Probing cluster structure of $sd$-shell nuclei using isoscalar monopole and dipole transitions
(アイソスカラー型単極・双極遷移でプローブする $sd$ 殻核のクラスター構造)

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March, 2017
Abstract

Background Cluster structure is well established in $p$-shell nuclei but clustering in mid $sd$-shell nuclei and that of heavier nuclei is still not established due to experimental and theoretical difficulties. Recently, isoscalar monopole transition was proposed as a probe for $J^π = 0^+$ cluster states and experimental difficulties are being overcame. Furthermore, the observed isoscalar dipole transition distributions of self-conjugate 4n nuclei show that many $1^-$ states exist near the $0^+$ states observed in isoscalar monopole transition distributions but their nature has been unclear.

Purpose I prove that the cluster states are strongly excited by isoscalar monopole and dipole transitions and provide reliable predictions of isoscalar monopole and dipole transitions of $^{24}\text{Mg}$ and $^{28}\text{Si}$ to clarify the nature of the $0^+$ and $1^-$ states observed in the isoscalar monopole and dipole transition distributions.

Method I derive analytical expression of isoscalar dipole transition matrix and estimate the magnitudes of isoscalar monopole and dipole transition metrics for $^{20}\text{Ne}$ using simple cluster model. I perform realistic calculations by antisymmetrized molecular dynamics for $^{24}\text{Mg}$ and $^{28}\text{Si}$ and numerically evaluate isoscalar monopole and dipole strengths and analyze the cluster structure of the excited states.

Results It is shown that the isoscalar dipole transition matrix between the ground state and the excited $1^-$ cluster states is comparable to the single particle strength and that its magnitude is naturally enhanced for parity-asymmetric cluster configuration and vanished for symmetric cluster configuration. Realistic calculations for $^{24}\text{Mg}$ and $^{28}\text{Si}$ show isoscalar monopole and dipole strengths to the cluster states are considerably large and comparable to the single particle strengths.

Conclusion I conclude that the cluster states with asymmetric configuration is strongly excited by isoscalar dipole transitions. Combined with isoscalar monopole transitions, the isoscalar monopole and dipole transitions are promising probes for asymmetric clusters.
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Chapter 1

Introduction

1.1 Clustering phenomena in nuclei

The ground states of atomic nuclei, which have shell structure (atom-like structure). On the other hand, in the excited states, cluster structure (molecule-like structure) appears. In cluster structure, groups of nucleons are strongly bound to form the subunits called clusters and the clusters are weakly bound each other (Fig.1.1).

Cluster structure has been payed special interest because it is a typical example of many-body correlations in nuclear system and hence it is indispensable to understand nuclear dynamics. $^{16}$O is one of the most famous example of nuclear clustering. The ground state of $^{16}$O has doubly-closed shell configuration in which nucleons occupy up to the $0p$-orbit and hence $^{16}$O is considered as an ideal nucleus that has shell structure. In the shell model picture, the parity of the first excited state should be negative because one nucleon in $0p$-orbit (negative-parity) must be promoted into $0d1s$-orbit (positive-parity) to excite the ground state. However, the actual parity of the first excited state is positive and the angular momentum is equal to 0. Its excitation energy is much smaller than that estimated from shell model which is typically 10 MeV. The existence of the first excited state is mysterious from shell model picture but it is naturally explained by cluster model [1]. If the first excited state can be regarded as the weakly bound state of $^{12}$C and $\alpha$ clusters, its excitation energy nearly equals to the threshold energy where $^{16}$O decays into $^{12}$C and $\alpha$ ($E_x = 7.2$ MeV) and its spin-parity must be $0^+$. In fact, the first excited state is located near the $\alpha$-decay threshold energy and has spin-parity of $0^+$ (Fig. 1.2). In addition to this, the observed large moment-of-inertia [2–6] and large $\alpha$ decay width [7, 8] also supports $^{12}$C+$\alpha$ clustering in the first excited state. Thus, clustering is a fundamental degree-of-freedom of nuclear excitation in light mass systems.
Ikeda and his collaborators conjectured that cluster states of self-conjugate $4n$ nuclei, in which neutron number $N$ and proton number $Z$ are even and equal, appear near threshold energies [9]. Because light self-conjugate $4n$ nuclei are tightly bound system compared to neighboring nuclei, they can be considered as subunits of system (clusters). If interaction between clusters is weak, the binding energy of cluster state is approximately given by the sum of binding energies of clusters. Therefore, cluster states are expected to appear around threshold energies of which the system decays into clusters. This is known as Ikeda threshold rule. From Ikeda threshold rule, systematical appearance of cluster states are expected in the self-conjugate $4n$ nuclei. Such clustering systematics are illustrated in Fig. 1.3, which is called Ikeda diagram. Here, the vertical axis indicates the energy from $n\alpha$-decay threshold energies and the values indicate the excitation energies.

Ikeda diagram suggests the following points. First, in light nuclei and the low-lying excited states, $\alpha$ cluster state dominates. By many experimental and theoretical efforts, $\alpha$-clustering phenomena in light nuclei were revealed in light nuclei. For examples, $2\alpha$, $3\alpha$, $^{12}\text{C}+\alpha$ and $^{16}\text{O}+\alpha$ cluster states are well established respectively in $^8\text{Be}$, $^{12}\text{C}$, $^{16}\text{O}$ and $^{20}\text{Ne}$ [10]. Recently, $3\alpha$ cluster state (the $0^+_2$ state in $^{12}\text{C}$) was proposed as Bose-Einstein condensation state of $3\alpha$-particles [11] and $n\alpha$-condensation has been extensively discussed in these days [13–16].

Second, with increase of mass number, the number of possible cluster configurations rapidly increases. In the nuclei heavier than $^{24}\text{Mg}$, non-$\alpha$ cluster structure such as $^{12}\text{C}+^{12}\text{C}$ cluster appears [17–31]. Therefore, heavier mass nuclei offers a rich variety of cluster structure. In addition to this, cluster states are also important
<table>
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<tr>
<th>Energy from threshold energies (MeV)</th>
<th>α particle</th>
<th>Ne</th>
<th>Ne</th>
<th>O</th>
<th>C</th>
<th>C</th>
<th>C</th>
<th>Ne</th>
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Figure 1.3: Ikeda diagram proposed in Ref. [9]. The values indicate the excitation energies of cluster-decay thresholds in unit of MeV.

from astrophysical view point. Because in the stellar environment \( \alpha \), \(^{12}\text{C}\) and \(^{16}\text{O}\) are the most abundant except for proton, the reactions that involve those nuclei are important. A typical example is the \( 0^+_2 \) state in \(^{12}\text{C}\), which is well-known 3\( \alpha \) cluster state. The existence of this state enables to produce the elements heavier than \(^{12}\text{C}\) via triple-alpha reaction in star [32, 33]. Similar to the \( 0^+_2 \) state of \(^{12}\text{C}\), the cluster states within Gamow window increase fusion reaction rate in order of magnitude and their decay modes determine reaction products (Fig. 1.4). In particular, \( \alpha \)- and \(^{12}\text{C}\) cluster states in \( sd\)-shell nuclei such as \(^{24}\text{Mg}\) and \(^{28}\text{Si}\) are of special interest because they are strongly related to He- and Carbon-burning processes [34, 35]. Therefore, the determination of the excitation energy and decay modes of the cluster states is very important for the understanding of the stellar processes.

Despite of these scientific interest in clustering phenomena and many experimental and theoretical efforts, the existence of cluster state is established only in the \( p\)-shell nuclei and clustering in mid \( sd\)-shell nuclei and in heavier nuclei is still unknown because of experimental and theoretical difficulties. Experimentally, conventional searches for cluster states have been done in direct method. For example, to search for \( \alpha \) cluster states, \( \alpha \) radiative capture reaction [36–39], \( \alpha \)-transfer rea-
Figure 1.4: Schematic figure that show α-fusion reaction. If α+\(^{20}\text{Ne}\) cluster states exist, the reaction rate increases in order of magnitude and the reaction products are determined by the decay mode of the cluster state.

...tion [40–42] and resonance scattering [37,43,44] were employed. By these reactions, α cluster states are considered to be selectively populated. However, the observation of cluster states in the direct method is difficult because of high level density and small cross section due to high Coulomb barrier.

From theoretical side, in sd-shell nuclei, description of a variety of cluster structure in a single theoretical framework and coexistence and mixing of mean-field and cluster structures are not easy.

1.2 Isoscalar monopole and dipole transitions and clustering

These difficulties are being overcame by new probe for cluster state. In this decade, it was found that cluster states are strongly populated by the IS monopole transition. The operator of IS monopole transition \(\hat{O}_{IS0}\) and IS monopole transition probability \(B(IS0)\) between initial state \(|i\rangle\) and final state \(|f\rangle\) are defined as

\[
\hat{O}_{IS0} = \sum_{i=1}^{A} (r_i - r_{\text{c.m.}})^2, \quad (1.1)
\]

\[
M_{IS0} = \langle f | \hat{O}_{IS0} | i \rangle, \quad (1.2)
\]

\[
B(IS0) = |M_{IS0}|^2. \quad (1.3)
\]
Table 1.1: The observed IS monopole transition probabilities $B(IS0)$ of the well-known cluster states. $B(IS0)$ is deduced from $B(E0)$ for $^{12}$C.

<table>
<thead>
<tr>
<th>Nuclide</th>
<th>$E_x$ (MeV)</th>
<th>$J^\pi$</th>
<th>$B(IS0)$ (fm$^4$)</th>
</tr>
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<tbody>
<tr>
<td>$^{11}$B [45]</td>
<td>8.56</td>
<td>$3/2^-$</td>
<td>$96 \pm 16$</td>
</tr>
<tr>
<td>$^{12}$C [47]</td>
<td>7.65</td>
<td>$0^+_2$</td>
<td>$116 \pm 5$</td>
</tr>
</tbody>
</table>

where $r_i$ is $i$-th nucleon coordinate and $r_{c.m.}$ is the center-of-mass coordinate of $A$-nucleon system. IS monopole operator does not change the spin-parity and isospin of the initial state. In the case of $N = Z$ nuclei, the excited $0^+$ states with $T = 0$ are populated by IS monopole transition. Experimentally, IS monopole transition probability $B(IS0)$ is extracted from the inelastic scattering cross section of isoscalar probe such as $\alpha$- and $d$. In the case of $N = Z$ nuclei, $B(IS0)$ can be deduced from electric monopole ($E0$) transition probability $B(E0)$ using relation $B(IS0) \approx 4B(E0)$, which is measured by electron-scattering experiment.

T. Kawabata [45] and Y. Kanada-En’yo [46] found that IS monopole transition strength between the ground state and the $2\alpha + t$ cluster state is enhanced in $^{11}$B and proposed IS monopole transition as a probe for clustering. Actually, the enhanced IS monopole strengths are observed for the well-known cluster states (see Table 1.1). The $0^+_2$ state in $^{12}$C is also famous to have large $B(IS0)$ [47]. Hence the IS monopole transition have been used as a probe for $J^\pi = 0^+$ cluster states. Many experiment have been performed for $N = Z$ nuclei and many low-lying $0^+$ states are found [48–50, 52–65]. In Fig. 1.5, the observed IS monopole strength distributions of $^{16}$O [52], $^{24}$Mg [57], and $^{28}$Si [55] are shown together with mean-field calculations. As clearly seen in Fig. 1.5, while the mean-field calculations succeed to describe the bulk properties of IS monopole distributions but fail to reproduce the low-lying peaks between 10-15 MeV. On the other hand, the low-lying peaks in the IS monopole distribution of $^{16}$O are reproduced well by $4\alpha$ cluster model calculation as reported in Ref. [70]. These results suggest the IS monopole transition is capable to search for unknown $0^+$ cluster states and indicate that both of cluster and mean-field aspects should be taken into account to understand IS monopole strength distribution. Similar to the case of $^{16}$O, the low-lying peaks in the IS monopole distributions of $^{24}$Mg and $^{28}$Si are expected to be cluster states but their characters are still not clarified.

In this thesis, I argue that IS (compressional) dipole transition can be utilized as a probe for the cluster states with $J^\pi = 1^-$. IS dipole operator $\hat{O}_{IS1}$ and transition


Figure 1.5: IS monopole distributions of $^{16}$O (upper panel), $^{24}$Mg (middle panel) and $^{28}$Si (lower panel). The observed data (histogram) and the mean-field calculation (solid line) are taken from Refs. [52,55,57].
probability \( B(IS1) \) are defined as

\[
\hat{O}_{IS1} = \sum_{i=1}^{A} (\mathbf{r}_i - \mathbf{r}_{c.m.})^2 \mathcal{Y}_{1\mu}(\mathbf{r}_i - \mathbf{r}_{c.m.}),
\]

(1.4)

\[
M_{IS1} = \langle f || \hat{O}_{IS1} || i \rangle,
\]

(1.5)

\[
B(IS1) = \frac{1}{2J_i + 1} |M_{IS1}|^2.
\]

(1.6)

where the spin-parity of the initial state is denoted by \( J_i \). Since IS dipole operator involves the solid spherical harmonics \( \mathcal{Y}_{\lambda\mu}(\mathbf{r}) = r^{\lambda} Y_{\lambda\mu}(\hat{r}) \), the spin-parity of the initial state is changed by \( 1^- \) and the excited \( 1^- \) states with \( T = 0 \) are populated by IS dipole transition in the case of \( N = Z \) nuclei. IS dipole transition is observed in \( \alpha\)- and \( d\)-inelastic scattering together with IS monopole transition. The observation of \( 1^- \) cluster states is very important to establish the existence of cluster states. If parity-asymmetric cluster structure exists, it appears as the pair of positive- and negative-parity states within small energy difference because of restoring parity-symmetry \cite{67}. This pair is called inversion-doublet. This situation is illustrated in Fig. 1.6. Reversely, the existence of inversion-doublet of \( 0^+ \) and \( 1^- \) states is a direct evidence of asymmetric clustering. Inversion-doublets of asymmetric cluster configurations have been observed in \(^{16}\text{O} + \alpha\) cluster of \(^{20}\text{Ne} \) and of course in \(^{12}\text{C} + \alpha\) cluster of \(^{16}\text{O} \) (Fig. 1.2). Therefore, the observation of both of \( 0^+ \) and \( 1^- \) cluster states is important to establish asymmetric clustering. As similar to the IS monopole distributions, the low-lying peaks are observed below 15 MeV in the IS dipole distributions of \( N = Z \) nuclei \cite{52–55,57–62,64,65}. In Fig. 1.7, IS dipole strength distributions of \(^{16}\text{O} \) \cite{52}, \(^{24}\text{Mg} \) \cite{57} and \(^{28}\text{Si} \) \cite{55} are displayed as examples.
Therefore, I conjecture that IS dipole transition strongly populates $1^-$ cluster states and that the observed peaks can be regarded as evidence for asymmetric clustering in those nuclei.

1.3 Purpose of this work

In this thesis, my purpose is to demonstrate that IS monopole and dipole transitions are a good probe for $0^+$ and $1^-$ cluster states. For this purpose, this thesis are organized as follows: In chapter 2, I explain why IS monopole and dipole transitions can be a good probe for cluster states using simple cluster model. To give more reliable results, I perform realistic calculations for $^{24}$Mg and $^{28}$Si. I explain theoretical framework called antisymmetrized molecule dynamics (AMD) [68, 69] to calculate excitation spectra and IS monopole and dipole strengths in chapter 3. I present my calculation results and discuss on cluster structure and IS monopole and dipole strengths of cluster states in $^{24}$Mg and $^{28}$Si in chapter 4. Finally, I summarize this thesis in chapter 5.
Figure 1.7: IS dipole distributions of $^{16}$O, $^{24}$Mg and $^{28}$Si. The observed data (histogram) and the mean-field calculation result (solid line) are taken from [52, 55, 57].
Chapter 2

Enhancement mechanism of isoscalar monopole and dipole strength to cluster states

In this chapter, I review the enhancement mechanism of IS monopole strength of \(0^+\) cluster states as explained by T. Yamada et. al. [70] and give analytical expression of IS dipole transition matrix element to \(1^-\) cluster states using \(^{16}\text{O}+\alpha\) cluster model of \(^{20}\text{Ne}\). From these results, it is shown that inversion-doublet of \(0^+\) and \(1^-\) are strongly excited by IS monopole and dipole transitions. These points are schematically summarized in Fig. 2.1.

2.1 Wave functions of ground state and excited cluster states

The ground state of \(^{20}\text{Ne}\) is dominated by \((0d1s)^4\) configuration on top of the closed-shell core \(^{16}\text{O}\). The shell model calculations [71–74] showed that the ground state of \(^{20}\text{Ne}\) is dominated by the \(SU(3)\) irreducible representation of \((\lambda, \mu) = (8, 0)\) in the Elliot’s \(SU(3)\) model [75]. Thus, the ground state wave function of \(^{20}\text{Ne}\) is given as

\[
\Phi(g.s.) = A \{ (0s)^4(0p)^{12}(0d1s)^4 \}_{J=0}^{(\lambda\mu)=(8,0)}
\]  

(2.1)
Figure 2.1: Cluster excitations induced by IS monopole and dipole transitions. Inversion doublet of $0^+$ and $1^-$ cluster states (nodal and angular excited cluster states expressed in Eqs. (2.5) and (2.6)) are strongly excited from the ground state (Eq. (2.4)) by IS monopole and dipole transitions.
Owing to Bayman-Bohr theorem [76], this shell model wave function is equivalent to 

\[ \Phi(g.s.) = \frac{c_0}{\sqrt{\mu N_0}} A \{ R_{N_00}(r) \phi_\alpha \phi_O \}, \]  

\[ c_0 = \sqrt{\frac{C_1! C_2!}{A!}}, \]  

\[ \mu_N = \langle \{ R_{N_00}(r) \phi_\alpha \phi_O \} | A \{ R_{N_00}(r) \phi_\alpha \phi_O \} \rangle. \]  

Here, \( \phi_\alpha \) and \( \phi_O \) are the internal wave functions of \( \alpha \) and \( {}^{16}\text{O} \) clusters. The internal wave functions are described by the harmonic oscillator wave functions with the oscillator parameter \( \nu = \frac{m \omega^2}{2\hbar} \). The wave function of inter-cluster motion is also expressed by the harmonic oscillator wave function \( R_{Nlm} = R_{Nl}(r) Y_{lm}(\hat{r}) \) but oscillator parameter is \( \nu' = \frac{C_1 C_2}{A} \nu \) with masses of clusters \( C_1 = 4 \) and \( C_2 = 16 \). The principle quantum number of the inter-cluster motion equals to the lowest Pauli-allowed value \( N_0 = 8 \). Hence, in the ground state, the \( \alpha \) and \( {}^{16}\text{O} \) clusters completely overlap (Fig. 2.1). This equivalence between shell model and cluster model wave function is called as duality or dual nature of shell and cluster. As emphasized in the Ref. [70], duality of shell and cluster implies that the degree-of-freedom of cluster excitation is embedded even in the pure shell model ground state. For example, the nodal excitation of the inter-cluster motion yields the excited \( 0^+ \) state (Fig. 2.1),

\[ \Phi(0^+_{ex}) = \sum_{N=N_0+2}^{\infty} e_N \frac{c_0}{\sqrt{\mu N}} A \{ R_{N00}(r) \phi_\alpha \phi_O \} \]  

where the nodal quantum number of inter-cluster motion \( n = (N - l)/2 \) is increased and the states with \( N \geq N_0 + 2 \) are coherently superposed with coefficients \( e_N \). Besides the nodal excitation, the angular excitation of the inter-cluster motion also takes place. For example, the angular excitation with \( \Delta l = 1 \) yields the excited \( 1^- \) state (Fig. 2.1),

\[ \Phi(1^-) = \sum_{N=N_0+1}^{\infty} f_N \frac{c_0}{\sqrt{\mu N}} A \{ R_{N10}(r) \phi_\alpha \phi_O \} \]  

where \( N \) must be equal to or greater than \( N_0 + 1 \). The angular excited state with \( \Delta l = 1 \) has negative parity and then only allowed for asymmetric cluster configuration \( C_1 \neq C_2 \). Therefore, the \( 1^- \) state has been regarded as evidence of of asymmetric clustering [67].
2.2 Cluster excitations induced by isoscalar monopole and dipole transitions

Using the wave function of the ground state and the excited cluster states in Eqs. (2.4), (2.5) and (2.6), I can prove that IS monopole and dipole transition matrix elements to the nodal and angular excited states are comparable to single particle estimates.

2.2.1 Isoscalar monopole transition and nodal excitation

Here, I review the enhancement mechanism of IS monopole strength to the nodal excited state as explained by Yamada et. al. [70]. IS monopole operator (Eq. (1.2)) can be rewritten by cluster-coordinate defined as in Fig. 2.2. In the case of $C_1 + C_2$ system composed from clusters with masses $C_1$ and $C_2$,

$$ \hat{O}_{IS0} = \sum_{i=1}^{A} (r_i - r_{c.m.})^2 \sum_{i \in C_1} \xi_i^2 + \sum_{i \in C_2} \xi_i^2 + \frac{C_1 C_2}{A} r^2, \quad (2.7) $$

where internal coordinates of each clusters $\xi_i$ and the inter-cluster coordinate $r$ are defined as in Fig. 2.2. The first and second terms in Eq. (2.7) represent the internal excitations of clusters and the third term induces the excitation of the inter-cluster motion because it can increase the nodal quantum number of inter-cluster wave function. The IS monopole matrix elements $M_{IS0}$ between the ground state and the
nodal excited state can be calculated as following.

\[ M_{IS0} = \langle \Phi(\text{g.s.}) | \hat{O}_{IS0} | \Phi(0^+_{ex}) \rangle, \quad (2.8) \]

\[ = \sum_{N=N_0+2} e_N \sqrt{\mu N_0 \mu N} \langle R_{N_0,00} \phi \phi_0 | \hat{O}_{IS0} | A \{ R_{N_0+2,00} \phi \phi_0 \} \rangle, \quad (2.9) \]

\[ = e_N \sqrt{\mu N_0 \mu N_0 + 2} \langle R_{N_0,00} \phi \phi_0 | \hat{O}_{IS0} | A \{ R_{N_0+2,00} \phi \phi_0 \} \rangle. \quad (2.10) \]

From the second line to the third line, I use the facts that \( \hat{O}_{IS0} \) increases the total principal quantum number of the bra state by 2 at most and it is orthogonal to the states with \( N > N_0 + 2 \). Using cluster coordinate expression of \( \hat{O}_{IS0} \) (Eq. (2.7)),

\[ \langle R_{N_0,00} \phi \phi_0 | A \{ \hat{O}_{IS0} R_{N+2,00} \phi \phi_0 \} \rangle = \langle R_{N_0,00} \phi \phi_0 | A \{ \frac{C_1 C_2}{A} \xi_2^2 R_{N_0+2,00} \phi \phi_0 \} \rangle \]

\[ + \langle R_{N_0,00} \phi \phi_0 | A \{ R_{N_0+2,00} \left( \sum_{i \in C_1} \xi_i^2 \phi_\alpha \right) \phi_0 \} \rangle + \langle R_{N_0,00} \phi \phi_0 | A \{ R_{N_0+2,00} \phi \left( \sum_{i \in C_2} \xi_i^2 \phi_\alpha \right) \} \rangle \quad (2.11) \]

The terms in the second line of Eq. have no contribution to \( M_{IS0} \) since the bra and ket states are orthogonal. It can be proven by counting the total principal quantum numbers of the bra and ket states. For example, denoting the principal quantum number for cluster \( C_i \) as \( N_{C_i} \), the total principal quantum number of bra state is \( N_0 + N_{C_1} + N_{C_2} \). On the other hand, the principal quantum number of ket state is equal to or greater than \( N_0 + 2 + N_{C_1} + N_{C_2} \), because the quanta of \( \sum_{i \in C_1} \xi_i^2 \phi_\alpha \) cannot be smaller than \( N_{C_1} \). Thus, the total principal quantum number for the ket state is larger than that of the bra state. Thus, the second and thirds terms of Eq. (2.2.1) vanishes and the contribution of the third term remains. It can be calculated
Here, I use the identity 
\[ r^2 \mathcal{R}_{N_0+2,00} = \sum_{N'} \mathcal{R}_{N'00} \langle \mathcal{R}_{N'00}|r^2|\mathcal{R}_{N_0+2,00} \rangle. \]
Finally, I obtain analytical expression of IS monopole transition matrix element for nodal excitation as

\[ \langle \Phi(0^+_{ex})|\hat{O}_{IS0}|\Phi(\text{g.s.}) \rangle = e_{N_0+2} \sqrt{\frac{\mu_{N_0}}{\mu_{N_0+2}}} \sqrt{(N_0 + 3)(N_0 + 2)} \]

From Eq. (2.17), I can estimate the magnitude of the IS monopole matrix element for $^{20}\text{Ne}$. Here, I use the oscillator parameter $\nu = 0.16 \text{ fm}^{-2}$, which minimize the ground state energy in AMD calculation and the coefficients $e_{N_0+1}$ is estimated from AMD calculation result [77]. For the other quantities, an analytical calculation can be done or experimental values are available. Putting those values to Eq. (2.17), the estimate for IS monopole transition matrix is

\[ M_{IS0} = 7.67 e_{10} = 5.48 \text{ fm}^2. \]

This value is comparable to Weisskopf estimate, which is approximated matrix element for single particle excitation,

\[ M_{IS0}^{WU} = \frac{3}{5} (1.2A^{\frac{1}{2}})^2 \approx 6.37 \text{ fm}^2. \]

Thus, the nodal excitation is induced and then excited $0^+$ states are strongly populated by IS monopole transition.
2.2.2 Isoscalar dipole transition and angular excitation

In the similar way to IS monopole transition, I can prove that IS dipole transition induces angular excitation and deduce analytical expression of the matrix element between the ground state and the angular excited states. IS dipole operator (Eq. (1.5)) is rewritten using the cluster coordinates defined in Fig. 2.2,

\[
\hat{O}_{IS1} = \sum_{i=1}^{A} (r_i - r_{c.m.})^2 Y_{1\mu}(r_i - r_{c.m.}),
\]

\[
= \sum_{i \in C_1} \xi_i^2 Y_{1\mu}^{(\xi_i)} + \sum_{i \in C_2} \xi_i^2 Y_{1\mu}^{(\xi_i)}
\]

\[- \sqrt{\frac{32\pi}{9}} \left\{ \frac{C_2}{A} \left[ \sum_{i \in C_1} Y_2(\xi_i) \otimes Y_1(r) \right] - \frac{C_1}{A} \left[ \sum_{i \in C_2} Y_2(\xi_i) \otimes Y_1(r) \right] \right\}_{1\mu}
\]

\[+ \frac{5}{3} \left( \frac{C_2}{A} \sum_{i \in C_1} \xi_i^2 - \frac{C_1}{A} \sum_{i \in C_2} \xi_i^2 \right) Y_{1\mu}(r) - \frac{C_1 C_2 (C_1 - C_2)}{A^2} r^2 Y_{1\mu}(r) \]  (2.20)

The proof of Eq. (2.20) is given in Appendix B. The terms in the first two lines of Eq. (2.20) involve internal excitation for clusters and the contribution from those internal excitation vanishes for \(^{20}\text{Ne}\) (see Appendix C). Since the terms in the last line of Eq. (2.20) include the solid spherical harmonics \(Y_{1\mu}(r)\), which increases the relative angular momentum \(l\) by 1, they induces the angular excitation into the inter-cluster motion. Therefore, IS dipole transition can populate the angular excited cluster states. Analytical evaluation of IS dipole matrix element \(M_{IS1}\) is possible using the wave functions of the ground state and the angular excited state in Eqs. (2.4) and (2.6). As similar to the case of IS monopole matrix element, we can evaluate each term as explained in Appendix C. As a result, the analytical expression of \(M_{IS1} = \langle \Phi(\text{g.s.})|\hat{O}_{IS1}|\Phi(1_{ex}) \rangle\) is derived as

\[
M_{IS1} = \langle \Phi(\text{g.s.})|\hat{O}_{IS1}|\Phi(1_{ex}) \rangle,
\]

\[
= \sqrt{\frac{3}{4\pi}} C_1 C_2 \frac{f_{N_0+1}}{A} \left[ \sqrt{\frac{\mu_{N_0+1}}{\mu_{N_0}}} \left\{ \frac{5}{3} \left( \langle r^2 \rangle_{C_1} - \langle r^2 \rangle_{C_2} \right) \langle R_{N_0} | r | R_{N_0+1,1} \rangle - \frac{(C_1 - C_2)}{A} \langle R_{N_0} | r^3 | R_{N_0+1,1} \rangle \right\} \right.
\]

\[- \left. \left. f_{N_0+3} \sqrt{\frac{\mu_{N_0+3}}{\mu_{N_0+1}}} \left( \frac{C_1 - C_2}{A} \langle R_{N_0} | r^3 | R_{N_0+3,1} \rangle \right) \right], \tag{2.21}
\]
where \( \langle r^2 \rangle_{C_1} \) and \( \langle r^2 \rangle_{C_2} \) are square of root-mean-square radius of clusters.

The matrix elements for \( r \) and \( r^3 \) are given as follows,

\[
\langle R_{N_0,0} | r | R_{N_0+1,0} \rangle = \sqrt{\frac{N_0 + 3}{4\nu'}} ,
\]

\[
\langle R_{N_0,0} | r^3 | R_{N_0+1,1} \rangle = \frac{3N_0 + 5}{4\nu'} \sqrt{\frac{N_0 + 3}{4\nu'}} ,
\]

\[
\langle R_{N_0,0} | r^3 | R_{N_0+3,1} \rangle = -\sqrt{(N_0 + 2)(N_0 + 5)} \frac{N_0 + 3}{4\nu'} \sqrt{\frac{N_0 + 3}{4\nu'}} .
\]

From Eq. (2.21) to Eq. (2.24), I find following properties: (1) The transition matrix is proportional to asymmetric term such as \( \langle r^2 \rangle_{C_1} - \langle r^2 \rangle_{C_2} \) and \( (C_1 - C_2)/A \). This means that the transition matrix is amplified for asymmetric cluster states and completely zero for symmetric cluster states. (2) For the cluster states dominated by \( 1\hbar\omega \) configuration, the first and second lines of Eq. (2.21) contribute dominantly to the transition matrix and third line mainly contributes for the cluster states dominated by \( 3\hbar\omega \) configuration. (3) The matrix element depends on the inverse of the oscillator parameter \( \nu' = \frac{C_1 C_2}{A} \nu \), which characterizes the size of the relative wave function. This means that the matrix element is amplified if the inter-cluster distance increases.

From Eq. (2.21), I obtained the analytical expression for IS dipole transition matrix element. As the same way as in IS monopole transition, I can estimate the magnitude of IS dipole matrix element for \(^{20}\text{Ne}\) from AMD calculation result [77].

\[
M_{IS1} = 3.08 f_0 - 7.36 f_{11} = 5.82 \text{ fm}^3 .
\]

This value is in the same order of Weisskopf estimate,

\[
M_{IS1}^{WU} = \frac{3}{4\pi} (1.2A^{1/3})^3 \simeq 8.44 \text{ fm}^3 .
\]

Thus, the angular excited cluster state has the strong IS dipole strength from the ground state which is comparable to Weisskopf estimates, even if the ground state is an ideal shell-model state. Single particle transition is usually fragmented into many excited states, only cluster states can have strong transition strengths. Furthermore, the IS monopole and dipole strengths are amplified if the ground state has cluster correlation [13,77]. It should be noted that the above proofs are also valid if system is composed from \( SU(3) \)-scalar (closed-shell) clusters such as \(^{40}\text{Ca} + \alpha \) cluster in \(^{44}\text{Ti}\). For the open-shell clusters such as \(^{12}\text{C} + \alpha \) cluster in \(^{16}\text{O}\), the contribution to IS dipole transition from internal excitation is not negligible. The two terms in the second line of Eq. (2.20) include isoscalar quadrupole excitations of clusters and the
matrix elements between the $0^+$ and $2^+$ states of the open-shell clusters are generally very large. Therefore, for the open-shell clusters, the contribution from the internal excitation of cluster can have a large contributions to IS dipole strength.

To end this chapter, I summarize the above discussions. By assuming the ground state is described by a $SU(3)$ shell-model wave function, I derived analytical expressions of the IS monopole and dipole transition matrix elements and demonstrated that the transition matrix are enhanced for the nodal and angular excited cluster states, which magnitudes are on the order of the Weisskopf estimates for $^{16}\text{O}+\alpha$ cluster. Therefore, the IS monopole and dipole transitions are good probes for parity-asymmetric clustering. However, in the realistic situation, the ground state wave function is deviated from an ideal $SU(3)$ shell-model wave function because of the strong spin-orbit interaction and the mixing between cluster and shell structures. Therefore, the strengths are deviated from the estimates made in this chapter. To make quantitative discussion on IS monopole and dipole transitions, realistic theoretical calculations are needed.

To provide reliable results for IS monopole and dipole transitions of $^{24}\text{Mg}$ and $^{28}\text{Si}$, I perform realistic calculation using antisymmetrized molecular dynamics (AMD) [68, 69]. AMD is one of theories that successfully explain nuclear properties of $sd$-shell nuclei including the excited cluster states. In the next chapter, I will explain how to calculate eigenenergies, eigenstates and IS monopole and dipole strengths.
Chapter 3

Framework of Antisymmetrized Molecular Dynamics

3.1 Hamiltonian and Wave function

The microscopic Hamiltonian of a nucleon system is used in antisymmetrized molecular dynamics (AMD)

\[ \hat{H} = \sum_i \frac{\hat{p}_i^2}{2m} - \hat{t}_{\text{c.m.}} + \sum_{i<j}^A \hat{v}_{\text{NN}}(ij) + \sum_{i<j}^Z \hat{v}_C(ij), \]  

(3.1)

where \( \hat{t}_{\text{c.m.}} \), \( \hat{v}_N \) and \( \hat{v}_C \) stand for the center-of-mass kinetic energy, Gogny D1S effective NN interaction [78], which successfully reproduces binding energies of stable and unstable nuclei, and the Coulomb interaction, respectively. The Coulomb interaction is approximated by a sum of seven Gaussians. The AMD variational wave function is an antisymmetrized product of the single particle wave packets projected onto the positive- or negative-parity states,

\[ \Phi^\pm = \frac{1 \pm \hat{P}_x}{2} A \{ \varphi_1, \varphi_2, \ldots \varphi_A \}, \]

(3.2)

\[ \varphi_i(r) = \exp \left[ - \sum_{\sigma=x,y,z} \nu_\sigma \left( r_\sigma - \frac{Z_{i\sigma}}{\sqrt{\nu_\sigma}} \right)^2 \right] \]

\[ \otimes (a_i \chi_\uparrow + b_i \chi_\downarrow) \otimes (\text{neutron or proton}), \]

(3.3)

where the single-particle wave function \( \varphi_i \) is represented by a deformed Gaussian wave packet [79], and the parameters \( \nu_\sigma, Z_{i\sigma}, a_i \) and \( b_i \) are determined in energy
variation or shifted-basis AMD method as explained in the next sections. In Fig. 3.1, I illustrate how shell and cluster structures are described by AMD wave function. I generate AMD wave function randomly and perform energy minimization. If the system favors shell structure, the wave packets are gathered around the center-of-mass of system \( \langle Z_i \rangle \approx 0 \) and then one-center structure is described. On the other hand, if cluster structure is favored, the wave packets are gathered around several points and then many-center structure is described.

3.2 Energy variation with constraints

To determine the parameters of AMD without \textit{a priori} assumption on nuclear structure, I perform the energy variation under constraints. The sum of energy and constraint potential \( V_c \)

\[
E' = \frac{\langle \Phi^\pi | \hat{H} | \Phi^\pi \rangle}{\langle \Phi^\pi | \Phi^\pi \rangle} + V_c 
\]  

(3.4)

is minimized using frictional cooling method,

\[
\frac{dX}{dt} = -\frac{\mu}{\hbar} \frac{\partial E'}{\partial X^*} \quad (\mu > 0),
\]

(3.5)

\[
X = \nu, Z_i, a_i, b_i \quad (i = 1 \ldots A). 
\]

(3.6)

In this study, a couple of constraint potential are introduced and each of them is used independently to describe mean-field or cluster aspects of \( ^{24}\text{Mg} \) and \( ^{28}\text{Si} \). Constraint on quadrupole deformation is used to describe the low-lying mean-field states.
of $^{24}\text{Mg}$ and $^{28}\text{Si}$. To describe the highly excited states of $^{24}\text{Mg}$, I introduce constraint on harmonic oscillator quanta and shifted-basis AMD method. The constraint on harmonic oscillator quanta dominantly yields cluster states and $1p1h$ excited states built on the deformed mean-field states are described in shifted-basis AMD method. For $^{28}\text{Si}$, I use $d$-constraint method to explicitly describe cluster states instead of the constraint on harmonic oscillator quanta since the number of possible configurations is enormous compared with $^{24}\text{Mg}$. The details of these constraints are described in the following.

### 3.2.1 Constraint on quadrupole deformation

To describe low-lying collective states, I introduce constraint on nuclear quadrupole deformation parameters $\beta$ and $\gamma$. The constraint potential is defined as

$$V_c = v_\beta (\langle \beta \rangle - \bar{\beta})^2 + v_\gamma (\langle \gamma \rangle - \bar{\gamma})^2.$$  \hspace{1cm} (3.7)

where $\langle \beta \rangle$ and $\langle \gamma \rangle$ are the quadrupole parameters of intrinsic wave function. They are defined as [80]

$$\langle \hat{x}^2 \rangle = R_0^2 \left[ 1 + \frac{\sqrt{5}}{\pi} \beta \cos \left( \gamma + \frac{2\pi}{3} \right) \right],$$  \hspace{1cm} (3.8)

$$\langle \hat{y}^2 \rangle = R_0^2 \left[ 1 + \frac{\sqrt{5}}{\pi} \beta \cos \left( \gamma - \frac{2\pi}{3} \right) \right],$$  \hspace{1cm} (3.9)

$$\langle \hat{z}^2 \rangle = R_0^2 \left[ 1 + \frac{\sqrt{5}}{\pi} \beta \cos \gamma \right],$$  \hspace{1cm} (3.10)

$$R_0^2 = \frac{1}{3} (\langle \hat{x}^2 \rangle + \langle \hat{y}^2 \rangle + \langle \hat{z}^2 \rangle),$$  \hspace{1cm} (3.11)

where $\langle \hat{x}^2 \rangle$, $\langle \hat{y}^2 \rangle$ and $\langle \hat{z}^2 \rangle$ are the expectation values of the those operators calculated from the intrinsic wave function and I define the internal coordinate such that the relation $\langle \hat{x}^2 \rangle \leq \langle \hat{y}^2 \rangle \leq \langle \hat{z}^2 \rangle$ is satisfied. Putting $v_\beta$ and $v_\gamma$ are sufficiently large, $\langle \beta \rangle$ and $\langle \gamma \rangle$ becomes equal to constraint condition $\bar{\beta}$ and $\bar{\gamma}$ after energy minimization. I denote the basis wave functions obtained by this method as $\Phi^\pm(\beta, \gamma)$. As discussed later, this constraint yields the deformed mean-field states which are important to describe the ground and low-lying states but highly excited cluster states are not obtained. Therefore, I introduce other constraints to describe cluster structure of $^{24}\text{Mg}$ and $^{28}\text{Si}$. 

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3.2.2 Constraints on harmonic oscillator quanta

For $^{24}\text{Mg}$, I introduce the extended version of the method used in Ref. [81] and impose the constraint on the expectation values of the harmonic oscillator quanta $N$, $\lambda$ and $\mu$ [84]. They are defined as

\begin{align}
N &= N_x + N_y + N_z, \\
\lambda &= N_z - N_y, \\
\mu &= N_y - N_x, \\
N_x &\leq N_y \leq N_z,
\end{align}

and $N_x, N_y$ and $N_z$ are the eigenvalues of of the 3 by 3 matrix,

\[ \eta_{\sigma\tau} = \langle \Phi^+ | \sum_{i=1}^{A} a_\sigma^\dagger(i) a_\tau(i) | \Phi^+ \rangle, \quad \sigma, \tau = x, y, z. \]

Here $a_\tau(i)$ is an ordinary annihilation operator of the harmonic oscillator acting on the $i$th nucleon, and the oscillator parameter $\hbar \omega$ is estimated from the ground state radius of $^{24}\text{Mg}$ and set to 12.6 MeV. The constraint potential is defined as,

\[ V_c = v_N(N - \overline{N})^2 + v_{Q_1}(Q_1 - \overline{Q_1})^2 + v_{Q_2}(Q_1 - \overline{Q_2})^2 \]

(3.17)

where $Q_1$ and $Q_2$ are a function of $\lambda, \mu$, which are introduced to avoid generating identical $SU(3)$ irreducible representation. The definitions of $Q_1$ and $Q_2$ and are given in Appendix A. As a measure of the particle-hole excitation, I introduce the quantity $\Delta N = N - N_0$ where $N_0$ is the lowest Pauli-allowed value equal to 28 in the case of $^{24}\text{Mg}$. Under the condition of the $\Delta N = 0, 2, 4, 6$ or 8, I put the constraints for all possible integer values of $\Delta N, \lambda$ and $\mu$. In other words, roughly speaking, I searched for the various many-particle-hole configurations within $8\hbar \omega$ excitation. I denote thus-obtained basis wave functions as $\Phi^+(\Delta N, \lambda, \mu)$. As discussed in the section 4.1.1, the basis wave functions obtained in this method describe a various kind of well-developed cluster structures as well as mean-field like mp mh excited states.

3.2.3 Constraint on relative distance between quasi-clusters

In the case of $^{28}\text{Si}$, application of the constraint on harmonic oscillator quanta is difficult because the number of possible configuration rapidly increases compared with
that of $^{24}\text{Mg}$. So, I focus on important cluster configurations to reduce the computational cost. For this purpose, I use $d$-constraint method [82], which is imposed on the distance between quasi clusters,

$$V_c = v_d(\langle d^2 \rangle - \bar{d}^2)^2$$

where $\langle d^2 \rangle$ is the squared distance between quasi clusters. The square distance between quasi clusters is defined as follows. First, I classify nucleons into two groups, which are called quasi clusters. For example, in the case of $^{24}\text{Mg} + \alpha$ quasi cluster configuration, I choose 2 proton wave packets and 2 neutron wave packets as the members of quasi $\alpha$ cluster and regard that the remaining 24 wave packets belong to quasi $^{24}\text{Mg}$ cluster. Then, I define the center-of-mass of these quasi-clusters as

$$R_\alpha = \frac{1}{4} \sum_{i \in \alpha} \text{Re} \left( \frac{Z_i}{\sqrt{\nu}} \right),$$

$$R^{24}\text{Mg} = \frac{1}{24} \sum_{i \in ^{24}\text{Mg}} \text{Re} \left( \frac{Z_i}{\sqrt{\nu}} \right),$$

and define $\langle d^2 \rangle$ as

$$\langle d^2 \rangle = |R_\alpha - R^{24}\text{Mg}|^2$$

As the same way, I define $\langle d^2 \rangle$ for $^{16}\text{O} + ^{12}\text{C}$ and $^{20}\text{Ne} + ^{8}\text{Be}$, which have the three lowest threshold energies together with $^{24}\text{Mg} + \alpha$, and perform the energy variation for each quasi cluster configurations. I denote the basis wave functions obtained by $d$-constraint method as $\Phi_{\text{Mg}+\alpha}^\pm (d)$, $\Phi_{\text{O}+\text{C}}^\pm (d)$ and $\Phi_{\text{Ne}+\text{Be}}^\pm (d)$. Since the constraint is imposed only on the relative distance between quasi-clusters $d$, the internal structures of quasi-clusters are optimized for the given $d$, i.e., the polarization of clusters are taken into account. As shown in the section 4.2.1, I can describe structure changes from mean-field structure and cluster structure as a function of $d$.

### 3.3 Shifted-basis AMD

To describe $1p1h$ excited states from the ground state of $^{24}\text{Mg}$, I further introduce an additional set of the basis wave functions $\Phi_{I\hat{S}0}^+ (\beta, \gamma)$ defined as,

$$\Phi_{I\hat{S}0}^+ (\beta, \gamma) = \left(1 - e^{-\mu \hat{O}_{I\hat{S}0}}\right) \Phi^+ (\beta, \gamma) \simeq \mu \hat{O}_{I\hat{S}0} \Phi^+ (\beta, \gamma),$$
where \( \mu \) is arbitrary small real number, \( \hat{O}_{IS0} \) is the IS monopole operator defined in Eq. 1.2. By definition, the wave function \( \Phi^{+}_{IS0}(\beta, \gamma) \) describes \( 1p1h \) \((2\hbar\omega)\) excited state built on \( \Phi^{+}(\beta, \gamma) \) by the IS monopole operator. In practical calculation, I introduce \( e^{-\mu\hat{O}_{IS0}}\Phi^{+}(\beta, \gamma) \) as basis wave function instead of \( \Phi^{+}_{IS0}(\beta, \gamma) \) since it is proportional to AMD wave function which parameters are shifted as \( \nu_{\sigma} \rightarrow \nu_{\sigma} + \mu \) and \( Z_{i\sigma} \rightarrow \sqrt{\nu_{\sigma}/(\nu_{\sigma} + \mu)}Z_{i\sigma} \). The inclusion of ”shifted” AMD wave function becomes equivalent to inclusion of \( \Phi^{+}_{IS0}(\beta, \gamma) \) after GCM calculation. This method is called Shifted-basis AMD [83, 84]. As explained later, the shifted-basis AMD is very effective to describe the Giant resonance.

### 3.4 Angular momentum projection and GCM

The basis wave functions obtained above methods are not the eigenstate of angular-momentum. Therefore, they are projected to the angular-momentum eigenstates and then superposed to take care of the mixing of the various configurations. For \(^{24}\text{Mg}\), I superpose three sets of wave functions \( \Phi^{+}(\beta, \gamma) \), \( \Phi^{+}(\Delta N, \lambda, \mu) \) and \( \Phi^{+}_{IS0}(\beta, \gamma) \) which are obtained by the energy variations with the constraints on the quadrupole deformation and harmonic oscillator quanta and shifted-basis AMD,

\[
\Psi^{0+}_n = \sum_{iK} g^{0+}_{n0}(\beta_i, \gamma_i) P^{0+}_{00}(\beta_i, \gamma_i) + \sum_{iK} g^{0+}_{n0}(\beta_i, \gamma_i) P^{0+}_{00}(\beta_i, \gamma_i) + \sum_{iK} g^{0+}_{n0}(\Delta N_i, \lambda_i, \mu_i) P^{0+}_{00}(\Delta N_i, \lambda_i, \mu_i) \quad (3.24)
\]

and for \(^{28}\text{Si}\), I superpose basis wave functions \( \Phi^{\pm}(\beta, \gamma) \), \( \Phi^{+}(\Delta N, \lambda, \mu) \) \( \Phi^{\pm}_{Mg+\alpha}(d) \), \( \Phi^{\pm}_{O+C}(d) \), and \( \Phi^{\pm}_{Ne+Be}(d) \), which are obtained respectively in the energy variations with the constraints on the quadrupole deformation and \( d \)-constraint for \(^{24}\text{Mg}+\alpha\), \(^{16}\text{O}+^{12}\text{C} \) and \(^{20}\text{Ne}+^{8}\text{Be} \) quasi-cluster configurations,

\[
\Psi^{J\pm}_n = \sum_{iK} g^{J\pm}_{nK}(\beta_i, \gamma_i) P^{J\pm}_{MK}(\beta_i, \gamma_i) + \sum_{iK} g^{J\pm}_{iK}(d_i) P^{J\pm}_{MK}\Phi^{\pm}_{Mg+\alpha}(d_i) + \sum_{iK} g^{J\pm}_{iK}(d_i) P^{J\pm}_{MK}\Phi^{\pm}_{O+C}(d_i) + \sum_{iK} g^{J\pm}_{K}(d_i) P^{J\pm}_{MK}\Phi^{\pm}_{Ne+Be}(d_i) \quad (3.25)
\]

where \( P^{J}_{MK} \) is the angular-momentum projection operator. In total, I superposed 524 and 384 basis wave functions respectively for \(^{24}\text{Mg}\) and \(^{28}\text{Si}\), and solved the Hill-Wheeler equation [85, 86] to obtain the eigenenergies \( E_n \) and wave functions \( \Psi^{J\pm}_n \) of the ground and excited states.
3.5 Isoscalar monopole and dipole strengths

Using the wave functions of the ground and excited $0^+$ and $1^-$ states directly, I derived the IS monopole and dipole matrix elements $M_n(IS0)$ and $M_n(IS1)$, reduced transition strengths $B(IS0)$ and $B(IS1)$.

\[ M_n(IS0) = \langle \Psi_n^{0+} | \hat{O}_{IS0} | \Psi_{g.s.}^{0+} \rangle, \quad \text{(3.26)} \]
\[ B(IS0; g.s. \rightarrow 0^+_n) = |M_n(IS0)|^2, \quad \text{(3.27)} \]

and

\[ M_n(IS1) = \langle \Psi_n^{1-} | \hat{O}_{IS1} | \Psi_{g.s.}^{0+} \rangle, \quad \text{(3.28)} \]
\[ B(IS1; g.s. \rightarrow 1^-_n) = |M_n(IS1)|^2. \quad \text{(3.29)} \]

For $^{24}$Mg, in order to compare with experimental data directly, I also define IS monopole strength function $S(E)$ and the energy non-weighted and weighted sums $m_k$ with $k = 0, 1, 3$,

\[ S(E) = \sum_n |M_n(IS0)|^2 E'_n \delta(E'_n - E), \quad \text{(3.30)} \]
\[ m_k = \int_0^\infty dE \sum_n |M_n(IS0)|^2 E'^{nk}_n \delta(E'_n - E), \quad \text{(3.31)} \]

where $E'_n$ stands for the excitation energy of the $n$th $0^+$ state, i.e. $E'_n = E_n - E_{g.s.}$. I also define partial energy non-weighted and weighted sum $m^*_k (k = 0, 1, 3)$ to compare with experimental data.

\[ m_k = \int_{9\text{MeV}}^{40\text{MeV}} dE \sum_n |M_n(IS0)|^2 E'^{nk}_n \delta(E'_n - E). \quad \text{(3.32)} \]

3.6 Cluster S-factor

To see the correlation between cluster configuration and IS monopole and dipole strengths, I introduce cluster S-factor as a measure of clustering of $^{28}$Si. For example, in the case of $^{16}$O+$^{12}$C configurations, cluster S-factor is defined as

\[ B_{ij} = \langle \Phi^\pi_{Mg+\alpha}(d_i) | \Phi^\pi_{Mg+\alpha}(d_i) \rangle, \quad \text{(3.33)} \]
\[ P_{O+C} = \sum_{ij} |\Phi^\pi_{O+C}(d_i) \rangle B_{ij}^{-1} \langle \Phi^\pi_{O+C}(d_j) |, \quad \text{(3.34)} \]
\[ S = \langle \Psi^\pi_n | P_{O+C} | \Psi^\pi_n \rangle \quad \text{(3.35)} \]
where $P_{O+C}$ is projector onto cluster model space defined by $\Phi_{O+C}^{J\pi}(d)$ and the basis wave functions with $d \leq 4.0$ fm are excluded from the model space to avoid contamination of mean-field states. Therefore, cluster S-factor is a good measure to know what extent a GCM wave functions is inside of $^{16}O+^{12}C$ cluster subspace. In the same manner, I define cluster S-factor for the other cluster configurations.
Chapter 4

Results and Discussions

In the chapter 2, using cluster model, I proved that IS monopole and dipole strengths to the cluster excited states are enhanced. The enhancements are ensured by duality of shell and cluster. However, in the realistic situation, SU(3)-symmetry is broken by strong $ls$-interaction and the cluster states are distorted and mixed with shell model states. Thus, the transition strength of the cluster states deviates from the estimates made in the previous section. Therefore, the theoretical analysis of the strength functions based on the realistic nucleon models is desired to clarify the nature of the observed resonances. In this chapter, I discuss the results of realistic calculation using AMD for $^{24}$Mg and $^{28}$Si. For $^{24}$Mg, I evaluate IS monopole strength function and analyze the structure of each $0^+$ states. I compare my results and the observed data directly and identify the cluster structure of the observed peaks. For $^{28}$Si, I calculate positive- and negative-parity states and propose the inversion-doublet bands of cluster states. Furthermore, I evaluate the IS monopole and dipole strengths to the excited states, I predict which cluster states are accessible via IS monopole and dipole transitions.

4.1 Isoscalar monopole strength and cluster structure of $^{24}$Mg

4.1.1 Results of the energy variation

Figure 4.1 (a) and (b) show the wave functions obtained by the constraint on the quadrupole deformation (Eq. 3.7) whose deformation parameters are $(\beta, \gamma) = (0.48, 22^\circ)$ and $(0.76, 35^\circ)$. After the GCM calculation, they become the dominant component of the ground and $0^+_2$ states, respectively. The centroids of the Gaussian
wave packets are gathered around the center-of-mass, describing triaxially deformed mean-field structure. As already discussed in the previous AMD work [80], the constraint on the quadrupole deformation generates this kind of the deformed mean-field structures [87,88], but no cluster structure.

On the other hand, energy variation with the constraint on the harmonic oscillator quanta (Eq. (3.17)) generates various kind of cluster configurations as well as single-particle excited configurations with approximate $\Delta N\hbar\omega$ excitations which are not accessible by the constraint on the quadrupole deformation. The panels (c)-(h) show the wave functions with various cluster configurations obtained by the constraint on the harmonic oscillator quanta, and they become the dominant component of the excited $0^+$ states corresponding to the prominent peaks in the IS monopole strength function $S(E)$ defined in Eq. (3.30) as discussed later. By the constraint of $\Delta N = 2$, $^{20}\text{Ne} + \alpha$ and $^{12}\text{C} + ^{12}\text{C}$ cluster states appear. As $\Delta N$ increases, the inter-cluster distance grows and the orientation of cluster changes depending on the combination of $\lambda$ and $\mu$. For example, the panels (c) and (d) show the $^{20}\text{Ne} + \alpha$ cluster configuration with $\Delta N = 4$ and 8, which mainly contribute to the $0^+_2$ and $0^+_5$ states, respectively. They have different orientation of $^{20}\text{Ne}$ cluster and inter-cluster distances (distance between the centroids of Gaussians describing clusters) are 3.0 and 3.3 fm, respectively. The panels (e), (f) and (g) show $^{12}\text{C} + ^{12}\text{C}$ cluster states with $\Delta N = 6$, 6 and 8, respectively. They have different orientations of the oblatelly deformed $^{12}\text{C}$ clusters, and inter-cluster distances are 3.5, 3.5 and 4.0 fm. By further increase of $\Delta N$, the $6\alpha$ cluster states appear as shown in the panels (h) and (i) which were obtained by the constraint of $\Delta N = 8$. In these configuration, 5$\alpha$ clusters locate at the vertex of a pentagon with side of 1.5 fm, and the last $\alpha$ cluster is 0.25 fm above it as shown in Fig. 4.1 (j). After the GCM calculation, the configurations displayed in the panel (h) and (i) generate two $0^+$ states above 20 MeV, respectively. Thus, by increasing the number of particle-hole, $^{24}\text{Mg}$ is clustered as illustrated in Ikeda diagram (Fig. 1.3) [9].

### 4.1.2 Properties of the calculated IS monopole transition strengths

The ground and excited $0^+$ states are calculated by the GCM with three different basis wave functions (a) $\Phi^+(\beta, \gamma)$, (b) $\Phi^+(\beta, \gamma)$ and $\Phi^+_{\text{IS}}(\beta, \gamma)$, (c) $\Phi^+(\beta, \gamma)$, $\Phi^+_{\text{IS}}(\beta, \gamma)$ and $\Phi^+(\Delta N, \lambda, \mu)$ where the basis wave functions $\Phi^+(\beta, \gamma)$, $\Phi^+(\Delta N, \lambda, \mu)$, $\Phi^+_{\text{IS}}(\beta, \gamma)$ are obtained respectively by the energy variations with constraints on the quadrupole deformation and harmonic oscillator quanta and shifted-basis AMD. The IS monopole transition strengths derived from these GCM wave functions are shown.
Figure 4.1: Intrinsic density distributions at the $z = 0$ plane obtained by constraint on the matter quadrupole deformation parameters ((a) and (b)) and the expectation values of the harmonic oscillator quanta ((c)-(i)). $5\alpha + \alpha$ cluster configuration of the panels (h) and (i) are illustrated in the panel (j). The crosses in each figure show the centroids of Gaussians describing nucleons. The contour lines are plotted in the interval of $0.2 \text{ fm}^{-3}$.
Figure 4.2: The isoscalar monopole strength function calculated with (a) $\Phi^+(\beta, \gamma)$, 
(b) $\Phi^+(\beta, \gamma) + \Phi^+_{IS0}(\beta, \gamma)$, (c) $\Phi^+(\beta, \gamma)$, $\Phi^+_{IS0}(\beta, \gamma)$ and $\Phi^+(\Delta N, \lambda, \mu)$. The solid 
line in the right panel shows the strength function smeared with Lorentzian with the 
width of 0.8 MeV. The vertical dashed lines indicate cluster decay threshold energies which are located at the observed binding energies.
Table 4.1: Calculated energy weighted sums $m_1$ (Eq. (3.31)) and $m_1^*$ (Eq. (3.32)) in fraction of the EWSR and the centroid energies of ISGMR in MeV evaluated from $m_1^*/m_0^*$ and $\sqrt{m_3^*/m_1^*}$. $m_1$, $m_1^*/m_0^*$ and $\sqrt{m_3^*/m_1^*}$ are calculated from the obtained states with $E = 9 – 40$ MeV excluding the $0^+_2$ state to compare with the experimental [51,57,58] and the other theoretical works [89].

<table>
<thead>
<tr>
<th>basis wave function</th>
<th>$m_1$</th>
<th>$m_1^*$</th>
<th>$m_1^<em>/m_0^</em>$</th>
<th>$\sqrt{m_3^<em>/m_1^</em>}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\Phi^+(\beta, \gamma)$</td>
<td>35</td>
<td>26</td>
<td>20.3</td>
<td>24.2</td>
</tr>
<tr>
<td>(b) $\Phi^+(\beta, \gamma)+\Phi^+_I(\beta, \gamma)$</td>
<td>116</td>
<td>101</td>
<td>25.6</td>
<td>29.3</td>
</tr>
<tr>
<td>(c) $\Phi^+(\beta, \gamma)+\Phi^+_I(\beta, \gamma)+\Phi^+(\Delta N, \lambda, \mu)$</td>
<td>103</td>
<td>90</td>
<td>22.2</td>
<td>25.2</td>
</tr>
<tr>
<td>exp. [51,57,58]</td>
<td>82 ± 9</td>
<td>21.9$^{+0.3}_{-0.2}$</td>
<td>24.7$^{+0.5}_{-0.3}$</td>
<td></td>
</tr>
<tr>
<td>QRPA [89]</td>
<td>94</td>
<td>20.57</td>
<td>20.57</td>
<td></td>
</tr>
</tbody>
</table>

in Fig. 4.2, and their energy weighted sums (Eq. (3.31)) and the centroid energies of Giant monopole resonance (GMR) are summarized in Tab. 4.1. With only the basis wave functions $\Phi^+(\beta, \gamma)$, the strength function (Fig. 4.2 (a)) fails to describe GMR, and the energy weighted sum $m_1$ amounts to only 35% of the sum rule (EWSR). Addition of the basis wave functions $\Phi^+_I(\beta, \gamma)$ (Fig. 4.2 (b)) greatly improves $m_1$ value (116% of EWSR), but overestimates the observed GMR centroid energy [51, 57, 58] because the GMR strength distributes widely in the region of $E > 30$ MeV. The inclusion of the basis wave functions $\Phi^+(\Delta N, \lambda, \mu)$ yields the reasonable strength function as shown in Fig. 4.2 (c). Namely various cluster and single-particle states with $\Delta N\hbar\omega$ excitation described by $\Phi^+(\Delta N, \lambda, \mu)$ lower the GMR position and enhance its strength. As a result, the strength function exhausts approximately 100% of EWSR and plausibly agrees with the experimental energy weighted sum and the GMR centroid energy observed in the energy range of $E = 9 – 40$ MeV. It is also noted that the quasi-particle random phase approximation (QRPA) with Gogny D1S interaction [89] also yielded similar values and qualitatively agrees with the present results and experiment with respect to the global structure of GMR.

From the comparison between the strength functions shown in Fig. 4.2 (b) and (c), I also see that not only the GMR strength ($E \geq 18$ MeV) but also the low-lying structure ($E \leq 18$ MeV) of the strength function is largely modified by the basis wave functions $\Phi_{\Delta N}$. For example, note that the prominent peak at 15.3 MeV in Fig. 4.2 (c) is completely missing in Fig. 4.2 (b). Based on the analysis of the wave functions corresponding to each peak, I can conclude that several prominent peaks are attributed to the cluster states and suggest that the cluster states shown in Fig. 4.1 can be populated and observed by their enhanced IS monopole transition strength. To see this point, I discuss the structure of the $0^+$ states relevant to the
prominent peaks in $S(E)$ in the following.

4.1.3 Cluster correlation in the ground state and structure of cluster states

The ground state is of course dominated by the mean-field structure and has the largest overlap (0.93) with the wave function shown in Fig. 4.1 (a). However, at the same time, it also has non-negligible overlaps with the cluster wave functions. For example, it has 0.26 and 0.40 overlaps with $^{20}$Ne+$\alpha$ and $^{12}$C+$^{12}$C cluster states shown in Fig. 4.1 (c) and (e), respectively. This result means following two points. The first is that the cluster correlation exists even in the ground state. The binding energy of the ground state increases from 198.3 MeV to 199.2 MeV by including $\Phi^+(\Delta N, \lambda, \mu)$ which indicates that the additional binding energy of 0.9 MeV is brought about by the cluster correlation. Secondly, it shows that the ground state has "duality nature" of the mean-field and clusters and that the degrees-of-freedom of cluster excitation are embedded in the ground state. This is an essential ingredient for the discussion of the that IS monopole transition from the ground to the excited cluster states as discussed in chapter 2 [76,90].

Cluster correlation is more important in the prominent peaks in Fig. 4.2 (c). The low-lying prominent peaks below 15 MeV in Fig. 4.2 (c), the 0$^+_2$, 0$^+_3$ and 0$^+_5$ states, have the mixing nature of the low-lying states below 15 MeV in Fig. 4.2 (b), and the $^{20}$Ne+$\alpha$, and $^{12}$C+$^{12}$C cluster states. The 0$^+_5$ state at 15.3 MeV, which is completely missing in Fig. 4.2 (b), is dominated by the $^{12}$C+$^{12}$C cluster states. Furthermore, 6$\alpha$ cluster states appear as the 0$^+_1$ and 0$^+_2$ states embedded in the GMR region.

The 0$^+_2$ state which appears as the lowest peak at 9.3 MeV in the strength function has the largest overlap (0.36) with the mean-field like wave function of Fig. 4.1 (b) which has larger quadrupole deformation parameter $\beta$ than the ground state. It can be regarded as the $\beta$-band built on the ground band, and hence, has large IS monopole transition strength as listed in Tab. 4.2. However, it also has 0.32 overlap with the $^{20}$Ne+$\alpha$ cluster wave function shown in Fig. 4.1 (c), as a result, it gains additional binding energy of 1.8 MeV which reduces the excitation energy from 10.2 MeV to 9.3 MeV by the inclusion of the cluster wave function.

Similar to the 0$^+_2$ state, the 0$^+_5$ state also has mixed nature of mean-field and $^{20}$Ne+$\alpha$ cluster states and the enhanced IS monopole strength. It has the largest overlap (0.30) with the wave function of the $^{20}$Ne+$\alpha$ cluster state shown in Fig. 4.1 (d) and the comparable overlap (0.25) with the $\Phi^+(\beta = 0.45, \gamma = 57^\circ)$.

The 0$^+_5$ state at 11.7 MeV has strong IS monopole transition strength (4 % of EWSR) and has the large overlaps with $^{12}$C+$^{12}$C, $^{20}$Ne+$\alpha$ and mean-field structures.
Table 4.2: Properties of the states obtained by GCM calculation with $\Phi^+(\beta, \gamma)$, $\Phi^+_{IS0}(\beta, \gamma)$ and $\Phi^+(\Delta N, \lambda, \mu)$. $E$, $\sqrt{\langle r_p^2 \rangle}$, $B(IS0)$ and $EB(IS0)$ are given in unit of MeV, fm, fm$^4$ and % of EWSR, respectively. The values in bracket are the observed values [91–94].

<table>
<thead>
<tr>
<th>State</th>
<th>$E$</th>
<th>$\sqrt{\langle r_p^2 \rangle}$</th>
<th>$B(IS0)$</th>
<th>$EB(IS0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^+_1$</td>
<td>0.0</td>
<td>3.06 (3.06)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0^+_2$</td>
<td>9.3 (6.4)</td>
<td>3.11</td>
<td>122 (180 ± 20)</td>
<td>6.1</td>
</tr>
<tr>
<td>$0^+_3$</td>
<td>11.7</td>
<td>3.08</td>
<td>59.3</td>
<td>3.7</td>
</tr>
<tr>
<td>$0^+_5$</td>
<td>13.2</td>
<td>3.06</td>
<td>31.1</td>
<td>2.2</td>
</tr>
<tr>
<td>$0^+_8$</td>
<td>15.3</td>
<td>3.11</td>
<td>77.8</td>
<td>6.4</td>
</tr>
<tr>
<td>$0^+_{15}$</td>
<td>21.8</td>
<td>3.14</td>
<td>21.6</td>
<td>2.5</td>
</tr>
<tr>
<td>$0^+_{24}$</td>
<td>24.9</td>
<td>3.28</td>
<td>8.90</td>
<td>1.2</td>
</tr>
</tbody>
</table>

The overlaps are 0.21, 0.19 and 0.16 respectively with $^{12}$C+$^{12}$C and $^{20}$Ne+$\alpha$ cluster wave functions in $\Phi^+(\Delta N, \lambda, \mu)$ and the mean-field wave function with $\Phi^+(\beta = 0.76, \gamma = 2.4^\circ)$. Thus, the $0^+_3$ states has mixing nature of mean-field and cluster states.

In contrast to the $0^+_2$, $0^+_3$ and $0^+_5$ state, the $0^+_8$ state at 15.3 MeV is dominated by $^{12}$C+$^{12}$C cluster states. It has 0.14 0.11 0.18 overlaps with the wave functions of Fig. 4.1 (e), (f) and (g). The overlaps with the other structures are less than 0.09 at most. The $0^+_8$ state has strong IS monopole transition strength (6 % of EWSR) but it is clearly missing in Fig. 4.2 (a), (b) and the QRPA calculation [89], which strongly suggests that the cluster nature of the $0^+_8$ state.

The 6$\alpha$ cluster states appear as the $0^+_{15}$ and $0^+_{24}$ states at 21.8 and 24.9 MeV, which are 7.6 and 4.5 below of the 6$\alpha$ threshold $E = 29.4$. This is owing to their compact pentagon-like configuration shown in Fig. 4.1 (h), which indicates that the interactions between $\alpha$ clusters are not negligible. The $0^+_{15}$ and $0^+_{24}$ states exhaust the 3 and 1 % of EWSR. Although they are embedded in GMR, I expect that these states can be identified because they should strongly decay by emitting $\alpha$-particles. The recent Hartree-Fock-Bogoliubov (HFB) calculation [14] predicted 6$\alpha$ cluster state as the local minimum of the energy surface. However, the configurations of 6$\alpha$ states obtained in the present result and the HFB calculation are completely different. The 6$\alpha$ state predicted by the HFB calculation has well-developed octahedral 6$\alpha$ configuration with 1.9 times larger radius than the ground state. On the other hand, the radii of 6$\alpha$ cluster states obtained in present result are not so large (Table 4.2). It should be noted that the description of developed 6$\alpha$ cluster states is difficult in
As discussed above, a variety of exotic cluster states can be seen in the IS monopole strength function. These cluster states exhaust 1-6% of EWSR and the sum of the strengths below 18 MeV reaches 20% of EWSR. Therefore, clustering is essential degree-of-freedom for $^{24}$Mg as well as collective and single-particle excitations.

Recently, T. Kawabata et al. [60] observed the IS monopole strength function of $^{24}$Mg using $\alpha$-inelastic scattering. Very narrow $0^+$ peaks are observed near the cluster decay threshold energies and the structure of the strength function is very similar to the present results as shown in Fig. 4.3. Thus, the prominent peaks below 18 MeV Fig. 4.2 (c) may correspond to the observed peaks below 16 MeV. However, to conclude this, the more qualitative discussion based on the cluster decay width is needed.
4.1.4 Summary for $^{24}$Mg

We investigate the $0^+$ cluster states populated by IS monopole transition of $^{24}$Mg with AMD. The collectivity and clustering of $^{24}$Mg are described by the basis wave function $\Phi^+(\beta, \gamma)$ and $\Phi^+(\Delta N, \lambda, \mu)$, respectively. The single particle excited states $\Phi^+_{IS0}(\beta, \gamma)$ are also introduced into the present calculation, which defined as the 1p1h ($2\hbar\omega$) excited states built on the basis wave function $\Phi^+(\beta, \gamma)$ by the IS monopole operator. Only with the basis wave functions $\Phi^+(\beta, \gamma)$ and $\Phi^+_{IS0}(\beta, \gamma)$, the calculated GMR centroid energies are higher than the observable ones. However, inclusion of $\Phi^+(\Delta N, \lambda, \mu)$ yields the reasonable strength function because various cluster and single-particle states with $\Delta N$ excitation in $\Phi^+(\Delta N, \lambda, \mu)$ lower the GMR position and enhance its strength. The calculated strength function exhausts almost 100% of EWSR and agree with the observed GMR properties.

The calculated IS monopole strength function indicates the presence of four prominent peaks below $E = 18$ MeV, the $0^+_2$, $0^+_3$, $0^+_5$, and $0^+_8$ states. The $0^+_2$, $0^+_3$ and $0^+_5$ states are generated by the mixing of mean-field, $^{20}$Ne+$\alpha$ and $^{12}$C+$^{12}$C cluster states. The $0^+_8$ state is dominated by $^{12}$C+$^{12}$C cluster states. Furthermore, we predicted two $6\alpha$ cluster states embedded in GMR region, which have the compact pentagon-like $6\alpha$ configuration. Thus, the calculated strength function shows appearance of a variety of exotic cluster states. The IS monopole transition strengths from the ground state to these cluster states are 1-6% of EWSR and the sum of the strength below 18 MeV reaches 20% of EWSR, which indicates clustering is an essential degree-of-freedom in the excited states of $^{24}$Mg. In the recent experiment [60], very similar structure to the present results is observed, especially in the below of GMR. It is suggested that the predicted low-lying states may correspond to the observed peaks. However, to conclude relationship between cluster structure and the observed peaks, the more quantitative discussions are needed.
Figure 4.4: The energy surfaces as functions of the quadrupole deformation parameters $\beta$ and $\gamma$ obtained by the energy variation with $\beta\gamma$-constraint and the angular momentum projection to the (a) $J^\pi = 0^+$ and (b) $J^\pi = 1^-$. Red circles show the energy minima or plateau.

4.2 Isoscalar monopole and dipole strengths and asymmetric cluster structure of $^{28}$Si

4.2.1 Result of energy variation

The energy surface of the $J^\pi = 0^+$ state obtained by energy variation with the constraint on the quadrupole deformation (Eq. 3.7) and the angular momentum projection is shown in Fig. 4.4 (a). There exist a couple of energy minima or plateau with different quadrupole deformations. The lowest energy minimum has an oblate shape of $(\beta, \gamma) = (0.36, 46^\circ)$ with the energy $E = -235.7$ MeV, whose intrinsic density distribution is shown in Fig. 4.5 (a). Around this global minimum, the energy surface is rather soft against both of $\beta$ and $\gamma$ deformation. The second lowest state is prolate deformed as seen in its intrinsic density distribution (Fig. 4.5 (b)) and locates at $(\beta, \gamma) = (0.5, 0^\circ)$ as a very shallow energy minimum. Those two energy minima indicate the oblate and prolate shape coexistence in this nucleus and yield the oblate deformed ground state and the prolate deformed $0^+_2$ state by the GCM calculation, respectively. By further increase of the deformation, the third energy minimum with strongly elongated shape (Fig. 4.5 (c)) appears at $(\beta, \gamma) = (0.85, 5^\circ)$. As discussed in Ref. [95], this configuration has the $(sd)^8(pf)^4$ configuration and becomes the dominant component of the $0^+_4$ state which is regarded as the superde-
Figure 4.5: Intrinsic matter density distributions of the minima on the positive- and negative-parity energy surfaces obtained by the constraint on quadrupole deformation (Eq (3.7)) shown in Fig. 4.4. The panels (a), (b) and (c) show the oblate, prolate and SD $J^\pi = 0^+$ minima, while the panels (d), (e) and (f) show the $J^\pi = 1^-$ minima generated by the 1p1h excitations from the positive-parity minima.

formed (SD) state.

The energy surface of the negative-parity $J^\pi = 1^-$ state shown in Fig. 4.4 (b) does not have clear local minima, but there are three shallow minima or plateau that are generated by the single-particle excitations. The global minimum is located at $(\beta, \gamma) = (0.43, 27^\circ)$ with the energy $E = -221.2$ MeV. This configuration has the density distribution (Fig. 4.5 (d)) similar to the oblate deformed ground state, because it is generated by the one-nucleon excitation from the ground state configuration. As shown in Fig. 4.5 (e) and (f), there also exist the prolate deformed and SD negative-parity minima that are generated by the 1p1h excitations from the corresponding positive-parity minima. They are respectively located at $(\beta, \gamma) = (0.60, 0^\circ)$ and $(0.73, 2^\circ)$ with the energies $E = -220.8$ and -219.1 MeV. In terms of the Nilsson orbit, the prolate minimum is generated by the nucleon excitation from the $[Nn_2ml\Omega^\pi] = [211 1/2^+]$ orbit to the $[330 1/2^-]$ orbit, while the negative-parity SD minimum is generated by the nucleon deexcitation from the $[330 1/2^-]$ orbit to the $[211 1/2^+]$ orbit.

As confirmed from the intrinsic density distributions shown in Fig. 4.5, the
Figure 4.6: Energy curves obtained by \( d \)-constraint (Eq. (3.18)). Panels (a), (b) and (c) respectively show the energy curves of the \( J^\pi = 0^+ \) and \( 1^- \) states having \( ^{24}\text{Mg} + \alpha \), \( ^{16}\text{O} + ^{12}\text{C} \) and \( ^{20}\text{Ne} + ^{8}\text{Be} \) configurations. Circles and Boxes in the figure show the overlap between these \( J^\pi = 0^+ \) cluster configurations and the energy minima on the \( \beta\gamma \) energy surface. Black circles in panels (a) and (c) shows the overlap between the oblate deformed minimum (ground state) and \( ^{24}\text{Mg} + \alpha(T) \), \( ^{20}\text{Ne} + ^{8}\text{Be} \) configurations, while the red boxes in the panel (a) shows the overlap between the SD minimum and \( ^{24}\text{Mg} + \alpha(A) \) configuration. The circles in the panel (b) show the overlap between the prolate deformed minimum and the \( ^{12}\text{C} + ^{16}\text{O} \) configuration.
energy variation with the constraint on the quadrupole deformation does not generate prominent cluster configurations, but mean-field configurations. On the other hand, the $d$-constraint yields various cluster configurations. Figure 4.6 shows the energy curves obtained by the $d$-constraint. In the previous AMD study [95], the $d$-constraint method was applied to the positive-parity states of the $^{24}$Mg + $\alpha$ and the $^{16}$O + $^{12}$C configurations. In addition to them, in the present study, I applied it to the $^{20}$Ne + $^{8}$Be configuration and investigated both of the positive- and negative-parity. As a result, I found that all of these cluster configurations appear in both parity states.

Figure 4.6 (a) shows the energy curves for $^{24}$Mg + $\alpha$ cluster configurations projected to the $J^\pi = 0^+$ and $1^-$ states. Because $^{24}$Mg cluster is prolately deformed, two different $^{24}$Mg + $\alpha$ configurations were obtained, in which the orientations of $^{24}$Mg cluster are different. In the configuration denoted by $^{24}$Mg + $\alpha$ (T), the longest axis of $^{24}$Mg is perpendicular to the inter-cluster coordinate between $^{24}$Mg and $\alpha$ clusters. As a result, total system is triaxially deformed as seen in its density distribution shown in Fig. 4.7 (a). It is noted that, when the inter-cluster distance becomes small, the wave function of this configuration with $J^\pi = 0^+$ becomes almost identical to that of the oblate minimum on the $\beta\gamma$ energy surface (i.e. the ground state). The overlap between the wave functions of the $^{24}$Mg + $\alpha$ (T) configuration and the oblate deformed ground state minimum (circles in Fig. 4.6 (a)) has the

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Figure 4.7: Intrinsic matter density distributions obtained by $d$-constraint (Eq. (3.18)). (a), (b), (c) and (d) respectively show the $^{24}$Mg + $\alpha$ (T), $^{24}$Mg + $\alpha$ (A), $^{16}$O + $^{12}$C and $^{20}$Ne + $^{8}$Be configurations with $J^\pi = 0^+$, while (e), (f), (g) and (h) show the $J^\pi = 1^-$ partner having the same configurations.
maximum value 0.96 at $d = 2.0$ fm, and their energies are very close to each other (-235.7 MeV and -234.5 MeV, respectively). Note that it does not necessarily mean that the ground state is clustered, but it means the equivalence of the cluster and shell model wave functions at small inter-cluster distance. This duality of shell and cluster is an essential ingredient for the enhanced monopole and dipole transitions discussed in the section 4.2.3.

The negative-parity $J^\pi = 1^- \rangle$ state with the same $^{24}\text{Mg} + \alpha \ (T)$ configuration (Fig. 4.7 (e)) appears at relatively high excitation energy that is approximately 15 MeV above the ground state. Different from the $J^\pi = 0^+$ state, this negative-parity state has small overlap with the negative-parity minima in the $\beta\gamma$ energy surface, which amounts to 0.20 at most. Other cluster configurations mentioned below also have small overlap with the mean-field configurations. This means that the constraint on the quadrupole deformation and $d$-constraint are describing the different class of the negative-parity states. Namely, the constraint on the quadrupole deformation yields the single-particle excited states built on the mean-filed, while the $d$-constraint yields the parity-asymmetric cluster states in which the relative motion between clusters have odd angular momenta.

In another $^{24}\text{Mg} + \alpha$ configuration denoted by $^{24}\text{Mg} + \alpha \ (A)$, the longest axis of the $^{24}\text{Mg}$ cluster is parallel to the inter-cluster coordinate, and hence, the system is axially deformed as shown in Fig. 4.7 (b). This configuration with $J^\pi = 0^+$ has large overlap with the SD configuration shown in Fig. 4.5 (c). The overlap between them amounts to 0.42 at the inter-cluster distance $d = 4.5$ fm. It is interesting to note that the negative-parity $J^\pi = 1^-$ state (Fig. 4.7 (f)) has almost the same intrinsic density distribution and almost the same energy with the positive-parity $J^\pi = 0^+$ state. It is because of the large inter-cluster distance of the $^{24}\text{Mg} + \alpha \ (A)$ configuration compared to the $^{24}\text{Mg} + \alpha \ (T)$ configuration. The $^{24}\text{Mg} + \alpha \ (A)$ configuration in negative-parity does not have corresponding state on $\beta\gamma$ energy surface having large overlap.

The $^{16}\text{O} + ^{12}\text{C}$ configuration with $J^\pi = 0^+$ appears approximately 10 MeV above the ground state with the inter-cluster distance $d = 2.5$ fm. At small inter-cluster distance, this configuration has large overlap with the prolate minimum located at $(\beta, \gamma) = (0.5, 0^\circ)$ on the $\beta\gamma$ energy surface (Fig. 4.5 (c)). The overlap amounts to 0.90 at $d = 2.5$ fm. The negative-parity $J^\pi = 1^-$ state has similar intrinsic density distribution to the positive-parity state as shown in Fig. 4.7 (g) and has the excitation energy close to its positive-parity partner. I also obtained another $^{16}\text{O} + ^{12}\text{C}$ configuration having different orientation of the $^{12}\text{C}$ which may corresponds to the highly excited $^{16}\text{O} + ^{12}\text{C}$ cluster states or molecular resonances [25–30]. However, its energy is rather high and is not discussed here. Finally, I explain the $^{20}\text{Ne} + ^8\text{Be}$
configuration. It has the triaxial intrinsic density distribution as shown in Fig. 4.7 (d) in which the longest axes of $^{20}$Ne and $^8$Be clusters are parallel to each other, but perpendicular to the inter-cluster coordinate. At small inter-cluster distance, this configuration with $J^\pi = 0^+$ becomes identical to the oblate minimum on $\beta\gamma$ energy surface (the ground state). Thus, the ground state, the $^{24}$Mg + $\alpha$ and $^{20}$Ne + $^8$Be cluster configurations have large overlap to each other at small inter-cluster distance. The negative-parity $J^\pi = 1^-$ state of $^{20}$Ne + $^8$Be configuration appears at approximately 17 MeV above the ground state with the inter-cluster distance $d = 3.0$ fm.

The result of the energy variation is summarized as follows. (1) The constraint on the quadrupole deformation yielded three mean-field configurations with $J^\pi = 0^+$ having oblate, prolate and SD shapes. The oblate minimum has the lowest energy and corresponds to the ground state, while the others constitute the excited $0^+$ states. (2) The $d$-constraint yielded prominent $^{24}$Mg + $\alpha$ (T) and (A), $^{16}$O + $^{12}$C and $^{20}$Ne + $^8$Be cluster configurations with large inter-cluster distance and smoothly connects them to the mean-filed states at small inter-cluster distance. These cluster configurations have large overlap with the mean-field configurations indicating that the prolate, oblate and SD minima on $\beta\gamma$ energy surface have duality of shell and cluster. Namely, the oblate deformed ground state has the duality of $^{24}$Mg + $\alpha$ (T) and $^{20}$Ne + $^8$Be cluster configurations. The prolate minimum has the duality of the $^{16}$O + $^{12}$C configuration and the SD minimum has the duality of the $^{24}$Mg + $\alpha$ (A) configuration. (3) All of the cluster configurations are accompanied by the negative-parity partner having almost the same intrinsic density distributions. These negative-parity states originate in their parity-asymmetric cluster configurations.

4.2.2 Excitation spectrum and clustering
Figure 4.8: Calculated and observed partial level scheme of $^{28}$Si. The levels shown by dashed lines are the averaged energies of the several states which have sizable cluster $S$-factors. Dotted lines show the $^{24}$Mg + $\alpha$, $^{16}$O + $^{12}$C and $^{20}$Ne + 2$\alpha$ threshold energies.
Overview of the spectrum

Figure 4.8 shows the energy levels obtained by the GCM calculation together with the corresponding observed states. In the figure, the rotational bands which have sizable amount of the cluster $S$-factors and overlaps with the cluster wave functions are shown. Other excited states are omitted except for the $\beta$-band built on the ground-state band. The detailed distribution of the cluster $S$-factors in the excited states is discussed in the section 4.2.3.

The results for the positive-parity states are consistent with the previous AMD study [95], and I find that the most of the positive-parity bands are accompanied by the negative-parity bands because of their duality of mean-field and parity-asymmetric clustering. The present result is briefly summarized as follows. The oblate minimum on the $\beta\gamma$ energy surface and the $^{24}\text{Mg}+\alpha\ (T)$ configuration are mixed to each other and generate a group of the rotational bands denoted by $^{24}\text{Mg}+\alpha\ (T)$ bands in Fig. 4.8. The oblate minimum is also mixed with the $^{20}\text{Ne}+{^8}\text{Be}$ configuration to generate a pair of the positive- and negative-parity bands denoted by $^{20}\text{Ne}+{^8}\text{Be}$ doublet. In a similar way, the prolate deformed minimum is mixed with the $^{16}\text{O}+{^{12}}\text{C}$ configuration, and the SD minimum is mixed with the $^{24}\text{Mg}+\alpha\ (A)$ configuration. As a result, they respectively generate pairs of the positive- and negative-parity bands, which are denoted by $^{16}\text{O}+{^{12}}\text{C}$ and $^{24}\text{Mg}+\alpha\ (A)$ doublets. In addition to them, the positive-parity band denoted by $\beta$-band is generated by the $\beta$ vibration of the ground-state band.

$^{24}\text{Mg}+\alpha\ (T)$ bands

A group of the rotational bands denoted by $^{24}\text{Mg}+\alpha\ (T)$ bands includes three positive-parity bands and two negative-parity bands. The positive-parity bands are the ground-state band ($K^\pi = 0^+$), $K^\pi = 2^+$ band and another $K^\pi = 0^+$ band built on the $0^+_1$ state at 18.2 MeV. Because of the parity-asymmetric clustering of the $^{24}\text{Mg}+\alpha\ (T)$ configuration, two negative-parity $K^\pi = 0^-$ bands built on the $1^-_1$ and $1^-_4$ are paired with the $K^\pi = 0^+$ bands to constitute two parity doublets which are denoted by the ground doublet and $^{24}\text{Mg}+\alpha\ (T)$ doublet in the figure.

The ground-state band is dominated by the oblatelly deformed mean-field configuration shown in Fig. 4.5 (a) whose overlap with the GCM wave function amounts to 0.84. The moment-of-inertia of the ground-state band and the electric quadrupole transition probability $B(E2)$ listed in Tab. 4.3 reasonably agree with the observed data, indicating that the deformed mean-field nature of the ground state is properly described by the present calculation. However, it must be noted that this ground-state band also has the large overlap with the $^{24}\text{Mg}+\alpha\ (T)$ configuration with small
Table 4.3: The calculated in-band $B(E2) (e^2 fm^4)$ strengths in the ground-state, $^{24}\text{Mg} + \alpha \text{ (T), } ^{20}\text{Ne} + ^8\text{Be, } ^{16}\text{O} + ^{12}\text{C, } ^{24}\text{Mg} + \alpha \text{ (A) doublets and in the } \beta\text{-band. The numbers in the parenthesis are the experimental data [96].}$

<table>
<thead>
<tr>
<th></th>
<th>ground</th>
<th>$^{24}\text{Mg} + \alpha \text{ (T)}$</th>
<th>$^{20}\text{Ne} + ^8\text{Be}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(E2; 2^+ \rightarrow 0^+)$</td>
<td>79.4 (67)</td>
<td>29.2</td>
<td>15.3</td>
</tr>
<tr>
<td>$B(E2; 4^+ \rightarrow 2^+)$</td>
<td>123 (83)</td>
<td>28.9</td>
<td>12.7</td>
</tr>
<tr>
<td>$B(E2; 3^- \rightarrow 1^-)$</td>
<td>94.9</td>
<td>24.9</td>
<td>18.3</td>
</tr>
<tr>
<td>$B(E2; 5^- \rightarrow 3^-)$</td>
<td>111</td>
<td>27.2</td>
<td>10.8</td>
</tr>
<tr>
<td></td>
<td>$^{16}\text{O} + ^{12}\text{C}$</td>
<td>$^{24}\text{Mg} + \alpha \text{ (A)}$</td>
<td>$\beta\text{-band}$</td>
</tr>
<tr>
<td>$B(E2; 2^+ \rightarrow 0^+)$</td>
<td>221</td>
<td>664</td>
<td>52.8</td>
</tr>
<tr>
<td>$B(E2; 4^+ \rightarrow 2^+)$</td>
<td>299 (150)</td>
<td>939</td>
<td>69.7</td>
</tr>
<tr>
<td>$B(E2; 3^- \rightarrow 1^-)$</td>
<td>244</td>
<td>409</td>
<td></td>
</tr>
<tr>
<td>$B(E2; 5^- \rightarrow 3^-)$</td>
<td>382</td>
<td>424</td>
<td></td>
</tr>
</tbody>
</table>

The inter-cluster distance, which amount to 0.96 for $d = 2.0 \text{ fm}$. This means that the ground-state band has a duality of the oblate shaped mean-field and $^{24}\text{Mg} + \alpha \text{ (T)}$ clustering. Therefore, the excitation of the inter-cluster motion between $^{24}\text{Mg}$ and $\alpha$ clusters yields excited bands with prominent clustering. The $1h\omega$ excitation of the inter-cluster motion yields the $K^\pi = 0^-$ band built on the $1^-$ state denoted by $1^-_1$ which is dominated by the negative-parity $^{24}\text{Mg} + \alpha \text{ (T)}$ configuration with $d = 2.5 \text{ fm}$. Hence, I assigned it as the partner of the ground-state band which constitutes the ground doublet, although the $1p1h$ mean-field configuration (Fig. 4.5 (d)) also has non-negligible contribution to this band. In addition to the ground doublet, the 2 and $3h\omega$ excitations of the inter-cluster motion yield $K^\pi = 0^\pm$ bands built on the $0^+_6$ state and a group of $1^-$ states denoted by $1^-_V$, which constitute another doublet denoted by $^{24}\text{Mg} + \alpha \text{ (T)}$ doublet. Because of the coupling with other non-cluster configurations, three $1^-$ states around 22 MeV have large $^{24}\text{Mg} + \alpha$ cluster $S$-factors, and their averaged energy is denoted by $1^-_V$ in Fig. 4.8. As the inter-cluster motion is largely excited, these bands have prominent cluster structure. The $0^+_6$ and $1^-_V$ states have large overlap with the configurations shown in Fig. 4.7 (a) and (e) which amount to 0.32 and 0.22 (averaged), respectively. In addition to these parity doublets, the triaxial deformation of the $^{24}\text{Mg} + \alpha \text{ (T)}$ configuration yields $K^\pi = 2^+$ band. Thus, the duality of the ground state yields two parity doublets and $K^\pi = 2^+$ band which are classified as $^{24}\text{Mg} + \alpha \text{ (T)}$ bands.
\[ ^{20}\text{Ne} + ^{8}\text{Be} \text{ doublet and } \beta\text{-band} \]

As discussed in the section 4.2.1, the oblate deformed minimum also has large overlap with the \(^{20}\text{Ne} + ^{8}\text{Be}\) configuration with small inter-cluster distance. Therefore, the excitation of the inter-cluster motion between \(^{20}\text{Ne}\) and \(^{8}\text{Be}\) clusters should yield a series of the cluster bands. In addition, the deformed mean-field aspect of the ground state can yield a different kind of excitation mode, \textit{i.e.} the \(\beta\) vibration. The GCM calculation showed that these two excitation modes strongly mix to each other to yield the \(0^+_2\) and \(0^+_5\) states. Their overlap with the configurations shown in Fig. 4.7 (d) amount to 0.13 and 0.18, respectively. Similarly to the \(^{24}\text{Mg} + \alpha\) configuration, the \(^{20}\text{Ne} + ^{8}\text{Be}\) configuration yields the \(1^-_{\text{II}}\) state which is paired with the \(0^+_5\) state to constitute the parity doublet denoted by \(^{20}\text{Ne} + ^{8}\text{Be}\) doublet.

\[ ^{16}\text{O} + ^{12}\text{C} \text{ doublet and } ^{24}\text{Mg} + \alpha \text{ (A) doublet} \]

In addition to the above-mentioned bands related to the ground state duality, there are other bands which are unrelated to the ground state. The prolate band is built on the \(0^+_3\) state at 10.0 MeV which has the large overlap with the prolate deformed local minimum shown in Fig. 4.5 (b). Combined with the oblately deformed ground state, this prolate deformed \(0^+_3\) state indicates the shape coexistence in the low-lying states of \(^{28}\text{Si}\). As discussed in the previous work [95], this prolate band has large overlap with the \(^{16}\text{O} + ^{12}\text{C}\) cluster configuration. Hence, it is concluded that the prolate band has the duality of the prolate deformed mean-field and \(^{16}\text{O} + ^{12}\text{C}\) clustering. The negative-parity band built on the \(1^-_{\text{II}}\) state also has the large overlap with the \(^{16}\text{O} + ^{12}\text{C}\) configuration (Fig. 4.7 (g)), and assigned as the partner of the positive-parity prolate band, that constitutes the \(^{16}\text{O} + ^{12}\text{C}\) doublet.

Another prolate deformed minimum \textit{i.e.} the SD minimum located at \((\beta, \gamma) = (0.85, 5^\circ)\) generates the SD band built on the \(0^+_4\) state at 12.7 MeV. This band has large overlap with the \(^{24}\text{Mg} + \alpha \text{ (A)}\) configuration shown in the Fig. 4.7 (f). There are two \(1^-\) states having large overlap with the negative-parity \(^{24}\text{Mg} + \alpha \text{ (A)}\) configuration and the mixing with the \(^{16}\text{O} + ^{12}\text{C}\) configuration. Their averaged energy is denoted by \(1^-_{\text{IV}}\) and the negative-parity band built on these states is associated with the positive-parity band. I denote this doublet as \(^{24}\text{Mg} + \alpha \text{ (A)}\) doublet. As discussed in the previous section, the \(^{16}\text{O} + ^{12}\text{C}\) and \(^{24}\text{Mg} + \alpha \text{ (A)}\) configurations do not have overlap with the oblate deformed ground state. Therefore, the \(^{16}\text{O} + ^{12}\text{C}\) and \(^{24}\text{Mg} + \alpha \text{ (A)}\) doublets are disconnected with the ground state.
Systematics of clustering and observed candidates

To summarize this section, the systematics of the cluster states explained above is schematically illustrated in Fig. 4.9. The ground state has duality of the oblate deformed mean-field, $^{24}$Mg + $\alpha$ (T) and $^{20}$Ne + $^8$Be cluster configurations. By the excitation of the inter-cluster motion between $^{24}$Mg and $\alpha$ clusters yields a group of the $^{24}$Mg + $\alpha$ (T) bands. The $^{20}$Ne + $^8$Be clustering also arise from the ground state duality and it is strongly mixed with the $\beta$-vibration mode which arises from the mean-field aspect of the ground state. Aside from these bands, the $^{16}$O + $^{12}$C and $^{24}$Mg + $\alpha$ (A) doublets exist and disconnected from the ground state, because of the orthogonality of their cluster configurations to the ground state.

Experimentally, the low-lying three positive-parity bands, i.e. the ground-state band, $\beta$-band and the prolate band, are assigned firmly and coincide with the present calculation, although the calculation slightly overestimates the energies of the excited bands. On the other hand, the experimental assignment of the negative-parity bands and high-lying bands are not established yet, and hence, the assignment of the cluster bands is still ambiguous. Many experiments have been performed to identify the cluster bands [25,37,38,41,44,96,97], and Fig. 4.8 shows the candidates of the cluster bands reported in Refs. [37, 44, 96], which energetically coincide with the present calculation. A couple of $0^+, 2^+$ and $1^-, 3^-$ states were reported around $E_x = 13$ MeV by the $\alpha$ transfer and radiative $\alpha$ capture reactions. They have relatively large $\alpha$ decay width, hence, can be regarded as the candidates of the $^{24}$Mg+$\alpha$ (T) or $^{24}$Mg+$\alpha$ (A) doublets. Furthermore, based on the analysis of the $^{24}$Mg($\alpha, \gamma$)
and $^{12}\text{C}(^{20}\text{Ne}, \alpha)^{28}\text{Si}$ reactions, another rotational band was suggested [96] and the authors assigned it to the SD band predicted by the previous AMD study [95].

### 4.2.3 Isoscalar monopole and dipole transitions

Here, I discuss that part of the clustering systematics summarized above can be detected by the IS monopole and dipole transitions from the ground state. To illustrate it, I first discuss the duality of shell and cluster. Then, I present the result of AMD calculation to show that the IS monopole and dipole transitions strongly yield cluster states.

#### Duality of shell and cluster

In the section 4.2.1 and 4.2.2, I have explained that the many low-lying positive-parity states have the duality of the mean-field (shell) and cluster. Here, I show that it is reasonably understood by the $SU(3)$ shell model [75] and the Bayman-Bohr theorem [76,90].

$^{28}\text{Si}$ has 12 nucleons in $sd$-shell on top of the $^{16}\text{O}$ core and its oblate deformed ground state can be approximated by the $(\lambda, \mu) = (0, 12)$ representation of $SU(3)$ shell model. Denoting the eigenstates of three-dimensional harmonic oscillator in the Cartesian representation $(n_xn_yn_z)$, it is written as

\[
(\lambda, \mu) = (0, 12) : (002)^4(011)^4(020)^4,
\]

where the configuration of 12 nucleons are explicitly shown, and the $^{16}\text{O}$ core which corresponds to $(000)^4(100)^4(010)^4(001)^4$ is omitted. In a same manner, the prolate deformed state is approximated by the $(\lambda, \mu) = (12, 0)$ representation,

\[
(\lambda, \mu) = (12, 0) : (002)^4(011)^4(101)^4,
\]

in which the orbit occupied by the last four nucleons is different from that in the ground state. The excitation of the last four nucleons into $pf$-shell yields the SD configuration which is given by the $(\lambda, \mu) = (20, 4)$ representation.

\[
(\lambda, \mu) = (20, 4) : (002)^4(011)^4(003)^4.
\]

The Bayman-Bohr theorem tells that these $SU(3)$ shell model wave functions are identical to the cluster model wave functions with the zero inter-cluster distance. First, I consider $^{24}\text{Mg} + \alpha$ cluster configurations. The triaxially deformed ground state of $^{24}\text{Mg}$ is given by the $(\lambda, \mu) = (8, 4)$ representation,

\[
(\lambda, \mu) = (8, 4) : (002)^4(011)^4,
\]

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where the longest deformation axis is $z$-axis. Now I consider the $^{24}\text{Mg} + \alpha$ configuration in which the $\alpha$ cluster is placed on the $y$-axis as illustrated in Fig. 4.10 (a). This configuration corresponds to the $^{24}\text{Mg} + \alpha$ (T) configuration obtained by AMD calculation. Since the orbits (000) and (010) are already occupied by the nucleons in $^{24}\text{Mg}$ cluster, the nucleons in the $\alpha$ cluster having (000)$^4$ configuration must occupy (020)$^4$ at zero inter-cluster distance $d = 0$. As a result, one sees that the $^{24}\text{Mg} + \alpha$ (T) cluster configuration becomes identical to the (0 12) representation for the ground state given by Eq. (4.2),

$$
\lim_{d \to 0} \Phi_{^{24}\text{Mg} + \alpha(T)}(d) = (002)^4(011)^4(020)^4.
$$

(4.5)

This clearly explains why the $^{24}\text{Mg} + \alpha$ (T) configuration obtained by the $d$-constraint becomes almost identical to the ground state configuration at small inter-cluster distance $d$. The placement of the $\alpha$ particle on $z$-axis corresponds the another configuration $^{24}\text{Mg} + \alpha$ (A) as shown in Fig. 4.10 (b). In this case, at zero inter-cluster distance, the nucleons in the $\alpha$ cluster occupy (003)$^4$ resulting in the SD configuration given by Eq. (4.3),

$$
\lim_{d \to 0} \Phi_{^{24}\text{Mg} + \alpha(A)}(d) = (002)^4(011)^4(003)^4.
$$

(4.6)
The other cluster configurations are also considered in the same way. The pro-
lately deformed ground state of $^{20}$Ne is given by the $(\lambda, \mu) = (8, 0)$ representation,

$$(\lambda, \mu) = (8, 0) : \text{(002)}^4, \quad (4.7)$$

where the symmetry axis of $^{20}$Ne is $z$-axis. The $^{20}$Ne + $^8$Be configuration corresponds
to the placement of $^8$Be cluster on $y$-axis where the symmetry axis of $^8$Be is also
$z$-axis (Fig. 4.10 (c)). At zero inter-cluster distance, the nucleons in the $^8$Be cluster
having $(000)^4 (001)^4$ configuration occupy $(020)^4 (011)^4$ due to the Pauli principle,
and one finds it is identical to the ground state configuration,

$$\lim_{d \to 0} \Phi_{\text{Ne+Be}}(d) = (002)^4 (011)^4 (020)^4. \quad (4.8)$$

The oblatelly deformed ground state of $^{12}$C is given by the $(\lambda, \mu) = (0, 4)$ representa-
tion,

$$(\lambda, \mu) = (0, 4) : \text{(000)}^4 (100)^4 (010)^4, \quad (4.9)$$

where the symmetry axis is $z$-axis. The $^{16}$O + $^{12}$C configuration corresponds the
placement of $^{16}$O and $^{12}$C clusters on $z$-axis as shown in Fig. 4.10 (d). At zero
inter-cluster distance, it is identical to the prolate deformed state,

$$\lim_{d \to 0} \Phi_{\text{O+C}}(d) = (002)^4 (011)^4 (101)^4. \quad (4.10)$$

Thus, considering the corresponding $SU(3)$ shell model wave function, the duality of
the mean-field and cluster configurations illustrated in Fig. 4.9 is clearly explained.
It is also noted that the duality of the shell and cluster in $^{28}$Si was also investigated
and found by the Skyrme Hartree-Fock calculation [98].

As discussed in Refs. [10, 70, 77], this duality of shell and cluster means that the
degree-of-freedom of $^{28}$Mg + $^\alpha$ (T) and $^{20}$Ne + $^8$Be cluster excitations are embedded
in the ground state. Therefore, the excitation of the inter-cluster motion embedded
in the ground state yields excited cluster states with pronounced $^{24}$Mg + $^\alpha$ (T) and
$^{20}$Ne + $^8$Be configurations. The important fact is that the IS monopole and dipole
transitions between the ground state and these excited cluster states are very strong,
and hence, these transitions are very good probe for the clustering. Indeed, if one
assumes that the ground state is a pure $SU(3)$ shell model state and the excited
cluster states are described by the cluster model wave function, it is possible to
analytically show the enhancement of the IS monopole and dipole transitions as
made in chapter 2 [70, 77]. However, in the case of $^{28}$Si, the ground state deviates

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Figure 4.11: (a) The cluster $S$-factors for the ground and excited $0^+$ states. (b) Same as (a) but for the excited $1^-$ states. (c) The IS monopole transition matrix from the ground state to the excited $0^+$ states. (d) The IS dipole transition matrix from the ground state to the excited $1^-$ states.

from a pure $SU(3)$ shell model state because of the strong influence of the spin-orbit interaction. In addition to this, the coupling between the cluster configurations and mean-field configurations in the excited states is not negligible. Therefore, the evaluation of the transition strengths by the realistic nuclear model is indispensable for the quantitative discussions. For this purpose, I present the results of the GCM calculation below.

**Isoscalar monopole and dipole transitions**

Using the wave functions of the ground and excited cluster states obtained by the GCM calculation, the IS monopole and dipole transition strengths are directly evaluated. The transition operators and matrix elements between the ground and excited states are given in Eqs. and . I also calculated the cluster $S$-factors of the ground and exited states to see how the clustering and IS monopole and dipole transitions are correlated to each other.

The results are shown in Fig. 4.11 where the panels (a) and (b) show the cluster $S$-factors of $0^+$ and $1^-$ states, while the panels (c) and (d) shows the IS monopole and dipole transition matrices. The calculated $S$-factors confirm the clustering systematics summarized in Fig. 4.9. Owing to the duality of the shell and cluster, the ground state has large $S$-factor for the $^{24}\text{Mg} + \alpha$ (T) configuration which amounts to 0.68 and has non-negligible $S$-factor for the $^{20}\text{Ne} + ^8\text{Be}$ configuration. The $0^+_6$ state is regarded as the pronounced $^{24}\text{Mg} + \alpha$ (T) cluster state, while the $0^+_5(0^+_2)$ state is regarded as the pronounced $^{20}\text{Ne} + ^8\text{Be}$ cluster states from their $S$-factors. One
also sees that the $1^-_I$ state and a group of $1^-_I$ state denoted by $1^-_V$ are the $^{24}\text{Mg} + \alpha$ (T) cluster states and paired with the ground state and the $0^+_6$ state, respectively, while the $1^-_H$ state should be paired with the $0^+_5$ state to constitute the $^{20}\text{Ne} + ^8\text{Be}$ doublet. In the same way, the $S$-factors clearly show that the $0^+_5$ and $1^-_H$ states are the $^{16}\text{O} + ^{12}\text{C}$ doublet, and the $0^+_4$ and $1^-_V$ states are the $^{24}\text{Mg} + \alpha$ (A) doublet although the $^{16}\text{O} + ^{12}\text{C}$ configuration is also mixed in the $1^-_V$ states.

It is very impressive to see that the IS monopole and dipole strengths shown in the panels (c) and (d) are very strongly correlated to the $S$-factors for the $^{24}\text{Mg} + \alpha$ (T) and $^{20}\text{Ne} + ^8\text{Be}$ configurations, but almost insensitive to the other cluster and non-cluster states except for a couple of $1^-_I$ states around 17 MeV. Because the IS monopole and dipole operators activate the degree-of-freedom of cluster excitation embedded in the ground state, the transition strengths to the $^{24}\text{Mg} + \alpha$ (T) and $^{20}\text{Ne} + ^8\text{Be}$ cluster states are as strong as the Weisskopf estimates which are given as,

$$M^{\text{IS}0}_{WU} = \frac{3}{5}(1.2A^{1/3})^2 \simeq 8.0 \text{ fm}^2 \quad (4.11)$$

$$M^{\text{IS}1}_{WU} = \sqrt{\frac{3}{16\pi}}(1.2A^{1/3})^3 \simeq 11.8 \text{ fm}^2. \quad (4.12)$$

Therefore, I can conclude that the IS monopole and dipole transitions are good probe to identify the $^{24}\text{Mg} + \alpha$ (T) and $^{20}\text{Ne} + ^8\text{Be}$ clustering.

Recently, an interesting and promising experimental data was reported by the measurement of the $^{28}\text{Si}(\alpha, \alpha')^{28}\text{Si}^*$ inelastic scattering [64, 65]. It was found that a couple of $0^+$ states above 9 MeV are strongly populated and deduced to have large IS monopole transition strengths. Hence, they are suggested as strong candidates of the $\alpha$ cluster states [65]. I expect that the detailed comparison of the IS monopole and dipole transition strengths between experiment and theory will reveal the clustering systematics in $^{28}\text{Si}$.

### 4.2.4 Summary for $^{28}\text{Si}$

I have investigated the clustering systematics in $^{28}\text{Si}$ based on the antisymmetrized molecular dynamics. It is found that the inversion doublet bands with various kinds of parity-asymmetric cluster configuration appears in the excited states, and the IS monopole and dipole transitions are good probe for $^{24}\text{Mg} + \alpha$ (T) and $^{20}\text{Ne} + ^8\text{Be}$ cluster states.

The energy variation by using $d$-constraint yielded various kinds of cluster configurations with positive- and negative-parity, while the constraint on the quadrupole
deformation yielded mean-field configurations. It is found that the cluster configurations become identical to the mean-field configurations at small inter-cluster distance because of the duality of mean-field (shell) and cluster. In particular, it is emphasized that the oblate deformed ground state has the duality of the $^{24}\text{Mg} + \alpha (T)$ and $^{20}\text{Ne} + ^8\text{Be}$ configurations.

The GCM calculation showed that a group of the $^{24}\text{Mg} + \alpha (T)$ and $^{20}\text{Ne} + ^8\text{Be}$ cluster bands are generated by the excitation of the inter-cluster motion embedded in the ground state. In addition to them, the prolate and SD bands have the duality of $^{16}\text{O} + ^{12}\text{C}$ and $^{20}\text{Ne} + ^8\text{Be}$ clustering, respectively. Because of their parity-asymmetric intrinsic configurations, they are accompanied by the negative-parity bands to constitute the inversion doublets.

Because of the duality of the ground state, it is numerically shown that the $^{24}\text{Mg} + \alpha (T)$ and $^{20}\text{Ne} + ^8\text{Be}$ cluster bands have enhanced IS monopole and dipole transition matrices which are as large as the Weisskopf estimates. On the other hand, other cluster states and non-cluster states are rather insensitive to the IS monopole and dipole transitions. Hence, I conclude that the $^{24}\text{Mg} + \alpha (T)$ and $^{20}\text{Ne} + ^8\text{Be}$ cluster bands can be identified from their enhanced transitions. I expect that more quantitative comparison with the experiments will reveal the clustering systematics in $^{28}\text{Si}$. 

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Chapter 5

Summary of this thesis

In this thesis, I discussed IS monopole and dipole transitions to cluster states. The observation of $1^-$ states is a key to prove parity-asymmetric structure. I have shown that IS dipole transition strongly populates those asymmetric cluster states and hence, it is regarded as a good probe for $1^-$ states.

First, I performed analytical calculations to estimate the magnitude of the IS monopole and dipole transition matrix between the ground states and the cluster states. By rewriting IS monopole and dipole operators, it was shown that the transitions brings about nodal and angular excitations into the inter-cluster motion. Therefore, the IS monopole and dipole transition have the potential to activate the degree-of-freedom of the cluster excitation embedded in the ground state.

By assuming the ground state is described by a shell model wave function, we derived an analytical expression of IS transition matrix and demonstrated that that transition matrix is enhanced and is on the order of the Weisskopf estimate, even if the ground state has an ideal shell model structure.

To provide realistic and reliable results for IS monopole and dipole strengths, I performed nuclear structure calculations for $^{24}$Mg and $^{28}$Si using AMD For $^{24}$Mg, I evaluated the IS monopole strength distribution and analyzed the cluster structure of the prominent peaks in the distribution. The properties of the calculated IS monopole strength distribution reasonably agree with the experimental data and mean-field calculation. By analyzing the structure of the prominent peaks, it was found that $^{20}$Ne+$\alpha$ and $^{12}$C+$^{12}$C cluster states appear around $E_x = 15$ MeV and the $6\alpha$ cluster states appears in the GMR region. Very similar structure to the present results was found in the recent experiment, especially in the below of GMR. It suggests that the prominent peaks observed around 15 MeV are originated from the $^{20}$Ne+$\alpha$ and $^{12}$C+$^{12}$C clustering. For $^{28}$Si, I calculated the positive- and negative-parity
energy levels and showed that various kind of asymmetric cluster states constitute to inversion-doublet bands. Because of the duality of the ground state, $^{24}\text{Mg}+\alpha(T)$ and $^{20}\text{Ne}+^{8}\text{Be}$ cluster bands have the enhanced IS monopole and dipole transition matrices which are as large as the Weisskopf estimates. On the other hand, the other cluster states and non-cluster states are rather insensitive against IS monopole and dipole transitions. Therefore, I conclude that the $^{24}\text{Mg}+\alpha(T)$ and $^{20}\text{Ne}+^{8}\text{Be}$ cluster bands can be identified from their enhanced transitions. These results are in good accordance with the experimental data and show the cluster states are strongly excited by IS monopole and dipole transitions despite deviation from ideal shell model structure in the ground state and mixing of shell and cluster structures. I conclude that IS monopole and dipole transitions are promising probes for parity-asymmetric clustering.
Appendices
Appendix A

Definition of $Q_1$ and $Q_2$ in constraint on harmonic oscillator quanta

H.O. $SU(3)$ irreducible representation is labeled by $\lambda$ and $\mu$. $\lambda$ and $\mu$ are related to the H.O. quantum numbers.

$$\lambda = N_z - N_y, \quad (A.1)$$
$$\mu = N_y - N_x. \quad (A.2)$$

where $N_x$, $N_y$ and $N_z$ are the H.O. quanta and satisfy $N_x \leq N_y \leq N_z$. Clearly, $\lambda$ and $\mu$ depend on the choice of space-coordinate axes. Because the condition $N_x \leq N_y \leq N_z$ is not always satisfied in variational calculation, $\lambda$ and $\mu$ are not suitable as constraint quantities. Instead of them, I introduce $Q_1$ and $Q_2$ as constraint quantities. I define $Q_1$ and $Q_2$ as coefficients of characteristic polynomial of $\eta'_{\sigma\tau} = \eta_{\sigma\tau} - \frac{N^3}{3}\delta_{\sigma\tau}$

$$\det \{\alpha I - \eta'\} = \alpha^3 + Q_1\alpha - Q_2. \quad (A.3)$$

where the term of the order of $\alpha^2$ vanishes because $\text{Tr} \eta' = 0$. Since the characteristic polynomial is invariant under orthogonal transformation of space-coordinate, $Q_1$ and $Q_2$ are independent with the definition of space-coordinate. By comparing both sides of Eq. (A.3), $Q_1$ and $Q_2$ are given in the following forms,

$$Q_1 = \text{Tr} \eta'^2, \quad (A.4)$$
$$Q_2 = \det \eta'. \quad (A.5)$$
$Q_1$ and $Q_2$ are also expressed with $\lambda$ and $\mu$.

\begin{align*}
Q_1 &= \frac{2}{3} (\lambda^2 + \lambda \mu + \mu^2), \quad &\text{(A.6)} \\
Q_2 &= \frac{1}{9} (\lambda - \mu)(2\lambda + \mu)(\lambda + 2\mu). \quad &\text{(A.7)}
\end{align*}

In practical calculation, I evaluate constraint conditions $Q_1$ and $Q_2$ from $\lambda$ and $\mu$ using Eqs. (A.6) and (A.7) and calculate $Q_1$ and $Q_2$ in energy variation using Eqs. (A.4) and (A.5).
Appendix B

Isoscalar dipole operator in cluster coordinate

I consider the $A$ nucleon system composed of the clusters with mass $C_1$ and $C_2$ ($C_1 + C_2 = A$), and give the representation of $\hat{O}_{IS1}$ in terms of the internal coordinates $\xi_i$ within each cluster and the inter-cluster coordinate $r$. The IS dipole operator is rewritten as,

$$\hat{O}_{IS1} = \sum_{i=1}^{A} (r_i - r_{\text{c.m.}})^2 \mathcal{Y}_{1\mu}(r_i - r_{\text{c.m.}})$$

$$= \sum_{i \in C_1} \left( \xi_i + \frac{C_2}{A} r \right)^2 \mathcal{Y}_{1\mu} \left( \xi_i + \frac{C_2}{A} r \right) + \sum_{i \in C_2} \left( \xi_i - \frac{C_1}{A} r \right)^2 \mathcal{Y}_{1\mu} \left( \xi_i - \frac{C_1}{A} r \right)$$

$$= \sum_{i \in C_1} \xi_i^2 \mathcal{Y}_{1\mu}(\xi_i) + \sum_{i \in C_2} \xi_i^2 \mathcal{Y}_{1\mu}(\xi_i) + \left( \frac{C_2}{A} \sum_{i \in C_1} \xi_i^2 - \frac{C_1}{A} \sum_{i \in C_2} \xi_i^2 \right) \mathcal{Y}_{1\mu}(r) - \frac{C_1 C_2 (C_1 - C_2)}{A^2} r^2 \mathcal{Y}_{1\mu}(r)$$

$$+ 2 \frac{C_2}{A} \sum_{i \in C_1} (\xi_i \cdot r) \mathcal{Y}_{1\mu}(\xi_i) - 2 \frac{C_1}{A} \sum_{i \in C_2} (\xi_i \cdot r) \mathcal{Y}_{1\mu}(\xi_i)$$

(B.1)

where I use the relations $R_{C_1} - r_{\text{c.m.}} = \frac{C_2}{A} r$, $R_{C_2} - r_{\text{c.m.}} = -\frac{C_1}{A} r$, $\sum_{i \in C_1} \xi_i = \sum_{i \in C_2} \xi_i = 0$ and $\mathcal{Y}_{1\mu}(a \alpha + b \beta) = a \mathcal{Y}_{1\mu}(a) + b \mathcal{Y}_{1\mu}(b)$. The terms in the last line of
Eq. (B.1) reads

\[(\xi_i \cdot \mathbf{r}) Y_{1\mu}(\xi_i) = -\frac{4\pi}{\sqrt{3}} [Y_1(\xi_i) \otimes [Y_1(\xi_i) \otimes Y_1(\mathbf{r})]_0]_{1\mu} \]

\[= -\frac{4\pi}{\sqrt{3}} \sum_{l=0,1,2} \sqrt{2l+1} \begin{pmatrix} 1 & 1 & l \\ 1 & 1 & 0 \end{pmatrix} [[Y_1(\xi_i) \otimes Y_1(\xi_i)]_l \otimes Y_1(\mathbf{r})]_{1\mu} \]

\[= \frac{1}{3} Y_{1\mu}(\mathbf{r}) - \sqrt{\frac{8\pi}{9}} [Y_2(\xi_i) \otimes Y_1(\mathbf{r})]_{1\mu} \quad (B.2) \]

Here, I utilize the identities \( \mathbf{a} \cdot \mathbf{b} = -\frac{4\pi}{\sqrt{3}} [Y_1(\mathbf{a}) \otimes Y_1(\mathbf{b})]_{00}, \) \([Y_1(\mathbf{a}) \otimes Y_1(\mathbf{a})]_{1\mu} = 0 \)
and \([Y_1(\mathbf{a}) \otimes Y_1(\mathbf{a})]_{2\mu} = \sqrt{\frac{3}{10\pi}} Y_{2\mu}(\mathbf{a}). \) Substituting Eq. (B.2) to (B.1), Eq.(2.20) is obtained.
Appendix C

Derivation of IS dipole matrix elements

I derive Eq.(2.21) from Eq. (2.20). First, the contributions from first line of the Eq.(2.20) vanishes in the case of the clusters are $SU(3)$-scalar. This is easily proven by counting the principal quantum numbers of the bra and ket states as the same way as the evaluation of IS monopole matrix element. For example, the first term in the Eq. (2.20) yields the matrix element proportional to

$$\left\langle R_{N_000}(r)\phi_{C_1}\phi_{C_2} \left| A\{R_{N10}(r) \left( \sum_{i \in C_1} \xi_i^2 Y_{10}(\xi_i)\phi_{C_1} \right) \phi_{C_2} \} \right. \rightangle$$

(C.1)

I denote the principal quantum number of $\phi_{C_1}$ and $\phi_{C_2}$ as $N_{C_1}$ and $N_{C_2}$. The principle quantum number of the bra state is $N_{C_1} + N_{C_2} + N_0$. On the other hand, that of the ket state is equal to or greater than $N_{C_1} + N_{C_2} + N + 1$ since $\sum_{i \in C_1} \xi_i^2 Y_{10}(\xi_i)$ induces at least $1\hbar\omega$ excitation of $\phi_{C_1}$. Because $N$ is equal to or greater than $N_0 + 1$, the principal quantum number of the ket state is larger than that of the bra state. Hence, this matrix element vanishes. In the same way, the first term in the second line of Eq. (2.20) yields

$$\left\langle R_{N_000}(r)\phi_{C_1}\phi_{C_2} \left| A\{R_{N10}(r) Y_{1-m}(r) \left( \sum_{i \in C_1} Y_{2-m}(\xi_i)\phi_{C_1} \right) \phi_{C_2} \} \right. \rightangle.$$  

(C.2)

The quantum number of $(\sum_{i \in C_1} Y_{2-m}(\xi_i)\phi_{C_1})$ is at least $N_{C_1} + 2$ because $\sum_{i \in C_1} Y_{2-m}(\xi_i)$ generates $2^+$ state of $SU(3)$-scalar cluster $\phi_{C_1}$ which includes at least $2\hbar\omega$ excitation. The principal quantum number of $R_{N10}(r) Y_{1-m}(r)$ is at least $N - 1$. So, the ket
state has larger principal quantum number than that of the bra state and the terms vanish.

The non-vanishing contribution comes from the third line of Eq. (2.20). The first term of the third line yields the matrix element proportional to

\[
\mathcal{R}_{N_000}(r)\phi_{C_1}\phi_{C_2}\left| A\{R_{N_{10}}(r)\mathcal{Y}_{1m}(r)\left(\sum_{i \in C_1} \xi_i^2 \phi_{C_1}\right) \phi_{C_2}\} \right.
\]

(C.3)

In the ket state, the IS monopole operator induces 0 or 2\(\hbar\omega\) excitations of \(\phi_{C_1}\).

\[
\sum_{i \in C_1} \xi_i^2 \phi_{C_1} = \phi_{C_1} \langle \phi_{C_1} | \sum_{i \in C_1} \xi_i^2 | \phi_{C_1} \rangle + (2\hbar\omega \text{ excited } 0^+ \text{ states}).
\]

(C.4)

and \(\mathcal{Y}_{1\mu}(r)\) induces angular excitation into inter-cluster motion with \(\pm 1\hbar\omega\), which changes the principal quantum of inter-cluster motion to \(N \pm 1\). In total, the possible principal quantum number of the ket state is \(N_{C_1} + N_{C_2} + N - 1\), \(N_{C_1} + N_{C_2} + N + 1\) or \(N_{C_1} + N_{C_2} + N + 3\). Since \(N\) is equal to or greater than \(N_0 + 1\) and the quantum number of the bra and ket states must be same, the non-zero contribution from this term only when \(N = N_0 + 1\) From Eq. (C.4) and the identities

\[
\mathcal{R}_{N_{10}}(r)\mathcal{Y}_{1m}(r) = \sqrt{\frac{1}{4\pi}} rR_{N_{10}}(r)Y_{00}(\hat{r}) + \sqrt{\frac{1}{5\pi}} rR_{N_{10}}(r)Y_{20}(\hat{r}),
\]

(C.5)

\[
rR_{N_{10}}(r) = \sum_{N'} \langle R_{N'0}|r|R_{N_{10}} \rangle R_{N'0}(r)
\]

(C.6)

Eq.(C.3) is calculated as,

\[
\sqrt{\frac{1}{4\pi}} \phi_1 \sum_{i \in C_1} \xi_i^2 \phi_1 \sum_{N'} \langle R_{N'0}(r)Y_{00}(\hat{r})\phi_{C_1}\phi_{C_2}\rangle \langle R_{N'0} | r | R_{N_0+1,1} \rangle
\]

\[
= \mu_{N_0} \sqrt{\frac{1}{4\pi}} C_1 \langle r_{C_1}^2 \rangle \langle R_{N_0+1,0} | r | R_{N_0+1,1} \rangle
\]

(C.7)

The last term in the third line of Eq. (2.20) yields

\[
\langle \mathcal{R}_{N_{000}}(r)\phi_{C_1}\phi_{C_2}| A\{R_{N_{10}}r^2\mathcal{Y}_{10}(r)\phi_{C_1}\phi_{C_2}\} \rangle
\]

(C.8)

where \(r^2\mathcal{Y}_{10}\) brings about the nodal and angular excitations of the inter-cluster motion with \(\pm 1\hbar\omega\) or \(\pm 3\hbar\omega\). Hence, the matrix element remains for \(N = N_0 + 1\) and
$N_0 + 3$ case. In the similar way, Eq. (C.8) becomes

$$
\mu_{N_0} \sqrt{\frac{1}{4\pi}} \langle R_{N_00}|r^3|R_{N_1}\rangle \quad (C.9)
$$

with $N = N_0 + 1$ or $N_0 + 3$. From those results, an analytical expression for the reduced matrix element is derived as Eq. (2.21).
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Chapter 6

Acknowledgements

The author appreciates with Prof. Kimura Prof. Horiuchi, Prof. Aikawa and Prof. Hirabayashi for their very kind advices and encouragements. The author thanks Prof. Taniguchi for fruitful discussions on clustering in $^{28}$Si and Prof. Kawaba for providing the φ-inelastic scattering data of $^{24}$Mg. The numerical calculations were performed on HIATACHI SR16000 at KEK and YITP. This work was supported by JSPS KAKENHI Grant Number JP1603654.