博士学位論文

Recognition of plane-to-plane map-germs and its application to projective differential geometry

（平面から平面への写像芽の認識問題
とその射影微分幾何学への応用）

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1 Introduction

In this thesis, we are interested in the local geometry of C^∞-maps between finite dimensional manifolds, in other words, the study of map-germs $\mathbb{R}^m, 0 \to \mathbb{R}^n, 0$. It is natural to identify two germs if they coincide through suitable local coordinate changes of the source and the target, that is usually called the $\mathcal{A}$-*equivalence* of map-germs. This notion is most important in Singularity Theory of Differentiable Maps: non-regular germs (and their $\mathcal{A}$-types) are called *singularities* of maps.

Throughout this thesis, we mainly treat with plane-to-plane map-germs of corank 1 under $\mathcal{A}$-equivalence. When we think of equivalence relation, there appears the "classification problem" first. Namely, one classifies all map-germs under the $\mathcal{A}$-equivalence from easier ones to more complicated ones step by step. A basic invariant which measures the complexity of germs is the codimension of $\mathcal{A}$-orbits in the space of all map-germs, called $\mathcal{A}$-*codimension* (denoted by $\mathcal{A}$-cod). In particular, plane-to-plane germs with $\mathcal{A}$-cod ≤ 2 are of type *regular*, *fold* and *cusp*, known as *stable-germs*, i.e., germs which are stable under small perturbations, that goes back to the pioneering work of H. Whitney [71], one of founders of singularity theory. Germs of $\mathcal{A}$-cod $= 3$ generically appear in one-parameter families of map-germs, and are classified into three types named by *lips*, *beaks* and *swallowtail*. Higher codimensional map-germs are also considered: J. H. Rieger classified the $\mathcal{A}$-types of plane-to-plane germs of of corank 1 with $\mathcal{A}$-cod ≤ 6 in [54, 55, 58]. The classification is done by detailed studies of $\mathcal{A}$-tangent spaces of map-germs via several infinitesimal criteria ($\mathcal{A}$-determinacy, Mather's lemma etc). In Chapter 2 below, we summarize the basics in singularity theory of map-germs and Rieger's $\mathcal{A}$-classification of plane-to-plane germs.

The $\mathcal{A}$-classification will invite various applications in differential geometry, and then we naturally encounter *recognition problem*: Suppose a plane-to-plane map-germ is given in a certain geometric setting or a germ is explicitly given using local coordinates, it is natural to ask

how shall we detect the $\mathcal{A}$-type which the germ belongs to ?

That is referred to as "recognition problem" (named in T. Gaffney's paper [22]). First, we should follow Rieger's procedure classifying germs, that is, we check the Taylor expansion of the given germ from low degree terms to higher degree ones subject to 'recognition trees' in [54]. However we soon meet a difficulty: the procedure (an algorithm) contains a lot of branches, at each of which we have to find some 'very nice coordinate changes' – some of such coordinate changes may not explicitly be given, while only just the existence can be certified by Mather's lemma etc. Thus in general it is NOT easy, especially for non-experts, to follow the classification procedure from the input of the given germ.

As an elementary approach to this problem, K. Saji [60] gave criteria for germs of $\mathcal{A}$-cod $= 3$ in a similar way to the criteria for stable germs presented essentially in the work of Whitney [71] (see also [61]). The criteria are described in terms of simple geometric operators (*intrincic derivatives*), and therefore, the method would be "user-friendly" for non-experts. In fact it is quite convenient for many applications of singularity theory to (classical) differential geometry.

The first purpose of this thesis is to develop geometric criteria suggested in Saji [60] to treat all $\mathcal{A}$-types in Rieger's list in [54]. An emphasis is that *equisingularity problem* (or topological $\mathcal{A}$-classification) of map-germs come into the picture, which has been treated in another paper of Rieger [55] . We obtain a complete set of the criteria for the recognition problem (Theorem 5.1): It consists of two sorts of conditions – *coordinate-free* conditions in terms of intrinsic derivatives etc and additional conditions in *coefficients of Taylor expansions* in a special form. The former condition detects the *specified jet* of *topological $\mathcal{A}$-type*, and the latter condition detects the $\mathcal{A}$-type provided the former condition is satisfied. The proof is to show first the invariance of former condition under $\mathcal{A}$-equivalence; and then to find the latter condition by describing explicitly 'very nice coordinate changes' which are hidden in Rieger's classification process. This will be done in Chapter 3.

Our second purpose is to present several concrete applications of our criteria to classical differential geometry of surfaces. In Chapter 4–6, we will mainly discuss projective differential geometry of surfaces in 3-space and binary differential equations associated to asymptotic curves. This theme was intensively investigated from the end of the 19th century to the early 20th century; in particular, Salmon, Cayley and Zeuthen developed the local study of the contact of a surface with planes and lines (cf. [66]). To this classical subject, rather recently, new interests and ideas have been brought from singularity theory, generic differential geometry and applied mathematics such as computer-vision [3, 4, 8, 6, 23, 30, 31, 32, 38, 33, 49, 50, 51, 53, 56, 68, 69]. Our standpoint in this thesis is to apply Rieger's $\mathcal{A}$-classification to projective differential geometry by using our criteria in Chapter 3. Especially, germs with high $\mathcal{A}$-codimension appear when we consider *central projections* of one or two parameter families of surfaces.

Notice that our approach is, in a sense, very traditional in singularity theory. It was originally R. Thom's idea and developed by I. Porteous [53] to study extrinsic geometry of surface by considering singularities of *height functions* and *distance squared functions* which measure the contact of surface with a plane and a sphere. On the other hand, the $\mathcal{A}$-type of the projection along a tangent line measures the contact of the surface with the line. The first attempt of an application of $\mathcal{A}$-classification to the geometry of surface might be seen in Gaffney-Ruas [23] (see also [22, 6, 8]). Remark that a similar result was given by Arnold-Platonova [1, 50, 51], though it was discussed in a different context from $\mathcal{A}$-classification. In this direction, refer also to D. Mond [38, 40], A. Nabarro [41, 42] where they classified map-germs between several dimensional spaces with an aim to study extrinsic geometry of surfaces embedded in some space. See also [9, 27, 30, 36, 57] about other classifications of map-germs and applications. Beside pure mathematics, singularity theory is regarded as an important tool in *computer vision*. A little bit before the works by Gaffney-Ruas [23] and by Arnold-Platonova [1, 50, 51], Koenderink-Doon dealt with shape recognition via apparent contours of projection using singularity theory of plane-to-plane maps (cf. [32]). Actually, Rieger himself wrote a paper [56] on this subject as an application of his $\mathcal{A}$-classification of more complicated singularities in [54, 55].

Chapter 4 is devoted to classification of singularities in *central projections* of surfaces. Suppose that we are given a smooth surface in 3-space. Look at it from a viewpoint (one's eye), then we get a smooth map from the surface to the plane (screen), that is called the *central projection.* It then gives rise to the classification problem of singularities arising in the central projection of generic surface, that was indeed treated in V. I. Arnold and O. A. Platonova (also O. P. Shcherbak and V. V. Goryunov) [1, 2, 26, 50, 62] in detail. Here we try to reconsider this problem as a typical case of *recognition problem* in $\mathcal{A}$-classification. Note that the freedom of the choice of viewpoint is just of 3-dimension, so we are given a 3-parameter family of plane-to-plane map-germs (locally). Thus we are concerned with detecting $\mathcal{A}$-type of map-germs arising in this particular geometric setting. It follows from Arnold-Platonova's theorem that some germs of $\mathcal{A}$-codimension 5 in Rieger's list *do not appear* generically in central projections, while the reason has not been quite clear from the context of $\mathcal{A}$-classification, as Rieger noted in his paper [54]; it is because Arnold-Platonova's argument stands on some different classification of diagrams of map-germs. Our criteria make the reason very clear – that is actually caused by the difference between coordinate-free conditions and Taylor coefficient conditions of our criteria. The basic line of our approach using the method of J. W. Bruce [8] is not so different from Platonova's [50], but we start from Rieger's classification and relate it to topological $\mathcal{A}$-classification. We present a new transparent proof of Arnold-Platonova's theorem within the $\mathcal{A}$-classification theory, moreover, in Theorem 4.4 below, we classify singularities which appear in central projections obtained by projecting moving surfaces with one-parameter (cf. Rieger [56] for *parallel projections*).

As a byproduct, we get an explicit stratification of jet space of Monge forms where each stratum corresponds to $\mathcal{A}$-types of central projections. In Chapter 5, we derive normal forms of jets of Monge forms via projective transformations on $\mathbb{P}^3$. A classically well-known fact (Tresse [67], Wilczynski [72]) is that at general (or the most non-degenerate) hyperbolic points of a surface, the Monge form is transformed to

$$xy + x^3 + y^3 + \alpha x^4 + \beta y^4 + \cdots$$

by some suitable projective transformation, where moduli parameters α, β are *primary* projective differential invariants. This is the simplest one. In fact, Platonova [38] gave a projective classification of jets of Monge forms with codimension ≤ 2, i.e. Monge forms at any point of a *generic* surface. Here we extend her classification to ones up to codimension ≤ 4; That is, strata of degenerate Monge forms at the transition moments appearing in an at most 2-parameter generic family of surfaces are explicitly captured. Our framework with Bruces's transversality theorem is a key setup. Also we carefully find normal forms with leading moduli parameters as primary differential invariants of the germ of surface, although Platonova's list only considers jets which does not contain moduli parameters. So our approach will be enable to bring new interests to this classical subject. In fact, for parabolic (and flat) points of surfaces, it seems that there has been almost no detailed studies on differential invariants in literature, thus our normal forms would be quite useful for further studies in local projective geometry at parabolic points.

Following Chapter 5, in Chapter 6, we study *parabolic and flecnodal* curves in a surface, which are invariant under projective transformations. For a generic surface, these invariant curves were well studied by Landis [33] based on the classification of Platonova [50]. For example, on a generic surface, the parabolic curve is smooth and the flecnodal curve is immersed with transversal self-intersections (nodes), and these two curves meet tangentially at *cusp of Gauss* (or *godron*). Then it would be a natural question to ask how they behave when a surface moves with parameter. The answers have been given in several different contexts. First we look at results from theory of *BDE (binary differential equation)*, which tells us about the behavior of nets of asymptotic curves and parabolic curves. Davydov [17, 18] and Bruce-Tari [11, 12, 13, 63, 65] has presented the topological classification of (families of) BDE. We then compare our classification of degenerate Monge forms with the classification of general BDE due to Davydov, Bruce, Tari etc. Note that considering just BDE does miss information about flecnodal curves, thus we also refer to Uribe-Vargas's [69] result. In part of his dissertation [69] Uribe-Vargas presented a complete list of generic 1-parameter bifurcations of flecnodal curves via a geometric approach using the dual surfaces and asymptotic BDE. We give a precise characterization of moduli parameters in our corresponding Monge form for each type of Uribe-Vargas's classification.

We finally deal with singular surface germs with *crosscap* in Chapter 7. Recently the crosscap has gotten attention in the context of differential geometry, and most works are done in Euclidean setting (cf. [70, 28, 29, 64, 48]). Our aim here is to reconsider results in Ph.D. thesis of West [70] from a viewpoint of projective or affine differential geometry. First, we deal with singularities of projections of generic crosscaps. West [70] classified singularities of orthogonal projections of generic crosscaps, where all corank 2 germs with $\mathcal{A}$-cod $= 4$, i.e. *the sharksfin* (for elliptic crosscaps) and *the deltoid* (for hyperbolic crosscaps), appear. On the other hand, corank 2 germs with $\mathcal{A}$-cod $= 5$ never appear in central projections of generic crosscaps (the similar thing happens in the case of regular surfaces: Arnold-Platonova's theorem [1, 50]). We show that corank 2 germs arising in central projections of generic crosscaps are just the sharksfin and the deltoid. Next we consider characteristic curves on crosscap. It is shown in West [70] that the parabolic curve does not approach to a hyperbolic crosscap point, while there are two smooth components of the curve passing through an elliptic crosscap point. We then consider flecnodal curves near an elliptic crosscap (the existence follows from the bifurcation diagram of sharksfin, see [73]). In fact, the parabolic and flecnodal curves meet tangentially at an elliptic crosscap, and we determine the *order of the contact* of the curves. It is remarkable that the order of contact of these two curves is a new invariant which exactly determines the type of corank 2 germs arising in central projections.

We end the Introduction by mentioning the author's works in progress related to this subject and his further research plans, the details which are not included in this thesis. Actually, our criteria for detecting $\mathcal{A}$-types of map-germs and their applications are widely open.

- Projective differential geometry of surfaces: In fact, our work gives a new insight to Wilczynski's works on projective differential geometry of surfaces in $\mathbb{P}^3$ estasblished in early 20th century. Our normal forms of Monge expression suggest a new direction of Cartan-type theory for a surface at parabolic points (cf. [47]). In the similar direction, the author is dealing with ruled surfaces in $\mathbb{P}^3$ with Deolindo Silva and Ohmoto [20], and generic surfaces in 4-space with Deolindo Silva [21].

- Bifurcation diagrams of map-germs: Our criteria are useful to find defining equations of strata of the bifurcation diagram of a given finitely determined germ. For instance, for plane-to-plane map-germs of corank 2 with $\mathcal{A}$-cod ≤ 5 the bifurcation diagrams have been completely examined in papers of the author with Yoshida and Ohmoto [73, 74].

- Recognition of map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^3, 0$: A similar approach as discussed in Chapter 3 can be considered in other dimensional cases. For instance, there have been existing $\mathcal{A}$-classification of map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ by Bruce [9], du Plessis [52], Marar-Tari [36], Hawes [27]. The author has already obtained several criteria for germs of corank 1 with $\mathcal{A}$-cod ≤ 4 in terms of intrinsic derivatives. As its application, we can classify singularities of focal surfaces arising in generic 1-parameter family of line congruences in $\mathbb{R}^3$. In particular, such families often appear in relation with integrable systems, e.g. breather surfaces with parameter are obtained as solutions of Sine-Gordon equation. This also suggests a new direction of application of singularity theory to classical differential geometry in integrable system.

- Recognition and Logarithmic vector fields of A_μ-discriminant: We solve recognition problem of plane-to-plane map-germs by directly constructing suitable coordinate changes, while Gaffney [22] solved it for a few germs in studying a finer algebraic structure of the corresponding $\mathcal{A}$-tangent space. These two approaches should be compared. It would also be reasonable to discuss about the problem in the context of Damon's $\mathcal{K}_D$-theory using the logarithmic vector fields along the A_μ-type discriminant of a stable unfolding. They are in progress by the author with Ohmoto and Wik Atique.

- Projections of crosscaps: As just mentioned, in Chapter 7, central projection of generic crosscap is discussed. In entirely the same way, central projection of parabolic crosscap has also been studied by the author with Yoshida and Ohmoto [74]. As another approach, let X be a standard crosscap, and consider $\mathcal{A}(X)$-equivalence of submersions $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$. The classification was given in West [70], and the bifurcation diagram is completely determined by the author with Barajas [5]. It would also be interesting to invent invariant theory of crosscaps in projective or affine differential geometry (cf. [47]), and compare that with our invariants gotten from singularity theoretic approach.

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2 Preliminary

In this chapter, we give a brief introduction to the Thom-Mather theory of singularities of maps. There are many good references, e.g. [25, 39, 43]. Also we review the classification of plane-to-plane map-germs due to J. H. Rieger [54, 55, 58], which is of particular interest in this thesis.

2.1 Map-germs and $\mathcal{A}$-equivalence

Definition 2.1 Let X be a topological space. Two subsets X_1 and X_2 of X *have the same germ at* $x_0 \in X$ if there is a neighborhood U of x_0 in X such that $X_1 \cap U = X_2 \cap U$.

This is evidently an equivalence relation. We call the equivalence class which contains X *the set germ of* X *at* x_0 , and denote it by (X, x_0).

Definition 2.2 Let N, P be topological spaces, $U_1, U_2 \subset N$ an open neighborhood of $x_0 \in N$. $f_1 : U_1 \to P$ and $f_2 : U_2 \to P$ *have the same germ at* x_0 , if there is a neighborhood $W \subset U_1 \cap U_2$ of x_0 such that $f_1 = f_2$ on W.

This is also an equivalence relation. We call the equivalence class of $f : U \to P$ *the map germ of* f *at* x_0, and denote it by $f : N, x_0 \to P$ or $f : N, x_0 \to P, f(x_0)$. When N, P are smooth manifolds and f is smooth at x_0, we say that $f : N, x_0 \to P$ is *smooth*.

In this thesis N and P are manifolds; we write a system of local coordinates of N as $x = (x_1, \cdots, x_n)$ centered at $x_0 \in N$; a system of local coordinates of P as $X = (X_1, \cdots, X_p)$ centered at $f(x_0) \in P$. Our main object is a smooth map-germ

$$f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0 \quad f \text{ is smooth.}$$

Let us introduce some notations. Let $\mathcal{E}_n := \{\varphi : \mathbb{R}^n, 0 \to \mathbb{R}, \varphi(0) : \varphi \text{ is smooth}\}$ be an $\mathbb{R}$-algebra with a unique maximal ideal $m_n := \{\varphi : \mathbb{R}^n, 0 \to \mathbb{R}, 0 : \varphi \text{ is smooth}\} = \langle x_1, \cdots, x_n \rangle_{\mathcal{E}_n}$. We denote

$$\mathcal{E}_n^p := \{f : \mathbb{R}^n, 0 \to \mathbb{R}^p, f(0) : f \text{ is smooth}\}$$

which is an $\mathcal{E}_n$-module. In particular, $m_n \mathcal{E}_n^p = \{f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0 : \text{smooth}\}$.

One can show that the composition of map germs is well-defined, so the next definition has the meaning.

Definition 2.3 ($\mathcal{A}$-equivalence) Let $f, g : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be smooth map germs. f and g are $\mathcal{A}$-*equivalent* ($f \sim_{\mathcal{A}} g$), if there exist diffeomorphism germs ϕ and ψ so that

$$\begin{tikzcd} \mathbb{R}^n, 0 \arrow[r, "f"] \arrow[d, "\phi"'] & \mathbb{R}^p, 0 \arrow[d, "\psi"] \\ \mathbb{R}^n, 0 \arrow[r, "g"'] & \mathbb{R}^p, 0 \end{tikzcd}$$

We are interested in the classification and the"recognition" of smooth map germs by $\mathcal{A}$-equivalence.

We denote by $\mathcal{A}.f$ the $\mathcal{A}$-orbit of f, and want to denote by $T\mathcal{A}(f)$ the 'tangent space' to the $\mathcal{A}$-orbit at f: Let $\xi : \mathbb{R}^n, 0 \to T\mathbb{R}^p$ be a smooth map-germ such that $\pi \circ \xi = f$ (where π is a projection of tangent vector bundle). We call ξ the *the vector field along* f or *infinitesimal deformation of* f, and denote the set of all the the vector field along f by $\theta(f)$. In an obvious way, $\theta(f)$ is a $\mathcal{E}_n$-module. For the identity maps $id_n : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$, $id_p : \mathbb{R}^p, 0 \to \mathbb{R}^p, 0$, we write $\theta(n) = \theta(id_n)$, $\theta(p) = \theta(id_p)$, which are the module of vector field-germs. We define $tf : \theta(n) \to \theta(f)$ by the map $\xi \mapsto df \circ \xi$, and $\omega f : \theta(p) \to \theta(f)$ by the map $\eta \mapsto \eta \circ f$. With these notations above, we define

$$T\mathcal{A}(f) := tf(m_n\theta_n) + \omega f(m_p\theta_p) \ \subset m_n\theta(f).$$

In fact, this space consists of all vectors $\frac{d}{dt}(\psi_t \circ f \circ \phi_t^{-1})\big|_{t=0}$ where ψ_t and ϕ_t are deformations of identity maps with $\psi_t(0) = 0$, $\phi_t(0) = 0$. We define the *$\mathcal{A}$-codimension of* f by

$$\mathrm{cod}(\mathcal{A}, f) := \dim_{\mathbb{R}} \frac{m_n\theta(f)}{T\mathcal{A}(f)}.$$

2.2 Determinacy

In this section, we introduce the notion of *determinacy* which is a key ingredient in the classification.

Definition 2.4 (jet) Let $U, V \subset \mathbb{R}^n$ be open and contain $p \in \mathbb{R}^n$, and $f : U \to \mathbb{R}^p$, $g : V \to \mathbb{R}^p$ smooth maps. We say that f and g *have the same r-jet at* $p \in \mathbb{R}^n$, if

$$\frac{\partial^{|\alpha|} f}{\partial x^{|\alpha|}}(p) = \frac{\partial^{|\alpha|} g}{\partial x^{|\alpha|}}(p)$$

for $0 \leq |\alpha| \leq r$ where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is the pair of positive integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Clearly this is an equivalence relation, and the equivalence class that contains f is denoted by $j^r f(p)$ and called the r-jet of f at p. $j^r f(p)$ is often identified with the truncated Talyor expansion $\sum_{0 \leq |\alpha| \leq r} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^{|\alpha|}}(p)(x-p)^\alpha$.

We define *the r-jet space* to be the set of r-jet of map-germs

$$J^r(n,p) := \{j^r f(0) | f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0 \ \text{ smooth}\}.$$

This is naturally identified with $m_n\mathcal{E}_n^p / m_n^{r+1}\mathcal{E}_n^p$, so $J^r(n,p)$ is regarded as a vector space with finite dimension.

Definition 2.5 Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a smooth map-germ. We say f *is r-determined for $\mathcal{A}$-equivalence*, or *r-$\mathcal{A}$-determined*, if $f \sim_{\mathcal{A}} g$ holds for any smooth map-germs $g : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ such that $j^r f(0) = j^r g(0)$. When f is r-determined for $r \in \mathbb{N}$, we say f *is finitely $\mathcal{A}$-determined.*

Theorem 2.6 *For a smooth map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, the followings hold:*

1. *If f is r-$\mathcal{A}$-determined, then $m_n^{r+1}\theta(f) \subset T\mathcal{A}(f)$.*
2. *If $m_n^r\theta(f) \subset T\mathcal{A}(f)$, then there exists a natural number $l(n, p, r)$ which depends only on n, p, r such that f is $l(n, p, r)$-determined for $\mathcal{A}$-equivalence.*
3. *f is finitely $\mathcal{A}$-determined if and only if there exists a natural number r such that $m_n^r\theta(f) \subset T\mathcal{A}(f)$.*
4. *f is finitely $\mathcal{A}$-determined if and only if cod$(\mathcal{A}, f) < \infty$.*

Definition 2.7 Let $f, g : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be smooth map germs. We say $j^r f(0)$ *is $\mathcal{A}^r$-equivalent to* $j^r g(0)$ ($j^r f(0) \sim_{\mathcal{A}^r} j^r g(0)$) if there exist diffeomorphism germs $\psi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and $\phi : \mathbb{R}^p, 0 \to \mathbb{R}^p, 0$ such that $j^r f(0) = j^r(\phi \circ g \circ \psi^{-1})(0)$.

Let $\mathcal{A}^r$ be the cartesian product of the spaces of r-jets of diffeomorphism germs of source and target, which becomes a Lie group. Naturally $\mathcal{A}^r$ acts on $J^r(n, p)$ (which is an algebraic action), and the orbit $\mathcal{A}^r(j^r f(0))$ is a locally closed semi-algebraic submanifold of $J^r(n, p)$. Hence we can think of $T\mathcal{A}^r(j^r f(0))$ – this is canonically isomorphic to $T\mathcal{A}(f)$ modulo $m_n^{r+1}\theta(f)$.

2.3 Unfoldings and stability

Definition 2.8 (1) An *unfolding* of a smooth map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is a map germ $\Phi : \mathbb{R}^n \times \mathbb{R}^k, (0, 0) \to \mathbb{R}^p, 0$ such that $\Phi(x, 0) = f(x)$.
(2) The unfolding Φ is *trivial* if there exit germs of diffeomorphisms $h : \mathbb{R}^n \times \mathbb{R}^k, (0, 0) \to \mathbb{R}^n \times \mathbb{R}^k, (0, 0)$ and $H : \mathbb{R}^p \times \mathbb{R}^d, (0, 0) \to \mathbb{R}^p \times \mathbb{R}^k, (0, 0)$ such that

1. $h(x, 0) = (x, 0)$ and $H(X, 0) = (X, 0)$.
2. The following diagram is commutative;

$$\begin{tikzcd}
\mathbb{R}^n \times \mathbb{R}^k, (0,0) \arrow[r, "(\Phi,\pi)"] \arrow[d, "h"'] & \mathbb{R}^p \times \mathbb{R}^k, (0,0) \arrow[r, "\pi'"] \arrow[d, "H"] & \mathbb{R}^k, 0 \arrow[d, "id"] \\
\mathbb{R}^n \times \mathbb{R}^k, (0,0) \arrow[r, "(f,\pi)"] & \mathbb{R}^p \times \mathbb{R}^k, (0,0) \arrow[r, "\pi'"] & \mathbb{R}^k, 0
\end{tikzcd}$$

   where $\pi : \mathbb{R}^n \times \mathbb{R}^k, (0, 0) \to \mathbb{R}^k, 0$ is the canonical projection.

(3) We say the map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is *stable* if every unfolding of f is trivial.

This definition is difficult to check the stability of mappings, but the next theorem makes the check easier.

Theorem 2.9 *For a smooth map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, the following conditions are equivalent:*

1. *f is stable.*

2. *f is infinitesimally stable: $tf(\theta(n)) + \omega f(\theta(p)) = \theta(f)$.*

3. *$tf(\theta(n)) + \omega f(\theta(p)) + f^* m_p \theta(f) = \theta(f)$.*

From Theorem 2.9 and 2.6, we get the following corollary:

Corollary 2.10 *A stable map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is finitely determined.*

Theorem 2.11 *Whether the smooth map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is stable or not depends only on $j^{p+1}f(0)$.*

2.4 Versality

Definition 2.12 Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be smooth, and $\Phi_i : \mathbb{R}^n \times \mathbb{R}^{k_i}, (0,0) \to \mathbb{R}^p, 0$ $(i = 1, 2)$ unfoldings of f. A triplet $(\tilde{s}, \tilde{t}, \varphi)$ is an *$\mathcal{A}$-morphism from Φ_1 to Φ_2* if $\varphi : \mathbb{R}^{k_1}, 0 \to \mathbb{R}^{k_2}, 0$ is a smooth map-germ, and $\tilde{s} : \mathbb{R}^n \times \mathbb{R}^{k_1}, (0,0) \to \mathbb{R}^n, 0$ and $\tilde{t} : \mathbb{R}^p \times \mathbb{R}^{k_1}, (0,0) \to \mathbb{R}^n, 0$ are unfoldings of id_n and id_p respectively such that

$$\Phi_1(x, \lambda) = \tilde{t}(\Phi_2(\tilde{s}(x, \lambda), \varphi(\lambda)), \lambda).$$

Definition 2.13 Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a smooth map germ, and $\Phi : \mathbb{R}^n \times \mathbb{R}^k, (0,0) \to \mathbb{R}^p, 0$ unfolding of f. Φ is called *$\mathcal{A}$-versal unfolding* if for any unfolding $\Psi : \mathbb{R}^n \times \mathbb{R}^\ell, (0,0) \to \mathbb{R}^p, 0$ of f, there exists an $\mathcal{A}$-morphism from Ψ to Φ.

Definition 2.14 Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a smooth map germ, and $\Phi : \mathbb{R}^n \times \mathbb{R}^k, (0,0) \to \mathbb{R}^p, 0$ unfolding of f. Φ is called *$\mathcal{A}$-infinitesimal versal unfolding* if

$$tf(\theta_n) + \omega f(\theta_p) + \sum_{i=1}^{k} \mathbb{R} \frac{\partial \Phi}{\partial \lambda_i}(x, 0) = \theta(f).$$

Theorem 2.15 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a smooth map germ, and $\Phi : \mathbb{R}^n \times \mathbb{R}^k, (0,0) \to \mathbb{R}^p, 0$ unfolding of f. Φ is $\mathcal{A}$-versal if and only if Φ is $\mathcal{A}$-infinitesimal versal.*

Theorem 2.16 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a smooth map germ. f is r-determined for $\mathcal{A}$-equivalence if and only if there exists an $\mathcal{A}$-versal unfolding of f.*

By the definition, the minimal number of parameters required for constructing an $\mathcal{A}$-versal unfolding of f is equal to dim $\theta(f)/T\mathcal{A}_e(f)$ where $T\mathcal{A}_e(f) = tf(\theta_n) + \omega f(\theta_p)$, that is often referred to as $\mathcal{A}_e$-codimension of f (extended $\mathcal{A}$-codimension). A versal unfolding with the minimal number of parameters is called an *$\mathcal{A}$-miniversal unfolding*.

Definition 2.17 Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a smooth map-germ, and $\Phi_i : \mathbb{R}^n \times \mathbb{R}^k, (0,0) \to \mathbb{R}^p, 0$ $(i = 1, 2)$ unfoldings of f. If there exists an $\mathcal{A}$-morphism from Φ_1 to Φ_2 such that the map-germs $\varphi : \mathbb{R}^k, 0 \to \mathbb{R}^k, 0$ between parameter spaces is a diffeomorphism, we say that Φ_1 is (*parametrically*) *$\mathcal{A}$-equivalent to* Φ_2, and write $\Phi_1 \sim_{\mathcal{A}} \Phi_2$.

Theorem 2.18 *Let* $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ *be a smooth map-germ, and* $F_1, F_2 : \mathbb{R}^n \times \mathbb{R}^k, (0,0) \to \mathbb{R}^p, 0$ *unfoldings of* f*. If* F_1 *and* F_2 *are* $\mathcal{A}$*-versal unfoldings of* f*, then* $F_1 \sim_{\mathcal{A}} F_2$*.*

2.5 $\mathcal{A}$-classification of plane-to-plane map-germs

All corank one singularities of plane-to-plane maps up to a certain codimension are classified as follows:

Theorem 2.19 (Rieger [54]) *All smooth map-germs* $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ *of corank 1 and* $\mathcal{A}$*-cod at most 6 are shown in Table 1.*

There are 29 types in Rieger's classification, which are indexed by

$$1, 2, 3, 4_k, \cdots, 11_{2k+1}, 12, 13, 15, \cdots, 19$$

with additional sign $\pm$. We use the same notation throughout this thesis. The type no.14: $(x, xy^2 + y^5)$ is not included in Table 1, since it has $\mathcal{A}$-codimension 7.

Remark 2.20 In fact Rieger also shows the classifications of all $\mathcal{A}$-simple smooth map-germs $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ of corank 1 in [54].

To obtain this classification, the procedure is roughly explained as follows: First, any corank one (non-regular) germ can be written by the form $(x, g(x,y))$ with $g \in m_2^2$. If (x, g) is not of type fold, then we may assume that the 2-jet of g is of the form xy or 0. If $g = xy + h.o.t$, then by a coordinate change of source we have k-jet $(x, xy + y^k)$, which is finitely determined. If $g \in m_2^3$, then the 3-jet of g is a combination of x^2y, xy^2, y^3 or 0. We continue this game step by step, and the process is built into a flowchart, called the *recognition tree* for $\mathcal{A}$-classification of plane-to-plane germs. At each step, we need to find some nice coordinate changes, but in almost all cases it is not easy to construct them in an explicit form. In such a case, instead, we use following well-established theorems, part of the heart of classification theory; then the existence of required coordinate changes can be certified. Notice that the resulting coordinate change might not be constructible.

Lemma 2.21 (Mather's Lemma) *Let* N *be a smooth manifold,* G *a Lie group,* $S \subset N$ *a connected submanifold. Then* S *is contained in a single orbit of* G *if and only if*

1. *for any* $x \in S$*,* $T_xS \subset T_x(G_x)$*;*
2. $dimT_x(G_x)$ *is independent of the choice of* $x \in S$*.*

Theorem 2.22 *Let* $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ *be a smooth map-germ, and suppose that*

$$m_n^\ell \theta(f) \subset tf(\theta_n) + f^* m_p(\theta_p) + m_n^{\ell+1}\theta(p) \quad \ell \geq 0$$

and

$$m_n^k \theta(f) \subset tf(\theta_n) + \omega f(\theta_p) + m_n^{k+\ell}\theta(p) \quad k \geq 0;$$

then f *is* $k + \ell$*-determined.*

$\mathcal{A}$ -cod	type	normal form
0	1 (regular)	(x, y)
1	2 (fold)	(x, y^2)
2	3 (cusp)	$(x, xy + y^3)$
3	4_2 (beaks and lips)	$(x, y^3 \pm x^2 y)$
	5(swallowtail)	$(x, xy + y^4)$
4	4_3 (goose)	$(x, y^3 + x^3 y)$
	6 (butterfly)	$(x, xy + y^5 + y^7)$
	11_5 (gulls)	$(x, xy^2 + y^4 + y^5)$
5	4_4 (ugly goose)	$(x, y^3 \pm x^4 y)$
	7 (elder butterfly)	$(x, xy + y^5)$
	11_7 (ugly gulls)	$(x, xy^2 + y^4 + y^7)$
	12	$(x, xy^2 + y^5 + y^6)$
	16	$(x, x^2 y + y^4 \pm y^5)$
	8 (unimodal)	$(x, xy + y^6 \pm y^8 + \alpha y^9)$
6	4_5	$(x, y^3 + x^5 y)$
	9	$(x, xy + y^6 + y^9)$
	$10^{\dagger}$ (bimodal)	$(x, xy + y^7 \pm y^9 + \alpha y^{10} + \beta y^{11})$
	11_9	$(x, xy^2 + y^4 + y^9)$
	13	$(x, xy^2 + y^5 \pm y^9)$
	15 (unimodal)	$(x, xy^2 + y^6 + y^7 + \alpha y^9)$
	17	$(x, x^2 y + y^4)$
	$18^{\dagger}$ (bimodal)	$(x, x^2 y + xy^3 + \alpha y^5 + y^6 + \beta y^7)$
	19 (unimodal)	$(x, x^3 y + \alpha x^2 y^2 + y^4 + x^3 y^2)$

Table 1: $\mathcal{A}$-classification up to $\mathcal{A}$-cod ≤ 6. $\dagger$: excluding exceptional values of the moduli

2.6 Topologically $\mathcal{A}$-classification

Two map-germs are *topologically $\mathcal{A}$-equivalent* if they commute via some homeomorphisms of source and target; that is the version where one just replaces diffeomorphisms for $\mathcal{A}$-equivalence by homeomorphisms. An unfolding F of a map-germ f is topologically trivial if it is topologically equivalent to the trivial unfolding. J. Damon gave a very useful theorem for finding topological triviality.

Theorem 2.23 (Damon [15]) *Let f be a finitely $\mathcal{A}$-determined map-germ having weighted homogeneous normal form in some coordinates. Let F be an unfolding of non-negative weight. Then F is topologically trivial.*

Apply this theorem to each in the list of Rieger's classification, then several different $\mathcal{A}$-types are combined into a single topological $\mathcal{A}$-type.

Theorem 2.24 (Rieger [55]) *Topologically $\mathcal{A}$-classification of plane-to-plane germs up to $\mathcal{A}$-codimension* 6 *is listed in Table 2 except for stable germs.*

topoligical type	$\mathcal{A}$ -type	normal form
$I_k^{\pm}$ $(k \geq 2)$	$4_k^{\pm}$	$(x, y^3 \pm x^k y)$
II_k $(4 \leq k \leq 6)$	$5-10$	$(x, xy + y^k)$
III_k $(k \geq 2)$	11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1})$
IV_5	$12, 13, (14)$	$(x, xy^2 + y^5)$
V_1	$16, 17$	$(x, x^2y + y^4)$

Table 2: Top. $\mathcal{A}$-classification with $\mathcal{A}$-codim ≤ 6 (stable germs and $15, 18, 19$ are omitted). IV_5 is denoted by IV in [55].

We provisionally call the weighted homogeneous part of each normal form in Table 2 *the specified jet* for the corresponding topological $\mathcal{A}$-type. It is actually the same as the normal form itself, except for the case 4_k and 11_{2k+1}; The specified jets for 4_k $(k \geq 3)$ and 11_{2k+1} $(k \geq 2)$ are (x, y^3) and $(x, xy^2 + y^4)$, respectively (specified jets for 4_k with $k = 1, 2$, i.e. cusp and lips/beaks, are the same as the normal forms). Note that the germs (x, y^3) and $(x, xy^2 + y^4)$ are not finitely $\mathcal{A}$-determined, thus we can not use Theorem 2.23; indeed 4_k and 11_{2k+1} for different k may have different topologically $\mathcal{A}$-types. However it is convenient to make all 4_k with $k \geq 3$ (resp. 11_k) into a group I_* (resp. III_*). Here we list up such *specified jets* of topological $\mathcal{A}$-types in Table 2 including $15(= IV_6), 18(= V_2), 19(= VI)$:

type	normal form
I_2	$(x, y^3 \pm x^2y)$
I_*	(x, y^3)
II_4	$(x, xy + y^4)$
II_5	$(x, xy + y^5)$
II_6	$(x, xy + y^6)$
III_*	$(x, xy^2 + y^4)$
IV_5	$(x, xy^2 + y^5)$
IV_6	$(x, xy^2 + y^6)$
V_1	$(x, x^2y + y^4)$
V_2	$(x, x^2y + xy^3)$ or (x, x^2y)
VI	$(x, y^4 + \alpha x^2y^2 + x^3y)$ or $(x, y^4 + \alpha x^2y^2)$

Table 3: Specified jets of topological $\mathcal{A}$-types (except for stable germs).

3 Criteria for map-germs

According to T. Gaffney [22],

"...recognition, which means finding criteria which will describe which germ on the list a given germ is equivalent to. Classification is usually easier than recognition".

As briefly reviewed in the previous chapter, J. H. Rieger solved the $\mathcal{A}$-classification problem of plane-to-plane germs of of corank 1 up to a certain codimenion. In [54] he gave an algorithm (called a 'recognition tree') for the recognition problem applied to a particular form of input $(x, g(x, y))$. This algorithm involves many steps, at each of which one has to find nice coordinate changes of source and target or use Mather's lemma etc, therefore the running process can be very technical. In particular, the process might not be constructible (i.e., required coordinate changes might not be explicitly given, although its existence is certified by Mather's lemma etc). That makes a major obstruction when we try to apply the classification result to some geometrical settings.

Our motivation is to find a "user-friendly" solution to the recognition problem. In this chapter, we present a complete set of criteria for detecting $\mathcal{A}$-types in Rieger's list. That is a useful package consisting of two-phased criteria (Table 4 and Table 5), which would easily be implemented in computer. The first one is about geometric conditions on 'specified jets' for *topological $\mathcal{A}$-types* in terms of *intrinsic derivatives* [71, 60, 61, 53, 35], and the second is about algebraic conditions on *Taylor coefficients* of germs with each specified jet, which are obtained by describing explicitly all the required coordinate changes of source and target of map-germs which are hidden in the classification process (Proposition 3.4, 3.6, 3.9, and 3.11).

This chapter is based on §3 in [31].

3.1 Main Theorem

Suppose that we are given a map-germ $f = (f_1, f_2) : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ with corank one, i.e., $\dim \ker df(0) = 1$. We always assume this condition below.

We define the *discriminant function* λ of $f = (f_1, f_2)$ by

$$\lambda(x, y) := \frac{\partial(f_1, f_2)}{\partial(x, y)}.$$

Denote by $H_\lambda(x, y)$ the Hesse matrix of λ at (x, y). We denote the singular point set of f around 0 by $S(f)$: it is defined by $\lambda = 0$. Note that $0 \in S(f)$ and for any point of $S(f)$ close to 0, it holds that $\dim \ker df = 1$.

We take an arbitrary vector field

$$\eta := \eta_1(x, y)\frac{\partial}{\partial x} + \eta_2(x, y)\frac{\partial}{\partial y}$$

on the source space so that the restriction $\eta|_{S(f)}$ to the singular point set spans $\ker df$. We call this vector field a *null vector field for* f. A differential operator η^k is defined by $\eta^k g := \eta(\eta^{k-1} g)$ for any function $g(x, y)$.

The followings are criteria for map-germs with $\mathcal{A}$-codimension ≤ 3. (The criteria for first two normal forms, the fold and the cusp, are implicitly due to H. Whitney [71], and explicitly described in [61]. Those for the latter two normal forms, the beaks/lips and the swallowtail, are by Saji [60]).

Theorem 3.1 (Whitney, Saji [71, 61, 60]) *$\mathcal{A}$-types of fold, cusp, lips, beaks and swallowtail are characterized by the following: for a corank one germ f,*

$$\begin{array}{lll}
f \sim_{\mathcal{A}} (x, y^2) & \Longleftrightarrow & \eta\lambda(0) \neq 0, \\
f \sim_{\mathcal{A}} (x, xy + y^3) & \Longleftrightarrow & d\lambda(0) \neq 0,\ \ \eta\lambda(0) = 0,\ \ \eta^2\lambda(0) \neq 0, \\
f \sim_{\mathcal{A}} (x, y^3 \pm x^2 y) & \Longleftrightarrow & d\lambda(0) = 0,\ \ \det H_\lambda(0) \neq 0,\ \ \eta^2\lambda(0) \neq 0, \\
f \sim_{\mathcal{A}} (x, xy + y^4) & \Longleftrightarrow & d\lambda(0) \neq 0,\ \ \eta\lambda(0) = \eta^2\lambda(0) = 0,\ \ \eta^3\lambda(0) \neq 0.
\end{array}$$

To detect $\mathcal{A}$-type of a given map-germ, we do not have to follow Rieger's recognition tree – Just we check the criteria in terms of λ and η at once. This is quite useful and it is easy to check the criteria by computer.

Our main aim is to extend this kind of characterization to other $\mathcal{A}$-types, as well topological $\mathcal{A}$-types, in Rieger's list up to $\mathcal{A}$-codimension 6 (Table 1 and Table 2).

The emphasis is that this is a *coordinate-free* expression of geometric characterization of $\mathcal{A}$-types of map-germs; the quantity $\eta^k\lambda$ is a variant of *intrinsic derivatives* in the sense of I. Porteous and H. Levine[35, 53]. Intrinsic derivatives were originally invented in order to introduce the Thom-Boardman singularity types, that was done around 1950-60's, and they took an important role in the fundamental studies on (C^∞ and C^0-) stable map-germs. Our results show that some variant of intrinsic derivatives is really useful also for characterizing $\mathcal{A}$-types of map-germs. Namely, our approach brings a flavor of classical differential geometry to the $\mathcal{A}$-classification theory of map-germs.

We state the main result of this thesis:

Theorem 3.2 *Specified jets of topologically $\mathcal{A}$-equivalent types of plane-to-plane germs in Table 2 are explicitly characterized by mean of geometric terms λ and η as in Table 4: Precisely saying, given a map-germ f of corank one, the jet $j^r f(0)$ is $\mathcal{A}^r$-equivalent to one of the specified r-jets listed in Table 4 if and only if the corresponding condition by λ and η for f in Table 4 is satisfied. A complete set of criteria for detecting $\mathcal{A}$-types in Table 1 is achieved by adding more finer invariants than λ and η, which are precisely described in Proposition 3.4, 3.6, 3.9, and 3.11 below (Table 5 is a brief summary of the criteria).*

Remark 3.3 For example, types $6, 7$ are topologically $\mathcal{A}$-equivalent, but not $\mathcal{A}$-equivalent. The condition in terms of λ and η in Table 4 detects whether a given germ is $\mathcal{A}$-equivalent to $(x, xy + y^5)$ in the level of 5-jets. Note that the germ $(x, xy + y^5)$ is 7-$\mathcal{A}$-determined. To distinguish between these two $\mathcal{A}$-types $6, 7$, we have to check a certain closed condition on 7-jets that is described in Proposition 3.4 (2) – this condition is written down in terms of coefficients of Taylor expansions, so it is not a *coordinate-free* expression. Remark that the geometric meaning of the condition in coefficients of Taylor expansions is not clear at the moment; in other words, coordinate-free description of the condition has not yet been found. On the other hand, the condition determines $\mathcal{A}$-codimension of corank one germs having the same specified jets, so it must distinguish some generators of the normal space $\theta(f)/T\mathcal{A}_e.f$. It would be reasonable to discuss the Taylor coefficients condition in the context of Damon's $\mathcal{K}_D$-theory where D is the A_μ-type discriminant of a stable unfolding of f. That will be considered somewhere else. In Chapter 4, we will discuss about an application of our criteria, then the key feature is just the difference between this additional condition of Taylor expansions and specified jet conditions in terms of λ and η.

3.2 Proof

The proof is divided into the following four cases:

(*case 0*) $d\lambda(0) \neq 0$;
(*case 1*) $d\lambda(0) = 0$ and $\mathrm{rk} H_\lambda(0) = 2$;
(*case 2*) $d\lambda(0) = 0$ and $\mathrm{rk} H_\lambda(0) = 1$;
(*case 3*) $d\lambda(0) = 0$ and $\mathrm{rk} H_\lambda(0) = 0$.

In fact, Table 4 is separated into these four cases by double lines. These cases deal with the same process in recognition trees Fig. 1–5 in [54]: Cases 0, 1, 3 correspond to Fig. 1, 3, 5, respectively, and case 2 corresponds to both Fig. 2 and 4 in [54].

For our simplicity, we omit the case of $\mathcal{A}$-cod ≤ 3, that is Theorem 3.1 [60]. In the following proof, we frequently use Rieger's results (e.g., $\mathcal{A}$-determinacy of germs), which should be referred to [54] (see also Table 5).

3.3 Case 0: $d\lambda \neq 0$ ($S(f)$ is smooth)

We deal with types $6 - 10$ of $\mathcal{A}$-cod $= 4, 5, 6$.

specified jet	$\mathcal{A}$-type	condition
$II_4 : (x, xy + y^4)$	5	$d\lambda(0) \neq 0$, $\eta\lambda(0) = \eta^2\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$
$II_5 : (x, xy + y^5)$	$6, 7$	$d\lambda(0) \neq 0$, $\eta\lambda(0) = \eta^2\lambda(0) = \eta^3\lambda(0) = 0$, $\eta^4\lambda(0) \neq 0$
$II_6 : (x, xy + y^6)$	$8, 9$	$d\lambda(0) \neq 0$, $\eta\lambda(0) = \cdots = \eta^4\lambda(0) = 0$ $\eta^5\lambda(0) \neq 0$
$II_7 : (x, xy + y^7)$	10	$d\lambda(0) \neq 0$, $\eta\lambda(0) = \cdots = \eta^5\lambda(0) = 0$, $\eta^6\lambda(0) \neq 0$
$I_2 : (x, y^3 \pm x^2y)$	$4_2^\pm$	$d\lambda(0) = 0$, $\det H_\lambda(0) \neq 0$, $\eta^2\lambda(0) \neq 0$
$III_* : (x, xy^2 + y^4)$	11_{odd}	$d\lambda(0) = 0, \det H_\lambda(0) < 0$, $\eta^2\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$
$IV_5 : (x, xy^2 + y^5)$	$12, 13$	$d\lambda(0) = 0, \det H_\lambda(0) < 0$, $\eta^2\lambda(0) = \eta^3\lambda(0) = 0$, $\eta^4\lambda(0) \neq 0$
$IV_6 : (x, xy^2 + y^6)$	15	$d\lambda(0) = 0, \det H_\lambda(0) < 0$, $\eta^2\lambda(0) = \eta^3\lambda(0) = \eta^4\lambda(0) = 0$, $\eta^5\lambda(0) \neq 0$
$I_* : (x, y^3)$	4_*	$d\lambda(0) = 0$, $\mathrm{rk}H_\lambda(0) = 1$, $\eta^2\lambda(0) \neq 0$
$V_1 : (x, x^2y + y^4)$	$16, 17$	$d\lambda(0) = 0, \mathrm{rk}H_\lambda(0) = 1$, $\eta^2\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$
$V_2 : (x, x^2y + xy^3)$, (x, x^2y)	18	$d\lambda(0) = 0$, $\mathrm{rk}H_\lambda(0) = 1$, $\eta^2\lambda(0) = \eta^3\lambda(0) = 0$
$VI : (x, y^4 + \alpha x^2y^2)$, $(x, y^4 + \alpha x^2y^2 + x^3y)$	19	$d\lambda(0) = 0$, $\mathrm{rk}H_\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$

Table 4: Geometric criteria for plane-to-plane germs with $\mathcal{A}$-codimension up to 6 (stable germs are omitted).

given germs	$\mathcal{A}$-type	r	condition
$II_5 : \left(x, xy + y^5 + \sum_{i+j\geq 6} a_{ij}x^iy^j\right)$	6	7	$a_{07} - \frac{5}{8}a_{06}^2 \neq 0$
	7	7	$a_{07} - \frac{5}{8}a_{06}^2 = 0$
$II_6 : \left(x, xy + y^6 + \sum_{i+j\geq 7} a_{ij}x^iy^j\right)$	8	8	$a_{08} - \frac{3}{5}a_{07}^2 \neq 0$
	9	9	$a_{08} - \frac{3}{5}a_{07}^2 = 0$, $a_{09} - \frac{7}{25}a_{07}^3 \neq 0$
$II_7 : \left(x, xy + y^7 + \sum_{i+j\geq 8} a_{ij}x^iy^j\right)$	$10^\dagger$	11	$a_{09} - \frac{7}{12}a_{08}^2 \neq 0$
$III_* : \left(x, xy^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j\right)$	11_5	5	$a_{05} \neq 0$
	11_7	7	$a_{05} = 0$, $a_{07} - 2a_{15} + 4a_{23} \neq 0$
$\left(x, xy^2 + y^4 + \sum_{i+j\geq 8} c_{ij}x^iy^j\right)$	11_9	9	$c_{09} - 2c_{17} \neq 0$
$IV_5 : \left(x, xy^2 + y^5 + \sum_{i+j\geq 6} a_{ij}x^iy^j\right)$	12	6	$a_{06} \neq 0$
	13	9	$a_{06} = 0$, $a_{09} - \frac{5}{2}a_{16} - \frac{5}{6}a_{07}^2 \neq 0$
$IV_6 : \left(x, xy^2 + y^6 + \sum_{i+j\geq 7} a_{ij}x^iy^j\right)$	15	9	$a_{07} \neq 0$
$I_* : \left(x, y^3 + \sum_{i+j\geq 4} a_{ij}x^iy^j\right)$	4_3	4	$a_{31} \neq 0$
	4_4	5	$a_{31} = 0, a_{41} - \frac{1}{3}a_{22}^2 \neq 0$
	4_5	6	$a_{31} = a_{41} - \frac{1}{3}a_{22}^2 = 0$, $a_{51} - \frac{2}{3}a_{32}a_{22} + \frac{1}{3}a_{13}a_{22}^2 \neq 0$
$V_1 : \left(x, x^2y + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j\right)$	16	5	$a_{05} \neq 0$
	17	5	$a_{05} = 0$
$V_2 : \left(x, x^2y + xy^3 + \sum_{i+j\geq 5} a_{ij}x^iy^j\right)$	$18^\dagger$	7	$a_{05} \neq \frac{3}{2}, \frac{9}{5}$, $a_{06}(5a_{05} - 9) - 15a_{14}a_{05} \neq 0$
$VI : \left(x, x^3y + \alpha x^2y^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j\right)$	19	5	$\Delta \neq 0$

Table 5: Complete criteria for plane-to-plane germs with $4 \leq \mathcal{A}$-cod≤ 6. Numbers in the third column mean determinacy-degrees of corresponding $\mathcal{A}$-types, i.e., each $\mathcal{A}$-type is r-$\mathcal{A}$-determined (see Rieger [54]). †: excluding exceptional values of the moduli

Proposition 3.4 *For a plane-to-plane map-germ f of corank one,*

(1) For $r \geq 5$,
$j^r f(0) \sim_{\mathcal{A}^r} (x, xy + y^r) \iff$

$$d\lambda(0) \neq 0, \ \ \eta^i \lambda(0) = 0 \ (1 \leq i \leq r-2), \ \ \eta^{r-1}\lambda(0) \neq 0.$$

(2) If we write $f = (x, xy + y^5 + \sum_{i+j \geq 6} a_{ij} x^i y^j)$,

$$\begin{aligned} a_{07} - \tfrac{5}{8} a_{06}^2 \neq 0 &\iff f \sim_{\mathcal{A}} (x, xy + y^5 \pm y^7) \cdots \boxed{6}, \\ a_{07} - \tfrac{5}{8} a_{06}^2 = 0 &\iff f \sim_{\mathcal{A}} (x, xy + y^5) \cdots \boxed{7}. \end{aligned}$$

(3) If we write $f = (x, xy + y^6 + \sum_{i+j \geq 7} a_{ij} x^i y^j)$,

$$\begin{aligned} a_{08} - \tfrac{3}{5} a_{07}^2 \neq 0 \qquad &\iff f \sim_{\mathcal{A}} (x, xy + y^6 \pm y^8 + \alpha y^9) \cdots \boxed{8}, \\ \begin{cases} a_{08} - \tfrac{3}{5} a_{07}^2 = 0 \\ a_{09} - \tfrac{7}{25} a_{07}^3 \neq 0 \end{cases} &\iff f \sim_{\mathcal{A}} (x, xy + y^6 + y^9) \cdots \boxed{9}. \end{aligned}$$

(4) If we write $f = (x, xy + y^7 + \sum_{i+j \geq 8} a_{ij} x^i y^j)$,

$$a_{09} - \tfrac{7}{12} a_{08}^2 \neq 0 \iff j^{11} f(0) \sim_{\mathcal{A}^{11}} (x, xy + y^7 \pm y^9 + \alpha y^{10} + \beta y^{11}),$$

and excluding exceptional values of α and β

$$f \sim_{\mathcal{A}} (x, xy + y^7 \pm y^9 + \alpha y^{10} + \beta y^{11}) \cdots \boxed{10}.$$

In order to prove 1 in Proposition 3.4, we need the next lemma based on Lemma 2.6 in [60]. Notice that λ is changed by multiplying a non-zero function when we take another coordinates, and also that there is an ambiguity to choose the null vector field η.

Lemma 3.5 *The conditions on the right hand side of 1 in Proposition 3.4 are independent from the choice of coordinates of the source and target and the choice of η.*

Proof : Let λ and η be the discriminant function and an arbitrary null vector field in coordinates x, y of source. Let $(\tilde{x}, \tilde{y}) = g(x, y)$ be a coordinate change. Then, the discriminant function $\tilde{\lambda}$ for $f \circ g^{-1}$ is written by

$$\tilde{\lambda}(\tilde{x}, \tilde{y}) = L(\tilde{x}, \tilde{y}) \cdot \lambda \circ g^{-1}(\tilde{x}, \tilde{y})$$

for some non-zero function L, and any null vector field for $f \circ g^{-1}$ is written by

$$\tilde{\eta} = a_1 \xi_1 + a_2 \xi_2$$

where $\xi_2 := dg(\eta)$ and ξ_1 is chosen so that ξ_1, ξ_2 form a basis at each point. Notice that $a_1 = 0$ on the locus $S(f \circ g^{-1})$ defined by $\lambda \circ g^{-1} = 0$ and a_2 is a non-zero function. From our assumption $d\lambda(0) \neq 0$, it follows that a_1 is divisible by $\lambda \circ g^{-1}$. Hence

$$\tilde{\eta}\tilde{\lambda} = (a_1\xi_1 + a_2\xi_2)(L \cdot \lambda \circ g^{-1}) = \alpha_0 \cdot \lambda \circ g^{-1} + \alpha_1 \cdot \eta(\lambda) \circ g^{-1}$$

for some function α_0 and $\alpha_1 = a_2 \cdot L$; Inductively, we see that

$$\tilde{\eta}^k(\tilde{\lambda}) = \alpha_{0,k} \cdot \lambda \circ g^{-1} + \sum_{i=1}^{k} \alpha_{i,k} \cdot \eta^i(\lambda) \circ g^{-1}$$

for some functions $\alpha_{i,k}$ $(0 \leq i \leq k-1)$ and $\alpha_{k,k} = a_2^k \cdot L$. Thus $\eta^i\lambda(0) = 0$ $(1 \leq i \leq k)$ if and only if $\tilde{\eta}^i\tilde{\lambda}(0) = 0$ $(1 \leq i \leq k)$. The claim immediately follows. $\square$

Proof of 1 in Proposition 3.4 : It is easily checked that for the r-jet $(x, xy+y^r)$, the condition in the right hand side holds. Thus the "only if" part of 1 follows from Lemma 3.5.

The "If" part is shown by finding a suitable coordinate change. Assume that the condition on the right hand side of 1 holds for f. Since f is of corank 1 at 0, we may write

$$f(x,y) = (x, \textstyle\sum_{i+j\geq 2} a_{ij}x^iy^j).$$

Take $\eta = \frac{\partial}{\partial y}$ and

$$\lambda(x,y) = \textstyle\sum_{i+j\geq 2} j \cdot a_{ij}x^iy^{j-1}.$$

For this choice of coordinates and η, by Lemma 3.5, we have

$$d\lambda(0) \neq 0, \;\; \eta\lambda(0) = \eta^2\lambda(0) = \cdots = \eta^{r-2}\lambda(0) = 0, \;\; \eta^{r-1}\lambda(0) \neq 0.$$

Then $a_{11} \neq 0$, $a_{02} = a_{03} = \cdots = a_{0\,r-1} = 0$, $a_{0r} \neq 0$. By some coordinate change, we have $j^rf(0) = (x, xy + y^r)$. $\square$

The following proof of the claim 2 uses a simple trick for eliminating a certain term in the normal form. This trick will implicitly appear several times in other cases.

Proof of 2 in Proposition 3.4 : It is easy to see that $f = (x, xy + y^5 + \sum_{i+j\geq 6} a_{ij}x^iy^j)$ is equivalent to

$$(x, xy + y^5 + a_{06}y^6 + a_{07}y^7 + O(8))$$

by the change of target $(X,Y) \mapsto (X, Y - a_{60}X^6 - \cdots)$ and the change of source

$$\bar{x} = x, \quad \bar{y} = y(1 + a_{51}x^4 + \cdots + a_{15}y^4 + a_{61}x^5 + \cdots + a_{16}y^5).$$

Since both 6-type and 7-type are 7-determined, our task is to eliminate the term y^6 in the second component, that is, we show

$$(x, xy + y^5 + cy^6 + dy^7) \sim_{\mathcal{A}^7} (x, xy + y^5 + (d - \tfrac{5}{8}c^2)y^7).$$

Write $xy+y^5+cy^6+dy^7 = xy+y^5(1+\alpha y)+\beta y^6+dy^7$ with $\alpha+\beta=c$. By the coordinate change so that

$$\bar{x}=x, \quad \bar{y}^5 = y^5(1+\alpha y),$$

the 7-jet has the form

$$(\bar{x}, \bar{x}(\bar{y}-\tfrac{1}{5}\alpha\bar{y}^2+\tfrac{4}{25}\alpha^2\bar{y}^3+h.o.t.)+\bar{y}^5+\beta(\bar{y}-\tfrac{1}{5}\alpha\bar{y}^2+h.o.t.)^6+d\bar{y}^7).$$

By the coordinate change

$$\begin{aligned} x &= \bar{x}(1-\tfrac{1}{5}\alpha\bar{y}+\tfrac{4}{25}\alpha^2\bar{y}^2+h.o.t.) \\ y &= \bar{y} \end{aligned}$$

the jet is written by

$$(x(1+\tfrac{1}{5}\alpha y-\tfrac{3}{25}\alpha^2y^2+h.o.t.),\ xy+y^5+\beta y^6+(d-\tfrac{6}{5}\alpha\beta)y^7).$$

Then, by the coordinate change of target

$$(X,Y)\to(X-\tfrac{1}{5}\alpha Y, Y),$$

we eliminate the term $\frac{1}{5}\alpha xy$ in the first component; Hence the jet becomes

$$\begin{pmatrix} x(1-\frac{3}{25}\alpha^2y^2+h.o.t.)-\frac{1}{5}\alpha(y^5+\beta y^6+(d-\frac{6}{5}\alpha\beta)y^7) \\ xy+y^5+\beta y^6+(d-\frac{6}{5}\alpha\beta)y^7 \end{pmatrix}.$$

Take $\tilde{x}$ to be the first component and $\tilde{y}=y$, then the jet is written by (after rewriting variables)

$$(x, xy(1+\tfrac{3}{25}\alpha^2y^2+h.o.t.)+y^5+(\tfrac{1}{5}\alpha+\beta)y^6+(d-\alpha\beta)y^7).$$

Now we choose $\alpha=\frac{5}{4}c$ and $\beta=-\frac{1}{4}c$ to kill the term y^6 in the second component. Finally by $\tilde{x}=x$ and $\tilde{y}=y(1+\frac{3}{25}\alpha^2y^2+h.o.t.)$, we obtain the form

$$(x, xy+y^5+(d-\tfrac{5}{8}c^2)y^7).$$

This completes the proof. $\square$

Proof of 3 in Proposition 3.4 : The proof is similar to that of the claim 2 just described above: First we eliminate the terms including x of order ≥ 7, and then we directly show that

$$(x, xy+y^6+cy^7+dy^8+ey^9)\sim_{\mathcal{A}^9}(x, xy+y^6+(d-\tfrac{3}{5}c^2)y^8+(e-\tfrac{7}{5}cd+\tfrac{14}{25}c^3)y^9).$$

In fact, rewriting variables as $\tilde{x},\tilde{y}$ of the germ in the left hand side, substitute

$$\begin{aligned} \tilde{x} &= x+\tfrac{c}{5}xy+\tfrac{c}{5}y^6-\tfrac{3c^3}{25}y^8+\tfrac{cd}{5}y^8+\tfrac{14c^4}{125}y^9-\tfrac{7c^2d}{25}y^9+\tfrac{ce}{5}y^9, \\ \tilde{y} &= y-\tfrac{c}{5}y^2+\tfrac{c^2}{25}y^3-\tfrac{c^3}{125}y^4+\tfrac{c^4}{625}y^5-\tfrac{c^5}{3125}y^6 \\ &\quad +\tfrac{c^6}{15625}y^7-\tfrac{c^7}{78125}y^8+\tfrac{2c^8}{390625}y^9, \end{aligned}$$

and take the coordinate change of the target

$$(X,Y) \mapsto \left(X - \tfrac{c}{5}Y, Y\right),$$

then we get the equivalence. Here $c = a_{07}$, $d = a_{08}$, $e = a_{09}$, and both 8-type and 9-type are 9-determined, thus we have the claim 3. $\Box$

Proof of 4 in Proposition 3.4 : Also in a similar way as above we see

$$(x, xy + y^7 + cy^8 + dy^9) \sim_{\mathcal{A}^9} (x, xy + y^7 + (d - \tfrac{7}{12}c^2)y^9).$$

In fact, it is achieved by

$$\begin{aligned}
\tilde{x} &= x + \tfrac{c}{6}xy + \tfrac{c}{6}y^7 - \tfrac{7c^3}{72}y^9 + \tfrac{cd}{6}y^9, \\
\tilde{y} &= y - \tfrac{c}{6}y^2 + \tfrac{c^2}{36}y^3 - \tfrac{c^3}{216}y^4 + \tfrac{c^4}{1296}y^5 - \tfrac{c^5}{7776}y^6 + \tfrac{c^6}{46656}y^7 \\
&\quad - \tfrac{c^7}{279936}y^8 - \tfrac{5c^8}{93312}y^9
\end{aligned}$$

and

$$(X,Y) \mapsto (X - \tfrac{c}{6}Y, Y).$$

In addtion, we easily get

$$(x, xy + y^7 \pm y^9 + O(10)) \sim_{\mathcal{A}} (x, xy + y^7 \pm y^9 + \alpha y^{10} + \beta y^{11} + O(12))$$

for some α, $\beta \in \mathbb{R}$. In Rieger [54], it is shown that the 10-type is 11-determined for generic α and β excluding some values explicitly given in [54, p.359]. This implies the claim 4. $\Box$

3.4 Case 1: $d\lambda(0) = 0$, $\mathrm{rk} H_\lambda(0) = 2$

We deal with types $11_{2k+1}\,(k = 2, 3, 4)$, $12, 13,\ 15$ of $\mathcal{A}$-cod $= 4, 5, 6$.

Proposition 3.6 *For a plane-to-plane map-germ f of corank one,*

(1) For $r \geq 4$,
$j^r f(0) \sim_{\mathcal{A}^r} (x, xy^2 + y^r) \iff$

$$d\lambda(0) = 0, \ \det H_\lambda(0) < 0, \ \eta^i\lambda(0) = 0 \ (2 \leq i \leq r-2), \ \eta^{r-1}\lambda(0) \neq 0.$$

(2) If we write $f = (x, xy^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$,

$$\begin{array}{ccl}
a_{05} \neq 0 & \iff & f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^5) \cdots \boxed{11_5}, \\
\left\{\begin{array}{l} a_{05} = 0 \\ a_{07} - 2a_{15} + 4a_{23} \neq 0 \end{array}\right. & \iff & f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^7) \cdots \boxed{11_7}, \\
a_{05} = a_{07} - 2a_{15} + 4a_{23} = 0 & \iff & j^7 f(0) \sim_{\mathcal{A}^7} (x, xy^2 + y^4).
\end{array}$$

Furthermore, if we write $f = (x, xy^2 + y^4 + \sum_{i+j\geq 8} c_{ij}x^iy^j)$,

$$c_{09} - 2c_{17} \neq 0 \iff f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^9) \cdots \boxed{11_9}.$$

(3) If we write $f = (x, xy^2 + y^5 + \sum_{i+j\geq 6} a_{ij}x^iy^j)$,

$$\begin{array}{lcl} a_{06} \neq 0 & \Longleftrightarrow & f \sim_{\mathcal{A}} (x, xy^2 + y^5 + y^6) \cdots \boxed{12}, \\ \left\{ \begin{array}{l} a_{06} = 0 \\ a_{09} - \frac{5}{2}a_{16} - \frac{5}{6}a_{07}^2 \neq 0 \end{array} \right. & \Longleftrightarrow & f \sim_{\mathcal{A}} (x, xy^2 + y^5 + y^9) \cdots \boxed{13}. \end{array}$$

(4) If we write $f = (x, xy^2 + y^6 + \sum_{i+j\geq 6} a_{ij}x^iy^j)$,

$$a_{07} \neq 0 \quad \Longleftrightarrow \quad f \sim_{\mathcal{A}} (x, xy^2 + y^6 + y^7 + \alpha y^9) \cdots \boxed{15}.$$

Note that in claim 1 of Proposition 3.6, $d\lambda(0) = 0$ implies $\eta\lambda(0) = 0$. As seen in the proof for 1 in Proposition 3.4 we need the following lemma for the proof of Proposition 3.6.

Lemma 3.7 *The conditions on the right hand side of 1 in Proposition 3.6 are independent from the choice of coordinates of the source and target and the choice of* η.

Proof : This is shown in entirely the same way as the proof of Lemma 3.5. Remark that if $a_1 = 0$ on $S(f \circ g^{-1}) = \{\lambda \circ g^{-1} = 0\}$, then a_1 is divisible by $\lambda \circ g^{-1}$. In fact, from the assumption that $d\lambda(0) = 0$ and $\det H_\lambda(0) < 0$, $S(f \circ g^{-1})$ has a node at the orgin, hence a_1 is divisible by the defining function of each smooth irreducible component. □

Proof of 1 in Proposition 3.6. The proof is similar to that of Proposition 3.4. "Only if" part is easy with the above lemma. "If " part is shown as follows.

Let f be of corank 1 at 0 and $d\lambda(0) = 0$, then we may put

$$f(x, y) = (x, a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \sum_{i+j\geq 4} a_{ij}x^iy^j).$$

Here we take $\eta = \frac{\partial}{\partial y}$.

Assume that the condition on the right hand side of 1 holds for f. Then $\eta^2\lambda(0) = 0$ implies $a_{03} = 0$, and $\det H_\lambda(0) < 0$ implies $a_{12} \neq 0$. By the coordinate change of the source

$$\tilde{x} = x, \quad \tilde{y}^2 + a\tilde{x}^2 = a_{12}y^2 + a_{21}xy$$

for a constant a, and a suitable change of the target, the germ is equivalent to

$$(x, xy^2 + \sum_{i+j\geq 4} b_{ij}x^iy^j)$$

for some b_{ij}. Moreover, writing the variables as $\tilde{x}, \tilde{y}$, substitute

$$\begin{aligned} &\tilde{x} = x \\ &\tilde{y} = y - \tfrac{1}{2}(b_{31}x^2 + b_{22}xy + b_{13}y^2), \end{aligned}$$

and apply a suitable coordinate change of the target, then we have

$$(x, xy^2 + b_{04}y^4 + O(5)).$$

Suppose $r = 4$ i.e. $\eta^3\lambda(0) \neq 0$, then we can use this condition for the above germ, which is allowed by Lemma 3.7; it leads to $b_{04} \neq 0$. Hence we have $j^4 f(0) \sim_{\mathcal{A}^4} (x, xy^2 + y^4)$. For the case $r > 4$, $j^4 f(0) \sim_{\mathcal{A}^4} (x, xy^2)$ holds, and repeat the above process to get $j^r f(0) \sim_{\mathcal{A}^r} (x, xy^2 + y^r)$. □

Proof of 2 in Proposition 3.6 : Let

$$f = (x, xy^2 + y^4 + \textstyle\sum_{i+j\geq 5} a_{ij}x^iy^j).$$

We may assume that $a_{i0} = 0$ for all i by coordinate changes of target. At first, writing the variables as $\tilde{x}, \tilde{y}$, by the coordinate change of the source

$$\begin{aligned}\tilde{x} &= x\\ \tilde{y} &= y - \tfrac{1}{2}(a_{41}x^3 + a_{32}x^2y + a_{23}xy^2 + a_{14}y^3),\end{aligned}$$

and a linear change of the target, f is equivalent to

$$(x, xy^2 + y^4 + a_{05}y^5 + \textstyle\sum_{i+j\geq 6} a'_{ij}x^iy^j)$$

for some a'_{ij} ($a'_{06} = a_{06} - 2a_{14}$ etc). Since $(x, xy^2 + y^4 + y^5)$ is 5-determined, $a_{05} \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^5).$$

Next we suppose $a_{05} = 0$. A similar coordinate change as above shows that f is equivalent to

$$(x, xy^2 + y^4 + (a_{06} - 2a_{14})y^6 + (a_{07} - 2a_{15} + 4a_{23})y^7 + \textstyle\sum_{i+j\geq 8} b_{ij}x^iy^j)$$

for some b_{ij}. Since the germ $(x, xy^2 + y^4 + y^7)$ is 7-determined, we want to eliminate the term y^6 from the second component of the right-hand side. By a similar argument as in the proof of 2 in Proposition 3.4, we have

$$(x, xy^2 + y^4 + cy^6 + dy^7) \sim_{\mathcal{A}^7} (x, xy^2 + y^4 + dy^7).$$

In fact this is achieved by an explicit coordinate change of source (writing variables as $\tilde{x}, \tilde{y}$ of the germ in the left hand side)

$$\begin{aligned}\tilde{x} &= x + cxy^2 + cy^4 + cdy^7\\ \tilde{y} &= y - \tfrac{c}{2}y^3 + \tfrac{3c^2}{8}y^5 - \tfrac{9c^3}{16}y^7\end{aligned}$$

and $(X, Y) \mapsto (X - cY, Y)$ of target. With this coordinate change and some adding coordinate change of source, f is equivalent to

$$(x, xy^2 + y^4 + (a_{07} - 2a_{15} + 4a_{23})y^7 + \textstyle\sum_{i+j\geq 8} b'_{ij}x^iy^j).$$

for some b'_{ij}. Hence $a_{07} - 2a_{15} + 4a_{23} \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^7).$$

Finally suppose $a_{07} - 2a_{15} + 4a_{23} = 0$; Put $f = (x, xy^2 + y^4 + \sum_{i+j\geq 8} c_{ij}x^iy^j)$ for some c_{ij}. We see

$$(x, xy^2 + y^4 + cy^8 + dy^9) \sim_{\mathcal{A}^9} (x, xy^2 + y^4 + dy^9)$$

by the change of source (writing variables as $\tilde{x}, \tilde{y}$ of the germ in the left hand side)

$$\begin{aligned}
\tilde{x} &= x - \tfrac{c}{2}x^2y^2 - \tfrac{c}{2}xy^4 + \tfrac{c^2}{4}x^3y^4 + \tfrac{c^2}{2}x^2y^6 + \tfrac{c^2}{4}xy^8 \\
\tilde{y} &= y + \tfrac{c}{4}xy^3 - \tfrac{c}{4}y^5 - \tfrac{c^2}{32}x^2y^5 - \tfrac{c^2}{16}xy^7 - \tfrac{5c^2}{8}y^9
\end{aligned}$$

and the change of target

$$(X, Y) \mapsto (X + \tfrac{c}{2}XY, Y).$$

Then it turns out that f is $\mathcal{A}$-equivalent to

$$(x, xy^2 + y^4 + (c_{09} - 2c_{17})y^9 + O(10)).$$

Since $(x, xy^2 + y^4 + y^9)$ is 9-determined, $c_{09} - 2c_{17} \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^9).$$

This completes the proof. $\square$

Remark 3.8 The terms c_{09} and c_{17} in the above proof are expressed in terms of the original coefficients a_{ij} of $(x, xy^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$:

$$\begin{aligned}
c_{17} &= a_{17} - 2a_{25} - 3a_{06}a_{23} - \tfrac{5}{2}a_{15}a_{14} + 4a_{33} + 9a_{14}a_{23} - 8a_{41}, \\
c_{09} &= a_{09} - 3a_{06}a_{15} + 6a_{06}a_{23} - a_{15}a_{14} + 2a_{14}a_{23}.
\end{aligned}$$

Hence $c_{09} - 2c_{17} = a_{09} - 2a_{17} + 4a_{25} - 8a_{33} - 3a_{06}a_{15} + 12a_{06}a_{23} + 4a_{15}a_{14} + 16a_{41} - 16a_{14}a_{23}$.

Proof of 3 and 4 in Proposition 3.6 : The proof is similar to that of 2 in Proposition 3.6. We directly show that

$$(x, xy^2 + y^5 + by^7 + cy^8 + dy^9) \sim_{\mathcal{A}^9} (x, xy^2 + y^5 + (d - \tfrac{5}{6}b^2)y^9).$$

by the coordinate change of the source:

$$\begin{aligned}
\tilde{x} = \quad & x + \tfrac{2b}{3}xy^2 + \tfrac{8bc}{45}x^2y^2 + \tfrac{32bc^2}{675}x^3y^2 + \tfrac{128bc^3}{10125}x^4y^2 + \tfrac{512bc^4}{151875}x^5y^2 \\
& + \tfrac{2048bc^5}{2278125}x^6y^2 + \tfrac{8192bc^6}{34171875}x^7y^2 + \tfrac{2b}{3}y^5 + \tfrac{8bc}{45}xy^5 + \tfrac{32bc^2}{675}x^2y^5 \\
& + \tfrac{128bc^3}{10125}x^3y^5 + \tfrac{512bc^4}{151875}x^4y^5 + \left(-\tfrac{5b^3}{9} + \tfrac{2bd}{3}\right)y^9, \\
\tilde{y} = \quad & y + \tfrac{2c}{15}xy + \tfrac{2c^2}{75}x^2y + \tfrac{4c^3}{675}x^3y + \tfrac{14c^4}{10125}x^4y + \tfrac{28c^5}{84375}x^5y \\
& + \tfrac{308c^6}{3796875}x^6y + \tfrac{152c^7}{11390625}x^7y - \tfrac{b}{3}y^3 - \tfrac{2bc}{15}xy^3 - \tfrac{2bc^2}{45}x^2y^3 \\
& - \tfrac{28bc^3}{2025}x^3y^3 - \tfrac{14bc^4}{3375}x^4y^3 - \tfrac{308bc^5}{253125}x^5y^3 - \tfrac{4004bc^6}{11390625}x^6y^3 \\
& - \tfrac{c}{5}y^4 - \tfrac{7c^2}{75}xy^4 - \tfrac{118c^3}{3375}x^2y^4 - \tfrac{121c^4}{10125}x^3y^4 - \tfrac{74c^5}{50625}x^4y^4 + \tfrac{b^2}{6}y^5 \\
& + \tfrac{b^2c}{9}xy^5 + \tfrac{b^2c^2}{135}x^2y^5 + \tfrac{14b^2c^3}{675}x^3y^5 + \tfrac{77b^2c^4}{10125}x^4y^5 + \tfrac{bc}{5}y^6 \\
& + \tfrac{11bc^2}{75}xy^6 + \tfrac{2bc^3}{27}x^2y^6 + \tfrac{107bc^4}{3375}x^3y^6 + \left(-\tfrac{5b^3}{54} - \tfrac{4c^2}{75}\right)y^7 \\
& + \left(-\tfrac{7b^3c}{81} + \tfrac{4c^3}{225}\right)xy^7 - \tfrac{7b^3c^2}{135}x^2y^7 + \tfrac{b^2c}{2}y^8 + \tfrac{49b^2c^2}{90}xy^8 \\
& + \left(-\tfrac{97b^4}{162} - \tfrac{26bc^2}{75}\right)y^9
\end{aligned}$$

and the coordinate change of the target:

$$(X,Y) \mapsto (X - \tfrac{2b}{3}Y,\ Y - \tfrac{4c}{15}XY + \tfrac{8bc}{45}Y^2).$$

On the other hand

$$(x, xy^2 + y^6 + y^7 + O(8)) \sim_{\mathcal{A}} (x, xy^2 + y^6 + y^7 + \alpha y^9 + O(10))$$

for $\alpha \in \mathbb{R}$, is shown by Rieger in the proof of Lemma 3.2.1:3 in [54]. This completes the proof. $\square$

3.5 Case 2: $d\lambda(0) = 0$, $\mathrm{rk}H_\lambda(0) = 1$

We deal with types 4_k $(k = 3, 4, 5), 16 - 18$ of $\mathcal{A}$-cod $= 4, 5, 6$.

Proposition 3.9 *For a plane-to-plane map-germ f of corank one,*

(1) $j^3f(0) \sim_{\mathcal{A}^3} (x, y^3) \iff$

$$d\lambda(0) = 0,\ \ \mathrm{rk}H_\lambda(0) = 1,\ \ \eta^2\lambda(0) \neq 0.$$

(2) If we write $f = (x, y^3 + \sum_{i+j\geq 4} a_{ij}x^iy^j)$,

$$\begin{aligned}
a_{31} \neq 0 \quad & \iff \quad f \sim_{\mathcal{A}} (x, y^3 + x^3y) \cdots \boxed{4_3}, \\
\begin{cases} a_{31} = 0 \\ a_{41} - \frac{1}{3}a_{22}^2 \neq 0 \end{cases} & \iff \quad f \sim_{\mathcal{A}} (x, y^3 \pm x^4y) \cdots \boxed{4_4}, \\
\begin{cases} a_{31} = a_{41} - \frac{1}{3}a_{22}^2 = 0 \\ a_{51} - \frac{2}{3}a_{32}a_{22} + \frac{1}{3}a_{13}a_{22}^2 \neq 0 \end{cases} & \iff \quad f \sim_{\mathcal{A}} (x, y^3 + x^5y) \cdots \boxed{4_5}.
\end{aligned}$$

(3) $j^4f(0) \sim_{\mathcal{A}^4} (x, x^2y + y^4) \iff$

$$d\lambda(0) = 0,\ \ \mathrm{rk}H_\lambda(0) = 1,\ \ \eta^2\lambda(0) = 0,\ \ \eta^3\lambda(0) \neq 0.$$

(4) If we write $f = (x, x^2y + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$*,*

$$a_{05} \neq 0 \iff f \sim_{\mathcal{A}} (x, x^2y + y^4 \pm y^5) \cdots \boxed{16},$$
$$a_{05} = 0 \iff f \sim_{\mathcal{A}} (x, x^2y + y^4) \quad \cdots \boxed{17}.$$

(5) $j^4f(0) \sim_{\mathcal{A}^4} (x, x^2y + xy^3)$ *or* $(x, x^2y) \iff$

$$d\lambda(0) = 0, \ \ \mathrm{rk}H_\lambda(0) = 1, \ \ \eta^2\lambda(0) = \eta^3\lambda(0) = 0.$$

(6) If we write $f = (x, x^2y + xy^3 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$*,*

$$\begin{cases} a_{05} \neq \frac{3}{2}, \frac{9}{5} \\ a_{06}(5a_{05} - 9) - 15a_{14}a_{05} \neq 0 \end{cases}$$

$$\iff j^7f(0) \sim_{\mathcal{A}^7} (x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7),$$

and excluding exceptional values of α *and* β*,*

$$f \sim_{\mathcal{A}} (x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7) \quad \cdots \boxed{18}.$$

Note that we exclude the type (x, x^2y), because it has codimension 7, while the type 18 has codimension 6.

Lemma 3.10 *The conditions on the right hand side of 1, 3 and 5 in Proposition 3.9 are independent from the choice of coordinates of the source and target and the choice of* η*.*

Proof : We use the same notation as in the proof of Lemma 3.5. It is easy to see that $\mathrm{rk}H_\lambda(0) = 1$ if and only if $\mathrm{rk}H_{\tilde{\lambda}}(0) = 1$. For other conditions, unlike the previous cases, a_1 may not be divided by $\lambda \circ g^{-1}$, however it holds that $d(\lambda \circ g^{-1})(0) = 0$ and $\xi_1\xi_2(\lambda \circ g^{-1})(0) = \xi_2\xi_1(\lambda \circ g^{-1})(0) = 0$. By using these equations, we can easily show the rest of claims about $\eta^i\lambda(0)$ and $\tilde{\eta}^i\tilde{\lambda}(0)$. □

Proof of 1, 3 and 5 in Proposition 3.9. We prove the "if " part (the converse is easy). Let f be of corank 1 at 0 and $d\lambda(0) = 0$, then we may put

$$f(x, y) = (x, a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \sum_{i+j\geq 4} a_{ij}x^iy^j).$$

Let $\eta = \frac{\partial}{\partial y}$. Now assume that the condition on the right hand side of the claim 1 holds. Then $\eta^2\lambda(0) \neq 0$ leads to $a_{03} \neq 0$. Writing variables as $\tilde{x}, \tilde{y}$, take the change of source

$$\tilde{x} = x, \quad \tilde{y} = y - \tfrac{a_{12}}{3a_{03}}x$$

and some change of target, then f is written by

$$(x, bx^2y + y^3 + O(4))$$

for a constant b. Here we use the remaining condition $\mathrm{rk}H_\lambda(0) = 1$ for the obtained germ, that is allowed by Lemma 3.10; it leads to $b = 0$. Hence

$$j^3 f(0) \sim_{\mathcal{A}^3} (x, y^3).$$

The claim 1 is proved.

Next we show the claim 3 and 5. Take $f(x,y)$ as above. We may put $a_{40} = 0$ by a suitable target change. By the condition $\mathrm{rk}H_\lambda(0) = 1$ and $\eta^2\lambda(0) = 0$, we have $a_{12} = a_{03} = 0$ and $a_{21} \neq 0$, therefore we may assume

$$f(x,y) = (x, x^2 y + \textstyle\sum_{i+j\geq 4} a_{ij} x^i y^j)$$

with replacing a_{ij} by a_{ij}/a_{21}. Suppose that $\eta^3\lambda(0) \neq 0$, then $a_{04} \neq 0$. Rewriting variables as $\tilde{x}, \tilde{y}$, by the change of source coordinates

$$\tilde{x} = x, \quad \tilde{y} = y - \tfrac{a_{13}}{4a_{04}} x,$$

we eliminate the term xy^3; then, with a coordinate change of the target, f is equivalent to

$$(x, x^2 y + a'_{31} x^3 y + a'_{22} x^2 y^2 + a_{04} y^4 + O(5)).$$

for some a'_{ij}. To eliminate other terms, rewriting variables again, substitute

$$\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= y - (a'_{31} xy + a'_{22} y^2); \end{aligned}$$

Then we have

$$j^4 f(0) \sim_{\mathcal{A}^4} (x, x^2 y + y^4).$$

Thus the claim 3 is proved. If $\eta^3\lambda(0) = 0$, then $a_{04} = 0$. With the same coordinate change as above we can eliminate the terms x^2y^2, x^3y from f. Hence, we have

$$j^4 f(0) \sim_{\mathcal{A}^4} (x, x^2 y) \ \text{ or } \ (x, x^2 y + xy^3),$$

according to whether $a_{13} = 0$ or not. The claim 5 is proved. This completes the proof. $\square$

Proof of 2 in Proposition 3.9. Let

$$f(x,y) = (x, y^3 + \textstyle\sum_{i+j\geq 4} a_{ij} x^i y^j).$$

Rewriting variables as $\tilde{x}, \tilde{y}$, substitute

$$\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= y - \tfrac{1}{3}(a_{22} x^2 + a_{13} xy + a_{04} y^2) \end{aligned}$$

with a target change, then f is equivalent to

$$(x, y^3 + a_{31} x^3 y + \textstyle\sum_{i+j\geq 5} {a'}_{ij} x^i y^j)$$

for some a'_{ij} where $a'{}_{41} = a_{41} - \frac{1}{3}a_{22}^2 - \frac{1}{3}a_{13}a_{31}$. Since 4_3-type is 4-determined, $a_{31} \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, y^3 + x^3 y).$$

Suppose $a_{31} = 0$. In entirely the same way as above, f is equivalent to

$$(x, y^3 + (a_{41} - \tfrac{1}{3}a_{22}^2)x^4 y + \textstyle\sum_{i+j\geq 6} b_{ij}x^i y^j)$$

for some b_{ij}. Since 4_4-type is 5-determined, $a_{41} - \frac{1}{3}a_{22}^2 \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, y^3 \pm x^4 y).$$

If $a_{41} - \frac{1}{3}a_{22}^2 = 0$, then f is equivalent to

$$(x, y^3 + (a_{51} - \tfrac{2}{3}a_{32}a_{22} + \tfrac{1}{3}a_{13}a_{22}^2)x^5 y + O(7)).$$

Since 4_5-type is 6-determined, the claim 2 follows. □

Proof of 4 in Proposition 3.9. Let

$$f(x, y) = (x, x^2 y + y^4 + \textstyle\sum_{i+j\geq 5} a_{ij}x^i y^j).$$

Rewrite variables, and substitute

$$\begin{aligned}\tilde{x} &= x\\ \tilde{y} &= y - \textstyle\sum_{i+j=5,\ i\geq 2} a_{ij}x^{i-2}y^j,\end{aligned}$$

then we see that f is equivalent to

$$(x, x^2 y + y^4 + a_{14}xy^4 + a_{05}y^5 + O(6)).$$

Now we show

$$(x, x^2 y + y^4 + cxy^4 + dy^5) \sim_{\mathcal{A}^5} (x, x^2 y + y^4 + dy^5).$$

This is explicitly given by $\tilde{x} = x$ and $\tilde{y} = y - \frac{c}{3}xy$ and the coordinate change of the target:

$$(X, Y) \mapsto \left(X, Y + \tfrac{c}{3}XY + \tfrac{c^2}{9}X^2 Y + \tfrac{c^3}{27}X^3 Y\right).$$

Since 16 and 17-types are 5-determined, the claim is proved. □

Proof of 6 in Proposition 3.9. At first we show that for $d \neq \frac{9}{5}$

$$\begin{aligned}&(x, x^2 y + xy^3 + cxy^4 + dy^5 + exy^5 + gy^6)\\ &\qquad \sim_{\mathcal{A}^6} (x, x^2 y + xy^3 + dy^5 + Pxy^5 + (g - \tfrac{15cd}{5d-9})y^6)\end{aligned}$$

where P is a constant. This is given by

$$\begin{aligned}
\tilde{x} = \;& x \\
\tilde{y} = \;& y - \tfrac{c}{-9+5d}x + \tfrac{3c^3}{(-9+5d)^3}x^2 - \tfrac{3c^2}{(-9+5d)^2}xy + \tfrac{3c}{-9+5d}y^2 \\
&+ \left(-\tfrac{9c^4}{(-9+5d)^4} + \tfrac{15c^4d}{(-9+5d)^4}\right)x^2y \\
&+ \left(\tfrac{18c^3}{(-9+5d)^3} - \tfrac{20c^3d}{(-9+5d)^3}\right)xy^2 \\
&+ \left(-\tfrac{9c^2}{(-9+5d)^2} + \tfrac{10c^2d}{(-9+5d)^2}\right)y^3 \\
&+ \left(\tfrac{378c^6}{(-9+5d)^6} - \tfrac{210c^6d}{(-9+5d)^6} - \tfrac{5c^4e}{(-9+5d)^4} + \tfrac{6c^5g}{(-9+5d)^5}\right)x^3y \\
&+ \left(-\tfrac{729c^5}{(-9+5d)^5} + \tfrac{405c^5d}{(-9+5d)^5} + \tfrac{10c^3e}{(-9+5d)^3} - \tfrac{15c^4g}{(-9+5d)^4}\right)x^2y^2 \\
&+ \left(\tfrac{783c^4}{(-9+5d)^4} - \tfrac{405c^4d}{(-9+5d)^4} - \tfrac{10c^2e}{(-9+5d)^2} + \tfrac{20c^3g}{(-9+5d)^3}\right)xy^3 \\
&+ \left(-\tfrac{459c^3}{(-9+5d)^3} + \tfrac{195c^3d}{(-9+5d)^3} + \tfrac{5ce}{-9+5d} - \tfrac{15c^2g}{(-9+5d)^2}\right)y^4
\end{aligned}$$

and a simple coordinate change of the target. Next we see

$$(x, x^2y + xy^3 + dy^5 + exy^5 + gy^6) \sim_{\mathcal{A}^6} (x, x^2y + xy^3 + dy^5 + gy^6)$$

for $d \neq \frac{3}{2}$. This follows from

$$\begin{aligned}
&\tilde{x} = x \\
&\tilde{y} = y + \tfrac{e}{2(3-2d)}xy + \tfrac{e^2}{4(3-2d)^2}x^2y + \tfrac{e^3}{8(3-2d)^3}x^3y - \tfrac{e}{3-2d}y^3 - \tfrac{5e^2}{4(3-2d)^2}xy^3
\end{aligned}$$

and $(X, Y) \mapsto \left(X, Y + \frac{e}{2(2d-3)}XY\right)$. Now let

$$f(x, y) = (x, x^2y + xy^3 + \textstyle\sum_{i+j\geq 5} a_{ij}x^iy^j).$$

A similar coordinate change as in the proof of 5 in Proposition 3.9 shows that f is equivalent to

$$(x, x^2y + xy^3 + a_{14}xy^4 + a_{05}y^5 + Qxy^5 + a_{06}y^6 + O(7))$$

where Q is a constant, and hence by the above argument, f is equivalent to

$$(x, x^2y + xy^3 + a_{05}y^5 + (a_{06} - \tfrac{15a_{14}a_{05}}{5a_{05}-9})y^6 + O(7))$$

for $a_{05} \neq \frac{3}{2}, \frac{9}{5}$. Finally by a similar coordinate change as above again, f is equivalent to

$$(x, x^2y + xy^3 + a_{05}y^5 + (a_{06} - \tfrac{15a_{14}a_{05}}{5a_{05}-9})y^6 + b_{16}xy^6 + b_{07}y^7 + O(8))$$

for some b_{ij}; Then xy^6 in the second component is killed, and moreover, if the coefficient of y^6 is not zero, then f is $\mathcal{A}$-equivalent to

$$(x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7 + O(8))$$

that follows from the same argument as in the proof of Proposition 3.2.2:2 in [54]. For generic values α and β, it implies the claim 6, by the determinacy result. This completes the proof. □

3.6 Case 3: $d\lambda(0) = 0$, $\mathrm{rk} H_\lambda(0) = 0$

Finally we deal with type 19 of $\mathcal{A}$-cod $= 6$.

Proposition 3.11 *For a plane-to-plane map-germ f of corank one,*

(1) $j^4 f(0) \sim_{\mathcal{A}^4} (x, x^3y + \alpha x^2y^2 + y^4)$ *or* $(x, \alpha x^2y^2 + y^4)$

$$\Longleftrightarrow d\lambda(0) = 0, \ \ \mathrm{rk}\, H_\lambda(0) = O, \ \ \eta^3\lambda(0) \neq 0.$$

(2) (Rieger [54]) *If we write* $f = (x, x^3y + \alpha x^2y^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$,

$$\Delta = 8\alpha c_{41} - 12c_{32} - 4\alpha^2 c_{23} + 4\alpha c_{14} + (3 + 2\alpha^3)c_{05} \neq 0$$

$$\Longleftrightarrow f \sim_{\mathcal{A}} (x, x^3y + \alpha x^2y^2 + y^4 + x^3y^2) \cdots \boxed{19}$$

Note that the $\mathcal{A}$-codimension of $(x, \alpha x^2y^2 + y^4)$ is greater than 6, so we exclude it, while the type 19 has codimension 6.

Lemma 3.12 *The condition on the right hand side of 1 in Proposition 3.11 are independent from the choice of coordinates of the source and target and the choice of η.*

Proof : It can be proved in the same way as the proof of Lemma 3.5. $\Box$

Proof of Proposition 3.11. The claim 2 is due to Rieger, that can be seen in the proof of Prop. 3.2.3.1 in [54]. We show 1. The proof is the same as that of Proposition 3.4, Proposition 3.6 and Proposition 3.9. The "only if" part is easy. We prove the "if " part. Let f be of corank 1 at 0, $d\lambda(0) = 0$ and $H_\lambda(0) = O$, then we may put

$$f(x, y) = (x, a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4 + O(5)).$$

Suppose that $\eta^3\lambda(0) \neq 0$, then $a_{04} \neq 0$. By the same source change with that in the proof of 4 in Proposition 3.9 we can eliminate xy^3; then f is equivalent to

$$(x, a'_{31}x^3y + a'_{22}x^2y^2 + y^4 + O(5)).$$

for some a'_{ij}. Hence we get

$$j^4 f(0) \sim_{\mathcal{A}^4} (x, \alpha x^2y^2 + y^4) \ \text{ or } \ (x, x^3y + \alpha x^2y^2 + y^4).$$

according to whether $a'_{31} = 0$ or not. $\Box$

3.7 More degenerate types

We should mention germs with $\mathcal{A}$-codimension ≥ 7. By putting a new equation instead of the last inequality in our criteria, we can get a condition which defines the union of $\mathcal{A}$-orbits with $\mathcal{A}$-codimension ≥ 7. For instance, put $\Delta = 0$ for the statement (2) in Proposition 3.11, and it defines the union of $\mathcal{A}$-orbits whose germs are $\mathcal{A}^5$-equivalent to $(x, x^3y + \alpha x^2y^2 + y^4)$.

For the latter application, it would be useful to define unions of degenerate $\mathcal{A}$-orbits as in the followings:

- $II_{\geq k}$ (for $k \geq 2$) : $d\lambda(0) \neq 0, \eta\lambda(0) = \cdots = \eta^{k-2}\lambda(0) = 0$;
- $IV_{\geq k}$ (for $k \geq 5$) : $d\lambda(0) = 0, \det H_\lambda(0) < 0, \eta^2\lambda(0) = \cdots = \eta^{k-2}\lambda(0) = 0$;
- $VI_1 : j^4 f \sim_{\mathcal{A}^4} (x, \beta x^3 y + y^4)$;
- $VI_2 : d\lambda(0) = 0, \mathrm{rk} H_\lambda(0) = 0, \eta^3\lambda(0) = 0$.

Remark that $II_{\geq 7}$, $IV_{\geq 6}$ and VI_2 consist of the complement of orbits in Table 4; and VI_1 is a proper orbit of VI which makes a sense in the latter application (see §5.3, 6.2, 6.3).

4 Application to projection of surface in 3-space

This chapter is devoted to an application of our criteria for plane-to-plane map-germs in Chapter 2. We deal with a classification problem of singularities arising in orthogonal/central projection of a surface in 3-space (cf. Arnold [1, 2], Bruce [8], Gaffney-Ruas [23, 22], Goryunov [26], Mond [40], Platonova [50], Rieger[56]). We look at this problem as a typical *$\mathcal{A}$-recognition problem* of plane-to-plane map-germs arising in a concrete geometric setting; we apply our geometric criteria for detecting $\mathcal{A}$-types to this setting, and generalize a well-known theorem due to Arnold and Platonova [1, 2, 50]. The most important feature of this application is the fact that our criteria consists of two different sorts of conditions: *coordinate-free* conditions for detecting specified jets and additional conditions of Taylor expansions. This clearly shows the reasoning why the classification of Arnold-Platonova defers from the Rieger's $\mathcal{A}$-classification of map-germs of $\mathcal{A}$-codimension greater than 4.

This chapter is based on §4 in [31].

4.1 Orthogonal and central projections

Assume that $\mathbb{R}^4$ is equipped with the standard inner product. Let $p = [\boldsymbol{p}] \in \mathbb{P}^3$, called a *viewpoint*, and let $W_p \subset \mathbb{R}^4$ denote the orthogonal complement to the vector $\boldsymbol{p} \in \mathbb{R}^4$. The *central projection* π_p is the map from $\mathbb{P}^3 - \{p\}$ to the projectivization $\mathbb{P}(W_p)$ given by

$$\pi_p([\boldsymbol{u}]) = \left[\boldsymbol{u} - \frac{(\boldsymbol{u}\cdot\boldsymbol{p})\boldsymbol{p}}{||\boldsymbol{p}||^2}\right] \in \mathbb{P}(W_p) \subset \mathbb{P}^3.$$

Restrict π_p to the open set $\mathbb{R}^3 \subset \mathbb{P}^3$. For $\boldsymbol{p} = (a, b, c, 1) \in \mathbb{R}^4$, set

$$A = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -c \\ -a & -b & -c & -1 \end{pmatrix}$$

and $\Phi_A : \mathbb{P}^3 \to \mathbb{P}^3$ the projective transformation defined by A. Obviously, $\Phi_A(p) = 0 \in \mathbb{R}^3$ and $\Phi_A(\mathbb{P}(W_p)) = \mathbb{P}^2$. We identify π_p with

$$\Phi_A \circ \pi_p : \mathbb{R}^3 - \{p\} \to \mathbb{P}^2, \quad (x, y, z) \mapsto [x - a : y - b : z - c].$$

If the viewpoint is at infinity, i.e. $\boldsymbol{p} = (p, 0)$ with $p = (a, b, c) \in \mathbb{R}^3$, then the projection is given by for $\boldsymbol{u} = (u, 1) \in \mathbb{R}^4$

$$\pi_p([\boldsymbol{u}]) = \left[\boldsymbol{u} - \frac{(\boldsymbol{u}\cdot\boldsymbol{p})\boldsymbol{p}}{||\boldsymbol{p}||^2}\right] = \left[u - \frac{(u\cdot p)p}{||p||^2} : 1\right] \in \mathbb{P}(W_p).$$

Hence it induces the *orthogonal projection* (or *parallel projection*) in $\mathbb{R}^3$ along the line generated by the vector p; if $a \neq 0$ and $v = b/a$, $w = c/a$, then we have by a linear transform on target $\mathbb{R}^2$

$$\pi_p : \mathbb{R}^3 \to \mathbb{R}^2, \quad \pi_p(x, y, z) = (y - vx, z - wx).$$

Let M be a surface in $\mathbb{R}^3$ ($\subset \mathbb{P}^3$) around the origin with the Monge form

$$z = f(x,y) = \sum_{i+j\geq 2} c_{ij} x^i y^j.$$

Take a viewpoint $p = (a,b,c)$ with $a \neq 0$, and then the central projection from p is locally written by

$$\varphi_{p,f} = \pi_p|_M : M \to \mathbb{R}^2, \quad (x,y) \mapsto \left(\frac{y-b}{x-a}, \frac{f(x,y)-c}{x-a} \right) = (X,Y)$$

using local coordinates (x,y) of M and $[1 : X : Y]$ of $\mathbb{P}^2$. If p is chosen to be at infinity in $\mathbb{P}^3$, then

$$\varphi_{p,f}(x,y) = (y - vx, f(x,y) - wx).$$

First we consider the classification of singularities arising in orthogonal projection of surface. The direction of orthogonal projection has two dimensional freedom; There is naturally produced a 2-parameter family of orthogonal projections, $M \times U \to \mathbb{R}^2$, where U is any small open subset of $\mathbb{R}^2$. Hence a naÏve guess is that any plane-to-plane germs of $\mathcal{A}$-cod ≤ 4 might appear generically in orthogonal projection of surface M at some points. In fact it is true.

Theorem 4.1 (Arnold [1], Gaffney-Ruas [23, 22], Bruce [8]) *For a generic surface M, singularities arising in orthogonal projections of M are $\mathcal{A}$-equivalent to the germs of $\mathcal{A}$-cod ≤ 4 in Table 1.*

Remark 4.2 We should remark about what the word "generic" means. A precise statement is as follows: there exists a residual subset of the space of all embeddings of M into $\mathbb{R}^3$ (equipped with C^∞-topology) so that for each element $\iota : M \hookrightarrow \mathbb{R}^3$ of this subset, any orthogonal projection $pr|_M : M \to \mathbb{R}^2$ admits only singularities of $\mathcal{A}$-cod ≤ 4 listed in Table 1. Below we abuse the word "generic" in the same manner for several similar situations; Perhaps that would not cause any confusion.

How about central projections of surfaces ? There is 3-dimensional freedom of the choice of viewpoint p; there is naturally produced a 3-parameter family of central projection, $M \times U \to \mathbb{P}^2$, where U is any small open subset of the complement $\mathbb{P}^3 - M$. Therefore we might have expected that any plane-to-plane germs of $\mathcal{A}$-cod ≤ 5 would appear in central projection generically. But it is not the case. Arnold and Platonova proved the following remarkable theorem [1, 50]:

Theorem 4.3 (Arnold [1], Platonova [50]) *For a generic surface M, and for any $p \in \mathbb{R}^3$ not lying on M, the germ $\varphi_p : M, x \to \mathbb{P}^2, \varphi_p(x)$ at any point $x \in M$ is $\mathcal{A}$-equivalent to one of the list of germs with $\mathcal{A}$-cod ≤ 5 in Table 1 except for $12, 16$ and unimodal type 8.*

So the three types $12, 16$ and 8 are excluded in the list of singularities arising in central projection of a generic surface, in other words, this geometric setting makes a strong restriction on the appearance of singularities of plane-to-plane germs of $\mathcal{A}$-cod $= 5$. Our criteria are applied to detecting $\mathcal{A}$-types of map-germs arising in this special geometric setting. Then we give not only a new transparent proof of Theorem 4.3 in the context of Rieger's classification but also some extension as stated in the following theorem:

Theorem 4.4 *For a generic one-parameter family of embeddings $M \times I \to \mathbb{R}^3$, $(x,t) \mapsto \iota_t(x)$, the central projection $\pi_p \circ \iota_t : M \to \mathbb{R}P^2$ for any t and any viewpoint p admit only $\mathcal{A}$-types with $\mathcal{A}$-cod ≤ 5 and types $12, 16, 8, 4_5, 9, 11_9, 13, 17, 19$ with $\mathcal{A}$-cod 6. Namely, each type of $10, 15, 18$ with $\mathcal{A}$-cod 6 does not appear generically.*

In Rieger [56], orthogonal projection of moving surfaces with one-parameter has been considered. Theorem 4.4 generalizes it in a much more general form. Theorems 4.3 and 4.4 will be proven in two steps; we first verify a version of Thom's transversality theorem, and second we give a desirable stratification of the jet spce of Monge forms.

4.2 Transversality theorem

We remark briefly about a variant of Thom's transversality theorem for proving the above theorems (see [8, §1] for details). Fix a metric of $\mathbb{R}^4$. For a point x_0 on a surface M, choose two smooth independent tangent vector fields and a smooth normal vector field in a neighborhood of x_0. This determines at each point near x_0 a linear (local) coordinate system (x, y, z), where the surface is given locally in Monge form $z = f(x, y)$. Let V_ℓ be the ℓ-jet space of the Monge form $z = f(x, y)$ at 0 (that is the space of polynomials of degree greater than 1 and less than or equal to ℓ). We then obtain a smooth map, the *Monge-Talyor map*, $\Theta : M, x \to V_p$, which assigns to each point x near x_0 the p-jet of f at x. The jet space V_p has a natural $GL(2) \times GL(1)$-action given by linear change of the xy-plane and non-zero scalar multiple of the z-axis, and obviously our strata of V_p are also invariants under this $GL(2) \times GL(1)$-action. Taking an open cover of M suitably, and using the same argument as in the proof of Theorem 1 in Bruce [8], we see that there is a residual subset in the space of smooth embeddings $M \to \mathbb{P}^3$ or smooth familiesof embeddings $M \times U \to \mathbb{P}^3$ with parameter space $U \subset \mathbb{R}^n$ (an open set) so that the corresponding Monge-Taylor map $\Theta : M \times U, (x_0, u) \to V_p$ will be transverse to our strata.

We then define $\Phi(p, j^\ell f(0)) \in J^\ell(2,2)$ to be the ℓ-jet of $\varphi_{p,f}(x,y) - \varphi_{p,f}(0,0)$ at the origin, and consider the following diagram:

$$\begin{array}{ccc} (\mathbb{P}^3 - M) \times V_\ell & \xrightarrow{\Phi} & J^\ell(2,2), \\ {\scriptstyle pr}\downarrow & & \\ V_\ell & & \end{array} \qquad \begin{array}{ccc} (p, j^\ell f(0)) & \xrightarrow{\Phi} & j^\ell \varphi_{p,f}(0). \\ {\scriptstyle pr}\downarrow & & \\ j^\ell f(0) & & \end{array}$$

$\operatorname{cod} G_W$	$\mathcal{A}$-cod	type W
0	0	1
	1	2
	2	3
1	3	$4_2, 5$
	4	4_3
2	4	$6, 11_5$
	5	$4_4, 7, 11_7$
3	5	$12, 16, 8$
	6	$4_5, 9, 11_9, 13, 17, 19$
4	6	$10, 15, 18$

Table 6: Codimension of G_W and $\mathcal{A}$-codimension of W

Note that $J^\ell(2,2)$ is stratified by $\mathcal{A}^\ell$-orbits (those strata of low codimension are given in Rieger's list). Therefore Φ induces a stratification of $(\mathbb{P}^3 - M) \times V_\ell$. Since any $\mathcal{A}^\ell$-orbit W is a semi-algebraic subset of $J^\ell(2,2)$, $\Phi^{-1}(W)$ and hence G_W turns out to be semi-algebraic. It immediately implies the following assertion:

Corollary 4.5 *(1) If codim $G_W \geq 3$, then for a generic embedded surface M, the central projection $\varphi_p : M \to \mathbb{R}P^2$ from any viewpoint p does not admit W-type singularity at any point of M. (2) For a generic s-parameter family of embeddings of M into $\mathbb{R}^3$, any central projection admits only singularties of type W with codim $G_W \leq 2 + s$.*

From Corollary 4.5, our main task for proving Theorem 4.4 is to determine $\operatorname{codim} G_W$ for all W in consideration. To do this, we describe explicitly the defining equations of G_W. We obtain the following result:

Proposition 4.6 *Table 6 is the list of* $\operatorname{codim} G_W$ *for all the map-germs of* $\mathcal{A}$-codim ≤ 6*, with ℓ large enough. In addition,* $\operatorname{codim} G_W \geq 4$ *holds for all the map-germs of* $\mathcal{A}$-cod ≥ 7.

Theorems 4.3 and 4.4 immediately follow from Proposition 4.6 and Corollary 4.5.

The proof will be done as follows. From now on we write

$$f(x,y) = \sum_{i+j\geq 2} c_{ij} x^i y^j.$$

For each $\mathcal{A}$-type in Table 1, we will apply our criteria in Chapter 2 to the plane-to-plane germ of the following form

$$\varphi_{p,f}(x,y) = \left(\frac{y-b}{x-a}, \frac{\sum c_{ij} x^i y^j - c}{x-a} \right)$$

Type	Rank of H_f	Normal form	Cod.
Elliptic	2	$x^2 + y^2$	0
Hyperbolic		xy	
Parabolic	1	y^2	1
Flat umbilic	0	0	2

Table 7: Stratification of of 2-jet space of Monge forms. Elliptic and hyperbolic types are distinguished by the sign of $\det H_f$. The fourth column shows codimension of each stratum in the 2-jet space.

or

$$\varphi_{p,f}(x, y) = \left(y - vx, \sum c_{ij}x^i y^j - wx\right).$$

Then we obtain a certain condition in variables

$$a, b, c \text{ (or } v, w), c_{20}, c_{11}, c_{02}, c_{30}, c_{21}, \cdots$$

so that $\varphi_{p,f}$ is $\mathcal{A}$-equivalent to the $\mathcal{A}$-type. That is nothing but the condition defining the semi-algebraic subset $\Phi^{-1}(W)$ in $\mathbb{R}^3 \times V_\ell$ (or $\mathbb{R}^2 \times V_\ell$) for the corresponding $\mathcal{A}^\ell$-orbit $W \subset J^\ell(2, 2)$ (with ℓ larger than the determinacy order). The condition consists of polynomial equations and inequalities. Simply we call the (system of) equations the *defining equation of* $\Phi^{-1}(W)$. By eliminating the variables a, b, c (or v, w) from the equation, we obtain the defining equation of G_W. The inequalities do not affect the codimension. In general the codimension of $\Phi^{-1}(W)$ is equal to that of W, therefore the main task is to check how the projection pr affects the defining equation of G_W. Notice that this process is equivalent to giving a stratification of the jest space of Monge-forms that is induced from the $\mathcal{A}$-classification via central projections. The strata are explicitly given in the next section.

4.3 Stratification of the jet space by central projections

It is well known that the 2-jet space of Monge forms is simply divided into four different classes corresponding to types of Hessian matrices as in Table 7. We can suppose that any 2-jet of Monge form is one of the normal forms in Table 7 by $GL(2) \times GL(1)$-action. Each type is also characterized by *asymptotic lines* (lines tangent to the surface with more than 2-point contact), that are invariant under projective transformations. The elliptic type has no asymptotic line; the hyperbolic type has two distinct asymptotic lines; the parabolic type has a unique asymptotic line; and any tangent line is asymptotic for the flat umbilical type. It is remarkable that central projections of surfaces give singularities other than the fold type only when view points are on asymptotic lines. Thus we consider central projections of each type for view points lying on asymptotic lines, and give a stratification of the higher jet space of Monge forms. Since an elliptic Monge form has no asymptotic line, we omit this case.

4.3.1 Hyperbolic type

Here we think of Monge forms of hyperbolic type: $f(x,y) = xy + \sum_{i+j\geq 3} c_{ij}x^iy^j$. In this form the x and y-axes are asymptotic lines. Thus we suppose that the view point $p = (a,0,0)$ is on the x-axis where $a \neq 0$ (possibly ∞). Note that only $\mathcal{A}$-types of case 0 appear in this setting: $d\lambda \neq 0$ and $\eta\lambda(0) = 0$ in Proposition 3.4 always hold. The following proposition gives strata corresponding to topological $\mathcal{A}$-types II_k ($k \geq 3$) in the jet space of Monge forms.

Proposition 4.7 *For the above Monge form of hyperbolic type and a viewpoint p lying on the x-axis, the central projection $\varphi_{p,f}$ is $\mathcal{A}^k$-equivalent to $II_k : (x, xy+y^k)$ for $k \geq 3$ if and only if*

$$c_{30} = c_{40} = \cdots = c_{k-1\ 0} = 0,\ c_{k0} \neq 0.$$

Thus codim $G_W = k - 3$ *for* $W = II_k$.

Proof: The proof is by direct calculations with criteria in Proposition 3.4.

The above stratum induced from the II_k-type can be more finely stratified by considering $\mathcal{A}$-orbits as W. The $\mathcal{A}$-orbits are distinguished according to the codition of special position of viewpoints on the asymptotic line. Note that equisingular types appear from II_k ($k \geq 5$).

$\boxed{II_5\text{-type}}$ Suppose that $c_{30} = c_{40} = 0$ and $c_{50} \neq 0$. Then the central projection $\varphi_{p,f}$ for viewpoint $p = (a,0,0)$ lying on the x-axis is $\mathcal{A}^5$-equivalent to II_5 : $(x, xy + y^5)$. The $\mathcal{A}$-orbits over the 5-jet: $(x, xy+y^5)$ are just 6 (butterfly) and 7 (elder butterfly)-types, and these are distinguished by an additional equation: $a_{07} - \frac{5}{8}a_{06}^2 = 0$ in our criterion.

With the above setting, $\varphi_{p,f}$ is $\mathcal{A}$-equivalent to $(x, xy+y^5+\sum_{i+j\geq 6} a_{ij}x^iy^j)$ where

$$a_{06} = -\frac{1}{c_{50}a}\left\{(5c_{21}c_{50} - 6c_{60})a - c_{50}\right\},$$

and

$$a_{07} = \frac{1}{c_{50}a^2}\left\{\begin{array}{l}(20c_{21}^2c_{50} - 5c_{31}c_{50} - 6c_{21}c_{60} + c_{70})a^2 \\ +(6c_{21}c_{50} + c_{60})a + c_{50}\end{array}\right\}.$$

(See Proof of 1 and 3 in Proposition 3.4). Here $a_{07} - \frac{5}{8}a_{06}^2 = 0$ gives an equation in the variable a with A, B, C depending only on c_{ij} :

$$Aa^2 + Ba + C = 0.$$

If one takes the viewpoint $p = (a,0,0)$ where a is a solution of the above equation, then the central projection is of 7-type. There are different states of special positions of viewpoints (Remark 4.8). In addition, A (also B, C) is independent from the other two equations $c_{30} = c_{40} = 0$; for instance, we see that there is a monomial $c_{31}c_{50}^2$ in A. Thus the variable a is solved in c_{ij}

generically, so G_7 (i.e. W is the orbit of type 7) is an open subset of G_6 (i.e. W is the orbit of type 6). Hence the defining equation of G_6 or G_7 is

$$c_{30} = c_{40} = 0$$

and

$$\text{codim } G_6 = \text{codim } G_7 = 2.$$

Remark 4.8 For a generic surface, hyperbolic points where the Monge form satisfies that $c_{30} = c_{40} = 0$ are isolated, since G_W has codimension 2 in V_ℓ. Look at such a point of the surface from a viewpoint lying on the a-axis ($b = c = 0$), then the central projection produces the butterfly singularity (6-type). However, there is an exception: from at most two points on the a-axis which are given by the solution $a = a(c_{ij})$ of the quadric equation, the central projection admits the elder-butterfly singularity (7-type). These exceptional points are called *h-focal points* ("h" for "hyperbolic") by Platonova [50]. Note that G_W can be stratified according to types of the equation $Aa^2 + Ba + C = 0$:

- If $C, B^2 - 4AC \neq 0$, there are 2 distinct or no h-focal points;
- If $C \neq 0$ and $B^2 - 4AC = 0$, there is a unique multiple h-focal point;
- If $C = 0$ and $B \neq 0$, there is a unique h-focal point;
- If $B = C = 0$ and $A \neq 0$, there is no h-focal point;
- If $A = B = C = 0$, the central projections give 7-types for viewpoints at any positions on the asymptotic line.

Note that h-focal points can be at the infinite when $A = 0$, and the quadratic equation has an impossible solution: $a = 0$ when $C = 0$.

$\boxed{II_6\text{-type}}$ Suppose that $c_{30} = c_{40} = c_{50} = 0$ and $c_{60} \neq 0$. Then a central projection $\varphi_{p,f}$ for p on the x-axis is $\mathcal{A}^6$-equivalent to $II_6 : (x, xy + y^6)$. The $\mathcal{A}$-orbits over $(x, xy + y^6)$ are 8, 9 and more degenerate orbits ($\mathcal{A}$-cod ≥ 7) whose germs are $\mathcal{A}^9$-equivalent to $(x, xy + y^6)$ (see Fig. 1 in [54]). They are distinguished by equations: $a_{08} - \frac{3}{5}a_{07}^2 = 0$ and $a_{09} - \frac{7}{25}a_{07}^3 = 0$ in our criteria (Proposition 3.4).

First, in the similar way to the previous case, the projection is written by $\varphi_{p,f} = (y, xy + y^6 + \sum_{i+j\geq 7} a_{ij}x^i y^j)$ in a suitable coordinate, and $a_{08} - \frac{3}{5}a_{07}^2 = 0$ gives

$$Aa^2 + Ba + C = 0$$

for some polynomials A, B, C in c_{ij} (e.g., A contains the monomial $c_{31}c_{60}^2$, so it is independent from other equations). Thus, generically, the projection is of type 8 from almost all view points lying on the asymptotic line (x-axis), and more degenerate types at exceptional points $(a, 0, 0)$ where a satisfyies $Aa^2 + Ba + C =$

0. As seen in Remark 4.8, the other states of special position is possible also in this case.

Let p be the special viewpoint. Then the projection is of type 9 if $a_{09} - \frac{7}{25}a_{07}^3 \neq 0$, while it is $\mathcal{A}^9$-equivalent to $(x, xy + y^6)$ ($\mathcal{A}$-cod ≥ 7) if the quantity is 0. Solve the variable a by the above $Aa^2 + Ba + C = 0$, and then the equation $a_{09} - \frac{7}{25}a_{07}^3 = 0$ yields a non-trivial equation, say $D(c_{ij}) = 0$, which is independent from other equations.

Hence the defining equation of G_8 or G_9 (i.e. W is the orbit of type 8 or 9) is

$$c_{30} = c_{40} = c_{50} = 0$$

and

$$\text{codim } G_8 = \text{codim } G_9 = 3,$$

while, for the orbit W whose germs are $\mathcal{A}^9$-equivalent to $(x, xy+y^6)$, the defining equation of G_W is

$$c_{30} = c_{40} = c_{50} = D(c_{ij}) = 0$$

and

$$\text{codim } G_W = 4.$$

Remark 4.9 The difference between 7-type and 8-type (although they have the same $\mathcal{A}$-codimension) is the difference between closed conditions $a_{07} - \frac{5}{8}a_{06}^2 = 0$ and $\eta^4\lambda(0) = 0$. As mentioned in Remark 4.8, the former condition on coefficients determines the position of viewpoint, while the geometric condition $\eta^4\lambda(0) = 0$ determines that of the Monge form. Also in the following other calculations, this kind of difference makes the difference of codim G_W.

$\boxed{II_{\geq 7}\text{-type}}$ Suppose that $c_{30} = c_{40} = c_{50} = c_{60} = 0$. Since the codimension of this stratum is already 4, we do not consider a further stratification. Remark that $\mathcal{A}$-singularities of case 0 with $\mathcal{A}$-cod ≥ 7 and no. 10 belong to the orbit of $II_{\geq 7}$-type. Therefore codim $G_W \geq 4$ for the above degenerate $\mathcal{A}$-orbits as W.

4.3.2 Parabolic type

Here we deal with Monge forms of parabolic type: We suppose $\mathrm{rk}H_f(0) = 1$ for Monge forms $z = f(x, y)$. Thus, through this subsection, we always consider the Monge form $f(x, y) = y^2 + \sum_{i+j\geq 3} c_{ij}x^i y^j$. In this form the x-axis is the unique asymptotic line. Thus we suppose that the view point $p = (a, 0, 0)$ is on the x-axis where $a \neq 0$ (possibly ∞). Note that $d\lambda(0) = 0$ always holds in this setting, hence $\mathcal{A}$-types of case $1, 2, 3$ appear. The next proposition gives strata in the jet space of parabolic Monge forms, where each stratum corresponds to a specified 3-jet of the central projection:

Proposition 4.10 *For a Monge form of the above form, and a view point p lying on the x-axis, the central projection $\varphi_{p,f}$ is $\mathcal{A}^3$-equivalent to*

Specified jet	Class	Pr_1	Pr_2	Extra doditions	Cod.
I_*	$\Pi^p_{I,1}$	4_2	4_3	$P \neq 0$	1
$c_{20} = 0$	$\Pi^p_{I,2}$	4_2	4_4	$P = 0,\ Q \neq 0$	2
$c_{30} \neq 0$	$\Pi^p_{I,3}$	4_2	4_5	$P = Q = 0,\ R \neq 0$	3
	$\Pi^p_{I,4}$	4_2	$4_{\geq 6}$	$P = Q = R = 0$	4
III_*	$\Pi^p_{c,1}$	11_5	11_7	$LB \neq 0$	2
$c_{20} = c_{30} = 0$		11_5	11_9	$L = 0,\ MB \neq 0$	3
$c_{40} \neq 0$		11_5	$11_{\geq 11}$	$L = M = 0,\ B \neq 0$	4
	$\Pi^p_{c,2}$	11_5	–	$B = 0,\ A \neq 0$	3
	$\Pi^p_{c,3}$	$11_{\geq 7}$	$\dagger 11_{\geq 9}$	$A = B = 0$	4
IV_5	$\Pi^p_{c,4}$	12	13	$NF \neq 0$	3
$c_{20} = c_{30} = c_{40} = 0$		$\dagger 12, 13, 14$	†	$NF = 0$	4
$c_{50} \neq 0$					
$IV_{\geq 6}$	$\Pi^p_{c,5}$	†15, etc.	†		4
$c_{20} = c_{30} = c_{40} = c_{50} = 0$					
$V_1(VI)$	$\Pi^p_{v,1}$	16	$17, 19$	$(I^2 - 4HJ)S\beta\Delta \neq 0$	3
$c_{20} = c_{30} = c_{21} = 0$		$\dagger 16, 17, 19$	†	$(I^2 - 4HJ)\beta\Delta = 0,$	4
$c_{40} \neq 0$		etc.		$S \neq 0$	
	$\Pi^p_{v,2}$		$\dagger VI_1$	$S = 0$	
$V_2(VI_2)$	$\Pi^p_{v,3}$	†18, etc.	†		4
$c_{20} = c_{30} = c_{21} = c_{40} = 0$					

Table 8: Stratification of parabolic Monge forms. The central projection of each type gives $\mathcal{A}$-type in the 3rd column (Pr_1) from almost all view points on the asymptotic line; while the projection gives a more degenerate $\mathcal{A}$-type in the 4th column (Pr_2) from some special view points. The fifth column means codimension of each stratum in the jet space. A stratum with codimension 4 is actually a union of strata which possibly give singularities with $\mathcal{A}$-cod ≥ 7 or degenerate states of special view points. † implies such degenerate cases.

- *type of goose series:* $(x, y^3 \pm x^2y)$ *or* (x, y^3) $(I_2$ *or* $I_*)$
  $\iff c_{30} \neq 0$ *(i.e. codim* $G_W = 1$*);*
- *type of gulls series:* (x, xy^2) $(III_*$ *or* $IV_{\geq 5})$
  $\iff c_{30} = 0$ *and* $c_{21} \neq 0$ *(i.e. codim* $G_W = 2$*);*
- *degenerate types:* (x, x^2y) *or* $(x, 0)$ $(V_1,\ V_2,\ VI$ *or* $VI_2)$
  $\iff c_{30} = c_{21} = 0$ *(i.e. codim* $G_W = 3$*).*

Proof : The proof is by direct calculations with criteria in Proposition 3.6, 3.9, 3.11.

In the following, we consider proper strata inside each stratum given in the above proposition, where the proper strata are distinguished by the difference of $\mathcal{A}$-types of central projections. The results are summed up in Table 8.

4.3.3 Parabolic type of goose series

$\boxed{I_k\text{-type}}$ Assume $c_{30} \neq 0$. Then the central projection $\varphi_{p,f}$ for a view point $p = (a, 0, 0)$ lying on the x-axis is $\mathcal{A}^3$-equivalent to $(x, \delta xy^2 + y^3)$, where $\delta = Ca + D$ for

$$C = -3c_{30}c_{12} + c_{21}^2, \quad D = 3c_{30}$$

(if p is at the infinity, that is, $\varphi_{p,f}$ is the orthogonal projection, $\delta = C/c_{30}^2$). Note that $(x, xy^2 \pm y^3)$ is $\mathcal{A}^3$-determined (4_2-type), and $\mathcal{A}$-orbits over the 3-jet (x, y^3) are 4_k-types ($k \geq 3$). Thus the projection is of type 4_2 (beaks/lips) from almost all viewpoints lying on the asymptotic line (x-axis), i.e. $Ca + D \neq 0$, while there is a unique viewpoint $p = (-D/C, 0, 0)$ (or at the infinity if $C = 0$) so that the projection is of type 4_3 or worse. This point divides the line into two half lines, each of which corresponds to viewpoints for either the lips singularity or the beaks singularity of the projection. Note that $C = D = 0$ if and only if $c_{30} = c_{21} = 0$, where the projection type becomes to be more degenerate (see §4.3.5).

Let p be the special view point. Then the degenerate projection is written by $\varphi_{p,f} = (x, y^3 + \sum_{i+j\geq 4} a_{ij}x^iy^j)$ in some local coordinates; we put

$$P = a_{13}, \;\; Q = a_{14} - \tfrac{1}{3}a_{22}^2, \;\; R = a_{15} - \tfrac{2}{3}a_{23}a_{22} + \tfrac{1}{3}a_{31}a_{22}^2.$$

It is shown in Proposition 3.11 that

- $P \neq 0 \iff \varphi_{p,f} \sim_{\mathcal{A}} 4_3$ (Goose type);
- $P = 0,\; Q \neq 0 \iff \varphi_{p,f} \sim_{\mathcal{A}} 4_4$ (Ugly goose type);
- $P = Q = 0,\; R \neq 0 \iff \varphi_{p,f} \sim_{\mathcal{A}} 4_5$;
- $P = Q = R = 0 \iff \varphi_{p,f} \sim_{\mathcal{A}} 4_{\geq 6}$.

Notice that a_{ij} are polynomials in c_{ij}, hence P, Q, R are so (they are independent from others). As a result

$$\operatorname{codim} G_{4_2} = \operatorname{codim} G_{4_3} = 1, \; \operatorname{codim} G_{4_4} = 2, \; \operatorname{codim} G_{4_5} = 3, \; \operatorname{codim} G_{4_{\geq 6}} = 4.$$

4.3.4 Parabolic type of gulls series

Assume $c_{30} = 0$ and $c_{21} \neq 0$, then the central projection is $\mathcal{A}^3$-equivalent to (x, xy^2) on this setting. The following proposition holds:

Proposition 4.11 *For the above Monge form of parabolic type and a view point p lying on the x-axis, the central projection $\varphi_{p,f}$ is $\mathcal{A}^k$-equivalent to $(x, xy^2 + y^k)$ for $k \geq 4$ if and only if*

$$c_{40} = \cdots = c_{k-1\,0} = 0, \; c_{k0} \neq 0$$

Proof: The proof is by direct calculations with criteria in Proposition 3.6.

$\boxed{III_*\text{-type}}$ Suppose that $c_{30} = 0$ and $c_{40} \neq 0$. Then the central projection $\varphi_{p,f}$ for a viewpoint $p = (a, 0, 0)$ lying on the x-axis is $\mathcal{A}^4$-equivalent to III_* :

(x, xy^2+y^4). $\mathcal{A}$-orbits over the 4-jet (x, xy^2+y^4) are 11_{2k+1}-types ($k \geq 2$), and conditions by coefficients of Taylor expansions in Proposition 3.6 distinguish these types. In the similar way to the hyperbolic case, the projection is written by $\varphi_{p,f} = (x, xy^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^i y^j)$ in some local coordinates. $a_{05} = 0$ in our criteria (see Proposition 3.6) gives a linear equation

$$Aa + B = 0$$

where

$$A = c_{21}^2 c_{50} + 4c_{12}c_{40}^2 - 2c_{21}c_{31}c_{40}, \quad B = c_{21}^2 c_{40} - 4c_{40}^2.$$

Generically, the variable a is solved in c_{ij} (see also Remark 4.12), hence the projection is of type 11_5 (gulls) from almost all viewpoints lying on the asymptotic line (x-axis), while there is a unique viewpoint $p = (-B/A, 0, 0)$ (or at the infinity, if $A = 0$) so that the projection is of type $11_{\geq 7}$.

Remark 4.12 As seen in Remark 4.8, we also have an exceptional point here. This exceptional point is called *p-focal* point ("p" for parabolic) by Platonova [50], and at this point the more degenerate singularity (11_{2k+1}-type ($k \geq 3$)) appears. Depending on the values of A and B, a state of the p-focal point changes:

- In case of $B \neq 0$, there exists a unique p-focal point at $p = (-B/A, 0, 0)$ (or at the infinity, if $A = 0$);
- In case of $B = 0$ and $A \neq 0$, there is no p-focal point – the projection is of type 11_5 from *any* viewpoint on the line;
- In case of $A = B = 0$, all points on the line are p-focal points – the type 11_{2k+1} ($k \geq 3$) is observed from any viewpoint on the line.

Suppose $B \neq 0$, and take p as the special view point (p-focal point). Proposition 3.6 says that

- $L := a_{07} - 2a_{15} + 4a_{23} \neq 0 \iff \varphi_{p,f} \sim_{\mathcal{A}} 11_7$ (Ugly gulls);
- $L = 0, \;\; M := \bar{a}_{09} - 2\bar{a}_{17} \neq 0 \iff \varphi_{p,f} \sim_{\mathcal{A}} 11_9$;
- $L = M = 0 \iff \varphi_{p,f} \sim_{\mathcal{A}} 11_{\geq 11}$,

where $\bar{a}_{ij}$ means the coefficient of the projection in suitable coordinates (see also Remark 3.8). Here L and M are polynomials with c_{ij} that are independent from other equations.

Thus

$$\operatorname{codim} G_{11_5} = \operatorname{codim} G_{11_7} = 2, \; \operatorname{codim} G_{11_9} = 3, \; \operatorname{codim} G_{11_{\geq 11}} = 4.$$

$\boxed{IV_5\text{-type}}$ Suppose that $c_{30} = c_{40} = 0$ and $c_{50} \neq 0$. Then the central projection $\varphi_{p,f}$ for a viewpoint $p = (a, 0, 0)$ lying on the x-axis is $\mathcal{A}^5$-equivalent to IV_5 :

(x, xy^2+y^5). $\mathcal{A}$-orbits over the 5-jet (x, xy^2+y^5) are $12, 13$ and 14-types (see Fig. 3 in Rieger [54]). Remark that $14 : (x, xy^2+y^5)$ is a 9-determined germ with $\mathcal{A}$-codimension 7 in [54]. As in the previous cases, the projection is written by $\varphi_{p,f} = (x, xy^2+y^5+\sum_{i+j\geq 6} a_{ij}x^iy^j)$ in some local coordinates. Here $a_{06} = 0$ in our criteria (see Proposition 3.6) gives a linear equation

$$Ea + F = 0$$

where E and F are polynomials with c_{ij}. Thus the projection is, generically, of type 12 from almost all viewpoints lying on the asymptotic line (x-axis) i.e. $Ea+F \neq 0$, while there is a unique viewpoint $p = (-F/E, 0, 0)$ (or at the infinity when $E = 0$) so that the projection is of type 13 or 14. Notice that other states of special position of view point can happen but in case of codimension 4.

Suppose $F \neq 0$, and take the special view point $p = (-F/E, 0, 0)$ (or at the infinity when $E = 0$). Put $N := a_{09} - \frac{5}{2}a_{16} - \frac{5}{6}a_{07}^2$, which is a polynomial with c_{ij} and independent from other equations. According to Proposition 3.6,

- $N \neq 0 \iff \varphi_{p,f} \sim_{\mathcal{A}} 13;$
- $N = 0 \iff \varphi_{p,f} \sim_{\mathcal{A}} 14.$

Thus

$$\operatorname{codim} G_{12} = \operatorname{codim} G_{13} = 3, \ \operatorname{codim} G_{14} = 4.$$

$\boxed{IV_{\geq 6}\text{-type}}$ Suppose that $c_{30} = c_{40} = c_{50} = 0$. Then the central projection $\varphi_{p,f}$ for p lying on the x-axis is $\mathcal{A}^5$-equivalent to (x, xy^2). Since the codimension of this stratum is already 4, codim $G_W \geq 4$ holds for any $\mathcal{A}$-orbit W over this 5-jet (x, xy^2). Note that the $\mathcal{A}$-type of no. 15 ($\mathcal{A}$-cod $= 6$) is this case.

4.3.5 More degenerate parabolic types

Finally, assume $c_{30} = c_{21} = 0$. Then the central projection $\varphi_{p,f}$ for a view point $p = (a, 0, 0)$ lying on the x-axis is $\mathcal{A}^3$-equivalent to $(x, \delta x^2 y)$ where $\delta = -c_{12}a+1$. Note that $\eta^3\lambda(0)$ in our criteria is a nonzero scalar multiple of c_{40}, hence the projection is of type V_1 or VI if $c_{40} \neq 0$; V_2 or VI_2 if $c_{40} = 0$ (see Proposition 3.9, 3.11 and §3.7).

$\boxed{V_1 \text{ or } VI\text{-type}}$ Assume $c_{30} = c_{21} = 0, \ c_{40} \neq 0$. Then the central projection $\varphi_{p,f}$ for $p = (a, 0, 0)$ lying on the x-axis is equivalent to V_1 (resp. VI) if $\delta \neq 0$ (resp. $\delta = 0$). That is, the projection is of type V_1 from almost all viewpoints lying on the asymptotic line (x-axis), while there is a unique viewpoint $p = (1/c_{12}, 0, 0)$ (or at the infinity when $c_{12} = 0$) so that the projection is of type VI. $\mathcal{A}$-orbits over V_1 are just 16 and 17; and those over VI are 19 and more degenerate types with $\mathcal{A}$-codimension ≥ 7 (see Fig. 4 and 5 in Rieger [54]).

If $\delta = -c_{12}a + 1 \neq 0$

$$\varphi_{p,f} \sim_{\mathcal{A}} (x, x^2y + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j),$$

and a_{05} determines whether the projection is of type 16 : $(x, x^2y + y^4 \pm y^5)$ or 17 : $(x, x^2y + y^4)$. Here $a_{05} = 0$ gives a quadratic equation

$$Ha^2 + Ia + J = 0$$

where H, I, J are polynomials with c_{ij}. As seen in Remark 4.8 for hyperbolic types, degenerate states of special view points are possible also for this case.

On the other hand, if we take the special view point $p = (1/c_{12}, 0, 0)$, the projection is $\mathcal{A}^4$-equivalent to $(x, \beta x^3y + \alpha x^2y^2 + y^4)$, where α, β depend on c_{ij}. Especially, α is a nonzero scalar multiple of

$$S := 3c_{31}^2 + 8c_{40}(c_{12}^2 - c_{22}).$$

Furthermore, if $\beta \neq 0$, the projection is written by

$$\varphi_{p,f} \sim_{\mathcal{A}} (x, x^3y + \bar{\alpha}x^2y^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j).$$

where a_{ij} depends on c_{ij} and $\bar{\alpha}$ is a scalar multiple of S. According to Proposition 3.11, the following holds:

$$\Delta := 8\alpha a_{41} - 12a_{32} - 4\alpha^2 a_{23} + 4\alpha a_{14} + (3 + 2\alpha^3)a_{05} \neq 0$$

$$\iff \varphi_{p,f} \sim_{\mathcal{A}} 19 : (x, x^3y + \alpha x^2y^2 + y^4 + x^3y^2).$$

To sum up, if $(I^2 - 4HJ)\beta\Delta \neq 0$, the projection is of type 16 from almost all viewpoints lying on the asymptotic line (x-axis), while there are some exceptional viewpoints: $p = (a, 0, 0)$ for a satisfying $Ha^2 + Ia + J = 0$ so that the projection is of type 17; and $p = (1/c_{12}, 0, 0)$ so that the projection is of type 19. Remark that more degenerate $\mathcal{A}$-types (over V_1 or VI) or degenerate states of special viewpoints are possible for Monge forms with codimension= 4 (i.e. $(I^2 - 4HJ)\beta\Delta = 0$). Hence

$$\operatorname{codim} G_{16} = \operatorname{codim} G_{17} = \operatorname{codim} G_{19} = 3,$$

and for any $\mathcal{A}$-orbit W with $\mathcal{A}$-cod ≥ 7 over VI

$$\operatorname{codim} G_W \geq 4.$$

Remark 4.13 For the latter chapter, we also consider the orbit VI_1 where 4-jets are equivalent to $(x, \beta x^3y+y^4)$ (VI_1 is a proper orbit of VI with $\alpha = 0$). On the above setting, $\alpha = 0$ gives an equation: $S = 3c_{02}c_{31}^2 + 8c_{40}(c_{12}^2 - c_{02}c_{22}) = 0$. It is seen that S determines type of singularity of the parabolic curve on the surface (see §5.3, 6.2, 6.3).

$\boxed{V_2 \text{ or } VI_2\text{-type}}$ Assume $c_{30} = c_{21} = c_{04} = 0$. Then the central projection $\varphi_{p,f}$ for p on the x-axis is equivalent to V_2 or VI_2. Thus, for any $\mathcal{A}$-orbit W over V_2 and VI_2 (e.g. no. 18), codim $G_W \geq 4$.

4.3.6 Flat umbilical type

Finally we deal with Monge forms of flat umbilical type. Then 3-jets are classified into $GL(2)$-orbits of $xy^2 \pm x^3$, xy^2, y^3 and 0 (see §5.5). Here we deal with types of $xy^2 \pm x^3$ (codimesion 3) and xy^2 (codimesion 4). Remark that all tangent lines at a flat umbilical point are asymptotic, hence we consider central projections from viewpoints $p = (a, b, 0)$ on the xy-plane (the argument below is also true for the case that p is at the infinity). The projection gives $\mathcal{A}$-types of case 1, 2 and 3, however they appear for higher codimansional Monge forms compared with the case of parabolic types. Thus it never affects the codimension of G_W calculated in the previous subsection for each $\mathcal{A}$-orbit.

(Π^f): If $D \neq 0$ (codimesion 3), then we may assume $j^3 f = x^2 y \pm y^3$. The sign in the normal form makes a difference. Let $f = xy^2 + x^3 + \cdots$. Then

- for general tangent lines ($a \neq 0, \pm\frac{1}{\sqrt{3}}b$), $\varphi_{p,f}$ is of type $4_2^{\pm}$;
- There are three exceptional tangent lines: $a = 0$ where $\varphi_{p,f}$ is of type 11_5 from almost all viewpoints on the line and of type 11_7 from some special points (it can be of type $11_{\geq 9}$ or $IV_{\geq 5}$ for the case of codimension 4); and $a = \pm\frac{1}{\sqrt{3}}b$, where $\varphi_{p,f}$ is of type 4_3 from almost all viewpoints on the line and of type 4_4 from some special points (it can be of type $4_{\geq 5}$ for the case of codimension 4).

Let $f = xy^2 - x^3 + \cdots$. Then

- for general tangent lines ($a \neq 0, \pm b$), $\varphi_{p,f}$ is of type 4_2^- (beaks only);
- If $a = 0$ or $\pm b$, $\varphi_{p,f}$ is of type 11_5 from almost all viewpoints on the line and of type 11_7 from some special points ($11_{\geq 9}$ or $IV_{\geq 5}$ are possible for the case of codimension ≥ 4).

(Π_2^f): If $D = 0$ (codimension 4), then we may assume $j^3 f = x^2 y$. In this case, from almost all viewpoints on the tangent plane ($ab \neq 0$), only 4_2^- is observed. On the line $a = 0$, $11_{\geq 5}$ or $IV_{\geq 5}$ are observed; on the line $b = 0$, V_1 or V_2 are observed.

5 Projective classification of jets of surfaces

In the previous chapter, we get an invariant stratification of the jet space of Monge forms which is induced from $\mathcal{A}$-classification of singularities of central projections. This chapter, we give a normal form for each stratum via projective transformations. The list of our normal forms is a natural extension of Platonova's result [50]. Only for non-degenerate hyperbolic points, the Monge form was examined by Wilczynski and followers in a different context of projective differential geometry; it is slightly surprising that for more than one hundred years, there has been missing so far a suitable treatment on Monge forms at more degenerate hyperbolic points, parabolic and flat points. Thus our result would give a new insight to this classical subject – in fact, our normal forms contain leading moduli parameters, and those parameters must be understood as certain primary projective differential invariants in (some generalized) sense of Wilczynski, or a framework of H. Cartan.

This chapter is based on a joint work with H. Sano, J. L. Deolindo Silva and T. Ohmoto [59].

5.1 Normal forms

We say that two germs or jets of surfaces are *projectively equivalent* if there is a projective transformation on $\mathbb{P}^3$ sending one to the other. A classically well-known fact is that at a general hyperbolic point of a surface, the jet of the Monge form is projectively equivalent to

$$xy + x^3 + y^3 + \alpha x^4 + \beta y^4 + \cdots$$

where moduli parameters α, β are primary projective differential invariants (those should be compared with the Gaussian/mean curvatures in the Euclidean case and the Pick invariant in the equi-affine case, cf. [47, 67, 72]).

Such expressions for generic surface germs were considered by Platonova[50]: Normal forms with codimension $s \leq 2$ in Tables 9, 10, 11 were given by her (see also [33, 3, 4, 26]). Our aim in this chapter is to expand Platonova's result: We give normal forms which represent strata of degenerate Monge forms (codimension ≥ 3) in the previous chapter.

Our main theorem is stated as follows:

Theorem 5.1 *Let M be a closed surface and U an open neighborhood of the origin in $\mathbb{R}^2$ (parameter space). There is a residual subset $\mathcal{O}$ of the space of 2-parameter families of smooth embeddings $\phi : M \times U \to \mathbb{P}^3$ equipped with C^∞-topology so that each $\phi \in \mathcal{O}$ satisfies that for arbitrary point $(x_0, u) \in M \times U$ the p-jet of Monge form of the surface $M_u(= \phi(M \times u))$ at x_0 is projectively equivalent to one of the normal forms at the origin given in Tables 9, 10, 11 with codimension s (≤ 4).*

In Tables, $\alpha, \beta, \cdots$ are moduli parameters and ϕ_k denotes arbitrary homogeneous polynomials of degree k. In our stratification, each stratum is determined

class	normal form	p	s	proj.
$\Pi^p_{\mathrm{I},1}$	$y^2+x^3+xy^3+\alpha x^4$	4	1	$\mathrm{I}_2\,(\mathrm{I}_3)$
$\Pi^p_{\mathrm{I},2}$	$y^2+x^3\pm xy^4+\alpha x^4+\beta y^5+x^2\phi_3$	5	2	$\mathrm{I}_2\,(\mathrm{I}_4)$
$\Pi^p_{c,1}$	$y^2+x^2y+\alpha x^4 \quad (\alpha\neq 0, \frac{1}{4})$	4	2	$\mathrm{III}_2\,(\mathrm{III}_3)$
$\Pi^p_{c,2}$	$y^2+x^2y+\frac{1}{4}x^4+\alpha x^5+y\phi_4\ (\alpha\neq 0)$	5	3	III_2
$\Pi^p_{c,4}$	$y^2+x^2y+x^5+y\phi_4$	5	3	IV_1
$\Pi^p_{\mathrm{I},3}$	$y^2+x^3+xy^5+\alpha x^4+\phi$	6	3	$\mathrm{I}_2\,(\mathrm{I}_5)$
$\Pi^p_{v,1}$	$y^2\pm x^4+\alpha x^3y+\beta x^2y^2 \quad (\beta\neq\pm\frac{3}{8}\alpha^2)$	4	3	$\mathrm{V}_1\,(\mathrm{VI})$
$\Pi^p_{c,3}$	$y^2+x^2y+\frac{1}{4}x^4+y\phi_4$	5	4	$\mathrm{III}_3\,(\mathrm{III}_4)$
$\Pi^p_{c,5}$	$y^2+x^2y\pm x^6+y(\phi_4+\phi_5)$	6	4	IV_2
$\Pi^p_{\mathrm{I},4}$	$y^2+x^3+\alpha x^4+\phi$	6	4	$\mathrm{I}_2\,(\mathrm{I}_6)$
$\Pi^p_{v,2}$	$y^2\pm x^4+\alpha x^3y\pm\frac{3}{8}\alpha^2x^2y^2$	4	4	$\mathrm{V}_1\,(\mathrm{VI}_1)$
$\Pi^p_{v,3}$	$y^2+x^5+y(\phi_3+\phi_4)$	5	4	$\mathrm{V}_2\,(\mathrm{VI}_2)$

Table 9: Parabolic case. In normal forms, ϕ_k denotes homogeneous polynomials of degree k (that is also in Tables 10, 11). For $\Pi^p_{\mathrm{I},k}$ $(k=3,4)$, we put $\phi=\beta y^5+\gamma y^6+x^2(\phi_3+\phi_4)$ for short. Double-sign $\pm$ corresponds in same order for each of $\Pi^p_{v,1}$ or $\Pi^p_{v,2}$. For $\Pi^p_{c,1}$, if $\alpha=1$, the normal form of 4-jet should be $y^2+x^2y+x^4+\beta x^3y$.

by its topological $\mathcal{A}$-type of projection along the asymptotic line. Recall that we actually dealt with the stratification corresponding to $\mathcal{A}$-singularities in the previous chapter. However the defining equation which distinguishes $\mathcal{A}$-types can be very complicated (the defining equation contains terms A, B, C, etc. that are big polynomials with c_{ij}). Thus we adopt topological $\mathcal{A}$-singularities as criteria of our stratification, which still gives us rich information in sense of projective differential geometry. The most interesting is the parabolic case (Table 9) – e.g. classes $\Pi^p_{\mathrm{I},k}$ can not be distinguished by using types of height functions, the parabolic curves and asymptotic curves, but can be so by the difference of singularity types of projection from a special isolated viewpoint. The same approach to the classification of surfaces in $\mathbb{P}^4$ is considered in [21].

Remark 5.2 In the case over $\mathbb{C}$, the elliptic case and the sign difference in Tables are omitted. Theorem 5.1 can be restated in the algebro-geometric context by mean of a Beltini-type theorem for the linear system of projective surfaces of degree greater than p.

Remark 5.3 In the normal forms listed in Tables, continuous moduli parameters $\alpha, \beta, \cdots$ and higher coefficients must be projective differential invariants in the sense of Sophus Lie; in fact, there is a similarity between our arguments and those in a modern theory of differential invariants due to Olver [47]. Besides, those leading parameters may have some particular geometric meanings. For instance, at a cusp of Gauss (that is a point of type $\Pi^p_{c,1}$ on the surface), the coefficient α of x^4 in the normal form coincides with the Uribe-Vargas cross-ratio invariant defined in [68]. Further, we will see that the coefficient of x^5 in the

class	normal form	p	s	proj.
$\Pi^h_{3,3}$	$xy + x^3 + y^3 + \alpha x^4 + \beta y^4$	4	0	$\mathrm{II}_3/\mathrm{II}_3$
$\Pi^h_{3,4}$	$xy + x^3 + y^4 + \alpha xy^3$	4	1	$\mathrm{II}_3/\mathrm{II}_4$
$\Pi^h_{3,5}$	$xy + x^3 + y^5 + \alpha xy^3 + x\phi_4$	5	2	$\mathrm{II}_3/\mathrm{II}_5$
$\Pi^h_{4,4}$	$xy + x^4 \pm y^4 + \alpha xy^3 + \beta x^3 y$	4	2	$\mathrm{II}_4/\mathrm{II}_4$
$\Pi^h_{3,6}$	$xy + x^3 + y^6 + \alpha xy^3 + x(\phi_4 + \phi_5)$	6	3	$\mathrm{II}_3/\mathrm{II}_6$
$\Pi^h_{4,5}$	$xy + x^4 + y^5 + \alpha xy^3 + \beta x^3 y + x\phi_4$	5	3	$\mathrm{II}_4/\mathrm{II}_5$
$\Pi^h_{3,7}$	$xy + x^3 + y^7 + \alpha xy^3 + x(\phi_4 + \phi_5 + \phi_6)$	7	4	$\mathrm{II}_3/\mathrm{II}_7$
$\Pi^h_{4,6}$	$xy + x^4 \pm y^6 + \alpha xy^3 + \beta x^3 y + x(\phi_4 + \phi_5)$	6	4	$\mathrm{II}_4/\mathrm{II}_6$
$\Pi^h_{5,5}$	$xy + x^5 + y^5 + \alpha xy^3 + \beta x^3 y + xy\phi_3$	5	4	$\mathrm{II}_5/\mathrm{II}_5$

Table 10: Hyperbolic case. There are two projection types with respect to distinct asymptotic lines. Here II_3 is of ordinary cusp.

class	normal form	p	s	proj.
Π^e	$x^2 + y^2$	2	0	fold
Π^f_1	$xy^2 \pm x^3 + \alpha x^3 y + \beta y^4$	4	3	$\mathrm{I}_2^{\pm\,\dagger}$
Π^f_2	$xy^2 + x^4 \pm y^4 + \alpha x^3 y$	4	4	$\mathrm{I}_2^{-\,\dagger}$

Table 11: Umbilical case (elliptic and flat umbilic cases). For $\Pi^e_{k,j}$ $(k, j \geq 4)$, we put $\phi = \beta x^3 y + \gamma xy^3$ for short. †: The projection type of Π^f_k $(k = 1, 2)$ are generically of type I_2, see §4.3.6 for the detail.

same Monge form corresponds to the position of a special viewpoint lying on the asymptotic line (§2.2). Also we discuss in the final section §4 a partial connection between our Monge forms and a topological classification of differential equations (BDE) defining nets of asymptotic curves (§1.3); indeed $\alpha, \beta, \cdots$ are related to initial moduli parameters of the BDE.

5.2 Proof of Theorem 5.1

5.2.1 Stratification by topological types of central projections

We briefly explain about stratification of Monge forms corresponding to topological classification of central projections. For parabolic types, the second column in Table 8 means the desired stratum that is defined by equations in the 1st and 4th column. For instance, a stratum of $\Pi^p_{c,2}$ is defined by $c_{30} = A = 0$, and the central projections from all viewpoints on the asymptotic line give III_2 $(= 11_5)$.

For hyperbolic types, we denote by $\Pi^h_{\ell,m}$ for $\ell, m \geq 3$ a stratum defined by

$$c_{30} = \cdots = c_{\ell-1\ 0} = c_{03} = \cdots = c_{0\ m-1} = 0, \quad c_{\ell 0}, c_{0m} \neq 0$$

for hyperbolic Monge forms $f(x,y) = xy + \sum_{i+j\geq 3} c_{ij} x^i y^j$. From Proposition 4.7, it is easily seen that central projections of a Monge form of type $\Pi^h_{\ell,m}$ from view points on the x-axis (or y-axis) give II_ℓ-type (resp. II_m-type).

Our aim in this chapter is to give plane normal forms via projective transformations to the above strata.

5.2.2 Projective equivalence

Let $0 \in \mathbb{R}^3 \subset \mathbb{P}^3$. A projective transformation on $\mathbb{P}^3$ preserving the origin and the xy-plane defines a diffeomorphism-germ $\Psi : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ of the form

$$\Psi(x,y,z) = \left(\frac{q_1(x,y,z)}{p(x,y,z)}, \frac{q_2(x,y,z)}{p(x,y,z)}, \frac{q_3(x,y,z)}{p(x,y,z)} \right)$$

where

$$q_1 = u_1 x + u_2 y + u_3 z, \quad q_2 = v_1 x + v_2 y + v_3 z, \quad q_3 = cz,$$

$$p = 1 + w_1 x + w_2 y + w_3 z.$$

Note that there are 10 independent parameters $u_1, \cdots, w_3$.

For Monge forms $z = f(x,y)$ and $z = g(x,y)$, we denote $j^k f \sim j^k g$ if k-jets of these surfaces at the origin are projectively equivalent. That is equivalent to that there is some projective transformation Ψ of the above form so that

$$F(x,y,g(x,y)) = o(k) \tag{1}$$

where $F(x,y,z) = q_3/p - f(q_1/p,\ q_2/p)$ and o means Landau's symbol.

We shall simplify jets of the Monge forms via projective transformations. Let $f(x,y) = \sum_{i+j\geq 2} c_{ij} x^i y^j$. For a simplified form $g(x,y)$, the equation (1) yields a number of algebraic equations in $u_1, \cdots, w_3$, and then our task is to find at least one solution in terms of c_{ij} (we use the software *Mathematica* for the computation).

5.3 Parabolic case

Suppose $c_{11} = c_{20} = 0$ and $c_{02} \neq 0$.

($\Pi^p_{\mathrm{I},k}$): Let $c_{30} \neq 0$, then we may assume that $j^3 f = y^2 + x^3$. Further,

$$j^4 f \sim y^2 + x^3 + c_{40}x^4 + c_{13}xy^3$$

by Ψ with

$$q_1 = x - \tfrac{1}{3}c_{22}z, \quad q_2 = y + \tfrac{1}{2}c_{31}z, \quad q_3 = z,$$
$$p = 1 + c_{31}y + (c_{04} + \tfrac{1}{4}c_{31}^2)z.$$

Let $j^4 f$ be of the form in the right hand side. For $c_{21} = c_{12} = 0$ in this form, we see that $C = 0$ and $D \neq 0$ where C and D are polynomials in previous section, thus the special viewpoint p is at the infinity. Then $\varphi_{p,f}$ is the parallel projection and

$$\varphi_{p,f} \sim_{\mathcal{A}} (y, x^3 + c_{40}x^4 + c_{13}xy^3 + h.o.t.).$$

For $c_{22} = 0$, we have $P = c_{13}$, $Q = c_{14}$ and $R = c_{15}$. Thus the conditions on P, Q and R lead to the normal forms of $\Pi^p_{\mathrm{I},k}$ $(1 \leq k \leq 4)$.

($\Pi^p_{c,k}$): Let $c_{30} = 0$ and $c_{21} \neq 0$, then we may assume $j^3 f = y^2 + x^2 y$. Then

$$j^4 f \sim y^2 + x^2y + c_{40}x^4 + c_{31}x^3y$$

by Ψ with

$$q_1 = x - \tfrac{1}{2}c_{13}z, \quad q_2 = y + c_{22}z, \quad q_3 = z,$$
$$p = 1 + 2c_{22}y + (c_{04} + c_{22}^2)z.$$

Furthermore, the term x^3y in the right hand side is killed[1] by a new Ψ with

$$q_1 = x - \tfrac{c_{31}}{4(1-c_{40})}y + \tfrac{c_{31}2}{32(1-c_{40})^2}z, \quad q_2 = y + \tfrac{3c_{31}^2}{8(1-c_{40})}z, \quad q_3 = z,$$

$$p = 1 - \tfrac{c_{31}}{2(1-c_{40})}y - \tfrac{c_{31}^2(12c_{40}-13)}{16(1-c_{40})^2}y - \tfrac{c_{31}^4(36c_{40}-37)}{256(1-c_{40})^3}z,$$

then we have the normal form of 4-jet:

$$\Pi^p_{c,1}: \quad y^2 + x^2y + c_{40}x^4.$$

As for the 5-jet, we define $\Pi^p_{c,2}$ by adding one more equation $B = 0$ and $\Pi^p_{c,3}$ by $A = B = 0$, where A, B are given in the previous subsection. Notice that the

[1] As an exception, if $c_{40} = 1$, we see by the same manner that x^3y can not be killed via any projective transformations, so the normal form is $y^2 + x^2y + x^4 + c_{31}x^3y$.

conditions are invariant under projective transformations. Therefore we may assume that $f = y^2 + x^2y + c_{40}x^4 + h.o.t.$, and then $B = 0$ implies $c_{40} = \frac{1}{4}$ and $A = c_{50}$. Hence we have the normal form for both classes:

$$y^2 + x^2y + \tfrac{1}{4}x^4 + c_{50}x^5 + y\phi_4 \ (= (y + x^2)^2 + c_{50}x^5 + y\phi_4).$$

Let $c_{30} = c_{40} = 0$ and $c_{21} \neq 0$, then we have $\Pi^p_{c,4}$ or $\Pi^p_{c,5}$ according to whether $c_{50} \neq 0$ or 0.

($\Pi^p_{v,k}$): Let $c_{30} = c_{21} = 0$, then we may assume that $j^3 f = y^2$. Further, if $c_{40} \neq 0$, we may assume $c_{40} = \pm 1$. Then we have the normal form

$$\Pi^p_{v,1} : \ \ y^2 \pm x^4 + \alpha x^3 y + \beta x^2 y^2$$

for some α, β by Ψ with $q_1 = x + u_1 y$, $q_2 = y$, $q_3 = z$ and

$$p = 1 + (\pm 1 + c_{13}u_1 + c_{22}u_1^2 + c_{31}u_1^3 \pm u_1^4)z,$$

where u_1 satisfies $c_{13} + 2c_{22}u_1 + 3c_{31}u_1^2 \pm 4u_1^3 = 0$. Let S be as in (VI) of the previous section. Then $\Pi^p_{v,2}$ is the case that

$$S = 3c_{02}c_{31}^2 + 8c_{40}(c_{12}^2 - c_{02}c_{22}) = 3\alpha^2 \pm 8\beta = 0.$$

If $c_{40} = 0$, we have $j^4 f = y^2 + y\phi_3$, that defines the class $\Pi^p_{v,3}$.

5.4 Hyperbolic case

Suppose $c_{11} \neq 0$ and $c_{20} = c_{02} = 0$.

($\Pi^h_{3,k}$): Let $c_{30}, c_{03} \neq 0$, then we may assume $j^3 f = xy + x^3 + y^3$. For the 4-jet, we obtain the normal form

$$\Pi^h_{3,3} : \ xy + x^3 + y^3 + \alpha x^4 + \beta y^4$$

by Ψ with $q_1 = x - \frac{c_{31}}{2}z$, $q_2 = y - \frac{c_{13}}{2}z$, $q_3 = z$ and $p = 1 - \frac{c_{13}}{2}x - \frac{c_{31}}{2}y + (c_{22} + \frac{c_{13}c_{31}}{4})z$.

Let $c_{30} = 0$ and $c_{03} \neq 0$. By a linear transformation, we may assume $j^3 f = xy + y^3$. Then

$$j^4 f \sim xy + y^3 + c_{40}x^4 + c_{31}x^3 y$$

by Ψ with

$$q_1 = x + c_{04}z, \ \ q_2 = y - \tfrac{1}{2}c_{13}z, \ \ q_3 = z,$$
$$p = 1 - \tfrac{1}{2}c_{13}x + c_{04}y + (c_{22} - \tfrac{1}{2}c_{13}c_{04})z.$$

If $c_{40} \neq 0$, we have

$$\Pi^h_{3,4} : \ xy + y^3 + x^4 + \alpha x^3 y.$$

If $c_{30} = c_{40} = 0$, we have the normal form of $\Pi^h_{3,k}$ ($k = 5, 6, 7$) as in Tables 9, 10, 11 according to whether c_{50} and/or c_{60} vanish or not.

($\Pi^h_{4,k}$): Let $c_{30} = c_{03} = 0$. Then

$$j^4 f \sim xy + c_{40}x^4 + \alpha x^3 y + \beta xy^3 + c_{04}y^4$$

by Ψ with

$$q_1 = x - c_{12}z, \quad q_2 = y - c_{21}z, \quad q_3 = z,$$

$$p = 1 + (c_{22} - 3c_{12}c_{21})z, \quad \text{with} \quad \alpha = c_{31} - c_{21}^2, \quad \beta = c_{13} - c_{12}^2.$$

This is the case that each of two asymptotic lines (x-axis and y-axis) has degenerate tangencies with the surface; the normal forms of $\Pi^h_{4,4}$, $\Pi^h_{4,5}$, $\Pi^h_{4,6}$ and $\Pi^h_{5,5}$ are immediately obtained.

5.5 Flat umbilical case

Suppose $c_{11} = c_{20} = c_{02} = 0$. Cubic forms of variables x, y are classified into $GL(2)$-orbits of $xy^2 \pm x^3$, xy^2, y^3 and 0. Let D be the discriminant of the cubic form:

$$D = c_{12}^2 c_{21}^2 - 4c_{03}c_{21}^3 - 4c_{12}^3 c_{30} + 18c_{03}c_{12}c_{21}c_{30} - 27c_{03}^2 c_{30}^2.$$

(Π^f): If $D \neq 0$, then we may assume $j^3 f = x^2 y \pm y^3$. Then the 4-jet can be transformed to

$$\Pi^f_1 : \; xy^2 \pm x^3 + \alpha x^3 y + \beta y^4$$

by Ψ with $q_1 = x$, $q_2 = y$, $q_3 = z$, $p = 1 + \frac{1}{2}c_{22}x + \frac{1}{2}c_{13}y$. Here $\alpha = -c_{13} + c_{31}$ and $\beta = c_{04}$. We may take either of α or β to be 1 unless both are 0.

If $D = 0$, then $j^3 f \sim x^2 y$ by some linear change. Furthermore, we may assume c_{40}, $c_{04}(= \beta) \neq 0$ generically. Then by the same Ψ above and a linear change, we have

$$\Pi^f_2 : \; xy^2 + x^4 \pm y^4 + \alpha x^3 y.$$

6 Parabolic curve and flecnodal curve on surface

In this chapter we consider bifurcations of *parabolic and flecnodal curves* (or nets of *asymptotic curves*). A hyperbolic point is said to be *flecnodal* if one asymptotic line admits more than 3-point contact with the surface. For a *generic* surface, the parabolic curve is the closure of the locus of points whose Monge form is of type $\Pi^p_{\mathrm{I},1}$, and the flecnodal curve is actually the closure of the locus of class $\Pi^h_{3,4}$; the parabolic and flecnodal curves meet each other tangentially at an ordinary cusp of Gauss, i.e. the class $\Pi^p_{c,1}$ [3, 6, 50, 33]. For a non-generic surface belonging to a family, these curves bifurcate as the parameter varies. Notice that our normal forms in Tables 9, 10 of codimension grater than 2 represent surface germs at transition moments of bifurcations. Our aim in this chapter is to compare normal forms in Tables 9, 10 with the following two different sort of results on parameter families of surfaces: First we consider asymptotic BDE which provides information about nets of asymptotic curves and parabolic curves. We determine the local diffeomorphic or topological types of asymptotic BDE for all normal forms in our tables, based on local classification results for BDE in [11, 12, 13, 14, 16, 17, 18, 63, 65]. However, the theory of general BDE is useless for analyzing the flecnodal curves, since the inflection points of integral curves (asymptotic curves) are not invariant under diffeomerphisms; the curves are more rigid invariant under projective transformations. Nevertheless, normal forms can provide certain geometric information of the flecnodal curves. We compare our result with a classification of Uribe-Vargas on local 1-parameter bifurcation of parabolic and flecnodal curves.

This chapter is based on joint works with H. Sano, J. L. Deolindo Silva, and T. Ohmoto [59, 20].

6.1 BDE of asymptotic curves

In this section, we establish a relation between normal forms in Theorem 5.1 and local bifurcations of nets of asymptotic curves. These curves are defined by a differential equation of particular form on the surface, which had been studied in classical literatures (e.g. [67, 72]). Here we take a modern approach from dynamical system. First we recall some basic notions in a general setting. A *binary differential equation* (BDE) in two variables x, y has the form

$$F: \quad a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0$$

with smooth functions a, b, c of x, y. We can regard a BDE as a map $\mathbb{R}^2 \to \mathbb{R}^3$ assigning $(x,y) \mapsto (a,b,c)$ and consider the C^∞-topology. A BDE defines a pair of directions at each point (x,y) in the plane. Put $\delta(x,y) := b(x,y)^2 - a(x,y)c(x,y)$. If $\delta > 0$, the BDE locally defines two transverse foliations. *The discriminant curve* of the BDE is defined by $\delta = 0$. At any point of the curve, the direction defined by BDE is unique, and the integral curve of BDE passing the point generically has a cusp. Two germs of BDEs F and G are equivalent if there is a local diffeomorphism in the xy-plane sending the integral curves of

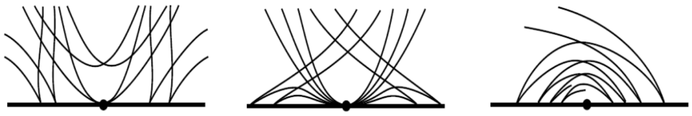

Figure 1: Folded singularities: saddle (left), node (center) and focus (right)

F to those of G. In the similar way, topological equivalence is defined. There have been known several classification results for germs of (families of) BDE [11, 12, 13, 14, 16, 17, 18, 63, 65].

One can separate BDE into two cases. The first case occurs when the functions a, b, c do not vanish at the origin at once, then the BDE is just an implicit differential equation (IDE). The second case is that all the coefficients of BDE vanish at the origin. Stable topological models of BDEs belong to the first case; it arises when the discriminant is smooth (or empty). If the unique direction at any point of the discriminant is transverse to it (i.e. integral curves form a family of cusps), then the BDE is stable and smoothly equivalent to $dy^2 + xdx^2 = 0$, that was classically known in Cibrario [14] and also Dara [16]. If the unique direction is tangent to the discriminant, then the BDE is stable and smoothly equivalent to $dy^2 + (-y + \lambda x^2)dx^2 = 0$ with $\lambda \neq 0, \frac{1}{16}$, that was shown in Davydov [17, 18]; the corresponding point in the plane is called a *folded singularity* – more precisely, a *folded saddle* if $\lambda < 0$, a *folded node* if $0 < \lambda < \frac{1}{16}$ and a *folded focus* if $\frac{1}{16} < \lambda$, see Figure 1.

In both cases, the topological classification of generic 1 and 2-parameter families of BDEs have been established in Bruce-Fletcher-Tari [11, 13] and Tari [63, 65], respectively. We will use those results later. Besides, we need a generic 3-parameter family of BDE studied in Oliver [46].

Consider the surface locally given by Monge form $(x, y, f(x, y))$ around the origin. Then asymptotic lines are defined by

$$f_{yy}dy^2 + 2f_{xy}dxdy + f_{xx}dx^2 = 0,$$

which we call *asymptotic BDE* for short. The discriminant curve is the same as the parabolic curve. We first discuss bifurcations of asymptotic BDE around parabolic and flat umbilical points.

Note that asymptotic BDEs form a particular class of general BDEs; the classification of asymptotic BDE is intimately related to the singularity type of the height function (i.e. the Monge form). The parabolic curve can be seen as the locus where the height function has $A_{\geq 2}$-singularities; when the height function has a $A_3^{\pm}$-singularity, the surface has an ordinary cusp of Gauss, which corresponds to the type $\Pi^p_{c,1}$ – in this case the asymptotic BDE has a folded saddle singularity (A_3^-) and has a folded node or focus singularity (A_3^+). The transitions in 1-parameter families occur generically in three ways at the following singularities of the height function: non-versal A_3, A_4 and D_4 (flat umbilical) [13]. For 2-parameter families, A_3, A_4, A_5 and D_5 singularities of the height function generically appear.

Combining these results with our Theorem 5.1, the following propositions are immediately obtained. Below, the Monge form is written by

$$f(x, y) = \sum_{2 \le i+j \le p} c_{ij}\, x^i y^j + o(p).$$

Proposition 6.1 *The following classes in Table 9 correspond to structurally stable types of BDE given in [14, 16, 17].*

1. $\Pi^p_{\mathrm{I},k}$ $(1 \le k \le 4)$: *the parabolic curve is smooth and the asymptotic direction on it is transverse to the curve; the asymptotic BDE is smoothly equivalent to*
$$dy^2 + x\, dx^2 = 0.$$

2. $\Pi^p_{c,1}$, $\Pi^p_{c,4}$, $\Pi^p_{c,5}$: *the parabolic curve is smooth and the asymptotic direction on it is transverse to the curve; the asymptotic BDE is smoothly equivalent to*
$$dy^2 + (-y + \lambda x^2)dx^2 = 0$$
with $\lambda = 6c_{40} - \frac{3}{2} \ne 0$, *where* c_{40} *is the coefficient of* x^4 *in the normal form.*

Proof : The results follow from the comments in the above. In case 2, i.e., $j^4 f = y^2 + x^2 y + c_{40} x^4$, the 2-jet of the asymptotic BDE is transformed to the above form via $x = \bar{x}$ and $y = -\frac{1}{2}\bar{x}^2 - \bar{y}$. □

Remark 6.2 As $c_{40} = 0$ for $\Pi^p_{c,4}$ and $\Pi^p_{c,5}$, we see $\lambda = -\frac{3}{2} < 0$, thus the asymptotic BDE has a folded saddle at the origin. The *folded saddle-node bifurcation* (cf. Fig.2 in [63]) occurs at $\lambda = 0$. That is the case of $c_{40} = \frac{1}{4}$, that corresponds to the classes $\Pi^p_{c,k}$ $(k = 2, 3)$ dealt below. However, another exceptional value $\lambda = \frac{1}{16}$ does not relate to our classification of Monge forms given by projection-types (Table 9). That is, the *folded node-forcus bifurcation* of asymptotic BDE occurs within the same class $\Pi^p_{c,1}$ (cf. Fig.3 in [63]) and $\lambda = \frac{1}{16}$ makes a condition on coefficients of order greater than 4 of the normal form, that is independent from the geometry of central projection of the surface.

Proposition 6.3 *The following classes correspond to some topological types of BDE with codimension* 1.

1. $\Pi^p_{v,1}$: *the Monge form has an* A_3*-singularity at the origin, at which the parabolic curve has a Morse singularity; the asymptotic BDE is topologically equivalent to the non-versal* $A_3^{\pm}$*-transitions with Morse type 1 in [13]*
$$dy^2 + (\pm x^2 \pm y^2)dx^2 = 0.$$

2. $\Pi^p_{c,2}$: *the Monge form has an A_4-singularity at the origin, at which the parabolic curve is smooth; the asymptotic BDE is topologically equivalent to the well-folded saddle-node type in [13, 18]*

$$dy^2 + (-y + x^3)dx^2 = 0,$$

provided $c_{50} \neq 0$ where c_{50} is the coefficient of x^5 in the normal form .

3. Π^f_1 : *the Monge form has a $D^\pm_4$-singularity at the origin, at which the parabolic curve has a Morse singularity; the asymptotic BDE is topologically equivalent to the bifurcation of star/1-saddle types in [11]*

$$\begin{array}{ll} D_4^+ : & ydy^2 - 2xdxdy - ydx^2 = 0 \ \text{(star)}; \\ D_4^- : & ydy^2 + 2xdxdy + ydx^2 = 0 \ \text{(1-saddle)}. \end{array}$$

Proof : For the normal form of $\Pi^p_{v,1}$ in Table 9, we see

$$c_{02} = 1, \ c_{40} \neq 0, \ c_{30} = c_{21} = 0, \ S = 3c_{02}c_{31}^2 + 8c_{40}(c_{12}^2 - c_{02}c_{22}) \neq 0.$$

In fact, these conditions characterize the class $\Pi^p_{v,1}$. Exactly the same conditions appear in [13, p.501, Case 1] as the condition for $A^\pm_3$-transition: $S \neq 0$ means the 2-jet $j^2\delta(0)$ is non-degenrate, thus the normal form follows from Theorem 2.7 (and Prop. 4.1) in [13]. For $\Pi^p_{c,2}$, coefficients of the normal form in Table 9 satisfy the condition for A_4-transition in [13, p.502, Case 2]:

$$c_{30} = 0, \ c_{21}^2 - 4c_{40} = 0, \ 4c_{50} - 2c_{31}c_{21} - c_{12}c_{21}^2 \neq 0.$$

Then the normal form of BDE is obtained ([13, Prop. 4.2], also see [18]). For the class Π^f_1, the asymptotic BDE is given in [11, Cor. 5.3]: indeed, for our normal form of Π^f_1, the parabolic curve is defined by $3x^2 - y^2 + 18\beta xy^2 + \cdots = 0$, hence it has a node at the origin for arbitrary $c_{31} = \alpha, c_{04} = \beta$. □

Remark 6.4 During the transition at a point of $\Pi^p_{v,1}$, the Morse bifurcation of the parabolic curve happens; in case of elliptic Morse bifurcation, a disc domain of hyperbolic points and two cusps of Gauss points are created (or disappear). Hence, a new component of the flecnodal curve must be created (or disappear) in the new disc of hyperbolic points, and it is tangent to the boundary of the disc (parabolic curve) at those cusps of Gauss points. Unexpectedly, the flecnodal curve has the form of figure eight (i.e. it has one self-intersection point), that was firstly discovered by F. Aicardi (see Uribe-Vargas [68, 69]). Proposition 6.3 (1) does not help for understanding the appearance of figure eight, because the equivalence of BDE does not preserve inflections of integral curves. The bifurcation of flecnodal curve is dealt in §6.2 with Uribe-Vargas's classification.

Proposition 6.5 *The following classes correspond to some topological types of BDE with codimension ≥ 2. Below, for each class, $\lambda_i \neq 0$ gives an open condition of moduli parameters c_{ij} of the normal form in Table 9.*

1. $\Pi^p_{v,2}$: *the Monge form has an A_3-singularity at the origin, at which the parabolic curve has a cusp singularity; the asymptotic BDE is topologically equivalent to the cusp type in [63]*

$$dy^2 + (\pm x^2 + y^3)dx^2 = 0,$$

provided $\lambda_1 := \pm 5c_{50}c_{31}^3 + 12c_{41}c_{31}^2 \pm 24c_{32}c_{31} + 32c_{23} \neq 0$.

2. $\Pi^p_{v,3}$: *the Monge form has an A_4-singularity at the origin, at which the parabolic curve has a Morse singularity; the asymptotic BDE is topologically equivalent to the non-transversal Morse type in [63]*

$$dy^2 + (xy + x^3)dx^2 = 0$$

provided $\lambda_2 := c_{31} \neq 0$.

3. $\Pi^p_{c,3}$: *the Monge form has an A_5-singularity at the origin, at which the parabolic curve is smooth; the asymptotic BDE is topologically equivalent to the folded degenerate elementary type in [63]*

$$dy^2 + (-y \pm x^4)dx^2 = 0,$$

provided $\lambda_3 := c_{60} - \frac{1}{2}c_{41} \neq 0$.

4. Π^f_2 : *the Monge form has a D_5-singularity at the origin, at which the parabolic curve has a cusp singularity; the asymptotic BDE is topologically equivalent to a cusp type 2 in [46]*

$$x^2dx^2 + 2ydxdy + xdy^2 = 0.$$

Proof : For each of the first three classes, it is easy to check that coefficients of the normal form in Table 9 satisfy the condition for general BDE with 2-parameters in Proposition 4.1 of [63]; the conditions $\lambda_1, \lambda_2, \lambda_3 \neq 0$ are found in [63, p.156], and the normal form of BDE are given in Theorem 1.1 of [63]. For the last class Π^f_2, the 1-jet of the asymptotic BDE is given by $j^1(a, b, c)(0) = (2x, 2y, 0)$ and the parabolic curve is defined by $-4y^2 + 24x^3 + \cdots = 0$, namely it has a cusp at the origin. Thus the corresponding BDE is equivalent to one of types described in Theorem 3.4 of [46]. □

Remark 6.6 In Oliver [46], BDE with the discriminant having a cusp are classified up to codimension 4, that generalizes a result for cusp types of codimension 2 in [65]. It is remarkable that the BDE for Π^f_2 in Proposition 6.5 is one of general BDE with codimension 5 in the space of all jets $j^p(a, b, c)(0)$ of BDE, while the stratum Π^f_2 has codimension 4 in the space of all jets $j^pf(0)$ of Monge forms.

6.2 Bifurcations of parabolic and flecnodal curves

Smooth and topological classifications of BDE do miss the geometry of flecnodal curves, although it is quite fruitful. In part of his dissertation [69] Uribe-Vargas presented 14 types of generic 1-parameter bifurcations of parabolic and flecnodal curves and of special elliptic points on the surface. That was done by a nice geometric argument using projective duality and BDE, however explicit normal forms are not given there. Thus we try to detect substrata in our classification of Monge forms which correspond to the transition moments of Uribe-Vargas's types.

Each type defines an invariant open subset of one of our class of codimension 3, or a proper invariant subset of a class of less codimension, which should be characterized by some particular relation of moduli parameters in the normal form. In fact we have already seen 8 types of degenerate parabolic and flat umbilical points in Uribe-Vargas's list as in all cases in Proposition 6.3 and an exceptional case in Proposition 6.1:

- $\Pi^p_{v,1}$ (4 types w.r.t. $\pm x^4$ and $\beta \lessgtr \frac{3}{8}\alpha^2$ in the normal form);
- $\Pi^p_{c,2}$ (the folded saddle-node type);
- Π^f_1 (star type and 1-saddle type);
- $\Pi^p_{c,4}$ (degenerate cusp of Gauss).

Remark 6.7 The corresponding strata in jet space have codimension 3. Take a generic family of surfaces passing through each stratum. Around a point of class $\Pi^p_{v,1}$, Morse bifurcations of the parabolic curve happen, while bifurcations of the flecnodal curve can not be analyzed from the classification of BDE in [13]. Such bifurcations have been determined completely in [69] and can also be checked by direct computations from our Monge forms. For $\Pi^p_{c,2}$, two folded saddle points are cancelled at this point, called in [13, Prop. 4.2] the folded saddle-node bifurcation. For Π^f_1, there are two types of degenerate flat umbilic points. For $\Pi^p_{c,4}$, parabolic and flecnodal curves meet tangentially, and a butterfly point comes across at the cusp of Gauss point. As an additional remark, for $\Pi^p_{c,1}$ with $\lambda = \frac{1}{16}$, the folded node-focus bifurcation of BDE appears as mentioned in Remark 6.2, but the parabolic and flecnodal curves do not bifurcate, so this case is not included.

The rest are divided into 5 types occurring at hyperbolic points and 1 type at an elliptic point; the normal forms for degenerate hyperbolic points are given below:

- $\Pi^h_{3,6}$ (degenerate butterfly type);
- $\Pi^h_{4,5}$ (swallowtail+butterfly);
- $\Pi^h_{3,5}$ with $\alpha = 0$ (Morse bifurcation; 'lips' and 'bec-à-bec');

- $\Pi^h_{4,4}$ with $\alpha\beta = 16$ (tacnode bifurcation).

In Table 10 (hyperbolic Monge forms), there are two strata of codimension 3. For the normal form of $\Pi^h_{3,6}$, the flecnodal curve is smooth, and the y-axis is tangent to the surface with 6-point contact. Through a generic bifurcation, two butterfly points ($\Pi^h_{3,5}$) are cancelled (or created) at this point. For $\Pi^h_{4,5}$, there are two irreducible components of the flecnodal curve, one of which has a butterfly at the origin. It bifurcates into one point of $\Pi^h_{4,4}$ and one butterfly point.

In strata of codimension 2, generic bifurcations of flecnodal curves may occur at some particular values of parameters appearing in the normal forms. First consider the class $\Pi^h_{3,k}$ ($k \geq 4$): $f(x,y) = xy + x^3 + \sum_{i+j\geq 4} c_{ij}\, x^i y^j$ ($\alpha = c_{13}$ in Table 10). We are now viewing the surface along lines close to the y-axis, so take parallel projection $\varphi : (x,y) \mapsto (x-uy, f(x,y)-vy)$. The flecnodal curve is just the locus of singular points (x,y) at which the projection for some (u,v) has the swallowtail singularities or more degenerate ones. Put $\lambda := \det d\varphi$ and $\eta := u\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ so that η spans the kernel of $d\varphi$ over $\lambda = 0$. Then the locus is defined to be the image via projection $(x,y,u,v) \mapsto (x,y)$ of curves given by three equations $\lambda = \eta\lambda = \eta\eta\lambda = 0$ (cf. [31]). By $\lambda = 0$ we eliminate v, and by $\eta\lambda = 0$, we solve u by implicit function theorem: $u = -c_{22}x^2 - 3c_{13}xy - 6c_{04}y^2 + o(2)$. Substitute them into $\eta\eta\lambda = 0$; we have the expansion of the equation of flecnodal curve:

$$c_{13}x + 4c_{04}y + c_{23}x^2 + 4c_{14}xy + 10c_{05}y^2 + h.o.t. = 0.$$

The curve is not smooth at the origin when $c_{13} = c_{04} = 0$, and it has a Morse singularity if $5c_{23}c_{05} - 2c_{14}^2 \neq 0$. Thus, we conclude that Morse bifurcations of flecnodal curves occur at a point of the class $\Pi^h_{3,5}$ with $\alpha = 0$ and generic homogeneous terms of order 5 (those are regarded as normal forms for 'lips' and 'bec-à-bec' in [69]).

Next, consider the class $\Pi^h_{4,4}$: $f(x,y) = xy + x^4 + y^4 + \alpha xy^3 + \beta x^3 y + o(4)$. In entirely the same way as seen above, for each of projections $\varphi_L : (x,y) \mapsto (x-uy, f(x,y)-vy)$ and $\varphi_R : (x,y) \mapsto (y-ux, f(x,y)-vx)$, we take λ and η, and solve $\lambda = \eta\lambda = \eta\eta\lambda = 0$. We then obtain the equation of each component of the flecnodal curve:

$$\alpha x + 4y + h.o.t = 0, \quad 4x + \beta y + h.o.t = 0.$$

Thus the tacnode bifurcation (tangency of two components) occurs at a point of the class $\Pi^h_{4,4}$ with $\alpha\beta = 16$. The condition exactly coincides with the one in Table 3.2 in Landis [33].

7 Projection of Crosscap

This chapter deal with the flat or affine differential geometry of singular surfaces with crosscaps in 3-space in relation with central projections from arbitrary viewpoints, that generalizes the study on singularities of projections of smooth surfaces as in the previous chapters.

7.1 Central projection of crosscaps

We quickly review a classification of germs of corank 2 [24, 58]. In [58] it is shown that a finitely $\mathcal{A}$-determined germ lying on the $\mathcal{K}$-orbits $I_{2,2}$ and $II_{2,2}$ is one of the following types

$$I_{2,2}^{\ell,m} : (x^2 + y^{2\ell+1}, y^2 + x^{2m+1}), \quad II_{2,2}^{n} : (x^2 - y^2 + x^{2n+1}, xy),$$

with $\mathcal{A}_e$-codimension $\ell + m$ and $2(n+1)$, respectively. They are $\mathcal{A}$-*simple orbits* of corank two, i.e., the cardinality of nearby $\mathcal{A}$-orbits is finite. Especially, $I_{2,2}^{1,1}$ and $II_{2,2}^{1}$ are called the sharksfin and the deltoid, respectively (named by Gibson-Hobbs [24]). The type next to the sharksfin and the deltoid is $I_{2,2}^{2,1}$ of $\mathcal{A}_e$-codimension 3. We call it the *odd-shaped sharksfin* throughout this paper. A generic 3-parameter family may meet the singularity type $I_{2,3} : (x^2, y^3)$, but it is not $\mathcal{A}$-simple: Indeed the $\mathcal{K}$-orbit $I_{2,3}$ is formed by a family of $\mathcal{A}$-orbits with the modality greater than one [58, Lem.2.3.3].

J. West [70] considered singularities of the parallel projection of a singular surface with crosscap: it is proved that when viewing the crosscap along its image tangent line, *the sharksfin* and *the deltoid* occur in general. We extend West's result to the case of central projection of crosscaps as follows:

Theorem 7.1 *For a generic crosscap at $x_0 \in M$ of a smooth map $\iota : M \to \mathbb{P}^3$, it holds that for arbitrary point p_0 lying on the image line of $d\iota(x_0)$ in the ambient space $\mathbb{P}^3$, the germ at x_0 of the central projection $\pi_{p_0} \circ \iota : M \to \mathbb{P}^2$ is $\mathcal{A}$-equivalent to either of sharksfin or deltoid.*

The proof will be given in §7.2. Note that it is not quite obvious that the odd-shaped sharksfin does not appear in general, that is the main point of the above theorem. Also the singularity type of $I_{2,3}$ does not appear, although it has codimension 5 in $J(2,2)$. This case is obvious: indeed a singularity of type $I_{2,3}$ occurs in projections when one views a *parabolic* crosscap, but it is not generic in the space of map-germs with crosscaps (see [74]).

Differential geometry of crosscaps has been studied by several authors, e.g., [70, 28, 29, 64, 48], while the counterpart in affine differential geometry has been less taken attention. A common particular feature is the *parabolic curve*: It is shown in West [70] that the parabolic curve does not approach to a hyperbolic crosscap point, while there are two smooth components of the curve passing through an elliptic crosscap point. Another important characteristic in affine differential geometry is the *flecnodal curve*. We show the existence of the flecnodal curve near an elliptic crosscap, like as the parabolic curve just as mentioned. In fact, the *order of contact of these two curves* is a new invariant which exactly characterizes *generic* elliptic crosscaps in Theorem 7.1.

Theorem 7.2 *The flecnodal curve approaches to any elliptic crosscap; it has two smooth components passing through the crosscap point. In the source space parametrizing the crosscap at $x_0 \in M$, each component of the flecnodal curve is tangent to a component of the parabolic curve at x_0 with odd contact order; in particular, both pairs of components of these curves have* 3*-point contact if and only if the crosscap is generic in the sense of Theorem 7.1, i.e., the projection of the crosscap along the image tangent line is of type sharksfin. The flecnodal curve does not approach to any hyperbolic crosscap.*

7.2 Proof of Theorem 7.1

First we consider the action of the subgroup with linear target changes $\mathrm{Diff}(\mathbb{R}^2, 0) \times GL_3 \subset \mathcal{A}_{2,3}$ on the ℓ-jet space $J^\ell(2, 3)$.

Let $f : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ be a germ of crosscap. Since f is of corank 1 and the 1-jet extension is transverse to Σ^1, by some linear change of the source and the target we may assume

$$f(x, y) = (x, xy + O(3), y^2 + \alpha x^2 + O(3))$$

where $\alpha = 1$, $\alpha = -1$ or $\alpha = 0$; We say that the crosscap f is *elliptic*, *hyperbolic* or *parabolic*, respectively ([70, 64]). By a further coordinate change of the source and a linear change of the target, we may write

$$f(x, y) = (x, xy + g(y), y^2 + \alpha x^2 + \phi(x, y)) + O(\ell + 1) \qquad (2)$$

for some $g(y) = \sum_{i\geq 4} d_i y^i$ and $\phi(x, y) = \sum_{i+j\geq 3} c_{ij} x^i y^j$ (cf. [70]). We call it the *affine normal form* of crosscap.

Let $\iota : M \to \mathbb{R}^3$ be a smooth map having a crosscap at $x_0 \in M$ with $\iota(x_0) \neq 0$, and $U \subset \mathbb{R}^3 - M$ a small neighborhood of the origin. We are concerned with the family of central projections

$$\varphi : M \times U \to \mathbb{R}P^2, \;\; \varphi(x, p) = \varphi_p(x) := \pi_p \circ \iota(x),$$

where $p = (u, v, w) \in U$ is a viewpoint. Of our particular interest is its corank 2 singularity.

Taking a local coordinates of M centered at x_0 and an affine transform of $\mathbb{R}^3$, we may assume that $\iota(x_0) = (1, 0, 0)$ and ι is locally written by

$$\iota(x, y) = \left(1 + x, xy + g(y), y^2 + \alpha x^2 + \phi(x, y)\right),$$

using the affine normal form (2); we have

$$\varphi(x, y, u, v, w) = \left(\frac{xy + g(y) - v}{1 + x - u}, \frac{y^2 + \alpha x^2 + \phi(x, y) - w}{1 + x - u}\right).$$

Clearly, $\varphi_p(x,y)$ has a corank 2 singularity if and only if the viewpoint p lies on the u-axis, so we now assume that $v=w=0$. Then, coefficients of the Taylor expansion of $\varphi_p(x,y)$ at the origin are expressed in terms of u, d_j, c_{ij}.

Let $\alpha=0$, i.e., take a parabolic crosscap; this yields a closed condition on the space of jets of all crosscap-germs ι, so it does not generically occur.

On the other hand, elliptic and hyperbolic crosscaps are generic. Let $\alpha=+1$. Substitute $x=\bar{x}+\bar{y}$ and $y=\bar{x}-\bar{y}$, then by some coordinate change of the source and a linear change of the target, we have

$$\varphi_p(x,y)=\left(x^2+\frac{A}{4(1-u)^2}y^3+O(4), y^2+\frac{B}{4(1-u)^2}x^3+O(4)\right)$$

where

$$A=c_{03}-c_{12}+c_{21}-c_{30},\ B=c_{03}+c_{12}+c_{21}+c_{30}.$$

Both A and B are not equal to zero if and only if $\varphi_p\sim_{\mathcal{A}}(x^2+y^3, y^2+x^3)$. $A=0$ or $B=0$ gives a relation among coefficients c_{ij}'s, that causes a closed condition on the space of jets of crosscap-germs ι, thus such a case does not generically occur. In case that $\alpha=-1$, we can easily see that $\varphi_p\sim_{\mathcal{A}}(xy, x^2-y^2+y^3)$ as long as $c_{03}\neq 0$.

This completes the proof of Theorem 7.1.

7.3 Affine geometry of crosscap

We deal with the affine geometry of crosscap given by the normal form (2) from singularity theory approach; we are concerned with two characteristic curves on a singular surface $\iota(M)\subset\mathbb{R}^3$ with a crosscap – the parabolic curve and the flecnodal curve. We prove Theorem 7.2, and show that these curves actually lie on the surface like as Fig.2.

As mentioned before, the parabolic curve and the flecnodal curve are characterized as the loci at which the parallel projection along asymptotic lines admit beaks and swallowtail singularity types, respectively. These curves do not approach to any hyperbolic crosscap, since the singularity of the projection along the tangent line at the crosscap is of type $II_{2,2}$ which has no adjacencies of beaks and swallowtail types. Thus we consider elliptic crosscaps from now on. Let $\varphi: M\times U\to\mathbb{R}^2$ be the parallel projection:

$$\varphi(x,y,v,w)=\big(xy-vx+g(y), x^2+y^2-wx+\phi(x,y)\big)$$

where $g(y)=d_4y^4+\cdots$ and $\phi(x,y)=c_{30}x^3+c_{21}x^2y+c_{12}xy^2+c_{03}y^3+\cdots$.

7.3.1 Parabolic curve

Put $\lambda=\det\varphi_{v,w}$, then the beaks singularity is characterized by three equations, $\lambda=\frac{\partial}{\partial x}\lambda=\frac{\partial}{\partial y}\lambda=0$. They provide an equation in x,y by eliminating v,w, which is the defining equation of the parabolic curve in the source xy-space:

$$\begin{array}{l} x^2 - y^2 + (3c_{03} + 2c_{21})x^2y + (c_{12} + 3c_{30})x^3 - 3c_{03}y^3 \\ +(-\frac{1}{4}c_{21}^2 + c_{22} + 3c_{12}c_{30} + 6c_{40})x^4 + (3c_{13} + c_{12}c_{21} + 9c_{03}c_{30} + 5c_{31})x^3y \\ +(6c_{04} + \frac{9}{2}c_{03}c_{21} + 3c_{22})x^2y^2 - 4d_4xy^3 - (\frac{9}{4}c_{03}^2 + 4c_{04})y^4 + h.o.t = 0. \end{array}$$

An alternative way is to compute the second fundamental form of the surface with respect to a fixed Euclidean metric; the equation $LN - M^2 = 0$ (the second fundamental form) gives the same answer. This curve has a nodal point at the origin and two analytic branches are expressed by

$$\begin{array}{rcl} y & = & y_{p,\pm}(x) \\ & = & \pm x + \frac{1}{2}(\pm c_{12} + 2c_{21} \pm 3c_{30})x^2 \\ & & +\frac{1}{8}\left(\begin{array}{l} \mp 9c_{03}^2 - 12c_{03}c_{12} \mp c_{12}^2 \mp 6c_{03}c_{12} \\ +4c_{12}c_{21} \pm 3c_{21}^2 \pm 6c_{12}c_{30} \mp 9c_{30}^2 \\ \pm 8c_{04} + 12c_{13} \pm 16c_{22} + 20c_{31} \pm 24c_{40} - 16d_4 \end{array}\right) x^3 + O(4). \end{array}$$

Indeed, the expansion up to order 3 is determined by the 4-jet of the above equation, and this is verified by substitution.

7.3.2 Flecnodal curve

Put $\eta = x\frac{\partial}{\partial x} - (y - v)\frac{\partial}{\partial y}$. The swallowtail singularity is characterized by three equations $\lambda = \eta\lambda = \eta\eta\lambda = 0$. It is however hard to obtain an equation in x, y from these equations by eliminating v, w. On the other hand, by a similar argument in the proof of Theorem 5.1 (also [34, 45]), we see that the solution in $xyvw$-space defines two smooth curve-germs at the origin. Then, from a bit long computation for power series solutions, we obtain the following expression of two branches in xy-plane:

$$\begin{array}{rcl} y & = & y_{i,\pm}(x) \\ & = & \pm x + \frac{1}{2}(\pm c_{12} + 2c_{21} \pm 3c_{30})x^2 \\ & & +\frac{1}{4}\left(\begin{array}{l} \mp 4c_{03}^2 - 5c_{03}c_{12} \mp 2c_{03}c_{21} + 3c_{12}c_{21} \\ \pm 2c_{21}^2 + c_{03}c_{30} \pm 4c_{12}c_{30} + c_{21}c_{30} \mp 4c_{30}^2 \\ \pm 4c_{04} + 6c_{13} \pm 8c_{22} + 10c_{31} \pm 12c_{40} - 8d_4 \end{array}\right) x^3 + O(4). \end{array}$$

Also v, w are expressed in x as

$$\begin{array}{rcl} v & = & \mp 2x + \frac{1}{2}(-2c_{03} \mp c_{12} + 4c_{21} \mp 7c_{30})x^2 \\ & & +\left(\begin{array}{l} -2c_{03}c_{12} \pm 3c_{03}c_{21} + c_{12}c_{21} \mp c_{21}^2 \\ -3c_{03}c_{30} \mp c_{12}c_{30} \pm 3c_{30}^2 \\ +2c_{13} \mp 4c_{22} + 6c_{31} \mp 8c_{40} + \frac{4}{3}d_4 \end{array}\right) x^3 + O(4) \\ w & = & 4x + (\mp c_{03} + 4c_{12} \mp 7c_{21} + 10c_{30})x^2 \\ & & +\left(\begin{array}{l} -3c_{03}^2 \pm 4c_{03}c_{12} + 2c_{12}^2 \mp 10c_{12}c_{21} + 7c_{21}^2 \\ \mp 6c_{03}c_{30} + 14c_{12}c_{30} \mp 12c_{21}c_{30} \\ +4c_{04} \mp 8c_{13} + 12c_{22} + \mp 16c_{31} \mp 20c_{40} \pm \frac{16}{3}d_4 \end{array}\right) x^3 + O(4), \end{array}$$

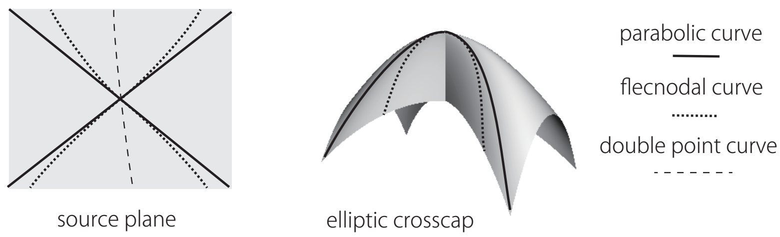

Figure 2: Two characteristic curves on a generic elliptic crosscap

where the double sign corresponds to the double sign in y above. A simple substitution verifies that the above parametrizations of y, v, w in x satisfy the defining equations $\lambda = \eta\lambda = \eta\eta\lambda = 0$ (up to 3-jets).

7.3.3 Proof of Theorem 7.2

Compare the flecnodal curve $y = y_{i,\pm}(x)$ and the parabolic curve $y = y_{p,\pm}(x)$ in xy-plane; they coincide up to degree 2, and the difference of cubic terms are surprisingly simplified:

$$y_{p,-} - y_{i,-} = \tfrac{1}{8}A^2x^3 + O(4), \quad y_{p,+} - y_{i,+} = -\tfrac{1}{8}B^2x^3 + O(4)$$

where $A = c_{03} - c_{12} + c_{21} - c_{30}$ and $B = c_{03} + c_{12} + c_{21} + c_{30}$. Thus from the proof of Theorem 7.1 given in the previous subsection, we see that the following properties are equivalent: (1) in the source plane parametrizing the crosscap, these smooth branches of the parabolic curve and the flecnodal curve have 3-point contact; (2) both A and B are not zero; (3) the elliptic crosscap is generic, i.e., the projection is of type sharksfin.

Moreover, in case that $A = 0$ and/or $B = 0$, the contact order becomes to be an odd number greater than three, since the flecnodal curve must be placed in the hyperbolic domain $LN - M^2 = x^2 - y^2 + \cdots < 0$ (see Fig.2).

As a remark, the parabolic curve and the flecnodal curve on the singular surface $\iota(M) \subset \mathbb{R}^3$ have 4-point contact at the crosscap point of generic elliptic type:

$$\begin{gathered}\iota(x, y_{i,\pm}(x)) - \iota(x, y_{p,\pm}(x)) = \\ (0, -\tfrac{1}{8}A^2x^4, \tfrac{1}{4}A^2x^4) + O(5), \quad (0, \tfrac{1}{8}B^2x^4, \tfrac{1}{4}B^2x^4) + O(5).\end{gathered}$$

This completes the proof of Theorem 7.2.

7.3.4 Non-generic elliptic crosscap

The order of tangency between (components of) the parabolic curve and the flecnodal curve defines a new affine differential invariant of crosscap: we conjecture that for elliptic crosscaps, the order of tangencies of these component

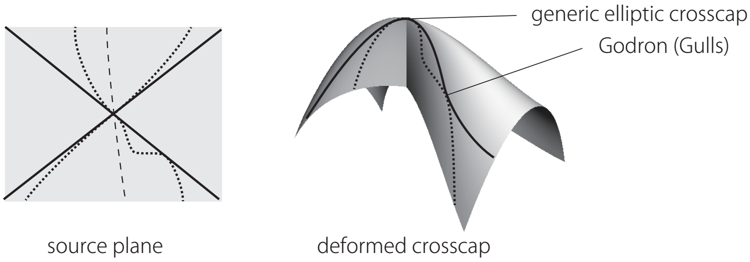

Figure 3: Non-generic elliptic crosscap is deformed to a generic one with producing a tangential point (gulls) of the parabolic curve and the flecnodal curve.

curves (in the source M) are $2\ell + 1$ and $2m + 1$, respectively, if and only if the projection along the image tangent line is $\mathcal{A}$-equivalent to the type $I_{2,2}^{\ell,m}$. Of course the case of $\ell = m = 1$ is proved in Theorem 7.2. It seems that our conjecture also relates to a result of [48, Thm. 2.3] on the torsion of parametrized parabolic curves near crosscap as space curves.

If $A = 0$ or $B = 0$ (i.e, ℓ or $m > 1$), then the projection surely has a worse singularity than the sharksfin $I_{2,2}^{1,1}$. As shown in Theorem 7.1, such a singularity does not appear generically, but for instance, the odd-shaped sharksfin $I_{2,2}^{2,1}$ can appear in projection when one generically deforms the singular surface with one parameter, say $M \times \mathbb{R} \to \mathbb{R}^3$, $(x, t) \mapsto \iota_t(x)$. Assume that the map ι_0 has such a non-generic elliptic crosscap point at $x_0 \in M$, and that ι_t for $t \neq 0$ has only generic elliptic crosscap. Then, in the source of ι_0, a branch of the flecnodal curve has 5-point contact with the parabolic curve at the point x_0. As the parameter t varies from 0, the non-generic crosscap breaks into a generic elliptic crosscap (at which the branches of parabolic and flecnodal curves meet each other with 3-point contact) and a tangential point of these two branches (2-point contact), see Fig.3. At the latter point, the projection along the asymptotic line has the singularity of gulls. This bifurcation of non-generic crosscap is nothing but the deformation of a generic section for the odd-shaped sharksfin (see [73]).

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