the work in [23]. We investigate SAW in a random environment, which is topologically regular, but random in energy landscape. The goal of this chapter is to achieve better understanding of how the introduction of a random environment changes the properties of the critical point and the critical exponent.

1.1 SAW in a general setting

SAW is a statistical-mechanical model for chain-like solvents, i.e., linear polymers we consider in this thesis. SAW was first introduced by Flory [36, 37] in order to investigate the behavior of polymer chains. Since then many physicists have much more conjectures that are believed to be true. Most of them are supported by numerical simulations and physical ideas that have not been fully justified mathematically.

We consider SAW on $\mathbb{Z}^d$. Let $\Omega(x, y)$ be the set of nearest-neighbor self-avoiding paths on $\mathbb{Z}^d$ from x to y, and let $\Omega(x) = \bigcup_{y\in\mathbb{Z}^d} \Omega(x, y)$. Denoting the length of ω by $|\omega|$ (i.e., $|\omega| = n$ for $\omega = (\omega_0, \ldots, \omega_n)$) and the energy cost of a bond between consecutive monomers by $h \in \mathbb{R}$, we define the susceptibility as

$$\chi_h = \sum_{\omega\in\Omega(x)} e^{-h|\omega|}, \tag{1.1.1}$$

which is independent of the location of the reference point $x \in \mathbb{Z}^d$. Two other key observables are the number of n-step SAWs and the two-point function:

$$c_n = \sum_{\omega\in\Omega(x)} \mathbf{1}_{\{|\omega|=n\}}, \qquad G_h(x) = \sum_{\omega\in\Omega(o,x)} e^{-h|\omega|}, \tag{1.1.2}$$

where o is the origin of $\mathbb{Z}^d$ and $\mathbf{1}_{\{\dots\}}$ is the indicator function. Obviously, we have

$$\chi_h = \sum_{n=0}^{\infty} e^{-hn} c_n = \sum_{x \in \mathbb{Z}^d} G_h(x). \tag{1.1.3}$$

Proposition 1.1.1 (Sub-additivity of SAW)**.** For any $m, n \in \mathbb{N}_0$, it holds that

$$c_{m+n} \leq c_m c_n. \tag{1.1.4}$$

Proof. By the translation invariance,

$$\begin{aligned} c_{m+n} &= \sum_{y \in \mathbb{Z}^d} \sum_{\omega \in \Omega(x;m)} \mathbf{1}_{\{\omega_m = y\}} \sum_{\eta \in \Omega(y;n)} \mathbf{1}_{\{\omega \circ \eta \in \Omega(x;m+n)\}} \\ &\leq \sum_{\omega \in \Omega(x;m)} \sum_{y \in \mathbb{Z}^d} \mathbf{1}_{\{\omega_m = y\}} \sum_{\eta \in \Omega(y;n)} 1 \\ &= c_m c_n, \end{aligned} \tag{1.1.5}$$

where we denote $\Omega(x;n) = \{\omega \in \Omega(x) : |\omega| = n\}$, and $\omega \circ \eta$ stands for the concatenation of ω and η. ■

Proposition 1.1.2 (The connective constant)**.** The connective constant μ is well-defined and moreover,

$$\mu = \lim_{n \to \infty} c_n^{1/n} = \inf_{n \in \mathbb{N}} c_n^{1/n}. \tag{1.1.6}$$

Proof. We fix $k \in \mathbb{N}$, then $n = km + r$ for some $m \in \mathbb{N}_0$ and $r < k$. By Proposition 1.1.1, we have

$$c_n^{1/n} \leq (c_k^m c_r)^{1/n} \leq c_k^{m/n} c_r^{1/n}, \tag{1.1.7}$$

which implies $\limsup_{n \to \infty} c_n^{1/n} \leq c_k^{1/k}$. By taking the $\liminf_{k \to \infty}$, we prove existence of the limit. Taking the infimum over $k \in \mathbb{N}$ gives us

$$\lim_{n \to \mathbb{N}} c_n^{1/n} = \inf_{k \in \mathbb{N}} c_k^{1/k}. \tag{1.1.8}$$

This completes the proof. ■

In the simple random walk case, the connective constant μ is equal to $2d$ since the random walker can choose $2d$ directions in each step. For the nearest-neighbor SAW model, we have $d^n \leq c_n \leq 2d(2d-1)^{n-1}$, which implies that $d \leq \mu \leq 2d - 1$. We obtain the lower bound by counting the walks that take steps only in the positive coordinate directions and the upper bound by the memory-1 (without immediate reversals) walks. For $d = 2$, the following rigorous bounds are known; $\mu \in [2.625622, 2.679193]$. The lower bound is due to Jensen [47] by counting bridges, and the upper bound is due to Pönitz and Tittmann [59] by counting finite memory SAWs. We also know the asymptotic behavior of the susceptibility around the critical point as follows.

$$\chi_h \underset{h \downarrow h_0}{\asymp} (h - h_0)^{-\gamma}. \tag{1.1.9}$$

The constant γ is believed to exist and be the rate of divergence of χ_h. γ is known as one of the critical exponents that have universality. It is predicted that for each dimension the constant γ satisfied that

$$c_n \underset{n \uparrow \infty}{\sim} A\, \mu^n n^{\gamma - 1}, \tag{1.1.10}$$

where constant A depends on dimension d. The predicted values of the critical exponent γ are

$$\gamma = \begin{cases} 1 & d = 1, \\ 43/32 & d = 2, \\ 1.162\ldots & d = 3, \\ 1 \text{ with logarithmic correction} & d = 4, \\ 1 & d \geq 5. \end{cases} \tag{1.1.11}$$

Note that in the simple random walk model $\gamma = 1$ for any dimension.

Proposition 1.1.3 (The critical point for SAW in a homogeneous setting)**.** It holds that for any dimension,

$$\chi_h < \infty \Leftrightarrow h > \log \mu. \tag{1.1.12}$$

Proof. Since $\mu = \inf_{n \in \mathbb{N}} c_n^{1/n}$ by Proposition 1.1.2,

$$\chi_h = \sum_{n=0}^{\infty} c_n e^{-hn} \geq \sum_{n=0}^{\infty} \left(\mu e^{-h}\right)^n = \frac{1}{1 - e^{-(h - \log \mu)}} \geq \frac{1}{h - \log \mu}, \tag{1.1.13}$$

which implies that $\chi_{h_0} = \infty$. We put $h = h_0 + \delta$ for any $\delta > 0$, then

$$\chi_h = \sum_{n=0}^{\infty} \frac{c_n}{\mu^n} e^{-\delta n}, \tag{1.1.14}$$

and this is finite. Therefore, by the monotonicity of χ_h,

$$\begin{aligned} \chi_h &< \infty \quad \text{if } h > h_0, \\ \chi_h &= \infty \quad \text{if } h < h_0. \end{aligned}$$

This completes the proof. ■

By the definition, we can also regard χ_h as a generating function of c_n. Therefore, the equation (1.1.3) directly shows that $\chi_h < \infty$ if and only if $h > \log \mu$. Then, we easily see that $h_0 = \log \mu$ is the critical point of the susceptibility χ_h. For $d = 1$, since $c_n = 2$, the critical point h_0 is equal to 0. For $d = 2$, on hexagonal lattice, it is known that $h_0 = \frac{1}{2}\log(2 + \sqrt{2})$ by [30]. Many other rigorous results on the behavior of χ_h and G_h around the critical point $h_0 = \log \mu$ have been proven, especially in high dimensions $d > 4$, with the help of the lace expansion [16, 53]. However, there still remain many challenging open problems in two and three dimensions. See [61] and the references therein.

1.2 SAW on random conductors on $\mathbb{Z}^d$

In recent years, various models of SAW in a quenched random environment have attracted much attention of chemists, physicists and mathematicians [21, 22, 45, 54]. It is natural to consider an inhomogeneous environment. SAW on a randomly diluted lattice has introduced by Chakrabarti and Kertész [21]. Le Doussal and Machta [29] investigate it by applying a renormalization method on a hierarchical lattice and show some conjectures. Lacoin [51] answers affirmatively to one of them by showing that, on an infinite super-critical percolation cluster in two dimensions, the quenched critical point (defined by the divergence of the quenched susceptibility) is strictly smaller than the annealed one (defined by the divergence of the annealed susceptibility).

1.2.1 The model and the thorems

In this section, we introduce a random environment to SAW. Let $\mathbb{B}^d$ denote the set of nearest-neighbor bonds in $\mathbb{Z}^d$, and let $\boldsymbol{X} = \{X_b\}_{b\in\mathbb{B}^d}$ be a collection of integrable random variables whose law $\mathbb{P}$ is translation-invariant and ergodic. From a physical point of view, X_b can be regarded as the magnitude of resistance of a conductor attached to a bond $b \in \mathbb{B}^d$, so it may be more natural to assume $X_b \geq 0$. However, the results in this section are all valid without this assumption $X_b \geq 0$. Given the environment $\boldsymbol{X}$ and the strength of randomness $\beta \geq 0$, we can define the quenched susceptibility at $x \in \mathbb{Z}^d$ as

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{\omega\in\Omega(x)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})}, \tag{1.2.1}$$

where

$$b_j \equiv b_j(\omega) = (\omega_{j-1}, \omega_j). \tag{1.2.2}$$

Because of the inhomogeneity of $\boldsymbol{X}$, the quenched susceptibility is no longer translation-invariant and does depend on the location of the reference point $x \in \mathbb{Z}^d$. Similarly to the homogeneous case, we also define

$$\hat{c}_{\beta,\boldsymbol{X}}(x;n) = \sum_{\omega \in \Omega(x)} e^{-\beta \sum_{j=1}^{|\omega|} X_{b_j}} \mathbf{1}_{\{|\omega|=n\}}, \tag{1.2.3}$$

$$\hat{G}_{h,\beta,\boldsymbol{X}}(x,y) = \sum_{\omega \in \Omega(x,y)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})}. \tag{1.2.4}$$

These quantities are reduced to χ_h, c_n and $G_h(y-x)$ respectively, when $\beta = 0$. Moreover,

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{n=0}^{\infty} e^{-hn} \hat{c}_{\beta,\boldsymbol{X}}(x;n) = \sum_{y \in \mathbb{Z}^d} \hat{G}_{h,\beta,\boldsymbol{X}}(x,y). \tag{1.2.5}$$

Since $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ is monotonic in h, we can define the quenched version of the critical point as

$$\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x) = \inf\{h \in \mathbb{R} : \hat{\chi}_{h,\beta,\boldsymbol{X}}(x) < \infty\}. \tag{1.2.6}$$

Our goal in this chapter is to understand how the random environment $\boldsymbol{X}$ affects the behavior of these quenched observables around its critical point. There are numerous examples in which the introduction of randomness alters the behavior of relevant observables. We mention those examples in preface.

As a first step to understand the properties of the random variable $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$, we consider the annealed case: we take the average of $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ over the environment $\boldsymbol{X}$ (before $n \to \infty$). Let

$$h^{\mathsf{a}}_{\beta} = \{h \in \mathbb{R} : \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] < \infty\}, \tag{1.2.7}$$

where $\mathbb{E}$ is the expectation for $\mathbb{P}$. Since $\mathbb{P}$ is translation-invariant, the annealed critical point h^{a}_{β} does not depend on the location of the reference point $x \in \mathbb{Z}^d$. We note that $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x) \le h^{\mathsf{a}}_{\beta}$ by the definition. In particular, if $\boldsymbol{X}$ is i.i.d. and the Laplace transform

$$\lambda_\beta = \mathbb{E}[e^{-\beta X_b}] \tag{1.2.8}$$

exists, then we can directly compute $\mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)]$ as

$$\mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)] = \sum_{\omega \in \Omega(x):|\omega|=n} \prod_{j=1}^{n} \mathbb{E}[e^{-\beta X_{b_j}}] = \lambda_\beta^n c_n, \tag{1.2.9}$$

and the annealed susceptibility $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]$ can be compute as

$$\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] = \sum_{n=0}^{\infty} e^{-hn} \mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)] = \sum_{n=0}^{\infty} e^{-(h-\log\lambda_\beta)n} c_n = \chi_{h-\log\lambda_\beta}. \tag{1.2.10}$$

Let $h = h_0 + \log \lambda_\beta$, then $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] < \infty$ if and only if $h > h_0 + \log \lambda_\beta$. Therefore, we obtain

$$h^{\mathsf{a}}_\beta = h_0 + \log \lambda_\beta. \tag{1.2.11}$$

By Jensen's inequality, we immediately see the lower bound as

$$h^{\mathsf{a}}_\beta \geq h_0 - \beta \mathbb{E}[X_b], \tag{1.2.12}$$

where the gap is $O(\beta^2)$ as $\beta \to 0$.

The following theorem is the main result of this section.

Theorem 1.2.1. *The quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ is almost surely a degenerate random variable.*

By the above theorem, we abbreviate $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ as $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}$. And we estimate the quenched critical point quantitatively.

Theorem 1.2.2. *For $d \geq 1$ and $\beta \geq 0$, we have $\mathbb{P}$ almost surely*

$$h_0 - \beta \mathbb{E}[X_b] \leq \hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}} \leq h^{\mathsf{a}}_\beta. \tag{1.2.13}$$

In particular, for $d = 1$ the lower bound is an equality.

Since $c_n = 2$ in $\mathbb{Z}^1$, $\mu = 1$, i.e., $h_0 = 0$. Let $h = -\beta \mathbb{E}[X_b] + \delta$ and $\Delta_j = X_{(x+j-1,x+j)} - \mathbb{E}[X_b]$. Then, we have

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = 1 + \sum_{n=1}^{\infty} e^{-\delta n} \Big(e^{-\beta \sum_{j=1}^{n} \Delta_j} + e^{-\beta \sum_{j=0}^{n-1} \Delta_{-j}} \Big). \tag{1.2.14}$$

By applying the individual ergodic theorem to those two sequences $\{\Delta_j\}_{j=1}^{\infty}$ and $\{\Delta_{-j}\}_{j=0}^{\infty}$, we can conclude that the above series almost surely converges if and only if $\delta > 0$.

For $d \geq 2$, however, since c_n grows exponentially, it is hard to control the speed of convergence along those SAWs at the same time. Because of this entropy effect, we strongly believe that the first inequality in (1.2.13) is a strict inequality. If β is large and $\mathbb{E}[X_b] > 0$, then the gap between the lower and upper bounds in (1.2.13) is large, and the inequality (1.2.13) is no longer effective. However in the following specific case, we may find a better bound. Suppose that $\mathbb{P}(X_b = 0)$ is bigger than the critical point for the oriented percolation on $\mathbb{Z}^d_+$. Then, there is almost surely an $\boldsymbol{X}$-free infinite oriented-percolation cluster $\mathcal{C}_x$ at some $x \in \mathbb{Z}^d_+$, in which the number of n-step directed paths from x grows exponentially in n [39, Theorem 3.1(2)]. The susceptibility $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ can be bounded below by restricting the sum over those directed paths in $\mathcal{C}_x$, implying existence of a β-independent positive lower bound on $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}$.

1.2.2 A qualitative study of the quenched critical point

We prove Theorem 1.2.1 by showing that the quenched critical point is a degenerate random variable that does not depend on the location of the reference point.

Recall that $\boldsymbol{X} = \{X_b\}_{b\in\mathbb{B}^d}$ is a collection of integrable (thus almost surely finite) random variables whose law $\mathbb{P}$ is translation-invariant and ergodic. Following the similar analysis to that in Lacoin [52], we first prove that the quenched critical point is independent of the location of the reference point.

Lemma 1.2.3. *The quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ is almost surely a constant function of $x \in \mathbb{Z}^d$.*

Proof. We will show that

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(u) \leq \hat{\chi}_{h,\beta,\boldsymbol{X}}(v)^2 + e^{h+\beta X_{(v,u)}}\hat{\chi}_{h,\beta,\boldsymbol{X}}(v) \tag{1.2.15}$$

holds for any pair of neighboring vertices $u, v \in \mathbb{Z}^d$. Since $X_{(u,v)}$ is almost surely finite, it implies that $\hat{\chi}_{h,\beta,\boldsymbol{X}}(u) < \infty$ if and only if $\hat{\chi}_{h,\beta,\boldsymbol{X}}(v) < \infty$. Repeated applications of this inequality to all neighboring vertices in $\mathbb{Z}^d$, we conclude that all vertices are in the same equivalent class, i.e., either $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) < \infty$ for all $x \in \mathbb{Z}^d$ or $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty$ for all $x \in \mathbb{Z}^d$. Therefore, $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ does not depend on $x \in \mathbb{Z}^d$, almost surely.

It remains to show (1.2.15). First, we split the sum into two as

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(u) = \sum_{\omega\in\Omega(u)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})}(\mathbf{1}_{\{v\in\omega\}} + \mathbf{1}_{\{v\notin\omega\}}). \tag{1.2.16}$$

Due to the sub-additivity and reversibility of SAW, the contribution from $\mathbf{1}_{\{v\in\omega\}}$ is bounded as

$$\begin{aligned}\sum_{\omega\in\Omega(u):v\in\omega} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})} &\leq \underbrace{\sum_{\omega\in\Omega(u,v)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j(\omega)})}}_{\hat{G}_{h,\beta,\boldsymbol{X}}(u,v)} \underbrace{\sum_{\eta\in\Omega(v)} e^{-\sum_{j=1}^{|\eta|}(h+\beta X_{b_j(\eta)})}}_{\hat{\chi}_{h,\beta,\boldsymbol{X}}(v)} \\ &= \hat{G}_{h,\beta,\boldsymbol{X}}(v,u)\ \hat{\chi}_{h,\beta,\boldsymbol{X}}(v) \\ &\leq \hat{\chi}_{h,\beta,\boldsymbol{X}}(v)^2. \end{aligned}\tag{1.2.17}$$

On the other hand, by adding an extra step from v to u, the contribution from $\mathbf{1}_{\{v\notin\omega\}}$ is bounded as

$$\begin{aligned}\sum_{\omega\in\Omega(u):v\notin\omega} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})} &= e^{h+\beta X_{(v,u)}} \sum_{\omega\in\Omega(u):v\notin\omega} e^{-(h+\beta X_{(v,u)})} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j(\omega)})} \\ &= e^{h+\beta X_{(v,u)}} \underbrace{\sum_{\bar\omega\in\Omega(v):\bar\omega_1=u} e^{-\sum_{j=1}^{|\bar\omega|}(h+\beta X_{b_j(\bar\omega)})}}_{\hat\chi_{h,\beta,\boldsymbol{X}}(v)} \\ &\le e^{h+\beta X_{(v,u)}}\hat\chi_{h,\beta,\boldsymbol{X}}(v), \end{aligned}\tag{1.2.18}$$

where we use the symmetry $X_{(u,v)} = X_{(v,u)}$. This completes the proof. ■

Henceforth we simply denote $\hat h^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ by $\hat h^{\mathsf{q}}_{\beta,\boldsymbol{X}}$.

Lemma 1.2.4. *The quenched critical point $\hat h^{\mathsf{q}}_{\beta,\boldsymbol{X}}$ is a degenerate random variable.*

Proof. Due to Lemma 1.2.3, the event $\{\hat h^{\mathsf{q}}_{\beta,\boldsymbol{X}} = h\}$ is translation-invariant for any $h \in \mathbb{R}$. Since $\mathbb{P}$ is ergodic, we can conclude that $\mathbb{P}(\hat h^{\mathsf{q}}_{\beta,\boldsymbol{X}} = h)$ is either zero or one. ■

1.2.3 A quantitative estimate of the quenched critical point

In this section, we prove Theorem 1.2.2. Recall that its reduction to an equality for $d = 1$ has already been mentioned soon after Theorem 1.2.2.

Upper Bound

Although it is trivial by the definition, the second inequality in (1.2.13) can be proven in the following indirect but heuristic way. First, by the Markov inequality, we have

$$\mathbb{P}\Big(\hat c_{\beta,\boldsymbol{X}}(x;n) \ge n^2\mathbb{E}[\hat c_{\beta,\boldsymbol{X}}(x;n)]\Big) \le \frac{1}{n^2}. \tag{1.2.19}$$

Then, by the Borel-Cantelli lemma, we can conclude that the opposite inequality $\hat c_{\beta,\boldsymbol{X}}(x;n) \le n^2\mathbb{E}[\hat c_{\beta,\boldsymbol{X}}(x;n)]$ holds for all but finitely many n, implying almost sure convergence of $\hat\chi_{h,\beta,\boldsymbol{X}}(x)$ for $h > h^{\mathsf{a}}_\beta$.

Remark 1.2.5. We may improve this upper bound to a strict inequality in two dimensions by adapting the idea of Lacoin [52]. In his setting (i.e., SAW on an infinite super-critical percolation cluster in $\mathbb{Z}^2$), it is proven that there are $b, \theta \in (0,1)$ such that

$$\mathbb{E}[\hat c_{\beta,\boldsymbol{X}}(x;n)^\theta] \le \big(b^n\mathbb{E}[\hat c_{\beta,\boldsymbol{X}}(x;n)]\big)^\theta. \tag{1.2.20}$$

Then, by the Markov inequality, we have

$$\mathbb{P}\Big(\hat{c}_{\beta,\boldsymbol{X}}(x;n) \geq n^{2/\theta} b^n \mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)]\Big) \leq \frac{1}{n^2}. \tag{1.2.21}$$

By the Borel-Cantelli lemma again, we conclude $\hat{h}_\beta^{\mathsf{q}} \leq h_\beta^{\mathsf{a}} - \log \frac{1}{b}$.

Analyzing fractional moments as in (1.2.20) has been a standard method to investigate disordered systems. To see how it is used in other settings, we refer to [63] for random walks in random environments and to [12, 13] for random pinning models.

Lower bound

In this section, we prove the first inequality in (1.2.13) by showing almost sure divergence of the quenched susceptibility at $h = h_0 - \beta\mathbb{E}[X_b] - \beta\delta$ for any $\beta > 0$ and $\delta > 0$. Let $\Delta_b = X_b - \mathbb{E}[X_b]$ and define

$$\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n) = \Big\{\omega \in \Omega(x;n) : \big|\tfrac{1}{n}\textstyle\sum_{j=1}^n \Delta_{b_j(\omega)}\big| < \delta\Big\}. \tag{1.2.22}$$

Using this random set, we can bound $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ at $h = h_0 - \beta\mathbb{E}[X_b] - \beta\delta$ as

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{\omega\in\Omega(x)} \frac{1}{\mu^{|\omega|}} e^{\beta|\omega|\left(\delta - \frac{1}{|\omega|}\sum_{j=1}^{|\omega|}\Delta_{b_j}\right)} \geq \sum_{n=1}^\infty \frac{1}{\mu^n}|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)|. \tag{1.2.23}$$

If there are infinitely many n such that $|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \frac{1}{2}c_n$, then, by $c_n \geq \mu^n$ (cf., (1.1.6) in Proposition 1.1.2), we obtain divergence of the susceptibility. Therefore,

$$\begin{aligned}
\mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}} = \infty) &\geq \underbrace{\mathbb{P}\Big(\hat{\chi}_{h,\beta,\boldsymbol{X}} = \infty \Big| \limsup_{n\to\infty}\big\{|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \tfrac{1}{2}c_n\big\}\Big)}_{1} \\
&\qquad \times \mathbb{P}\Big(\limsup_{n\to\infty}\big\{|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \tfrac{1}{2}c_n\big\}\Big) \\
&\geq \lim_{n\to\infty} \mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \tfrac{1}{2}c_n\big).
\end{aligned} \tag{1.2.24}$$

To complete the proof, it suffices to show that the rightmost limit is positive since $\mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty)$ is either zero or one. Here we use the Paley-Zygmund (PZ) inequality [57] as follow. For a random variable $Z \geq 0$ whose second moment is finite and for $\varepsilon \in (0,1)$,

$$\mathbb{P}(Z \geq \varepsilon\mathbb{E}[Z]) \geq (1-\varepsilon)^2 \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}. \tag{1.2.25}$$

Let $Z = |\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)|$. Notice that, by the definition and ergodicity, we can bound $\mathbb{E}\big[|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)|\big]$ from below as

$$\mathbb{E}\big[|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)|\big] = \sum_{\omega\in\Omega(x;n)} \mathbb{P}\Big(\big|\tfrac{1}{n}\textstyle\sum_{j=1}^{n}\Delta_{b_j(\omega)}\big| < \delta\Big) \geq c_n\big(1-o(1)\big). \quad (1.2.26)$$

Using this and the trivial inequality $\mathbb{E}\big[|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)|^2\big] \leq c_n^2$, we obtain

$$\lim_{n\to\infty} \mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \tfrac{1}{2}c_n\big) \geq \frac{1}{4} > 0, \quad (1.2.27)$$

as required. ■

Remark 1.2.6. We have the following much simpler proof of (1.2.27). First, by the trivial inequality $|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \leq c_n$, we obtain

$$\begin{aligned}\mathbb{E}\big[|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)|\big] &\leq \frac{1}{2}c_n\,\mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| < \tfrac{1}{2}c_n\big) + c_n\,\mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \tfrac{1}{2}c_n\big)\\ &= \frac{1}{2}c_n\Big(1+\mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \tfrac{1}{2}c_n\big)\Big). \quad (1.2.28)\end{aligned}$$

Combining this with (1.2.26), we can readily conclude $\mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \tfrac{1}{2}c_n\big) \geq 1-o(1)$.

1.2.4 Another application of the PZ inequality

The PZ inequality is often applied to the second-moment method. It has also been a standard tool to investigate the disordered systems. We show below that the PZ inequality is used to investigate the critical behavior for SAW on i.i.d. random conductors. From now on, we assume that $\lambda_\beta < \infty$ for all $\beta \geq 0$.

Proposition 1.2.7. *Suppose that*

$$B_1 \equiv \mathbb{E}\Big[\sum_{y\in\mathbb{Z}^d} \hat{G}_{h,\beta,\boldsymbol{X}}(x,y)^2\Big] < \infty \quad (1.2.29)$$

and

$$B_2 \equiv \mathbb{E}\Big[\sum_{y,z\in\mathbb{Z}^d} \hat{G}_{h,\beta,\boldsymbol{X}}(x,z)\,\hat{G}_{h,\beta,\boldsymbol{X}}(z,y)^2\,\hat{G}_{h,\beta,\boldsymbol{X}}(y,x)\Big] < \infty \quad (1.2.30)$$

hold uniformly in $h > h^{\mathsf{a}}_\beta$. Then, for any slowly-varying function $L(h) \downarrow 0$ as $h \downarrow h^{\mathsf{a}}_\beta$, we have

$$\liminf_{h\downarrow h^{\mathsf{a}}_\beta} \mathbb{P}\Big(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) \geq \frac{L(h)}{h-h^{\mathsf{a}}_\beta}\Big) \geq 1-O(\beta^2). \quad (1.2.31)$$

Although the above result is conditional and still weak to establish a decisive conclusion, it provides an evidence to support the belief that, in high dimensions, the coincidence $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}} = h^{\mathsf{a}}_{\beta}$ occurs and the critical exponent for $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$, if it exists, is bounded below by its mean-field value 1. For SAW in a homogeneous environment, the conditions (1.2.29)–(1.2.30) (in fact, the former implies the latter because $B_2 \le B_1^2$, which is a result of translation invariance and the Cauchy-Schwarz inequality) are known to hold in dimensions $d > 4$, via the lace expansion [16, 53]. The lace expansion yields a convolution equation for the two-point function, which is applicable in both homogeneous and inhomogeneous settings. In the current random setting, however, because of the lack of translation invariance, we have not been able to fully control the lace-expansion coefficients. This is under investigation in an ongoing project.

Proof of Proposition 1.2.7. First, by replacing Z in (1.2.25) by $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$, we have

$$\mathbb{P}\Big(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) \ge \varepsilon \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]\Big) \ge (1-\varepsilon)^2 \frac{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2}{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2]}. \tag{1.2.32}$$

Since $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] = \chi_{h-\log\lambda_\beta}$ (cf., (1.2.10)) and $\chi_h \ge (h-h_0)^{-1}$ for all $h > h_0$ (cf., (1.1.6)), we have $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] \ge (h-h^{\mathsf{a}}_{\beta})^{-1}$ for all $h > h^{\mathsf{a}}_{\beta}$. Replacing ε in (1.2.32) by a slowly-varying function $L(h) \downarrow 0$ as $h \downarrow h^{\mathsf{a}}_{\beta}$, we can conclude (1.2.31) as soon as we can show

$$\frac{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2] - \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2}{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2} \le O(\beta^2), \tag{1.2.33}$$

in the neighborhood of h^{a}_{β}.

To prove (1.2.33) under the assumptions (1.2.29)–(1.2.30), we introduce the notation

$$H_{\boldsymbol{X}}(\omega) = -\sum_{j=1}^{|\omega|} \big(h + \beta X_{b_j(\omega)}\big). \tag{1.2.34}$$

Let $\boldsymbol{Y} = \{Y_b\}_{b\in\mathbb{B}^d}$ be an independent copy of $\boldsymbol{X}$. Then, we obtain

$$\begin{aligned}&\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2] - \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2\\ &= \sum_{\omega,\eta\in\Omega(x)} \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega)}\mathbb{E}_{\boldsymbol{Y}}\Big[e^{H_{\boldsymbol{X}}(\eta)} - e^{H_{\boldsymbol{Y}}(\eta)}\Big]\Big].\end{aligned} \tag{1.2.35}$$

By the telescopic-sum representation, we can decompose $e^{H_{\boldsymbol{X}}(\eta)} - e^{H_{\boldsymbol{Y}}(\eta)}$ as

$$e^{H_{\boldsymbol{X}}(\eta)} - e^{H_{\boldsymbol{Y}}(\eta)} = \sum_{j=1}^{|\eta|} e^{H_{\boldsymbol{X}}(\eta_{<j})} e^{-h}\Big(e^{-\beta X_{b_j(\eta)}} - e^{-\beta Y_{b_j(\eta)}}\Big) e^{H_{\boldsymbol{Y}}(\eta_{>j})}, \tag{1.2.36}$$

where $\eta_{<j} = (\eta_0, \dots, \eta_{j-1})$ and $\eta_{>j} = (\eta_{j+1}, \dots, \eta_{|\eta|})$, with the convention $H_{\boldsymbol{X}}(\varnothing) = 0$. Substituting this back into (1.2.35) and changing variables from $\eta_{<j}$ to $\eta^{(1)}$, from η_j to a bond b, and from $\eta_{>j}$ to $\eta^{(2)}$, we obtain

$$\begin{aligned}&\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2] - \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2 \\ &= \sum_{\substack{\omega\in\Omega(x) \\ \eta^{(1)}\circ b\circ\eta^{(2)}\in\Omega(x)}} \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega)+H_{\boldsymbol{X}}(\eta^{(1)})}\mathbb{E}_{\boldsymbol{Y}}\Big[e^{-h}\Big(e^{-\beta X_b} - e^{-\beta Y_b}\Big)e^{H_{\boldsymbol{Y}}(\eta^{(2)})}\Big]\Big],\end{aligned} \tag{1.2.37}$$

where $\eta^{(1)} \circ b \circ \eta^{(2)}$ is the concatenation of those three paths, whose lengths are not fixed any more (due to the sum over j). Since b is not contained in $\eta^{(2)}$, Y_b is independent of $H_{\boldsymbol{Y}}(\eta^{(2)})$, hence

$$\begin{aligned}\mathbb{E}_{\boldsymbol{Y}}\Big[\Big(e^{-\beta X_b} - e^{-\beta Y_b}\Big)e^{H_{\boldsymbol{Y}}(\eta^{(2)})}\Big] &= \mathbb{E}_{\boldsymbol{Y}}\Big[e^{-\beta X_b} - e^{-\beta Y_b}\Big]\ \mathbb{E}_{\boldsymbol{Y}}\Big[e^{H_{\boldsymbol{Y}}(\eta^{(2)})}\Big] \\ &= \Big(e^{-\beta X_b} - \lambda_\beta\Big)\ \mathbb{E}_{\boldsymbol{Y}}\Big[e^{H_{\boldsymbol{Y}}(\eta^{(2)})}\Big].\end{aligned} \tag{1.2.38}$$

Substituting this back into (1.2.37) yields

$$\begin{aligned}&\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2] - \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2, \\ &= e^{-h} \sum_{\substack{\omega\in\Omega(x) \\ \eta^{(1)}\circ b\circ\eta^{(2)}\in\Omega(x)}} \mathbb{E}\Big[\underbrace{e^{H_{\boldsymbol{X}}(\omega)+H_{\boldsymbol{X}}(\eta^{(1)})}\Big(e^{-\beta X_b} - \lambda_\beta\Big)}_{0 \text{ if } b\notin\omega}\Big]\ \mathbb{E}_{\boldsymbol{Y}}\Big[e^{H_{\boldsymbol{Y}}(\eta^{(2)})}\Big] \\ &\le e^{-2h}\Big(\lambda_{2\beta} - \lambda_\beta^2\Big)\ \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] \sum_{\substack{\omega^{(1)}\circ b\circ\omega^{(2)}\in\Omega(x) \\ \eta^{(1)}\circ b\in\Omega(x)}} \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^{(1)})+H_{\boldsymbol{X}}(\omega^{(2)})+H_{\boldsymbol{X}}(\eta^{(1)})}\Big],\end{aligned} \tag{1.2.39}$$

where the restricted sum over $\eta^{(2)}$ is bounded above by $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]$, which is translation invariant and independent of $x \in \mathbb{Z}^d$.

Next, we investigate the remaining sum

$$\sum_{\substack{\omega^1\circ b\circ\omega^{(2)}\in\Omega(x) \\ \eta^{(1)}\circ b\in\Omega(x)}} \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^{(1)})+H_{\boldsymbol{X}}(\omega^{(2)})+H_{\boldsymbol{X}}(\eta^{(1)})}\Big]\big(\mathbf{1}_{\{\omega^{(2)}\cap\eta^{(1)}=\varnothing\}} + \mathbf{1}_{\{\omega^{(2)}\cap\eta^{(1)}\neq\varnothing\}}\big). \tag{1.2.40}$$

Due to the independence among the variables in $\boldsymbol{X}$, the contribution from

$\mathbf{1}_{\{\omega^{(2)}\cap\eta^{(1)}=\varnothing\}}$ is bounded by

$$\begin{aligned}
&\sum_{\substack{\omega^{(1)}\circ b\circ\omega^{(2)}\in\Omega(x)\\ \eta^{(1)}\circ b\in\Omega(x)}} \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^{(1)})+H_{\boldsymbol{X}}(\eta^{(1)})}\Big]\ \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^{(2)})}\Big]\\
&\qquad\le \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] \sum_{\substack{\omega^{(1)}\circ b\in\Omega(x)\\ \eta^{(1)}\circ b\in\Omega(x)}} \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^{(1)})+H_{\boldsymbol{X}}(\omega^{(2)})}\Big]\\
&\qquad\le \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]\, 2dB_1. \qquad (1.2.41)
\end{aligned}$$

To bound the contribution from $\mathbf{1}_{\{\omega^{(2)}\cap\eta^{(1)}\neq\varnothing\}}$ in (1.2.40), we split $\omega^{(2)}$ as $\omega^{(3)}\circ\omega^{(4)}$ at the last visit to $\eta^{(1)}$, so that $\omega^{(4)}\cap\eta^{(1)}=\{\omega_0^{(4)}\}$. Then, by using the independence among the variables in $\boldsymbol{X}$, we can bound the sum over $\omega^{(4)}$ by $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]$. As a result, the contribution from $\mathbf{1}_{\{\omega^{(2)}\cap\eta^{(1)}\neq\varnothing\}}$ is bounded by

$$\begin{aligned}
&\sum_{y\in\mathbb{Z}^d}\sum_{\substack{\omega^{(1)}\circ b\circ\omega^{(3)}\in\Omega(x,y)\\ \omega^{(4)}\in\Omega(y)}} \mathbf{1}_{\{\omega^{(1)}\circ b\circ\omega^{(3)}\circ\omega^{(4)}\in\Omega(x)\}} \sum_{\substack{\eta^{(3)}\in\Omega(x,y)\\ \eta^{(4)}\circ b\in\Omega(y)}} \mathbf{1}_{\{\eta^{(3)}\circ\eta^{(4)}\circ b\in\Omega(x)\}}\mathbf{1}_{\{\omega^{(4)}\cap(\eta^{(3)}\circ\eta^{(4)})=\{y\}\}}\\
&\qquad\qquad \times \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^{(1)})+H_{\boldsymbol{X}}(\omega^{(3)})+H_{\boldsymbol{X}}(\eta^{(3)})+H_{\boldsymbol{X}}(\eta^{(4)})}\Big]\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^{(4)})}\Big]\\
&\le \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] \sum_{y\in\mathbb{Z}^d}\sum_{\substack{\omega^{(1)}\circ b\circ\omega^{(3)}\in\Omega(x,y)\\ \eta^{(3)}\in\Omega(x,y)\\ \eta^{(4)}\circ b\in\Omega(y)}} \mathbf{1}_{\{b\notin\eta^{(3)}\}}\,\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^{(1)})+H_{\boldsymbol{X}}(\omega^{(3)})+H_{\boldsymbol{X}}(\eta^{(3)})+H_{\boldsymbol{X}}(\eta^{(4)})}\Big]\\
&= e^h\lambda_\beta^{-1}\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] \sum_{y,z\in\mathbb{Z}^d}\sum_{\substack{\omega^{(1)}\in\Omega(x,z)\\ b\circ\omega^{(3)}\in\Omega(z,y)\\ \eta^{(3)}\in\Omega(x,y)\\ \eta^{(4)}\in\Omega(y,z)}} \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^{(1)})+H_{\boldsymbol{X}}(b\circ\omega^{(3)})+H_{\boldsymbol{X}}(\eta^{(3)})+H_{\boldsymbol{X}}(\eta^{(4)})}\Big]\\
&= e^h\lambda_\beta^{-1}\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]B_2. \qquad (1.2.42)
\end{aligned}$$

Finally, by summarizing (1.2.39)–(1.2.42), we arrive at

$$\frac{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2]-\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2}{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2} \le e^{-2h}(2dB_1+e^h\lambda_\beta^{-1}B_2)(\underbrace{\lambda_{2\beta}-\lambda_\beta^2}_{O(\beta^2)}), \qquad (1.2.43)$$

which proves (1.2.33). This completes the proof of Proposition 1.2.7 ■

1.3 Phase transition for SAW on random conductors on a tree

This model we treat in this section is SAW on a tree with random conductors, which can be regarded as a directed polymer model on a disordered tree.

According to a classical theorem by Kahane and Peyrière [49] and Biggins [14], it is known that there exists a transition behavior in this model (see [55]). Derrida and Spohn [27] prove that in each phase, there exists some critical parameter inducing a qualitative change for the behavior of polymers showing that a directed polymer on a tree with disorder can be reduced to the study of nonlinear partial differential equations of reaction-diffusion type. Buffet, Patrick, and Pulé give another proof based on the study of martingale in [7]. In this section we give another probabilistic approach to the quenched critical point by applying the fractional moment estimate.

1.3.1 The model and the theorem

We consider an SAW path ω on a degree-ℓ tree $\mathbb{T}^\ell$. We denote by $|\omega|$ the length of SAW ω and by $\Omega(x;n)$ the set of n-step SAWs from $x \in \mathbb{T}^\ell$. We also denote by $\mathbb{B}^\ell$ the set of nearest-neighbor bonds on $\mathbb{T}^\ell$, we define the set of random conductors $\boldsymbol{X} = \{X_b\}_{b\in\mathbb{B}^\ell}$ as a collection of i.i.d. random variables whose probability law is denoted by $\mathbb{P}$. Since $c_n = \ell(\ell-1)^{n-1}$ on $\mathbb{T}^\ell$, we obtain $\mu = \ell - 1$.

Similarly to the $\mathbb{Z}^d$ case, we define the quenched susceptibility at $x \in \mathbb{T}^\ell$ by

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{\omega\in\Omega(x)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})}, \tag{1.3.1}$$

where $h \in \mathbb{R}$, $\beta \geq 0$ and $b_j \equiv b_j(\omega) = (\omega_{j-1}, \omega_j)$. Since $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ is monotonic in h, we can define the quenched critical point by

$$\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x) = \inf\{h \in \mathbb{R} : \hat{\chi}_{h,\beta,\boldsymbol{X}}(x) < \infty\}. \tag{1.3.2}$$

Recall that we prove on $\mathbb{Z}^d$ that $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ is independent of the reference point x and it is a degenerate random variable in the previous section. It is valid for the case that $\{X_b\}$ is a collection of integrable random variables whose law $\mathbb{P}$ is translation-invariant and ergodic. From now on, we simply write the quenched critical point by $\hat{h}^{\mathsf{q}}_\beta$.

We compute the annealed susceptibility $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]$ as

$$\mathbb{E}\big[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)\big] = \sum_{n=0}^{\infty} c_n\, \lambda_\beta^n\, e^{-hn} = \chi_{h-\log\lambda_\beta}, \tag{1.3.3}$$

and we have already known the annealed critical point is $h^{\mathsf{a}}_\beta = \log\mu + \log\lambda_\beta$. As in Section 1.2, we have

$$\log\mu - \beta\mathbb{E}[X_b] \leq \hat{h}^{\mathsf{q}}_\beta \leq h^{\mathsf{a}}_\beta, \tag{1.3.4}$$

where the gap between these two bounds is $O(\beta^2)$ when β is small enough.

In a directed polymer model on a disordered tree, it is known that there exists a transition behavior. The critical parameter that divides the system in two phases is known explicitly. According to Kahane and Peyrière [49] and Biggins [14], we introduce the quenched partition function of the directed polymer model on a disordered tree.

$$Z_n = \frac{1}{c_n} \sum_{\omega \in \Omega(x;n)} e^{-\sum_{j=1}^{n} (\beta X_{b_j} + \log \lambda_\beta)}. \tag{1.3.5}$$

They prove that this partition function Z_n is a positive martingale with respect to $\mathcal{F}_n(x) = \sigma(X_b : b = (u,v) \in \mathbb{B}^\ell, |u-x| \leq n, |v-x| \leq n)$ for $x \in \mathbb{T}^\ell$ through computing as follows.

$$\begin{aligned}
\mathbb{E}[Z_{n+1}|\mathcal{F}_n(x)] &= \mathbb{E}\Big[\frac{1}{c_{n+1}} \sum_{\omega \in \Omega(x;n+1)} e^{-\sum_{j=1}^{n+1} (\beta X_{b_j} + \log \lambda_\beta)} |\mathcal{F}_n(x)\Big] \\
&= \sum_{y \in \mathbb{T}^\ell} \frac{1}{c_n \mu} \sum_{\substack{\omega \in \Omega(x;n) \\ \omega_n = y}} e^{-\sum_{j=1}^{n} (\beta X_{b_j} + \log \lambda_\beta)}\, \mathbb{E}\Big[\sum_{\substack{z \in \mathbb{T}^\ell \\ |z-y|=1}} e^{-(\beta X_{(y,z)} + \log \lambda_\beta)}\Big] \\
&= Z_n.
\end{aligned} \tag{1.3.6}$$

By the martingale convergence theorem and Kolmogorov's 0-1 law, there exists a non-negative random variable $Z_\infty := \lim_{n\to\infty} Z_n$ and the probability $\mathbb{P}(Z_\infty = 0)$ is equal to either 0 or 1. We set the quenched free energy $\mathrm{F}(\beta)$, which is equal to the quenched critical point $\hat{h}^{\mathsf{q}}_\beta$.

$$\mathrm{F}(\beta) := \lim_{n\to\infty} \frac{1}{n} \log(e^{nh^{\mathsf{a}}_\beta} Z_n). \tag{1.3.7}$$

For $\beta \geq 0$, we define the function

$$f(\beta) = h^{\mathsf{a}}_\beta - \beta\Big(\frac{d}{d\beta} h^{\mathsf{a}}_\beta\Big). \tag{1.3.8}$$

Since

$$\begin{aligned}
\frac{d}{d\beta} f(\beta) &= \frac{d}{d\beta} h^{\mathsf{a}}_\beta - \Big\{\frac{d}{d\beta} h^{\mathsf{a}}_\beta + \beta\Big(\frac{d^2}{d\beta^2} h^{\mathsf{a}}_\beta\Big)\Big\} \\
&= -\beta\Big\{\frac{\mathbb{E}[X^2 e^{-\beta X}]}{\lambda_\beta} - \Big(\frac{\mathbb{E}[X e^{-\beta X}]}{\lambda_\beta}\Big)^2\Big\} < 0 \quad \text{for } \beta > 0,
\end{aligned} \tag{1.3.9}$$

the function $f(\beta)$ is decreasing in β. Let β_c be the positive root of $f(\beta) = 0$ if there exists, $\beta_c = \infty$ otherwise. Kahane and Peyrière [49] and Biggins [14] show that

$$\begin{aligned}
\mathbb{P}(Z_\infty > 0) = 1 \;\Leftrightarrow\; \beta < \beta_c \; (f(\beta) > 0), \\
\mathbb{P}(Z_\infty = 0) = 1 \;\Leftrightarrow\; \beta \geq \beta_c \; (f(\beta) \leq 0).
\end{aligned} \tag{1.3.10}$$

For $\beta < \beta_c$, we call the weak disorder regime, and for $\beta > \beta_c$, the strong disorder regime. Derrida and Spohn [27] prove that the free energy is

$$\mathrm{F}(\beta) = \begin{cases} h^{\mathsf{a}}_{\beta} & \text{if } \beta \le \beta_c, \\ \frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c} & \text{if } \beta > \beta_c. \end{cases} \tag{1.3.11}$$

They prove this result through the study of nonlinear partial differential equations of diffusion-reaction type and derive β_c from the minimal speed of traveling wave solutions. Buffet, Patrick and Pulé [7] also prove (1.3.11) by the martingale argument.

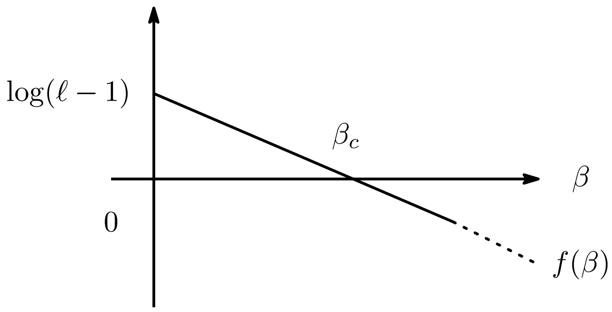

Figure 1.1: The function $f(\beta)$ is convex and there exists β_c such that $f(\beta_c) = 0$.

We can find a close connection between the partition function Z_n and the quenched susceptibility $\hat{\chi}_{h,\beta,\boldsymbol{X}}$.

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{n=0}^{\infty} c_n \, \lambda^n_{\beta} \, e^{-hn} \, Z_n. \tag{1.3.12}$$

Recall that the free energy $\mathrm{F}(\beta)$ is equal to the quenched critical point $\hat{h}^{\mathsf{q}}_{\beta}$.

The following statement is the main theorem of this section.

Theorem 1.3.1. *For $\ell > 3$,*

$$\hat{h}^{\mathsf{q}}_{\beta} = \begin{cases} h^{\mathsf{a}}_{\beta} & \textit{if } \beta \le \beta_c, \\ \frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c} & \textit{if } \beta > \beta_c, \end{cases} \tag{1.3.13}$$

where $\beta_c = \theta_c \beta$ and we obtain θ_c by optimizing the function $\log r(\theta) = h^{\mathsf{a}}_{\theta\beta} - \theta h^{\mathsf{a}}_{\beta}$.

Note that the case $\ell = 2$ is equivalent to the case $\mathbb{Z}$. Since on $\mathbb{Z}$, $c_n = 2$ and these two SAW paths are independent, it is proven that $\hat{h}^{\mathsf{q}}_{\beta} = -\beta\mathbb{E}[X_b]$

on $\mathbb{Z}$ by the strong law of large numbers. On $\mathbb{Z}^{d\geq 2}$, however, since c_n grows exponentially, it is hard to control the speed of convergence along the SAWs at the same time. Because of the entropy effect, we strongly believe that the first inequality in (1.3.4) is a strictly inequality. The exact value of quenched critical point on $\mathbb{Z}^{d\geq 2}$ remains an open problem.

1.3.2 In the weak disorder regime

First we prove that $\hat{h}^{\mathsf{q}}_\beta = h^{\mathsf{a}}_\beta$ for $\beta \leq \beta_c$. We show that $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty$ at $h = h^{\mathsf{a}}_\beta - \delta$ for any $\beta \in [0, \beta_c)$ and $\delta > 0$. In the weak disorder regime, Z_n almost surely converges to some positive random variable Z_∞. Then, for $\varepsilon > 0$, there exists an almost surely finite random variable $N = N(\omega, \boldsymbol{X}, \varepsilon) \in \mathbb{N}$ such that for $n \geq N$, $|Z_n - Z_\infty| < \varepsilon$. Therefore, for $h = h^{\mathsf{a}}_\beta - \delta$,

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{n=0}^{\infty} \frac{c_n}{\mu^n} e^{\delta n} Z_n \geq \frac{\ell}{\ell-1}(Z_\infty - \varepsilon) \sum_{n=N}^{\infty} e^{\delta n} = \infty. \tag{1.3.14}$$

This implies that $\hat{h}^{\mathsf{q}}_\beta = h^{\mathsf{a}}_\beta$ for $\beta \leq \beta_c$.

As an immediate consequence from (1.3.10) and (1.3.13), we show that for $\ell \geq 3$, the critical exponent of the quenched susceptibility is almost surely equal to 1 in the weak disordered regime. We consider the quenched susceptibility at $h = h^{\mathsf{a}}_\beta + \delta$ for any $\beta \in [0, \beta_c)$ and $\delta > 0$, which is given by

$$\hat{\chi}_{h,\beta,\boldsymbol{X}} = \sum_{n=0}^{\infty} \frac{c_n}{\mu^n} e^{-\delta n} Z_n. \tag{1.3.15}$$

Therefore, $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ is bounded from above as

$$\begin{aligned} \hat{\chi}_{h,\beta,\boldsymbol{X}} &\leq \sum_{n=0}^{N-1} \frac{c_n}{\mu^n} e^{-\delta n} Z_n + \sum_{n=N}^{\infty} \frac{c_n}{\mu^n} e^{-\delta n}(Z_\infty + \varepsilon) \\ &\leq \frac{\ell N}{\ell - 1}\Big(\max_{0\leq n \leq N-1} Z_n\Big) + \frac{\ell(Z_\infty + \varepsilon)}{(\ell-1)\, e^{\delta N}} \, \frac{1}{1-e^{-\delta}}. \end{aligned} \tag{1.3.16}$$

and is bounded from below as

$$\hat{\chi}_{h,\beta,\boldsymbol{X}} \geq \sum_{n=N}^{\infty} \frac{c_n}{\mu^n} e^{-\delta n}(Z_\infty - \varepsilon) = \frac{\ell(Z_\infty - \varepsilon)}{(\ell-1)\, e^{\delta N}} \, \frac{1}{1-e^{-\delta}}. \tag{1.3.17}$$

By (1.3.16) and (1.3.17), there exist random variables $0 < c < C < \infty$ depending on ω, $\boldsymbol{X}$ and ε such that

$$\frac{c}{h - h^{\mathsf{a}}_\beta} \leq \hat{\chi}_{h,\beta,\boldsymbol{X}}(x) \leq \frac{C}{h - h^{\mathsf{a}}_\beta}, \qquad \text{as } h \downarrow h^{\mathsf{a}}_\beta. \tag{1.3.18}$$

1.3.3 In the strong disorder regime

In this section, we split the proof of Theorem 1.3.1 in two parts. First we will give an upper bound of the quenched critical point by bounding fractional moments of Z_n and optimizing the choice of the fraction. Second we will bound the quenched critical point from below by the same value that we find as the upper bound.

The upper bound for the quenched critical point

First, we prove the following proposition to bound the quenched critical point from above.

Proposition 1.3.2. *For $\ell \geq 3$ and $\beta > \beta_c$, it holds that*

$$\hat{h}^{\mathsf{q}}_{\beta} \leq \frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c}, \quad \mathbb{P}\text{-}a.s., \tag{1.3.19}$$

and the critical parameter β_c is given by $\theta_c\beta$ where $\theta_c \in (0,1)$ is the value that minimizes the function $\log r(\theta)$, where $r(\theta)$ is defined by (1.3.26)*.*

Our strategy for this proposition is to estimate the rate of convergence of the martingale Z_n. Recall that we have already known the convergence of Z_n to zero in the strong disorder regime. In this section, we denote Z_n by $Z_n^{(x)}$ to emphasize the starting point x. We introduce another martingale defined by

$$\widetilde{Z}_n^{(y)} = \frac{1}{(\ell-1)^n} \sum_{\eta \in \widetilde{\Omega}(y;n)} e^{-\sum_{j=1}^n (\beta X_{b_j(\eta)} + \log \lambda_\beta)}, \tag{1.3.20}$$

where $\widetilde{\Omega}(y;n)$ is the set of SAWs on a Cayley tree with branching ratio $\ell - 1$. Therefore, we have

$$Z_n^{(x)} = \sum_{\substack{y \in \mathbb{T}^\ell \\ |x-y|=1}} \frac{e^{-\beta X_{(x,y)}}}{\ell \lambda_\beta} \widetilde{Z}_{n-1}^{(y)}, \tag{1.3.21}$$

$$\widetilde{Z}_{n-1}^{(y)} = \sum_{\substack{z \in \mathbb{T}^\ell \setminus \{x\} \\ |y-z|=1}} \frac{e^{-\beta X_{(y,z)}}}{(\ell-1)\lambda_\beta} \widetilde{Z}_{n-2}^{(z)}. \tag{1.3.22}$$

We apply a trivial inequality: for every $\theta \in (0,1)$, $(\sum_n a_n)^\theta \leq \sum_n (a_n)^\theta$ to (1.3.21), and take expectation. Due to the transitivity of a homogeneous degree tree and the i.i.d. property of $\boldsymbol{X}$, we obtain

$$\mathbb{E}[Z_n^\theta] \leq \sum_{\substack{y \in \mathbb{T}^\ell \\ |x-y|=1}} \mathbb{E}\Big[\Big(\frac{e^{-\beta X_{(x,y)}}}{\ell \lambda_\beta}\Big)^\theta\Big] \mathbb{E}[\widetilde{Z}_{n-1}^\theta] \leq \ell^{1-\theta} \frac{\lambda_{\theta\beta}}{\lambda_\beta^\theta} \mathbb{E}[\widetilde{Z}_{n-1}^\theta], \tag{1.3.23}$$

and

$$\mathbb{E}[\widetilde{Z}^{\theta}_{n-1}] \le \sum_{\substack{z_1\in\widetilde{\mathbb{T}}^{\ell}\\ |y-z_1|=1}} \mathbb{E}\Big[\Big(\frac{e^{-\beta X_{(y,z_1)}}}{(\ell-1)\lambda_\beta}\Big)^{\theta}\Big]\mathbb{E}[\widetilde{Z}^{\theta}_{n-2}] \le (\ell-1)^{1-\theta}\,\frac{\lambda_{\theta\beta}}{\lambda_\beta^{\theta}}\,\mathbb{E}[\widetilde{Z}^{\theta}_{n-2}]$$

$$\le \cdots \le \Big\{(\ell-1)^{1-\theta}\,\frac{\lambda_{\theta\beta}}{\lambda_\beta^{\theta}}\Big\}^{n-1}. \tag{1.3.24}$$

Substituting (1.3.24) into (1.3.23), we have

$$\mathbb{E}[Z^{\theta}_{n}] \le \ell^{1-\theta}\,\frac{\lambda_{\theta\beta}}{\lambda_\beta^{\theta}}\Big\{(\ell-1)^{1-\theta}\,\frac{\lambda_{\theta\beta}}{\lambda_\beta^{\theta}}\Big\}^{n-1} = \Big(\frac{\ell}{\ell-1}\Big)^{1-\theta} r(\theta)^n, \tag{1.3.25}$$

where

$$r(\theta) = (\ell-1)\mathbb{E}\Big[\Big(\frac{e^{-\beta X_b}}{(\ell-1)\lambda_\beta}\Big)^{\theta}\Big]. \tag{1.3.26}$$

Therefore, by the definition of the annealed critical point h^{a}_β, we have

$$\log r(\theta) = \log\mu + \log\lambda_{\theta\beta} - \theta\Big(\log\mu + \log\lambda_\beta\Big) = h^{\mathsf{a}}_{\theta\beta} - \theta h^{\mathsf{a}}_{\beta}. \tag{1.3.27}$$

We will show that $\mathbb{E}[Z^{\theta}_n]$ decays exponentially, i.e., $r(\theta) < 1$ for some $\theta \in (0,1)$, we consider the function $\log r(\theta)$, and compute the first and second derivatives of it.

$$\begin{aligned}\frac{d}{d\theta}(\log r(\theta)) &= -\beta\frac{\mathbb{E}[Xe^{-\theta\beta X}]}{\lambda_{\theta\beta}} - h^{\mathsf{a}}_\beta\\ &= \beta\Big(\frac{d}{d\beta}h^{\mathsf{a}}_\beta\big|_{\beta=\theta\beta}\Big) - h^{\mathsf{a}}_\beta\\ &= \frac{1}{\theta}\Big(h^{\mathsf{a}}_{\theta\beta} - f(\theta\beta)\Big) - h^{\mathsf{a}}_\beta,\end{aligned} \tag{1.3.28}$$

and

$$\frac{d^2}{d\theta^2}(\log r(\theta)) = \beta^2\Big\{\frac{\mathbb{E}[X^2e^{-\theta\beta X}]}{\lambda_{\theta\beta}} - \Big(\frac{\mathbb{E}[Xe^{-\theta\beta X}]}{\lambda_{\theta\beta}}\Big)^2\Big\} \ge 0. \tag{1.3.29}$$

Thus, we can say that the function $\log r(\theta)$ is convex. Since

$$\frac{d}{d\theta}(\log r(1)) = \beta\Big(\frac{d}{d\beta}h^{\mathsf{a}}_\beta\Big) - h^{\mathsf{a}}_\beta = -f(\beta) > 0 \tag{1.3.30}$$

by (1.3.28), $\log r(0) = \log(\ell-1) > 0$ and $\log r(1) = 0$ (see Figure 1.2), there exists $\theta_1 \in (0,1)$ such that $\log r(\theta_1) = 0$. Therefore, for $\theta \in (\theta_1, 1)$, we conclude that $\mathbb{E}[Z^{\theta}_n]$ is exponentially decaying in the strong disorder regime.

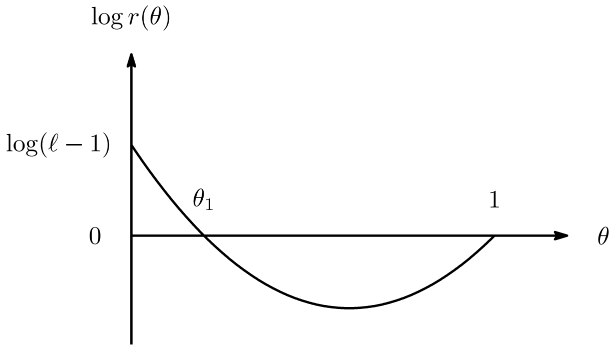

Figure 1.2: The function $\log r(\theta)$ is convex and there exists $\theta_1 \in (0,1)$ such that $\log r(\theta_1) = 0$. For $\theta \in (\theta_1, 1)$, the function $\log r(\theta)$ is strictly negative.

We show that the quenched susceptibility $\chi_{h,\beta,\boldsymbol{X}}(x)$ is almost surely finite for $h > h_\beta^{\mathsf{a}} - \frac{1}{\theta}\log\frac{1}{r(\theta)}$ and $\theta \in (\theta_1, 1)$. For $\theta \in (\theta_1, 1)$, $h = h_\beta^{\mathsf{a}} - \frac{1}{\theta}\log\frac{1}{r(\theta)} + \delta$ and $\delta > 0$, we have

$$\begin{aligned}\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) &= \sum_{n=0}^{\infty} c_n\, \lambda_\beta^n\, e^{-hn}\, Z_n \\ &= \frac{\ell}{\ell-1}\sum_{n=0}^{\infty} e^{-\delta n}\, r(\theta)^{-n/\theta}\, Z_n. \end{aligned} \tag{1.3.31}$$

By Markov's inequality, for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}(Z_n \geq (r(\theta)+\varepsilon)^{n/\theta}) &= \mathbb{P}(Z_n^\theta \geq (r(\theta)+\varepsilon)^n) \\ &\leq \frac{\mathbb{E}[Z_n^\theta]}{(r(\theta)+\varepsilon)^n} \leq \Big(\frac{\ell}{\ell-1}\Big)^{1-\theta}\Big(\frac{r(\theta)}{r(\theta)+\varepsilon}\Big)^n. \end{aligned} \tag{1.3.32}$$

Then, by the Borel-Cantelli lemma, the event $\{Z_n < (r(\theta)+\varepsilon)^{n/\theta}\}$ occurs for all but for finitely many n. We can control $\varepsilon > 0$ depending on $\delta > 0$ for the summation in (1.3.31) to be finite as

$$e^{-\delta n}\, r(\theta)^{-n/\theta}\, Z_n \leq \exp\Big\{-n\Big(\delta - \frac{1}{\theta}\log\big(1+\frac{\varepsilon}{r(\theta)}\big)\Big)\Big\}. \tag{1.3.33}$$

so that $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ is almost surely finite if we choose $\varepsilon < r(\theta)e^{\theta\delta}$. This implies that for any $\theta \in (\theta_1, 1)$,

$$\hat{h}_\beta^{\mathsf{q}} \leq h_\beta^{\mathsf{a}} + \frac{1}{\theta}\log r(\theta) < h_\beta^{\mathsf{a}}. \tag{1.3.34}$$

To optimize an upper bound above, we compute a derivation of $r(\theta)$ and

$f(\beta)$.

$$\begin{aligned}\frac{\theta}{d\theta}\Big(\frac{1}{\theta}\log r(\theta)\Big) &= -\frac{1}{\theta^2}\log r(\theta) + \frac{1}{\theta}\frac{d}{d\theta}(\log r(\theta)) \\ &= -\frac{1}{\theta^2}\Big\{h^{\mathsf{a}}_{\theta\beta} - \theta\beta\Big(\frac{d}{d\beta}h^{\mathsf{a}}_{\beta}\big|_{\beta=\theta\beta}\Big)\Big\} \\ &= -\frac{1}{\theta^2}f(\theta\beta). \end{aligned} \tag{1.3.35}$$

Therefore, we have

$$\frac{\theta}{d\theta}\Big(\frac{1}{\theta}\log r(\theta)\Big)\begin{cases} <0 & \text{if } \theta\beta<\beta_c, \\ =0 & \text{if } \theta\beta=\beta_c, \\ >0 & \text{if } \theta\beta>\beta_c. \end{cases} \tag{1.3.36}$$

For $\theta_c = \frac{\beta_c}{\beta} \in (\theta_1, 1)$[1], we have the upper bound on the quenched critical point $\hat{h}^{\mathsf{q}}_{\beta}$, i.e.,

$$\hat{h}^{\mathsf{q}}_{\beta} \leq h^{\mathsf{a}}_{\beta} + \frac{1}{\theta_c}\log r(\theta_c) = \frac{\beta}{\beta_c}h^{\mathsf{a}}_{\beta_c} \tag{1.3.39}$$

We finish the proof of Proposition 1.3.2. ■

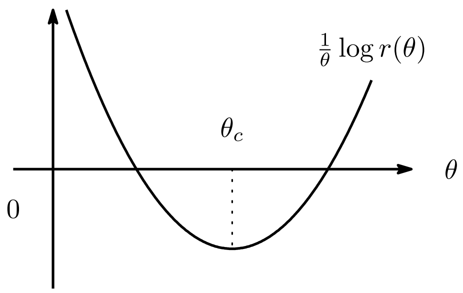

Figure 1.3: The function $\frac{1}{\theta}\log r(\theta)$ is convex and takes the minimum value when $\theta_c\beta = \beta_c$.

[1]We check that $\theta_c > \theta_1$. This is because $\frac{1}{\beta}h^{\mathsf{a}}_{\beta}$ is convex since

$$\frac{\partial}{\partial\beta}\Big(\frac{1}{\beta}h^{\mathsf{a}}_{\beta}\Big) = -\frac{1}{\beta^2}f(\beta)\begin{cases} <0 & \beta<\beta_c, \\ =0 & \beta=\beta_c, \\ >0 & \beta>\beta_c. \end{cases} \tag{1.3.37}$$

Therefore,

$$\log r(\theta_c) = h^{\mathsf{a}}_{\beta_c} - \frac{\beta_c}{\beta}h^{\mathsf{a}}_{\beta} < 0 = \log r(\theta_1). \tag{1.3.38}$$

This implies that $\theta_c > \theta_1$.

The lower bound for the quenched critical point

To prove that $\hat{h}^{\mathsf{q}}_\beta = \frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c}$, we need to show that for $\ell > 3$, $\hat{h}^{\mathsf{q}}_\beta$ is almost surely greater than $\frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c}$ in the strong disorder regime.

Proposition 1.3.3. *For $\ell \geq 3$ and $\beta > \beta_c$, it holds that*

$$\hat{h}^{\mathsf{q}}_\beta \geq \frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c}, \quad \mathbb{P}\text{-}a.s. \tag{1.3.40}$$

Proof of Proposition 1.3.3. We refer the proof in [7]. For lighter notations, we define $S_n(\omega) = \sum_{j=1}^n X_{b_j(\omega)}$, $\langle \cdot \rangle_\beta = \frac{\sum_{\omega\in\Omega(x;n)} \cdot e^{-\beta S_n(\omega)}}{\sum_{\omega\in\Omega(x;n)} e^{-\beta S_n(\omega)}}$, and

$$\mathrm{F}_n(\beta) = \frac{1}{n\beta} \log \sum_{\omega\in\Omega(x;n)} e^{-\beta S_n(\omega)}. \tag{1.3.41}$$

We give the expression of the quenched susceptibility again.

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{n=0}^{\infty} c_n \lambda_\beta e^{-nh} Z_n = \frac{\ell}{\ell-1} \sum_{n=0}^{\infty} e^{-n(h-\beta \log \mathrm{F}_n(\beta))}. \tag{1.3.42}$$

To show that $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty$ holds almost surely for $h = \frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c}$, it suffices to show that $\liminf_{n\to\infty} \mathrm{F}_n(\beta) \geq \frac{1}{\beta_c} h^{\mathsf{a}}_{\beta_c}$ holds almost surely. Note that $\liminf_{n\to\infty} \mathrm{F}_n(\beta) \geq \frac{1}{\beta_c} h^{\mathsf{a}}_{\beta_c}$ implies $\limsup_{n\to\infty} \mathbb{P}(Z_n \geq r(\theta_c)^{n/\theta_c}) > 0$[2].

First we will show that $\mathrm{F}_n(\beta)$ is decreasing in β. For $\beta_1 \geq \beta_2$, it holds that

$$\sum_{\omega\in\Omega(x;n)} \Big(\frac{e^{-\beta_2 S_n(\omega)}}{\sum_{\omega\in\Omega(x;n)} e^{-\beta_2 S_n(\omega)}} \Big)^{\beta_1/\beta_2} \leq 1, \tag{1.3.45}$$

[2]Since the event $\{\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty\}$ is translation-invariant in x, and the probability measure $\mathbb{P}$ is ergodic,

$$\mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty) > 0 \ \Rightarrow\ \mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty) = 1. \tag{1.3.43}$$

To prove that $\hat{h}^{\mathsf{q}}_\beta \geq \frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c}$, we show that $\mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty) > 0$ for $h = \frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c}$. Then we define the event $A_n = \{Z_n \geq r(\theta_c)^{n/\theta_c}\}$. If $Z_n = r(\theta_c)$, then $\hat{\chi}_{h,\beta,\boldsymbol{X}}$ diverges.

$$\begin{aligned}\mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty) &= \underbrace{\mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty | \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k)}_{=1} \mathbb{P}(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k) \\ &\geq \limsup_{n\to\infty} \mathbb{P}(A_n). \end{aligned} \tag{1.3.44}$$

Therefore, to prove that $\mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty) > 0$, it suffices to show that the rightmost limit in (1.3.44) is positive.

and then,

$$\Big(\sum_{\omega\in\Omega(x;n)} e^{-\beta_1 S_n(\omega)}\Big)^{1/\beta_1} \le \Big(\sum_{\omega\in\Omega(x;n)} e^{-\beta_2 S_n(\omega)}\Big)^{1/\beta_2}. \tag{1.3.46}$$

We can say that $\mathrm{F}_n(\beta)$ is decreasing in β. Next we will show the convexity of $\mathrm{F}_n(\beta)$. Since by the decreasingness of $\mathrm{F}_n(\beta)$, it holds that $\mathrm{F}'_n(\beta) \le 0$, we have

$$\mathrm{F}''_n(\beta) = -\frac{1}{2n\beta}\mathrm{F}'_n(\beta) + \frac{1}{n\beta}\Big(\langle S_n^2\rangle_\beta - \langle S_n\rangle_\beta^2\Big) \ge 0. \tag{1.3.47}$$

This implies that $\mathrm{F}_n(\beta)$ is a convex function in β. Finally by this convexity of $\mathrm{F}_n(\beta,\omega)$, for any $\varepsilon > 0$ independent of n and β we have

$$\mathrm{F}_n(\beta) \ge \mathrm{F}'_n(\beta_c-\varepsilon)(\beta-(\beta_c-\varepsilon)) + \mathrm{F}_n(\beta_c-\varepsilon). \tag{1.3.48}$$

Recall that we have already know $\lim_{n\to\infty}\mathrm{F}_n(\beta) = \frac{1}{\beta}h^{\mathsf{a}}_\beta$ in the weak disorder regime and since $\frac{1}{\beta}h^{\mathsf{a}}_\beta$ is differentiable in β, we have

$$\mathrm{F}'_n(\beta) \to \frac{d}{d\beta}\Big(\frac{1}{\beta}h^{\mathsf{a}}_\beta\Big) = -\frac{1}{\beta^2}h^{\mathsf{a}}_\beta + \frac{1}{\beta}\Big(\frac{d}{d\beta}h^{\mathsf{a}}_\beta\Big) = -\frac{1}{\beta^2}f(\beta). \tag{1.3.49}$$

We choose ε independently of n and β so that it holds that $\lim_{\varepsilon\to 0}\lim_{n\to\infty}\mathrm{F}'_n(\beta_c) = 0$ as $n\to\infty$. Therefore, we can conclude that $\liminf_{n\to\infty}\mathrm{F}_n(\beta) \ge \frac{1}{\beta_c}h^{\mathsf{a}}_{\beta_c}$. The result follows. ■

Chapter 2

The pinning model

The pinning model was introduced for the study of the behavior of linear polymers by Fisher [35]. Linear polymers are chemical compounds consisting of monomers. The polymer chain is defined as a Markov chain on $\mathbb{Z}$, and we consider a renewal process which is given as a sequence of inter-arrival times of the Markov chain.

2.1 In a homogeneous setting

Let τ be a renewal process with $\tau_0 = 0$ and assume that τ is non-terminating, that is, $\mathbb{P}(\tau_1 < \infty) = 1$. The distribution of τ is given by

$$K(n) := \mathbb{P}(\tau_1 = n) = \frac{L(n)}{n^{1+\alpha}}, \quad \alpha > 0, \tag{2.1.1}$$

for all $n \in \mathbb{N}$, where $L(n)$ is a slowly varying function (See [15]) and the parameter α is positive. If $\alpha = 1/2$, the Markov chain represents return times of a 1-dimensional simple random walk. We denote the probability measure and its expectation by $\mathbb{P}$ and $\mathbb{E}$ respectively, and define the expected waiting time $\mu := \mathbb{E}[\tau_1]$ (note that μ is not the connective constant in this chapter). If $\alpha < 1$, μ is infinite, and otherwise. Then, we give the definitions of the polymer measure $\mathbb{P}_{n,h}$ and the partition function $Z_{n,h}$ respectively by the followings.

$$\frac{\mathrm{d}\mathbb{P}_{n,h}}{\mathrm{d}\mathbb{P}}(\tau) = \frac{1}{Z_{n,h}} e^{h\sum_{k=1}^{n}\delta_k}\delta_n, \quad n \in \mathbb{N}, \quad h \in \mathbb{R}, \tag{2.1.2}$$

where the partition function $Z_{n,h}$ is given by

$$Z_{n,h} := \mathbb{E}\left[e^{h\sum_{k=1}^{n}\delta_k}\delta_n\right], \quad Z_{0,h} = 1. \tag{2.1.3}$$

In (2.1.3), the function δ_n is an indicator function $\mathbf{1}_{\{n\in\tau\}}$. The definition (2.1.2) implies that a Markov chain obtains e^h at every renewal (see Figure 2.1). Therefore, if $h < 0$, a Markov chain penalized by $e^h < 1$ at every renewal. So that it is dominant that paths do not visit the origin and we regard that the Markov chain tends to delocalize from the origin. From this observation, we have a phase transition between the localized phase and the delocalized phase. We define a critical point h_c that separates these phases and it is predicted that $h_c = 0$. Note that if $\sum_n K(n) < 1$, then we put $\widetilde{K}(n) := \frac{K(n)}{1-K(\infty)}$ and replace h by $h + \log(1 - K(\infty))$. So that the critical point can be $h_c = -\log(1-K(\infty)) = -\log \sum_n K(n)$. We also define he free energy $\mathrm{F} : \mathbb{R} \mapsto [0, \infty)$ by

$$\sum_{n\in\mathbb{N}} e^{-n\mathrm{F}(h)+h} K(n) = 1, \tag{2.1.4}$$

when such a solution exists, i.e., for $h \geq 0$. When we can not solve this equation, i.e., for $h < 0$, we set $\mathrm{F}(h) = 0$. It is well-known the phase transition is characterized by the free energy F.

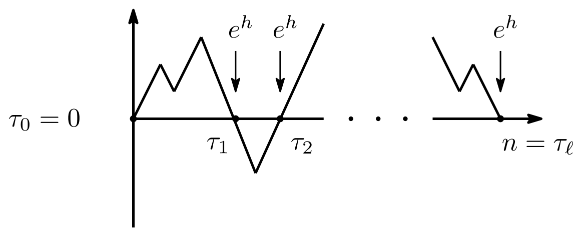

Figure 2.1: A sample of a Markov chain and the renewal process τ pinned at n visit at n at ℓ-th renewal. The Markov chain obtains a weight e^h in each renewal.

We consider the relation between the free energy and the partition function (see Figure 2.1). By a simple computation,

$$\begin{aligned} Z_{n,h} &= \sum_{k=1}^{n} \sum_{\substack{\ell\in\mathbb{N}^k \\ |\ell|=n}} \prod_{j=1}^{k} K(\ell_j) e^h = \sum_{k=1}^{n} \sum_{\substack{\ell\in\mathbb{N}^k \\ |\ell|=n}} \prod_{j=1}^{k} e^{\mathrm{F}(h)\ell_j} \widetilde{K}_h(\ell_j) \\ &= e^{n\mathrm{F}(h))} \sum_{k=1}^{n} \sum_{\substack{\ell\in\mathbb{N}^k \\ |\ell|=n}} \prod_{j=1}^{k} \widetilde{K}_h(\ell_j) = e^{n\mathrm{F}(h)} \mathbb{P}(n \in \tilde{\tau}^{(h)}), \end{aligned} \tag{2.1.5}$$

where the law $\widetilde{K}_h$ is defined by

$$\widetilde{K}_h(n) := e^{-n\mathrm{F}(h)+h} K(n). \tag{2.1.6}$$

By the definition (2.1.6), $\tilde{\tau}^{(h)}$ is also a renewal process with the density $\widetilde{K}_h$. Then we have the following relation between the free energy $\mathrm{F}(h)$ and the partition function $Z_{n,h}$.

$$\lim_{n\to\infty}\frac{1}{n}\log Z_{n,h}=\mathrm{F}(h). \tag{2.1.7}$$

This expression is often used for the definition of the free energy. (2.1.7) can be proved based on the renewal theorem (See [41] or [42]). By (2.1.5), it is sufficient to show $\log\mathbb{P}(n\in\tilde{\tau}^{(h)})=o(n)$. For $h<0$, we have trivial bounds as $e^hK(n)\le\mathbb{P}(n\in\tilde{\tau}^{(h)})\le 1$. For $h>0$, by the renewal theorem, $\mathbb{P}(n\in\tilde{\tau}^{(h)})$ goes to $\mathbb{E}[\tilde{\tau}_1^{(h)}]$ as $n\to\infty$. To understand the critical phenomenon, we often focus on the behavior of the free energy at the critical point h_c.

Theorem 2.1.1 ([41, 42]). *If $h_c=-\log\sum_n K(n)$ (i.e. $K(\infty)>0$), we have that*

$$\mathrm{F}(h)\underset{h\searrow h_c}{\sim}C(K)\times\begin{cases} h-h_c & \alpha>1\ , \\ \frac{h-h_c}{\log(h-h_c)} & \alpha=1\ , \\ (h-h_c)^{1/\alpha} & \alpha\in(0,1)\ , \end{cases} \tag{2.1.8}$$

where

$$C(K)=\begin{cases} \sum_n K(n)/\sum_n nK(n) & \alpha>1\ , \\ 1/L(n) & \alpha=1\ , \\ \left\{\alpha\sum_n K(n)/\big(L(n)\Gamma(1-\alpha)\big)\right\}^{1/\alpha} & \alpha\in(0,1)\ . \end{cases} \tag{2.1.9}$$

Proof. We give the rough sketch of the proof of Theorem 2.1.1 (refer to [41, 42] for more detail). First, we introduce the function φ as follows.

$$\varphi(x)=1-\sum_{n=1}^{\infty}e^{-nx}K(n). \tag{2.1.10}$$

We compute $\varphi(x)$. Let $\bar{K}(n):=\sum_{j\ge n}K(j)$, then

$$\varphi(x)=1-\sum_{n=1}^{\infty}e^{-nx}\big(\bar{K}(n-1)-\bar{K}(n)\big)=\big(1-e^{-x}\big)\sum_{n=0}^{\infty}e^{-nx}\bar{K}(n). \tag{2.1.11}$$

By the renewal theorem, we have the asymptotic behavior of $\varphi(x)$ as follows.

$$\varphi(x)\underset{x\downarrow 0}{\sim}\begin{cases} x\mathbb{E}[\tau_1] & \text{if } \alpha>1, \\ L(n)\ x\log\frac{1}{x} & \text{if } \alpha=1, \\ \frac{\Gamma(1-\alpha)}{\alpha}x^{\alpha} & \text{if } \alpha<1 \end{cases} \tag{2.1.12}$$

We have $\mathrm{F}(h)\downarrow 0$ for $h\to h_c$. Recall that by the definition of the free energy (2.1.4), we have $\varphi(\mathrm{F}(h))=1-e^{-h}$. Therefore, replacing x by $\mathrm{F}(h)$, the proof is completed. ■

2.2 The general pinning model

The pinning model with i.i.d. disorder is regarded as a general model but it has so many heuristic arguments and methods to analyze the disordered systems. In order to make the argument lighter, we replace (2.1.1) by

$$K(n) := \mathbb{P}(\tau_1 = n) \underset{n\nearrow\infty}{\sim} \frac{c_K}{n^{1+\alpha}}, \quad \alpha > 0, \tag{2.2.1}$$

for all $n \in \mathbb{N}$, where $c_K > 0$ is a normalization constant. We assume that ω is a sequence of i.i.d. random variables satisfying the followings.

$$\mathbb{E}_\omega[\omega_1] = 0,\ \mathbb{E}_\omega[\omega_1^2] = 1,\ \mathbb{E}_\omega[e^{\beta\omega_1}] \equiv M_\beta < \infty, \tag{2.2.2}$$

where we denote the probability measure and its expectation by $\mathbb{P}_\omega$ and $\mathbb{E}_\omega$ respectively. For $\beta \geq 0$, $h \in \mathbb{R}$ and a fixed realization of ω, we can respectively define the probability measure $\mathbb{P}_\omega$ and the quenched partition function $Z_{n,\omega}$ by

$$\frac{\mathrm{d}\mathbb{P}_\omega}{\mathrm{d}\mathbb{P}}(\tau) = \frac{1}{Z_{n,\omega}} e^{\sum_{k=1}^n (h+\beta\omega_k)\delta_k} \delta_n, \tag{2.2.3}$$

and

$$Z_{n,\omega} := Z_{n,\beta,h} = \mathbb{E}\Big[e^{\sum_{k=1}^n (h+\beta\omega_k)\delta_k} \delta_n\Big]. \tag{2.2.4}$$

We also define the annealed counterparts by

$$\frac{\mathrm{d}\mathbb{P}^{\mathsf{a}}_\omega}{\mathrm{d}\mathbb{P}\times\mathbb{P}_\omega}(\tau) = \frac{1}{Z^{\mathsf{a}}_{n,\omega}} e^{\sum_{k=1}^n (h+\beta\omega_k)\delta_k} \delta_n, \tag{2.2.5}$$

and

$$Z^{\mathsf{a}}_{n,\omega} = Z^{\mathsf{a}}_{n,\beta,h} = \mathbb{E}_\omega[Z_{n,\omega}]. \tag{2.2.6}$$

By the Fubini-Tonelli theorem and the i.i.d property of ω, we can compute the annealed partition function as

$$\begin{aligned} Z^{\mathsf{a}}_{n,\omega} &= \mathbb{E}_\omega\mathbb{E}[e^{\sum_{k=1}^n (h+\beta\omega_n)\delta_k} \delta_n] \\ &= \mathbb{E}[e^{\sum_{k=1}^n (h+\log M_\beta)\delta_k} \delta_n] = Z_{h+\log M_\beta}. \end{aligned} \tag{2.2.7}$$

Therefore, the annealed free energy is given as the homogeneous one.

$$\mathrm{F}^{\mathsf{a}}(\beta, h) = \lim_{n\to\infty} \log Z^{\mathsf{a}}_{n,\omega} = \mathrm{F}(0, h + \log M_\beta) \tag{2.2.8}$$

The annealed critical point h_β^{a} is $h_c(0) - \log M_\beta$, where we write by $h_c(0)$ the critical point in homogeneous case. On the other hand, the quenched free energy is defined by the following limit

$$\mathrm{F}(\beta, h) := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_\omega[\log Z_{n,\omega}] = \sup_{n\in\mathbb{N}_0} \frac{1}{n}\mathbb{E}_\omega[\log Z_{n,\omega}], \tag{2.2.9}$$

for all $h \in \mathbb{R}$ and $\beta \geq 0$. The existence of this limit and the last equality follows from the super-additivity of the sequence $\{\mathbb{E}_\omega[\log Z_{n,\omega}]\}_{n\in\mathbb{N}}$, i.e., For $m = 1, \cdots, n-1$,

$$\begin{aligned}
\log Z_{n,\omega} &\geq \log \mathbb{E}\left[e^{\sum_{k=1}^{n}(h+\beta\omega_n)\delta_k}\delta_n\delta_m\right] \\
&= \log \mathbb{E}\left[e^{\sum_{k=1}^{m}(h+\beta\omega_n)\delta_k}\delta_m\right] + \log \mathbb{E}\left[e^{\sum_{k=m+1}^{n}(h+\beta\omega_n)\delta_k}\delta_n | m \in \tau\right] \\
&= \log Z_{m,\omega} + \log Z_{n-m,\theta^m\omega},
\end{aligned} \tag{2.2.10}$$

where θ is a shift by one step to the left. By the definition (2.2.9), we can see some straightforward properties of the quenched free energy.

- The quenched free energy $\mathrm{F}(\beta, h)$ is non-decreasing in both h and β.
- The function $(\beta, h) \mapsto \mathrm{F}(\beta, h)$ is a convex since $\mathrm{F}(\beta, h)$ is the limit of a sequence of convex functions.

The following property called the self-averaging holds for the quenched free energy.

Proposition 2.2.1. *$\mathbb{P}$-almost surely and in L^1 we have that for every β and h,*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_{n,\omega} = \mathrm{F}(\beta, h). \tag{2.2.11}$$

Note that the assumption of i.i.d. environment is not essential for this proposition (In Section 2.3 we can also prove this property in another setting). The proof of Proposition 2.2.1 is a direct consequence of the super-additivity (2.2.10) and the sub-additive ergodic theorem (see [50]), however we put another proof to show the basic argument of the pinning models.

Proof. By the super-additivity (2.2.10), letting $n = km + r$,

$$\frac{1}{n}\log Z_{n,\omega} \geq \frac{m}{n}\sum_{j=0}^{\frac{n}{m}-1} \frac{1}{m}\log Z_{m,\theta^{jm}\omega}. \tag{2.2.12}$$

Applying the strong law of large numbers to the lefthand-side of (2.2.12), we have

$$\liminf_{n\to\infty} \frac{1}{n} \log Z_{n,\omega} \geq \mathbb{E}_\omega\Big[\frac{1}{m}\log Z_{m,\theta^{jm}\omega}\Big]. \tag{2.2.13}$$

Therefore, by the definition (2.2.9), we obtain

$$\mathrm{F}(\beta,h) = \limsup_{m\to\infty} \frac{1}{m}\mathbb{E}_\omega[\log Z_{m,\theta^{jm}\omega}] \leq \liminf_{n\to\infty}\frac{1}{n}\log Z_{n,\omega}. \tag{2.2.14}$$

On the other hand, by decomposing the partition function, we have

$$Z_{n,\omega} = \underbrace{Z_{m,\omega}Z_{n-m,\theta^m\omega}}_{m\in\tau} + \underbrace{\sum_{i=0}^{m-1} Z_{i,\omega}\sum_{j-m+1}^{n} K(j-i)e^{h+\beta\omega_j}Z_{n-j,\theta^j\omega}}_{m\notin\tau}. \tag{2.2.15}$$

There exists a constant $c>0$ (see (2.2.1)) such that for every i,j, and m,

$$\frac{K(j-i)}{K(m-i)K(j-m)} \leq c\{(j-m)\wedge(m-i)\}^{1+\alpha} \leq c\{(n-m)\wedge m\}^{1+\alpha}. \tag{2.2.16}$$

Substituting (2.2.16) into (2.2.15), then taking logarithms and deviding by n, we have

$$\frac{1}{n}\log Z_{n,\omega} \leq \frac{m}{n}\sum_{j=0}^{\frac{n}{m}-1}\Big(\frac{1}{m}\log Z_{m,\theta^{jm}\omega} + \frac{1}{m}(c+\beta|\omega_{jm}|+(1+\alpha)\log m)\Big). \tag{2.2.17}$$

Therefore, using the same argument as estimating a lower bound, the result follows. ■

By the monotonicity in β and Jensen's inequality, we have

$$\mathrm{F}(h) = \mathrm{F}(0,h) \leq \mathrm{F}(\beta,h) \leq \mathrm{F}^{\mathsf{a}}(\beta,h). \tag{2.2.18}$$

This directly yields

$$h_\beta^{\mathsf{a}} = h_c - \log M_\beta \leq \hat{h}_\beta^{\mathsf{q}} \leq h_c. \tag{2.2.19}$$

Therefore, we can define these two phases

$$\mathcal{L} = \{(\beta,h) : h > \hat{h}_\beta^{\mathsf{q}}\}, \quad \mathcal{D} = \{(\beta,h) : h \leq \hat{h}_\beta^{\mathsf{q}}\}. \tag{2.2.20}$$

The region $\mathcal{L}$ is called the localized phase and the region $\mathcal{D}$ is called the delocalized phase.

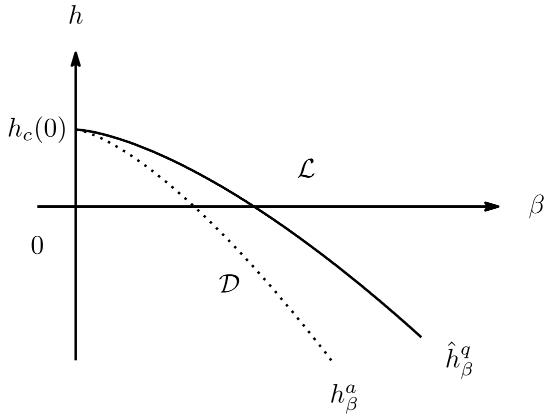

Figure 2.2: The critical curve $\beta \mapsto \hat{h}^{\mathtt{q}}_\beta$ that separates $\mathcal{L}$ and $\mathcal{D}$ is convex and decreasing in β. Recall that h_c is the homogeneous critical point and it is positive if τ is not terminating.

The curvature of the quenched critical point close to $\beta = 0$ has been studied with big interests on the pinning model. In fact, it is known that if $\alpha < \frac{1}{2}$, the quenched and annealed critical points coincide for small β. We call this regime the irrelevant disorder regime. In the irrelevant disorder regime, i.e., for $\alpha \in (0, \frac{1}{2})$, the asymptotic behavior of the quenched free energy is given as

$$\lim_{h \downarrow \hat{h}^{\mathtt{q}}_\beta} \frac{\log \mathrm{F}(\beta, h)}{\log(h - \hat{h}^{\mathtt{q}}_\beta)} = \frac{1}{\alpha} < 2. \tag{2.2.21}$$

Recalling (2.1.8) in Theorem 2.1.1, the behavior of the quenched free energy around its critical point is the same as the one of the homogeneous free energy. On the other hand, it is known that if disorder is relevant, i.e., for $\alpha > \frac{1}{2}$ and $\beta > 0$, the critical behavior differs from the one of the homogeneous system. In the relevant disorder regime, it has been known that

$$\liminf_{h \downarrow \hat{h}^{\mathtt{q}}_\beta} \frac{\log \mathrm{F}(\beta, h)}{\log(h - \hat{h}^{\mathtt{q}}_\beta)} \geq 2, \tag{2.2.22}$$

for every $\beta > 0$. Comparing with the curvature in the irrelevant disorder regime, the quenched free energy in the relevant regime is smoother than that in the irrelevant disorder regime. We call this phenomenon smoothing effect. The smoothing effect for the free energy plays very important role to understand the relevant disorder regime. A smoothing inequality (2.2.22) is proven for general cases (see [17]).

The fractional moment method is known for the useful method to estimate the quenched critical point. This method has been standard in the disordered systems. In [41, 42] the following theorem is proven by the fractional moment method.

Theorem 2.2.2 ([41, 42]). *For small $\beta > 0$, there exists a constant c such that*

$$\hat{h}^{\mathsf{q}}_{\beta} - h^{\mathsf{a}}_{\beta} \geq \begin{cases} c\beta^2 & \alpha > 1 \\ c\beta^2/(\log(1+\frac{1}{\beta}))^2 & \alpha = 1 \\ c\beta^{\frac{2\alpha}{2\alpha-1}} & \alpha \in (\frac{1}{2}, 1) \\ \exp\left\{\frac{-1}{c\beta^4}\right\} & \alpha = \frac{1}{2} \end{cases} \tag{2.2.23}$$

These constants are given, for example the case $\alpha > 1$. Alexander and Zygouras [5] and den Hollander [26] also give the constant. Berger, Caravenna, Poisat, Sun and Zygouras prove the asymptotic behavior below in [10].

Theorem 2.2.3 ([10]).

$$\hat{h}^{\mathsf{q}}_{\beta} \underset{\beta \searrow 0}{\sim} (-\frac{1}{2} + \frac{1}{2\mu}\frac{\alpha}{\alpha+1})\beta^2 \ , \ \alpha > 1 \ . \tag{2.2.24}$$

Recently, Caravenna, Toninelli and Torri [19] have proven the asymptotic behavior of the quenched critical point in the case $\alpha \in (1/2, 1)$. They use the scaling limit and consider the continuous pinning model. This may be a next standard method to understand the relevant regime of the disordered systems. Many rigorous results about the quenched critical point and the behavior of the quenched free energy have been obtained, however, it still remain challenging open problems on the quenched case.

2.2.1 The pinning model on a Markovian environment

Here we introduce the pinning model with Markov disorder as an example of short-range correlated model, which is studied in my master thesis. We consider the renewal process τ as in the i.i.d. environment. Let $\omega = \{\omega_n\}_{n\geq 0}$ be a sequence of Markovian random variables satisfying that

$$\mathbb{P}_{\omega}(\omega_j = +1|\omega_{j-1}) = \frac{1+\varepsilon\omega_{j-1}}{2}, \quad \mathbb{P}_{\omega}(\omega_j = -1|\omega_{j-1}) = \frac{1-\varepsilon\omega_{j-1}}{2}, \tag{2.2.25}$$

where $\varepsilon > 0$ is regarded as the strength of memory. In this environment, the potential at n is affected by the status at the previous step. We define the quenched and annealed partition functions and free energies as follows.

$$Z_{n,\omega} = \mathbb{E}[e^{\sum_{k=1}^{n}(h+\beta\omega_k\omega_{k-1})\delta_k}\delta_n], \qquad Z^{\mathsf{a}}_{n,\omega} = \mathbb{E}_{\omega}[Z_{n,\omega}], \tag{2.2.26}$$

and

$$\mathrm{F}(\beta, h) = \lim_{n\to\infty}\frac{1}{n}\log Z_{n,\omega}, \quad \mathrm{F}^{\mathsf{a}}(\beta, h) = \lim_{n\to\infty}\frac{1}{n}\log Z^{\mathsf{a}}_{n,\omega}. \tag{2.2.27}$$

We note that the Hamiltonian of the quenched partition function has non-linear part $\omega_k\omega_{k-1}$. If $\beta = 0$, we can reduce this model to the homogeneous case. The following lemma plays important role of this model.

Lemma 2.2.4. *For any $n \in \mathbb{N}$,*

$$\mathbb{P}_\omega(\omega_n = s_n, \cdots, \omega_1 = s_1) = \prod_{j=0}^{n}(1+\varepsilon\tilde{s}_j)\mathbb{P}_\eta(\eta_j = \tilde{s}_j), \tag{2.2.28}$$

where $\omega_{k+1}\omega_k = \eta_k$ and $\tilde{s}_j = s_{j+1}s_j$ with $\tilde{s}_0 = 0$. Therefore, $\eta = \{\eta_n\}_{n\geq 0}$ is a sequence of i.i.d. random variables on $\{\pm 1\}$ with $\eta_0 = \omega_0 = 0$, $\mathbb{P}_\eta$ is the Bernoulli measure for η, and

$$\mathbb{P}_\eta(\eta_j = +1) = \mathbb{P}_\eta(\eta_j = -1) = \frac{1}{2}. \tag{2.2.29}$$

Proof . The proof is based on the total probability formula and Markovian property.

$$\begin{aligned}\mathbb{P}_\omega(\omega_n = s_n, \cdots, \omega_1 = s_1) &= \mathbb{P}_\omega(\omega_n = s_n|\omega_{n-1} = s_{n-1}) \\ &\quad \cdots \mathbb{P}_\omega(\omega_2 = s_2|\omega_1 = s_1)\mathbb{P}_\omega(\omega_1 = s_1) \\ &= \frac{1+\varepsilon s_n s_{n-1}}{2} \cdots \frac{1+\varepsilon s_2 s_1}{2}\frac{1}{2} \\ &= \left(\frac{1}{2}\right)^n \prod_{j=1}^{n}(1+\varepsilon s_j s_{j-1}). \end{aligned} \tag{2.2.30}$$

Let $\eta = \{\eta_n\}_{n\geq 0}$ be the sequence of i.i.d. random variables on $\{\pm 1\}$ which satisfy that

$$\mathbb{P}_\eta(\eta_n = +1) = \mathbb{P}_\eta(\eta_n = -1) = \frac{1}{2}. \tag{2.2.31}$$

Therefore,

$$\mathbb{P}_\omega(\omega_n = s_n, \cdots, \omega_1 = s_1) = \prod_{j=0}^{n}(1+\varepsilon\tilde{s}_j)\mathbb{P}_\eta(\eta_j = \tilde{s}_j), \tag{2.2.32}$$

where $\tilde{s}_j = s_{j+1}s_j$ with $\tilde{s}_0 = 0$. We complete the proof. ■

In the annealed case, we can compute its critical point by the lemma above.

$$\begin{aligned} Z^{\mathsf{a}}_{n,\omega} &= \mathbb{E}\big[e^{h\sum_{k=1}^n \delta_k}\mathbb{E}_\omega[e^{\beta\sum_{k=1}^n \omega_k\omega_{k-1}\delta_k}]\delta_n\big] \\ &= \mathbb{E}\big[e^{h\sum_{k=1}^n \delta_k}\mathbb{E}_\eta\big[\prod_{k=1}^{n}(1+\varepsilon\eta_k)e^{\beta\eta_k\delta_k}\big]\delta_n\big] \\ &= \mathbb{E}\big[e^{h\sum_{k=1}^n \delta_k}\prod_{k=1}^{n}\mathbb{E}_\eta\big[(1+\varepsilon\eta_k)e^{\beta\eta_k}\big]^{\delta_k}\delta_n\big] \end{aligned} \tag{2.2.33}$$

Since η takes value on $\{\pm 1\}$, $\mathbb{E}_\eta[(1+\varepsilon\eta_k)e^{\beta\eta_k}] = \cosh\beta + \varepsilon\sinh\beta$. Therefore, we have

$$h_\beta^{\mathsf{a}} = -\log M_{\varepsilon,\beta}. \tag{2.2.34}$$

where $M_{\varepsilon,\beta} = \cosh\beta + \varepsilon\sinh\beta$. In an i.i.d. environment, if ω takes value on $\{\pm 1\}$, then the annealed critical point is $h_\beta^{\mathsf{a}} = -\log M_\beta \sim -\frac{1}{2}\beta^2$ as $\beta \downarrow 0$. However, in a Markovian environment, the annealed critical point (2.2.34) is asymptotically equal to $\varepsilon\beta$. About the quenched case, the results is same as those in general case. For example, we can prove the existence and the self-averaging property of the quenched free energy. We define the shifted quenched partition function by,

$$Z_{n-m,\theta^m\omega} = \mathbb{E}_\omega\left[e^{\sum_{k=m+1}^{n}(h+\beta\omega_k\omega_{k-1})\delta_k}\delta_n\right], \tag{2.2.35}$$

for $m < n$. The following lemma implies an estimate for the gap by the shift θ^m.

Lemma 2.2.5. *For every $\beta \geq 0$, $h \in \mathbb{R}$, and for $m, n \in \mathbb{N}$ $(m < n)$,*

$$\mathbb{E}_\omega\Big[\log Z_{n-m,\omega}\Big] - \beta \leq \mathbb{E}_\omega\Big[\log Z_{n-m,\theta^m\omega}\Big] \leq \mathbb{E}_\omega\Big[\log Z_{n-m,\omega}\Big] + \beta. \tag{2.2.36}$$

Proof. We can prove this lemma straightforwardly. Isolating one step, we have

$$Z_{n-m,\theta^m\omega} = \mathbb{E}_\omega\left[e^{h+\sum_{k=m+2}^{n}(h+\beta\omega_k\omega_{k-1})\delta_k}e^{\beta\omega_{m+1}\omega_m\delta_{m+1}}\delta_n\right]. \tag{2.2.37}$$

We easily show that $e^{\beta\omega_1\omega_0\delta_1} = 1$ and $e^{-\beta} \leq e^{\beta\omega_{m+1}\omega_m\delta_{m+1}} \leq e^\beta$. Since ω is a sequence of Markovian random variables, we conclude this lemma. ■

By Lemma 2.2.5, we can say the super-additivity for $\{\mathbb{E}_\omega[\log Z_{n,\omega}]\}$. Therefore, we can prove the existence and the self-averaging property of the quenched free energy same as the general case. The critical point may differ, however it is known that the asymptotic behavior of the free energy does not affect by inducing a Markovian property to the environment.

2.3 The pinning model on renewal set

This section is based on the joint work with Dimitris Cheliotis and Julien Poisat. We study the pinning model on a random environment defined by a renewal set. This model belongs neither to the class of the short-range correlations studied in Poisat [58, Theorems 2.1 and 2.2] and Berger [8, Theorems 2.2 and 2.3], nor to the class of strong disorder introduced in Berger [8, Theorem 2.5], [9, Theorems 1.5, 1.6 and 2.9] and Berger and Lacoin [11, Theorem 1.5]. The long-range correlated disorder model has been treated in [58] and [8]. They assume the long-range correlated Gaussian environment with power-low decaying correlation function. The model we consider in this section belongs to the class of the long-range correlations but the power of decaying is not clear.

2.3.1 Introduction of the pinning model on renewal set

As in Section 2.1, τ is defined as a renewal process with $\tau_0 = 0$, and we assume that the distribution of τ is given by (2.1.1). Let $\hat{\tau}$ be another (recurrent) renewal process with $\hat{\tau}_0 = 0$ independent of τ. Similarly to the renewal process τ, we define its inter-arrival law as

$$\hat{K}(n) := \hat{\mathbb{P}}(\hat{\tau}_1 = n) = \frac{c_{\hat{K}}}{n^{1+\hat{\alpha}}}, \quad \hat{\alpha} > 0, \tag{2.3.1}$$

for all $n \in \mathbb{N}$, where $c_{\hat{K}}$ is a normalization constant. The results in this section is still valid for the case that we put a slowly varying function $\hat{L}(n)$ instead of $c_{\hat{K}}$, that is, for all $n \in \mathbb{N}$,

$$\hat{K}(n) := \hat{\mathbb{P}}(\hat{\tau}_1 = n) = \frac{\hat{L}(n)}{n^{1+\hat{\alpha}}}, \quad \hat{\alpha} > 0. \tag{2.3.2}$$

In this section we use both definitions for making the argument lighter. Similarly to τ, we denote by $\hat{\mathbb{P}}$ and $\hat{\mathbb{E}}$ the probability measure and its expectation respectively generated by $\hat{K}$. We also set the expected waiting time of $\hat{\tau}$ by

$$\hat{\mu} := \hat{\mathbb{E}}[\hat{\tau}_1], \tag{2.3.3}$$

which may be finite or infinite. The quenched polymer measure $\mathbb{P}_{n,\beta,h}$ and the quenched partition function $Z_{n,\hat{\tau}}$ are respectively defined by

$$\frac{\mathrm{d}\mathbb{P}_{n,\beta,h}}{\mathrm{d}\mathbb{P}}(\tau) = \frac{1}{Z_{n,\hat{\tau}}} e^{\sum_{k=1}^{n}(h+\beta\hat{\delta}_k)\delta_k}\delta_n, \tag{2.3.4}$$

and

$$Z_{n,\hat{\tau}} := Z_{n,\beta,h} := \mathbb{E}\Big[e^{\sum_{k=1}^{n}(h+\beta\hat{\delta}_k)\delta_k}\delta_n\Big], \quad Z_{0,\hat{\tau}} = 1, \tag{2.3.5}$$

for $n \in \mathbb{N}$, $\beta \geq 0$, and $h \in \mathbb{R}$, where $\hat{\delta}_k = \mathbf{1}_{\{k \in \hat{\tau}\}}$. Recall that $\mathbb{P}$ and $\mathbb{E}$ are the probability measure and the expectation for the renewal process τ respectively and $\delta_k = \mathbf{1}_{\{k \in \tau\}}$. The annealed counterparts are also defined by

$$\frac{\mathrm{d}\mathbb{P}^{\mathsf{a}}_{n,\beta,h}}{\mathrm{d}\mathbb{P} \times \hat{\mathbb{P}}}(\tau) = \frac{1}{Z^{\mathsf{a}}_{n,\hat{\tau}}} e^{\sum_{k=1}^{n}(h+\beta\hat{\delta}_k)\delta_k}\delta_n, \tag{2.3.6}$$

and

$$Z^{\mathsf{a}}_{n,\hat{\tau}} := Z^{\mathsf{a}}_{n,\beta,h} = \hat{\mathbb{E}}\mathbb{E}\Big[e^{\sum_{k=1}^{n}(h+\beta\hat{\delta}_k)\delta_k}\delta_n\Big], \quad Z^{\mathsf{a}}_{0,\hat{\tau}} = 1, \tag{2.3.7}$$

for $n \in \mathbb{N}$, $\beta \geq 0$, and $h \in \mathbb{R}$. We denote by $\widetilde{\tau}$ the intersection of the two renewals, which is itself a renewal process.

2.3.2 Annealed behavior on renewal set

We begin with the annealed model and show some results on the annealed case. The first result on the annealed model is the existence of the annealed free energy which characterizes the rate of the annealed partition function $Z^{\mathsf{a}}_{n,\hat{\tau}}$. Similarly to the general case, the annealed free energy $\mathrm{F}^{\mathsf{a}}(\beta, h)$ must be defined by

$$\mathrm{F}^{\mathsf{a}}(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log Z^{\mathsf{a}}_{n,\hat{\tau}}. \tag{2.3.8}$$

The following proposition shows that this limit is well-defined and is finite, non-negative. To simplify the proof, we introduce the fully-pinned partition function by

$$Z^{\mathsf{a,c}}_{n,\hat{\tau}} := \hat{\mathbb{E}}\mathbb{E}\left[e^{\sum_{k=1}^{n}(h+\beta\hat{\delta}_k)\delta_k}\delta_n\hat{\delta}_n\right]. \tag{2.3.9}$$

Proposition 2.3.1 (The existence of the free energy). *For all $\beta \geq 0$ and $h \in \mathbb{R}$, the annealed free energy* (2.3.8) *exists and it is finite and non-negative.*

These basic properties of the annealed free energy $\mathrm{F}^{\mathsf{a}}(\beta, h)$ are the same as those for the i.i.d. case, i.e., the function $(\beta, h) \mapsto \mathrm{F}^{\mathsf{a}}(\beta, h)$ is convex, continuous and non-decreasing (see Figure 2.3). Therefore, we can define the annealed critical point h^{a}_β for all β.

$$h^{\mathsf{a}}_\beta = \inf\{h \in \mathbb{R} : \mathrm{F}^{\mathsf{a}}(\beta, h) > 0\}. \tag{2.3.10}$$

It is obvious that $\mathrm{F}^{\mathsf{a}}(\beta, h) = 0 \Leftrightarrow h \leq h^{\mathsf{a}}_\beta$ by this definition.

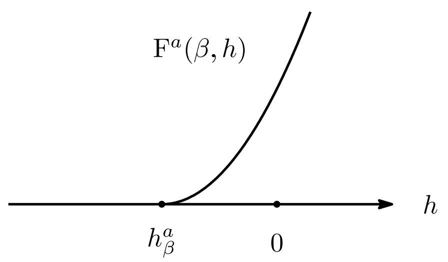

Figure 2.3: The annealed free energy is convex and increasing. The critical point h^{a}_β is negative, i.e., less than the homogeneous critical point.

Before we show the next result, we observe the annealed critical point by comparing it with the critical point for the homogeneous one. Due to the proof of Proposition 2.3.1, we have the expression of the fully-pinned annealed free energy which is equal to $\mathrm{F}^{\mathsf{a}}(\beta, h)$.

$$\mathrm{F}^{\mathsf{a}}(\beta, h) = \mathrm{F}^{\mathsf{a,c}}(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log Z^{\mathsf{a,c}}_{n,\hat{\tau}}. \tag{2.3.11}$$

So that we consider $Z^{\mathsf{a,c}}_{n,\hat\tau}$ instead of $Z^{\mathsf{a}}_{n,\hat\tau}$. Then, we transform the fully-pinned partition function $Z^{\mathsf{a,c}}_{n,\beta,h}$ as

$$Z^{\mathsf{a,c}}_{n,\hat\tau} = \hat{\mathbb{E}}\mathbb{E}\left[e^{\sum_{k=1}^n (h+\beta\hat\delta_k)\delta_k}\delta_n\hat\delta_n\right] = \hat{\mathbb{E}}\mathbb{E}_h\left[e^{\beta\sum_{k=1}^n \hat\delta_k\delta_k}\delta_n\hat\delta_n\right]. \tag{2.3.12}$$

Here we introduce another interarrival law

$$K_h(n) := e^h K(n), \quad K_h(\infty) = 1 - e^h, \tag{2.3.13}$$

therefore, we can regard $Z^{\mathsf{a,c}}_{n,\hat\tau}$ as a homogeneous partition function under $\hat{\mathbb{P}} \times \mathbb{P}_h$. By the definition of the homogeneous critical point, the annealed critical point h^{a}_β satisfies that

$$\beta = -\log\sum_{n\in\mathbb{N}} \hat K(n) K_{h^{\mathsf{a}}_\beta}(n). \tag{2.3.14}$$

To make the argument below clear, we introduce the notation $p(h)$ by

$$p(h) = \sum_{n\in\mathbb{N}} \hat K(n) K_h(n) = \hat{\mathbb{P}} \times \mathbb{P}_h(\widetilde\tau^{(h)}_1 < \infty), \tag{2.3.15}$$

where $\widetilde\tau^{(h)}$ is a renewal process under $\hat{\mathbb{P}} \times \mathbb{P}_h$. Then, we can simply rewrite (2.3.14) as

$$\beta = -\log p(h^{\mathsf{a}}_\beta). \tag{2.3.16}$$

So that $\mathrm{F}^{\mathsf{a}}(\beta, h) > 0$ if and only if $\beta > -\log p(h)$. We also introduce the notation $\mathcal{I}(h)$ by

$$\begin{aligned}\mathcal{I}(h) &:= \hat{\mathbb{E}}\mathbb{E}_h[|\widetilde\tau|] \\ &= \sum_{k=0}^{\infty} \mathbb{P}_h(k\in\tau)\hat{\mathbb{P}}(k\in\hat\tau).\end{aligned} \tag{2.3.17}$$

We have the following relation between $p(h)$ and $\mathcal{I}(h)$ by the renewal theorem.

$$p(h) = \hat{\mathbb{P}} \times \mathbb{P}_h(\widetilde\tau_1 < \infty) = 1 - \hat{\mathbb{P}} \times \mathbb{P}_h(\widetilde\tau_1 = \infty) = 1 - \frac{1}{\mathcal{I}(h)}. \tag{2.3.18}$$

Therefore, we have

$$\beta = -\log(1 - \mathcal{I}(h^{\mathsf{a}}_\beta)^{-1}) \Leftrightarrow \mathcal{I}(h^{\mathsf{a}}_\beta) = \frac{1}{1-e^{-\beta}}. \tag{2.3.19}$$

Remark 2.3.2. In this remark, we use the definition (2.3.2) for $\hat{\tau}$. Let

$$\beta_0 := -\log p(0), \tag{2.3.20}$$

then by the relation (2.3.18) and the argument below, we have that β_0 is positive if and only if $\mathcal{I}(0)$ is finite. Therefore, we see that

$$\beta_0 = \begin{cases} > 0 & \text{if } \alpha + \hat{\alpha} < 1, \\ = 0 & \text{if } \alpha + \hat{\alpha} > 1. \end{cases} \tag{2.3.21}$$

We argue that β_0 is positive if and only if $\mathcal{I}(0)$ is finite. Let

$$\mathcal{I}(h; n) := \sum_{k=0}^{n} \mathbb{P}_h(k \in \tau)\hat{\mathbb{P}}(k \in \hat{\tau}), \tag{2.3.22}$$

then $\mathcal{I}(h) = \lim_{n\to\infty} \mathcal{I}(h; n)$ by (2.3.17). For $\widetilde{\tau}$ to be recurrent, it is necessary that both τ and $\hat{\tau}$ are recurrent. By the renewal theorem (see (2.4.3) in Section 2.4), $\widetilde{\tau}$ is recurrent if and only if

(i) $\alpha + \hat{\alpha} > 1$,

(ii) $\alpha, \hat{\alpha} \in (0, 1)$ with $\alpha + \hat{\alpha} = 1$ and $\sum_{n\geq 1} \frac{1}{nL(n)\hat{L}(n)} = +\infty$,

(iii) $\alpha = 0$, $\hat{\alpha} = 1$ ($\alpha = 1$, $\hat{\alpha} = 0$) and $\sum_{n\geq 1} \frac{\hat{L}(n)}{nr_n^2\hat{\mu}_n} + \infty$,

where $r_n = \mathbb{P}(\tau_1 > n)$ and $\hat{\mu}_n = \mathbb{E}[\hat{\tau}_1 \wedge n] \to \hat{\mu}$ as $n \to \infty$. In the case (i), we can easily see that $\mathcal{I}(0)$ is infinite. By Alexander and Berger [2], it is known that the renewal mass function of $\widetilde{\tau}$ satisfies

$$\mathbb{P}(n \in \tau)\hat{\mathbb{P}}(n \in \hat{\tau}) = \frac{L^*(n)}{n^{\theta^*}}, \tag{2.3.23}$$

for some $\theta^* \geq 0$ and slowly-varying function L^*. Especially, if both τ and $\hat{\tau}$ are recurrent, then $\theta^* = 2 - (\alpha \wedge 1) - (\hat{\alpha} \wedge 1)$. In both cases, (ii) and (iii),

$$\mathcal{I}(0) = \sum_{k=0}^{\infty} \frac{L^*(k)}{k}. \tag{2.3.24}$$

Therefore, in the case $\alpha + \hat{\alpha} = 1$, whether $\mathcal{I}(0)$ is finite or not depends on the slowly-varying functions $L, \hat{L}$.

Consequently, the annealed critical point h_β^{a} can be expressed as follow.

Proposition 2.3.3 (The annealed critical curve)**.** *The annealed critical curve is*

$$h_\beta^{\mathsf{a}} = \begin{cases} \mathcal{I}^{-1}\left(\frac{1}{1-e^{-\beta}}\right) & \text{if } \beta > \beta_0, \\ 0 & \text{if } 0 \leq \beta \leq \beta_0. \end{cases} \tag{2.3.25}$$

By the property of the function $\mathcal{I}$, we can see that h^{a}_{β} is infinitely differentiable in $[0,\infty)\setminus\{\beta_0\}$ and the function $\beta \mapsto h^{\mathsf{a}}_{\beta}$ is concave since $(\beta,h)\mapsto \mathrm{F}^{\mathsf{a}}(\beta,h)$ is convex. By Jensen's inequality, we have

$$Z^{\mathsf{a}}_{n,\beta,h} \geq \mathbb{E}\Big[e^{\sum_{k=1}^{n}(h+\beta\hat{\mathbb{E}}[\hat{\delta}_k])\delta_k}\delta_n\Big] = \mathbb{E}\Big[e^{\sum_{k=1}^{n}(h+\beta\hat{\mathbb{P}}(k\in\hat{\tau}))\delta_k}\delta_n\Big]. \tag{2.3.26}$$

Therefore, by the renewal theorem and the fact that $h_c = 0$, we have

$$h^{\mathsf{a}}_{\beta} \leq -\frac{\beta}{\hat{\mu}}. \tag{2.3.27}$$

Moreover, we have the following result about the annealed critical point h^{a}_{β}.

Theorem 2.3.4 (The annealed critical point)**.** *If $\hat{\alpha} > 1$ and as $\beta \downarrow 0$, then there exists a constant $c_a > 0$ such that*

$$h^{\mathsf{a}}_{\beta} = -\frac{\beta}{\hat{\mu}} - c_a\beta^{\gamma^{\mathsf{a}}}\big(1+o(1)\big), \tag{2.3.28}$$

where

$$\gamma^{\mathsf{a}} = \begin{cases} \hat{\alpha} & \text{if } \alpha > 1 \text{ and } \hat{\alpha} < 2, \\ \frac{\alpha+\hat{\alpha}-1}{\alpha} & \text{if } \alpha \in (0,1) \text{ and } \hat{\alpha} < 1+\alpha, \\ 2 & \text{otherwise.} \end{cases} \tag{2.3.29}$$

The next proposition is about the behavior of the annealed critical point around β_0.

Proposition 2.3.5 (The annealed critical curve)**.** *If $\beta_0 > 0$, then there exists a constant $c \in (0,\infty)$ such that*

$$h^{\mathsf{a}}_{\beta} \sim c(\beta-\beta_0)^{1\vee\frac{\alpha}{1-\alpha-\hat{\alpha}}} \tag{2.3.30}$$

as $\beta \downarrow \beta_0$.

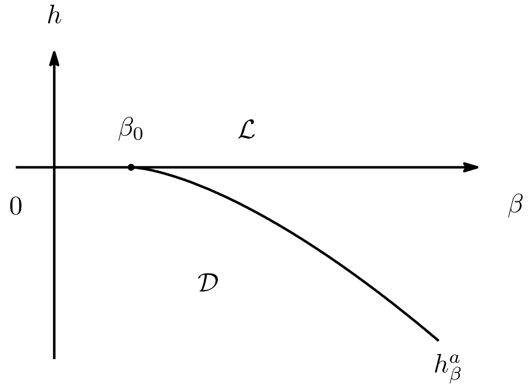

Figure 2.4: The annealed critical curve $\beta \mapsto h^{\mathsf{a}}_\beta$ is convex and decreasing. The critical point is $\beta_0 = -\log p(0)$ and slope of h^{a}_β at β_0 might be positive or equal to zero, depending on α and $\hat{\alpha}$ (See Remark 2.3.2 and Proposition 2.3.5). $\mathcal{L}$ is the localized phase and $\mathcal{D}$ is the delocalized phase respectively.

Our last result for the annealed model is about the critical exponent for the annealed free energy.

Theorem 2.3.6 (The annealed critical exponent)**.** *Let $\beta > 0$. There exists a constant $C = C(\beta) \in (0, \infty)$ such that*

$$\frac{1}{C}\,\varepsilon^{\nu^{\mathsf{a}}_\beta} \leq \mathrm{F}^{\mathsf{a}}(\beta, h^{\mathsf{a}}_\beta + \varepsilon) \leq C\,\varepsilon^{\nu^{\mathsf{a}}_\beta} \tag{2.3.31}$$

for all $\varepsilon \in (0,1)$, with

$$\nu^{\mathsf{a}}_\beta := \begin{cases} \frac{1}{\alpha_{\mathrm{eff}}} \vee 1 & \text{if } \beta > \beta_0, \\ \frac{1}{\alpha} \vee 1 & \text{if } 0 \leq \beta \leq \beta_0, \end{cases} \tag{2.3.32}$$

where $\alpha_{\mathrm{eff}} := \alpha + \{(1-\hat{\alpha}) \vee 0\}$.

The case $\hat{\alpha} = 0$ is not covered by this theorem, due to Lemma 2.4.5 in Section 2.4. When $\nu^{\mathsf{a}}_\beta = \infty$, we keep only the second inequality in (2.3.31) and it means that $\mathrm{F}^{\mathsf{a}}(\beta, h^{\mathsf{a}}_\beta + \varepsilon)$ vanishes faster than any power of ε.

The existence of the annealed free energy

From this section, we will prove the results on the annealed model. First, we prove the Proposition 2.3.1.

Proof of Proposition 2.3.1. First, we prove the existence of the limit $\lim_{n\to\infty} \frac{1}{n} \log Z^{\mathsf{a;c}}_{n,\hat{\tau}}$ and its finiteness and non-negativity.

(i) *The existence of the limit*:
Since the sequence $\{\log Z^{\mathsf{a,c}}_{n,\hat\tau}\}_{n\in\mathbb{N}}$ is super-additive, there exists the limit $\lim_{n\to\infty}\frac{1}{n}\log Z^{\mathsf{a,c}}_{n,\hat\tau}$, and it is defined by

$$\lim_{n\to\infty}\frac{1}{n}\log Z^{\mathsf{a,c}}_{n,\hat\tau}=\sup_{n\in\mathbb{N}}\frac{1}{n}\log Z^{\mathsf{a,c}}_{n,\hat\tau}=:\mathrm{F}^{\mathsf{a,c}}(\beta,h). \tag{2.3.33}$$

(ii) *Finiteness and non-negativity*:
The following bounds of $Z^{\mathsf{a,c}}_{n,\hat\tau}$ are trivial.

$$\underbrace{e^{h+\beta}K(n)\hat K(n)}_{\tau\text{ and }\hat\tau\text{ return only at 0 and }n}\le Z^{\mathsf{a,c}}_{n,\hat\tau}\le\underbrace{e^{|h+\beta|n}}_{\tau\text{ and }\hat\tau\text{ return every step in }[0,n]}. \tag{2.3.34}$$

This implies the limit $\lim_{n\to\infty}\frac{1}{n}\log Z^{\mathsf{a,c}}_{n,\hat\tau}$ is finite and non-negative.

In the rest of the proof, we show that $Z^{\mathsf{a}}_{n,\hat\tau}$ is not different from $Z^{\mathsf{a,c}}_{n,\hat\tau}$ so much and the limits of them coincide each other. Let R be the last visit of $\widetilde\tau$. If $R>n$, then we say that $\hat\tau$ does not return. Now we decompose $Z^{\mathsf{a,c}}_{n,\hat\tau}$ into two, up to R and from R to n.

$$\begin{aligned}
Z^{\mathsf{a}}_{n,\hat\tau}&=\hat{\mathbb{E}}\mathbb{E}\left[e^{\sum_{k=1}^n(h+\beta\hat\delta_k)\delta_k}\delta_n\right]\\
&=\sum_{r=0}^n\hat{\mathbb{E}}\mathbb{E}\left[e^{\sum_{k=1}^r(h+\beta\hat\delta_k)\delta_k}e^{\sum_{k=r+1}^n(h+\beta\hat\delta_k)\delta_k}\mathbf{1}_{\{R=r\}}\delta_n\right]\\
&=\sum_{r=0}^n\hat{\mathbb{E}}\mathbb{E}\left[e^{\sum_{k=1}^r(h+\beta\hat\delta_k)\delta_k}\delta_r\hat\delta_r\right]\hat{\mathbb{E}}\mathbb{E}\left[e^{\sum_{k=r+1}^n(h+\beta\hat\delta_k)\delta_k}\mathbf{1}_{\{\widetilde\tau\cap[r+1,n]=\phi\}}\delta_n\middle|r\in\widetilde\tau\right]\\
&=\sum_{r=0}^n Z^{\mathsf{a,c}}_{r,\hat\tau}\ \hat{\mathbb{E}}\mathbb{E}\left[e^{\sum_{k=r+1}^n(h+\beta\hat\tau_k)\delta_k}\mathbf{1}_{\{\widetilde\tau\cap[r+1,n]=\phi\}}\delta_n\middle|r\in\widetilde\tau\right].
\end{aligned} \tag{2.3.35}$$

Then we estimate $\hat{\mathbb{E}}\mathbb{E}[e^{\sum_{k=r+1}^n(h+\beta\hat\tau_k)\delta_k}\mathbf{1}_{\{\widetilde\tau\cap[r+1,n]=\phi\}}\delta_n|r\in\widetilde\tau]$ by using translation invariance of $\widetilde\tau$, the super-additivity of $\{\log Z^{\mathsf{a,c}}_{r,\hat\tau}\}$ and the monotonicity of $\mathrm{F}^{\mathsf{a,c}}(\beta,h)$ with respect to β. By (2.3.33), we have $Z^{\mathsf{a,c}}_{r,\beta,h}\le e^{r\mathrm{F}^{\mathsf{a,c}}(\beta,h)}$.

$$\begin{aligned}
&\hat{\mathbb{E}}\mathbb{E}\left[e^{\sum_{k=r+1}^n(h+\beta\hat\delta_k)\delta_k}\mathbf{1}_{\{\widetilde\tau\cap[r+1,n]=\phi\}}\delta_n\middle|r\in\widetilde\tau\right]=\hat{\mathbb{E}}\mathbb{E}\left[e^{h\sum_{k=r+1}^n\delta_k}\delta_n\middle|r\in\widetilde\tau\right]\\
&=Z^{\mathsf{a,c}}_{n-r,0,h}\frac{1}{\hat{\mathbb{P}}(n-r\in\hat\tau)}\le e^{(n-r)\mathrm{F}^{\mathsf{a,c}}(0,h)}\frac{1}{\hat{\mathbb{P}}(n-r\in\hat\tau)}\\
&\le e^{(n-r)\mathrm{F}^{\mathsf{a,c}}(\beta,h)}\frac{1}{\hat{\mathbb{P}}(n-r\in\hat\tau)}.
\end{aligned} \tag{2.3.36}$$

Combining with the trivial bound $Z^{\mathsf{a,c}}_{n,\hat\tau}\le Z^{\mathsf{a}}_{n,\hat\tau}$, therefore we have

$$Z^{\mathsf{a,c}}_{n,\hat\tau}\le Z^{\mathsf{a}}_{n,\hat\tau}\le\sum_{r=0}^n Z^{\mathsf{a,c}}_{r,\hat\tau}\ e^{(n-r)\mathrm{F}^{\mathsf{a,c}}(\beta,h)}\frac{1}{\hat{\mathbb{P}}(n-r\in\hat\tau)}=e^{n\mathrm{F}^{\mathsf{a,c}}(\beta,h)}\sum_{r'=0}^n\frac{1}{\hat{\mathbb{P}}(r'\in\hat\tau)} \tag{2.3.37}$$

The last sum increases polynomially in n. This completes the proof. ■

The annealed critical curve

Next, we prove the Proposition 2.3.3.

Proof of Proposition 2.3.3. Recall that by the definition of the annealed free energy (see the argument below Figure 2.3), we have

$$\begin{aligned} \mathrm{F}^{\mathsf{a}}(\beta, h) > 0 &\Leftrightarrow \beta > -\log p(h), \\ \mathrm{F}^{\mathsf{a}}(\beta, h) = 0 &\Leftrightarrow \beta \leq -\log p(h). \end{aligned} \tag{2.3.38}$$

For $h = 0$, $\mathrm{F}^{\mathsf{a}}(\beta, 0) > 0$ if and only if $\beta > \beta_0$ and otherwise. We devide the proof into two cases $\beta \leq \beta_0$ and $\beta > \beta_0$.

1): $\beta \leq \beta_0$
For $h \leq 0$, we have $\mathrm{F}^{\mathsf{a}}(\beta, h) \leq \mathrm{F}^{\mathsf{a}}(\beta, 0) = 0$ by the monotonicity of the annealed free energy in h (see Figure 2.3) and the fact mentioned above (by (2.3.38)). On the other hand, for $h > 0$, we have $\mathrm{F}^{\mathsf{a}}(\beta, h) \geq \mathrm{F}^{\mathsf{a}}(0, h) > 0$ since τ is a recurrent renewal. Thus, we conclude that $h^{\mathsf{a}}_\beta = 0$.

2): $\beta > \beta_0$
For $h \geq 0$, we have $\mathrm{F}^{\mathsf{a}}(\beta, h) \geq \mathrm{F}^{\mathsf{a}}(\beta, 0) > 0$ by the monotonicity in h and the choice of β (again (2.3.38)). We have

$$\hat{\mathbb{E}}\mathbb{E}_h\left[e^{\beta \sum_{k=1}^n \hat{\delta}_k \delta_k}\right] \leq Z^{\mathsf{a}}_{n,\hat{\tau}} \leq Z^{\mathsf{a,c}}_{n,\hat{\tau}} \tag{2.3.39}$$

The upper and lower bounds in (2.3.39) give the same free energy, which is positive if and only if $\beta > -\log p(h)$. This implies that $\mathcal{I}(h) > (1 - e^{-\beta})^{-1}$. By the choice of β, we have

$$\mathcal{I}(0) > \frac{1}{1 - e^{-\beta}} > 1, \tag{2.3.40}$$

which means that $(1 - e^{-\beta})^{-1}$ is in the range of $\mathcal{I}$. The proof is completed. ■

Next, we prove Theorem 2.3.4 by analyzing the function $\mathcal{I}$.

Proof of Theorem 2.3.4. Let us write

$$h^{\mathsf{a}}_\beta = -\frac{\beta}{\hat{\mu}}(1 + \varepsilon_\beta), \tag{2.3.41}$$

with $\varepsilon_\beta \to 0$ as $\beta \downarrow 0$. Note that $\alpha > 1$ implies $\beta_0 = 0$ as in Remark 2.3.2. From Proposition 2.3.3, we obtain, on the one hand,

$$\mathcal{I}(h^{\mathsf{a}}_\beta) = \frac{1}{1-e^{-\beta}} = \frac{1}{\beta}\Big(1 + \frac{1}{2}\beta + o(\beta)\Big). \tag{2.3.42}$$

and on the other hand, from Lemma 2.3.7 below, we obtain

$$\begin{aligned} \mathcal{I}(h^{\mathsf{a}}_\beta) &= \frac{1}{\hat{\mu}}\frac{1}{1-e^{h^{\mathsf{a}}_\beta}} + c(-h^{\mathsf{a}}_\beta)^{\gamma^{\mathsf{a}}-2} \\ &= \frac{1}{\beta}\Big(1 - \varepsilon_\beta + \frac{1}{2\hat{\mu}}\beta + c\beta^{\gamma^{\mathsf{a}}-1} + o(\beta) + o(\varepsilon_\beta)\Big). \end{aligned} \tag{2.3.43}$$

By comparing (2.3.42) and (2.3.43), the result follows. ■

We use the definition (2.3.1) in the following lemmas and their proofs but it is not essential.

Lemma 2.3.7. *Suppose $\hat{\alpha} > 1$. As $h \uparrow 0$,*

$$\mathcal{I}(h) - \frac{1}{\hat{\mu}}\frac{1}{1-e^h} \sim \begin{cases} c|h|^{\hat{\alpha}-2} & \text{if } \alpha > 1 \text{ and } \hat{\alpha} < 2, \\ c|h|^{\frac{\hat{\alpha}-1}{\alpha}-1} & \text{if } \alpha \in (0,1) \text{ and } \hat{\alpha} < \alpha + 1, \\ c & \text{otherwise,} \end{cases} \tag{2.3.44}$$

where the constant c is given as $\frac{c_{\hat{K}}}{\hat{\mu}^2\hat{\alpha}(\hat{\alpha}+1)}$.

Proof of Lemma 2.3.7. By the definition of $\mathcal{I}$,

$$\begin{aligned} \mathcal{I}(h) &= \sum_{n\in\mathbb{N}_0} \mathbb{P}_h(n \in \tau)\hat{\mathbb{P}}(n \in \hat{\tau}) = \sum_{n,k\in\mathbb{N}_0} e^{hk}\mathbb{P}(\tau_k = n)\hat{\mathbb{P}}(n \in \hat{\tau}) \\ &= \sum_{k\in\mathbb{N}_0} e^{hk}\mathbb{P}\times\hat{\mathbb{P}}(\tau_k \in \hat{\tau}). \end{aligned} \tag{2.3.45}$$

Then, for $h < 0$,

$$\begin{aligned} \mathcal{I}(h) - \frac{1}{\hat{\mu}}\frac{1}{1-e^h} &= \sum_{k\in\mathbb{N}_0} e^{hk}\mathbb{P}\times\hat{\mathbb{P}}(\tau_k \in \hat{\tau}) - \frac{1}{\hat{\mu}}\sum_{k\in\mathbb{N}_0} e^{hk} \\ &= \sum_{k\in\mathbb{N}_0} e^{hk}\Big(\mathbb{P}\times\hat{\mathbb{P}}(\tau_k \in \hat{\tau}) - \frac{1}{\hat{\mu}}\Big) = \sum_{k\in\mathbb{N}_0} e^{hk}\mathbb{E}\Big[\hat{\mathbb{P}}(\tau_k \in \hat{\tau}) - \frac{1}{\hat{\mu}}\Big]. \end{aligned} \tag{2.3.46}$$

By Lemma 2.3.8, we have

$$\mathbb{E}\Big[\hat{\mathbb{P}}(\tau_k \in \hat{\tau}) - \frac{1}{\hat{\mu}}\Big] \sim \frac{c_{\hat{K}}}{\hat{\mu}^2\hat{\alpha}(\hat{\alpha}-1)}\mathbb{E}[\tau_k^{1-\hat{\alpha}}], \quad k \to \infty. \tag{2.3.47}$$

On the other hand, for $\hat{\alpha} \in (0,1)$, from Proposition 2.4.3,

$$\mathbb{E}\Big[\mathbb{P}(\tau_k \in \hat{\tau})\Big] \sim \frac{c_{\hat{\alpha}}}{c_{\hat{K}}}\mathbb{E}[\tau_k^{\hat{\alpha}-1}], \quad k \to \infty. \tag{2.3.48}$$

In both case, the result follows from Lemma 2.4.4 and the standard Tauberian arguments. ∎

Lemma 2.3.8. *(i) If $\hat{\alpha} > 1$, then*

$$\mathbb{E}\Big[\hat{\mathbb{P}}(\tau_k \in \hat{\tau}) - \frac{1}{\hat{\mu}}\Big] \sim \frac{c_{\hat{K}}}{\hat{\mu}^2\hat{\alpha}(\hat{\alpha}-1)} \begin{cases} \mu^{1-\hat{\alpha}}k^{1-\hat{\alpha}} & \text{if } \alpha > 1, \\ \mathbb{E}[X_\alpha^{1-\hat{\alpha}}]k^{\frac{1-\hat{\alpha}}{\alpha}} & \text{if } \alpha \in (0,1), \end{cases} \tag{2.3.49}$$

as $k \to \infty$, where X_α is an α-stable random variable [1].

(ii) If $\hat{\alpha} \in (0,1)$, then

$$\mathbb{E}\Big[\hat{\mathbb{P}}(\tau_k \in \hat{\tau}) - \frac{1}{\hat{\mu}}\Big] \sim \frac{c_{\hat{\alpha}}}{c_{\hat{K}}} \begin{cases} \mu^{\hat{\alpha}-1}k^{\hat{\alpha}-1} & \text{if } \alpha > 1, \\ \mathbb{E}[X_\alpha^{\hat{\alpha}-1}]k^{\frac{\hat{\alpha}-1}{\alpha}} & \text{if } \alpha \in (0,1), \end{cases} \tag{2.3.52}$$

as $k \to \infty$.

Proof of Lemma 2.3.8.

(i) From the renewal convergence estimates in Frenk [38], for $\hat{\alpha} > 1$, we obtain

$$\begin{aligned} \hat{\mathbb{P}}(n \in \hat{\tau}) - \frac{1}{\hat{\mu}} &\sim \frac{1}{\hat{\mu}^2(\hat{\alpha}-1)} n\hat{\mathbb{P}}(\hat{\tau}_1 > n) \\ &\sim \frac{c_{\hat{K}}}{\hat{\mu}^2\hat{\alpha}(\hat{\alpha}-1)} n^{1-\hat{\alpha}}, \quad n \to \infty. \end{aligned} \tag{2.3.53}$$

Then, we have $\mathbb{P}$-almost surely

$$\hat{\mathbb{P}}(\tau_k \in \hat{\tau}) - \frac{1}{\hat{\mu}} \sim \frac{c_{\hat{K}}}{\hat{\mu}^2\hat{\alpha}(\hat{\alpha}-1)}\tau_k^{1-\hat{\alpha}}, \quad k \to \infty, \tag{2.3.54}$$

[1] The distributions whose characteristic functions are given by the following are called stable laws.

$$\exp\{itc - b|t|^\alpha(1 + i\kappa\,\mathrm{sgn}(t)w_\alpha(t))\}, \tag{2.3.50}$$

where b, c are some constants, $\kappa \in [-1,1]$ and

$$w_\alpha(t) = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \text{if } \alpha \neq 1, \\ \frac{2}{\pi}\log|t| & \text{if } \alpha = 1. \end{cases} \tag{2.3.51}$$

Commonly, α is called the index. The α-stable random variable is defined as a random variable which has stable law with index α. For more detail, refer to [46] and references therein.

and

$$\mathbb{E}\Big[\hat{\mathbb{P}}(\tau_k \in \hat{\tau}) - \frac{1}{\hat{\mu}}\Big] \sim \frac{c_{\hat{K}}}{\hat{\mu}^2 \hat{\alpha}(\hat{\alpha}-1)} \mathbb{E}[\tau_k^{1-\hat{\alpha}}], \quad k \to \infty. \tag{2.3.55}$$

The result follows.

(ii) From Proposition 2.4.3, we have $\mathbb{P}$-almost surely

$$\mathbb{P}(\tau_k \in \hat{\tau}) \sim \frac{c_{\hat{\alpha}}}{c_{\hat{K}}} \tau_k^{\hat{\alpha}-1}, \quad k \to \infty, \tag{2.3.56}$$

and with the same argument as in (i),

$$\mathbb{E}\Big[\mathbb{P}(\tau_k \in \hat{\tau})\Big] \sim \frac{c_{\hat{\alpha}}}{c_{\hat{K}}} \mathbb{E}[\tau_k^{\hat{\alpha}-1}], \quad k \to \infty. \tag{2.3.57}$$

The result follows from Lemma 2.4.4. ■

Remark 2.3.9. *If $\hat{\alpha} = 1$, then the sequence $\{\hat{\mathbb{P}}(\tau_k \in \hat{\tau}) - \frac{1}{\hat{\mu}}\}$ belongs to the de Haan class $\Pi(\hat{L})$, which is defined as follow. Given a slowly varying function $\hat{L}$, a sequence A_n belongs to the de Haan class $\Pi(\hat{L})$ if A_n satisfies for $t > 0$,*

$$\lim_{n\to\infty} \frac{A_{\lfloor tn \rfloor} - A_n}{\hat{L}(n)} = \log t. \tag{2.3.58}$$

We prove Proposition 2.3.5 by using Lemma 2.3.8.

Proof of Proposition 2.3.5. Since $\beta_0 > 0$, $\mathcal{I}(0)$ is finite. Then, we may write

$$\mathcal{I}(h) - \mathcal{I}(0) = \sum_{k\in\mathbb{N}_0} (e^{hk-1}) \mathbb{E}[\hat{\mathbb{P}}(\tau_k \in \hat{\tau})], \tag{2.3.59}$$

and since $\alpha, \hat{\alpha} < 1$, we have $\mathbb{E}[\hat{\mathbb{P}}(\tau_k \in \hat{\tau})] \sim ck^{\frac{\hat{\alpha}-1}{\alpha}}$ by Lemma 2.3.8. The result follows by the standard Tauberian arguments. ■

The asymptotic behavior of the annealed free energy

Proof of Theorem 2.3.6. We prove this theorem using the analogue of the proof of Theorem 2.1.1.

1): $\beta > -\log p(0)$ $(h_\beta^{\mathtt{a}} < 0)$

Lower Bound: First, we transform the annealed partition function similarly to Proposition 2.3.3.

$$Z^{\mathsf{a}}_{n,\beta,h^{\mathsf{a}}_\beta+\varepsilon} \geq \hat{\mathbb{E}}\mathbb{E}\Big[e^{\sum_{k=1}^n (h^{\mathsf{a}}_\beta+\varepsilon+\beta\hat{\delta}_k)\delta_k}\delta_n\hat{\delta}_n\Big] = \hat{\mathbb{E}}\mathbb{E}_{h^{\mathsf{a}}_\beta}\Big[e^{\varepsilon\beta\sum_{k=1}^n \delta_k}e^{\beta\sum_{k=1}^n \hat{\delta}_k\delta_k}\delta_n\hat{\delta}_n\Big]$$
$$\geq \hat{\mathbb{E}}\mathbb{E}_{h^{\mathsf{a}}_\beta}\Big[e^{(\beta+\varepsilon)\sum_{k=1}^n \hat{\delta}_k\delta_k}\delta_n\hat{\delta}_n\Big] =: \widetilde{\mathbb{E}}_\beta\Big[e^{\varepsilon\sum_{k=1}^n \widetilde{\delta}_k}\widetilde{\delta}_n\Big], \tag{2.3.60}$$

where $\widetilde{\mathbb{E}}_\beta$ stands for the expectation with respect to the law

$$K_\beta(n) := e^\beta \hat{\mathbb{P}} \times \mathbb{P}_{h^{\mathsf{a}}_\beta}(\widetilde{\tau}_1 = n), \quad K_\beta(\infty) = 1 - e^{-\beta}. \tag{2.3.61}$$

Note that $\widetilde{\mathbb{E}}_\beta = \hat{\mathbb{E}}\mathbb{E}$ when $0 \leq \beta \leq -\log p(0)$ since $h^{\mathsf{a}}_\beta = 0$, and by Lemma 2.4.5, the renewal function $n \mapsto \hat{\mathbb{P}} \times \mathbb{P}_{h^{\mathsf{a}}_\beta}(\widetilde{\tau}_1 = n)$ satisfies (2.4.12). Under the law K_β, the annealed partition function $Z^{\mathsf{a}}_{n,\beta,h^{\mathsf{a}}_\beta+\varepsilon}$ can be regard as the homogeneous partition function. Then, we introduce the function $\varphi_\beta(x)$ and compute it.

$$\varphi_\beta(x) := 1 - \sum_{n=1}^\infty e^{-nx}K_\beta(n) = 1 - e^\beta \hat{K}_\beta(e^{-x}), \tag{2.3.62}$$

where $\hat{K}_\beta$ is a generating function of K_β i.e. $\hat{K}_\beta(z) := \sum_{n=1}^\infty K(n)z^n$. Let $u_\beta(n)$ be a renewal function for inter-arrival law K_β, and we have

$$u_\beta(n) = \mathbf{1}_{\{n=0\}} + \sum_{j=1}^n K_\beta(j)u_\beta(n-j). \tag{2.3.63}$$

Then,

$$\hat{u}_\beta(z) = 1 + \hat{K}_\beta(z)\hat{u}_\beta(z), \quad \hat{K}_\beta(z) = 1 - \frac{1}{\hat{u}_\beta(z)}. \tag{2.3.64}$$

Since $\hat{K}_\beta(1) = \sum_{n=1}^\infty K_\beta(n) = e^{-\beta}$, therefore we have

$$\varphi_\beta(x) = 1 - e^\beta\Big(1 - \frac{1}{\hat{u}_\beta(e^{-x})}\Big) = e^\beta\Big(e^{-\beta} - 1 + \frac{1}{\hat{u}_\beta(e^{-x})}\Big)$$
$$= e^\beta\Big(\frac{1}{\hat{u}_\beta(e^{-x})} - \frac{1}{\hat{u}_\beta(1)}\Big) = \frac{e^\beta}{\hat{u}_\beta(e^{-x})\hat{u}_\beta(1)}\Big(\hat{u}_\beta(1) - \hat{u}_\beta(e^{-x})\Big)$$
$$= \frac{e^\beta}{\hat{u}_\beta(e^{-x})\hat{u}_\beta(1)}A_\beta(e^{-x}), \tag{2.3.65}$$

with

$$A_\beta(z) := \Big(\hat{u}_\beta(1) - \hat{u}_\beta(z)\Big) = (1-z)\sum_{n=1}^\infty u_\beta(n)\sum_{k=0}^{n-1} z^k$$
$$= (1-z)\sum_{k=0}^\infty z^k \sum_{n=k+1}^\infty u_\beta(n). \tag{2.3.66}$$

The claim in Lemma 2.4.6 shows the asymptotic behavior of A_β. Therefore, the lower bound follows.

Upper Bound: Take a $\beta_1 \in (-\log p(0), \beta)$, and let $\varepsilon > 0$ be small so that $h^{\mathsf{a}}_\beta + \varepsilon < h^{\mathsf{a}}_{\beta_1}$. By the continuity of h^{a} we have that there is $\beta_\varepsilon \in (\beta_1, \beta)$ so that $h^{\mathsf{a}}_\beta + \varepsilon = h^{\mathsf{a}}_{\beta_\varepsilon}$. And by the mean value theorem, there is $\xi_\varepsilon \in (\beta_\varepsilon, \beta)$ with $h^{\mathsf{a}}_\beta - h^{\mathsf{a}}_{\beta_\varepsilon} = \frac{d}{d\beta} h^{\mathsf{a}}_{\xi_\varepsilon}(\beta - \beta_\varepsilon)$. Thus $\beta - \beta_\varepsilon = c(\beta, \varepsilon)\,\varepsilon$ with $c(\beta,\varepsilon) = -1/\frac{d}{d\beta} h^{\mathsf{a}}_{\xi_\varepsilon} \to -1/\frac{d}{d\beta} h^{\mathsf{a}}_\beta > 0$ as $\varepsilon \to 0^+$, as we know from Proposition 2.3.3 and the regularity properties of $\mathcal{I}$. Then, since $h^{\mathsf{a}}_\beta + \varepsilon < 0$,

$$\begin{aligned}
Z^{\mathsf{a,c}}_{n,\beta,h^{\mathsf{a}}_\beta+\varepsilon} &= \hat{\mathbb{E}}\mathbb{E}\Big[e^{\sum_{k=1}^n (h^{\mathsf{a}}_\beta + \varepsilon + \beta\hat{\delta}_k)\delta_k}\delta_n\hat{\delta}_n\Big] = \hat{\mathbb{E}}\mathbb{E}_{h^{\mathsf{a}}_\beta+\varepsilon}\Big[e^{\beta\sum_{k=1}^n \hat{\delta}_k\delta_k}\delta_n\hat{\delta}_n\Big] \\
&=: \hat{\mathbb{E}}\mathbb{E}_{h^{\mathsf{a}}_{\beta_\varepsilon}}\Big[e^{(\beta_\varepsilon + c_{\beta,\varepsilon}\varepsilon)\sum_{k=1}^n \hat{\delta}_k\delta_k}\delta_n\hat{\delta}_n\Big] =: \widetilde{\mathbb{E}}_{\beta_\varepsilon}\Big[e^{c_{\beta,\varepsilon}\varepsilon\sum_{k=1}^n \widetilde{\delta}_k}\widetilde{\delta}_n\Big] \\
&\le \widetilde{\mathbb{E}}_{\beta_\varepsilon}\Big[e^{c\varepsilon\sum_{k=1}^n \widetilde{\delta}_k}\widetilde{\delta}_n\Big],
\end{aligned} \tag{2.3.67}$$

with $c = 2c(\beta, 0+) > 0$ and $\widetilde{\mathbb{E}}_{\beta_\varepsilon}$ the mean value with respect to the renewal defined in (2.3.61) above with β_ε in place of β. The result now follows on applying Lemma 2.4.6 with $I = [\beta_1, \beta]$ and with the role of K_γ played by the law $n \mapsto \mathbb{P}_{h^{\mathsf{a}}_\gamma} \times \hat{\mathbb{P}}(\widetilde{\tau}_1 = n)$. The assumptions of the lemma are satisfied due to Lemma 2.4.5 because $J := h^{\mathsf{a}}_I$ is a compact subset of $(-\infty, 0)$ as h^{a} is continuous, decreasing, with $h^{\mathsf{a}}_{\beta_1} < 0$.

2): $\beta \le -\log p(0)$ $(h^{\mathsf{a}}_\beta = 0)$

Lower Bound: Since

$$Z^{\mathsf{a}}_{n,\beta,\varepsilon} \ge \hat{\mathbb{E}}\mathbb{E}\Big[e^{\varepsilon\sum_{k=1}^n \delta_k}\delta_n\hat{\delta}_n\Big] = \mathbb{E}\Big[e^{\varepsilon\sum_{k=1}^n \delta_k}\delta_n\Big]\hat{\mathbb{P}}(n \in \hat{\tau}), \tag{2.3.68}$$

the lower bound follows from Theorem 2.1 in [41].

Upper Bound: We assume that $\varepsilon \in (0, \beta)$. Pick $\bar{p} > 1$ so that $\bar{p}(\beta - \varepsilon) \le -\log p(0)$, and let $q > 1$ be defined by $\bar{p}^{-1} + q^{-1} = 1$. Then

$$\begin{aligned}
Z^{\mathsf{a,c}}_{n,\beta,\varepsilon} &= \hat{\mathbb{E}}\mathbb{E}\Big[e^{\sum_{k=1}^n (\varepsilon + \beta\hat{\delta}_k)\delta_k}\delta_n\hat{\delta}_n\Big] \le \mathbb{E}\hat{\mathbb{E}}\Big[e^{2\varepsilon\sum_{k=1}^n \delta_k}e^{(\beta-\varepsilon)\sum_{k=1}^n \delta_k\hat{\delta}_k}\delta_n\hat{\delta}_n\Big] \\
&\le \mathbb{E}\hat{\mathbb{E}}\Big[e^{2q\varepsilon\sum_{k=1}^n \delta_k}\delta_n\Big]^{1/q}\mathbb{E}\hat{\mathbb{E}}\Big[e^{\bar{p}(\beta-\varepsilon)\sum_{k=1}^n \delta_k\hat{\delta}_k}\delta_n\hat{\delta}_n\Big]^{1/\bar{p}},
\end{aligned} \tag{2.3.69}$$

The quantity $\mathbb{E}\hat{\mathbb{E}}[e^{\bar{p}(\beta-\varepsilon)\sum_{k=1}^n \delta_k\hat{\delta}_k}\delta_n\hat{\delta}_n]$ is the partition function at $\bar{p}(\beta-\varepsilon)$ for the homopolymer defined by the renewal $\widetilde{\tau}$, which has zero free energy for all parameters in $(-\infty, -\log p(0)]$. Thus, the required bound follows again from Theorem 2.1 in [41]. ■

2.3.3 Quenched behavior on a renewal set

In this section we discuss the quenched case, which is ongoing work. First, we state the quenched free energy.

Proposition 2.3.10 (The quenched free energy)**.** *The sequence*

$$\frac{1}{n}\log Z_n \tag{2.3.70}$$

converges almost surely and in L^1 to a constant $\mathrm{F}(\beta, h)$*.*

We briefly explain the proof of Proposition 2.3.10. Note that when $\beta = 0$, this is the homogeneous pinning model, for which we know that the free energy $\mathrm{F}(0, h)$ exists. Therefore, we assume $\beta > 0$. In the case of $\hat{\mathbb{E}}[\hat{\tau}_1] = \infty$,

$$\mathbb{E}\left[e^{h\sum_{k=1}^{n}\delta_k}\delta_n\right] \leq Z_n \leq e^{\beta|\hat{\tau}\cap[n]|}\mathbb{E}\left[e^{h\sum_{k=1}^{n}\delta_k}\delta_n\right]. \tag{2.3.71}$$

Since $|\hat{\tau}\cap[n]|/n \to 0$ as $n \to \infty$, the last inequality show that the limit we are interested in exists and equals $\mathrm{F}(0, h)$. On the other hand, in the case of $\hat{\mathbb{E}}[\hat{\tau}_1] < \infty$, we apply Kingman's subadditive ergodic theorem (see [50]). Note that we need careful observation when $\hat{\tau}$ is stationary or not. Due to this proposition, we can define the quenched critical point $\hat{h}^{\mathsf{q}}_\beta$.

The main interest of the disordered systems is considering the quenched case, the criterion between relevant and irrelevant, smoothing, and estimating the difference between the quenched and annealed critical points. However, we have not yet understood how the two renewal processes affect each other. Recently, Alexander and Berger [2] show the distribution of $\tau \cap \hat{\tau}$ for all $\alpha, \hat{\alpha} \geq 0$. This result must be useful and essential for understanding the quenched case of our model. As in Section 1.3 in Chapter 1, we strongly believe that the quenched case in this system must have the weak and strong disorder regimes, that is, there exists $\beta_c \geq 0$ such that

$$\hat{h}^{\mathsf{q}}_\beta \begin{cases} = h^{\mathsf{a}}_\beta & \text{if } \beta \in [0, \beta_c), \\ > h^{\mathsf{a}}_\beta & \text{if } \beta > \beta_c. \end{cases} \tag{2.3.72}$$

In August 2016, Alexander and Berger [4] write a paper on this model and they prove that in the case where $\tau \cap \hat{\tau}$ is recurrent, or transient with $\alpha + \hat{\alpha} = 1$, the quenched and annealed critical points are equal (both equal to 0 in the recurrent case).

2.4 Results from the renewal theory

We present some well-known results on renewal theorem (for more detail and the proofs, refer to [6, 14]).

Proposition 2.4.1. *If τ is a renewal with $\tau_0 = 0$, then*

$$\mathbb{E}[|\tau|] = \frac{1}{\mathbb{P}(\tau_1 = \infty)}, \tag{2.4.1}$$

where $|\tau|$ is the cardinality of the set of renewal times.

Proposition 2.4.2. *If τ is transient with regular varying return time distribution, then*

$$\mathbb{P}(n \in \tau) \sim \frac{\mathbb{P}(\tau_1 = n)}{\mathbb{P}(\tau_1 = \infty)^2} \tag{2.4.2}$$

as $n \to \infty$.

Proposition 2.4.3. *If τ is a recurrent renewal with first return time distribution K as in* (2.1.1) *then*

$$\mathbb{P}(n \in \tau) \sim \begin{cases} \frac{K(n)}{\bar{K}(n)^2} & \text{if } \alpha = 0, \\ \frac{c_\alpha}{L(n)n^{1-\alpha}} & \text{if } \alpha \in (0,1), \\ \sum_{j=0}^{n} \bar{K}(j) & \text{if } \alpha = 1, \end{cases} \tag{2.4.3}$$

as $n \to \infty$, where $\bar{K}(j) = \sum_{i=j+1}^{\infty} K(i)$ for each $i \geq 0$ and $c_\alpha = \frac{\alpha \sin(\pi\alpha)}{\pi}$. Note that when $\alpha = 0$, the sequence $\{\bar{K}(n)\}_{n\geq 1}$ is slowly varying, when $\alpha = 1$, the sequence $\{\sum_{j=0}^{n} \bar{K}(j)\}_{n\geq 1}$ is slowly varying.

The proof of this proposition is in [56, Theorem 1.1] for $\alpha = 0$, in [40, Theorem 1.1] for $\alpha \in (0,1)$, in [15, Theorem 8.7.5] for $\alpha = 1$, while for $\alpha > 1$ it is the renewal theorem, refer to [6].

Lemma 2.4.4. *If $r > 0$, then*

$$\mathbb{E}[\tau_k^{-r}] \sim \begin{cases} (\mu k)^{-r} & \text{if } \alpha > 1, \\ \mathbb{E}[X_\alpha^{-r}]k^{-r/\alpha} & \text{if } \alpha \in (0,1), \end{cases} \tag{2.4.4}$$

as $k \to \infty$, where X_α is an α-stable random variable.

Proof of Lemma 2.4.4. If $\alpha > 1$, the result follows by bounded convergence theorem. Since,

$$\tau_k = \tau_k - \tau_{k-1} + \tau_{k-1} - \cdots - \tau_1 + \tau_1 = \sum_{i=1}^{k} (\tau_i - \tau_{i-1}), \tag{2.4.5}$$

τ_k is a summation of i.i.d. sequence. By the renewal theorem, $(\frac{\tau_k}{k})^{-r}$ converges $\mathbb{P}$-a.s. to μ^{-r} (it may be proven by Markov's inequality as in the case $\alpha \in (0,1)$ below and the Borel-Cantelli lemma) and is bounded from above by 1.

If $\alpha \in (0,1)$, we use that $\frac{\tau_k}{k^{1/\alpha}}$ converges to an α-stable r.v. X_α. The only complication is that $(\frac{\tau_k}{k^{1/\alpha}})^{-r}$ is not bounded, but the result still holds by uniform integrability, namely, by Exercise 3.2.5 in [31], it is enough to show that for some $\gamma > r$ we have

$$\sup_{k\geq 1} \mathbb{E}[(\frac{\tau_k}{k^{1/\alpha}})^{-\gamma}] < \infty. \tag{2.4.6}$$

To show this, first note that

$$\mathbb{E}[(\frac{\tau_k}{k^{1/\alpha}})^{-\gamma}] = \int_0^\infty \mathbb{P}(\{\tau_k/k^{1/\alpha}\}^{-\gamma} > t)\,dt = \int_0^\infty \mathbb{P}(\tau_k < k^{1/\alpha}t^{-1/\gamma})\,dt. \tag{2.4.7}$$

With the use of Markov's inequality, the probability inside the integral is bounded as

$$\begin{aligned}\mathbb{P}(\tau_k < k^{1/\alpha}t^{-1/\gamma}) &= \mathbb{P}(e^{-\theta\tau_k} < e^{-\theta k^{1/\alpha}t^{-1/\gamma}}) \leq \mathbb{E}[e^{-\theta\tau_k}]e^{\theta k^{1/\alpha}t^{-1/\gamma}} \\ &= \mathbb{P}[e^{-\theta\tau_1}]^k e^{\theta k^{1/\alpha}t^{-1/\gamma}} = e^{-k(-\log M(\theta) - \theta k^{1/\alpha - 1}t^{-1/\gamma})},\end{aligned} \tag{2.4.8}$$

where we use i.i.d. property of τ for the equality in the second line and denote by $M(\theta)$ the moment generating function $\mathbb{E}[e^{-\theta\tau_1}]$. We define the function Λ by

$$\Lambda(\theta, x) = -\log M(\theta) - \theta x, \tag{2.4.9}$$

for $x > 0$. Then, $\Lambda''(\theta,x) = -(\frac{M''(\theta)}{M(\theta)} - (\frac{M'(\theta)}{M(\theta)})^2) \leq 0$, $\Lambda(0,x) = 0$, and $\Lambda(\theta,x) \to -\infty$ as $\theta \to \infty$. Therefore, there exists θ_c such that $\Lambda(\theta,x) \leq \Lambda(\theta_c,x)$ for any $\theta \geq 0$. A standard Tauberian argument [34, (5.22) of Chapter XIII] shows that there exists a constant $C > 0$ such that $M(\theta)$ satisfies $M(\theta) \geq \exp(-C|\theta|^\alpha)$ for all $\theta \geq 0$. This implies, for $x > 0$, the following bound

$$\begin{aligned}\Lambda(\theta_c, x) &= \{-\theta_c x - \log M(\theta_c)\} \\ &\leq \{-\theta_c x + C|\theta_c|^\alpha\} = C_1 x^{-\alpha/(1-\alpha)} =: \Lambda^*(x),\end{aligned} \tag{2.4.10}$$

where $C_1 := (1-\alpha)C^{(1-\alpha)^{-1}}\alpha^{\alpha(1-\alpha)^{-1}} > 0$. We can regard $\Lambda^*(x)$ as the rate function for the sequence $\{\tau_k\}_{k\geq 1}$.

$$\begin{aligned}\mathbb{P}(\tau_k < k^{1/\alpha}t^{-1/\gamma}) &\leq e^{-k\Lambda^*(k^{1/\alpha-1}t^{-1/\gamma})} \\ &= \exp\{-C_1 t^{\frac{\alpha}{\gamma(1-\alpha)}}\}.\end{aligned} \tag{2.4.11}$$

Consequently, the probability in (2.4.8) is bounded from above. The result follows from (2.4.7) and (2.4.11). ■

Lemma 2.4.5. *For any compact subset J of $(-\infty, 0)$ there are constants $0 < c_1 < c_2$ and a slowly varying function $\tilde{L}$ such that*

$$c_1 \le \frac{\mathbb{P}_h \times \hat{\mathbb{P}}(n \in \widetilde{\tau})}{\tilde{L}(n)/n^{1+\alpha_{\text{eff}}}} \le c_2, \tag{2.4.12}$$

for all $h \in J$ and $n \ge 1$.

Proof of Lemma 2.4.5. The probability in the display equals

$$\mathbb{P}_h(n \in \tau)\hat{\mathbb{P}}(n \in \hat{\tau}). \tag{2.4.13}$$

The second probability according to Proposition 2.4.3 is asymptotically equivalent to $L_1(n)/n^{(1-\hat{\alpha})^+}$ for some slowly varying function L_1. Then the probability $\mathbb{P}_h(n \in \tau)$ is bounded below by $K_h(n) = e^h L(n) n^{-(1+\alpha)}$, while for an upper bound we let $h_0 = \sup J < 0$ and use the fact $\mathbb{P}(\tau_k = n) \le k^c K(n)$ for all $n, k \ge 1$ for some constant $c > 0$ (see [41, Lemma A.5]) to get

$$\mathbb{P}_h(n \in \tau) = \sum_{k=1}^{n} e^{hk} \mathbb{P}(\tau_k = n) \le K(n) \sum_{k=1}^{n} e^{h_0 k} k^c, \tag{2.4.14}$$

for all $h \in J$. It is crucial here that the compact set J does not include 0 so that $h_0 < 0$. ■

Lemma 2.4.6. *Let I be a compact subset of $(0, \infty)$, $\{K_\gamma : \gamma \in I\}$ a family of renewal interarrival laws with $K_\gamma(\infty) = 1 - e^{-\gamma}$ and renewal function u_γ for each $\gamma \in I$, and such that there are $\alpha, c_1, c_2 > 0$ and slowly varying function L so that*

$$c_1 \frac{L(n)}{n^{1+\alpha}} \le u_\gamma(n) \le c_2 \frac{L(n)}{n^{1+\alpha}}, \tag{2.4.15}$$

for all $n \ge 1$ and $\gamma \in I$. Let F_γ be the free energy corresponding to the homopolymer defined by K_γ. Then there are $C_1, C_2 > 0$ and slowly varying function $\hat{L}$ so that

$$C_1 \le \frac{F_\gamma(\gamma + h)}{h^{\frac{1}{\alpha} \vee 1} \hat{L}(1/h)} \le C_2, \tag{2.4.16}$$

for all $h \in (0, 1]$ and $\gamma \in I$. For $\alpha = 0$, (2.4.16) means that for $h \downarrow 0$, $F_\gamma(\gamma + h)$ vanishes faster than any polynomial.

Recall that F_γ is zero exactly in $(-\infty, \gamma]$ and positive elsewhere.

Proof of Lemma 2.4.6. For $h > 0$, $F_\gamma(\gamma + h)$ is the unique solution in x of the equation

$$\sum_{n=1}^{\infty} K_\gamma(n) e^{-nx} = e^{-(\gamma+h)}, \tag{2.4.17}$$

which we write as

$$\Psi_\gamma(x) = 1 - e^{-h}, \tag{2.4.18}$$

with

$$\Psi_\gamma(x) = 1 - e^{\gamma} \sum_{n=1}^{\infty} K_\gamma(n) e^{-nx}. \tag{2.4.19}$$

Now for any function $f : \mathbb{N} \to [0, \infty)$, we define $\hat{f}(z) = \sum_{n=1}^{\infty} f(n) z^n$ for all $z \in [0, 1]$. Then the equality

$$u_\gamma(n) = \mathbf{1}_{\{n=0\}} + \sum_{j=1}^{n} K_\gamma(j) u_\gamma(n-j) \tag{2.4.20}$$

gives

$$\hat{K}_\gamma(z) = 1 - \frac{1}{\hat{u}_\gamma(z)}. \tag{2.4.21}$$

In particular, $e^{-\gamma} = 1 - 1/\hat{u}_\gamma(1)$, so that

$$\Psi_\gamma(x) = 1 - e^{\gamma} \hat{K}_\gamma(e^{-x}) = e^{\gamma} \left(\frac{1}{\hat{u}_\gamma(e^{-x})} - \frac{1}{\hat{u}_\gamma(1)} \right) = \frac{e^{\gamma}}{\hat{u}_\gamma(e^{-x}) \hat{u}_\gamma(1)} A_\gamma(e^{-x}), \tag{2.4.22}$$

with

$$A_\gamma(z) := \hat{u}_\gamma(1) - \hat{u}_\gamma(z) = (1 - z) \sum_{n=1}^{\infty} u_\gamma(n) \sum_{k=0}^{n-1} z^k = (1 - z) \sum_{k=0}^{\infty} z^k \sum_{n=k+1}^{\infty} u_\gamma(n). \tag{2.4.23}$$

Of interest to us is the behavior of Ψ_γ around 0, and thus of A_γ around 1. The following claims addresses the issue. Let $m := \sum_{n=1}^{\infty} L(n)/n^{\alpha}$.

Claim . *(a) If $\alpha = 0$, then there are $0 < C_3 < C_4$ so that*

$$C_3 \le \frac{A_\gamma(z)}{L_0((1 - z)^{-1})} \le C_4, \tag{2.4.24}$$

for all $z \in [1/2, 1]$, where L_0 is the slowly varying function defined in (2.4.30).

(b) If $\alpha \in (0,1)$, then there are $0 < C_3 < C_4$ so that

$$C_3 \le \frac{A_\gamma(z)}{(1-z)^\alpha L((1-z)^{-1})} \le C_4, \tag{2.4.25}$$

for all $z \in [1/2, 1]$.

(c) If $\alpha = 1$ and $m = \infty$, then there is a slowly varying function $\widetilde{L}$ and $0 < C_3 < C_4$ so that

$$C_3 \le \frac{A_\gamma(z)}{(1-z) L_1((1-z)^{-1})} \le C_4, \tag{2.4.26}$$

for all $z \in [1/2, 1]$.

(d) If $m < \infty$ then there are $0 < C_3 < C_4$ so that

$$C_3 \le \frac{A_\gamma(z)}{1-z} \le C_4, \tag{2.4.27}$$

for all $z \in [1/2, 1]$.

Proof of the claim. By the bounds we have on u_γ, it suffices to examine the behavior of

$$Q(z) := \sum_{k=0}^{\infty} z^k \sum_{n=k+1}^{\infty} \frac{L(n)}{n^{1+\alpha}}. \tag{2.4.28}$$

Denote by q_k the coefficient of z^k in this power series. It is $q_k \sim \frac{L(k)}{\alpha k^\alpha}$, for $\alpha > 0$, by Proposition 1.5.10 in [15], while for $\alpha = 0$, q_k is slowly varying (Proposition 1.5.9.b in [15]). Thus

$$\sum_{k=1}^{r} q_k \begin{cases} \sim r q_r & \text{if } \alpha = 0, \\ \sim \frac{L(r) r^{1-\alpha}}{\alpha(1-\alpha)} & \text{if } \alpha \in (0,1), \\ \text{is slowly varying} & \text{if } \alpha = 1 \text{ and } m = \infty, \\ \to m & \text{if } m < \infty. \end{cases} \tag{2.4.29}$$

These follow from Proposition 1.5.8 and Proposition 1.5.9(a) in [15]. Then parts (a)-(c) of the claim follow from Corollary 1.7.3 in [15], while for the case $m < \infty$ we just note that $Q(z)$ maps $[1/2, 1]$ to a compact set of $(0, \infty)$. The corollary specifies that for L_0 in (2.4.24) we can take

$$L_0(y) = q_{[y]}, \tag{2.4.30}$$

for all $y \in [0, \infty)$. ∎

We continue with the proof of Lemma 2.4.6. The Claims and (2.4.22) imply that there are constants $0 < C_5 < C_6$ and slowly varying function L_2 so that

$$C_5 \leq \frac{\Psi_\gamma(x)}{x^{\alpha\wedge 1} L_2(1/x)} \leq C_6, \tag{2.4.31}$$

for all $x \in (0,1]$ and $\gamma \in I$. Let $C > 0$ fixed. For $\alpha > 0$, by Proposition 1.5.15 in [15], there is a slowly varying function $\hat{L}$ so that a solution $x_C(h)$ of $x^{\alpha\wedge 1} L_2(1/x) = (1 - e^{-h})/C$ is asymptotically equivalent to a constant multiple of $h^{1\vee 1/\alpha}\hat{L}(1/h)$ as $h \to 0^+$ ($\hat{L}$ is the same for all C). If $\alpha = 0$, let $x_C(h)$ be the smallest solution of $L_0(1/x) = (1 - e^{-h})/C$. Then, $x_C(h) = 1/L_0^{-1}((1 - e^{-h})/C)$ with L_0^{-1} an obviously defined "inverse" of L_0. It is easy to see, by bounding below q_k by $\sum_{n=k+1}^{\infty} L(n)/n^{1+\varepsilon}$ for any $\varepsilon > 0$, that each x_C goes to zero faster than any power of h. For each $\gamma \in I$, the solution of (2.4.18) is between $x_{C_5}(h), x_{C_6}(h)$, and this finishes the proof of the lemma. ■

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Summary

We consider self-avoiding walk (SAW) on random conductors and the pinning models in this thesis. Through considering these models we take a look at the phase transitions and the critical phenomena for the disordered systems.

SAW on random conductors

In the study of SAW on random conductors, we can show the quenched critical point is independent of the reference point and is a degenerate random variable although the quenched susceptibility does depend on both the reference point and the environment. On $\mathbb{Z}^1$, we know the exact value of the quenched critical point, but on $\mathbb{Z}^{d\geq 2}$, we only estimate it from above and below by applying the first and second moment estimate. Under the condition that the lattice is homogeneous degree tree, we have known that the quenched and annealed critical points coincide for small β, i.e., in the weak disorder regime, and have the exact value of the quenched critical point in the strong disorder regime. Derrida and Spohn have proven by associating with solution for a nonlinear partial differential equation in [27], and Baffet, Patrick and Pulé have applied the martingale theory in [7]. We also provide the exact value of the quenched critical point by applying the fractional moment method. However, on $\mathbb{Z}^{d\geq 2}$, we can not apply these methods because we do not have the martingale structure on $\mathbb{Z}^{d\geq 2}$.

Open Problem 1. *On $\mathbb{Z}^{d\geq 2}$, what is the quenched critical point?*
Can we find more sharp bounds than (1.2.13) *in Theorem 1.2.1?*

On a tree graph, we can prove that the quenched susceptibility diverges at the critical point, but we have not known its behavior yet.

(i) $\mathbb{Z}^d$

$$\log \mu - \beta \mathbb{E}[X] \leq \hat{h}^{\mathsf{q}}_{\beta} \leq h^{\mathsf{a}}_{\beta}. \tag{2.4.32}$$

(ii) $\mathbb{Z}^1$

$$\hat{h}^{\mathsf{q}}_{\beta} = \underbrace{\log \mu}_{=0} - \beta \mathbb{E}[X]. \tag{2.4.33}$$

(iii) $\mathbb{Z}^2$

$$\log \mu - \beta \mathbb{E}[X] \leq \hat{h}^{\mathsf{q}}_{\beta} < h^{\mathsf{a}}_{\beta}. \tag{2.4.34}$$

(iv) Homogeneous tree

$$\hat{h}^{\mathsf{q}}_{\beta} = \begin{cases} h^{\mathsf{a}}_{\beta} & \text{if } \beta \leq \beta_c, \\ \frac{\beta}{\beta_c} h^{\mathsf{a}}_{\beta_c} & \text{if } \beta > \beta_c. \end{cases} \tag{2.4.35}$$

To compute or estimate the quenched critical point on $\mathbb{Z}^{d\geq 2}$ and to understand the behavior of the quenched susceptibility around its critical point are ongoing works.

Pinning model on renewal set

In the study of the pinning model, we consider the model that belongs to the class of the long-range correlations. By observing the interaction between two renewals, we obtain some information on the annealed case

(i) the annealed critical curve

(ii) the annealed critical point

(iii) the asymptotic behavior of the annealed critical point

(iv) the asymptotic behavior of the annealed free energy around its critical point

The difficulty of this model lies in how two renewal processes affect each other. We have not completely understood it yet. Especially, we have few knowledge about the quenched case.

Open Problem 2 (Irrelevant regime)**.** *For what α and $\hat{\alpha}$, the quenched and annealed critical point coincides for small β?*

Open Problem 3 (Relevant regime)**.** *How the smoothing inequality is?*

Open Problem 4 (Relevant regime)**.** *If in the relevant regime, how much the quenched critical point differs from the annealed one?*

As in SAW on random conductors, we also want to know the critical parameter that divides the weak and strong disorder regimes. This is being done by Alexander and Berger [4]. For the problem about the interaction of two renewals, Alexander and Berger show some results in [2], for the problem about the quenched critical point estimate, Caravenna, Sun, and Zygouras give a hopeful method in [18]. The knowledge for the problems above is accumulating. In August. 2016, Alexander and Berger [4] write a paper on this model and they prove that in the case where $\tau \cap \hat{\tau}$ is recurrent, or transient with $\alpha + \hat{\alpha} = 1$, the quenched and annealed critical points are equal (both equal to 0 in the recurrent case).

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